A Note on Risk Attitude for Delta People

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I have put this draft together to clarify some discussion in my office hours in the last couple of weeks. I want to make crystal clear the right answer to what infinitely risk averse and infinitely risk preferring people would have as a certain equivalent. Consider a delta person, whose u-curve is of course

$$u(x) = -e^{-\gamma x}$$

The inverse, needed to compute the certain equivalent is

$$x = -\frac{1}{\gamma}\ln(-u)$$

Consider an uncertain prospect with prices ordered from least preferred to best preferred

$$x_1 < x_2 < x_3 < \dots < x_n$$

and with probabilities

$$p_1, p_2, p_3, \dots p_n$$

For such a delta person facing such a prospect, the expected utility is

$$\langle u \rangle = -\sum_{i=1}^{n} p_i\, e^{-\gamma x_i}$$

To calculate the certain equivalent, we calculate the inverse

$$\tilde{x} = -\frac{1}{\gamma}\ln\left(\sum_{i=1}^{n} p_i\, e^{-\gamma x_i}\right)$$

1 Risk Averse Delta Person

Consider first a risk averse delta person, i.e., $\gamma>0$. In this situation, it is clear based on the ordering of the prizes that

$$\gamma x_1 < \gamma x_2 < \gamma x_3 < \dots < \gamma x_n$$

If we multiply by -1, we reverse the order of the inequalities

$$-\gamma x_1 > -\gamma x_2 > -\gamma x_3 > \ldots > -\gamma x_n$$

Exponentiation is monotonic in the arguments, so that

$$e^{-\gamma x_1} > e^{-\gamma x_2} > e^{-\gamma x_3} > \ldots > e^{-\gamma x_n}$$

As we pass γ to the infinite limit, the worst prize has the biggest term. The larger γ gets, the more it dominates and obscures all the other terms

$$-\langle u \rangle = \sum_{i=1}^{n} p_i\, e^{-\gamma x_i} \to p_1\, e^{-\gamma x_1}$$

Substitute this into the certain equivalent equation

$$\begin{aligned}
\tilde{x} &\to -\frac{1}{\gamma}\ln\left(\sum_{i=1}^{n} p_i\, e^{-\gamma x_i}\right) = -\frac{1}{\gamma}\ln\left(p_1\, e^{-\gamma x_1}\right) \\
&= -\frac{1}{\gamma}\ln(p_1) - \frac{1}{\gamma}\ln\left(e^{-\gamma x_1}\right) \\
&= -\frac{\ln(p_1)}{\gamma} - \frac{-\gamma x_1}{\gamma} \\
&= -\frac{\ln(p_1)}{\gamma} + x_1
\end{aligned}$$

As γ passes to the infinite limit, the first term vanishes, meaning that

$$\tilde{x} \to x_1 \text{ as } \gamma \to \infty$$

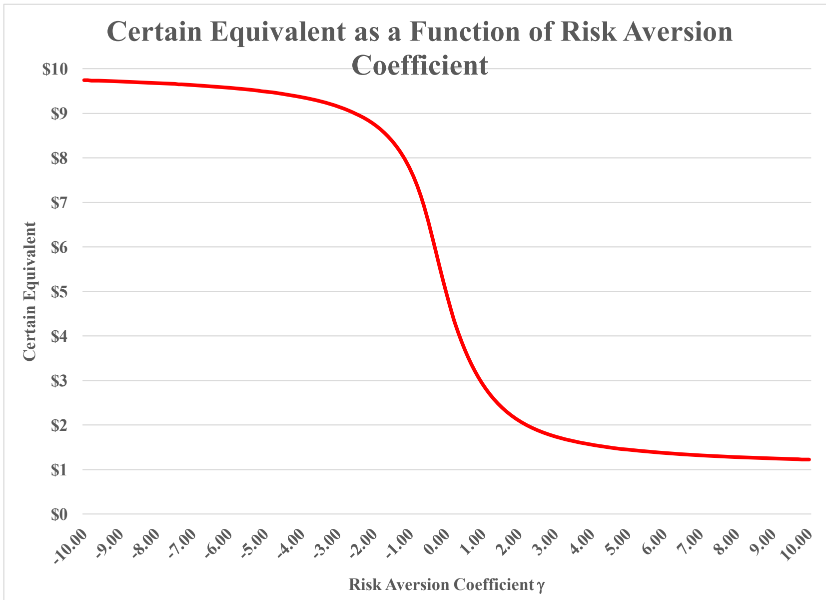
For an infinitely risk avers individual, the certain equivalent goes to the WORST PRIZE. It does not go to probability times worst prize. It trends toward the worst prize.

2 Risk Preferring Delta Person

The proof is exactly reversed for infinitely negative risk aversion coefficient γ. For an infinitely risk averse decision maker, the price becomes the best prize (to the exclusion of all others).

3 Plot

We postulate prizes 1,2,…,10 for a lottery with 10 outcomes. We have generated probabilities of the prizes at random. We have plotted the certain equivalent as a function of the risk aversion coefficient in the following diagram, varying from minus infinity at the left to plus infinity at the right. Notice that the infinitely risk averse person at the right will have a certain equivalent approaching the worst prize of 1, and the infinitely risk preferring person will have a certain equivalent approaching the best prize of 10. This is precisely what we have proven here. There have been some discussions in my office in which we did not fully and properly clarify this. The infinitely risk averse person only cares about the worst prize and not about its probability of occurrence. The infinitely risk preferring person only cares about the best prize and not about its probability of occurrence. The certain equivalent is clearly approaching the worst prize at the lower right, and that worst prize is not affected by its probability of occurrence.



Some of the conversations in my office argued that the certain equivalent for an infinitely risk averse delta person should be adjusted for the probability of the worst prize. That is not true, as is proven here. It IS the worst prize, not the probability of the worst prize times the worst prize.