CHAPTER

12

Sensitivity Analysis

CHAPTER CONCEPTS

After reading this chapter, you will be able to explain the following concepts:

- Sensitivity analysis
- Risk sensitivity profile
- Value of clairvoyance sensitivity

12.1 INTRODUCTION

Sensitivity analysis enables us to investigate how the decision will change if we change certain numbers in the decision basis. By doing that, we can determine whether additional effort should be expended in increasing the precision of the numbers we have used. **Sensitivity analysis** is an important feature of professional decision analysis, where we must continually refine and focus our attention on the important aspects of the problem. We shall discuss the subject of sensitivity analysis in further sections. However, let us now illustrate its use in the party problem.

12.2 KIM'S SENSITIVITY TO PROBABILITY OF SUNSHINE

Suppose that Kim is concerned about how sensitive her decision is to the probability of Sunshine she assigns; she has currently assigned 0.4. She may want to know whether small changes in this probability will change the best alternative and/or substantially change the value of the party to her. She also may be interested in a sensitivity analysis to this probability because she expects to receive additional information and wants to know how she should adapt her party strategy in the face of a new probability of Sunshine. Knowing how sensitive the decision is to this probability also helps Kim to decide how hard she needs to work on the probability assignment to accurately represent her belief.

To explore this issue, we let p be the probability of sunshine that she assigns and consider how the analysis of the problem will depend on p. In Figure 12.1, we draw Kim's decision tree for a probability of sunshine p replacing the value of 0.4 we used in Figure 9.5.

The u-values of the three alternatives are easily computed. The u-value of the Outdoor alternative is simply p.

The u-value of the Porch alternative is

$$0.950p + 0.323(I - p) = 0.323 + 0.627p$$

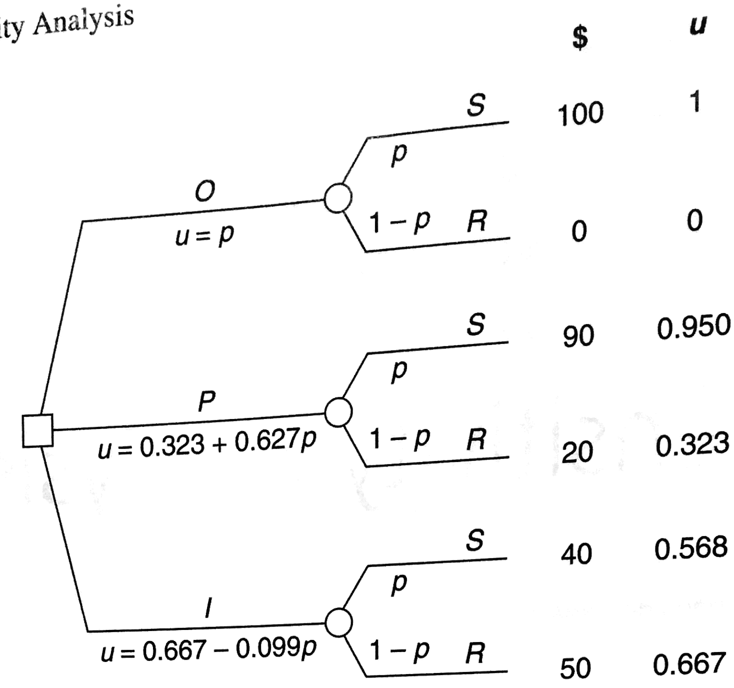

FIGURE 12.1 Kim's Decision Tree for *p*

The u-value of the Indoor alternative is

$$0.568p + 0.667(I - p) = 0.667 - 0.099p$$

The u-value for each alternative is shown by the branch for that alternative in the figure. Figure 12.2 shows a plot of these u-values versus p, which is the probability of sunshine as p ranges from 0 to 1.

Each alternative plots as a straight line that joins the u-value of the alternative with $p = 0$ to the u-value of the alternative with $p = 1$. The best alternative to follow is the one with the highest u-value for the particular p. Thus, we see that the best alternative when $p = 0$ is the Indoor alternative. The Indoor line remains highest until it crosses the Porch line at a value of $p = 0.47$. Then the Porch line is highest until it crosses the Outdoor line at $p = 0.87$. So, for probabilities of sunshine less than 0.47, the Indoor alternative is best; for probabilities of sunshine greater than 0.87, the Outdoor alternative is best; and otherwise the Porch alternative is best. Note that the value of probability of sunshine that Kim originally assigned, 0.4, leads her to follow the Indoor alternative, as we found in Figure 10.7. You can see that Kim will only move to the Outdoor alternative when she is very sure that the weather will be sunny.

One easy way to draw Figure 12.2 is to create on the right-hand side of the figure a dollar scale that is distorted according to Kim's u-curve of Figure 10.4. For example, we plot the point corresponding to \$50 on the right of Figure 12.2 by observing from Figure 10.4 that \$50 corresponds to a u-value of 0.667. Once we have constructed this distorted dollar scale, we can draw the lines corresponding to each alternative by simply connecting the end points that correspond to the dollar values. For example, the Porch line connects the point corresponding to \$20 when $p = 0$ to the point corresponding to \$90 when $p = 1$.

We can also use this figure to show the u-value of free clairvoyance. Recall that if Kim receives free clairvoyance on the weather and finds that the weather will be sunny, then she will have a party outdoors with a value of \$100. If she finds that the weather will be rainy, then she will have her party indoors with a value of \$50. Consequently, if the probability of sunshine is p,

The u-value of free clairvoyance is

$$pu(100) + (1 - p)u(50)$$

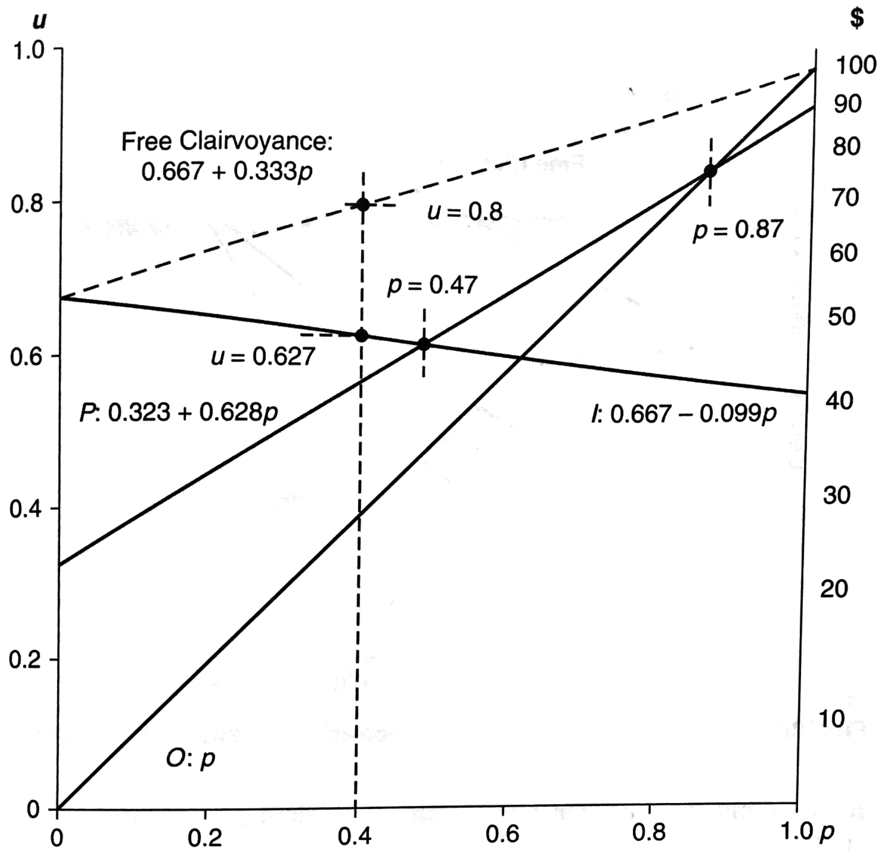

FIGURE 12.2 **Kim's u-Value Sensitivity to Probability of Sun, p**

This is the line that connects the point corresponding to \$50 when $p = 0$ to the point corresponding to \$100 when $p = 1$. In Figure 12.2, it is shown as a dotted line. Note that it connects the highest point on the $p = 0$ axis to the highest point on the $p = 1$ axis. It is clear that the alternative consisting of making a decision *after* receiving free clairvoyance is better than any of the other alternatives in the problem without clairvoyance, unless $p = 0$ or $p = 1$.

12.3 CERTAIN EQUIVALENT SENSITIVITY

From Figure 12.2, we can determine the u-value for any probability of sunshine for any given alternative. By reading from the distorted dollar scale, we can also obtain the corresponding **certain equivalent**.

However, it will be more convenient to perform this operation on the whole figure, so we obtain a plot of Kim's certain equivalent sensitivity for probability of sunshine p, as shown in Figure 12.3.

Note that these plots showing how the certain equivalent of each alternative depends on the probability of sunshine are not straight lines because of the distortion in the dollar scale of Figure 12.2. We see, for example, that in accordance with Figure 10.7, for Kim's original probability of sunshine $p = 0.4$, the Indoor, Porch, and Outdoor alternatives have certain equivalents of \$45.83, \$40.60, and \$25.73, respectively.

Figure 12.3 also shows that, as the probability of sunshine varies from 0.2 to 0.6, Kim's certain equivalent will be much less sensitive to this change if she follows the Indoor alternative than it would be if she followed the Porch alternative. For the Indoor alternative, there is only about a \$4 difference in certain equivalents over this range, while for the Porch alternative there is a \$24 difference.

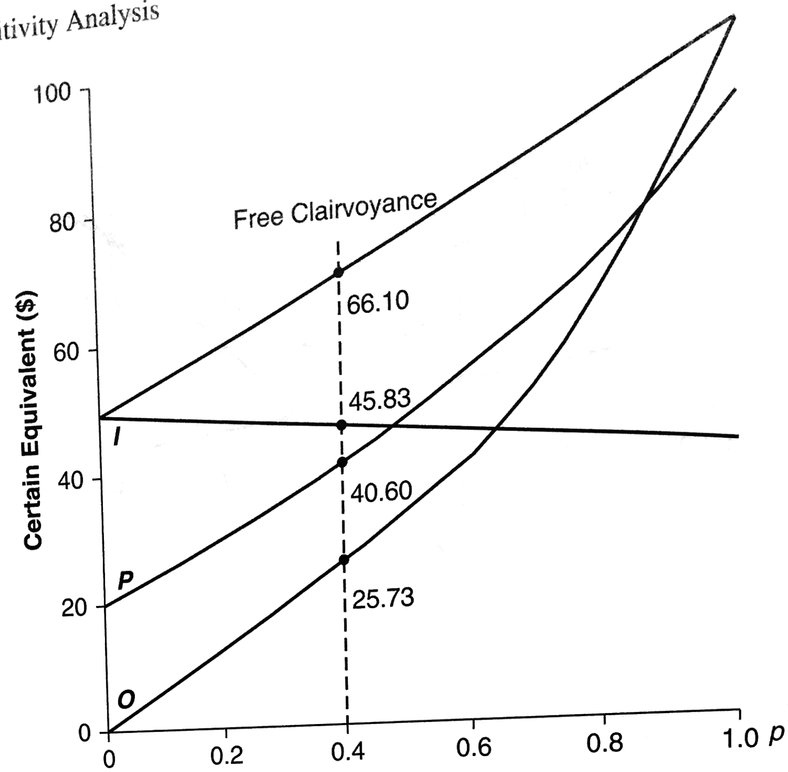

FIGURE 12.3 Kim's Certain Equivalent Sensitivity to Probability of Sun, *p*

Also, at this point, we see that her certain equivalent for free clairvoyance is \$66.10. Since Kim satisfies the delta property, this means that the value to her of free clairvoyance is \$66.10 − \$45.83, or \$20.27, which is in agreement with our earlier results.

12.4 VALUE OF CLAIRVOYANCE SENSITIVITY TO PROBABILITY OF SUNSHINE

Because Kim wishes to accept the delta property, we can refer to Figure 12.3 to define her value of clairvoyance (VOC) for any probability of sunshine p. Specifically, we need to compute the difference between the certain equivalent for free clairvoyance and the certain equivalent for the best alternative. The result of the computation appears in Figure 12.4.

We observe that there will be no value of clairvoyance when p is either 0 or 1, because at these points, Kim would have clairvoyance. Otherwise, in this problem, there will be a positive value to free knowledge of the weather.

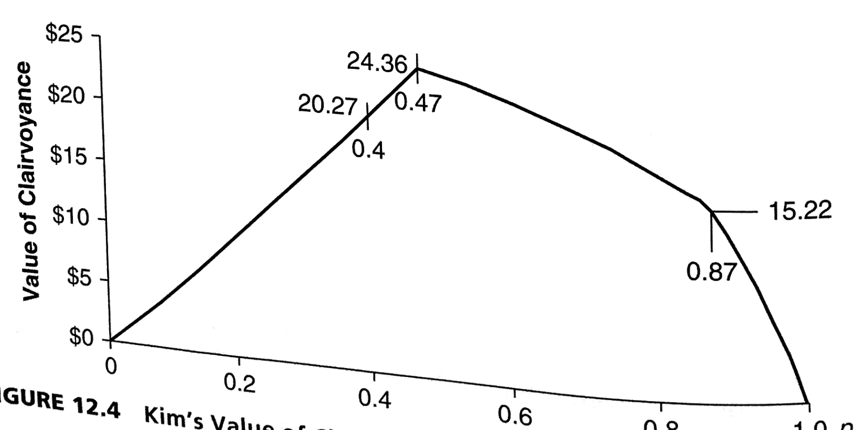

FIGURE 12.4 Kim's Value of Clairvoyance Sensitivity to Probability of Sun, *p*

As p increases from 0, the value of clairvoyance also increases, reaching a peak of about \$24.36 at $p = 0.47$. Note that the value is \$20.27 at $p = 0.4$. When p exceeds 0.47, the value of clairvoyance falls, and when p exceeds 0.87, it falls even more rapidly. For probabilities of sunshine that range from about 0.3 to 0.85, we can see that clairvoyance will be worth at least \$15.

12.5 JANE'S SENSITIVITY TO PROBABILITY OF SUNSHINE

We can perform this analysis even more easily for the risk-neutral Jane. Figure 12.5 shows Jane's decision tree for the general probability of sunshine p, in correspondence with Figure 10.11.

Since Jane's certain equivalents are simply her *e*-values of monetary values, we find her certain equivalent for the Outdoor alternative is simply $100p$.

For the Porch alternative, her certain equivalent is

$$90p + 20(1 - p) = 20 + 70p$$

For the Indoor alternative, it is

$$40p + 50(1 - p) = 50 - 10p$$

We plot these certain equivalents in Figure 12.6.

The certain equivalent for each alternative is a straight line connecting the dollar values for $p = 0$ and $p = 1$. Again, as p increases from 0 through 1, the best alternative will change from Indoor to Porch to Outdoor; however, the points of change are different from those for Kim. Jane will change from the Indoor to the Porch alternative when p crosses 0.375, and from the Porch to the Outdoor alternative when p crosses 0.667.

Thus, she is more willing than Kim to follow the more exposed alternatives for lower probabilities of sunshine. We see that for Jane, the value of $p = 0.4$ corresponds to the Porch region. Her certain equivalents are shown for the Indoor, Porch, and Outdoor alternatives for $p = 0.4$ as \$48, \$46, and \$40.

If Jane is given free clairvoyance, her certain equivalent is represented in Figure 12.6 by the dotted line that connects the \$100 value she will receive when $p = 1$ by having the party outdoors, and the \$50 value she will receive when $p = 0$ by having the party indoors. When $p = 0.4$, this line shows that Jane's certain equivalent of free clairvoyance will be \$70. Since

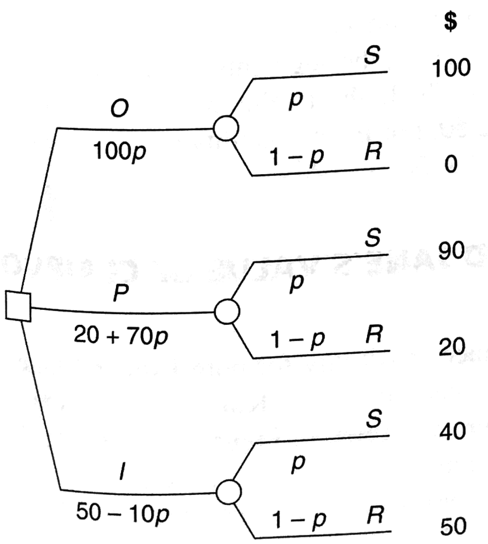

FIGURE 12.5 Jane's Decision Tree for General Probability of Sun, *p*

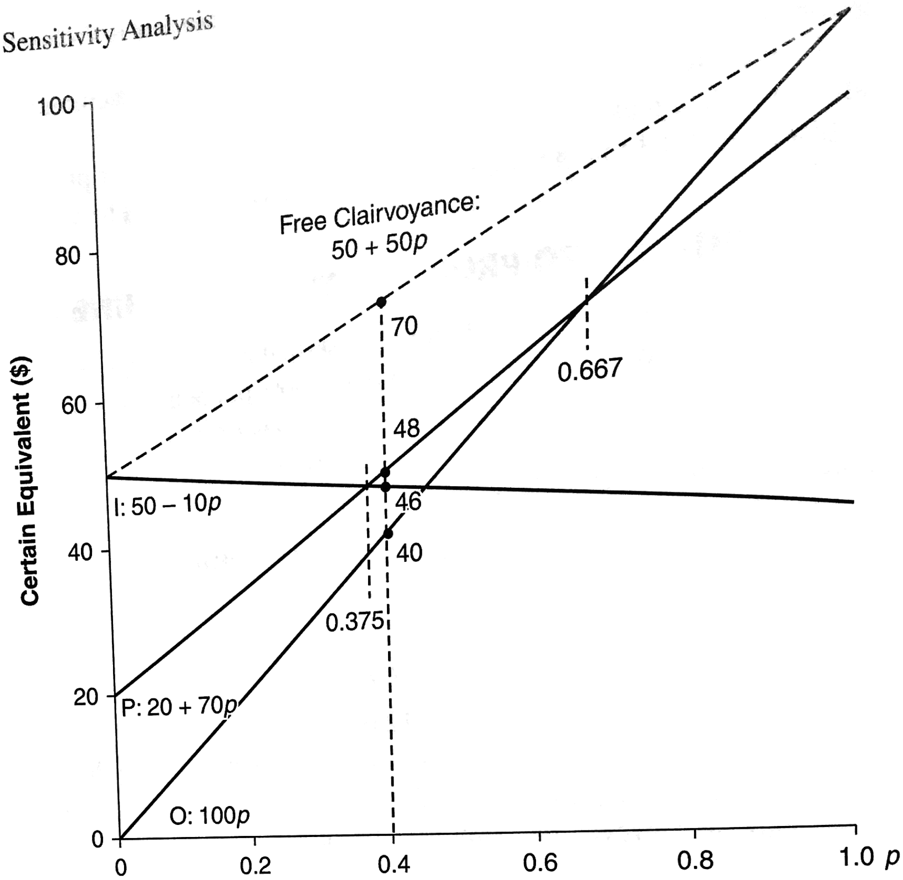

FIGURE 12.6 Jane's Certain Equivalent Sensitivity to Probability of Sun, *p*

Jane is risk-neutral, she satisfies the delta property. Consequently, we can obtain her value of clairvoyance by subtracting the $48 certain equivalent of her otherwise best alternative, having the party on the porch, from the $70 certain equivalent of having free clairvoyance. The result is the $22 value of clairvoyance for Jane that we computed originally in Figure 10.16, at $p = 0.4$.

To obtain Jane's value of clairvoyance sensitivity to p, as shown in Figure 12.7, we can perform this same computation. Specifically, we subtract the certain equivalent of the best alternative for a particular p from the certain equivalent of having free clairvoyance at the same p.

Just as for Kim, Jane will have no value of clairvoyance when $p = 0$ or $p = 1$, because she will already have clairvoyance at these points. Otherwise, clairvoyance will have a positive value to Jane. As you can see from the construction, the segments of Jane's value of clairvoyance sensitivity curve will all be straight lines. The highest value of clairvoyance occurs at the crossover point $p = 0.375$, and here it is $22.50. For the probability of sunshine 0.4, the value of clairvoyance is $22, as we have observed.

12.6 COMPARISON OF KIM'S AND JANE'S VALUE OF CLAIRVOYANCE SENSITIVITIES

We can now compare the value of clairvoyance sensitivity for both Kim and Jane. Note that, from Figure 12.4, when the probability of sunshine is 0.4, Kim's value of clairvoyance is $20.27, whereas Jane's value of clairvoyance, as shown in Figure 12.7, is $22. In this case, Jane, as the risk-neutral person, is willing to pay more than Kim for clairvoyance. But would this always hold true?

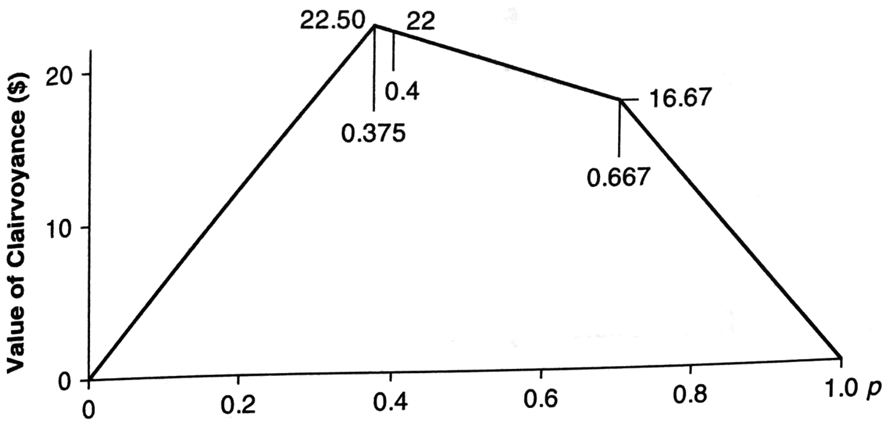

FIGURE 12.7 Jane's Value of Clairvoyance Sensitivity to Probability of Sun, *p*

While many people think so, the answer is no. Compare the two figures when $p = 0.5$. At this point Kim's value of clairvoyance is about \$24, while Jane's is \$20. The order of their values is reversed for this different probability of sun.

When $p = 0.5$, both Kim and Jane will follow the Porch alternative and be more exposed to the weather. Kim, as a risk-averse person, is more concerned about this and, consequently, is willing to pay more for clairvoyance regarding the weather. However, when $p = 0.4$, Kim has already switched to the Indoor alternative, while Jane is still following the Porch alternative. Kim, as we found in the sensitivity analysis, is now much less exposed to the weather, so her value of clairvoyance becomes less than Jane's. Yet we note that when $p = 0.3$, both Kim and Jane are following the Indoor alternative—the least exposed—yet Jane is willing to pay more (\$18) for clairvoyance than Kim (\$14.87). Even in such a simple problem as this, intuition can often be misleading.

By superimposing the value of clairvoyance sensitivities for both Kim and Jane, as shown in Figure 12.8, we observe that these two people who share everything but risk attitude will, in fact, have quite different dependence of value of clairvoyance on probability of the Sun *p*.

***Note:** In Figure 12.8, we see that Jane will be willing to pay more for clairvoyance for values of p below 0.423, and that Kim will be willing to pay more for clairvoyance for values of p above 0.423. It is clear that even in this simplest nontrivial decision problem with three alternatives, you cannot make any generalizations about whether increasing risk-aversion will increase or decrease the value of clairvoyance.*

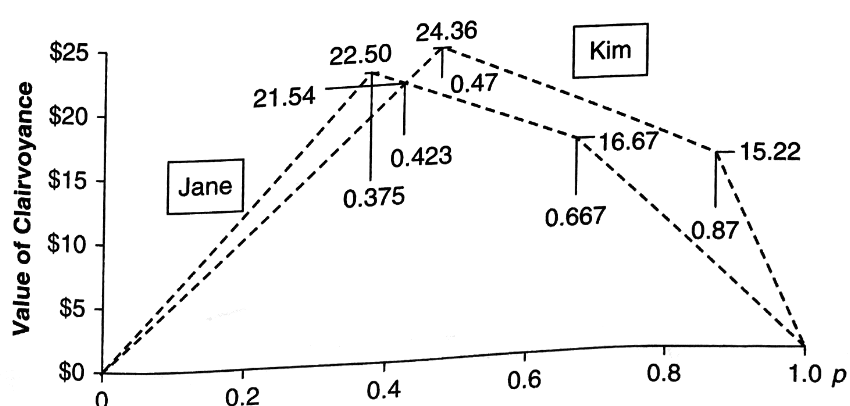

FIGURE 12.8 Value of Clairvoyance Sensitivity to *p* for Kim and Jane

12.7 RISK SENSITIVITY PROFILE

So far, we have conducted sensitivity analysis only to the probability of sunshine in the party problem. In principle, we can change any of the parameters of the decision situation. The **risk sensitivity profile** is a plot of the certain equivalent of a deal as a function of the risk-aversion coefficient. While such a plot applies strictly only to the preferences of deltapeople, it is still useful in obtaining a feel for the effect of risk attitude on certain equivalent in general.

EXAMPLE 12.1 Risk Sensitivity Profile

To examine the concept, let us consider the two deals shown in Figure 12.9.

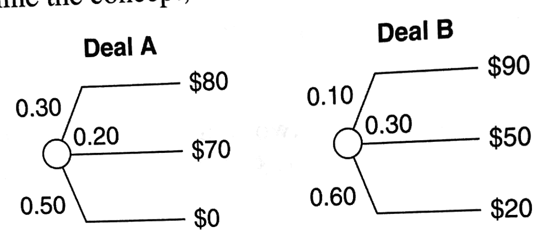

FIGURE 12.9 Examining Risk Sensitivity Profiles

Deal A is a 30% chance of \$80, a 20% chance of \$70, and otherwise nothing. Deal B is a 10% chance of \$90, a 30% chance of \$50, and a 60% chance of \$20. If someone were given the choice between the deals, we can imagine a thought that might arise. "With Deal A, I have a 50% chance of nothing, but with deal B, I am assured of at least \$20 and may win as much as \$90." When we examine the deals from the point of view of a risk-neutral person, we find that Deal A has a certain equivalent (*e*-value of dollars) of \$38, whereas Deal B has a certain equivalent of \$36. Therefore, as the risk-aversion coefficient increases, it will be interesting to see if and when Deal B becomes preferable to Deal A.

The risk sensitivity profile of Figure 12.10 reveals the answer.

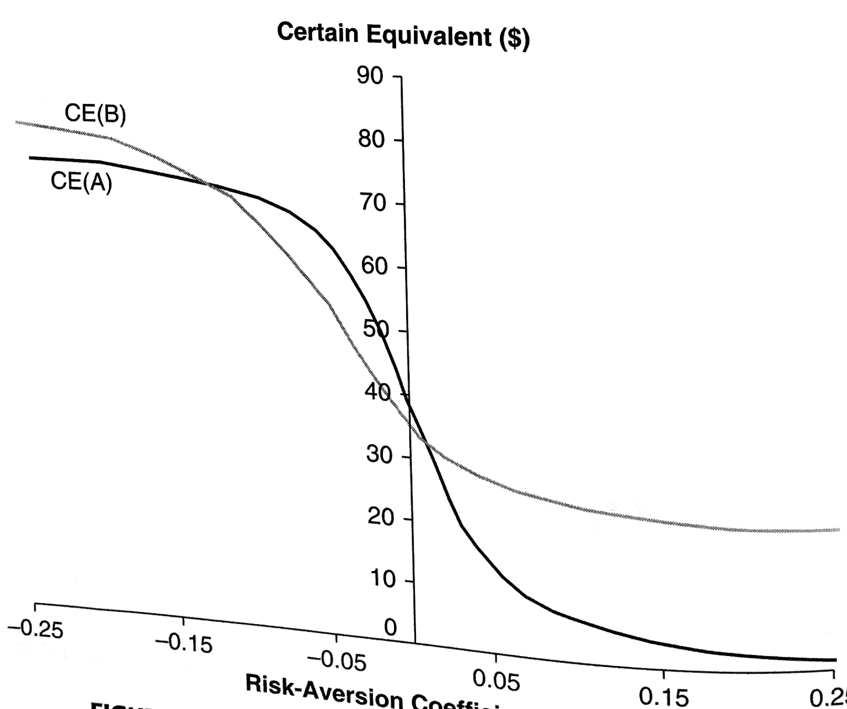

FIGURE 12.10 Risk Sensitivity Profiles with Deals A and B

In Figure 12.10, we have plotted the certain equivalents of the deals against both negative and positive values of the risk-aversion coefficient. We recall that a risk-aversion coefficient of 0 corresponds to risk-neutrality, and we observe that the curves representing each deal are equal to the e-values of the deals when $\gamma = 0$. As γ increases above 0, the certain equivalents of the deals fall. We find that the curves cross and the deals have the same certain equivalent of \$35.00 when $\gamma = 0.00415$ or ρ equals \$241; for positive values of ρ larger than \$241, Deal A will be preferred.

***Note** When the risk-aversion coefficient γ becomes negative, the certain equivalent of the deal increases, the more negative it becomes.*

Of special interest are the limits that the risk sensitivity profiles for each deal approach as γ becomes large and positive or large and negative.

As γ becomes larger and larger in the positive direction, the certain equivalent approaches the smallest payoff in the deal, regardless of its probability.

Extreme risk averters act as if they were sure to receive the worst result. Therefore, in this case, the risk sensitivity profile for deal A approaches 0 as γ becomes large and positive, whereas for Deal B, the profile approaches \$20.

Similarly, as γ becomes large and negative—regardless of probability—the certain equivalent of a deal approaches the largest payoff.

Extreme risk preferrers act as if they are sure to win the highest payoff. The risk sensitivity profiles for large and negative γ show that the certain equivalent of Deal A approaches \$80 and that of Deal B approaches \$90.

12.7.1 Risk Sensitivity in the Party Problem

Speaking of Kim, as we show in Figure 12.11, we can construct the risk sensitivity profile for the deals provided by the location alternatives in the party problem. This figure only shows the

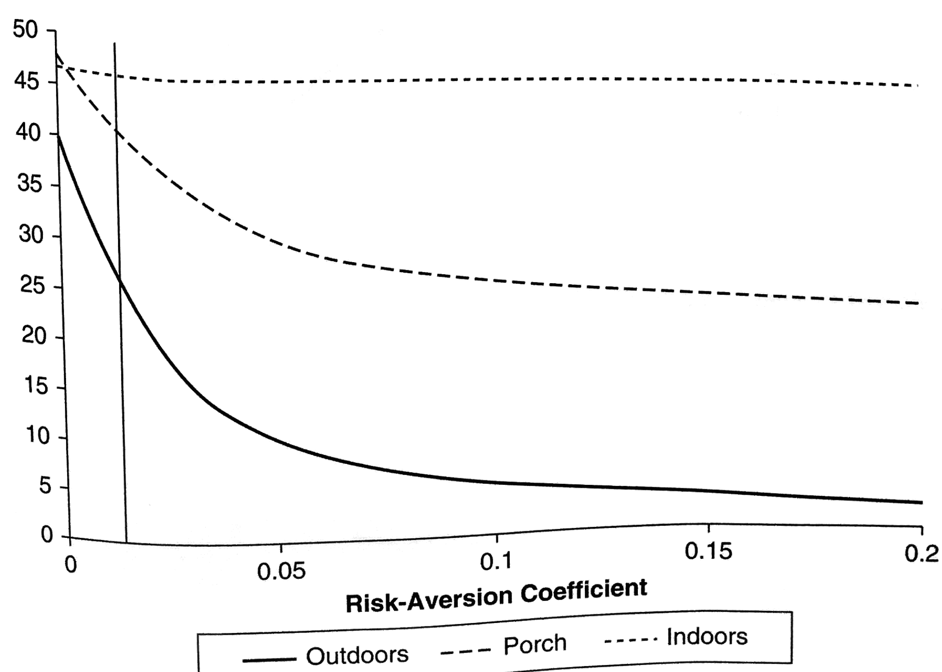

FIGURE 12.11 Kim's Risk Sensitivity Profile for Party Problem

portion of the profile representing risk-aversion, since this is the region of practical interest. When $\gamma = 0$, the curves for the certain equivalents of the Porch, Indoor, and Outdoor alternatives have the values \$48, \$46, and \$40 corresponding to the *e*-values of dollars of risk-neutral Jane. The vertical line at $\gamma = 0.01386$ or $\gamma = \$72.13$ represents the portion of the profile describing Kim's preferences. Here, the alternatives are ranked in the order Indoor, Porch, and Outdoor with certain equivalents \$45.83, \$40, and \$25, respectively. From the profile, we observe that no risk-averse deltaperson will ever prefer the Outdoor alternative to one of the other two, as long as all agree on dollar values and probabilities.

Risk sensitivity profiles are one way to explore the possibilities for creating deals that two or more people will find satisfying. Thus, we see from the profile that although Jane and Kim disagree on where to have the party, Jane would agree with Kim to have the party Indoors instead of on the Porch if Kim paid her about \$2, and that such a payment would still leave Kim with a certain equivalent of about \$44—three dollars more than if she had to settle for a Porch party. Of course, such calculations are only a guide to the possibilities. They rely on complete candor in information and preference, both of which are often absent during actual negotiations.

12.8 SUMMARY

Sensitivity analysis is a useful tool that determines how the decision may change if we change certain numbers of the problem.

KEY TERMS

- Sensitivity analysis
- Certain equivalent
- Risk sensitivity profile