

Nonhomogeneous Equations

a) $y'' + 2y' + y = x$

Complimentary

$$y'' + 2y' + y = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r_1 = r_2 = -1$$

$$y_c = c_1 e^{-x} + c_2 x e^{-x}$$

Particular

$$f(x) = x$$

$$f'(x) = 1$$

$$y_p = Ax + B$$

$$y_p' = A$$

$$y_p'' = 0$$

$$y_p'' + 2y_p' + y_p = x$$

$$0 + 2A + Ax + B = x$$

$$A = 1 \quad 2A + B = 0$$

$$2 + B = 0$$

$$B = -2$$

$$y = c_1 e^{-x} + c_2 x e^{-x} + x - 2$$

$$y_p = x - 2$$

b) $y'' + 5y' + 6y = 3x^2$

Complimentary

$$y'' + 5y' + 6y = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2) = 0$$

$$r_1 = -3 \quad r_2 = -2$$

$$y_c = c_1 e^{-3x} + c_2 e^{-2x}$$

Particular

$$f(x) = 3x^2$$

$$f'(x) = 6x$$

$$f''(x) = 6$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' + 5y_p' + 6y_p = 3x^2$$

$$2A + 10Ax + 5B + 6Ax^2 + 6Bx + 6C = 3x^2$$

$$6A = 3 \quad 10A + 6B = 0 \quad 2A + 5B + 6C = 0$$

$$A = \frac{1}{2} \quad \frac{5}{2} + 3B = 0 \quad 1 - \frac{25}{6} + 6C = 0$$

$$B = -\frac{5}{6}$$

$$6C = \frac{19}{6}$$

$$C = \frac{19}{36}$$

$$y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{x^2}{2} - \frac{5x}{6} + \frac{19}{36}$$

$$y_p = \frac{1}{2}x^2 - \frac{5}{6}x + \frac{19}{36}$$

$$c) y'' - 7y' + 6y = e^x$$

Complimentary

$$y'' - 7y' + 6y = 0$$

$$r^2 - 7r + 6 = 0$$

$$(r-6)(r-1) = 0$$

$$r_1 = 6 \quad r_2 = 1$$

$$y_c = c_1 e^{6x} + c_2 e^x$$

$$y = c_1 e^{6x} + c_2 e^x - \frac{1}{5} x e^x$$

Particular

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$y_p = A e^x * y_2 = e^x$$

$$y_p = A x e^x$$

$$y_p' = A e^x + A x e^x$$

$$y_p'' = A e^x + A e^x + A x e^x$$

$$y_p'' - 7y_p' + 6y_p = e^x$$

$$2A e^x + A x e^x - 7A e^x - 7A x e^x + 6A x e^x = e^x$$

$$-5A e^x = e^x$$

$$-5A = 1$$

$$A = -\frac{1}{5}$$

$$y_p = -\frac{1}{5} x e^x$$

$$d) y'' - 2y' + 7y = x e^{2x}$$

Complimentary

$$y'' - 2y' + 7y = 0$$

$$r^2 - 2r + 7 = 0$$

$$r = \frac{2 \pm \sqrt{-24}}{2} = 1 \pm \sqrt{6} i$$

$$y_c = e^x [c_1 \sin(\sqrt{6} x) + c_2 \cos(\sqrt{6} x)]$$

Particular

$$f(x) = x e^{2x}$$

$$f'(x) = e^{2x} + 2x e^{2x}$$

$$y_p = A e^{2x} + B x e^{2x}$$

$$y_p' = 2A e^{2x} + B e^{2x} + 2B x e^{2x}$$

$$y_p'' = 4A e^{2x} + 2B e^{2x} + 2B e^{2x} + 4B x e^{2x}$$

$$y_p'' - 2y_p' + 7y_p = x e^{2x}$$

$$4A e^{2x} + 4B e^{2x} + 4B x e^{2x} - 2(2A e^{2x} + B e^{2x} + 2B x e^{2x}) + 7(A e^{2x} + B x e^{2x}) = x e^{2x}$$

$$= x e^{2x}$$

$$4A e^{2x} + 4B e^{2x} + 4B x e^{2x} - 4A e^{2x} - 2B e^{2x} - 4B x e^{2x} + 7A e^{2x} + 7B x e^{2x} = x e^{2x}$$

$$2B e^{2x} + 7A e^{2x} + 7B x e^{2x} = x e^{2x}$$

$$2B + 7A = 0$$

$$7B = 1$$

$$\frac{2}{7} = -7A$$

$$B = \frac{1}{7}$$

$$A = -\frac{2}{49}$$

$$y = e^x [c_1 \sin(\sqrt{6} x) + c_2 \cos(\sqrt{6} x)] - \frac{2}{49} e^{2x} + \frac{1}{7} x e^{2x}$$

$$y_p = -\frac{2}{49} e^{2x} + \frac{1}{7} x e^{2x}$$

1) 1e) $2y'' + 5y = \sin x$

Complimentary

$$2y'' + 5y = 0$$

$$2r^2 + 5 = 0$$

$$r^2 = -\frac{5}{2}$$

$$r = \pm \sqrt{\frac{5}{2}} i = \pm \frac{\sqrt{10}}{2} i$$

$$y_c = c_1 \sin\left(\frac{\sqrt{10}}{2} x\right) + c_2 \cos\left(\frac{\sqrt{10}}{2} x\right)$$

Particular

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$y_p = A \sin x + B \cos x$$

$$y_p' = A \cos x - B \sin x$$

$$y_p'' = -A \sin x - B \cos x$$

$$2y_p'' + 5y_p = \sin x$$

$$-2A \sin x - 2B \cos x + 5A \sin x + 5B \cos x = \sin x$$

$$3A \sin x + 3B \cos x = \sin x$$

$$3A = 1$$

$$3B = 0$$

$$A = \frac{1}{3}$$

$$B = 0$$

$$y_p = \frac{1}{3} \sin x$$

$$y = c_1 \sin\left(\frac{\sqrt{10}}{2} x\right) + c_2 \cos\left(\frac{\sqrt{10}}{2} x\right) + \frac{1}{3} \sin x$$

1P) $3y'' - 2y' + 4y = \cos(2x)$

Complimentary

$$3y'' - 2y' + 4y = 0$$

$$3r^2 - 2r + 4 = 0$$

$$r = \frac{2 \pm \sqrt{-44}}{6} = \frac{1}{3} \pm \frac{\sqrt{11}}{3} i$$

$$y_c = e^{x/3} \left[c_1 \sin\left(\frac{\sqrt{11}}{3} x\right) + c_2 \cos\left(\frac{\sqrt{11}}{3} x\right) \right]$$

Particular

$$f(x) = \cos(2x)$$

$$f'(x) = -2 \sin(2x)$$

$$y_p = A \sin(2x) + B \cos(2x)$$

$$y_p' = 2A \cos(2x) - 2B \sin(2x)$$

$$y_p'' = -4A \sin(2x) - 4B \cos(2x)$$

$$3y_p'' - 2y_p' + 4y_p = \cos(2x)$$

$$-12A \sin(2x) - 12B \cos(2x) - 4A \cos(2x) + 4B \sin(2x) + 4A \sin(2x) + 4B \cos(2x) = \cos(2x)$$

$$(4B - 8A) \sin(2x) + (-4A - 8B) \cos(2x) = \cos(2x)$$

$$4B - 8A = 0$$

$$-4A - 8B = 1$$

$$8A = 4B$$

$$-4A - 16A = 1$$

$$B = 2A$$

$$-20A = 1$$

$$B = \frac{1}{10}$$

$$A = -\frac{1}{20}$$

$$y_p = -\frac{1}{20} \sin(2x) + \frac{1}{10} \cos(2x)$$

$$y = e^{x/3} \left[c_1 \sin\left(\frac{\sqrt{11}}{3} x\right) + c_2 \cos\left(\frac{\sqrt{11}}{3} x\right) \right] - \frac{\sin(2x)}{20} + \frac{\cos(2x)}{10}$$

$$1a) y'' + y = \sin x$$

Complementary

$$y'' + y = 0$$

$$r^2 + 1 = 0$$

$$r^2 = -1$$

$$r = \pm i$$

$$y_c = c_1 \sin x + c_2 \cos x$$

Particular

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$y_p = A \sin x + B \cos x$$

$$y_p = A x \sin x + B x \cos x$$

$$y_p' = A \sin x + A x \cos x + B \cos x - B x \sin x$$

$$y_p'' = A \cos x + A \cos x - A x \sin x - B \sin x - B \sin x - B x \cos x$$

$$y_p'' + y_p = \sin x$$

$$2A \cos x - A x \sin x - 2B \sin x - B x \cos x + A x \sin x + B x \cos x = \sin x$$

$$2A \cos x - 2B \sin x = \sin x$$

$$y = c_1 \sin x + c_2 \cos x - \frac{1}{2} x \cos x$$

$$2A = 0$$

$$-2B = 1$$

$$A = 0$$

$$B = -\frac{1}{2}$$

$$y_p = -\frac{1}{2} x \cos x$$

$$1b) y''' - y' = 3e^x$$

Complementary

$$y''' - y' = 0$$

$$r^3 - r = 0$$

$$r(r^2 - 1) = 0$$

$$r_1 = 0, r_2 = 1, r_3 = -1$$

$$y_c = c_1 + c_2 e^x + c_3 e^{-x}$$

Particular

$$f(x) = 3e^x$$

$$f'(x) = 3e^x$$

$$y_p = A e^x$$

$$y_p = A x e^x$$

$$y_p' = A e^x + A x e^x$$

$$y_p'' = A e^x + A e^x + A x e^x$$

$$y_p''' = A e^x + A e^x + A e^x + A x e^x$$

$$y_p''' - y_p' = 3e^x$$

$$3A e^x + A x e^x - A e^x - A x e^x = 3e^x$$

$$2A e^x = 3e^x$$

$$2A = 3$$

$$A = \frac{3}{2}$$

$$y_p = \frac{3}{2} x e^x$$

$$y = c_1 + c_2 e^x + c_3 e^{-x} + \frac{3}{2} x e^x$$

i) $2y^{(4)} - 8y'' = 2$

Complimentary

$$2y^{(4)} - 8y'' = 0$$

$$2r^4 - 8r^2 = 0$$

$$2r^2(r^2 - 4) = 0$$

$$r_1 = r_2 = 0 \quad r_3 = 2 \quad r_4 = -2$$

$$y_c = C_1 + C_2 x + C_3 e^{2x} + C_4 e^{-2x}$$

Particular

$$f(x) = 2$$

$$f'(x) = 0$$

$$y_p = A$$

$$y_p = A$$

$$y_p = Ax^2$$

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$y_p''' = 0$$

$$y_p^{(4)} = 0$$

$$2y_p^{(4)} - 8y_p'' = 2$$

$$0 - 8(2A) = 2$$

$$-16A = 2$$

$$A = \frac{-2}{16} = -\frac{1}{8}$$

$$y_p = -\frac{1}{8}x^2$$

$$y = C_1 + C_2 x + C_3 e^{2x} + C_4 e^{-2x} - \frac{x^2}{8}$$

ii) $y^{(4)} - 4y'' + 4y = \sin x + e^x$

Complimentary

$$y^{(4)} - 4y'' + 4y = 0$$

$$r^4 - 4r^2 + 4 = 0$$

$$(r^2 - 2)^2 = 0$$

$$(r - \sqrt{2})^2 (r + \sqrt{2})^2 = 0$$

$$r_1 = r_2 = \sqrt{2} \quad r_3 = r_4 = -\sqrt{2}$$

$$y_c = C_1 e^{\sqrt{2}x} + C_2 x e^{\sqrt{2}x} + C_3 e^{-\sqrt{2}x} + C_4 x e^{-\sqrt{2}x}$$

Particular

$$f(x) = \sin x + e^x$$

$$f'(x) = \cos x + e^x$$

$$y_p = A \sin x + B \cos x + C e^x$$

$$y_p' = A \cos x - B \sin x + C e^x$$

$$y_p'' = -A \sin x - B \cos x + C e^x$$

$$y_p''' = -A \cos x + B \sin x + C e^x$$

$$y_p^{(4)} = A \sin x + B \cos x + C e^x$$

$$y_p^{(4)} - 4y_p'' + 4y_p = \sin x + e^x$$

$$A \sin x + B \cos x + C e^x + 4A \sin x + 4B \cos x - 4C e^x + 4A \sin x$$

$$+ 4B \cos x + 4C e^x = \sin x + e^x$$

$$9A \sin x + 9B \cos x + C e^x = \sin x + e^x$$

$$9A = 1 \quad 9B = 0 \quad C = 1$$

$$A = \frac{1}{9} \quad B = 0$$

$$y_p = \frac{1}{9} \sin x + e^x$$

$$y = C_1 e^{\sqrt{2}x} + C_2 x e^{\sqrt{2}x} + C_3 e^{-\sqrt{2}x} + C_4 x e^{-\sqrt{2}x} + \frac{1}{9} \sin x + e^x$$

$$k) y''' - 3y'' + 3y' - y = x^2 e^x$$

Complimentary

$$y''' - 3y'' + 3y' - y = 0$$

$$r^3 - 3r^2 + 3r - 1 = 0$$

$$(r-1)^3 = 0$$

$$r_1 = r_2 = r_3 = 1$$

$$y_c = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Particular

$$f(x) = x^2 e^x$$

$$f'(x) = 2x e^x + x^2 e^x$$

$$f''(x) = 2e^x + 2x e^x + 2x e^x + x^2 e^x$$

$$y_p = A e^x + B x e^x + C x^2 e^x$$

$$y_p = A x^3 e^x + B x^4 e^x + C x^5 e^x$$

$$y_p' = 3A x^2 e^x + A x^3 e^x + 4B x^3 e^x + B x^4 e^x + 5C x^4 e^x + C x^5 e^x$$

$$y_p'' = 6A x e^x + 3A x^2 e^x + 3A x^2 e^x + A x^3 e^x + 12B x^2 e^x + 4B x^3 e^x + 4B x^3 e^x + B x^4 e^x + 20C x^3 e^x + 5C x^4 e^x + 5C x^4 e^x + C x^5 e^x$$

$$= 6A x e^x + 6A x^2 e^x + A x^3 e^x + 12B x^2 e^x + 8B x^3 e^x + B x^4 e^x + 20C x^3 e^x + 10C x^4 e^x + C x^5 e^x$$

$$y_p''' = 6A e^x + 6A x e^x + 12A x e^x + 6A x^2 e^x + 3A x^2 e^x + A x^3 e^x + 24B x e^x + 12B x^2 e^x + 24B x^2 e^x + 8B x^3 e^x$$

$$+ 4B x^3 e^x + B x^4 e^x + 60C x^2 e^x + 20C x^3 e^x + 40C x^3 e^x + 10C x^4 e^x + 5C x^4 e^x + C x^5 e^x$$

$$= 6A e^x + 18A x e^x + 9A x^2 e^x + A x^3 e^x + 24B x e^x + 36B x^2 e^x + 12B x^3 e^x + B x^4 e^x$$

$$+ 60C x^2 e^x + 60C x^3 e^x + 15C x^4 e^x + C x^5 e^x$$

$$y_p''' - 3y_p'' + 3y_p' - y_p = x^2 e^x$$

$$6A e^x + (18A + 24B) x e^x + (9A + 36B + 60C) x^2 e^x + (A + 12B + 60C) x^3 e^x + (B + 15C) x^4 e^x + C x^5 e^x$$

$$- 3[6A x e^x + (6A + 12B) x^2 e^x + (A + 8B + 20C) x^3 e^x + (B + 10C) x^4 e^x + C x^5 e^x]$$

$$+ 3[3A x^2 e^x + (A + 4B) x^3 e^x + (B + 5C) x^4 e^x + C x^5 e^x] - (A x^3 e^x + B x^4 e^x + C x^5 e^x) = x^2 e^x$$

$$6A e^x + (18A + 24B - 18A) x e^x + (9A + 36B + 60C - 18A - 3(6A + 12B) + 3A) x^2 e^x + (A + 12B + 60C - 3A - 24B - 60C + 3A + 12B - A) x^3 e^x$$

$$+ (B + 15C - 3B - 30C + 3B + 15C - B) x^4 e^x + (C - 3C + 3C - C) x^5 e^x = x^2 e^x$$

$$6A e^x + 24B x e^x + 60C x^2 e^x = x^2 e^x$$

$$6A = 0 \quad 24B = 0 \quad 60C = 1$$

$$A = 0 \quad B = 0 \quad C = \frac{1}{60}$$

$$y_p = \frac{1}{60} x^3 e^x$$

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + \frac{1}{60} x^3 e^x$$

$$2a) y'' + 6y' + 9y = x^2 - 2x \quad y(0) = 0 \quad y'(0) = -1$$

Complementary

$$y'' + 6y' + 9y = 0$$

$$r^2 + 6r + 9 = 0$$

$$(r+3)^2 = 0$$

$$r_1 = r_2 = -3$$

$$y_c = c_1 e^{-3x} + c_2 x e^{-3x}$$

Particular

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$y_p'' + 6y_p' + 9y_p = x^2 - 2x$$

$$2A + 6(2Ax + B) + 9(Ax^2 + Bx + C) = x^2 - 2x$$

$$2A + 12Ax + 6B + 9Ax^2 + 9Bx + 9C = x^2 - 2x$$

$$9A = 1$$

$$A = \frac{1}{9}$$

$$12A + 9B = -2$$

$$\frac{4}{3} + 9B = -2$$

$$9B = -\frac{10}{3}$$

$$B = -\frac{10}{27}$$

$$2A + 6B + 9C = 0$$

$$\frac{2}{9} + \frac{-20}{9} + 9C = 0$$

$$9C = -2$$

$$C = -\frac{2}{9}$$

$$y_p = \frac{1}{9}x^2 - \frac{10}{27}x - \frac{2}{9}$$

$$y = c_1 e^{-3x} + c_2 x e^{-3x} + \frac{1}{9}x^2 - \frac{10}{27}x - \frac{2}{9}$$

$$0 = c_1 + c_2(0) + \frac{1}{9}(0) - \frac{10}{27}(0) - \frac{2}{9}$$

$$0 = c_1 - \frac{2}{9}$$

$$c_1 = \frac{2}{9}$$

$$y = \frac{2}{9}e^{-3x} + c_2 x e^{-3x} + \frac{1}{9}x^2 - \frac{10}{27}x - \frac{2}{9}$$

$$y' = -\frac{2}{3}e^{-3x} + c_2 e^{-3x} - 3c_2 x e^{-3x} + \frac{2}{9}x - \frac{10}{27}$$

$$-1 = -\frac{2}{3} + c_2 + 0 + 0 - \frac{10}{27}$$

$$\frac{1}{27} = c_2$$

$$y = \frac{2}{9}e^{-3x} + \frac{1}{27}x e^{-3x} + \frac{1}{9}x^2 - \frac{10}{27}x - \frac{2}{9}$$

$$2b) y'' - y' = \sin(2x)$$

$$y(0)=1 \quad y'(0)=0$$

Complementary

Particular

$$y'' - y' = 0$$

$$y_p = A \sin(2x) + B \cos(2x)$$

$$r^2 - r = 0$$

$$y_p' = 2A \cos(2x) - 2B \sin(2x)$$

$$r(r-1)=0$$

$$y_p'' = -4A \sin(2x) - 4B \cos(2x)$$

$$r_1 = 0, r_2 = 1$$

$$y_p'' - y_p' = \sin(2x)$$

$$y_c = c_1 + c_2 e^x$$

$$-4A \sin(2x) - 4B \cos(2x) - 2A \cos(2x) + 2B \sin(2x) = \sin(2x)$$

$$-4A + 2B = 1 \quad -4B - 2A = 0$$

$$8B + 2B = 1$$

$$2A = -4B$$

$$10B = 1$$

$$A = -2B$$

$$B = \frac{1}{10}$$

$$A = -\frac{2}{10}$$

$$A = -\frac{1}{5}$$

$$y_p = -\frac{1}{5} \sin(2x) + \frac{1}{10} \cos(2x)$$

$$y = c_1 + c_2 e^x - \frac{1}{5} \sin(2x) + \frac{1}{10} \cos(2x)$$

$$y' = c_2 e^x - \frac{2}{5} \cos(2x) - \frac{1}{5} \sin(2x)$$

$$1 = c_1 + c_2 - 0 + \frac{1}{10}$$

$$0 = c_2 - \frac{2}{5} - 0$$

$$\frac{9}{10} = c_1 + c_2$$

$$c_2 = \frac{2}{5}$$

$$\frac{9}{10} = c_1 + \frac{2}{5}$$

$$\frac{1}{2} = c_1$$

$$y = \frac{1}{2} + \frac{2}{5} e^x - \frac{1}{5} \sin(2x) + \frac{1}{10} \cos(2x)$$

$$2c) y'' + 3y' = \cos x$$

$$y(0) = 1 \quad y'(0) = 1$$

Complimentary

Particular

$$y'' + 3y' = 0$$

$$r^2 + 3r = 0$$

$$r(r+3) = 0$$

$$r_1 = 0 \quad r_2 = -3$$

$$y_c = c_1 + c_2 e^{-3x}$$

$$y_p = A \sin x + B \cos x$$

$$y_p' = A \cos x - B \sin x$$

$$y_p'' = -A \sin x - B \cos x$$

$$y_p'' + 3y_p' = \cos x$$

$$-A \sin x - B \cos x + 3A \cos x - 3B \sin x = \cos x$$

$$-A - 3B = 0 \quad -B + 3A = 1$$

$$A = -3B \quad -B - 9B = 1$$

$$A = \frac{2}{10} \quad -10B = 1$$

$$B = -\frac{1}{10}$$

$$y_p = \frac{2}{10} \sin x - \frac{1}{10} \cos x$$

$$y = c_1 + c_2 e^{-3x} + \frac{2}{10} \sin x - \frac{1}{10} \cos x$$

$$1 = c_1 + c_2 + 0 - \frac{1}{10}$$

$$\frac{11}{10} = c_1 + c_2$$

$$\frac{11}{10} = c_1 - \frac{1}{30}$$

$$\frac{43}{30} = c_1 = \frac{4}{3}$$

$$y' = -3c_2 e^{-3x} + \frac{2}{10} \cos x + \frac{1}{10} \sin x$$

$$1 = -3c_2 + \frac{2}{10} + 0$$

$$\frac{7}{10} = -3c_2$$

$$c_2 = -\frac{7}{30}$$

$$y = \frac{4}{3} - \frac{7}{30} e^{-3x} + \frac{2}{10} \sin x - \frac{1}{10} \cos x$$

$$1) \quad 2d) \quad y^{(5)} - y''' = 2 \quad y(0) = y'(0) = y''(0) = 0 \quad y'''(0) = y^{(4)}(0) = 1$$

Complimentary

$$y^{(5)} - y''' = 0$$

$$r^5 - r^3 = 0$$

$$r^3(r^2 - 1) = 0$$

$$r_1 = r_2 = r_3 = 0 \quad r_4 = 1 \quad r_5 = -1$$

$$y_c = C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 e^{-x}$$

Particular

$$y_p = A$$

$$y_p = Ax$$

$$y_p = Ax^2$$

$$y_p = Ax^3$$

$$y_p' = 3Ax^2$$

$$y_p'' = 6Ax$$

$$y_p''' = 6A$$

$$y_p^{(4)} = 0$$

$$y_p^{(5)} = 0$$

$$y_p^{(5)} - y_p''' = 2$$

$$0 - 6A = 2$$

$$-6A = 2$$

$$A = -\frac{1}{3}$$

$$y_p = -\frac{1}{3}x^3$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 e^{-x} - \frac{1}{3}x^3$$

$$0 = C_1 + 0 + 0 + C_4 + C_5 + 0$$

$$0 = C_1 + C_4 + C_5$$

$$0 = C_1 + 2 - 1$$

$$C_1 = -1$$

$$y'' = 2C_3 + C_4 e^x + C_5 e^{-x} - 2x$$

$$0 = 2C_3 + C_4 + C_5 - 0$$

$$0 = 2C_3 + C_4 + C_5$$

$$0 = 2C_3 + 2 - 1$$

$$-1 = 2C_3$$

$$C_3 = -\frac{1}{2}$$

$$y' = C_2 + 2C_3 x + C_4 e^x - C_5 e^{-x} - x^2$$

$$0 = C_2 + 0 + C_4 - C_5 - 0$$

$$0 = C_2 + C_4 - C_5$$

$$0 = C_2 + 2 + 1$$

$$C_2 = -3$$

$$y''' = C_4 e^x - C_5 e^{-x} - 2$$

$$1 = C_4 - C_5 - 2$$

$$C_4 = 3 + C_5$$

$$C_4 = 3 - 1$$

$$C_4 = 2$$

$$y^{(4)} = C_4 e^x + C_5 e^{-x}$$

$$1 = C_4 + C_5$$

$$1 = 3 + 2C_5$$

$$-2 = 2C_5$$

$$C_5 = -1$$

$$y = -1 - 3x - \frac{1}{2}x^2 + 2e^x - e^{-x} - \frac{1}{3}x^3$$