Math 181: Exam 1 solutions

1. Determine if each of the following equations has a (real) solution x. If there is, find the solution. If not, say why not. If your answer is a whole number, write it as a whole number. Otherwise, you may leave your answer in terms of natural logarithms. [4 points ea]

(a) $1^{x+1} = 2^{x^2}$ [note: $2^{x^2} = 2^{(x^2)}$]
$x = 0$ is the only solution.

(b) $2^{x+1} = 4^{2x+5}$
taking natural logs gives

$$(x+1)\ln 2 = (2x+5)\ln 4$$
$$(x+1)\ln 2 = 2(2x+5)\ln 2$$
$$(x+1) = 2(2x+5)$$
$$3x + 9 = 0$$
$$x = -3$$

(c) $e^{x+1} = -1$
No solutions. e^{x+1} is always positive

(d) $4^{x^2} = 5^{x+1}$

$x^2 \ln 4 = (x+1)\ln 5$
$x^2 - \frac{\ln 5}{\ln 4}x - \frac{\ln 5}{\ln 4} = 0$
$x = \frac{\frac{\ln 5}{\ln 4} \pm \sqrt{\left(\frac{\ln 5}{\ln 4}\right)^2 + 4\frac{\ln 5}{\ln 4}}}{2}$

2. This problem concerns the function $f(x) = \frac{4x+1}{x-3}$.

(a) Find the *domain of* $f(x)$.
All $x \neq 3$

(b) Find any *vertical asymptotes* of $y = f(x)$.
$x = 3$

(c) Find any *horizontal asymptotes* of $y = f(x)$.
$y = 4$

(d) Compute $\displaystyle\lim_{x\to 3^+} \frac{4x+1}{x-3}$.
$+\infty$

(e) Compute $\displaystyle\lim_{x\to 3^-} \frac{4x+1}{x-3}$.
$-\infty$

(f) Compute $\displaystyle\lim_{x\to \infty} \frac{4x+1}{x-3}$.
4

(g) Find the inverse function of $f(x)$.
$x(y-3) = 4y+1$ 1pt
$(x-4)y = 3x+1$
$y = \frac{3x+1}{x-4}$ 2pt

(h) Find the range of $f(x)$.
All real $y \neq 4$

3 points each part.

3. The population of the planet Romulus was 3 billion in stardate 41000. In star date 41100, the population was 4 billion. Assume that the population grows exponentially. (In each part, you may leave your answer in terms of logarithms, exponentials, and fractions. *Do not* attempt to use a calculator to produce a numerical answer.)

(a) Write the exponential model $P = P_0 a^t$ that best fits this data.

(b) Use your model to estimate the population of Romulus in stardate 42000.

(c) Find how long it will take the population to double.

(a) Let t =stardates from 41000. Then, with P in billions, $P(0) = P_0 = 3$, and $P(100) = P_0 a^{100} = 3a^{100} = 4$. So $a = (4/3)^{1/100}$. Thus the model is

$$P = 3(4/3)^{t/100}.$$

(b) It will be $3(4/3)^{1000/100} = 3(4/3)^{10}$. (c) We want to solve $(4/3)^{t/100} = 2$. Taking natural logarithms gives

$$\frac{t}{100}\ln(4/3) = \ln(2) \implies t = \frac{100\ln 2}{\ln(4/3)}.$$

4. Evaluate the limits. If they are infinite, write $+\infty, -\infty$ where appropriate.

(a) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x-2}$

(b) $\displaystyle\lim_{x\to\infty} \sin x$

(c) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+3}}{2x-1}$

(d) $\displaystyle\lim_{t\to 0} \frac{(2t+1)^{-1}-1}{t}$

(e) $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2+x-2}$

3pts ea.

(a) $2/3$, (b) DNE, (c) $1/2$, (d) -2, (e) $-1/3$

5. Consider the rational function

$$f(x) = \frac{(x-1)^2}{(x^2-1)(x+2)^2}.$$

Compute the following limits: (2pts ea)

(a)

(i) $\lim_{x\to 1^-} f(x)$
0

(ii) $\lim_{x\to 1^+} f(x)$
0

(iii) $\lim_{x\to 1} f(x)$
0

(iv) $\lim_{x\to -1^+} f(x)$
$-\infty$

(v) $\lim_{x\to -1^-} f(x)$
$+\infty$

(vi) $\lim_{x\to -1} f(x)$
DNE

(vii) $\lim_{x\to -2^+} f(x)$
$+\infty$

(viii) $\lim_{x\to -2^-} f(x)$
$+\infty$

(ix) $\lim_{x\to -2} f(x)$
$+\infty$

Consistent but wrong worth 1/2. Correct but not consistent worth 1/2.

(b) Find all horizontal asymptotes of $f(x)$.

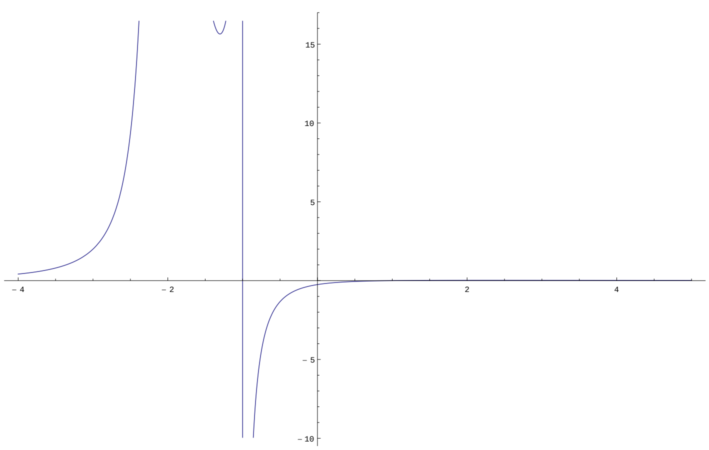
$y = 0$ (3pts)

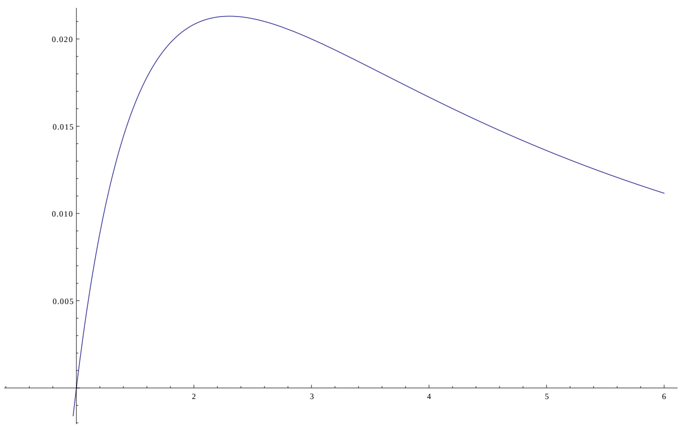
(c) Sketch the graph of $y = \frac{(x-1)^2}{(x^2-1)(x+2)^2}$, showing the correct behavior at the zeros, poles, removable singularities, and proper end behavior

9 points. 2 points for each of poles and zeros. 2 points for end behavior. 1 point for the removable singularity.

Near $x = 1$, $f(x) \approx +(x-1)$, so the graph would cross, but $x = 1$ is a removable singularity. (3pts)

It is likely that errors in part (a) will lead to inconsistent answers in part (c), and so a meaningless graph. I do not check consistency, only correctness.





6. (a) State the squeeze theorem in your own words. Draw a picture illustrating the conclusion of the theorem.

If the graph of $f(x)$ lies between the graphs of an upper and lower envelope, and both envelopes have the same limit at a point, then $f(x)$ also has a limit at that point, and the value of the limit is the same as the common value of the limits of the envelopes.

The statement is worth 6 points. There are two hypotheses and two conclusions, each worth a point. A supporting illustration showing an omitted hypothesis or conclusion is worth 1/2 points for each omitted part, for a maximum of 2 points total.

(b) If $4x - 9 \leq f(x) \leq x^2 - 4x + 7$ for $x \geq 0$, find (with proof) $\lim_{x\to 4} f(x)$.

Value of limit: 7 (worth 2pt), squeezed between $\pm\sqrt{x}$ (2pt), proper invocation of squeeze theorem (2pt). (I was quite strict in the grading for the proper invocation of the theorem.)

7. Beginning with the graph of $y = x^2$, we perform some geometric operations. For (a)–(f), pick the correct formula from the second column.

(a) The graph is reflected in the line $y = x$.

(b) The graph is moved to the left by one unit, and then stretched horizontally by a factor of 3.

(c) The graph is moved vertically upwards by one unit, and then compressed horizontally by a factor of 1/3.

(d) The graph is moved one unit to the left, and then reflected in the y-axis.

(e) The graph is moved to the left by one unit, and then stretched vertically by a factor of 3.

(f) The graph is moved to the left by one unit, and then reflected in the x-axis.

(1) $y = 3(x-1)^2$

(2) $y = (x-1)^2/3$

(3) $y = 3(x+1)^2$

(4) $y = (x/3+1)^2$

(5) $y = (3x-1)^2$

(6) $y = (x+1)^2/3$

(7) $y = -(x+1)^2$

(8) $y = (-x+1)^2$

(9) $y = -x^2$

(10) The resulting graph is not a function.

(11) None of the above.

8. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. Also, suppose that $f(2) = 8$. Is it possible that $f(3) < 6$? If so, give an example of such a function. If not, explain why. Support your answer with a sketch if desired.

It is not possible. Since f is a continuous function, and $f(2) > 6$ and $f(3) < 6$, there would be a solution to $f(x) = 6$ between 2 and 3 by the intermediate value theorem. But the given conditions rule that possibility out.