A Categorical Framework for Testing Generalised Tree automata

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Summary

- Motivation: equivalence query in **automata learning**

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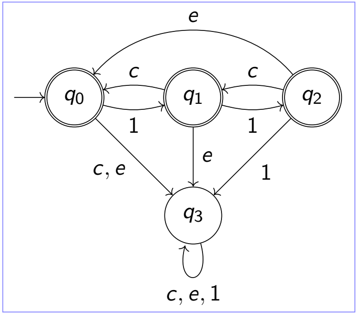
- Provide a **categorical framework** for **testing black-box systems**
- Generalise the gist of the completeness proof for DFAs/Mealy machines to a categorical level
- Based on recent work [KR25]

Outline

1. Introduction
   - Conformance Testing
   - The W-method
2. Further generalisation
   - Automata
   - Main result
3. Conclusion

Idea of conformance testing



(a) Specification

(b) Implementation

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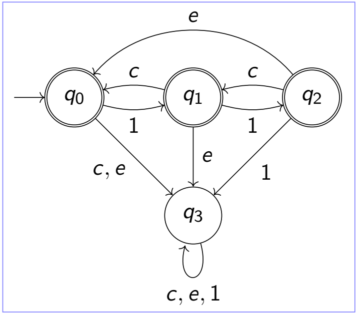

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Are they equivalent?

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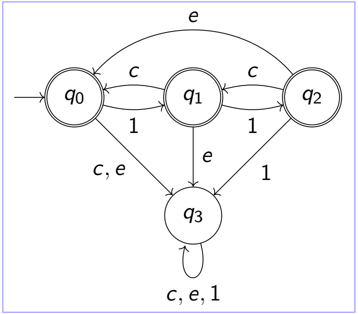

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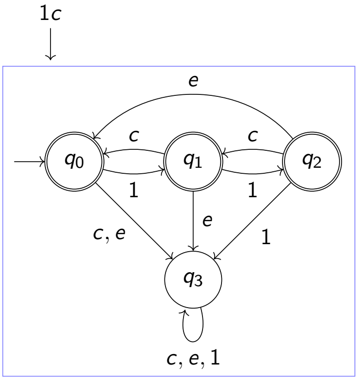

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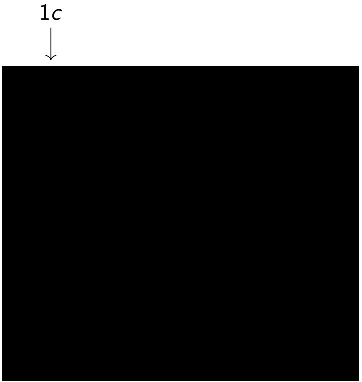

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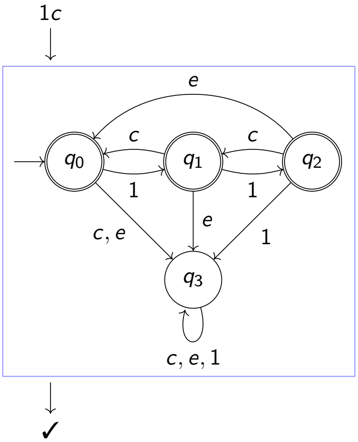

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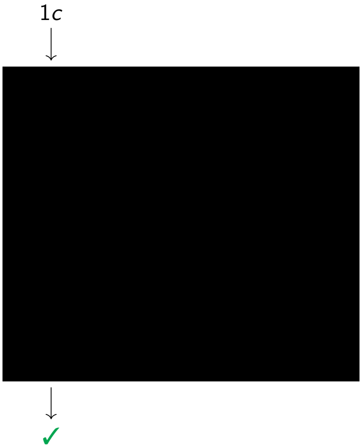

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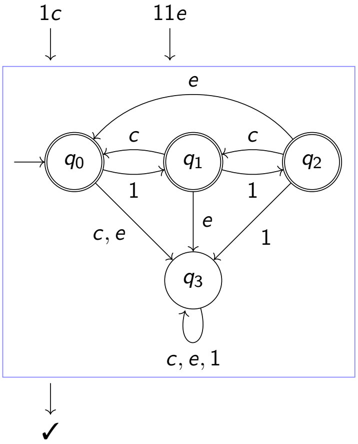

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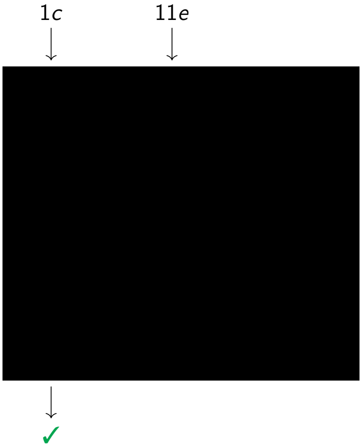

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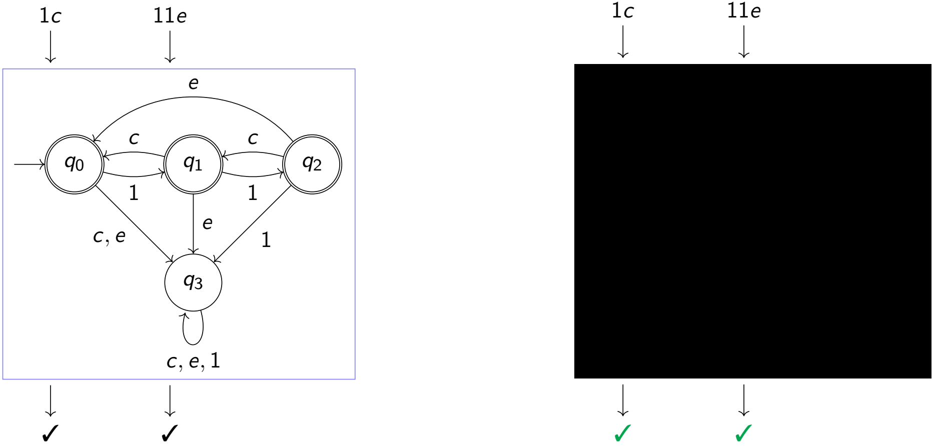

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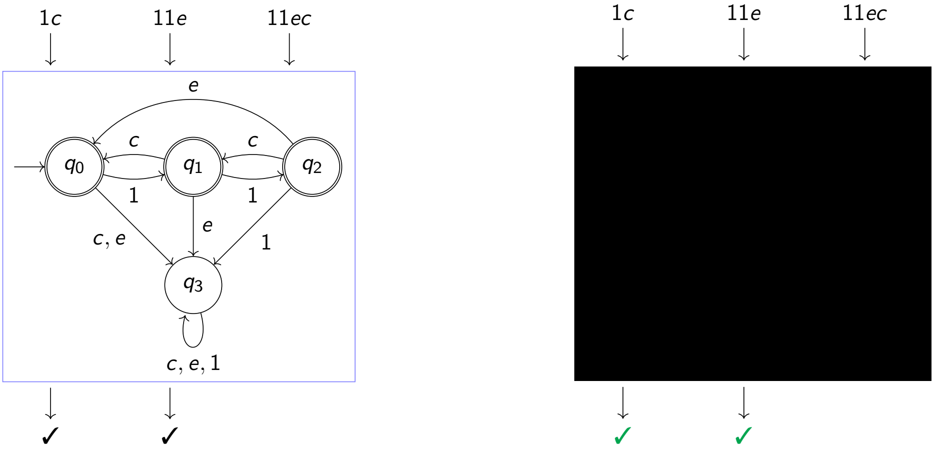

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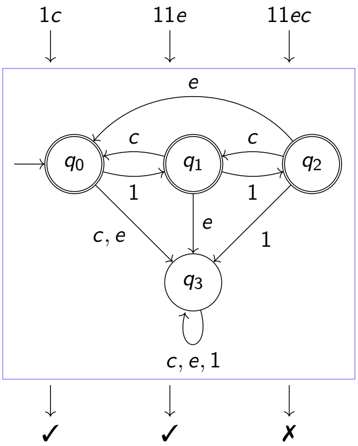

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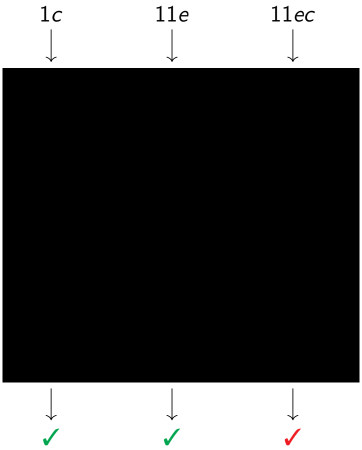

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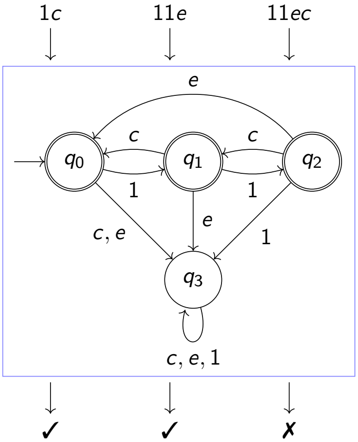

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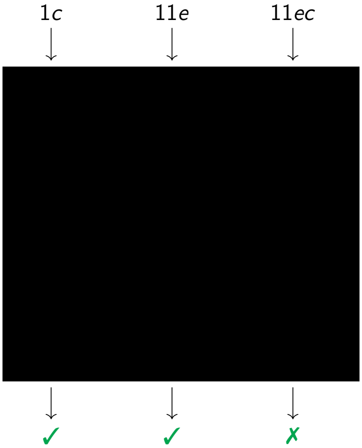

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What if we pass all the tests?

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- More formally, no **complete test suite** exists

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A test suite $T \subseteq \Sigma^*$ is **complete** for $\mathcal{S}$ with respect to a fault domain $\mathcal{U}$ if $\mathcal{L}_\mathcal{S} \cap T = \mathcal{L}_\mathcal{M} \cap T$ implies $\mathcal{L}_\mathcal{S} = \mathcal{L}_\mathcal{M}$ for all $\mathcal{M} \in \mathcal{U}$.

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How do we construct complete test suites?

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Theorem

Suppose $\mathcal{S}$ has n states. Then $T^k_{P,W}$ is complete for $\mathcal{S}$ with respect to $\mathcal{U}_{n+k}$.

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This lemma was generalised to a categorical setting in [KR25]; we improve on that work

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- **tree automata**: $FX = \sum_{f \in \Sigma} X^{\mathrm{ar}(f)}$ for an algebraic signature Σ

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Let $\mathcal{S}$ and $\mathcal{M}$ be two automata. Suppose c is a weak state cover for $\mathcal{M}$ and w is a characterisation morphism for $\mathcal{S}$. Let $t = c^{+} \cdot w$. Then $B_{\mathcal{S}} \circ t = B_{\mathcal{M}} \circ t$ implies $B_{\mathcal{S}} = B_{\mathcal{M}}$.

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- $c^{+} = [c, \gamma_I \circ Fc]$: obtained by extending inputs in c with one symbol
- $a \cdot b = m \circ (a \otimes b)$: concatenation of the inputs in a and in b

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- Implement algorithms for concrete models

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