Name: ________________

Section: ________________

R·I·T School of Mathematical Sciences

29 - Second Order Applications with Laplace Transforms

MATH 211

1. A mass of 2kg is attached to a spring with spring constant 8N/m. An external force given by $f(t) = 4u(t-2)$ acts on the spring which is initially stretched 1 m and then released with no initial velocity. Find the equation of motion $x(t)$ assuming no damping.

$$mx'' + bx' + kx = f(t)$$
$$2x'' + 8x = 4u(t-2)$$
$$x'' + 4x = 2u(t-2)$$

$$\mathscr{L}\{x''\} + 4\mathscr{L}\{x\} = 2\mathscr{L}\{u(t-2)\}$$
$$s^2\mathscr{L}\{x\} - s\underbrace{x(0)}_{=1} - \underbrace{x'(0)}_{=0} + 4\mathscr{L}\{x\} = \frac{2e^{-2s}}{s}$$
$$s^2\mathscr{L}\{x\} - s + 4\mathscr{L}\{x\} = \frac{2e^{-2s}}{s}$$
$$(s^2+4)\mathscr{L}\{x\} = s + \frac{2e^{-2s}}{s}$$
$$\mathscr{L}\{x\} = \frac{s}{s^2+4} + \frac{2e^{-2s}}{s(s^2+4)}{}_{\star 1}$$
$$\mathscr{L}\{x\} = \frac{s}{s^2+4} + e^{-2s}\left[\frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+4}\right)\right]$$
$$\mathscr{L}\{x\} = \frac{s}{s^2+4} + \frac{1}{2}e^{-2s}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]_{\star 2}$$

$$\boxed{x = \cos(2t) + \frac{1}{2}\left[1 - \cos(2(t-2))\right]u(t-2)}$$

$\star 1$

$$\frac{2}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$$
$$2 = As^2 + 4A + Bs^2 + Cs$$
$$A + B = 0 \quad C = 0 \quad 4A = 2$$
$$B = -\frac{1}{2} \quad \leftarrow A = \frac{1}{2}$$

$\star 2$

from e^{-2s}

$$F(s) = \frac{1}{s} - \frac{s}{s^2+4}, \quad a = 2$$
$$f(t) = 1 - \cos(2t)$$
$$f(t-2) = 1 - \cos(2(t-2))$$

2. A simple pendulum rotates around a point, Q. The pendulum rod is $l = \frac{1}{2}$ feet long and is released from rest at $\theta(0) = \pi/3$ radians. An electromagnet giving off a force of 2 lb is mounted directly below the mass when at equilibrium and is shut off after 3 seconds.

$$f(t) = 2 - 2u(t-3)$$

$$\theta'' + \frac{g}{l}\theta = f(t)$$

$$\theta'' + 64\theta = 2 - 2u(t-3)$$

$$\mathscr{L}\{\theta''\} + 64\mathscr{L}\{\theta\} = 2\mathscr{L}\{1\} - 2\mathscr{L}\{u(t-3)\}$$

$$s^2\mathscr{L}\{\theta\} - s\theta(0) - \theta'(0) + 64\mathscr{L}\{\theta\} = \frac{2}{s} - \frac{2e^{-3s}}{s}$$

$$s^2\mathscr{L}\{\theta\} - \frac{\pi}{3}s + 64\mathscr{L}\{\theta\} = \frac{2}{s} - \frac{2e^{-3s}}{s}$$

$$(s^2+64)\mathscr{L}\{\theta\} = \frac{2}{s} + \frac{\pi}{3}s - \frac{2e^{-3s}}{s}$$

$$\mathscr{L}\{\theta\} = \frac{2}{s(s^2+64)}_{\star 1} + \frac{\pi}{3}\left[\frac{s}{s^2+64}\right] - \frac{2e^{-3s}}{s(s^2+64)}_{\star 1}$$

$$\mathscr{L}\{\theta\} = \frac{1}{32}\left(\frac{1}{s}\right) - \frac{1}{32}\left(\frac{s}{s^2+64}\right) + \frac{\pi}{3}\left(\frac{s}{s^2+64}\right) - e^{-3s}\left[\frac{1}{32}\left(\frac{1}{s}\right) - \frac{1}{32}\left(\frac{s}{s^2+64}\right)\right]$$

$$\mathscr{L}\{\theta\} = \frac{1}{32}\left(\frac{1}{s}\right) + \left(\frac{\pi}{3} - \frac{1}{32}\right)\left(\frac{s}{s^2+64}\right) - e^{-3s}\left[\frac{1}{32}\left(\frac{1}{s}\right) - \frac{1}{32}\left(\frac{s}{s^2+64}\right)\right]_{\star 2}$$

$$\boxed{\theta(t) = \frac{1}{32} + \left(\frac{\pi}{3} - \frac{1}{32}\right)\cos(8t) - \left[\frac{1}{32} - \frac{1}{32}\cos(8(t-3))\right]u(t-3)}$$

$\star 1$

$$\frac{2}{s(s^2+64)} = \frac{A}{s} + \frac{Bs+C}{s^2+64}$$

$$2 = As^2 + 64A + Bs^2 + Cs$$

$$A + B = 0 \qquad C = 0 \qquad 64A = 2$$

$$B = -\frac{1}{32} \qquad \leftarrow A = \frac{1}{32}$$

$\star 2$

$$F(s) = \frac{1}{32}\left(\frac{1}{s}\right) - \frac{1}{32}\left(\frac{s}{s^2+64}\right), \quad a = 3$$

$$f(t) = \frac{1}{32}(1) - \frac{1}{32}\cos(8t)$$

$$f(t-3) = \frac{1}{32} - \frac{1}{32}\cos(8(t-3))$$