11 - First Order Linear Equations II

MATH 211

Find the general solution to the following first order linear equation.

$$x\frac{dy}{dx} - y = x$$

$$\frac{dy}{dx} - \frac{1}{x}\ y = 1$$

$$e^{\int P(x)\ dx} = e^{\int -\frac{1}{x}\ dx} = e^{-\ln|x|} = e^{\ln|x^{-1}|} = x^{-1} = \frac{1}{x}$$

$$\frac{1}{x}\left[\frac{dy}{dx} - \frac{1}{x}\ y\right] = \frac{1}{x}$$

$$\frac{d}{dx}\left[y \cdot \frac{1}{x}\right] = \frac{1}{x}$$

$$\int\ d\left[y \cdot \frac{1}{x}\right] = \int \frac{1}{x}\ dx$$

$$\frac{y}{x} = \ln|x| + C$$

Find the general solution to the following first order linear equation.

$$\frac{dr}{d\theta} = \csc\theta - r\cot\theta$$

$$\frac{dr}{d\theta} + r\cot\theta = \csc\theta$$

$$e^{\int P(\theta)\ d\theta} = e^{\int \cot\theta\ d\theta} = e^{\ln|\sin\theta|} = \sin\theta$$

$$\sin\theta\left[\frac{dr}{d\theta} + r\cot\theta\right] = \csc\theta\sin\theta$$

$$\frac{d}{d\theta}\left[r\sin\theta\right] = 1$$

$$\int\ d\left[r\sin\theta\right] = \int 1\ d\theta$$

$$r\sin\theta = \theta + C$$

Find the general solution to the following first order linear equation.

$$y' + \left(\frac{2}{x^2-1}\right) y = \frac{x+1}{x-1}$$

$$\frac{2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$$

$$2 = A(x-1) + B(x+1)$$

$$2 = Ax - A + Bx + B$$

$$A + B = 0 \qquad -A + B = 2$$

$$-A = B \rightarrow \qquad B + B = 2$$

$$2B = 2$$

$$A = -1 \qquad \leftarrow B = 1$$

$$\frac{2}{x^2-1} = \frac{-1}{x+1} + \frac{1}{x-1}$$

$$e^{\int P(x)dx} = e^{\int 2/(x^2-1)dx}$$

$$= e^{\int (-1/(x+1)+1/(x-1))dx}$$

$$= e^{-\ln(x+1)+\ln(x-1)}$$

$$= e^{\ln\left(\frac{x-1}{x+1}\right)}$$

$$= \frac{x-1}{x+1}$$

$$\left(\frac{x-1}{x+1}\right)\left[y' + \left(\frac{2}{x^2-1}\right) y\right] = \left(\frac{x+1}{x-1}\right)\left(\frac{x-1}{x+1}\right)$$

$$\frac{d}{dx}\left[y\left(\frac{x-1}{x+1}\right)\right] = 1$$

$$\int d\left[y\left(\frac{x-1}{x+1}\right)\right] = \int dx$$

$$y\left(\frac{x-1}{x+1}\right) = x + C$$

Solve the following initial value problem.

$$\frac{dx}{dt} + \frac{4x}{t} = \frac{1}{t}, \; x(1) = 0$$

$$e^{\int P(t)\ dt} = e^{\int 4/t\ dt} = e^{4\ln t} = e^{\ln t^4} = t^4$$

$$t^4\left[\frac{dx}{dt} + \frac{4x}{t}\right] = \frac{1}{t}\cdot t^4$$

$$\frac{d}{dt}\left[x\ t^4\right] = t^3$$

$$\int d\left[x\ t^4\right] = \int t^3\ dt$$

$$x\ t^4 = \frac{t^4}{4} + C$$

$$0 = \frac{1}{4} + C$$

$$C = -\frac{1}{4}$$

$$x\ t^4 = \frac{t^4}{4} - \frac{1}{4}$$

$$4x = 1 - \frac{1}{t^4}$$