Name: ______________________

Section: ______________________

R·I·T School of Mathematical Sciences

28 - First Order Applications with Laplace Transforms

MATH 211

1. A boat with a mass of 10 slugs is being towed at 3 feet per second. The tow rope is then cut and the motor, that exerts a force of 20 lb on the boat, is started one minute later. If the water exerts a retarding force that numerically equals twice the velocity, what is the velocity of the boat 3 minutes later?

$$mv' = F - kv,\ v(0) = 3(60) = 180$$
$$mv' + kv = F$$
$$10v' + 2v = 20u(t-1)$$
$$5v' + v = 10u(t-1)$$
$$5\left[s\mathscr{L}\{v\} - v(0)\right] + \mathscr{L}\{v\} = \mathscr{L}\{10u(t-1)\}$$
$$5\left[s\mathscr{L}\{v\} - 180\right] + \mathscr{L}\{v\} = \frac{10e^{-s}}{s}$$
$$5s\mathscr{L}\{v\} - 900 + \mathscr{L}\{v\} = \frac{10e^{-s}}{s}$$
$$(5s+1)\mathscr{L}\{v\} = \frac{10e^{-s}}{s} + 900$$
$$\mathscr{L}\{v\} = e^{-s}\left[\frac{10}{s(5s+1)}\right]_{\star_1} + \frac{900}{5s+1}$$
$$\mathscr{L}\{v\} = e^{-s}\left[\frac{10}{s} - \frac{50}{5s+1}\right]_{\star_2} + \frac{180}{s+1/5}$$
$$v = \left[10 - 10e^{-(t-1)/5}\right]u(t-1) + 180e^{-t/5}$$

$$\star_1\ \frac{10}{s(5s+1)} = \frac{A}{s} + \frac{B}{5s+1}$$
$$10 = 5As + A + Bs$$
$$5A + B = 0 \qquad A = 10$$
$$50 + B = 0$$
$$B = -50$$
$$\frac{10}{s(5s+1)} = \frac{10}{s} + \frac{-50}{5s+1}$$

$$\star_2 F(s) = \frac{10}{s} - \frac{10}{s+1/5}$$
$$f(t) = 10 - 10e^{-t/5}$$
$$f(t-1) = 10 - 10e^{-(t-1)/5}$$
$$f(t-1)u(t-1) = \left[10 - 10e^{-(t-1)/5}\right]u(t-1)$$

2. Suppose that in a simple RL circuit the resistance is 10 Ω and the inductance is 2 H. If a battery gives a voltage of $E(t) = \sin t\ u(t-\pi)$ V and the switch is closed when $t = 0$ so the current starts at 3 mA, find the current as a function of time.

$$Li' + Ri = E(t)$$

$$2i' + 10i = \sin t\ u(t-\pi)$$

$$2\left[s\mathscr{L}\{i\} - i(0)\right] + 10\mathscr{L}\{i\} = \mathscr{L}\{\sin t\ u(t-\pi)\}_{\star_1}$$

$$2s\mathscr{L}\{i\} - 2\left(\frac{3}{1000}\right) + 10\mathscr{L}\{i\} = -e^{-\pi s}\cdot\frac{1}{s^2+1}$$

$$(2s+10)\mathscr{L}\{i\} = -e^{-\pi s}\cdot\frac{1}{s^2+1} + \frac{3}{500}$$

$$\mathscr{L}\{i\} = -e^{-\pi s}\cdot\frac{1}{(s^2+1)(2s+10)} + \frac{3}{500}\cdot\frac{1}{2s+10}$$

$$\mathscr{L}\{i\} = -\frac{1}{2}e^{-\pi s}\cdot\frac{1}{(s^2+1)(s+5)}_{\star_2} + \frac{3}{1000}\cdot\frac{1}{s+5}$$

$$i = -\frac{1}{2}\cdot\frac{1}{26}\left[\cos t - 5\sin t + e^{-5(t-\pi)}\right]\ u(t-\pi) + \frac{3}{1000}e^{-5t}$$

$$i = -\frac{1}{52}\left[\cos t - 5\sin t + e^{-5(t-\pi)}\right]\ u(t-\pi) + \frac{3}{1000}e^{-5t}$$

$$\star_1\ a = \pi,\ f(t) = \sin t$$

$$f(t+\pi) = \sin(t+\pi) = -\sin t$$

$$\mathscr{L}\{f(t+\pi)\} = -\mathscr{L}\{\sin t\} = -\frac{1}{s^2+1}$$

$$e^{-\pi s}\mathscr{L}\{f(t+\pi)\} = -e^{-\pi s}\cdot\frac{1}{s^2+1}$$

$$\star_2\ a = \pi,\ F(s) = \frac{1}{(s^2+1)(s+5)} = \frac{As+B}{s^2+1} + \frac{C}{s+5}$$

$$1 = As^2 + 5As + Bs + 5B + Cs^2 + C$$

$A + C = 0$	$5A + B = 0$	$5B + C = 1$
$C = -A$	$B = -5A$	$5(-5A) - A = 1$
		$-26A = 1$
		$A = -\frac{1}{26}$
$C = \frac{1}{26}$	$B = \frac{5}{26}$	

$$F(s) = \frac{(-1/26)\,s + 5/26}{s^2+1} + \frac{1/26}{s+5}$$

$$F(s) = \frac{1}{26}\left[-\frac{s}{s^2+1} + \frac{5}{s^2+1} + \frac{1}{s+5}\right]$$

$$f(t) = \frac{1}{26}\left[-\cos t + 5\sin t + e^{-5t}\right]$$

$$f(t-\pi) = \frac{1}{26}\left[-\cos(t-\pi) + 5\sin(t-\pi) + e^{-5(t-\pi)}\right]$$

$$f(t-\pi)\ u(t-\pi) = \frac{1}{26}\left[\cos t - 5\sin t + e^{-5(t-\pi)}\right]\ u(t-\pi)$$

3. Pure water flows at a constant rate of 5 liters per minute into a large tank that initially held 125 liters of brine solution in which was dissolved 1 kilogram of salt. The solution inside the tank is kept well stirred and flows out at the same rate. At $t = 5$ minutes, 4 kilograms of salt are instantaneously dumped into the tank. Find a formula for the amount of salt in the tank as a function of time t. How much salt is in the tank after 10 minutes?

$$\frac{dx}{dt} = 0 - \frac{5x}{125} + 4\delta(t-5)$$

$$x' + \frac{1}{25}x = 4\delta(t-5)$$

$$s\mathscr{L}\{x\} - x(0) + \frac{1}{25}\mathscr{L}\{x\} = 4\mathscr{L}\{\delta(t-5)\}$$

$$s\mathscr{L}\{x\} - 1 + \frac{1}{25}\mathscr{L}\{x\} = 4e^{-5s}$$

$$\left(s + \frac{1}{25}\right)\mathscr{L}\{x\} = 4e^{-5s} + 1$$

$$\mathscr{L}\{x\} = 4e^{-5s} \cdot \underbrace{\frac{1}{s + {}^{1}\!/_{25}}}_{\star} + \frac{1}{s + {}^{1}\!/_{25}}$$

$$x = 4e^{-(t\,-\,5)/25}\,u(t-5) + e^{-t/25}$$

$$\boxed{\begin{gathered} \star\ a = 5,\ F(s) = \frac{1}{s + {}^{1}\!/_{25}} \\ f(t) = e^{-t/25} \\ f(t-5) = e^{-(t\,-\,5)/25} \\ f(t-5)\,u(t-5) = e^{-(t\,-\,5)/25}\,u(t-5) \end{gathered}}$$