5 - Iterated Integrals

MATH 211

Evaluate the following integral. [Hint: Make the substitution $t = \sqrt{x}$ and rewrite the integral.]

$$\int_0^1 e^{\sqrt{x}}\ dx$$

$$t = \sqrt{x}$$

$$t^2 = x$$

$$2t\ dt = dx$$

$$x = 0 \implies t = \sqrt{0} = 0$$

$$x = 1 \implies t = \sqrt{1} = 1$$

$$\int_0^1 e^{\sqrt{x}}\ dx = \int_0^1 e^t \cdot 2t\ dt = 2\int_0^1 te^t\ dt$$

$$\begin{array}{ll} u = t & dv = e^t\ dt \\ du = dt & v = e^t \end{array}$$

$$\begin{aligned}
&= 2\left[te^t\Big|_0^1 - \int_0^1 e^t\ dt \right] \\
&= 2\left[(1)e^1 - 0 - \left[e^t\right]_0^1 \right] \\
&= 2\left[e - (e - 1)\right] \\
&= 2\left[e - e + 1\right] \\
&= 2
\end{aligned}$$

Evaluate the following iterated integral.

$$\int_0^1 \int_0^{x+1} \frac{1}{x^2+1}\ dy\ dx$$

$$= \int_0^1 \left[\frac{1}{x^2+1} \cdot y \right]_0^{x+1}\ dx$$

$$= \int_0^1 \frac{1}{x^2+1} \cdot (x+1)\ dx$$

$$= \int_0^1 \left(\frac{x}{x^2+1} + \frac{1}{x^2+1} \right)\ dx$$

$$= \int_0^1 \frac{x}{x^2+1}\ dx + \int_0^1 \frac{1}{x^2+1}\ dx$$

$$u = x^2+1$$

$$du = 2x\ dx$$

$$\frac{1}{2}\ du = x\ dx$$

$$x = 0 \implies u = 0+1 = 1$$

$$x = 1 \implies u = 1+1 = 2$$

$$= \int_1^2 \frac{1}{u} \cdot \frac{1}{2}\ du + \int_0^1 \frac{1}{x^2+1}\ dx$$

$$= \frac{1}{2} \int_1^2 \frac{1}{u}\ du + \int_0^1 \frac{1}{x^2+1}\ dx$$

$$= \frac{1}{2}\ \ln|u|\big|_1^2 + \arctan x\big|_0^1$$

$$= \frac{1}{2}\left[\ln 2 - \ln 1\right] + \arctan 1 - \arctan 0$$

$$= \frac{1}{2}\ln 2 - 0 + \frac{\pi}{4} - 0$$

$$= \ln\sqrt{2} + \frac{\pi}{4}$$