Lecture 26: Three theorems about the derivative

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^{*}These slides may incorporate material from Hughes-Hallet, et al, "Calculus", Wiley

Fermat's theorem

Theorem

Suppose that c is a maximum of a differentiable function f(x) in the interval (a, b), then f'(c) = 0.

Proof.

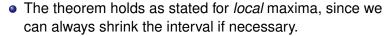
Since *c* is a maximum, $f(x) \le f(c)$ for all x < c. We have

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le 0$$

and also

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge 0.$$

So f'(c) = 0.



Rolle's theorem

Theorem

Suppose that f is a continuous function on [a,b] that is differentiable on the open interval (a,b). If f(a)=f(b), then there is a $c \in (a,b)$ such that f'(c)=0.

Proof.

By the extreme value theorem, f the function f has an absolute maximum and an absolute minimum in [a, b]. If both absolute extrema are at the endpoints then, since f(a) = f(b), they must be equal and so f is constant, and therefore f' = 0 at every point. Otherwise, there is an absolute extremum f'(c) = 0.

^aWhich is very difficult to prove

The Mean Value Theorem

Theorem

If f is continuous on $a \le x \le b$ and differentiable on a < x < b, then there exists a number c with a < c < b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof.

Let $g(x) = f(x) - (f(b) - f(a)) \frac{x-a}{b-a}$. Then g(a) = f(a) and g(b) = f(a) as well. So g(b) = g(a). By Rolle's theorem, g'(c) = 0 for some $c \in (a, b)$. Written out, this is

$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

as required.



An example

- Let f(x) = 1/x on [1, 3].
- The slope of the secant line of f(x) is

$$\frac{f(3) - f(1)}{3 - 1} = \frac{1/3 - 1}{2} = -\frac{1}{3}$$

- The derivative of f(x) is $f'(x) = -1/x^2$.
- So, with $c = \sqrt{3}$, we have f'(c) = -1/3 is the slope of the secant line.
- In this case, we can find *c* by solving the equation

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

• The mean value theorem tells us that this equation *must* have a solution, even if it doesn't tell us how to find it.

Interpretations

- Geometrical interpretation: The secant line to the graph has the same slope as the tangent line at x = c.
- Physical interpretation: At some point of the domain of a function f, the instantaneous rate of change is equal to the average rate of change of f over its domain.
- Example: You drive on route 90 West from the Corning exit to the PA state line. The total distance is 133.6 miles, and it takes you exactly 2 hours. Your average speed was 133.6/2 = 66.8 mph.
- At some point during the journey, your speed was exactly 66.8 mph.

Several corollaries

We have the *constant function theorem*:

Corollary

Suppose f'(x) = 0 throughout an interval [a, b], then f(x) is constant.

Proof.

If there were two points, say c and d, where $f(c) \neq f(d)$, then at some x between c and d, $f'(x) = \frac{f(c) - f(d)}{c - d} \neq 0$. This is ruled out by hypothesis, so we must have f(c) = f(d) for all c, d.

Corollary

If f'(x) = g'(x) throughout [a, b], then f(x) = g(x) + C for some constant C.

Proof.

Apply the constant function theorem to f(x) - g(x).

The increasing function theorem

Theorem

Suppose that f is continuous on $a \le x \le b$ and differentiable on a < x < b.

- If f'(x) > 0 on a < x < b, then f is increasing on $a \le x \le b$.
- If f'(x) < 0 on a < x < b, then f is decreasing on $a \le x \le b$.

Proof.

- Suppose that f'(x) > 0 on a < x < b
- Pick x_1, x_2 with $a \le x_1 < x_2 \le b$
- We'll show that $f(x_1) < f(x_2)$
- We want to show $f(x_2) f(x_1) > 0$
- The MVT implies that there is a point c between x_1 and x_2 such that $f'(c)(x_2 x_1) = f(x_2) f(x_1) > 0$.



An application

- Let $f(x) = \sin x + 2x + 1$. Show that f(x) = 0 has a *unique* solution.
- We have $f(-2\pi) < 0$ and f(0) = 1 > 0, so there is a solution between -2π and 0 by the IVT
- Also $f'(x) = \cos x + 2 > 0$. So f is an increasing function.
- Hence the graph of y = f(x) can only cross the x-axis at a single point.

The racetrack principle

Theorem

Let f and g be two continuous functions on $a \le x \le b$, differentiable on a < x < b, and that $f'(x) \le g'(x)$ for a < x < b. Hypothesis: Horse f always runs slower than horse g

- If f(a) = g(a) then $f(x) \le g(x)$ for $a \le x \le b$. If horses f and g start the race at the same time, then horse f always runs behind horse g.
- 2 If g(b) = f(b) then $f(x) \ge g(x)$ for $a \le x \le b$. If horses f and g finish the race at the same time, then horse f would have needed a head start

Example

Show that $e^x \ge 1 + x$ for all values of x. Let f(x) = 1 + x, $f(x) = e^x$. For $x \ge 0$, $f'(x) \le g'(x)$ so use 1. For $x \le 0$, $f'(x) \ge g'(x)$ so use 2.