R·I·T School of Mathematical Sciences

First Order Linear Equations

MATH 211

1. Find the general solution to the following differential equations.

   (a) $\frac{dy}{dx} = 5y$

   $$\frac{dy}{dx} - 5y = 0$$
   $$e^{-5x}\left[\frac{dy}{dx} - 5y\right] = 0 \cdot e^{-5x}$$
   $$\frac{d}{dx}\left[ye^{-5x}\right] = 0 \qquad e^{\int P(x)dx} = e^{-\int 5dx} = e^{-5x}$$
   $$\int d\left[ye^{-5x}\right] = \int 0dx$$
   $$ye^{-5x} = C$$

   (b) $\frac{dy}{dx} + y = e^{3x}$

   $$\frac{dy}{dx} + y = e^{3x}$$
   $$e^{x}\left[\frac{dy}{dx} + y\right] = e^{3x}e^{x}$$
   $$\frac{d}{dx}[ye^{x}] = e^{4x} \qquad e^{\int P(x)dx} = e^{\int 1dx} = e^{x}$$
   $$\int d[ye^{x}] = \int e^{4x}\ dx$$
   $$ye^{x} = \frac{1}{4}e^{4x} + C$$

(c) $y' + 3x^2y = x^2$

$$\frac{dy}{dx} + 3x^2y = x^2$$

$$e^{x^3}\left[\frac{dy}{dx} + 3x^2y\right] = x^2e^{x^3}$$

$$e^{\int P(x)dx} = e^{\int 3x^2dx} = e^{x^3}$$

$$\frac{d}{dx}\left[ye^{x^3}\right] = x^2e^{x^3}$$

$$\boxed{\begin{aligned} \star\ u &= x^3 \\ du &= 3x^2dx \\ \frac{1}{3}du &= x^2dx \end{aligned}}$$

$$\int d[ye^{x^3}] = \int x^2e^{x^3}dx_\star$$

$$ye^{x^3} = \frac{1}{3}\int e^u du$$

$$ye^{x^3} = \frac{1}{3}e^{x^3} + C$$

(d) $x^2y' + xy = 1$

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2}$$

$$x\left[\frac{dy}{dx} + \frac{1}{x}y\right] = \frac{1}{x^2}\cdot x$$

$$\frac{d}{dx}[yx] = \frac{1}{x}$$

$$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int d[yx] = \int \frac{1}{x}dx$$

$$yx = \ln|x| + C$$

(e) $xy' - y = x^2\sin x$

$$\frac{dy}{dx} - \frac{1}{x}y = x\sin x$$

$$\frac{1}{x}\left[\frac{dy}{dx} - \frac{1}{x}y\right] = (x\sin x)\cdot\frac{1}{x}$$

$$\frac{d}{dx}\left[y\left(\frac{1}{x}\right)\right] = \sin x$$

$$e^{\int P(x)dx} = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = e^{\ln(x^{-1})} = x^{-1} = \frac{1}{x}$$

$$\int d\left[\frac{y}{x}\right] = \int \sin x dx$$

$$\frac{y}{x} = -\cos x + C$$

(f) $x^2y' + x(x+2)y = e^x$

$$\frac{dy}{dx} + \left(\frac{x+2}{x}\right)y = \frac{e^x}{x^2}$$

$$x^2e^x\left[\frac{dy}{dx} + \left(\frac{x+2}{x}\right)y\right] = \frac{e^x}{x^2}\cdot x^2e^x$$

$$\frac{d}{dx}[yx^2e^x] = e^{2x}$$

$$\int d[yx^2e^x] = \int e^{2x}dx$$

$$yx^2e^x = \frac{1}{2}e^{2x} + C$$

$$\begin{aligned} e^{\int P(x)dx} &= e^{\int \left(1+\frac{2}{x}\right)dx} \\ &= e^{(x+2\ln x)} \\ &= e^x\cdot e^{\ln(x^2)} \\ &= x^2e^x \end{aligned}$$

(g) $xy' + (1+x)y = e^{-x}\sin(2x)$

$$\frac{dy}{dx} + \left(\frac{1+x}{x}\right)y = \frac{e^{-x}\sin(2x)}{x}$$

$$xe^x\left[\frac{dy}{dx} + \left(\frac{1+x}{x}\right)y\right] = \frac{e^{-x}\sin(2x)}{x}\cdot xe^x$$

$$\frac{d}{dx}[yxe^x] = \sin(2x)$$

$$\int d[yxe^x] = \int \sin(2x)dx$$

$$yxe^x = -\frac{1}{2}\cos(2x) + C$$

$$\begin{aligned} e^{\int P(x)dx} &= e^{\int \left(\frac{1}{x}+1\right)dx} \\ &= e^{(\ln x+x)} \\ &= e^{\ln x}\cdot e^x \\ &= xe^x \end{aligned}$$

(h) $y\ dx - 4(x+y^6)\ dy = 0$

$$y\frac{dx}{dy} - 4x - 4y^6 = 0$$

$$y\frac{dx}{dy} - 4x = 4y^6$$

$$\frac{dx}{dy} - \left(\frac{4}{y}\right)x = 4y^5$$

$$\frac{1}{y^4}\left[\frac{dx}{dy} - \left(\frac{4}{y}\right)x\right] = 4y^5\cdot\frac{1}{y^4}$$

$$\frac{d}{dy}\left[x\cdot\frac{1}{y^4}\right] = 4y$$

$$\int d\left[\frac{x}{y^4}\right] = \int 4ydy$$

$$\frac{x}{y^4} = 2y^2 + C$$

$$\begin{aligned} e^{\int P(y)dy} &= e^{-4\int \frac{1}{y}dy} \\ &= e^{-4\ln y} \\ &= e^{\ln(y^{-4})} \\ &= y^{-4} \\ &= \frac{1}{y^4} \end{aligned}$$

(i) $\cos x\frac{dy}{dx}+(\sin x)y=1$

$$\frac{dy}{dx}+(\tan x)y=\sec x$$

$$\sec x\left[\frac{dy}{dx}+(\tan x)y\right]=\sec x\sec x$$

$$\frac{d}{dx}[y\sec x]=\sec^2 x$$

$$\int d[y\sec x]=\int \sec^2 x dx$$

$$y\sec x=\tan x+C$$

$$e^{\int P(x)dx}=e^{\int \tan x dx}$$
$$=e^{\ln\sec x}$$
$$=\sec x$$

(j) $(x+2)^2y'=5-8y-4xy$

$$(x+2)^2\frac{dy}{dx}+4(x+2)y=5$$

$$\frac{dy}{dx}+\left(\frac{4}{x+2}\right)y=\frac{5}{(x+2)^2}$$

$$(x+2)^4\left[\frac{dy}{dx}+\left(\frac{4}{x+2}\right)y\right]=\frac{5}{(x+2)^2}\cdot(x+2)^4$$

$$\frac{d}{dx}[y(x+2)^4]=5(x+2)^2$$

$$\int d[y(x+2)^4]=5\int (x+2)^2dx$$

$$y(x+2)^4=\frac{5}{3}(x+2)^3+C$$

$$e^{\int P(x)dx}=e^{\int \frac{4}{x+2}dx}$$
$$=e^{4\ln(x+2)}$$
$$=e^{\ln(x+2)^4}$$
$$=(x+2)^4$$

(k) $\frac{dr}{d\theta}=\cos\theta-r\sec\theta$

$$\frac{dr}{d\theta}+(\sec\theta)r=\cos\theta$$

$$(\sec\theta+\tan\theta)\left[\frac{dr}{d\theta}+(\sec\theta)r\right]=\cos\theta(\sec\theta+\tan\theta)$$

$$\frac{d}{d\theta}\left[r(\sec\theta+\tan\theta)\right]=1+\sin\theta$$

$$\int d[r(\sec\theta+\tan\theta)]=\int (1+\sin\theta)d\theta$$

$$r(\sec\theta+\tan\theta)=\theta-\cos\theta+C$$

$$e^{\int P(\theta)d\theta}=e^{\int \sec\theta d\theta}$$
$$=e^{\ln|\sec\theta+\tan\theta|}$$
$$=\sec\theta+\tan\theta$$

(l) $(x^2-1)y' + 2y = (x+1)^2$

$$e^{\int P(x)dx} = e^{\int \frac{2}{(x-1)(x+1)}dx}_{\star}$$
$$= e^{\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx}$$
$$= e^{\ln|x-1| - \ln|x+1|}$$
$$= e^{\ln\left|\frac{x-1}{x+1}\right|}$$
$$= \frac{x-1}{x+1}$$

$$\frac{dy}{dx} + \left(\frac{2}{x^2-1}\right)y = \frac{(x+1)^2}{(x-1)(x+1)}$$
$$\frac{x-1}{x+1}\left[\frac{dy}{dx} + \left(\frac{2}{x^2-1}\right)y\right] = \frac{x+1}{x-1}\cdot\frac{x-1}{x+1}$$
$$\frac{d}{dx}\left[\left(\frac{x-1}{x+1}\right)y\right] = 1$$
$$\int d\left[\left(\frac{x-1}{x+1}\right)y\right] = \int 1dx$$
$$\left(\frac{x-1}{x+1}\right)y = x + C$$

$$\boxed{\begin{array}{c} \star\ \dfrac{2}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1} \\ 2 = Ax + A + Bx - B \\ 2 = (A+B)x + (A-B) \\ \begin{array}{cc} A+B=0 & A-B=2 \\ B=-A & A+A=2 \\ & 2A=2 \\ & A=1 \\ B=-1 & \end{array} \\ \dfrac{2}{(x-1)(x+1)} = \dfrac{1}{x-1} - \dfrac{1}{x+2} \end{array}}$$

2. Solve the following initial value problems.

(a) $\frac{dy}{dx} = x + 5y,\ y(0) = 3$

$$\frac{dy}{dx} - 5y = x$$

$$e^{-5x}\left[\frac{dy}{dx} - 5y\right] = xe^{-5x}$$

$$\frac{d}{dx}\left[ye^{-5x}\right] = xe^{-5x}$$

$$\int d[ye^{-5x}] = \int xe^{-5x}dx_{\star}$$

$$ye^{-5x} = -\frac{1}{5}xe^{-5x} + \frac{1}{5}\int e^{-5x}dx$$

$$ye^{-5x} = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$$

$$y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$$

$$3 = 0 - \frac{1}{25} + C$$

$$\frac{76}{25} = C$$

$$y = -\frac{x}{5} - \frac{1}{25} + \frac{76}{25}e^{5x}$$

$$e^{\int P(x)dx} = e^{-\int 5dx}$$
$$= e^{-5x}$$

$$\boxed{\star \quad \begin{array}{ll} u = x & dv = e^{-5x}dx \\ du = dx & v = -\frac{1}{5}e^{-5x} \end{array}}$$

(b) $\frac{dy}{dx} = 2x - 3y,\ y(0) = \frac{1}{3}$

$$\frac{dy}{dx} + 3y = 2x$$

$$e^{3x}\left[\frac{dy}{dx} + 3y\right] = 2xe^{3x}$$

$$\frac{d}{dx}[ye^{3x}] = 2xe^{3x}$$

$$\int d[ye^{3x}] = 2\int xe^{3x}dx_{\star}$$

$$ye^{3x} = 2\left[\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}dx\right]$$

$$e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}$$

$$\boxed{\star \quad \begin{matrix} u = x & dv = e^{3x}dx \\ du = dx & v = \frac{1}{3}e^{3x} \end{matrix}}$$

$$ye^{3x} = \frac{2}{3}xe^{3x} - \frac{2}{9}e^{3x} + C$$

$$y = \frac{2}{3}x - \frac{2}{9} + Ce^{-3x}$$

$$\frac{1}{3} = 0 - \frac{2}{9} + C$$

$$C = \frac{5}{9}$$

$$y = \frac{2}{3}x - \frac{2}{9} + \frac{5}{9}e^{-3x}$$

(c) $xy' + y = e^x,\ y(1) = 2$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = \frac{e^x}{x}$$

$$x\left[\frac{dy}{dx} + \left(\frac{1}{x}\right)y\right] = \frac{e^x}{x}\cdot x$$

$$\frac{d}{dx}[yx] = e^x$$

$$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$\int d[yx] = \int e^x dx$$

$$yx = e^x + C$$

$$2(1) = e^1 + C$$

$$2 - e = C$$

$$yx = e^x + 2 - e$$

(d) $y' + (\tan x)y = \cos^2 x,\ \ y(0) = -1$

$$\frac{dy}{dx} + (\tan x)y = \cos^2 x$$

$$\sec x \left[\frac{dy}{dx} + (\tan x)y\right] = \cos^2 x \cdot \sec x$$

$$\frac{d}{dx}[y \sec x] = \cos x$$

$$\int d[y \sec x] = \int \cos x dx$$

$$e^{\int P(x)dx} = e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$$

$$y \sec x = \sin x + C$$

$$-1 \sec 0 = \sin 0 + C$$

$$-1 = 0 + C$$

$$C = -1$$

$$y \sec x = \sin x - 1$$