Multivariable Calculus Practice

MATH 211

Functions of Several Variables:

1. Evaluate the given functions at the indicated points.

   (a) $f(x,y) = x^2 - 5y + y^2,\ (2,-2)$

   $$\begin{aligned} f(2,-2) &= 2^2 - 5(-2) + (-2)^2 \\ &= 4 + 10 + 4 \\ &= 18 \end{aligned}$$

   (b) $f(r,t) = r - \dfrac{r - 2t^2 - 5t}{r+t},\ (r, t+k)$

   $$\begin{aligned} f(r,t+k) &= r - \frac{r - 2(t+k)^2 - 5(t+k)}{r + (t+k)} \\ &= r - \frac{r - 2t^2 - 4tk - 2k^2 - 5t - 5k}{r+t+k} \\ &= \frac{r^2 + rt + rk - r + 2t^2 + 4tk + 2k^2 + 5t + 5k}{r+t+k} \end{aligned}$$

   (c) $f(x,y) = 3x^3 - x^2y + 5y^2,$ find $f(3x^2, x) - f(x,x)$

   $$\begin{aligned} f(3x^2,x) - f(x,x) &= \big(3(3x^2)^3 - (3x^2)^2x + 5x^2\big) - \big(3x^3 - x^2x + 5x^2\big) \\ &= 3(27x^6) - (9x^4)x + 5x^2 - 3x^3 + x^3 - 5x^2 \\ &= 81x^6 - 9x^5 - 2x^3 \end{aligned}$$

2. Evaluate the following limits, if they exist.

(a) $\lim\limits_{(x,y)\to(2,1)} (x+3y^2)$

$$= 2+3$$
$$= 5$$

(b) $\lim\limits_{(x,y)\to(2,4)} \dfrac{x+y}{x-y}$

$$= \frac{2+4}{2-4}$$
$$= \frac{6}{-2}$$
$$= -3$$

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x+y}{x^2-y^2}$

$$= \frac{0}{0}$$
$$= \lim_{(x,y)\to(0,0)} \frac{x+y}{(x+y)(x-y)}$$
$$= \lim_{(x,y)\to(0,0)} \frac{1}{x-y}$$
$$= \frac{1}{0} \text{ undefined}$$

(d) $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy-1}{1-xy}$

$$= \frac{1-1}{1-1} = \frac{0}{0}$$
$$= \lim_{(x,y)\to(1,1)} \frac{-(1-xy)}{1-xy}$$
$$= -1$$

Partial Derivatives:

1. Find the first partial derivatives of the following functions.

(a) $z = \dfrac{x^2}{y} - 2xy$

$$z_x = \frac{2x}{y} - 2y$$
$$z_y = -\frac{2x^2}{y^2} - 2x$$

(b) $z = xe^{2y}$

$$z_x = e^{2y}$$
$$z_y = 2xe^{2y}$$

(c) $f(x,y) = \dfrac{2+\cos x}{1-\sec(3y)}$

$$f_x(x,y) = \frac{-\sin x}{1-\sec(3y)}$$
$$f_y(x,y) = -(2+\cos x)(1-\sec(3y))^{-2}(0-\sec(3y)\tan(3y)\cdot 3)$$
$$= \frac{3(2+\cos x)\sec(3y)\tan(3y)}{(1-\sec(3y))^2}$$

(d) $z = \left(x^2 + xy^3\right)^4$

$$z_x = 4(x^2 + xy^3)^3(2x + y^3)$$
$$z_y = 4(x^2 + xy^3)^3(0 + 3xy^2) = 12xy^2(x^2 + xy^3)^3$$

(e) $f(x, y) = \ln(x^2 + y)$

$$f_x(x, y) = \frac{1}{x^2 + y} \cdot 2x = \frac{2x}{x^2 + y}$$
$$f_y(x, y) = \frac{1}{x^2 + y}$$

(f) $z = e^x \cos(xy) + e^{-2x} \tan y$

$$z_x = [e^x] \cos(xy) + e^x [-\sin(xy) \cdot y] + \left[-2e^{-2x}\right] \tan y$$
$$= e^x \cos(xy) - ye^x \sin(xy) - 2e^{-2x} \tan y$$
$$z_y = e^x [-\sin(xy) \cdot x] + e^{-2x} \sec^2 y$$
$$= -xe^x \sin(xy) + e^{-2x} \sec^2 y$$

2. Find the second and mixed partial derivatives of the following functions.

(a) $z = 2xy^3 - 3x^2y$

$$z_x = 2y^3 - 6xy$$
$$z_{xx} = -6y$$
$$z_{xy} = 6y^2 - 6x$$

$$z_y = 6xy^2 - 3x^2$$
$$z_{yy} = 12xy$$
$$z_{yx} = 6y^2 - 6x$$

(b) $f(x, y) = y \ln(x + 2y)$

$$f_x(x, y) = y\left(\frac{1}{x + 2y}\right)(1 + 0)$$
$$= \frac{y}{x + 2y}$$
$$f_{xx}(x, y) = -y(x + 2y)^{-2}(1 + 0)$$
$$= -\frac{y}{(x + 2y)^2}$$
$$f_{xy}(x, y) = \frac{(x + 2y)[1] - y[0 + 2]}{(x + 2y)^2}$$
$$= \frac{x}{(x + 2y)^2}$$

$$f_y(x, y) = y\left(\frac{1}{x + 2y}\right)(0 + 2) + \ln(x + 2y)$$
$$= \frac{2y}{x + 2y} + \ln(x + 2y)$$
$$f_{yy}(x, y) = \frac{(x + 2y)[2] - 2y[0 + 2]}{(x + 2y)^2} + \frac{1}{x + 2y} \cdot 2$$
$$= \frac{2x}{(x + 2y)^2} + \frac{2}{x + 2y}$$
$$f_{yx}(x, y) = -2y(x + 2y)^{-2}(1 + 0) + \frac{1}{x + 2y} \cdot 1$$
$$= -\frac{2y}{(x + 2y)^2} + \frac{x + 2y}{(x + 2y)^2}$$
$$= \frac{x}{(x + 2y)^2}$$

(c) $z = \dfrac{x}{y} + e^x \sin y$

$$z_x = \frac{1}{y} + e^x \sin y$$
$$z_{xx} = e^x \sin y$$
$$z_{xy} = -\frac{1}{y^2} + e^x \cos y$$

$$z_y = -\frac{x}{y^2} + e^x \cos y$$
$$z_{yy} = \frac{2x}{y^3} - e^x \sin y$$
$$z_{yx} = -\frac{1}{y^2} + e^x \cos y$$

(d) $z = \dfrac{1+\cos y}{1+x^2}$

$$\begin{aligned}
z_x &= -(1+\cos y)(1+x^2)^{-2}(2x) \\
&= -\frac{2x(1+\cos y)}{(1+x^2)^2} \\
z_{xx} &= \frac{-(1+x^2)^2[2(1+\cos y)] + 2x(1+\cos y)[2(1+x^2)(2x)]}{(1+x^2)^4} \\
&= \frac{(6x^2-2)(1+\cos y)}{(1+x^2)^3} \\
z_{xy} &= -\frac{2x(-\sin y)}{(1+x^2)^2} \\
&= \frac{2x\sin y}{(1+x^2)^2} \\
z_y &= \frac{-\sin y}{1+x^2} \\
z_{yy} &= \frac{-\cos y}{1+x^2} \\
z_{yx} &= -[-\sin y(1+x^2)^{-2}(2x)] \\
&= \frac{2x\sin y}{(1+x^2)^2}
\end{aligned}$$

3. Find the total differential for the following functions.

(a) $f(x,y) = 2x^2 - y^2 + 3x$

$$\begin{aligned}
f_x(x,y) &= 4x+3 \\
f_y(x,y) &= -2y \\
df(x,y) &= f_x(x,y)dx + f_y(x,y)dy \\
df(x,y) &= (4x+3)dx - 2ydy
\end{aligned}$$

(b) $f(x,y) = xe^y - y^2$

$$\begin{aligned}
f_x(x,y) &= e^y \\
f_y(x,y) &= xe^y - 2y \\
df(x,y) &= f_x(x,y)dx + f_y(x,y)dy \\
df(x,y) &= e^y dx + (xe^y - 2y)dy
\end{aligned}$$

(c) $z = \sin(xy) - y\cos x$

$$\begin{aligned}
z_x &= \cos(xy)\cdot y + y\sin x \\
z_y &= \cos(xy)\cdot x - \cos x \\
dz &= z_x dx + z_y dy \\
dz &= (y\cos(xy) + y\sin x)dx + (x\cos(xy) - \cos x)dy
\end{aligned}$$

4. Examine the function for any relative extrema.

(a) $f(x,y) = x^2 + y^2 - 2x - 6y + 10$

$$f_x(x,y) = 2x - 2 \qquad f_y(x,y) = 2y - 6$$
$$0 = 2x - 2 \qquad 0 = 2y - 6$$
$$x = 1 \qquad y = 3$$

Critical Point: $(1,3)$

$$f_{xx}(x,y) = 2 > 0 \qquad f_{yy}(x,y) = 2 \qquad f_{xy}(x,y) = 0$$
$$f_{xx}(1,3) = 2 \qquad f_{yy}(1,3) = 2 \qquad f_{xy}(1,3) = 0$$

$$\begin{aligned} D &= f_{xx}(1,3)f_{yy}(1,3) - [f_{xy}(1,3)]^2 \\ &= (2)(2) - 0^2 \\ &= 4 > 0 \end{aligned}$$

$$f(1,3) = 1 + 9 - 2 - 18 + 10 = 0$$

There is a relative minimum at $(1,3,0)$.

(b) $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 2$

$$\begin{aligned} f_x(x,y) &= 2x + y + 3 \\ 0 &= 2x + y + 3 \\ y &= -2x - 3 \\ y &= -2(-3) - 3 \\ y &= 6 - 3 \\ y &= 3 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= x + 2y - 3 \\ 0 &= x + 2y - 3 \\ 0 &= x + 2(-2x - 3) - 3 \\ 0 &= x - 4x - 6 - 3 \\ 3x &= -9 \\ x &= -3 \end{aligned}$$

Critical Point: $(-3, 3)$

$$\begin{aligned} f_{xx}(x,y) &= 2 > 0 \\ f_{xx}(-3,3) &= 2 \end{aligned}$$

$$\begin{aligned} f_{yy}(x,y) &= 2 \\ f_{yy}(-3,3) &= 2 \end{aligned}$$

$$\begin{aligned} f_{xy}(x,y) &= 1 \\ f_{xy}(-3,3) &= 1 \end{aligned}$$

$$\begin{aligned} D &= f_{xx}(-3,3)f_{yy}(-3,3) - [f_{xy}(x,y)]^2 \\ &= (2)(2) - 1^2 \\ &= 4 - 1 \\ &= 3 > 0 \end{aligned}$$

$$f(-3,3) = 9 - 9 + 9 - 9 - 9 + 2 = -7$$

There is a relative minimum at $(-3, 3, -7)$.

(c) $f(x,y) = x^4 - 4x + y^2$

$$f_x(x,y) = 4x^3 - 4$$
$$0 = 4x^3 - 4$$
$$x^3 = 1$$
$$x = 1$$

$$f_y(x,y) = 2y$$
$$0 = 2y$$
$$y = 0$$

Critical Point: $(1,0)$

$$f_{xx}(x,y) = 12x^2$$
$$f_{xx}(1,0) = 12 > 0$$

$$f_{yy}(x,y) = 2$$
$$f_{yy}(1,0) = 2$$

$$f_{xy}(x,y) = 0$$
$$f_{xy}(1,0) = 0$$

$$\begin{aligned} D &= f_{xx}(1,0)f_{yy}(1,0) - [f_{xy}(1,0)]^2 \\ &= (12)(2) - 0 \\ &= 24 > 0 \end{aligned}$$

$$f(1,0) = 1 - 4 + 0 = -3$$

There is a relative minimum at $(1,0,-3)$.

(d) $f(x,y) = xy + \frac{1}{x} + \frac{8}{y}$

$$\begin{aligned} f_x(x,y) &= y - \frac{1}{x^2} \\ 0 &= \frac{x^2y-1}{x^2} \\ 0 &= x^2y - 1 \\ 0 &= \left(\frac{8}{y^2}\right)^2 y - 1 \\ 1 &= \frac{64}{y^3} \\ y^3 &= 64 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} f_y(x,y) &= x - \frac{8}{y^2} \\ 0 &= \frac{xy^2-8}{y^2} \\ 0 &= xy^2 - 8 \\ x &= \frac{8}{y^2} \\ x &= \frac{8}{16} \\ x &= \frac{1}{2} \end{aligned}$$

$$\text{Critical Point: } \left(\frac{1}{2}, 4\right)$$

$$f_{xx}(x,y) = \frac{2}{x^3} \qquad f_{xx}\left(\frac{1}{2},4\right) = \frac{2}{(1/2)^3} = 16 > 0$$

$$f_{yy}(x,y) = \frac{16}{y^3} \qquad f_{yy}\left(\frac{1}{2},4\right) = \frac{16}{4^3} = \frac{1}{4}$$

$$f_{xy}(x,y) = 1 \qquad f_{xy}\left(\frac{1}{2},4\right) = 1$$

$$\begin{aligned} D &= f_{xx}\left(\frac{1}{2},4\right) f_{yy}\left(\frac{1}{2},4\right) - \left[f_{xy}\left(\frac{1}{2},4\right)\right]^2 \\ &= (16)(1/4) - 1 \\ &= 4 - 1 \\ &= 3 > 0 \end{aligned}$$

$$f\left(\frac{1}{2},4\right) = 2 + 2 + 2 = 8$$

There is a relative minimum at $\left(\frac{1}{2}, 4, 8\right)$.

5. The centripetal acceleration of a particle moving in a circle is $a = \frac{v^2}{r}$, where v is the velocity and r is the radius of the circle. Approximate the change in the acceleration if v increases by 0.1m/sec from 4m/sec and r decreases by 0.2m from 10m.

Homework Solutions

6. The temperature distribution in a square 16 foot room containing a heating element and vent fan is given by

$$T(x,y) = 72 - 2x^2 - y^2 + 8x + 2y$$

where the point (x,y) is your location on the floor of that room and temperature is given in Fahrenheit. Find the location in the room where the temperature is at its highest. What is the maximum temperature?

$$\begin{aligned} T_x(x,y) &= -4x + 8 \\ 0 &= -4x + 8 \\ 4x &= 8 \\ x &= 2 \end{aligned}$$

$$\begin{aligned} T_y(x,y) &= -2y + 2 \\ 0 &= -2y + 2 \\ 2y &= 2 \\ y &= 1 \end{aligned}$$

Critical Point: $(2,1)$

$$T_{xx}(x,y) = -4 \qquad T_{yy}(x,y) = -2 \qquad T_{xy}(x,y) = 0$$

$$T_{xx}(2,1) = -4 < 0 \qquad T_{yy}(2,1) = -2 \qquad T_{xy}(2,1) = 0$$

$$D = T_{xx}(2,1)T_{yy}(2,1) - [T_{xy}(2,1)]^2 = (-4)(-2) - 0^2 = 8 > 0$$

$$T(2,1) = 72 - 2(2)^2 - 1^2 + 8(2) + 2(1) = 72 - 8 - 1 + 16 + 2 = 81$$

The maximum temperature is 81°F and it occurs at $(2,1)$.

7. The voltage V is across a resistance R. The current i is given by $i = V/R$. Find the approximate change in the current if the voltage changes from 220 V to 225 V and the resistance changes from 20 Ω to 21 Ω.

$$i(V,R) = \frac{V}{R}$$

$$i_V(V,R) = \frac{1}{R} \implies i_V(220,20) = \frac{1}{20}$$

$$i_R(V,R) = -\frac{V}{R^2} \implies i_R(220,20) = -\frac{220}{(20)^2} = -\frac{220}{400} = -\frac{11}{20}$$

$$di = i_V(220,20)dV + i_R(220,20)dR$$

$$\begin{aligned} di &= \left(\frac{1}{20}\right)(5) + \left(-\frac{11}{20}\right)(1) \\ &= \frac{5}{20} - \frac{11}{20} \\ &= -\frac{6}{20} \\ &= -\frac{3}{10} \text{ A} \end{aligned}$$

8. An electronic manufacturer determines that the profit P (in dollars) obtained by producing x units of a iPhone circuit board and y units of a TI-$n$spire calculator circuit board is approximated by the model

$$P(x,y) = 14x + 10y - \frac{1}{2}x^2 - \frac{1}{2}xy - \frac{1}{2}y^2 - 10$$

Find the production level of each that produces the maximum profit.

Recitation Solutions

9. A cylindrical rod made from aluminum is being heated in a vat of boiling water which compromises the dimensional and rigidity of the rod. The rod's radius increases by 0.1 mm from 2 mm and the length increases by 0.25 mm from 20 mm. Find the approximate change in the volume of the cylindrical rod.

$$V(r,l) = \pi r^2 l$$

$$dr = \frac{1}{10} \qquad dl = \frac{1}{4}$$

$$V_r(r,h) = 2\pi r l \qquad V_l(r,h) = \pi r^2$$

$$V_r(2,20) = 2\pi(2)(20) = 80\pi \qquad V_l(2,20) = \pi(2)^2 = 4\pi$$

$$dV = V_r(2,20)dr + V_l(2,20)dl$$

$$\begin{aligned} dV &= (80\pi)\left(\frac{1}{10}\right) + (4\pi)\left(\frac{1}{4}\right) \\ &= 8\pi + \pi \\ &= 9\pi \text{ mm}^3 \end{aligned}$$

Double Integration:

1. Evaluate the following integrals.

(a) $\displaystyle\int_0^1 \int_2^3 \sqrt{x+4y}\ dx\ dy$

$$= \int_0^1 \int_{2+4y}^{3+4y} \sqrt{u_1}\ du_1\ dy$$
$$= \int_0^1 \int_{2+4y}^{3+4y} u_1^{1/2}\ du_1\ dy$$
$$= \int_0^1 \left[\frac{2}{3}u_1^{3/2}\right]_{2+4y}^{3+4y} dy$$
$$= \frac{2}{3}\int_0^1 \left[(3+4y)^{3/2} - (2+4y)^{3/2}\right] dy$$
$$= \frac{2}{3}\left[\int_0^1 (3+4y)^{3/2}\ dy - \int_0^1 (2+4y)^{3/2}\ dy\right]$$
$$= \frac{2}{3}\left[\int_3^7 u_2^{3/2}\ du_2 - \int_2^6 u_3^{3/2}\ du_3\right]$$
$$= \frac{2}{3}\left[\frac{2}{5}u_2^{5/2}\Big|_3^7 - \frac{2}{5}u_3^{5/2}\Big|_2^6\right]$$
$$= \frac{4}{15}\left[(7^{5/2} - 3^{5/2}) - (6^{5/2} - 2^{5/2})\right]$$
$$= \frac{4}{15}\left[7^{5/2} - 3^{5/2} - 6^{5/2} + 2^{5/2}\right]$$

$$u_1 = x + 4y$$
$$du_1 = (1+0)\ dx$$
$$du_1 = dx$$
$$x = 2 \implies u_1 = 2+4y$$
$$x = 3 \implies u_1 = 3+4y$$

$$u_2 = 3+4y$$
$$du_2 = 4\ dy$$
$$\frac{1}{4}\ du_2 = dy$$
$$y = 0 \implies u_2 = 3$$
$$y = 1 \implies u_2 = 7$$

$$u_3 = 2+4y$$
$$du_3 = 4\ dy$$
$$\frac{1}{4}\ du_3 = dy$$
$$y = 0 \implies u_3 = 2$$
$$y = 1 \implies u_3 = 6$$

(b) $\displaystyle\int_0^{\sqrt{\pi/2}} \int_0^{x^2} x\cos y\ dy\ dx$

$$= \int_0^{\sqrt{\pi/2}} [x\sin y]_0^{x^2}\ dx$$
$$= \int_0^{\sqrt{\pi/2}} x\sin(x^2)\ dx$$
$$= \frac{1}{2}\int_0^{\pi/2} \sin u\ du$$
$$= \frac{1}{2}[-\cos u]_0^{\pi/2}$$
$$= -\frac{1}{2}[\cos(\pi/2) - \cos(0)]$$
$$= -\frac{1}{2}[0-1]$$
$$= \frac{1}{2}$$

$$u = x^2$$
$$du = 2x\ dx$$
$$\frac{1}{2}\ du = dx$$
$$x = 0 \implies u = 0$$
$$x = \sqrt{\frac{\pi}{2}} \implies u = \frac{\pi}{2}$$

2. Use a double integral to find the area of the indicated region.

(a) $R: \{y = x,\ y = 2x,\ x = 3\}$

$$V = \int_0^3 \int_x^{2x} dy\ dx$$

(b) $R : \{y = \sin x,\ y = \cos x,\ x = 0,\ x = \pi/4\}$

$$V = \int_0^{\pi/4} \int_{\sin x}^{\cos x} \ dy\ dx$$

(c) $R : \{y = 4 - x,\ y = x,\ y = 0\}$

$$V = \int_0^{2} \int_{y}^{4-y} \ dx\ dy$$

(d) $R : \left\{y = 4 - x^2,\ y = 3x\right\}$

$$V = \int_{-4}^{1} \int_{3x}^{4-x^2} \ dy\ dx$$

3. Use a double integral to find the volume of the solid bounded

(a) above by the surface $z = 9 - x^2 - y^2$ and below by $R : \{y = x,\ y = 2x,\ x = 3\}$.

$$V = \int_0^{3} \int_{x}^{2x} (9 - x^2 - y^2)\ dy\ dx$$

(b) above by the surface $z = e^{x+2y}$ and below by $R : \{y = \sin x,\ y = \cos x,\ x = 0,\ x = \pi/4\}$

$$V = \int_0^{\pi/4} \int_{\sin x}^{\cos x} e^{x+2y}\ dy\ dx$$

(c) above by the surface $z = \sqrt{x + 2y}$ and below by $R : \{y = 4 - x,\ y = x,\ y = 0\}$

$$V = \int_0^{2} \int_{y}^{4-y} \sqrt{x + 2y}\ dx\ dy$$

(d) above by the surface $z = y + 5$ and below by $R : \left\{y = 4 - x^2,\ y = 3x\right\}$

$$V = \int_{-4}^{1} \int_{3x}^{4-x^2} (y + 5)\ dy\ dx$$