Name: _____

Section: _____

R·I·T School of Mathematical Sciences

Homework 6

MATH 211

1. A simple pendulum rotates around a point, Q. The pendulum rod is $l = 4.9$ meters long and is released from rest at $\theta(0) = \pi/6$ radians. Find the equation of angular motion of the pendulum.

$$\theta'' + \frac{g}{l}\theta = 0$$

$$\theta'' + \frac{9.8}{4.9}\theta = 0$$

$$\theta'' + 2\theta = 0$$

$$r^2 + 2 = 0$$

$$r^2 = -2$$

$$r = \pm\sqrt{2}$$

$$\theta(t) = c_1 \sin\left(\sqrt{2}t\right) + c_2 \cos\left(\sqrt{2}t\right)$$

$$\frac{\pi}{6} = c_1 \sin(0) + c_2 \cos(0)$$

$$c_2 = \frac{\pi}{6}$$

$$\theta'(t) = \sqrt{2}c_1 \cos\left(\sqrt{2}t\right) - \sqrt{2}c_2 \sin\left(\sqrt{2}t\right)$$

$$0 = \sqrt{2}c_1 \cos(0) - \sqrt{2}c_2 \sin(0)$$

$$c_1 = 0$$

$$\theta(t) = \frac{\pi}{6} \cos\left(\sqrt{2}t\right)$$

2. A $\frac{1}{2}$ slug mass is attached to a spring with spring constant 2 lb/ft. The mass is initially released from rest from a point 2 feet below equilibrium position and the subsequent motion takes place in a medium that offers a damping force numerically equal to 2 times the instantaneous velocity. Find the equation of motion of the mass driven by an external force equal to $f(t) = 25\cos t$.

$$mx'' + bx' + kx = f(t)$$

$$\frac{1}{2}x'' + 2x' + 2x = 25\cos t$$

$$x'' + 4x' + 4x = 50\cos t$$

$$x'' + 4x' + 4x = 0$$
$$r^2 + 4r + 4 = 0$$
$$(r+2)^2 = 0$$
$$r_1 = r_2 = -2$$
$$x_c = c_1e^{-2t} + c_2te^{-2t}$$

$$x_p = A\sin t + B\cos t$$
$$x_p' = A\cos t - B\sin t$$
$$x_p'' = -A\sin t - B\cos t$$
$$x_p'' + 4x_p' + 4x_p = 50\cos t$$
$$-A\sin t - B\cos t + 4\left(A\cos t - B\sin t\right) + 4\left(A\sin t + B\cos t\right) = 50\cos t$$
$$-A\sin t - B\cos t + 4A\cos t - 4B\sin t + 4A\sin t + 4B\cos t = 50\cos t$$
$$(3A - 4B)\sin t + (4A + 3B)\cos t = 50\cos t$$

$3A - 4B = 0$ $\qquad$ $4A + 3B = 50$

$3A = 4B$

$A = \frac{4}{3}B$

$$4\left(\tfrac{4}{3}B\right) + 3B = 50$$
$$\tfrac{16}{3}B + \tfrac{9}{3}B = 50$$
$$\tfrac{25}{3}B = 50$$
$$B = 6$$

$A = \frac{4}{3}(6)$

$A = 8$

$$x_p = 8\sin t + 6\cos t$$

$$x = c_1e^{-2t} + c_2te^{-2t} + 8\sin t + 6\cos t$$

$$x' = 2c_1e^{2t} - 2c_2te^{-2t} + c_2e^{-2t} + 8\cos t - 6\sin t$$

$$2 = c_1e^0 + c_2(0)e^0 + 8\sin(0) + 6\cos(0)$$
$$2 = c_1 + 6$$
$$c_1 = -4$$

$$0 = 2c_1e^0 - 2c_2(0)e^0 + c_2e^0 + 8\cos(0) - 6\sin(0)$$
$$0 = 2c_1 + c_2 + 8$$
$$0 = 2(-4) + c_2 + 8$$
$$0 = -8 + c_2 + 8$$
$$c_2 = 0$$

$$x(t) = -4e^{2t} + 8\sin t + 6\cos t$$

3. Find the charge and current as functions of time for a circuit with inductance 1H, resistance 100Ω, capacitance 0.0004F and impressed voltage 25V, if the initial current and charge are both zero.

$$Lq'' + Rq' + \frac{q}{C} = E(t)$$

$$q'' + 100q' + \frac{q}{(1/2500)} = 30$$

$$q'' + 100q' + 2500q = 30$$

$$q'' + 100q' + 2500q = 0$$
$$m^2 + 100m + 2500 = 0$$
$$(m+50)^2 = 0$$
$$m_1 = m_2 = -50$$
$$q_c = c_1 e^{-50t} + c_2 te^{-50t}$$

$$q_p = A$$
$$q_p' = 0$$
$$q_p'' = 0$$
$$q_p'' + 100q_p' + 2500q_p = 25$$
$$0 + 0 + 2500A = 25$$
$$A = \frac{25}{2500}$$
$$A = \frac{1}{100}$$
$$q_p = \frac{1}{100}$$

$$q(t) = c_1 e^{-50t} + c_2 te^{-50t} + \frac{1}{100}$$

$$q' = -50c_1 e^{-50t} + c_2 e^{-50t} - 50c_2 te^{-50t}$$

$$0 = c_1 e^0 + c_2(0)e^0 + \frac{1}{100}$$
$$c_1 = -\frac{1}{100}$$

$$0 = -50c_1 e^0 + c_2 e^0 - 50(0)e^0$$
$$0 = -50c_1 + c_2$$
$$c_2 = 50\left(-\frac{1}{100}\right)$$
$$c_2 = -\frac{1}{2}$$

$$q(t) = -\frac{1}{100}e^{-50t} - \frac{1}{2}te^{-50t} + \frac{1}{100}$$

4. A beam of length 10m is embedded at both ends. Find the deflection of the beam if a load of $\omega(x) = 2EIx$ is uniformly distributed along its length.

$$EIy^{(4)} = \omega(x)$$
$$EIy^{(4)} = 12EIx$$
$$y^{(4)} = 12x$$
$$y_c = c_1 + c_2x + c_3x^2 + c_4x^3$$
$$\cancel{y_p = Ax + B}$$
$$y_p = Ax^5 + Bx^4$$
$$y_p' = 5Ax^4 + 4Bx^3$$
$$y_p'' = 20Ax^3 + 12Bx^2$$
$$y_p''' = 60Ax^2 + 24Bx$$
$$y_p^{(4)} = 120Ax + 24B$$
$$y_p^{(4)} = 12x$$
$$120Ax + 24B = 12x$$

$$120A = 12 \qquad 24B = 0$$
$$A = \tfrac{1}{10} \qquad B = 0$$

$$y_p = \frac{1}{10}x^5$$

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{x^5}{10}$$

$$y = c_1 + c_2x + c_3x^2 + c_4x^3 + \tfrac{x^5}{10}$$
$$0 = c_1 + 0 + 0 + 0 + 0$$
$$c_1 = 0$$
$$0 = c_1 + 10c_2 + 100c_3 + 1000c_4 + \tfrac{100{,}000}{10}$$
$$0 = 100c_3 + 1000c_4 + 10{,}000$$
$$0 = c_3 + 10c_4 + 100$$
$$c_3 = -10c_4 - 100$$

$$y' = c_2 + 2c_3x + 3c_4x^2 + \tfrac{x^4}{2}$$
$$0 = c_2 + 0 + 0 + 0$$
$$c_2 = 0$$
$$0 = c_2 + 20c_3 + 300c_4 + \tfrac{10{,}000}{2}$$
$$0 = 20c_3 + 300c_4 + 5{,}000$$
$$0 = 2c_3 + 30c_4 + 500$$
$$0 = 2\left(-10c_4 - 100\right) + 30c_4 + 500$$
$$0 = -20c_4 - 200 + 30c_4 + 500$$
$$0 = 10c_4 + 300$$
$$c_4 = -30$$

$$c_3 = 300 - 100$$
$$c_3 = 200$$

$$y = 200x^2 - 30x^3 + \frac{x^5}{10}$$