Name: ______________________

Section: ______________________

R·I·T School of Mathematical Sciences

30 - Laplace Transforms Applications Practice

MATH 211

1. A 1 kg toy rocket is launched with initial velocity of 6 m/s^2. A motor with a force of 12 N The motor burns out and stops producing force at $t = 10$ seconds. If the rocket experiences drag numerically equal to twice the velocity, find the velocity as a function of time.

$$mv' = F - kv$$

$$v' = 12 - 12u(t-10) - 2v$$

$$v' + 2v = 12 - 12u(t-10)$$

$$s\mathscr{L}\{v\} - v(0) + 2\mathscr{L}\{v\} = \mathscr{L}\{12\} - 12\mathscr{L}\{u(t-10)\}$$

$$s\mathscr{L}\{v\} - 6 + 2\mathscr{L}\{v\} = \frac{12}{s} - \frac{12e^{-10s}}{s}$$

$$(s+2)\mathscr{L}\{v\} = \frac{12}{s} + 6 - \frac{12e^{-10s}}{s}$$

$$\mathscr{L}\{v\} = \frac{12}{s(s+2)} + \frac{6}{s+2} - e^{-10s}\left[\frac{12}{s(s+2)}\right]$$

$$\mathscr{L}\{v\} = \frac{6}{s} - \frac{6}{s+2} + \frac{6}{s+2} - e^{-10s}\left[\frac{6}{s} - \frac{6}{s+2}\right]$$

$$\mathscr{L}\{v\} = \frac{6}{s} - e^{-10s}\left[\frac{6}{s} - \frac{6}{s+2}\right]$$

$$v(t) = 6 - \left[6 - 6e^{-2(t-10)}\right]u(t-10)$$

$$\boxed{\begin{gathered} \frac{12}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \\ 12 = As + 2A + Bs \\ \begin{array}{ll} A + B = 0 & 2A = 12 \\ B = -A & A = 6 \\ B = -6 & \end{array} \end{gathered}}$$

$$\boxed{\begin{gathered} F(s) = \frac{6}{s} - \frac{6}{s+2},\ a = 10 \\ f(t) = 6 - 6e^{-2t} \\ f(t-10) = 6 - 6e^{-2(t-10)} \\ f(t-10)u(t-10) = \left[6 - 6e^{-2(t-10)}\right]u(t-10) \end{gathered}}$$

2. A brine solution containing 0.25 kg of salt per liter flows at a constant rate of 4 liters per minute into a large tank that initially held 40 liters of pure water. The solution inside the tank is kept well stirred and flows out at the same rate. At $t = 4$ minutes, 2 kilograms of salt are instantaneously dumped into the tank. Find a formula for the amount of salt in the tank as a function of time t.

$$x' = \text{rate in} - \text{rate out}$$

$$x' = \left(\frac{1}{4}\right)(4) - \frac{4x}{40} + 2\delta(t-4)$$

$$x' = 1 - \frac{x}{10} + 2\delta(t-4)$$

$$x' + \frac{1}{10}x = 1 + 2\delta(t-4)$$

$$s\mathscr{L}\{x\} - x(0) + \frac{1}{10}\mathscr{L}\{x\} = \mathscr{L}\{1\} + 2\mathscr{L}\{\delta(t-4)\}$$

$$s\mathscr{L}\{x\} + \frac{1}{10}\mathscr{L}\{x\} = \frac{1}{s} + 2e^{-4s}$$

$$\left(s + \frac{1}{10}\right)\mathscr{L}\{x\} = \frac{1}{s} + 2e^{-4s}$$

$$\mathscr{L}\{x\} = \frac{1}{s\left(s+\frac{1}{10}\right)} + 2e^{-4s}\left(\frac{1}{s+\frac{1}{10}}\right)$$

$$\mathscr{L}\{x\} = \frac{10}{s} - \frac{10}{s+\frac{1}{10}} + 2e^{-4s}\left(\frac{1}{s+\frac{1}{10}}\right)$$

$$x = 10 - 10e^{-t/10} + 2e^{-(t-4)/10}u(t-4)$$

$$\frac{1}{s\left(s+\frac{1}{10}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{10}}$$

$$1 = As + \frac{A}{10} + Bs$$

$$A + B = 0 \qquad \frac{A}{10} = 1$$

$$B = -A \qquad A = 10$$

$$B = -10$$

$$F(s) = \frac{1}{s+\frac{1}{10}}, \quad a = 4$$

$$f(t) = e^{-t/10}$$

$$f(t-4) = e^{-(t-4)/10}$$

$$f(t-4)u(t-4) = e^{-(t-4)/10}u(t-4)$$

3. A brine solution containing 0.2 kg of salt per liter flows at a constant rate of 5 liters per minute into a large tank that initially held 100 liters of pure water. The solution inside the tank is kept well stirred and flows out at the same rate. At $t = 2$ minutes, a constant supply of solution with concentration of 6 kg/L begins flowing into the tank. Find a formula for the amount of salt in the tank as a function of time t.

$$x' = \text{rate in} - \text{rate out}$$

$$x' = \left(\frac{1}{5}\right)(5) + 6u(t-2) - (5)\left(\frac{x}{100}\right)$$

$$x' = 1 + 6u(t-2) - \frac{x}{20}$$

$$x' + \frac{x}{20} = 1 + 6u(t-2)$$

$$s\mathscr{L}\{x\} - x(0) + \frac{1}{20}\mathscr{L}\{x\} = \mathscr{L}\{1\} + 6\mathscr{L}\{u(t-2)\}$$

$$s\mathscr{L}\{x\} + \frac{1}{20}\mathscr{L}\{x\} = \frac{1}{s} + \frac{6e^{-2s}}{s}$$

$$\left(s + \frac{1}{20}\right)\mathscr{L}\{x\} = \frac{1}{s} + \frac{6e^{-2s}}{s}$$

$$\mathscr{L}\{x\} = \frac{1}{s\left(s+\frac{1}{20}\right)} + 6e^{-2s}\left[\frac{1}{s\left(s+\frac{1}{20}\right)}\right]$$

$$\mathscr{L}\{x\} = \frac{10}{s} - \frac{10}{s+\frac{1}{10}} + 6e^{-2s}\left[\frac{10}{s} - \frac{10}{s+\frac{1}{10}}\right]$$

$$x = 10 - 10e^{-t/10} + 6\left[10 - 10e^{-(t-2)/10}\right]u(t-2)$$

$$x = 10 - 10e^{-t/10} + \left[60 - 60e^{-(t-2)/10}\right]u(t-2)$$

$$\frac{1}{s\left(s+\frac{1}{10}\right)} = \frac{A}{s} + \frac{B}{s+\frac{1}{10}}$$

$$1 = As + \frac{A}{10} + Bs$$

$$\begin{aligned} A + B &= 0 \qquad & \tfrac{A}{10} &= 1 \\ B &= -A & A &= 10 \end{aligned}$$

$$F(s) = \frac{10}{s} - \frac{10}{s+\frac{1}{10}}, \quad a = 2$$

$$f(t) = 10 - 10e^{-t/10}$$

$$f(t-2) = 10 - 10e^{-(t-2)/10}$$

$$f(t-2)u(t-2) = \left[10 - 10e^{-(t-2)/10}\right]u(t-2)$$

4. A simple pendulum 8 ft long is pulled to a starting position of $\pi/4$ radians and released from rest. 3 seconds after the pendulum begins its motion, an electromagnet giving of a constant force of $10t$ lb directly below the pendulum (in equilibrium) is turned on, creating a force which alters the pedulums motion. Find the equation of angular motion of the pendulum. [You may use 32 ft/sec^2 for gravity.]

$$\theta'' + \frac{g}{l}\theta = f(t)$$

$$\theta'' + \frac{32}{8}\theta = 10tu(t-3)$$

$$\theta'' + 4\theta = 10tu(t-3)$$

$$s^2\mathscr{L}\{\theta\} - s\theta(0) - \theta'(0) + 4\mathscr{L}\{\theta\} = 10\mathscr{L}\{tu(t-3)\}$$

$$s^2\mathscr{L}\{\theta\} - \frac{\pi}{4}s - 0 + 4\mathscr{L}\{\theta\} = 10e^{-3s}\left[\frac{1}{s^2} + \frac{3}{s}\right]$$

$$s^2\mathscr{L}\{\theta\} + 4\mathscr{L}\{\theta\} = 10e^{-3s}\left[\frac{1+3s}{s^2}\right] + \frac{\pi}{4}s$$

$$(s^2+4)\mathscr{L}\{\theta\} = 10e^{-3s}\left[\frac{1+3s}{s^2}\right] + \frac{\pi}{4}s$$

$$\mathscr{L}\{\theta\} = 10e^{-3s}\left[\frac{3s+1}{s^2(s^2+4)}\right] + \frac{\pi}{4}\cdot\frac{s}{s^2+4}$$

$$\mathscr{L}\{\theta\} = 10e^{-3s}\left[\frac{3}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{s}{s^2+4}\right) - \frac{1}{4}\left(\frac{1}{s^2+4}\right)\right] + \frac{\pi}{4}\cdot\frac{s}{s^2+4}$$

$$\theta = 10\left[\frac{3}{4} + \frac{t-3}{4} - \frac{3}{4}\cos\left(2(t-3)\right) - \frac{1}{8}\sin\left(2(t-3)\right)\right]u(t-3) + \frac{\pi}{4}\cos\left(2t\right)$$

$$f(t) = t,\ a = 3$$

$$f(t+3) = t+3$$

$$\mathscr{L}\{f(t+3)\} = \mathscr{L}\{t+3\} = \frac{1}{s^2} + \frac{3}{s}$$

$$e^{-3s}\mathscr{L}\{f(t+3)\} = e^{-3s}\left[\frac{1}{s^2} + \frac{3}{s}\right]$$

$$\frac{3s+1}{s^2(s^2+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

$$3s+1 = As(s^2+4) + B(s^2+4) + (Cs+D)s^2$$

$$3s+1 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Ds^2$$

$$\begin{array}{cccc} A+C=0 & B+D=0 & 4A=3 & 4B=1 \\ C=-\frac{3}{4} & D=-\frac{1}{4} & A=\frac{3}{4} & B=\frac{1}{4} \end{array}$$

$$F(s) = \frac{3}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{s}{s^2+4}\right) - \frac{1}{4}\left(\frac{1}{s^2+4}\right),\ a = 3$$

$$F(s) = \frac{3}{4}\left(\frac{1}{s}\right) + \frac{1}{4}\left(\frac{1}{s^2}\right) - \frac{3}{4}\left(\frac{s}{s^2+4}\right) - \frac{1}{8}\left(\frac{2}{s^2+4}\right)$$

$$f(t) = \frac{3}{4} + \frac{t}{4} - \frac{3}{4}\cos\left(2t\right) - \frac{1}{8}\sin\left(2t\right)$$

$$f(t-3) = \frac{3}{4} + \frac{t-3}{4} - \frac{3}{4}\cos\left(2(t-3)\right) - \frac{1}{8}\sin\left(2(t-3)\right)$$

$$f(t-3)u(t-3) = \left[\frac{3}{4} + \frac{t-3}{4} - \frac{3}{4}\cos\left(2(t-3)\right) - \frac{1}{8}\sin\left(2(t-3)\right)\right]u(t-3)$$

5. A 1 kg mass is attached to a spring with constant 16 N/m. The spring is stretched 1 m past equilibrium and then released from rest. At $t = 3$ seconds, an electromagnet is switched on, producing a force of $f(t) = 32$ N to continue the springs motion. Find the equation of motion of the spring, assuming the absence of damping.

$$mx'' + bx' + kx = f(t)$$

$$x'' + 16x = 32u(t-3)$$

$$s^2\mathscr{L}\{x\} - sx(0) - x'(0) + 16\mathscr{L}\{x\} = 32\mathscr{L}\{u(t-3)\}$$

$$s^2\mathscr{L}\{x\} - s + 16\mathscr{L}\{x\} = \frac{32e^{-3s}}{s}$$

$$(s^2+16)\mathscr{L}\{x\} = \frac{32e^{-3s}}{s} + s$$

$$\mathscr{L}\{x\} = e^{-3s}\left[\frac{32}{s(s^2+16)}\right] + \frac{s}{s^2+16}$$

$$\mathscr{L}\{x\} = 2e^{-3s}\left[\left(\frac{1}{s}\right) - 2\left(\frac{s}{s^2+16}\right)\right] + \frac{s}{s^2+16}$$

$$\mathscr{L}\{x\} = 2e^{-3s}\left[\frac{1}{s} - \frac{s}{s^2+16}\right] + \frac{s}{s^2+16}$$

$$x(t) = 2[1 - cos(4(t-3))]u(t-3) + cos(4t)$$

$$\boxed{\begin{gathered}\frac{32}{s(s^2+16)} = \frac{A}{s} + \frac{Bs+C}{s^2+16}\\ 32 = As^2 + 16A + Bs^2 + Cs\\ \begin{array}{lll} A+B=0 & C=0 & 16A=32\\ B=-A & & A=2\\ B=-2 & & \end{array}\end{gathered}}$$

$$\boxed{\begin{gathered}F(s) = \frac{1}{s} - \frac{s}{s^2+16},\quad a=3\\ f(t) = 1-\cos(4t)\\ f(t-3) = 1-\cos(4(t-3))\\ f(t-3)u(t-3) = [1-\cos(4(t-3))]\,u(t-3)\end{gathered}}$$

6. An RL circuit contains an inductance of 2 H and a resistance of 6 Ω. A battery is attached giving off a constant voltage of 12 V and a second power supply produces a voltage of 12 V starting at $t = 3$ seconds. Find the current as a function of time if the initial current is 2 A.

$$\begin{gathered} Li' + Ri = E(t) \\ 2i' + 6i = 12 + 12u(t-3) \\ i' + 3i = 6 + 6u(t-3) \end{gathered}$$

$$\begin{aligned} s\mathscr{L}\{i\} - i(0) + 3\mathscr{L}\{i\} &= \mathscr{L}\{6\} + 6\mathscr{L}\{u(t-3)\} \\ s\mathscr{L}\{i\} - 2 + 3\mathscr{L}\{i\} &= \frac{6}{s} + \frac{6e^{-3s}}{s} \\ (s+3)\mathscr{L}\{i\} &= \frac{6}{s} + 2 + \frac{6e^{-3s}}{s} \\ \mathscr{L}\{i\} &= \frac{6}{s(s+3)} + \frac{2}{s+3} + e^{-3s}\left[\frac{6}{s(s+3)}\right] \\ \mathscr{L}\{i\} &= \frac{2}{s} - \frac{2}{s+3} + \frac{2}{s+3} + e^{-3s}\left[\frac{2}{s} - \frac{2}{s+3}\right] \\ \mathscr{L}\{i\} &= \frac{2}{s} + e^{-3s}\left[\frac{2}{s} - \frac{2}{s+3}\right] \\ i(t) &= 2 + \left[2 - 2e^{-3(t-3)}\right]u(t-3) \end{aligned}$$

$$\boxed{\begin{gathered} \frac{6}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \\ 6 = As + 3A + Bs \\ \begin{array}{ll} A + B = 0 & 3A = 6 \\ B = -A & A = 2 \\ B = -2 & \end{array} \end{gathered}}$$

$$\boxed{\begin{gathered} F(s) = \frac{2}{s} - \frac{2}{s+3}, \quad a = 3 \\ f(t) = 2 - 2e^{-3t} \\ f(t-3) = 2 - 2e^{-3(t-3)} \\ f(t-3)u(t-3) = \left[2 - 2e^{-3(t-3)}\right]u(t-3) \end{gathered}}$$

7. An RLC circuit contains an inductance of 1 H, resistance of 8 Ω and capacitance of $\frac{1}{20}$ F. A battery gives off a voltage of $E(t) = 4e^{-4t}$ V. Find the charge as a function of time if the initial current and charge are both zero. You MUST use Laplace Transforms to solve!

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

$$q'' + 8q' + 20q = 4e^{-4t}$$

$$s^2\mathscr{L}\{q\} - sq(0) - q'(0) + 8\left[s\mathscr{L}\{q\} - q(0)\right] + 20\mathscr{L}\{q\} = 4\mathscr{L}\{e^{-4t}\}$$

$$\begin{aligned} s^2\mathscr{L}\{q\} + 8s\mathscr{L}\{q\} + 20\mathscr{L}\{q\} &= \frac{4}{s+4} \\ (s^2 + 8s + 20)\mathscr{L}\{q\} &= \frac{4}{s+4} \\ \mathscr{L}\{q\} &= \frac{4}{(s+4)(s^2+8s+20)} \\ \mathscr{L}\{q\} &= \frac{1}{s+4} - \frac{s+4}{s^2+8s+20} \\ \mathscr{L}\{q\} &= \frac{1}{s+4} - \frac{s+4}{s^2+8s+16+4} \\ \mathscr{L}\{q\} &= \frac{1}{s+4} - \frac{s+4}{(s+4)^2+4} \\ q(t) &= e^{-4t} - e^{-4t}\cos(2t) \end{aligned}$$

$$\frac{4}{(s+4)(s^2+8s+20)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+8s+20}$$

$$4 = As^2 + 8As + 20A + Bs^2 + 4Bs + Cs + 4C$$

$$\begin{array}{ccc} A+B=0 & 8A+4B+C=0 & 20A+4C=4 \\ B=-A & 8A-4A+1-5A=0 & C=1-5A \\ B=-1 & A=1 & C=-4 \end{array}$$

8. An RLC circuit contains an inductance of 1 H, resistance of 4 Ω and capacitance of $\frac{1}{13}$ F. A battery gives off a voltage of $E(t) = 26$ V. Find the charge as a function of time if the initial current and charge are both zero. You MUST use Laplace Transforms to solve!

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

$$q'' + 4q' + 13q = 26$$

$$s^2\mathscr{L}\{q\} - sq(0) - q'(0) + 4\left[s\mathscr{L}\{q\} - q(0)\right] + 13\mathscr{L}\{q\} = \frac{26}{s}$$

$$s^2\mathscr{L}\{q\} + 4s\mathscr{L}\{q\} + 13\mathscr{L}\{q\} = \frac{26}{s}$$

$$(s^2 + 4s + 13)\mathscr{L}\{q\} = \frac{26}{s}$$

$$\mathscr{L}\{q\} = \frac{26}{s(s^2 + 4s + 13)}$$

$$\mathscr{L}\{q\} = \frac{2}{s} - 2\left(\frac{s+4}{s^2 + 4s + 13}\right)$$

$$\mathscr{L}\{q\} = \frac{2}{s} - 2\left(\frac{s+4}{s^2 + 4s + 4 + 9}\right)$$

$$\mathscr{L}\{q\} = \frac{2}{s} - 2\left(\frac{s+2+2}{(s+2)^2 + 9}\right)$$

$$\mathscr{L}\{q\} = \frac{2}{s} - 2\left(\frac{s+2}{(s+2)^2 + 9} + \frac{2}{(s+2)^2 + 3^2}\right)$$

$$\mathscr{L}\{q\} = \frac{2}{s} - 2\left(\frac{s+2}{(s+2)^2 + 9} + \frac{2}{3} \cdot \frac{3}{(s+2)^2 + 3^2}\right)$$

$$q = 2 - 2\left(e^{-2t}\cos(3t) + \frac{2}{3}e^{-2t}\sin(3t)\right)$$

$$q = 2 - 2e^{-2t}\cos(3t) - \frac{4}{3}e^{-2t}\sin(3t)$$

$$\frac{26}{s(s^2 + 4s + 13)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s + 13}$$

$$26 = As^2 + 4As + 13A + Bs^2 + Cs$$

$$\begin{array}{ccc} A + B = 0 & 4A + C = 0 & 13A = 26 \\ B = -A & C = -4A & A = 2 \\ B = -2 & C = -8 & \end{array}$$