Name: ____________________

Section: ____________________

R·I·T School of Mathematical Sciences

14 - Kirchhoff's Law

MATH 211

An RLC circuit consists of a voltage source, a resistor, an inductor and a capacitor. The current as a function of time is the rate at which the charge flows through the circuit. The sum of the voltages across the circuit is given by

$$V = V_L + V_R + V_C$$

where L represents inductance (H), R represents resistance (Ω) and C represents capacitance (F).

Ohm's Law finds the voltage drop across a resistor. State V_R.

$$V_R = iR$$

The voltage drop across an inductor is equal to the product of the inductance and the rate of change of the current with respect to time. State V_L.

$$V_L = L\,\frac{di}{dt}$$

The voltage drop across the capacitor is equal to the ratio of the charge q and capacitance. Find V_C.

$$V_C = \frac{q}{C}$$

Rewrite the sum of the voltages as a differential equation.

$$V = L\,\frac{di}{dt} + iR + \frac{q}{C}$$

$$L\,\frac{di}{dt} + Ri + \frac{q}{C} = E(t)$$

1. Suppose that in a simple RL circuit the resistance is 10 Ω and the inductance is 2 H. If a battery gives a constant voltage of 30 V and the switch is closed when $t = 0$ so the current starts at 0 A, find the current as a function of time. Then, find the limiting value of the current.

$$L\,\frac{di}{dt} + Ri = E(t)$$

$$2\,\frac{di}{dt} + 10i = 30$$

$$\frac{di}{dt} + 5i = 15$$

$$e^{\int 5\ dt} = e^{5t}$$

$$e^{5t}\left[\frac{di}{dt} + 5i\right] = 15e^{5t}$$

$$\frac{d}{dt}\left[ie^{5t}\right] = 15e^{5t}$$

$$\int d\left[ie^{5t}\right] = 15\int e^{5t}\ dt$$

$$ie^{5t} = \frac{15}{5}e^{5t} + C$$

$$i = 3 + Ce^{-5t}$$

$$i(0) = 0$$

$$0 = 3 + Ce^{0}$$

$$0 = 3 + C$$

$$C = -3$$

$$i(t) = 3 - 3e^{-5t}$$

$$\lim_{t\to\infty} i(t) = \lim_{t\to\infty}\left(3 - 3e^{-5t}\right) = 3 - 3(0) = 3\text{ A}$$

2. The rate of change of charge with respect to time is equal to the current. Write the derivative representation of this statement. Then, rewrite the differential equation on the first page as a model for the charge flow over a circuit with respect to time.

$$\frac{dq}{dt} = i$$

$$\frac{d^2q}{dt^2} = \frac{di}{dt}$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

3. Consider an RC circuit where the resistance is 10 Ω and the capacitance is 10 μF. If a battery gives a voltage of $E(t) = e^{-t}$ V and the switch is closed such that the initial charge in the circuit is 0 C, find the charge and current as functions of time.

$$R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

$$10\frac{dq}{dt} + 10^5 q = e^{-t}$$

$$\frac{dq}{dt} + 10,000q = e^{-t}$$

$$e^{\int 10,000 \ dt} = e^{10,000t}$$

$$e^{10,000t}\left[\frac{dq}{dt} + 10,000q\right] = e^{-t}e^{10,000t}$$

$$\frac{d}{dt}\left[qe^{10,000t}\right] = e^{9,999t}$$

$$\int d\left[qe^{10,000t}\right] = \int e^{9,999t} \ dt$$

$$qe^{10,000t} = \frac{1}{9,999}e^{9,999t} + K$$

$$q = \frac{1}{9,999}e^{-t} + Ke^{-10,000t}$$

$$0 = \frac{1}{9,999}e^{0} + Ke^{0}$$

$$K = -\frac{1}{9,999}$$

$$q(t) = \frac{1}{9,999}e^{-t} - \frac{1}{9,999}e^{-10,000t}$$

$$i(t) = q'(t) = -\frac{1}{9,999}e^{-t} + \frac{10,000}{9,999}e^{-10,000t}$$

4. Find the current in a simple circuit with inductance 3 H, resistance 12 Ω and voltage source $E(t) = 3\sin t$ V if the initial current is $i(0) = 0$ A.

$$L\frac{di}{dt} + Ri = E(t)$$

$$3\frac{di}{dt} + 12i = 3\sin t$$

$$\frac{di}{dt} + 4i = \sin t$$

$$e^{\int 4\ dt} = e^{4t}$$

$$e^{4t}\left[\frac{di}{dt} + 4i\right] = e^{4t}\sin t$$

$$\frac{d}{dt}\left[ie^{4t}\right] = e^{4t}\sin t$$

$$\int d\left[ie^{4t}\right] = \int e^{4t}\sin t\ dt$$

$$ie^{4t} = -\frac{1}{17}e^{4t}\cos t + \frac{4}{17}e^{4t}\sin t + C$$

$$i = -\frac{1}{17}\cos t + \frac{4}{17}\sin t + Ce^{-4t}$$

$$0 = -\frac{1}{17}(1) + \frac{4}{17}(0) + C(1)$$

$$C = \frac{1}{17}$$

$$i(t) = -\frac{1}{17}\cos t + \frac{4}{17}\sin t + \frac{1}{17}e^{-4t}$$

$$u = e^{4t} \qquad dv = \sin t\ dt$$
$$du = 4e^{4t}\ dt \qquad v = -\cos t$$

$$\int e^{4t}\sin t\ dt = -e^{4t}\cos t + 4\int e^{4t}\cos t\ dt$$

$$u = e^{4t} \qquad dv = \cos t\ dt$$
$$du = 4e^{4t}\ dt \qquad v = \sin t$$

$$\int e^{4t}\sin t\ dt = -e^{4t}\cos t + 4\left[e^{4t}\sin t - 4\int e^{4t}\sin t\ dt\right]$$

$$\int e^{4t}\sin t\ dt = -e^{4t}\cos t + 4e^{4t}\sin t - 16\int e^{4t}\sin t\ dt$$

$$17\int e^{4t}\sin t\ dt = -e^{4t}\cos t + 4e^{4t}\sin t + C$$

$$\int e^{4t}\sin t\ dt = -\frac{1}{17}e^{4t}\cos t + \frac{4}{17}e^{4t}\sin t + C$$