Title of the original version: Scale-free networks obtained from d-primes via visibility algorithms

Title of the revised version: A numerical study on the regularity of d-primes via informational entropy and visibility algorithms

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Dear Editor,

The paper was modified by considering the comments and suggestions described in the reviewer's report (in *italic*). Other alterations were also made to improve the manuscript. The relevant modifications in red are mentioned in the following point-by-point summary of the changes. Phrases of the original version are in blue.

Answer to the Reviewer:

First of all, I would like to acknowledge your consideration of my previous suggestions in the new version of your paper entitled "Scale-free networks obtained from d-primes via visibility algorithms". I think that these modifications have improved the quality of the paper but, unfortunately, there are still important questions that have to be solved before having my approval for its publication in Complexity.

Please find below the comments I would like to share with you and ask for a reasonable solution:

1. The first point has to do with the denomination of the numbers that have d divisors (including 1 and itself). It must be stressed that the name prime, that is given to those numbers that have no divisors different from 1 and themselves, makes reference to their basic nature because, as it is stated by the Fundamental Theorem of Arithmetic, all natural numbers can be decomposed in terms of them. Since this is not the case with d-primes for $d > 2$, I wonder whether this denomination is adequate. Instead, I suggest to define these numbers as d-divisors, i.e. natural numbers that admit d different divisors. In particular, 2-divisors are prime numbers because they share elementary arithmetic properties.

We do understand the issue raised by the reviewer; however, we preferred to keep d-prime throughout the paper for three reasons: 1) d-prime can be considered as a generalization of the concept of prime number, in which d specifies the number of divisors; 2) the more natural notation d_d for d-divisor may be confusing; 3) d-prime is a more appealing denomination.

Please note that we changed the title to mention the analysis performed by using informational entropy.

2. In relation with the previous point, I wonder about the convenience of defining a d-divisors by considering only its prime divisors. For instance, according to this alternative definition, the number 36 would be classified as 4-divisors, $36 = 2^2 3^2$*, in contrast to the authors definition that would define this number as 8-divisors. Likely, this definition would allow to find theoretical results related to the Fundamental Theorem of Arithmetic. Have the authors try this approach?*

Please observe that this alternative definition can imply results quite different from those reported in this manuscript. This suggestion, however, is very nice.

3. In the way of obtaining any theoretical result, let me share with you this classical result, that likely you already know, but that could be explored to obtain a relationship between the terms of the sequences $p_d(n)$*.*

"If $d = a\ b\ c\ ...$*, then it is well known that the numbers with d divisors are of the form* $p^{a-1}q^{b-1}r^{c-1}..., p, q, r, ...$ *being prime numbers."*

For instance, for $d = 4$*, all 4-divisors can be written as* $p\ q$ *and* p^3 *with* p *and* q *primes. From table 1, we know that*

$$p_4(n) = \{2\,3; 2^3; 2\,5; 7\,2; 3\,5; 7\,3; 11\,2; 13\,2; 3^3; 11\,3; 17\,2; 7\,5\}$$

Could any relation between the sequences of first order differences of consecutive d-divisors be deduced from this result?

The derivation of analytic relations for d-primes is a golden dream. This task, however, is far from being trivial. Indeed, this task seems to be as difficult as to find a formula to generate the exact sequence of prime numbers, but such a formula cannot even exist!

For instance, consider the sequence above for $d = 4$. One of the main difficulties is to determine if the n-th number is pq or p^3. This sequence shows that $n = 2$ and $n = 9$ are p^3. Which is the value of n corresponding to the next p^3? Also, from a given value of n (for instance, $n = 701$), is the corresponding number pq or p^3? Analytical answers are not evident, even for the particular case $d = 4$. Hence, as a first approach, numerical experiments were performed in order to investigate how the gap sequences depend on d.

4. In my opinion, the basic properties of prime numbers (2-divisors) should be used as a reference to compare with. I miss this viewpoint in the paper, though the new version includes a relationship between the sequence $p_d(n)$ *and* $p_2(n)$ *when d is not 2-divisors. I would suggest to apply this sequence as a reference for the rest of d-divisors sequences and, if you consider it suitable, to cite this work: Matsushita, R. and Da Silva, S. (2016) "A Power Law Governing Prime Gaps". Open Access Library Journal, 3: e2989. Here, the authors already conjecture about the power law* *behavior of prime gaps (although, other authors conjecture about a Poisson distribution). As a matter of fact, contrary to what the authors state, the independence of gamma with d could inform about a possible universal law behind d-divisors sequences. In my opinion, this question requires further investigation.*

As the reviewer already observed, relations between the sequences $p_d(n)$ and $p_2(n)$ are presented in the paper. Note that a goal here is try to understand the properties of d-primes; the case $d = 2$ is just a particular case. Obviously, we do acknowledge the relevance of this case.

In addition, we thank the reviewer for the reference mentioned above and included it in the following comment in page 6:

Power laws have been found in the distribution of prime gaps [22] and in a myriad of contexts, such as psychiatric ward [23] and financial crashes [23].

This reference was added:

[22] R. Matsushita and S. da Silva, "A Power Law Governing Prime Gaps," *Open Access Library Journal*, vol. 3, article ID e2989, 2016.

5. Related to the previous point, I would like to point out that the power law fit, proposed by the authors in figures 1 and 2, should be checked. In particular, the curve that is depicted in Figure 1, obtained after applying the horizontal visibility algorithm, do you think that is well fitted by a straight line? Besides, I do not see the reason to restrict the figures to the cases $d = 4$ *(Fig. 1) and* $d = 5$ *(Fig. 2). I wonder why not to show the curves for the rest of d-values, in particular for* $d = 2$*. In so doing, the comparison between all the curves would provide additional information to that shown in Table 4. Indeed, with only these fittings one could wonder about the scale free nature of the corresponding visibility graphs, which is one of the main aims of the paper.*

We agreed with the reviewer and included <u>all plots</u> for $d \in \{2, 3, ..., 11\}$ (please, see new Figures 1 and 2 of the revised version). Also, the following phrases were included in page 6:

In a log-log plot, $\log P(k) = \log A - \gamma \log k$; thus, the power law dependence is transformed into a linear relation between $\log P(k)$ and $\log k$. At least as a first approximation, this relation can be taken as linear in most plots shown in both figures; that is, a power-law form for $P(k)$ can be considered a plausible model for these plots. Possible exceptions are the NV plots for $d = \{7, 11\}$.

In page 8:

Fluctuations observed around the fitted straight lines shown in both figures can be effects of the finite size of the gap sequences $x_d(n)$ used in the numerical experiments [25].

This reference was also added:

[25] A. D. Broido and A. Clauset, "Scale-free networks are rare," *Nature Communications*, vol. 10, Article ID 1017, 2019.

6. Concerning the summary presented in Table 4, I suggest to present this table in a different way.... Another possibility could be splitting up this table into two tables, one for each of the visibility algorithms. In my opinion, and I know this contradicts author's statement, the average connectivity could not be used to discard d-divisors. As it can be seen, the average connectivity is similar for all d-values in the horizontal visibility graph. Certainly, it seems that this average connectivity depends on d for the natural visibility graphs, being similar for d-values that are not prime and, according with the authors, it increases for d-values that are prime numbers. In my opinion, this result should be investigated and explained, at least, heuristically.

We agreed with the reviewer and split Table 4 into two tables (Tables 4 and 5).

About the issue raised by the reviewer concerning the average connectivity: Please note that a conjecture of our work is $\langle k \rangle$ in NV plots distinguishes d that is prime from d that is not prime; $\langle k \rangle$ in HV plots distinguishes d odd from d even. This conjecture was based on the numerical results found in our computational experiments.

7. The authors state that: "Note that $P(k)$ for d-primes has smaller slope than the slope of $P_{rand}(k)$." However, I do not see this in figures 10 and 11. Indeed, it is just the contrary, isn't it?

The equation of a straight line is usually written as $y = mx + c$, in which m is the slope. In Figures 1 and 2, m_{rand} for $P_{rand}(k)$ and m for $P(k)$ obey the relation $m < m_{rand}$ as we had written; however, observe that both m and m_{rand} are negative numbers.

In summary, despite the subject under study is, in my opinion, of evident interest, I think that the paper requires further improvements to be considered for publication in Complexity.

Please note that this is the first paper which investigates the properties of d-primes (which includes the usual prime numbers). Also, this investigation is based on informational entropy and two visibility graphs, which is an original approach. All results presented here are new, including the idea of exploring d-primes.

The authors would like to sincerely express gratitude to the reviewer for this second round of suggestions.