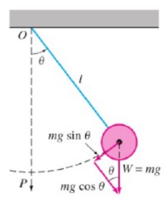
Name: ______________________

Section: ______________________

R·I·T School of Mathematical Sciences

20 - Pendulum

MATH 211



A simple pendulum is a special case of a physical pendulum which consists of a rod of length l to which a mass m is attached at one end. The angular acceleration of the mass is given by

$$a = \frac{d^2 s}{dt^2} = l\frac{d^2\theta}{dt^2} \text{ since } s = l\theta$$

By Newton's Second Law, we have the force $F_1 = ma = ml\frac{d^2\theta}{dt^2}$ and the magnitude of the tangential component of the force due to the weight is $F_2 = mg\sin\theta$. But we can linearly approximate for very small values of θ, so $\sin\theta \approx \theta$.
Since the sum of the forces is equal to zero, we have

$$F_1 + F_2 = 0$$

$$ml\theta'' + mg\theta = 0$$

which simplifies to

$$\theta'' + \frac{g}{l}\theta = 0$$

An 8 inch long pendulum rod has a mass of 0.25 kg attached. The pendulum to an initial angle of $\theta = \frac{\pi}{12}$ and released from rest. Find its equation of angular motion.

$$\theta'' + \frac{32.2}{{}^2\!/_3}\theta = 0$$

$$\theta'' + \frac{483}{10}\theta = 0$$

$$\lambda^2 + \frac{483}{10} = 0$$

$$\lambda^2 = -\frac{483}{10}$$

$$\lambda = \pm\sqrt{-\frac{483}{10}}$$

$$\lambda = \pm\frac{\sqrt{4830}}{10}i$$

$$\theta = c_1 \sin\left(\frac{\sqrt{4830}}{10}t\right) + c_2 \cos\left(\frac{\sqrt{4830}}{10}t\right)$$

$$\theta(0) = \frac{\pi}{12}$$

$$\theta = c_1 \sin\left(\frac{\sqrt{4830}}{10}t\right) + c_2 \cos\left(\frac{\sqrt{4830}}{10}t\right)$$

$$\frac{\pi}{12} = c_1(0) + c_2(1)$$

$$c_2 = \frac{\pi}{12}$$

$$\theta'(0) = 0$$

$$\theta' = \frac{\sqrt{4830}}{10}c_1 \cos\left(\frac{\sqrt{4830}}{10}t\right) - \frac{\sqrt{4830}}{10}c_2 \sin\left(\frac{\sqrt{4830}}{10}t\right)$$

$$0 = \frac{\sqrt{4830}}{10}c_1(1) - \frac{\sqrt{4830}}{10}c_2(0)$$

$$c_1 = 0$$

$$\theta(t) = \frac{\pi}{12}\cos\left(\frac{\sqrt{4830}}{10}t\right)$$

Now, consider a pendulum with an external force of $f(t) = 2\sin t$ driving its motion. The length of the pendulum's rod is 50 cm. The pendulum's initial angular displacement is $\frac{\pi}{30}$ and the pendulum is released from rest. Find the equation of motion of the pendulum. [Note: Since we need to balance the forces, we now have $F_1 + F_2 = F$.]

$$\theta'' + \frac{g}{l}\theta = f(t)$$

$$\theta'' + \frac{9.81}{0.5}\theta = 2\sin t$$

$$\theta'' + \frac{981}{50}\theta = 2\sin t$$

Complementary:

$$\theta'' + \frac{981}{50}\theta = 0$$

$$\lambda^2 + \frac{981}{50} = 0$$

$$\lambda^2 = -\frac{981}{50}$$

$$\lambda = \pm\sqrt{-\frac{981}{50}}$$

$$\lambda = \pm\frac{\sqrt{1962}}{10}i$$

$$\theta_c = c_1 \sin\left(\frac{\sqrt{1962}}{10}t\right) + c_2 \cos\left(\frac{\sqrt{1962}}{10}t\right)$$

Particular:

$$\theta_p = A\sin t + B\cos t$$

$$\theta_p' = A\cos t - B\sin t$$

$$\theta_p'' = -A\sin t - B\cos t$$

$$\theta_p'' + \frac{981}{50}\theta_p = 2\sin t$$

$$-A\sin t - B\cos t + \frac{981}{50}\left(A\sin t + B\cos t\right) = 2\sin t$$

$$\frac{931}{50}A\sin t + \frac{931}{50}B\cos t = 2\sin t$$

$$\tfrac{931}{50}A = 2 \qquad \tfrac{931}{50}B = 0$$

$$A = \tfrac{100}{931} \qquad B = 0$$

$$\theta_p = \frac{100}{931}\sin t$$

$$\theta = \theta_c + \theta_p$$

$$\theta = c_1 \sin\left(\frac{\sqrt{1962}}{10}t\right) + c_2 \cos\left(\frac{\sqrt{1962}}{10}t\right) + \frac{100}{931}\sin t$$

$$\theta = c_1 \sin\left(\frac{\sqrt{1962}}{10}t\right) + c_2 \cos\left(\frac{\sqrt{1962}}{10}t\right) + \frac{100}{931}\sin t$$

$$\frac{\pi}{30} = c_1(0) + c_2(1) + 0$$

$$c_2 = \frac{\pi}{30}$$

$$\theta' = \frac{\sqrt{1962}}{10}c_1 \cos\left(\frac{\sqrt{1962}}{10}t\right) - \frac{\sqrt{1962}}{10}c_2 \sin\left(\frac{\sqrt{1962}}{10}t\right) + \frac{100}{931}\cos t$$

$$0 = \frac{\sqrt{1962}}{10}c_1(1) - \frac{\sqrt{1962}}{10}c_2(0) + \frac{100}{931}(1)$$

$$\frac{\sqrt{1962}}{10}c_1 = -\frac{100}{931}$$

$$c_1 = -\frac{500\sqrt{1962}}{913,311}$$

$$\theta = -\frac{500\sqrt{1962}}{913,311}\sin\left(\frac{\sqrt{1962}}{10}t\right) + \frac{\pi}{30}\cos\left(\frac{\sqrt{1962}}{10}t\right) + \frac{100}{931}\sin t$$