9 - Separation of Variables

MATH 211

Solve the following equation for $\frac{dy}{dx}$ in terms of x.

$$\frac{dy}{dx} + x^2 \frac{dy}{dx} = x$$

$$\left(1 + x^2\right) \frac{dy}{dx} = x$$

$$\frac{dy}{dx} = \frac{x}{1 + x^2}$$

Now, find y as a function of x.

$$dy = \frac{x}{1 + x^2}\, dx$$

$$\int dy = \int \frac{x}{1 + x^2}\, dx$$

$$u = 1 + x^2$$

$$du = 2x\, dx$$

$$\frac{1}{2}\, du = dx$$

$$\int dy = \frac{1}{2} \int \frac{1}{u}\, du$$

$$y = \frac{1}{2} \ln |u| + C$$

$$y = \frac{1}{2} \ln \left(1 + x^2\right) + C$$

$$y = \ln \sqrt{1 + x^2} + C$$

Solve the following separable equation.

$$\frac{1}{y}\,\frac{dy}{dx} + yx^2 \ln x = 0,\ y(1) = 1$$

$$\frac{1}{y}\,\frac{dy}{dx} = -yx^2 \ln x$$

$$-\frac{1}{y^2}\,dy = x^2 \ln x\,dx$$

$$-\int \frac{1}{y^2}\,dy = \int x^2 \ln x\,dx$$

$$-\int y^{-2}\,dy = (\ln x)\left(\frac{x^3}{3}\right) - \int \left(\frac{x^3}{3}\right)\left(\frac{1}{x}\right)dx$$

$u = \ln x \qquad dv = x^2\,dx$
$du = \frac{1}{x}\,dx \qquad v = \frac{x^3}{3}\,dx$

$$y^{-1} = \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\,dx$$

$$\frac{1}{y} = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C$$

$$1 = 0 - \frac{1}{9} + C$$

$$C = \frac{10}{9}$$

$$\frac{1}{y} = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + \frac{10}{9}$$