Complex Analysis: Exercises

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1 Preliminaries to Complex Analysis

Exercise 1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1,\ z_2 \in \mathbb{C}$.

(b) $1/z = \overline{z}$.

(c) $\mathrm{Re}(z) = 3$.

(d) $\mathrm{Re}(z) > c$, (resp., $\geq c$) where $c \in \mathbb{R}$.

(e) $\mathrm{Re}(az + b) > 0$ where $a,\ b \in \mathbb{C}$.

(f) $|z| = \mathrm{Re}(z) + 1$.

(g) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Exercise 2. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product in $\mathbb{R}^2$. In other words, if $Z = (x_1, y_1)$ and $W = (x2, y2)$, then

$$\langle Z, W \rangle = x_1 x_2 + y_1 y_2.$$

Similarly, we may define a Hermitian inner product $(\cdot, \cdot)$ in $\mathbb{C}$ by

$$(z, w) = z\overline{w}$$

.

The term Hermitian is used to describe the fact that $(\cdot, \cdot)$ is not symmetric, but rather satisfies the relation

$$(z, w) = (w, z) \quad \text{for all } z, w \in C.$$

Show that

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \mathrm{Re}(z, w),$$

where we use the usual identification $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$.

Exercise 3. With $\omega = se^{i\varphi}$, where $s \geq 0$ and $\varphi \in \mathbb{R}$, solve the equation $zn = \omega$ in $\mathbb{C}$ where n is a natural number. How many solutions are there?

Exercise 4. Show that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\succ$ between complex numbers so that:

(i) For any two complex numbers z, w, one and only one of the following is true: $z \succ w$, $w \succ z$, or $z = w$.

(ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \succ z_2$ implies $z_1 + z_3 \succ z_2 + z_3$.

(iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \succ 0$, then $z_1 \succ z_2$ implies $z_1 z_3 \succ z_2 z_3$.

[Hint: First check if $i \succ 0$ is possible.]

Exercise 5. A set Ω is said to be **pathwise connected** if any two points in Ω can be joined by a (piecewise-smooth) curve entirely contained in Ω. The purpose of this exercise is to prove that an *open* set Ω is pathwise connected if and only if Ω is connected.

(a) Suppose first that Ω is open and pathwise connected, and that it can be written as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint non-empty open sets. Choose two points $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$ and let γ denote a curve in Ω joining w_1 to w_2. Consider a parametrization $z : [0,1] \to \Omega$ of this curve with $z(0) = w_1$ and $z(1) = w_2$, and let

$$t^* = \sup_{0 \le t \le 1} \{t : z(s) \in \Omega_1 \quad \text{for all } 0 \le s < t\}.$$

Arrive at a contradiction by considering the point $z(t^*)$.

(b) Conversely, suppose that Ω is open and connected. Fix a point $w \in \Omega$ and let $\Omega_1 \subset \Omega$ denote the set of all points that can be joined to w by a curve contained in Ω. Also, let $\Omega_2 \subset \Omega$ denote the set of all points that cannot be joined to w by a curve in Ω. Prove that both Ω_1 and Ω_2 are open, disjoint and their union is Ω. Finally, since Ω_1 is non-empty (why?) conclude that $\Omega = \Omega_1$ as desired.

The proof actually shows that the regularity and type of curves we used to define pathwise connectedness can be relaxed without changing the equivalence between the two definitions when Ω is open. For instance, we may take all curves to be continuous, or simply polygonal lines.[1]

Exercise 6. Let Ω be an open set in $\mathbb{C}$ and $z \in \Omega$. The **connected component** (or simply the **component**) of Ω containing z is the set $\mathcal{C}_z$ of all points w in Ω that can be joined to z by a curve entirely contained in Ω.

1. Check first that $\mathcal{C}_z$ is open and connected. Then, show that $w \in \mathcal{C}_z$ defines an equivalence relation, that is: (i) $z \in \mathcal{C}_z$, (ii) $w \in \mathcal{C}_z$ implies $z \in \mathcal{C}_w$, and (iii) if $w \in \mathcal{C}_z$ and $z \in \mathcal{C}_\zeta$, then $w \in \mathcal{C}_\zeta$.

   Thus Ω is the union of all its connected components, and two components are either disjoint or coincide.
2. Show that Ω can have only countably many distinct connected components.
3. Prove that if Ω is the complement of a compact set, then Ω has only one unbounded component.

[Hint: For (b), one would otherwise obtain an uncountable number of disjoint open balls. Now, each ball contains a point with rational coordinates. For (c), note that the complement of a large disc containing the compact set is connected.]

Exercise 7. The family of mappings introduced here plays an important role in complex analysis. These mappings, sometimes called **Blaschke factors**, will reappear in various applications in later chapters.

(a) Let z, w be two complex numbers such that $\overline{z}w \neq 1$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

[Hint: Why can one assume that z is real? It then suffices to prove that

$$(r-w)(r-\overline{w}) \le (1-rw)(1-r\overline{w})$$

with equality for appropriate r and $|w|$.]

(b) Prove that for a fixed w in the unit disc $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(i) F maps the unit disc to itself (that is, $F : \mathbb{D} \to \mathbb{D}$), and is holomorphic.

[1] A polygonal line is a piecewise-smooth curve which consists of finitely many straight line segments.

(ii) F interchanges 0 and w, namely $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \to \mathbb{D}$ is bijective. [Hint: Calculate $F \circ F$.]

Exercise 8. Suppose U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to C$ are two functions that are differentiable (in the real sense, that is, as functions of the two real variables x and y), and $h = g \circ f$, then

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial z}$$

and

$$\frac{\partial h}{\partial \overline{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{z}}\frac{\partial \overline{f}}{\partial \overline{z}}.$$

This is the complex version of the chain rule.

Exercise 9. Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \quad \text{where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Exercise 10. Show that

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} = 4\frac{\partial}{\partial \overline{z}}\frac{\partial}{\partial z} = \triangle,$$

where $\triangle$ is the **Laplacian**

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Exercise 11. Use Exercise 10 to prove that if f is holomorphic in the open set Ω, then the real and imaginary parts of f are **harmonic**; that is, their Laplacian is zero.

Exercise 12. Consider the function defined by

$$f(x + iy) = \sqrt{|x||y|}, \qquad \text{whenever } x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.

Exercise 13. Suppose that f is holomorphic in an open set Ω. Prove that in any one of the following cases:

(a) $\mathrm{Re}(f)$ is constant;

(b) $\mathrm{Im}(f)$ is constant;

(c) $|f|$ is constant;

one can conclude that f is constant.

Exercise 14. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$. Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

Exercise 15. Abel's theorem. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Prove that

$$\lim_{r\to 1, r<1} \sum_{n=1}^{\infty} r^n a_n = \sum_{n=1}^{\infty} a_n.$$

[Hint: Sum by parts.] In other words, if a series converges, then it is Abel summable with the same limit. For the precise definition of these terms, and more information on summability methods, we refer the reader to Book I, Chapter 2.

Exercise 16. Determine the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when:

(a) $a_n = (\log n)^2$

(b) $a_n = n!$

(c) $a_n = n^2/(4n+3n)$

(d) $a_n = (n!)^3/(3n)!$ [Hint: Use Stirling's formula, which says that $n! \sim cn^{n+1/2}e^{-n}$ for some $c > 0$..]

(e) Find the radius of convergence of the **hypergeometric series**

$$F(\alpha, \beta, \gamma; z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n$$

Here $\alpha, \beta \in \mathbb{C}$ and $\gamma = 0, -1, -2, \cdots$.

(f) Find the radius of convergence of the Bessel function of order r:

$$J_r(z) = \left(\frac{z}{2}\right)^r \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+r)!} \left(\frac{z}{2}\right)^{2n},$$

where r is a positive integer.

Exercise 17. Show that if $\{a_n\}_{n=0}^{\infty}$ is a sequence of non-zero complex numbers such that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L,$$

then

$$\lim_{n\to\infty} |a_n|^{1/n} = L.$$

In particular, this exercise shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

Exercise 18. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

[Hint: Write $z = z_0 + (z - z_0)$ and use the binomial expansion for z_n.]

Exercise 19. Prove the following:

(a) The power series $\sum nz^n$ does not converge on any point of the unit circle.

(b) The power series $\sum z^n/n^2$ converges at every point of the unit circle.

(c) The power series z^n/n converges at every point of the unit circle except $z = 1$. [Hint: Sum by parts.]

Exercise 20. Expand $(1-z)^{-m}$ in powers of z. Here m is a fixed positive integer. Also, show that if

$$(1-z)^{-m} = \sum_{n=0}^{\infty} a_n z^n,$$

then one obtains the following asymptotic relation for the coefficients:

$$a_n \sim \frac{1}{(m-1)!} n^{m-1} \quad \text{as } n \to \infty$$

Exercise 21. Show that for $|z| < 1$, one has

$$\frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \cdots + \frac{z^{2^n}}{1-z^{2^{n+1}}} + \cdots = \frac{z}{1-z},$$

and

$$\frac{z}{1+z} + \frac{2z^2}{1+z^2} + \cdots + \frac{2^k z^{2^k}}{1+z^{2^k}} + \cdots = \frac{z}{1-z}.$$

Justify any change in the order of summation.

[Hint: Use the dyadic expansion of an integer and the fact that $2^{k+1} - 1 = 1 + 2 + 2^2 + \cdots + 2^k$.]

Exercise 22. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of positive integers. A subset $S \subset \mathbb{N}$ is said to be in arithmetic progression if

$$S = a, a+d, a+2d, a+3d, \ldots$$

where $a, d \in \mathbb{N}$. Here d is called the step of S.

Show that $\mathbb{N}$ cannot be partitioned into a finite number of subsets that are in arithmetic progression with distinct steps (except for the trivial case $a = d = 1$).

[Hint: Write $\sum_{n \in \mathbb{N}} z^n$ as a sum of terms of the type $\frac{z^a}{1-z^d}$.]

Exercise 23. Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ e^{-1/x^2} & \text{if } x > 0. \end{cases}$$

Prove that f is indefinitely differentiable on $\mathbb{R}$, and that $f^{(n)}(0) = 0$ for all $n \geq 1$. Conclude that f does not have a converging power series expansion $\sum_{n=0}^{\infty} a_n x^n$ for x near the origin.

Exercise 24. Let γ be a smooth curve in $\mathbb{C}$ parametrized by $z(t) : [a, b] \to C$. Let γ^- denote the curve with the same image as γ but with the reverse orientation. Prove that for any continuous function f on γ

$$\int_{\gamma} f(x)\, dz = -\int_{\gamma^-} f(X)\, dz\,.$$

Exercise 25. The next three calculations provide some insight into Cauchy's theorem, which we treat in the next chapter.

(a) Evaluate the integrals

$$\int_{\gamma} z^n\, dz$$

for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

(b) Same question as before, but with γ any circle not containing the origin.

(c) Show that if $|a| < r < |b|$, then

$$\int_\gamma \frac{1}{(z-a)(z-b)}\, dz = \frac{2\pi i}{a-b},$$

where γ denotes the circle centered at the origin, of radius r, with the positive orientation.

Exercise 26. Suppose f is continuous in a region Ω. Prove that any two primitives of f (if they exist) differ by a constant.

2 Cauchy's Theorem and Its Applications

Exercise 27. Prove that

$$\int_0^\infty \sin(x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \frac{\sqrt{2\pi}}{4}.$$

These are the **Fresnel integrals**. Here, $\int_0^\infty$ is interpreted as $\lim_{R\to\infty} \int_0^R$.

[Hint: Integrate the function $e - z^2$ over the path in Figure 14. Recall that $\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}$.]

Exercise 28. Show that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

[Hint: The integral equals $\frac{1}{2i} \int_{-\infty}^\infty \frac{e^{ix}-1}{x}\, dx$. Use the indented semicricle.]

Exercise 29. Evaluate the integrals

$$\int_0^\infty e^{-ax} \cos bx\, dx \quad \text{and} \quad \int_0^\infty e^{-ax} \sin bx\, dx, \quad a > 0$$

by integrating e^{-Ax}, $A = \sqrt{a^2+b^2}$, over an appropriate sector with angle ω, with $\cos\omega = a/A$.

Exercise 30. Prove that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^\infty e^{-\pi x^2} e^{2\pi i x\xi}\, dx$.

Exercise 31. Suppose f is continuously *complex* differentiable on Ω, and $T \subset \Omega$ is a triangle whose interior is also contained in Ω. Apply Green's theorem to show that

$$\int_T f(x)\, dz = 0.$$

This provides a proof of Goursat's theorem under the additional assumption that f' is continuous.

[Hint: Green's theorem says that if (F, G) is a continuously differentiable vector field, then

$$\int_T F\, dx + G\, dy = \int_{\text{Interior of } T} \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\right) dx\, dy\,.$$

For appropriate F and G, one can then use the Cauchy-Riemann equations.]

Exercise 32. Let Ω be an open subset of $\mathbb{C}$ and let $T \subset \Omega$ be a triangle whose interior is also contained in Ω. Suppose that f is a function holomorphic in Ω except possibly at a point w inside T. Prove that if f is bounded near w, then

$$\int_T f(x)\, dz = 0.$$

Exercise 33. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic. Show that the diameter $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ of the image of f satisfies

$$2\,|f'(0)| \leq d.$$

Moreover, it can be shown that equality holds precisely when f is linear, $f(z) = a_0 + a_1 z$.

Note. In connection with this result, see the relationship between the diameter of a curve and Fourier series described in Problem 1, Chapter 4, Book I.

[Hint: $2f'(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)-f(-\zeta)}{\zeta^2}\,d\zeta$ whenever $0 < r < 1$.]

Exercise 34. If f is a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ with

$$|f(z)| \leq A(1+|z|)^{\eta}, \quad \eta \text{ a fixed real number}$$

for all z in that strip, show that for each integer $n \geq 0$ there exists $A_n \geq 0$ so that

$$\left|f^{(n)}(x)\right| \leq A_n(1+|x|)^{\eta}, \quad \text{for all } x \in \mathbb{R}.$$

[Hint: Use the Cauchy inequalities.]

Exercise 35. Let Ω be a bounded open subset of $\mathbb{C}$, and $\varphi : \Omega \to \Omega$ a holomorphic function. Prove that if there exists a point $z_0 \in \Omega$ such that

$$\varphi(z_0) = z_0 \quad \text{and} \quad \varphi'(z_0) = 1$$

then φ is linear.

[Hint: Why can one assume that $z_0 = 0$? Write $\varphi(z) = z + a_n z_n + O(z^{n+1})$ near 0, and prove that if $\varphi_k = \varphi \cdot \ldots \cdot \varphi$ (where φ appears k times), then $\varphi_k(z) = z + k a_n z_n + O(z^{n+1})$. Apply the Cauchy inequalities and let $k \to \infty$ to conclude the proof. Here we use the standard O notation, where $f(z) = O(g(z))$ as $z \to 0$ means that $|f(z)| \leq C\,|g(z)|$ for some constant C as $|z| \to 0$.]

Exercise 36. Weierstrass's theorem states that a continuous function on $[0, 1]$ can be uniformly approximated by polynomials. Can every continuous function on the closed unit disc be approximated uniformly by polynomials in the variable z?

Exercise 37. Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathrm{R}e^{i\varphi})\,\mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right)d\varphi\,.$$

(b) Show that

$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}.$$

[Hint: For the first part, note that if $w = R^2 2/\overline{z}$, then the integral of $f(\zeta)/(\zeta - w)$ around the circle of radius R centered at the origin is zero. Use this, together with the usual Cauchy integral formula, to deduce the desired identity.]

Exercise 38. Let u be a real-valued function defined on the unit disc $\mathbb{D}$. Suppose that u is twice continuously differentiable and harmonic, that is,

$$\triangle u(x, y) = 0$$

for all $(x, y) \in \mathbb{D}$.

(a) Prove that there exists a holomorphic function f on the unit disc such that

$$\operatorname{Re}(f) = u.$$

Also show that the imaginary part of f is uniquely defined up to an additive (real) constant. [Hint: From the previous chapter we would have $f'(z) = 2\,\partial u\,/\,\partial z$. Therefore, let $g(z) = 2\,\partial u\,/\,\partial z$ and prove that g is holomorphic. Why can one find F with $F' = g$? Prove that $\operatorname{Re}(F)$ differs from u by a real constant.]

(b) Deduce from this result, and from Exercise 11, the Poisson integral representation formula from the Cauchy integral formula: If u is harmonic in the unit disc and continuous on its closure, then if $z = re^{i\theta}$ one has

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)u(\varphi)\,d\varphi$$

where $P_r(\gamma)$ is the Poisson kernel for the unit disc given by

$$P_r(\gamma) = \frac{1 - r^2}{1 - 2r\cos\gamma + r^2}.$$

Exercise 39. Suppose f is an analytic function defined everywhere in $\mathbb{C}$ and such that for each $z_0 \in \mathbb{C}$ at least one coefficient in the expansion

$$f(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$$

is equal to 0. Prove that f is a polynomial.

[Hint: Use the fact that $c_n n! = f^{(n)}(z_0)$ and use a countability argument.]

Exercise 40. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Show that if

$$\sum_{n=0}^{\infty} a_n z^n$$

denotes the power series expansion of f in the open unit disc, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0.$$

Exercise 41. Suppose f is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if $|f(z)| = 1$ whenever $|z| = 1$, then f is constant.

[Hint: Extend f to all of $\mathbb{C}$ by $f(z) = 1/\overline{f(1/\overline{z})}$ whenever $|z| > 1$, and argue as in the Schwarz reflection principle.]

Exercise 42. Here are some examples of analytic functions on the unit disc that cannot be extended analytically past the unit circle. The following definition is needed. Let f be a function defined in the unit disc $\mathbb{D}$, with boundary circle $\mathbb{C}$. A point w on $\mathbb{C}$ is said to be regular for f if there is an open neighborhood U of w and an analytic function g on U, so that $f = g$ on $\mathbb{D} \cap U$. A function f defined on $\mathbb{D}$ cannot be continued analytically past the unit circle if no point of $\mathbb{C}$ is regular for f.

(a) Let

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \quad \text{for } |z| < 1.$$

Notice that the radius of convergence of the above series is 1. Show that f cannot be continued analytically past the unit disc. [Hint: Suppose $\theta = 2\pi p/2^k$, where p and k are positive integers. Let $z = re^{i\theta}$; then $\left|f(re^{i\theta}\right| \to \infty$ as $r \to 1$.]

(b) * Fix $0 < \alpha < \infty$. Show that the analytic function f defined by

$$f(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n} \quad \text{for } |z| < 1$$

extends continuously to the unit circle, but cannot be analytically continued past the unit circle. [Hint: There is a nowhere differentiable function lurking in the background. See Chapter 4 in Book I.]

Exercise 43. * Let

$$F(z) = \sum_{n=1}^{\infty} d(n) z^n \quad \text{for } |z| < 1$$

where $d(n)$ denotes the number of divisors of n. Observe that the radius of convergence of this series is 1. Verify the identity

$$\sum_{n=1}^{\infty} d(n) z^n = \sum_{n=1}^{\infty} \frac{z^n}{1 - z^n}.$$

Using this identity, show that if $z = r$ with $0 < r < 1$, then

$$|F(r)| \geq c \frac{1}{1-r} \log\left(1/(1-r)\right)$$

as $r \to 1$. Similarly, if $\theta = 2\pi p/q$ where p and q are positive integers and $z = re^{i\theta}$, then

$$\left|F(re^{i\theta})\right| \geq c_{p/q} \frac{1}{1-r} \log(1/(1-r))$$

as $r \to 1$. Conclude that F cannot be continued analytically past the unit disc.

Exercise 44. Morera's theorem states that if f is continuous in $\mathbb{C}$, and $\int_T f(z)\,dz = 0$ for all triangles T, then f is holomorphic in $\mathbb{C}$. Naturally, we may ask if the conclusion still holds if we replace triangles by other sets.

(a) Suppose that f is continuous on $\mathbb{C}$, and

$$\int_C f(z)\,dz = 0 \tag{16}$$

for every circle $\mathbb{C}$. Prove that f is holomorphic.

(b) More generally, let Γ be any toy contour, and $\mathcal{F}$ the collection of all translates and dilates of Γ. Show that if f is continuous on $\mathbb{C}$, and

$$\int_\gamma f(z)\,dz = 0 \quad \text{for all } \gamma \in \mathcal{F}$$

then f is holomorphic. In particular, Morera's theorem holds under the weaker assumption that $\int_T f(z)\,dz = 0$ for all equilateral triangles.

[Hint: As a first step, assume that f is twice real differentiable, and write $f(z) = f(z_0) + a(z - z_0) + b\overline{(z - z_0)} + O(|z - z_0|^2)$ for z near z_0. Integrating this expansion over small circles around z_0 yields $\partial f \,/\, \partial \overline{z} = b = 0$ at z_0. Alternatively, suppose only that f is differentiable and apply Green's theorem to conclude that the real and imaginary parts of f satisfy the Cauchy-Riemann equations.

In general, let $\varphi(w) = \varphi(x, y)$ (when $w = x + iy$) denote a smooth function with $0 \leq \varphi(w) \leq 1$, and $\int_{\mathbb{R}^2} \varphi(w)\,dV(w) = 1$, where $dV(w) = dx\,dy$, and $\int$ denotes the usual integral of a function of two variables in $\mathbb{R}^2$. For each $\epsilon > 0$, let $\varphi_\epsilon(z) = \epsilon^{-2}\varphi(\epsilon^{-1}z)$, as well as

$$f_\epsilon(z) = \int_{\mathbb{R}^2} f(z - w)\varphi_\epsilon(w)\,dV(w),$$

where the integral denotes the usual integral of functions of two variables, with $dV(w)$ the area element of $\mathbb{R}^2$. Then f_ϵ is smooth, satisfies condition (16), and $f_\epsilon \to f$ uniformly on any compact subset of $\mathbb{C}$.]

Exercise 45. Prove the converse to Runge's theorem: if K is a compact set whose complement if not connected, then there exists a function f holomorphic in a neighborhood of K which cannot be approximated uniformly by polynomial on K.

[Hint: Pick a point z_0 in a bounded component of K^c, and let $f(z) = 1/(z - z_0)$. If f can be approximated uniformly by polynomials on K, show that there exists a polynomial p such that $|(z - z_0)p(z) - 1| < 1$. Use the maximum modulus principle (Chapter 3) to show that this inequality continues to hold for all z in the component of K^c that contains z_0.]

Exercise 46. * There exists an entire function F with the following "universal" property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^\infty$ of positive integers, so that

$$\lim_{n\to\infty} F(z + N_k) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$.

(a) Let $p_1, p_2, \cdots$ denote an enumeration of the collection of polynomials whose coefficients have rational real and imaginary parts. Show that it suffices to find an entire function F and an increasing sequence $\{M_n\}$ of positive integers, such that

$$|F(z) - p_n(z - M_n)| < \frac{1}{n} \quad \text{whenever } z \in D_n, \tag{17}$$

where D_n denotes the disc centered at M_n and of radius n. [Hint: Given h entire, there exists a sequence $\{n_k\}$ such that $\lim_{k\to\infty} p_{n_k}(z) = h(z)$ uniformly on every compact subset of $\mathbb{C}$.]

(b) Construct F satisfying (17) as an infinite series

$$F(z) \sum_{n=1}^{\infty} u_n(z)$$

where $u_n(z) = p_n(z - M_n)e^{-c_n(z-M_n)^2}$, and the quantities $c_n > 0$ and $M_n > 0$ are chosen appropriately with $c_n \to 0$ and $M_n \to \infty$. [Hint: The function e^{-z^2} vanishes rapidly as $|z| \to \infty$ in the sectors $\{|\arg z| < \pi/4 - \delta\}$ and $\{|\pi - \arg z| < \pi/4 - \delta\}$.]

In the same spirit, there exists an alternate "universal" entire function G with the following property: given any entire function h, there is an increasing sequence $\{N_k\}_{k=1}^\infty$ of positive integers, so that

$$\lim_{k\to\infty} D^{N_k} G(z) = h(z)$$

uniformly on every compact subset of $\mathbb{C}$. Here $D^j G$ denotes the j^{th} (complex) derivative of G.