CSE 015: Discrete Mathematics
Homework #7
Solution

Chapter 2.2

16. Let A and B be sets. Show that

    (a) $(A \cap B) \subseteq A$
    Solution.
    If x is in $A \cap B$, then perforce it is in A (by definition of intersection).

    (c) $A - B \subseteq A$
    Solution.
    If x is in $A - B$, then perforce it is in A (by definition of intersection).

22. Show that if A and B are sets with $A \subseteq B$, then

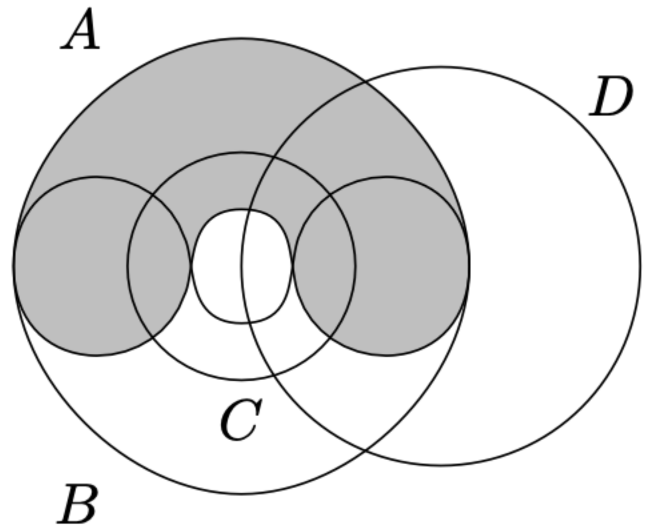
    (a) $A \cup B = B$.
    Solution.
    It is always the case that $B \subseteq A \cup B$, so it remains to show that $A \cup B \subseteq B$.But this is clear because if $x \subseteq A \cup B$, then either $x \in A$, in which case $x \in B$ (because we are given $A \subseteq B$) or $x \in B$; in either case $x \in B$.

    (b) $A \cap B = B$.
    Solution.
    It is always the case that $A \cap B \subseteq A$, so it remains to show that $A \subseteq A \cap B$. But this is clear because if $x \in A$,then $x \in B$ as well (because we are given $A \subseteq B$), so $x \in A \cap B$.

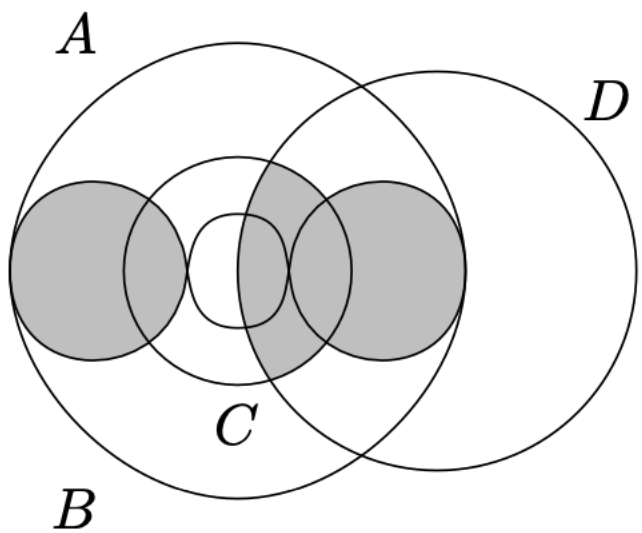
30. Draw the Venn diagrams for each of these combinations of the sets A, B, C, and D.

    (a) $(A \cap B) \cup (C \cap D)$
    (b) $\overline{A} \cup \overline{B} \cup \overline{C} \cup \overline{D}$

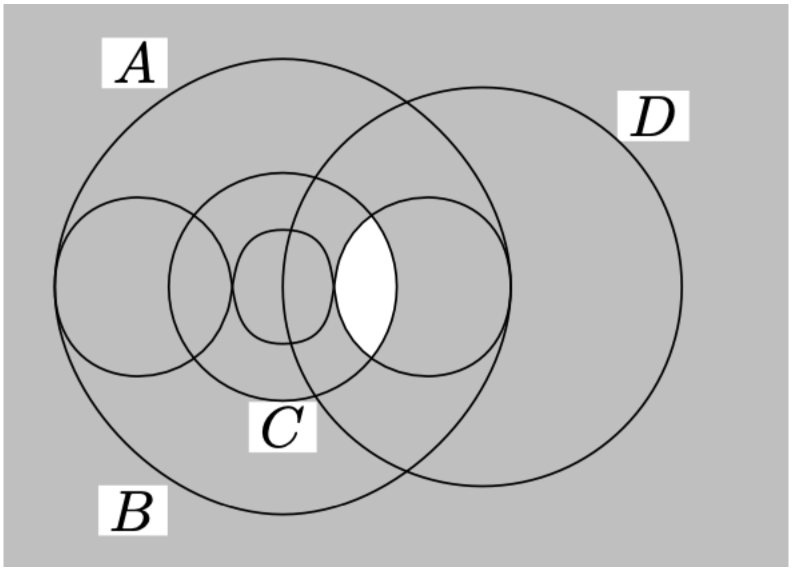
    Solution. Here is a Venn diagram that can be used for four sets. Notice that sets A and B are not convex in this picture. We have shaded set A. Notice that each of the 16 different combinations are



represented by a region.



(a)

(b) 

54. Let $A_i = \{\ldots, -2, -1, 0, 1, \ldots, i\}$. Find

    (a) $\bigcup_{i=1}^{n} A_i$

    (b) $\bigcap_{i=1}^{n} A_i$

    Solution. We note that these sets are increasing, that is, $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$. Therefore, the union of any collection of these sets is just the one with the largest subscript, and the intersection is just the one with the smallest subscript.

    (a) $A_i = \{\ldots, -2, -1, 0, 1, \ldots, n\}$

    (b) $A_i = \{\ldots, -2, -1, 0, 1\}$

Chapter 2.3

2. Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

    (a) $f(n) = \pm n$.
    Solution.
    This is not a function because the rule is not well-defined. We do not know whether $f(3) = 3$ or $f(3) = -3$. For a function, it cannot be both at the same time.

    (b) $f(n) = \sqrt{n^2 + 1}$
    Solution.
    This is a function. For all integers n, $\sqrt{n^2 + 1}$ is a well-defined real number.

10. Determine whether each of these functions from $\{a, b, c, d\}$ to itself is one-to-one.

    (a) $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$
    Solution.
    This is one-to-one.

(b) $f(a) = b,\ f(b) = b,\ f(c) = d,\ f(d) = c$

Solution.

This is not one-to-one, since b is the image of both a and b.

14. Determine whether $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if

(a) $f(m, n) = 2m - n$

Solution.

This is clearly onto, since $f(0, -n) = n$ for every integer n.

(b) $f(m, n) = m^2 - n^2$

Solution.

This is not onto, since, for example, 2 is not in the range. To see this, if $m^2 - n^2 = (m-n)(m+n) = 2$, then m and n must have same parity (both even or both odd). In either case, both $m - n$ and $m + n$ are then even, so this expression is divisible by 4 and hence cannot equal 2.

22. Determine whether each of these functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

(a) $f(x) = -3x + 4$

(b) $f(x) = -3x^2 + 7$

Solution.

If we can find an inverse, the function is a bijection. Otherwise we must explain why the function is not on-to-one or not onto.

(a) This is a bijection since the inverse function is $f^{-1}(x) = (4 - x)/3$

(b) This is not one-to-one since $f(17) = f(-17)$, for instance. It is also not onto, since the range is the interval $(-\infty, 7]$. For example, 42548 is not in the range.

38. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution.

We have $(f \circ g)(x) = f(g(x)) = f(x+2) = (x+2)^2 + 1 = x^2 + 4x + 5$, whereas $(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = x^2 + 1 + 2 = x^2 + 3$. Note that they are not equal.

40. Let $f(x) = ax + b$ and $g(x) = cx + d$, where a, b, c, and d are constants. Determine necessary and sufficient conditions on the constants a, b, c, and d so that $f \circ g = g \circ f$.

Solution.

Forming the compositions we have $(f \circ g)(x) = acx + ad + b$ and $(g \circ f)(x) = cax + cb + d$. These are equal if and only if $ad + b = cb + d$. In other words, equality holds for all 4-tuples (a, b, c, d) for which $ad + b = cb + d$.