Name: ______________________

Section: ______________________

Date: ______________________

R·I·T School of Mathematical Sciences

Kirchhoff's Law

MATH 211 - 01

Consider the RLC circuit with an inductance of 1 H, a resistance of 100 Ω, a capacitance of 400 μF and a voltage source of 30 V. The initial current is 2 A and the initial charge is 0 C.

1. Set up the differential equation whose solution models the charge in the circuit at time t.

$$Lq'' + Rq' + \frac{1}{C}q = E$$
$$q'' + 100q' + \frac{1000000}{400}q = 30$$
$$q'' + 100q' + 2500q = 30$$

2. Find the transient solution.

$$q'' + 100q' + 2500q = 0$$
$$m^2 + 100m + 2500 = 0$$
$$(m + 50)(m + 50) = 0$$
$$m_1 = m_2 = -50$$
$$q_c(t) = c_1e^{-50t} + c_2te^{-50t}$$

3. State the linearly independent solutions from the transient solution and verify that they satisfy the differential equation.

$$q_1 = e^{-50t}$$
$$q_1' = -50e^{-50t}$$
$$q_1'' = 2500e^{-50t}$$
$$q_1'' + 100q_1' + 2500q_1 = 0$$
$$2500e^{-50t} + 100(-50e^{-50t}) + 2500e^{-50t} = 0$$
$$2500e^{-50t} - 5000e^{-50t} + 2500e^{-50t} = 0$$
$$0 = 0$$

$$q_2 = te^{-50t}$$
$$q_2' = e^{-50t} - 50te^{-50t}$$
$$q_2'' = -50e^{-50t} - 50e^{-50t} + 2500te^{-50t}$$
$$q_2'' + 100q_2' + 2500q_2 = 0$$
$$-100e^{-50t} + 2500te^{-50t} + 100e^{-50t} - 5000te^{-50t} + 2500te^{-50t} = 0$$
$$0 = 0$$

4. Is the transient solution considered to be overdamped, critically damped or underdamped?

Critically Damped

5. Find the steady-state solution.

$$q_p = A$$
$$q_p' = q_p'' = 0$$
$$q_p'' + 100q_p' + 2500q_p = 30$$
$$0 + 0 + 2500A = 30$$
$$A = \frac{3}{250}$$
$$q_p = \frac{3}{250}$$

6. Write the general solution to the differential equation.

$$q(t) = c_1 e^{-50t} + c_2 t e^{-50t} + \frac{3}{250}$$

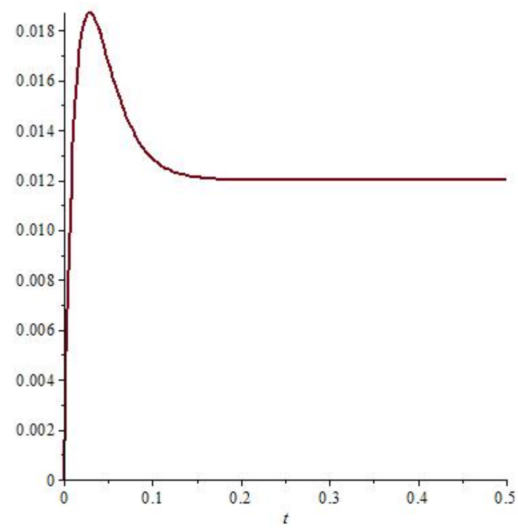
7. Find the charge as a function of time.

$$i(t) = q'(t) = -50c_1 e^{-50t} + c_2 e^{-50t} - 50c_2 t e^{-50t}$$

$$q(0) = 0$$
$$0 = c_1 + \frac{3}{250}$$
$$c_1 = -\frac{3}{250}$$

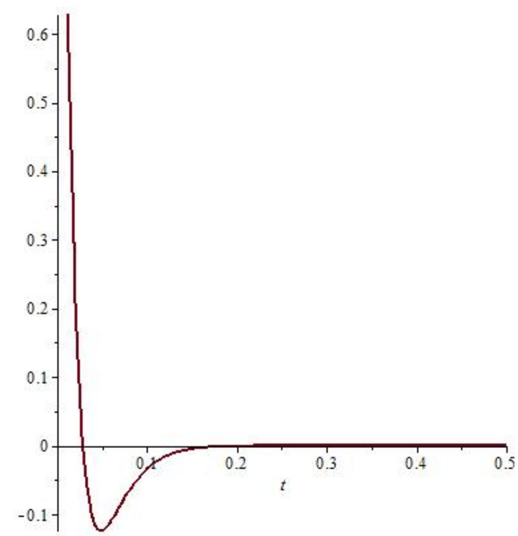
$$i(0) = 2$$
$$2 = -50c_1 + c_2$$
$$2 = -50\left(-\frac{3}{250}\right) + c_2$$
$$2 = \frac{3}{5} + c_2$$
$$c_2 = \frac{7}{5}$$

$$q(t) = -\frac{3}{250} e^{-50t} + \frac{7}{5} t e^{-50t} + \frac{3}{250}$$



8. Find current as a function of time.

$$i(t) = \frac{3}{5}e^{-50t} - 70te^{-50t} + \frac{7}{5}e^{-50t}$$



9. Find the limiting factor of the charge as a function of time.

$$\lim_{t\to\infty} q(t) = \lim_{t\to\infty}\left(-\frac{3}{250}e^{-50t} + \frac{7}{5}te^{-50t} + \frac{3}{250}\right) = \lim_{t\to\infty}\left(\frac{-3 + 7(50)t + 3e^{50t}}{250e^{50t}}\right)$$

$$= \lim_{t\to\infty}\left(\frac{350 + 3(50)e^{50t}}{250(50)e^{50t}}\right) = \lim_{t\to\infty}\left(\frac{3(50)(50)e^{50t}}{250(50)(50)e^{50t}}\right) = \frac{3}{250}$$

10. Now, consider the voltage source of $E(t) = 30e^{-50t}$. What would the steady state solution be here?

$$q_p = At^2e^{-50t}$$
$$q_p' = 2Ate^{-50t} - 50At^2e^{-50t}$$
$$q_p'' = 2Ae^{-50t} - 100Ate^{-50t} - 100Ate^{-50t} + 2500At^2e^{-50t}$$
$$q_p'' + 100q_p' + 2500q_p = 30e^{-50t}$$
$$2Ae^{-50t} - 200Ate^{-50t} + 2500At^2e^{-50t} + 100(2Ate^{-50t} - 50At^2e^{-50t}) + 2500At^2e^{-50t} = 30e^{-50t}$$
$$2Ae^{-50t} - 200Ate^{-50t} + 2500At^2e^{-50t} + 200Ate^{-50t} - 5000At^2e^{-50t} + 2500At^2e^{-50t} = 30e^{-50t}$$
$$2Ae^{-50t} = 30e^{-50t}$$
$$2A = 30$$
$$A = 15$$
$$q_p = 15t^2e^{-50t}$$

11. Write the new general solution to the differential equation.

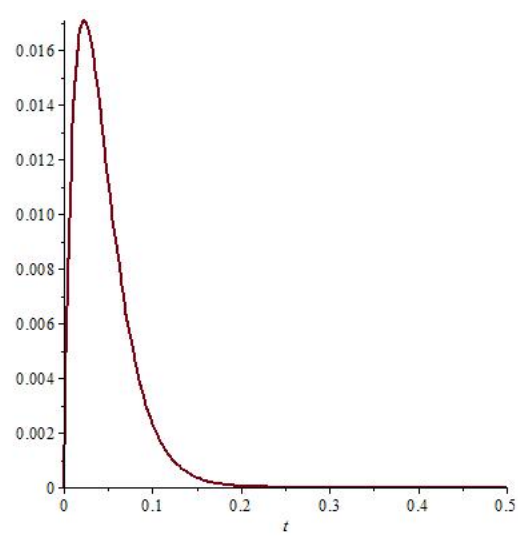
$$q(t) = c_1e^{-50t} + c_2te^{-50t} + 15t^2e^{-50t}$$

12. Now, find charge and current as functions of time.

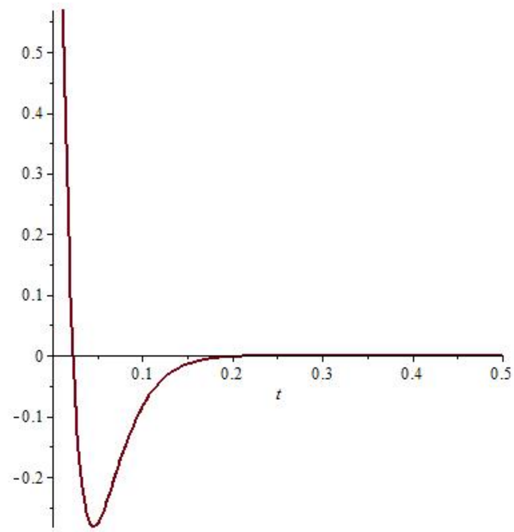
$$i(t) = q'(t) = -50c_1e^{-50t} + c_2e^{-50t} - 50c_2te^{-50t} + 30te^{-50t} - 750t^2e^{-50t}$$

$$q(0) = 0$$
$$0 = c_1$$

$$i(0) = 2$$
$$2 = -50c_1 + c_2$$
$$2 = c_2$$
$$q(t) = 2te^{-50t} + 15t^2e^{-50t}$$



$$i(t) = 2e^{-50t} - 100te^{-50t} + 30te^{-50t} - 750t^2e^{-50t}$$
$$i(t) = 2e^{-50t} - 70te^{-50t} - 750t^2e^{-50t}$$



13. Find the limiting factor of the current in the circuit.

$$\lim_{t\to\infty} i(t) = \lim_{t\to\infty}\left(\frac{2-70t-750t^2}{e^{50t}}\right) = \lim_{t\to\infty}\left(\frac{-70-1500t}{50e^{50t}}\right) = \lim_{t\to\infty}\left(\frac{-1500}{2500e^{50t}}\right) = 0$$