Name: ____________________

Section: ____________________

R·I·T School of Mathematical Sciences

22 - Kirchhoff's Law

MATH 211

State the first order differential equation whose solution models the current over an RLC circuit as a function of time (Kirchhoff's Law).

$$L\frac{di}{dt} + Ri + \frac{q}{C} = E(t)$$

The current in a circuit is equal to the rate of change of the charge with respect to time. Write the relationship between current i and charge q.

$$\frac{dq}{dt} = i$$

$$\frac{d^2q}{dt^2} = \frac{di}{dt}$$

Write Kirchhoff's Law in the second order. What will the solution to this differential equation represent?

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

$$Lq''(t) + Rq'(t) + \frac{q(t)}{C} = E(t)$$

The solution to this equation represents the charge as a function of time.

Find the current as a function of time for a circuit in which $L = 2$ H, $R = 20\ \Omega$, $C = 2000\ \mu$F and $E(t) = 2e^{-5t}$ V, if the initial charge is $\frac{1}{225}$ C and the initial current is 0 A.

$$\boxed{C = 2000\ \mu\text{F} = 2000 \times 10^{-6}\ \text{F} = \frac{1}{500}\ \text{F}}$$

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

$$2q'' + 20q' + \frac{q}{1/500} = 2e^{-5t}$$

$$2q'' + 20q' + 500q = 2e^{-5t}$$

$$q'' + 10q' + 250q = e^{-5t}$$

Complementary:

$$q'' + 10q' + 250q = 0$$

$$m^2 + 10m + 250 = 0$$

$$m = \frac{-10 \pm \sqrt{10^2 - 4(1)(25)}}{2(1)}$$

$$m = -5 \pm 15i$$

$$q_c = e^{-5t}\left[c_1 \sin(15t) + c_2 \cos(15t)\right]$$

Particular:

$$q_p = Ae^{-5t}$$

$$q_p' = -5Ae^{-5t}$$

$$q_p'' = 25Ae^{-5t}$$

$$q_p'' + 10q_p' + 250q_p = e^{-5t}$$

$$25Ae^{-5t} + 10\left(-5Ae^{-5t}\right) + 250Ae^{-5t} = e^{-5t}$$

$$225Ae^{-5t} = e^{-5t}$$

$$225A = 1$$

$$A = \frac{1}{225}$$

$$q_p = \frac{1}{225}e^{-5t}$$

$$q = q_c + q_p$$

$$q = e^{-5t}\left[c_1 \sin(15t) + c_2 \cos(15t)\right] + \frac{1}{225}e^{-5t}$$

$$q = e^{-5t}\left[c_1 \sin(15t) + c_2 \cos(15t)\right] + \frac{1}{225}e^{-5t}$$

$$\frac{1}{225} = (1)\left[c_1(0) + c_2(1)\right] + \frac{1}{225}(1)$$

$$\frac{1}{225} = c_2 + \frac{1}{225}$$

$$c_2 = 0$$

$$q = e^{-5t}\left[c_1 \sin(15t) + 0\right] + \frac{1}{225}e^{-5t}$$

$$q' = -5e^{-5t}c_1 \sin(15t) + 15e^{-5t}c_1 \cos(15t) - \frac{1}{45}e^{-5t}$$

$$0 = -5c_1(0) + 15c_1(1) - \frac{1}{45}$$

$$15c_1 = \frac{1}{45}$$

$$c_1 = \frac{1}{675}$$

$$q = \frac{1}{675}e^{-5t}\sin(15t) + \frac{1}{225}e^{-5t}$$

$$i = -\frac{1}{135}e^{-5t}\sin(15t) + \frac{1}{45}e^{-5t}\cos(15t) - \frac{1}{45}e^{-5t}$$

Consider the RLC circuit with inductance 1 H, resistance 100 Ω and a capacitance of 400 μF with a voltage source of 30 V. The initial current is 2A and the initial charge is 0 C. Find the charge as a function of time.

$$C = 400\ \mu\text{F} = 400 \times 10^{-6}\ \text{F} = \frac{1}{2500}\ \text{F}$$

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

$$q'' + 100q' + \frac{q}{1/2500} = 30$$

$$q'' + 100q' + 2500q = 30$$

Complementary:

$$q'' + 100q' + 2500q = 0$$
$$m^2 + 100m + 2500 = 0$$
$$(m+50)^2 = 0$$
$$m_1 = m_2 = -50$$
$$q_c = c_1 e^{-50t} + c_2 t e^{-50t}$$

Particular:

$$q_p = A$$
$$q_p' = 0$$
$$q_p'' = 0$$
$$q_p'' + 100q_p' + 2500q_p = 30$$
$$0 + 0 + 2500A = 30$$
$$A = \frac{3}{250}$$
$$q_p = \frac{3}{250}$$

$$q = q_c + q_p$$

$$q = c_1 e^{-50t} + c_2 t e^{-50t} + \frac{3}{250}$$

$$q = c_1 e^{-50t} + c_2 t e^{-50t} + \frac{3}{250}$$
$$0 = c_1 e^0 + c_2(0)e^0 + \frac{3}{250}$$
$$0 = c_1 + \frac{3}{250}$$
$$c_1 = -\frac{3}{250}$$

$$q = -\frac{3}{250}e^{-50t} + c_2 t e^{-50t} + \frac{3}{250}$$
$$q' = \frac{3}{5}e^{-50t} - 50c_2 t e^{-50t} + c_2 e^{-50t}$$
$$2 = \frac{3}{5}e^0 - 50c_2(0)e^0 + c_2 e^0$$
$$2 = \frac{3}{5} + c_2$$
$$c_2 = \frac{7}{5}$$

$$q = -\frac{3}{250}e^{-50t} + \frac{7}{5}te^{-50t} + \frac{3}{250}$$