Name: ______________________

Section: ______________________

Date: ______________________

R·I·T School of Mathematical Sciences

Spring Mass Systems

MATH 211 - 01

A 4 kg mass stretches a spring 2 meters. The spring is stretched 1 meter past equilibrium and released from rest. (For this exercise, let $g = 10\text{m/s}^2$.)

1. Use Hooke's Law to determine the strength of the spring (spring constant).

$$F = ma = 4(10) = 40N$$

$$F = kx$$
$$40 = k(2)$$
$$k = 20$$

2. Set up the differential equation whose solution will model the motion of the spring if there is no damping or external force exerted.

$$mx'' + bx' + kx = f(t)$$

$$4x'' + 20x = 0$$
$$x'' + 5x = 0$$

3. Does the differential equation suggest damped or undamped motion? Free or driven motion?

Free Undamped Motion

4. Find linearly independent solutions to the differential equation and verify that each satisfies the differential equation.

$$\lambda^2 + 5 = 0$$

$$\lambda^2 = -5$$
$$\lambda = \pm\sqrt{5}i$$

$$x_1 = \sin(\sqrt{5}t)$$
$$x_1' = \sqrt{5}\cos(\sqrt{5}t)$$
$$x_1'' = -5\sin(\sqrt{5}t)$$
$$x_1'' + 5x_1 = 0$$
$$-5\sin(\sqrt{5}t) + 5\sin(\sqrt{5}t) = 0$$
$$0 = 0$$

$$x_2 = \cos(\sqrt{5}t)$$
$$x_2' = -\sqrt{5}\sin(\sqrt{5}t)$$
$$x_2'' = -5\cos(\sqrt{5}t)$$
$$x_2'' + 5x_2 = 0$$
$$-5\cos(\sqrt{5}t) + 5\cos(\sqrt{5}t) = 0$$
$$0 = 0$$

5. Write the linear combination of linearly independent solutions.

$$x(t) = c_1 \sin(\sqrt{5}t) + c_2 \cos(\sqrt{5}t)$$

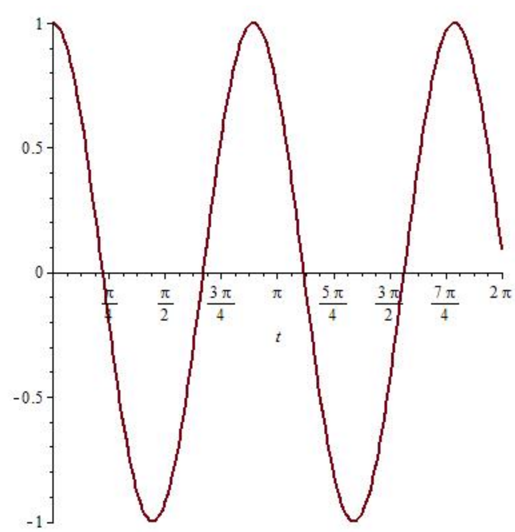
6. Solve the homogeneous IVP.

$$x'(t) = \sqrt{5}c_1 \cos(\sqrt{5}t) - \sqrt{5}c_2 \sin(\sqrt{5}t)$$

$$x(0) = 1$$
$$1 = c_1(0) + c_2(1)$$
$$c_2 = 1$$

$$x'(0) = 0$$
$$0 = \sqrt{5}c_1(1) - \sqrt{5}c_2(0)$$
$$0 = \sqrt{5}c_1$$
$$c_1 = 0$$

$$x(t) = \cos(\sqrt{5}t)$$



7. State whether the equation of motion represents motion that is underdamped, critically damped or overdamped.

Underdamped

8. Now, consider adding a damping force numerically equal to 18 times the instantaneous velocity. Set up the new differential equation.

$$4x'' + 18x' + 20x = 0$$

$$2x'' + 9x' + 10x = 0$$

9. Solve the new homogeneous IVP.

$$\lambda = \frac{-9 \pm \sqrt{81-80}}{2(2)} = \frac{-9 \pm 1}{4}$$

$$\lambda_1 = \frac{-10}{4} = \frac{-5}{2}$$

$$\lambda_2 = -2$$

$$x(t) = c_1 e^{-5t/2} + c_2 e^{-2t}$$

$$x'(t) = -\frac{5}{2} c_1 e^{-5t/2} - 2c_2 e^{-2t}$$

$$x(0) = 1$$

$$1 = c_1 + c_2$$

$$1 = -\frac{4}{5} c_2 + c_2$$

$$1 = \frac{1}{5} c_2$$

$$c_2 = 5$$

$$x'(0) = 0$$

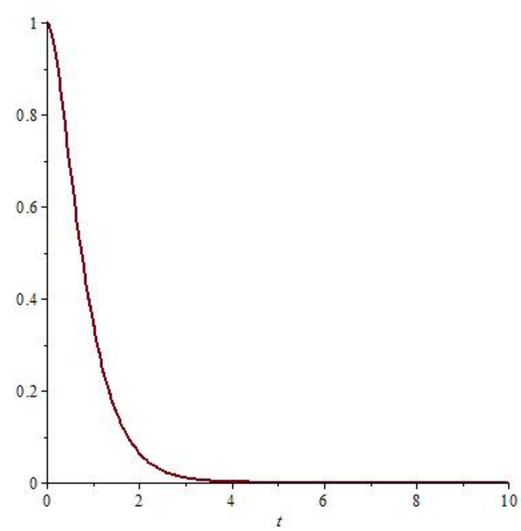
$$0 = -\frac{5}{2} c_1 - 2c_2$$

$$\frac{5}{2} c_1 = -2c_2$$

$$c_1 = -\frac{4}{5} c_2$$

$$c_1 = -4$$

$$x(t) = -4e^{-5t/2} + 5e^{-2t}$$



10. State whether the equation of motion represents motion that is underdamped, critically damped or overdamped.

Overdamped

11. Lastly, consider that an external force of $f(t) = e^{-2t}$ is required to stretch the spring the 1 meter past equilibrium before release. Set up the new differential equation.

$$4x'' + 18x' + 20x = e^{-2t}$$

12. Find the transient solution for the spring mass system.

$$x_c(t) = c_1 e^{-5t/2} + c_2 e^{-2t}$$

13. Find the steady-state solution for the spring mass system.

$$x_p(t) = Ate^{-2t}$$

$$x_p' = Ae^{-2t} - 2Ate^{-2t}$$

$$x_p'' = -2Ae^{-2t} - 2Ae^{-2t} + 4Ate^{-2t}$$

$$4x_p'' + 18x_p' + 20x_p = e^{-2t}$$

$$4\left(-4Ae^{-2t} + 4Ate^{-2t}\right) + 18\left(Ae^{-2t} - 2Ate^{-2t}\right) + 20Ate^{-2t} = e^{-2t}$$

$$-16Ae^{-2t} + 16Ate^{-2t} + 18Ae^{-2t} - 36Ate^{-2t} + 20Ate^{-2t} = e^{-2t}$$

$$2Ae^{-2t} = e^{-2t}$$

$$2A = 1$$

$$A = \frac{1}{2}$$

$$x_p = \frac{1}{2}te^{-2t}$$

14. Write the general solution to the nonhomogeneous differential equation.

$$x(t) = c_1 e^{-5t/2} + c_2 e^{-2t} + \frac{1}{2}te^{-2t}$$

15. Solve the IVP.

$$x'(t) = -\frac{5}{2}c_1e^{-5t/2} - 2c_2e^{-2t} + \frac{1}{2}e^{-2t} - te^{-2t}$$

$$x(0) = 1$$
$$1 = c_1 + c_2 + 0$$
$$1 = c_1 + c_2$$
$$c_1 = 1 - c_2$$
$$c_1 = 1 - 4$$
$$c_1 = -3$$

$$x'(0) = 0$$
$$0 = -\frac{5}{2}c_1 - 2c_2 + \frac{1}{2} - 0$$
$$\frac{5}{2}c_1 + 2c_2 = \frac{1}{2}$$
$$\frac{5}{2}(1 - c_2) + 2c_2 = \frac{1}{2}$$
$$5 - 5c_2 + 4c_2 = 1$$
$$5 - c_2 = 1$$
$$c_2 = 4$$

$$x(t) = -3e^{-5t/2} + 4e^{-2t} + \frac{1}{2}te^{-2t}$$