CSE 015: Discrete Mathematics
Homework #8
Solution

Chapter 2.3

54. Show that if x is a real number and n is an integer, then

    (a) $x \leq n$ if and only if $\lceil x \rceil \leq n$.

    Solution.

    The "if" direction is trivial, since $x \leq \lceil x \rceil$. For the other direction, suppose that $x \leq n$. Since n is an integer no smaller than x, and $\lceil x \rceil$ is by definition the smallest such integer, clearly $\lceil x \rceil \leq n$.

    (b) $n \leq x$ if and only if $n \leq \lfloor x \rfloor$.

    Solution.

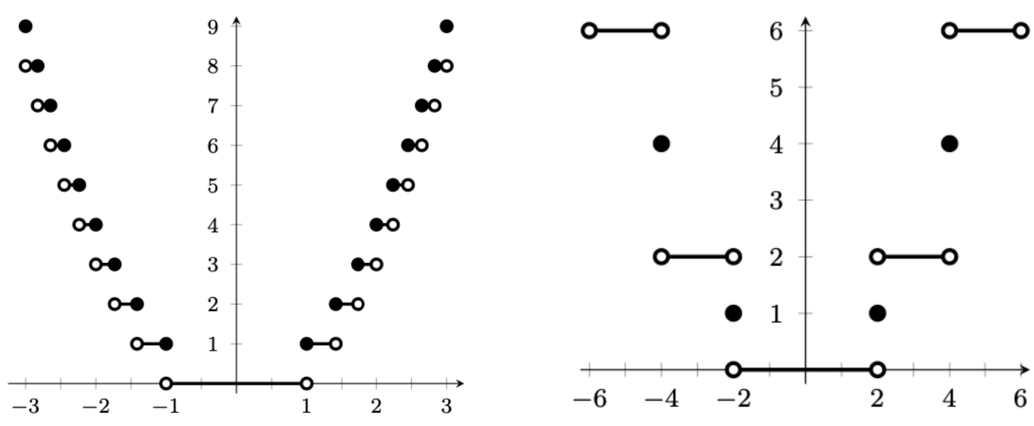
    The "if" direction is trivial, since $\lfloor x \rfloor \leq x$. For the other direction, suppose that $n \leq x$. Since n is an integer not exceeding x, and $\lfloor x \rfloor$ is by definition the largest such integer, clearly $n \leq \lfloor x \rfloor$.

70. Draw graphs of each of these functions.

    (d) $f(x) = \lfloor x^2 \rfloor$

    Solution.

    The basic shape is the parabola, $y = x^2$. However, because of the greatest integer function, the curve is broken into steps, with jumps at $x = \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, \ldots$ Note the symmetry around the y-axis.



(a) $f(x) = \lfloor x^2 \rfloor$ (b) $f(x) = \lceil \frac{x}{2} \rceil \lfloor \frac{x}{2} \rfloor$

(e) $f(x) = \lceil \frac{x}{2} \rceil \lfloor \frac{x}{2} \rfloor$

Solution.

The basic shape is the parabola, $y = x^2/4$. However, because of the step functions, the curve is broken into steps. For x an even integer, $f(x) = x^2/4$, since the terms inside the floor and ceiling function symbols are integers. Note how these are isolated point, as in Exercise 69f.

Chapter 2.4

10. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

(a) $a_n = -2a_{n-1}$, $a_0 = -1$

Solution. We simply plug $n = 0, 1, 2, 3, 4, 5$ using the initial conditions for the first few and then the recurrence relation.

$a_0 = -1$, $a_1 = -2a_0 = 2$, $a_2 = -2a_1 = -4$, $a_3 = -2a_2 = 8$, $a_4 = -2a_3 = -16$, $a_5 = -2a_4 = 32$

(b) $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$

Solution.

We simply plug $n = 0, 1, 2, 3, 4, 5$ using the initial conditions for the first few and then the recurrence relation.

$a_0 = 2$, $a_1 = -1$, $a_2 = a_1 - a_0 = -3$, $a_3 = a_2 - a_1 = -2$, $a_4 = a_3 - a_2 = 1$, $a_5 = a_4 - a_3 = 3$

12. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if

(c) $a_n = (-4)^n$

Solution.

$-3a_{n-1} + 4a_{n-2} = -3 \cdot (-4)^{n-1} + 4 \cdot (-4)^{n-2} = (-4)^{n-2}\left((-3)(-4) + 4\right) = (-4)^{n-2} \cdot 16 = (-4)^{n-2}(-4)^2 = (-4)^n = a_n$

(d) $a_n = 2(-4)^n + 3$.

Solution.

$-3a_{n-1}+4a_{n-2} = -3\cdot\big(2(-4)^{n-1} + 3\big)+4\cdot\big(2(-4)^{n-2} + 3\big) = (-4)^{n-2}\left((-6)(-4) + 4 \cdot 2\right)-9+12 = (-4)^{n-2} \cdot 32 + 3 = (-4)^{n-2}(-4)^2 \cdot 2 + 3 = 2 \cdot (-4)^n + 3 = a_n$

16. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.

In the iterative approach, we write an in terms of a_{n-1} , then write a_{n-1} in terms of a_{n-2} (using the recurrence relation with $n-1$ plugged in for n), and so on. When we reach the end of this procedure, we use the given initial value of a_0. This will give us an explicit formula for the answer or it will give us a finite series, which we then sum to obtain an explicit formula for the answer.

(a) $a_n = -a_{n-1}, a_0 = 5$

Solution.

$a_n = -a_{n-1} = (-1)^2 a_{n-2} = \ldots = (-1)^n a_{n-n} = (-1)^n a_0 = 5 \cdot (-1)^n$

(b) $a_n = a_{n-1} + 3, a_0 = 1$

Solution.

$a_n = 3+a_{n-1} = 3+3+a_{n-2} = 2\cdot3+a_{n-2} = 3\cdot3+a_{n-3} = \ldots = n\cdot3+a_{n-n} = n\cdot3+a_0 = 3n+1$

40. $\sum_{k=99}^{200} k^3$ (Use table 2 from the textbook)

Solution.

$\sum_{k=99}^{200} k^3 = \sum_{k=1}^{200} k^3 - \sum_{k=1}^{98} k^3$

Chapter 2.5

12. Show that if A and B are sets and $A \subset B$ then $|A| \leq |B|$.

Solution.

The definition of $|A| \leq |B|$ is that there is a one-to-one function from A to B. In this case the desired function is just $f(x) = x$ for each $x \in A$.

22. Suppose that A is a countable set. Show that the set B is also countable if there is an onto function f from A to B.

Solution.

If $A = \emptyset$, then the only way for the conditions to be met are that $B = \emptyset$ as well, and we are done. So assume that A is nonempty. Let f be the given onto function from A to B, and let $g : \mathbb{Z}^+ \to A$ be an onto function that establishes the countability of A. (If A is finite rather than countably infinite, say of cardinality k, then the function g will be defined so that $g(1), g(2), \ldots, g(k)$ will list the elements of A, and $g(n) = g(1)$ for $n > k$). We need to find an onto function from $\mathbb{Z}^+$ to B. The function $f \circ g$ does the trick, because the composition of two onto functions is onto (Exercise 33b in Section 2.3).

28. Show that the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Solution.

We can think of $\mathbb{Z}^+ \times \mathbb{Z}^+$ as the countable union of countable sets, where the ith set in the collection, for $i \in \mathbb{Z}^+$, is $\{(i, n) | n \in \mathbb{Z}^+\}$. The statement now follows from Exercise27.