Name: ______________________

Section: ______________________

R·I·T School of Mathematical Sciences

19 - Nonhomogeneous Equations III

MATH 211

Find the general solutions to the following nonhomogeneous equations.

1. $4y'' + 9y = x^2 + 2$

Complementary:

$$4y'' + 9y = 0$$

$$4r^2 + 9 = 0$$

$$4r^2 = -9$$

$$r^2 = -\frac{9}{4}$$

$$r = \pm\sqrt{-\frac{9}{4}}$$

$$r = \pm\frac{3}{2}i$$

$$y_c = c_1 \sin\left(\frac{3}{2}x\right) + c_2 \cos\left(\frac{3}{2}x\right)$$

Particular:

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$4y_p'' + 9y_p = x^2 + 2$$

$$4(2A) + 9(Ax^2 + Bx + C) = x^2 + 2$$

$$9Ax^2 + 9Bx + 8A + 9C = x^2 + 2$$

$$\begin{array}{ccc} 9A = 1 & 9B = 0 & 8A + 9C = 2 \\ A = \frac{1}{9} & B = 0 & \frac{8}{9} + 9C = 2 \\ & & 9C = \frac{10}{9} \\ & & C = \frac{10}{81} \end{array}$$

$$y_p = \frac{1}{9}x^2 + \frac{10}{81}$$

$$y = y_c + y_p$$

$$y = c_1 \sin\left(\frac{3}{2}x\right) + c_2 \cos\left(\frac{3}{2}x\right) + \frac{1}{9}x^2 + \frac{10}{81}$$

2. $r'' - 8r' + 16r = \sin\theta$

Complementary:

$$r'' - 8r' + 16r = 0$$

$$m^2 - 8m + 16 = 0$$

$$(m-4)(m-4) = 0$$

$$m_1 = m_2 = 4$$

$$r = c_1 e^{4\theta} + c_2 \theta e^{4\theta}$$

Particular:

$$r_p = A\sin\theta + B\cos\theta$$

$$r_p' = A\cos\theta - B\sin\theta$$

$$r_p'' = -A\sin\theta - B\cos\theta$$

$$r_p'' - 8r_p' + 16r_p = \sin\theta$$

$$-A\sin\theta - B\cos\theta - 8\left(A\cos\theta - B\sin\theta\right)$$
$$16\left(A\sin\theta + B\cos\theta\right) = \sin\theta$$

$$-A\sin\theta - B\cos\theta - 8A\cos\theta + 8B\sin\theta$$
$$+16A\sin\theta + 16B\cos\theta = \sin\theta$$

$$(-A + 8B + 16A)\sin\theta +$$
$$(-B - 8A + 16B)\cos\theta = \sin\theta$$

$$15A + 8B = 1 \qquad 15B - 8A = 0$$
$$8A = 15B$$
$$15\left(\tfrac{15B}{8}\right) + 8B = 1 \qquad \leftarrow A = \tfrac{15B}{8}$$
$$\tfrac{225B}{8} + 8B = 1$$
$$\tfrac{289B}{8} = 1$$
$$B = \tfrac{8}{289} \rightarrow \qquad A = \tfrac{15}{8} \cdot \tfrac{8}{289}$$
$$A = \tfrac{15}{289}$$

$$r_p = \frac{15}{289}\sin\theta + \frac{8}{289}\cos\theta$$

$$r = r_c + r_p$$

$$r = c_1 e^{4\theta} + c_2 \theta e^{4\theta} + \frac{15}{289}\sin\theta + \frac{8}{289}\cos\theta$$

Solve the following IVPs.

1. $i'' + 3i' + 2i = e^{-t},\ i(0) = -2,\ i'(0) = 0$

Complementary:

$$i'' + 3i' + 2i = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r_1 = -2,\ r_2 = -1$$

$$i_c = c_1 e^{-2t} + c_2 e^{-t}$$

Particular:

$$\cancel{i_p = Ae^{-t}}$$

$$i_p = Ate^{-t}$$

$$i_p' = Ae^{-t} - Ate^{-t}$$

$$i_p'' = -Ae^{-t} - Ae^{-t} + Ate^{-t}$$

$$i_p'' + 3i_p' + 2i_p = e^{-t}$$

$$-2Ae^{-t} + Ate^{-t} + 3\left(Ae^{-t} - Ate^{-t}\right)$$
$$+2Ate^{-t} = e^{-t}$$

$$-2Ae^{-t} + \cancel{Ate^{-t}} + 3Ae^{-t} - \cancel{3Ate^{-t}}$$
$$+\cancel{2Ate^{-t}} = e^{-t}$$

$$Ae^{-t} = e^{-t}$$

$$A = 1$$

$$i_p = te^{-t}$$

$$i = i_c + i_p$$

$$i = c_1 e^{-2t} + c_2 e^{-t} + te^{-t}$$

$$i = c_1 e^{-2t} + c_2 e^{-t} + te^{-t}$$

$$-2 = c_1 + c_2 + 0$$

$$-2 = c_1 + 1 - 2c_1$$

$$-3 = -c_1$$

$$c_1 = 3 \rightarrow$$

$$i = -2c_1 e^{-2t} - c_2 e^{-t} + e^{-t} - te^{-t}$$

$$0 = -2c_1 - c_2 + 1 - 0$$

$$\leftarrow c_2 = 1 - 2c_1$$

$$c_2 = 1 - 2(3)$$

$$c_2 = -5$$

$$i = 3e^{-2t} - 5e^{-t} + te^{-t}$$

2. $x'' + 10x' + 25x = 2e^{-5t} + 75t,\ x(0) = -3,\ x'(0) = 4$

Complementary:

$$x'' + 10x' + 25x = 0$$
$$r^2 + 10r + 25 = 0$$
$$(r+5)^2 = 0$$
$$r_1 = r_2 = -5$$
$$x_c = c_1 e^{-5t} + c_2 t e^{-5t}$$

Particular:

$$\cancel{x_p = Ae^{-5t} + Bt + C}$$
$$\cancel{x_p = Ate^{-5t} + Bt + C}$$
$$x_p = At^2e^{-5t} + Bt + C$$
$$x_p' = 2Ate^{-5t} - 5At^2e^{-5t} + B$$
$$x_p'' = 2Ae^{-5t} - 20Ate^{-5t} + 25At^2e^{-5t}$$
$$x_p'' + 10x_p' + 25x_p = 2e^{-5t} + 75t$$

$$2Ae^{-5t} - 20Ate^{-5t} + 25At^2e^{-5t} + 10(2Ate^{-5t} - 5At^2e^{-5t} + B) + 25(At^2e^{-5t} + Bt + C) = 2e^{-5t} + 75t$$

$$2Ae^{-5t} - \cancel{20Ate^{-5t}} + \cancel{25At^2e^{-5t}} + \cancel{20Ate^{-5t}} - \cancel{50At^2e^{-5t}} + 10B + \cancel{25At^2e^{-5t}} + 25Bt + 25C = 2e^{-5t} + 75t$$

$$2Ae^{-5t} + 10B + 25Bt + 25C = 2e^{-5t} + 75t$$

$$2A = 2 \qquad 10B + 25C = 0 \qquad 25B = 75$$
$$A = 1 \qquad 2B + 5C = 0 \qquad B = 3$$
$$2(3) + 5C = 0$$
$$6 = -5C$$
$$C = -\frac{6}{5}$$

$$x_p = t^2e^{-5t} + 3t - \frac{6}{5}$$

$$x = x_c + x_p$$

$$x = c_1e^{-5t} + c_2te^{-5t} + t^2e^{-5t} + 3t - \frac{6}{5}$$

$$x = c_1e^{-5t} + c_2te^{-5t} + t^2e^{-5t} + 3t - \frac{6}{5}$$
$$-3 = c_1 + 0 + 0 + 0 - \frac{6}{5}$$
$$c_1 = -\frac{9}{5}$$

$$x' = -5c_1e^{-5t} + c_2e^{-5t} - 5c_2te^{-5t} + 2te^{-5t} - 5t^2e^{-5t} + 3$$

$$4 = -5c_1 + c_2 - 0 + 0 - 0 + 3$$
$$4 = -5\left(-\frac{9}{5}\right) + c_2 + 3$$
$$4 = c_2 + 12$$
$$c_2 = -8$$

$$x = -\frac{9}{5}e^{-5t} - 8te^{-5t} + t^2e^{-5t} + 3t - \frac{6}{5}$$