R·I·T School of Mathematical Sciences

Separation of Variables

MATH 211

1. Find the general solution to the following differential equations.

   (a) $\frac{dy}{dx} = \sin(5x)$

   $$\begin{aligned} dy &= \sin(5x)\ dx \\ \int\ dy &= \int \sin(5x)\ dx \\ y &= -\frac{1}{5}\cos(5x) + C \end{aligned}$$

   (b) $dx + e^{3x}\ dy = 0$

   $$\begin{aligned} dx &= -e^{3x}\ dy \\ e^{-3x}\ dx &= -dy \\ \int e^{-3x}\ dx &= -\int\ dy \\ -\frac{1}{3}e^{-3x} &= -y + C \end{aligned}$$

   (c) $x\frac{dy}{dx} = 4y$

   $$\begin{aligned} xdy &= 4ydx \\ \frac{1}{y}\ dy &= \frac{4}{x}\ dx \\ \int \frac{1}{y}\ dy &= 4\int \frac{1}{x}\ dx \\ \ln|y| &= 4\ln|x| + C \\ \ln|y| &= \ln x^4 + C \end{aligned}$$

(d) $\frac{dy}{dx} = e^{3x+2y}$

$$dy = e^{3x}e^{2y}\ dx$$
$$e^{-2y}\ dy = e^{3x}\ dx$$
$$\int e^{-2y}\ dy = \int e^{3x}\ dx$$
$$-\frac{1}{2}e^{-2y} = \frac{1}{3}e^{3x} + C$$

(e) $y \ln x\ \frac{dx}{dy} = \left(\frac{y+1}{x}\right)^2$

$$y \ln x\ dx = \frac{(y+1)^2}{x^2}\ dy$$
$$x^2 \ln x\ dx = \frac{y^2+2y+1}{y}\ dy$$
$$\int x^2 \ln x\ dx_{\star} = \int \left(y + 2 + \frac{1}{y}\right)\ dy$$

$$\boxed{\star \quad \begin{array}{ll} u = \ln x & dv = x^2\ dx \\ du = \frac{1}{x}\ dx & v = \frac{x^3}{3} \end{array}}$$

$$\frac{x^3 \ln x}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x}\ dx = \int \left(y + 2 + \frac{1}{y}\right)\ dy$$
$$\frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\ dx = \frac{y^2}{2} + 2y + \ln|y| + C$$
$$\frac{x^3 \ln x}{3} - \frac{x^3}{9} = \frac{y^2}{2} + 2y + \ln|y| + C$$

(f) $\csc y\ dx + \sec^2 x\ dy = 0$

$$\csc y\ dx = -\sec^2 x\ dy$$
$$\cos^2 x\ dx = -\sin y\ dy$$
$$\int \cos^2 x\ dx = -\int \sin y\ dy$$
$$\frac{1}{2}\int (1 + \cos(2x))\ dx = -\int \sin y\ dy$$
$$\frac{1}{2}\left(x + \frac{1}{2}\sin(2x)\right) = \cos y + C$$

(g) $\sin(3x)\ dx + 2y\cos^3(3x)\ dy = 0$

$$\sin(3x)\ dx = -2y\cos^3(3x)\ dy$$

$$\frac{\sin(3x)}{\cos^3(3x)}\ dx = -2y\ dy$$

$$\int \frac{\sin(3x)}{\cos^3(3x)}\ dx_{\star} = -2\int y\ dy$$

$$\boxed{\begin{gathered}\star\ u = \cos(3x)\\ du = -3\sin(3x)\ dx\\ -\frac{1}{3}\ du = \sin(3x)\ dx\end{gathered}}$$

$$-\frac{1}{3}\int \frac{1}{u^3}\ du = -y^2 + C$$

$$-\frac{1}{3}\int u^{-3}\ du = -y^2 + C$$

$$-\frac{1}{3}\left(\frac{u^{-2}}{-2}\right) = -y^2 + C$$

$$\frac{1}{6\cos^2(3x)} = -y^2 + C$$

$$\frac{1}{6}\sec^2(3x) = -y^2 + C$$

(h) $(e^y+1)^2 e^{-y}\ dx + (e^x+1)^3 e^{-x}\ dy = 0$

$$(e^y+1)^2 e^{-y}\ dx = -(e^x+1)^3 e^{-x}\ dy$$

$$\frac{e^x}{(e^x+1)^3}\ dx = -\frac{e^y}{(e^y+1)^2}\ dy$$

$$\int \frac{e^x}{(e^x+1)^3}\ dx_{\star_1} = -\int \frac{e^y}{(e^y+1)^2}\ dy_{\star_2}$$

$$\boxed{\begin{gathered}\star_1\ u = e^x + 1\\ du = e^x\ dx\end{gathered}} \quad \boxed{\begin{gathered}\star_1\ w = e^y + 1\\ dw = e^y\ dy\end{gathered}}$$

$$\int \frac{1}{u^3}\ du = \int \frac{1}{w^2}\ dw$$

$$\int u^{-3}\ du = \int w^{-2}\ dw$$

$$\frac{u^{-2}}{-2} = \frac{w^{-1}}{-1} + C$$

$$-\frac{1}{2(e^x+1)^2} = -\frac{1}{e^y+1} + C$$

(i) $y' = x\sqrt{1-y^2}$

$$\frac{dy}{dx} = x\sqrt{1-y^2}$$

$$dy = x\sqrt{1-y^2}\ dx$$

$$\frac{1}{\sqrt{1-y^2}}\ dy = x\ dx$$

$$\int \frac{1}{\sqrt{1-y^2}}\ dy = \int x\ dx$$

$$\sin^{-1} y = \frac{x^2}{2} + C$$

(j) $(e^x + e^{-x})y' = y^2$

$$(e^x + e^{-x})\frac{dy}{dx} = y^2$$

$$(e^x + e^{-x})\ dy = y^2\ dx$$

$$\frac{1}{y^2}\ dy = \frac{1}{e^x + e^{-x}}\ dx$$

$$\int y^{-2}\ dy = \int \frac{e^x}{(e^x)^2 + 1}\ dx_\star$$

$$\boxed{\begin{aligned} \star_1\ u &= e^x \\ du &= e^x\ dx \end{aligned}}$$

$$\frac{y^{-1}}{-1} = \int \frac{1}{u^2+1}\ du$$

$$-\frac{1}{y} = \tan^{-1} u + C$$

$$-\frac{1}{y} = \tan^{-1} e^x + C$$

2. Solve the following initial value problems.

(a) $\frac{dx}{dt} = 4(x^2+1),\ x(\pi/4) = 1$

$$\begin{aligned} dx &= 4(x^2+1)\ dt \\ \frac{1}{x^2+1}\ dx &= 4\ dt \\ \int \frac{1}{x^2+1}\ dx &= 4\int\ dt \\ \tan^{-1} x &= 4t + C \\ \tan^{-1} 1 &= 4\left(\frac{\pi}{4}\right) + C \\ \frac{\pi}{4} &= \pi + C \\ C &= -\frac{3\pi}{4} \\ \tan^{-1} x &= 4t - \frac{3\pi}{4} \end{aligned}$$

(b) $y' = \frac{y^2-1}{x^2-1},\ y(2) = 2$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y^2-1}{x^2-1} \\ \frac{1}{y^2-1}\ dy &= \frac{1}{x^2-1}\ dx \\ \int \frac{1}{y^2-1}\ dy &= \int \frac{1}{x^2-1}\ dx\star \end{aligned}$$

$$\star\ \frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x-1)(x+1)}$$

$$1 = Ax + A + Bx - B$$

$$1 = (A+B)x + (A-B)$$

$$\begin{array}{ll} A+B=0 & A-B=1 \\ B=-A & A+A=1 \\ & 2A=1 \\ & A=\frac{1}{2} \\ B=-\frac{1}{2} & \end{array}$$

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}$$

$$\frac{1}{2}\int\left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$\ln|y-1| - \ln|y+1| = \ln|x-1| - \ln|x+1| + C$$
$$\ln\left|\frac{y-1}{y+1}\right| = \ln\left|\frac{x-1}{x+1}\right| + C$$
$$e^{\ln\left|\frac{y-1}{y+1}\right|} = e^{\ln\left|\frac{x-1}{x+1}\right| + C}$$
$$e^{\ln\left|\frac{y-1}{y+1}\right|} = e^{\ln\left|\frac{x-1}{x+1}\right|} e^{C}$$
$$\frac{y-1}{y+1} = C\left(\frac{x-1}{x+1}\right)$$
$$\frac{2-1}{2+1} = C\left(\frac{2-1}{2+1}\right)$$
$$C = 1$$
$$\frac{y-1}{y+1} = \frac{x-1}{x+1}$$

(c) $x^2 \frac{dy}{dx} = y - xy, \ y(-1) = -1$

$$x^2 \ dy = y(1-x) \ dx$$
$$\frac{1}{y} \ dy = \frac{1-x}{x^2} \ dx$$
$$\int \frac{1}{y} \ dy = \int \left(x^{-2} - \frac{1}{x}\right) \ dx$$
$$\ln|y| = \frac{x^{-1}}{-1} - \ln|x| + C$$
$$\ln|y| = -\frac{1}{x} - \ln|x| + C$$
$$\ln|-1| = -\frac{1}{-1} - \ln|-1| + C$$
$$0 = 1 - 0 + C$$
$$C = -1$$
$$\ln|y| = -\frac{1}{x} - \ln|x| - 1$$