# Modular Forms in Type IIB Superstring Theory

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## DECLARATION

This piece of work is a result of my own work and I have complied with the Department's guidance on multiple submission and on the use of AI tools. Material from the work of others not involved in the project has been acknowledged, quotations and paraphrases suitably indicated, and all uses of AI tools have been declared.

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# Chapter 1

# Introduction

The enchanting world of mathematical physics presents a landscape where profound ideas from number theory and quantum physics not only coexist but also collaborate to unveil the deep-seated symmetries of our universe. This thesis explores one such intersection—modular forms in type IIB string theory—venturing into a realm where mathematical elegance meets the raw complexity of theoretical physics. The following work aims to elucidate the intricate dance between the modular symmetry inherent in certain mathematical functions and their role in describing the symmetries of physical theories, specifically in the context of string theory.

The core of this investigation centers around modular forms, mathematical structures characterized by their invariance under the action of the modular group $SL(2, \mathbb{Z})$, and their applications in the theoretical framework of string theory. Modular forms serve as more than just aesthetically pleasing mathematical constructs; they encode significant physical properties such as symmetry and duality in string theory, influencing the formulation of scattering amplitudes and the low-energy behavior of string interactions.

This report is structured to provide a comprehensive analysis starting with a deep dive into the theory of modular forms, exploring their genesis from simple lattices to their profound implications in higher-dimensional moduli spaces. Chapter 2 sets the stage by introducing the modular group and its action on the complex upper half-plane, followed by an exploration of both holomorphic and non-holomorphic modular forms, and culminating in the detailed study of Eisenstein series and their Fourier expansions.

Transitioning from pure mathematics to the physics of strings, Chapter 3 delves into string theory, particularly focusing on Type IIB superstrings. It discusses how the modularity in mathematical functions is mirrored in physical phenomena through the symmetries of the string theory, particularly examining the role of modular forms in string perturbation theory and the resulting physical implications, such as the computation of scattering amplitudes and the low-energy expansion of string theory amplitudes.

Chapter 4 synthesizes these ideas by studying modular constraints in the low-energy expansion of string theory amplitudes. It rigorously addresses how modular forms constrain the coefficients in the low-energy expansion and the profound implications of these constraints for supersymmetry and quantum gravity, particularly in the "Non-renormalisation theorems" we derive.

Building on this foundation, Chapter 5 explores modular graph functions, extending the discussion to include how these functions contribute to our understanding of multi-loop string amplitudes, thereby showcasing the practical utility of modular forms in higher-order corrections in string theory.

Our main source for this report are the D'hoker Kaidi notes [1]. We also heavily cite Eisenstein Series and Automorphic Representations: with Applications in String Theory [2] Fleig, Gustafsson, Kleinschmidt, and Persson. We also heavily use the SageX review [3] by Dorigoni, Green, and Wen. We note when a certain section follows parts of these books closely.

In conclusion, this thesis not only demonstrates the rich interplay between modular forms and string theory but also underscores the broader implications of this synergy, such as insights into quantum gravity and the unification of forces. Through a meticulous blend of mathematical rigor and physical insight, this work aims to contribute significantly to our understanding of the universe's fundamental fabric, guided by the symmetries and dualities that shape it.

# Chapter 2

# Modular Forms

In this chapter we introduce Modular Forms, embarking on a journey through a fascinating landscape where the realms of continuous and discrete mathematics converge. Modular forms, which emerge from the intricate dance of symmetry and structure in the complex plane, serve as a bridge between seemingly disparate mathematical worlds. Our primary reference are the lecture notes by D'hoker and Kaidi [1] and we mostly follow their presentation. We shall colloquially refer to these as "the D'hoker Kaidi notes" when needed. We shall cite other resources when used. Some secondary sources used extensively are [2] [4] [3]. We start with the modular group $SL(2, \mathbb{Z})$ and its action on the complex upper half-plane $\mathbb{H}$ (called the Poincare upper half plane), introducing Mobius transformations that lay the groundwork for defining modular functions and forms. We motivate the elucidating properties of modular forms by deriving them through two lenses. The first is as an invariant form and the second is as an inhomogeneous function of a lattice. We then give a concrete example that will be of paramount importance to us throughout this report: the Eisenstein series, deriving it through both lenses of modular forms. We then investigate the Modular group further and define its generators and fundamental domain, which elucidates the modular groups beautiful and interesting properties. We then generalise modular forms to non-holomorphic modular forms and through a modular form called the Poincare series we define the non-holomorphic modular form and its variations. Finally we define and derive the Fourier expansion of the non-holomorphic Eisenstein series before finally defining some extensions of modular forms: the generalised non-holomorphic Eisenstein series and Automorphic forms.

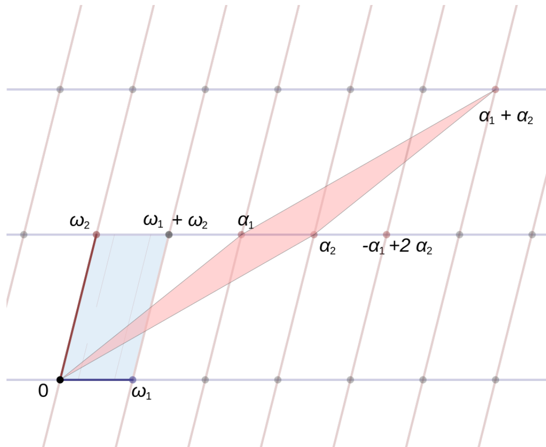


Figure 2.1: $\omega_1$ and $\omega_2$ are the periods. $\alpha_1$ and $\alpha_2$ are an equivalent pair of periods. [5]

## 2.1 The Lattice

The modular group $SL(2,\mathbb{Z})$ is the group of all two by two matrices with integer coefficients such that their determinant is one:

$$\gamma \in SL(2,\mathbb{Z}) \implies \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \tag{2.1}$$

It's easy to check that an action of this group on the upper half plane $\mathbb{H}$ (where $\mathrm{Im}(\tau) > 0$) is given by the mobius transformation [4]

$$\gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad \gamma \in SL(2,\mathbb{Z}) \tag{2.2}$$

A function $f : \mathbb{H} \to \mathbb{C}$ is called a modular function if it is invariant under the action of $SL(2,\mathbb{Z})$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.3}$$

A modular function must also be meromorphic everywhere including at $i\infty$.

Consider a lattice in the complex plane (Figure 2.1) such that it is spanned by two complex numbers $\omega_1$ and $\omega_2$ called the periods. We consider functions of the periods $\omega_1$ and $\omega_2$ [1]. Consider such a function $g(\omega_1, \omega_2)$. It is obviously invariant under re-scaling:

$$g(\omega_1, \omega_2) = g(\lambda\omega_1, \lambda\omega_2) \tag{2.4}$$

[1] An elliptic function is a function of $z$ that is periodic with respect to its lattice, defined as $h(z) = h(z + \omega_1) = h(z + \omega_2)$.

We can change $\omega_1 \to c\omega_2 + d\omega_1$ and $\omega_2 \to a\omega_2 + b\omega_1$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(z)$, (which is $SL(2, \mathbb{Z})$ but with $\det = \pm 1$) to get

$$g(c\omega_2 + d\omega_1, a\omega_2 + b\omega_1) = g(\omega_1, \omega_2) \tag{2.5}$$

As this represents the invariant transformation under $\gamma \in SL(2, \mathbb{Z})$:

$$\begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \gamma \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} = \begin{pmatrix} a\omega_2 + b\omega_1 \\ c\omega_2 + d\omega_1 \end{pmatrix} \tag{2.6}$$

Notice we can reduce to one variable by defining $\omega_1 = 1$ and $\omega_2 = \tau$:

$$f(\tau) = g(1, \tau) \qquad g(\omega_1, \omega_2) = f\left(\frac{\omega_2}{\omega_1}\right) \tag{2.7}$$

Where $\frac{\omega_2}{\omega_1} = \tau$. The previous functional relations clearly imply

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \tag{2.8}$$

We have assumed $\tau$ can be any complex number with non-zero imaginary part. It's convenient to assume $\mathrm{Im}(\tau) > 0$ as we can always get negative values of $\tau$ by changing $\tau \to -\tau$ since it doesn't change the lattice. So we only consider functions on the upper half plane $\mathbb{H}$. This means

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = +1 \tag{2.9}$$

Since if the determinant is negative one the matrix

$$\det \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.10}$$

changes $\tau \to -\tau$. We therefore only consider $SL(2, \mathbb{Z})$. Lets now find some of these functions that are holomorphic. This is actually a difficult problem so an easier one is to find some modular forms.

## 2.2 Modular Forms

A Modular form is a complex function $f(\tau)$ of weight $w$ that is holomorphic in $\mathbb{H}$ including at $i\infty$ satisfying the functional relation:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w f(\tau) \qquad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{2.11}$$

This is quite an abstract definition so we shall motivate two different approaches for modular forms forms. Namely that they correspond to:

1. Invariant forms $f(\tau)(d\tau)^{\frac{w}{2}}$
2. Homogeneous functions of lattices with periods $< \omega_1, \omega_2 >$

The functional equation will be derived from these two approaches.

**Invariant forms**

Modular functions are difficult to find. Instead of

$$f(\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.12}$$

Lets consider an invariant one-form (where $d\tau$ is the differential of $\tau$ and $f(\tau)$ satisfies the holomorphicity conditions for modular forms):

$$f(\tau)d\tau = f\left(\frac{a\tau + b}{c\tau + d}\right) d\left(\frac{a\tau + b}{c\tau + d}\right) = f\left(\frac{a\tau + b}{c\tau + d}\right) \frac{d\tau}{(c\tau + d)^2} \tag{2.13}$$

This gives us the functional equation:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 f(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.14}$$

Which is a modular form of weight 2 (referring to the exponent of $(c\tau + d)$. To get modular forms of different weights we consider $f(\tau)(d\tau)^{\frac{w}{2}}$ invariant [2] under $SL(2,\mathbb{Z})$. This defines a modular form of weight w as a holomorphic function acting on the upper half plane $\mathbb{H}$ which obeys the following transformation property:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^w f(\tau) \qquad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \tag{2.15}$$

which we call a Modular form of weight $w$. [3]

**Homogeneous functions of lattices**

On the lattice $\Lambda$ from the previous chapter we now define $g$ as a homogeneous function of the lattice meaning

$$g(\lambda\omega_1, \lambda\omega_2) = \lambda^{-w} g(\omega_1, \omega_2) \tag{2.17}$$

Instead of $g$ being invariant under re-scaling it is multiplied by the constant $\lambda^{-w}$. We still have from before

$$g(c\omega_2 + d\omega_1, a\omega_2 + b\omega_1) = g(\omega_1, \omega_2) \tag{2.18}$$

and can re-scale to

$$f(\tau) = g(1, \tau) \tag{2.19}$$

[2] $(d\tau)^{\frac{w}{2}}$ is just a formal symbol for the transformation

[3] Note we can now find modular functions by dividing two modular forms of equal weight: $f_1(\tau)(d\tau)^{\frac{w}{2}}$ and $f_2(\tau)(d\tau)^{\frac{w}{2}}$ i.e

$$\frac{f_1(\tau)(d\tau)^{\frac{w}{2}}}{f_2(\tau)(d\tau)^{\frac{w}{2}}} = \frac{f_1(\tau)}{f_2(\tau)} \tag{2.16}$$

This transforms the functional equation into:

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{w}f(\tau)\qquad\text{for all }\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})\tag{2.20}$$

What can we learn from the fact modular forms can be defined from these two different perspectives? The two definitions of modular forms—one emphasizing their role as invariant forms $f(\tau)(d\tau)^{\frac{w}{2}}$ in differential geometric and analytic way and the other highlighting their algebraic and geometric nature as homogeneous functions of lattices—offer a comprehensive view of their mathematical essence. This synthesis showcases the unified framework within which modular forms operate, bridging complex analysis, geometry, and algebra in a cohesive narrative.

### 2.2.1 Example: Eisenstein Series

Now let's show a concrete example of a modular form derived from the above two definitions. There are different definitions of Eisenstein series in the literature.[4] Here we aim to show how Eisenstein series are central to the definitions of Modular forms whilst defining them in a language familiar to the string theory literature.

#### Eisenstein Series as Invariant Forms

Consider the invariant form definition of modular forms $f(\tau)(d\tau)^{\frac{w}{2}}$ and lets construct $(d\tau)^{\frac{w}{2}}$ such that is invariant under $SL(2,\mathbb{Z})$.

$$\sum_{\gamma}d\left(\frac{a\tau+b}{c\tau+d}\right)^{\frac{w}{2}}\qquad\text{where }\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})\tag{2.21}$$

However this sum is not convergent. This is because as we have a subgroup of all elements invariant under $\alpha=\begin{pmatrix}1&n\\0&1\end{pmatrix}$ which take $\tau\to\tau+n$ we get $d(\frac{\tau+n}{0+1})$ $=d\tau$. This means every term in the sum occurs an infinite number of times since they are all invariant under $\alpha$. So if we instead define

$$\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})\mod\begin{pmatrix}1&n\\0&1\end{pmatrix}\tag{2.22}$$

We only count each term once giving the Eisenstein Series:

$$E_w(\tau)=\sum_{(c,d)}\frac{d\tau}{(c\tau+d)^{w}}\tag{2.23}$$

[4]Different authors omit or include a factor of $\frac{1}{2}$, or sum over either integers that are co-prime or all pairs of integers that are not zero, leading to a factor of $\zeta(2s)$ being present.

where $(c,d)$ are coprime [5]. Note $w \geq 4$ and even so $E_w(\tau)$ is absolutely convergent. We can check $E_w$ is a modular form as it satisfies the transformation property:

$$E_w(\gamma\tau) = (c\tau + d)^w E_w(\tau). \tag{2.24}$$

which we leave as an exercise for the reader.

### Eisenstein Series as homogeneous functions of a lattice

Consider the lattice $\Lambda$ over $\mathbb{C}$ shown in Figure 2.1 defined by the periods $\omega_1, \omega_2$. We want to find a homogeneous function of this lattice. A natural candidate is the sum over the periods defined by

$$G_w(\omega_1, \omega_2) = G_w(\Lambda) = \sum_{\omega \in \Lambda'} \frac{1}{\omega^w} = \sum_{[(m,n)\neq(0,0)]\in\mathbb{Z}} \frac{1}{(m\omega_1 + n\omega_2)^w} \tag{2.25}$$

Since $G_w$ depends on the lattice itself and not the periods representation of the lattice it is invariant under $SL(2,\mathbb{Z})$ giving

$$G_w(\omega_1', \omega_2') = G_w(\omega_1, \omega_2) \tag{2.26}$$

Where $\omega_1$' and $\omega_2$' are transformations of $\omega_1$ and $\omega_2$ under $SL(2,\mathbb{Z})$. As $G_w$ is clearly a homogeneous function of the lattice so we have

$$G_w(\lambda\omega_1, \lambda\omega_2) = \lambda^{-w} G_w(\omega_1, \omega_2) \tag{2.27}$$

Again by re-scaling $\tau = \frac{\omega_2}{\omega_1}$ and defining $\omega_1 = 1$ we get

$$G_w(\tau) = \omega_1^w G_w(\omega_1, \omega_2) = \sum_{[(m,n)\neq(0,0)]\in\mathbb{Z}} \frac{1}{(m + n\tau)^w} \tag{2.28}$$

As before the two expressions above imply the modular functional relation

$$G_w(\gamma\tau) = \left(\frac{\omega_1'}{\omega_1}\right)^w G_w(\tau) = (c\tau + d)^w G_w(\tau) \tag{2.29}$$

### Normalised Eisenstein Series

We now have two similar but different summations that we refer to as Eisenstein Series. If we consider $(m,n) = (hc, hd)$ with $(c,d)$ co-prime and $h \geq 1$ we get a relation between the two.

$$G_w(\tau) = \zeta(w) E_w(\tau) \tag{2.30}$$

where $\zeta(w) = \sum_{h\geq 1} \frac{1}{h^w}$ is the Riemann Zeta function. In other words we have factored out the greatest common divisor of $m$ and $n$ from $G_w$.

[5]This is because the elements of $\gamma$ are equivalent to choosing a pair (c, d) with them being co prime due to the mod. If c and d are co prime we can extend it to a matrix $\in SL(2,\mathbb{Z})$ which is unique up to multiplication of an element of $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

## 2.3 The Fundamental Domain

The modular group $SL(2,\mathbb{Z})$ is well known to be generated by the following matrices:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{2.31}$$

What this means is that with combinations of these two matrices and their inverses we can get every element of the Modular Group $SL(2,\mathbb{Z})$. For $S$ we have $(a,b,c,d) = (0,-1,1,0)$ so that for $S$ under the action of $SL(2,\mathbb{Z})$ on $H$

$$S\tau = S\left(\frac{a\tau + b}{c\tau + d}\right) = T\left(\frac{0 * \tau - 1}{1 * \tau + 0}\right) = \frac{-1}{\tau} \tag{2.32}$$

For $T$ we have $(a,b,c,d) = (1,1,0,1)$ so for $T$ under the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$

$$T\tau = T\left(\frac{a\tau + b}{c\tau + d}\right) = T\left(\frac{1 * \tau + 1}{0 * \tau + 1}\right) = \tau + 1 \tag{2.33}$$

This implies $f(\tau) = f(\tau + 1)$ meaning $f$ is periodic under $\tau \to \tau + 1$. The group actions of $S$ and $T$ are very simple, and combinations of these matrices represent combinations of translations and inversion plus mirroring. To show $f(\tau)$ is a modular function we now only need to check its a function on $S$ and $T$, that is that

$$f(\tau + 1) = f(\tau), \qquad f(-\frac{1}{\tau}) = \tau^w f(\tau) \tag{2.34}$$

A result of these symmetries is that for a modular form we only need to know its values in the "Fundamental Domain". In Figure 2.2 we see that for the fundamental domain (for a modular form, in this case $E_6$) transformations under $T$ and $T^{-1}$ shift the function by one unit to the right and left respectively. A transformation of the fundamental domain under $S$ is an inversion (mirroring on the unit circle), which itself can then be transformed under $T$ to shift the inversion to the right and left. We can repeat the shifting and mirroring under $S$ and $T$ to define the function on entirely on the complex plane.

**Fourier Expansion of Modular Forms**

As for Modular Forms $f(\tau + 1) = f(\tau)$, (that is periodic in the real direction with period one) we can Fourier expand it:

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n q^n \qquad q = e^{2\pi i\tau} \tag{2.35}$$

Where $q$ are the modes of the function and $a_n$ are the Fourier coefficients. We can decompose the Fourier expansion into two parts: the zero Fourier mode (the constant term) and the non zero Fourier modes (non constant terms). We

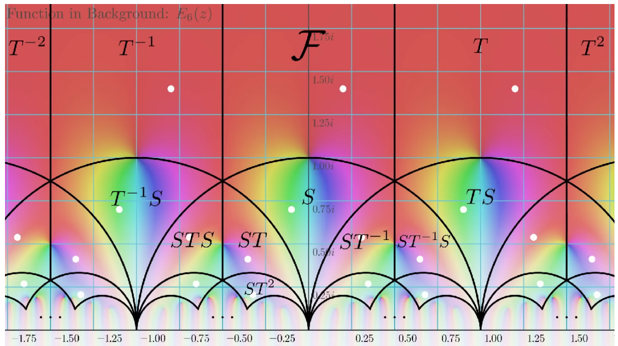


Figure 2.2: For the Modular Form $E_6$, $\mathcal{F}$ is its fundamental domain. Sections labeled the transformation $T$, $T^{-1}$, $S$, and combinations of those transformations show the fundamental domain under those transformations [6].

can write $q = e^{2\pi i\tau} = e^{2\pi i\tau_1}e^{2\pi i\tau_2}$ to compute the Fourier coefficients [6]:

$$a_n e^{-2\pi n\tau_2} = \int_0^1 e^{-2\pi i n\tau_1} f(\tau) d\tau_2 \tag{2.36}$$

The nature of $f(\tau)$ give us information about the Fourier coefficients. Recall if $f$ is a modular function then it is meromorphic everywhere including at $i\infty$. This means there are only a finite number of non-zero Fourier coefficients $a_n$ for $n < 0$ at $i\infty$.

If $f$ is a modular form, it is then holomorphic everywhere including at $i\infty$ meaning it's Fourier coefficients are all zero for $n < 0$ implying $f$ is finite at $i\infty$. One can actually define modular functions and forms of weight $w$ in this way, showing the power of the Fourier expansion.

## 2.4 Non-holomorphic Modular Forms

So far we have only considered holomorphic modular forms. We define non-holomorphic modular forms $f^{(w,\hat{w})}(\tau)$ [7] (where $\tau = \tau_1 + i\tau_2$ is from the normalised lattice with periods $<1, \tau>$) as functions following the transformation law under $SL(2,\mathbb{Z})$ as [3]

$$f^{(w,\hat{w})}(\gamma\tau) = (c\tau + d)^w (c\bar{\tau} + d)^{\hat{w}} f^{(w,\hat{w})}(\tau) \tag{2.37}$$

where $w$ and $\hat{w}$ are the holomorphic and anti-holomorphic modular weights respectively. Our aim in this section is to derive the "Non-holomorphic Eisenstein series". To do this we define another way of constructing Modular Forms: via Poincare series.

[6]Mathematically much of the arithmetic information is contained in the Fourier coefficients of modular forms

[7]A holomorphic function can only be a function of $\tau$. A function of $\tau$ and $\bar{\tau}$ is not holomorphic since it does not obey the Cauchy Riemann equations

### Poincare Series

Starting with a meromorphic function $\sigma$ we sum over its images under the modular group to construct a function that is obviously manifestly invariant under the modular group. In practicality this often leads to an infinite sum, so we instead sum over the images under the cosets $\Gamma_\infty \backslash SL(2,\mathbb{Z})$ (where $\Gamma_\infty$ is the subgroup that has an infinite number of elements.) This gives the modular function $f(\tau)$ defined by a Poincare series.

$$f(\tau) = \sum_{\gamma \in \Gamma_\infty \ SL(2,\mathbb{Z})} \sigma(\gamma\tau) \tag{2.38}$$

where $\sigma$ is the "seed" of the Poincare series. We can repeat the analysis to obtain modular forms by using a seed that is a differential form of weight $w$.

### Non-holomorphic Eisenstein Series

We compute the action of $SL(2,\mathbb{Z})$ on the seed function $\sigma(\tau) = \tau_2^s$. First we compute the imaginary part for general $\tau$ under the action:

$$\Im\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{adi\tau_2 - bci\tau_2}{|c\tau + d|^2} = \frac{\tau_2}{|c\tau + d|^2} \tag{2.39}$$

where we used $ad - bc = 1$ in latter equality. Now lets sum the seed function over the modular group $SL(2,\mathbb{Z})$:

$$\sum_{SL(2,\mathbb{Z})} \tau_2^s \tag{2.40}$$

Unfortunately this sum is divergent because Equation 2.39 does not depend on $b$. As $T : \tau \to \tau + 1$ and as $b$ is non zero in the $T$ transformation there is an infinite sequence of shifts of the fundamental domain, all giving the same contribution so they sum to infinity. We therefore take the following sum:

$$\sum_{\Gamma_\infty \backslash SL(2,\mathbb{Z})} \tau_2^s \qquad \Gamma_\infty = \begin{pmatrix} \pm 1 & m \\ 0 & \pm 1 \end{pmatrix} \tag{2.41}$$

where $\Gamma_\infty$ is the Borel subgroup $B(\mathbb{Z})$ with $m \in \mathbb{Z}$. We define the Non-holomorphic Eisenstein series as

$$E_s(\tau) = \sum_{B \backslash SL(2,\mathbb{Z})} \sigma(\tau) = \sum_{\text{gcd(p, q)}} \frac{y^s}{|p + q\tau|^{2s}} \tag{2.42}$$

where the seed $\sigma(\tau)$ is just the $(\Im(\tau))^s$. Therefore $E_s(\tau)$ is invariant under $SL(2,\mathbb{Z})$ and we call it a "non-holomorphic modular form".

$$\begin{aligned} E_s(\tau) &= \sum_{(m,n) \neq (0,0)} \frac{\tau_2^s}{\pi^s |m + n\tau|^{2s}} = \zeta(2s) \sum_{c,d} \frac{\tau_2^s}{\pi^s |d + c\tau|^{2s}} \\ &= \frac{\zeta(2s)}{\pi^s} \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbb{Z}} (\text{Im}\gamma\tau)^s \end{aligned} \tag{2.43}$$

where $(c, d)$ are co prime.

The curved space Laplacian is defined as

$$\nabla_\tau^2 = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2) \tag{2.44}$$

Since the Laplace operator is invariant, we can compute the Laplace operator acting on the non-holomorphic Eisenstein series $E_s(\tau)$ by computing the Laplace operator on the seed function:

$$\tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)\tau_2^s = s(s-1)\tau_2 \tag{2.45}$$

So the seed function is an eigenfunction of the Laplace operator $\nabla_\tau$.

$$\nabla_\tau^2 E_s(\tau) = s(s-1)E_s(\tau) \tag{2.46}$$

## 2.5 Fourier Expansion of Eisenstein Series

We follow the derivation given in the D'hoker Kaidi notes [1]. We can Fourier expand the series in terms of its constant zero mode and non zero modes. Consider for the non-holomorphic Eisenstein Series the integral representation for the summands,

$$\Gamma(s)E_s(\tau) = \int_0^\infty \frac{dt}{t} t^s \left( \sum_{m,n\in\mathbb{Z}} exp(-\pi t|m + n\tau|^2/\tau_2) - 1 \right) \tag{2.47}$$

where the negative one removes the zero mode. We manipulate the exponential in the integrand using a technique called Poisson resummation:

$$\begin{aligned} &\sum_{m,n\in\mathbb{Z}} \exp\left(-\pi t \frac{|m+n\tau|^2}{\tau_2}\right) \\ &= \sqrt{\frac{\tau_2}{t}} \sum_{m,n\in\mathbb{Z}} \exp\left(2\pi i m n \tau_1\right) \exp\left(-\pi m^2 \frac{\tau_2}{t} - \pi n^2 t \tau_2\right). \end{aligned} \tag{2.48}$$

Splitting the sum over $m, n$ into contributions for $mn \neq 0$ and $mn = 0$, we have:

$$\Gamma(s)E_s(\tau) = A_s(\tau) + B_s(\tau) \tag{2.49}$$

where,

$$A_s(\tau) = \sqrt{\tau_2} \sum_{m\neq 0} \sum_{n\neq 0} e^{2\pi i m n \tau_1} \int_0^\infty \frac{dt}{t} t^{s-1} e^{-\pi\tau_2\left(\frac{m^2}{t} + n^2 t\right)} \tag{2.50}$$

$$B_s(\tau) = \int_0^\infty \frac{dt}{t} t^s \left( -1 + 2\sum_{n=1}^\infty \sqrt{\frac{\tau_2}{t}} e^{-\pi n^2 t \tau_2} + \sum_{m\in\mathbb{Z}} \sqrt{\frac{\tau_2}{t}} e^{-\pi m^2 \frac{\tau_2}{t}} \right) \tag{2.51}$$

The integral $A_s(\tau)$ is manifestly holomorphic in $s$ throughout $\mathbb{C}$, and may be computed by performing the $t$-integral in terms of a $K$-Bessel function,

$$A_s(\tau) = 2\sqrt{\tau_2} \sum_{m,n\neq 0} \left|\frac{m}{n}\right|^{s-\frac{1}{2}} e^{2\pi i m n \tau_1} K_{s-\frac{1}{2}}(2\pi\tau_2|mn|) \tag{2.52}$$

The integral $B_s(\tau)$ may be computed by Poisson re-summing the $m$-sum in the integrand,

$$B_s(\tau) = \sum_{n\neq 0} \int_0^\infty dt\, t^{s-1} \left(\sqrt{\frac{\tau_2}{t}} e^{-\pi n^2 t \tau_2} + e^{-\pi n^2 \frac{t}{\tau_2}}\right) \tag{2.53}$$

which is readily evaluated in terms of the Riemann $\zeta$-function or associated $\xi$-function,

$$B_s(\tau) = 2\Gamma(s)\pi^{-s}\tau_2^s\zeta(2s) + 2\Gamma\left(s-\frac{1}{2}\right)\pi^{\frac{1}{2}-s}\tau_2^{1-s}\zeta(2s-1) \tag{2.54}$$

$$= \tau_2^s\xi(2s) + \tau_2^{1-s}\xi(2s-1) \tag{2.55}$$

Assembling both contributions, we obtain the Fourier series decomposition for $E_s(\tau)$,

$$\begin{aligned}\Gamma(s)E_s(\tau) &= 2\Gamma(s)\pi^{-s}\zeta(2s)\tau_2^s + 2\Gamma\left(s-\frac{1}{2}\right)\pi^{\frac{1}{2}-s}\zeta(2s-1)\tau_2^{1-s} \\ &\quad +4\sqrt{\tau_2}\sum_{N\neq 0}|N|^{\frac{1}{2}-s}\sigma_{2s-1}(|N|)e^{2\pi i N\tau_1}K_{s-\frac{1}{2}}(2\pi\tau_2|N|)\end{aligned} \tag{2.56}$$

where $\sigma_\alpha(n)$ is defined as the sum of positive divisors $d$ of $n$ raised to the power $\alpha$.

In other words for the non-holomorphic Eisenstein series the Fourier expansion is [3]

$$E_s(\tau) = \sum_{k\in\mathbb{Z}} F_k(s;\tau_2)e^{2\pi i k\tau_1} \tag{2.57}$$

where the zero mode is given by

$$F_0(s;\tau_2) = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\sqrt{\pi}\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\pi^s\Gamma(s)}\tau_2^{1-s} \tag{2.58}$$

and the non-zero modes are given by

$$F_k(s;\tau_2) = \frac{4}{\Gamma(s)}|k|^{s-\frac{1}{2}}\sigma_{1-2s}(|k|)\sqrt{\tau_2}K_{s-\frac{1}{2}}(2\pi|k|\tau_2), \quad k\neq 0, \tag{2.59}$$

## 2.6 Generalised Non-Holomorphic Eisenstein Series

In a previous section we showed the non-holomorphic Eisenstein series is the solution to a certain eigenvalue equation. This along with $SL(2,\mathbb{Z})$ invariance actually defines the Eisenstein series.

We ask if we can define other such functions this way: as solutions to different classes of eigenvalue equations. As we shall see later in string theory other such functions do in fact arise. We define the generalised non-holomorphic Eisenstein series modular function as $\varepsilon(s_1,s_2;r;\tau,\bar{\tau})$[3] such that the inhomogeneous Laplace eigenvalue equation is satisfied:

$$[\Delta_\tau - r(r+1)]\varepsilon(s_1,s_2;r;\tau,\bar{\tau}) = -E(s_1;\tau,\bar{\tau})E(s_2;\tau,\bar{\tau}) \tag{2.60}$$

An example that will be relevant later is $s_1=\frac{3}{2}, s_2=\frac{3}{2}, r=3$ where it satisfies the inhomogeneous Laplace equation:

$$(\Delta_\tau - 12)\varepsilon_6(\tau) = -(2\zeta(3)E_{\frac{3}{2}}(\tau))^2 \tag{2.61}$$

The solution can be expressed as a Poincare series:

$$\varepsilon_6(\tau) = \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbb{Z})} \Phi(\gamma\tau) \tag{2.62}$$

where

$$\Phi(\tau) = a_0(\tau_2) + \sum_{n\neq 0} a_n(\tau_2)e^{2\pi i n\tau_1} \tag{2.63}$$

Where $a_n(\tau_2)$ is linear in $K_0$ and $K_1$ Bessel functions. We can calculate its Fourier modes as before

$$\varepsilon_6(\tau) = \sum_k \hat{f}_k(\tau_2)e^{2\pi i k\tau_1} \tag{2.64}$$

Its zero Fourier mode is given by a sum of rational multiple zeta values.

$$\hat{f}_0(\tau_2) \tag{2.65}$$
$$= \frac{2}{3}\zeta(3)^2\tau_2^3 + \frac{4}{3}\zeta(2)\zeta(3)\tau_2 + \frac{8}{5}\zeta(2)^2\tau_2^{-1} + \frac{4}{27}\zeta(6)\tau_2^{-3} + O(e^{-4\pi\tau_2}) \tag{2.66}$$

## 2.7 Automorphic forms

In our final section we define a generalisation of Modular forms called Automorphic forms. They obey the following conditions:

1. Invariant under the action of a discrete subgroup $\Gamma \subset G : f(\gamma\cdot g) = f(g)$ for all $\gamma\in\Gamma$ where $G$ is a real lie group (in our case $\Gamma$ is $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$).

2. Satisfy eigenvalue differential equations under the action of the ring of $G$-invariant differential operators.

3. Have well-behaved growth conditions.

This generalisation lends itself to our further study of string theory, where the second and third conditions come from supersymmetric and physical constraints. A keen eyed reader may notice the second condition lends itself to our study of the non-holomorphic Eisenstein series.

# Chapter 3

# String Theory

String theory posits that all fundamental particles are one-dimensional strings that can be either open or closed (as shown in Figure 3.1). In this chapter we motivate a supersymmetric version of string theory called "Type IIB Superstring theory". Closed superstrings in Type IIB at low energy should reproduce Einstein gravity. We aim to find the string theoretic corrections to Einstein gravity by finding the low energy expansion of Type IIB superstring theory. We first define the classical action of string theory as a functional of the area of its world sheet, then giving its quantised and supersymmetric form. We also motivate Type IIB superstring theory and its low energy effective action: Type IIB supergravity. From this we show its action is $SL(2,\mathbb{R})$ invariant and upon quantisation how this symmetry group is restricted from the continuous $SL(2,\mathbb{R})$ group to the discrete subgroup $SL(2,\mathbb{Z})$ we call S-Duality. We then review the general theory of scattering amplitudes and from superstring perturbation theory, give the closed form expression of the tree-level, one-loop, and two-loop contributions to the amplitude for four graviton scattering. This then allows us to calculate the low energy or $\alpha'$ expansion in terms of higher derivative corrections, and then compare the explicit forms of the corrections in terms of multiple zeta values to string perturbation theory.

We aim to introduce the necessary string theoretic concepts where modularity appears. Although we endeavour to be self contained where possible and motivate the mathematical and physical details, we must necessarily be light on details. For more in-depth explanations, any good introductory text on string theory will do. We recommend "Basic Concepts in String Theory" by Blumenhagen [7] and Polchinski's two volumes "String Theory" [8]. The chapter mostly follows the exposition of Chapter 12 and 14 from the "D'Hoker Kaidi notes" [1] with other results necessarily cited.

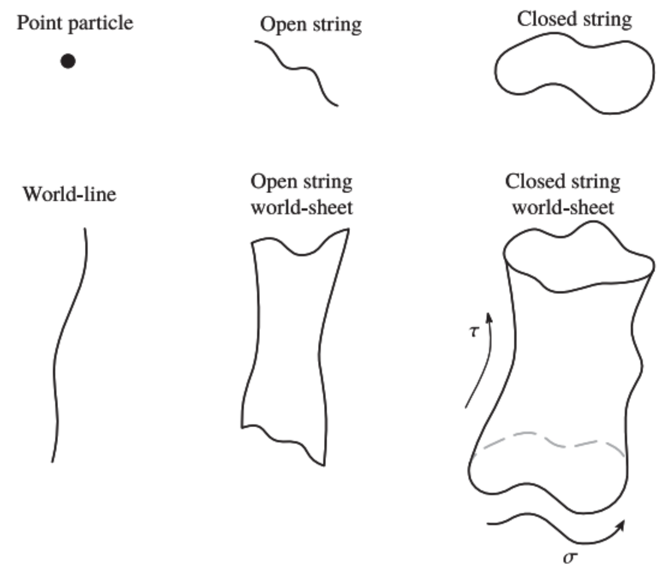


Figure 3.1: A point particle, an open string, and a closed string with their world-sheets, parameterised by $\sigma$ and $\tau$ [1]

## 3.1 Basic Concepts

In space-time a one-dimensional string traces out a world sheet as seen in Figure 3.1. We want to define an action of the theory so we can motivate scattering amplitudes. We do this by first defining the world sheet to be a Lorentzian manifold $\Sigma$ embedded in space-time, itself a Lorentzian manifold $M$ of dimension $D$. String theory can then be described as the dynamics of the embedding map:

$$X : \Sigma \to M \tag{3.1}$$

on coordinate maps $X^\mu$ with $\mu = 0, \dots, D-1$. We then define the metric on $M$

$$G = G_{\mu v} dx^\mu \otimes dx^v \tag{3.2}$$

where $x^\mu$ are coordinates on $M$. The metric $G$ on $M$ induces another metric $\gamma$ on $\Sigma$

$$\gamma = X^*(G) = \gamma_{mn} d\sigma^m \otimes d\sigma^n, \quad \gamma_{mn} = G_{\mu\nu}(X) \frac{\partial X^\mu}{\partial \sigma^m} \frac{\partial X^\nu}{\partial \sigma^n}. \tag{3.3}$$

Where the action of the theory can be given by the area of the world-sheet.

$$\text{Area}(X(\Sigma)) = \int_\Sigma d^2\sigma \sqrt{|\det \gamma|} \tag{3.4}$$

The square root makes quantisation difficult so we fix a new metric $g$ on $\Sigma$ that is independent of $X$.

$$g = g_{mn}(\sigma) d\sigma^m \otimes d\sigma^n, \tag{3.5}$$

This gives us famous the Polyakov action:

$$S_P[g,G,X] = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{g}g^{mn}\partial_m X^\mu \partial_n X^\nu G_{\mu\nu}(X), \tag{3.6}$$

where $\sqrt{g} = \sqrt{\det g}$ and $\alpha' = l_s^2$ which is the square of the length of the string. The equations of motion for the area and Polyakov action can be shown to be equivalent [2].

We can add more terms to the Polyakov action representing different background fields. Two important fields are the dilaton field $\phi(X)$, and the two-form $B = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu$ on $M$ which induces a two-form on $\Sigma$ given by $X^*(B) = \frac{1}{2}B_{mn}d\sigma^m \wedge d\sigma^n$. The action contributions corresponding to these fields are given by

$$S_{\text{dilaton}} = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{g}\phi(X)R^{(2)}(g) \tag{3.7}$$

and

$$S_B = \frac{i}{2\pi\alpha'}\int_\Sigma X^*(B) \tag{3.8}$$

where $R^{(2)}$ is the Ricci curvature scalar of the world-sheet.

### 3.1.1 Supersymmetry

Upon quantisation, the spectrum of the Polyakov action is made of tachyons [7] which are particles which move faster than light and violate causality. This threatens string theory as a sensible theory of the physical world. Luckily, we can fix this by introducing supersymmetry. For the bosonic fields $X^\mu(\sigma,\tau)$ we include their "superpartners" $\Psi^\mu(\sigma,\tau)$ which are just two dimensional Majorana spinors and space-time vectors [7]. The important point is that supersymmetry allows us to impose further boundary conditions on the strings (see below) that will give rise to an interesting and elucidating mathematical structure.

Using light cone coordinates $\sigma_\pm = \tau \pm \sigma$ we can write the supersymmetric version of the Polyakov action

$$S_P = \frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma(\partial_+ X^\mu \partial_- X_\mu - i\psi_L \cdot \partial_+\psi_L - i\psi_R \cdot \partial_-\psi_R) \tag{3.9}$$

where $\rho^a$ are world-sheet Dirac matrices. $\chi^a$ is a Rarita–Schwinger field, the world-sheet gravitino (superpartner of $\gamma_{ab}$) is a space-time scalar.

We now motivate Type IIB Superstring theory. Upon deriving the equations of motion from the above action for closed strings we get an expansion $X^\mu$ and $\psi^\mu$ in terms of left and right moving oscillation modes.[1] As $\sigma$ is taken in the range $[0, 2\pi]$, we obviously have for $\psi_L$ and $\psi_R$ two consistent boundary conditions: periodic or anti periodic. The periodic boundary condition is called

[1] For open strings the left and right moving degrees are equivalent (with Neumann boundary conditions).

the "Ramond" sector and the antiperiodic is called the "Neveu-Schwarz" sector which are denoted by (R) and (NS) respectively.

We can choose the chirality for the left moving and right moving modes on the string by defining the string to have NS-NS, R-R, NS-R, or R-NS boundary conditions. Choosing the chirality to be the same (that is NS-NS and R-R boundary conditions so that both sectors contribute states with the same chirality) for both the left moving and right moving modes results in Type IIB string theory [7].[2] Type IIB has $N = (2, 0)$ supersymmetry.[3]

### 3.1.2 Type IIB Supergravity and S-Duality

The low energy effective action of Type IIB Superstring Theory is Type IIB Supergravity [9]. We focus on the bosonic particles, of which we get two types with the listed corresponding fields

1. The NS-NS bosons: the metric $G_{\mu v}$, the two-form $B_2$, and the dilaton $\phi$. The field strength is given by $H_3 = dB_2$
2. The R-R bosons: $C_0$, $C_2$, and $C_4$. The field strength is given by $F_{r+1} = dC_r$

The action for the theory (in the string frame) is given by the following three terms [8]

$$\begin{aligned} S_{NS} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G_S}\left[e^{-2\phi}(R_S + 4\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}|H_3|^2)\right], \\ S_R &= -\frac{1}{4\kappa_{10}^2}\int d^{10}x\sqrt{-G_S}\left[|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right], \\ S_{CS} &= -\frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3. \end{aligned} \tag{3.10}$$

Where $G_S$ and $R_S$ are the space-time metric and Ricci scalar in the string frame, $\kappa_{10}$ is the ten dimensional version of Newtons constant and $\tilde{F}_3$ and $\tilde{F}_5$ are given by

$$\begin{aligned} \tilde{F}_3 &= F_3 - C_0 \wedge H_3 \\ \tilde{F}_5 &= F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \end{aligned} \tag{3.11}$$

We also have the constraint

$$*\tilde{F}_5 = \tilde{F}_5 \tag{3.12}$$

Transforming $G_S$ to the Einstein frame we get

$$G_{\mu v} = (G_E)_{\mu v} = e^{-\frac{\phi}{2}}(G_S)_{\mu v} \tag{3.13}$$

[2]Choosing opposite chirality for the modes results in a non-chiral theory called Type IIA string theory.

[3]This means that it has two left-handed world-sheet supersymmetry generators or two irreducible real spin representations.

and the corresponding Ricci scalar $R$. Defining the axion-dilaton field

$$\tau = C_0 + ie^{-\phi} \tag{3.14}$$

on the Poincare upper half plane $\mathbb{H}$ and

$$G_3 = \tilde{F}_3 - ie^{-\phi} H_3 = F_3 - \tau H_3 \tag{3.15}$$

we get [7]

$$\begin{aligned} S_{IIB} =& \frac{1}{2\kappa_{10}^2} \int d^{10}x\sqrt{-G}(R - \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{2(\Im(\tau))^2} - \frac{|G_3|^2}{2\Im(\tau)} - \frac{|\tilde{F}_5|^2}{4}) \\ &+ \frac{1}{8i\kappa_{10}^2} \int \frac{1}{\Im(\tau)} C_4 \wedge G_3 \wedge \bar{G_3} \end{aligned} \tag{3.16}$$

**S-Duality**

The action $S_{IIB}$ is invariant under $SL(2, \mathbb{R})$ given by

$$\tau \to \gamma(\tau) = \frac{a\tau + b}{c\tau + d}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \tag{3.17}$$

Where the fields, the field strengths, and metrics transform as

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \quad G_{\mu v} \to G_{\mu v}, \quad \tilde{F}_5 \to \tilde{F}_5 \tag{3.18}$$

The $SL(2, \mathbb{R})$ symmetry in supergravity in 10 dimensions is broken to $SL(2, \mathbb{Z})$ upon quantisation [7]. To see this consider a fundamental string carrying a single unit charge of the $B_2$ field. Under the above transformations we get $d$ units of the $B_2$ field, which are quantised and is therefore be integer. The largest subgroup of $SL(2, \mathbb{R})$ with $d \in \mathbb{Z}$ is [7]

$$\begin{pmatrix} a & \alpha b \\ \frac{c}{\alpha} & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. We have that $\alpha \in \mathbb{R}$ can be absorbed by re-scaling $C_2$, leaving the discrete subgroup $SL(2, \mathbb{Z})$. This symmetry is called S-Duality.

So why do we care about S-Duality? Take the $SL(2, \mathbb{Z})$ generator $S$ from Chapter 2. We have the transformation rule $\tau \to -\frac{1}{\tau}$. When $\tau_1 = C_0 = 0$ we get $e^{-\phi} = \tau_2 \to \frac{1}{\tau_2} = e^{\phi}$ where $e^{\phi_0} = g_s$ which is the string coupling constant where $\phi_0 = \lim_{X\to\infty} \phi$ is the constant mode of the dilaton.

The main point to take from this is that we care about S-Duality since it connects the weak coupling limit to the strong coupling limit in Type IIB, a regime which is usually computationally out of our reach.

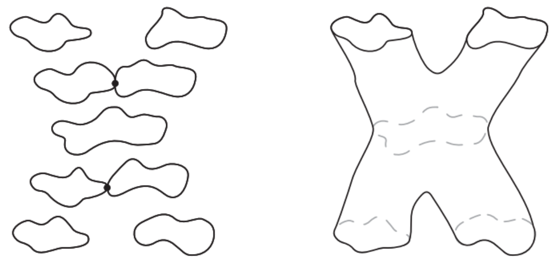

Figure 3.2: On the left we have two open strings interacting. On the right we have two closed string interacting. [2]

## 3.2 General Theory of Scattering Amplitudes

Now that we have a general theory of string theory (specifically type IIB) we want to study the dynamics of the theory, in particular its interactions.

Let's motivate a reason for modelling particles as string rather than zero point particles. In quantum field theory, we want to describe two gravitons moving through space interacting at a point, then moving away from each other. This presents difficulties in the calculation due to the interaction at a point. In string theory interactions can be described as the dynamics of the world sheets of both of the strings as shown in Figure 3.2. We call this process scattering, and we compute scattering amplitudes in which its square represents the probability of the interaction. The scattering amplitude depends on the strength of the interaction given by the string coupling constant $g_s$ and the length of the string $\alpha' = l_s^2$. Unfortunately we know of no closed form solution for the four graviton scattering amplitude in string theory so we perform an approximation via a series expansion. There are two types of expansions we shall use.

1. String perturbation theory: We sum over all the the configurations of the world sheet (in increasing genus as shown in Figure 3.3, given the initial and final state data) via a series expansion in powers of the small string coupling constant $g_s$. The probability for the process to occur is given by the absolute value square of the amplitude.

2. Low energy expansion: we consider the energies of all the scattering states to be small in units of $\alpha'$. This then leads to an expansion in derivatives of the fields as higher orders in $\alpha'$ are accompanied by higher powers of curvature and thus are higher derivatives of the metric whose fluctuations are the graviton. To leading order, as the squared masses of the massive string states are of order $\frac{1}{\alpha'}$ this expansion omits the effects of the massive string states. The massless states reproduce supergravity.

Let us now investigate these two expansions in more detail.

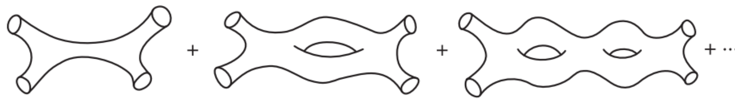


Figure 3.3: In string perturbation theory we calculate the scattering amplitude $A$ as a topological expansion in powers of $g_s$ [1]

## 3.3 String Perturbation Theory

We calculate the amplitude $A$ in string perturbation theory using the path integral formulation. We weigh each possible world sheet by $e^{-S}$ for an action $S$. Each world-sheet is a Riemann surface $(\Sigma, g)$. It is obvious that string amplitudes must be independent of the coordinates $\Sigma$ and $M$ so we require invariance under the diffeomorphism groups Diff($\Sigma$) and Diff($M$). We encode the information of the incoming and outgoing states in terms of vertex operators. We think of the data of the vertex operators as being encoded at the vertex insertion points on the world-sheet as seen in Figure 3.4.

We can separate $\phi_0$ from the dilaton action [2]

$$\frac{1}{4\pi}\int_{\Sigma} d^2\sigma\sqrt{g}\phi_0 R^{(2)} = \phi_0\chi(\Sigma) \tag{3.19}$$

where $\chi(\Sigma) = 2 - 2h$ where h is the genus of the world-sheet. As $g_s = e^{\phi_0}$ we have that each world-sheet topology is weighted by $g_s^{-\chi(\Sigma)} = g_s^{2h-2}$. We therefore get that the scattering amplitude $A$ is given by

$$A = \sum_{h=0}^{\infty} g_s^{2h-2} A^{(h)} + \text{non-perturbative effects} \tag{3.20}$$

where $A_h$ is the genus $h$ (also called $h$-loop) contribution to $A$ given by

$$A^{(h)} = \int_{\text{Maps }(\Sigma,\, M)X\ \text{Met}(\Sigma)} \frac{DXDg}{\text{Vol}(\text{Diff} \propto \text{Weyl})} V_1 \cdots V_n e^{-S[g,G,X]} \tag{3.21}$$

Where $V_i$ are the vertex operators for each asymptotic state and Vol(Diff $\propto$ Weyl) is an infinite volume of diffeomorphisms and symmetries[4] so we get a finite answer. Met($\Sigma$) is the space of all metrics on $\Sigma$ and $DXDg$ is the integration measure. The non-perturbative terms however can't be reached by superstring perturbation theory. They are of the form $\exp\left(-\frac{1}{g_s}\right)$ [2].

The derivations of certain $A^{(h)}$ is extremely involved and won't be given in this report. The important point to take from this is the general concept of summing over different genus world-sheets and the fact that there are non-perturbative effects. We shall now look at superstring perturbation theory expressions for four graviton scattering in Type IIB superstring theory.

[4]Weyl transformations are an invariant of the Polyakov action [2]

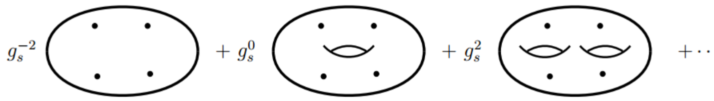

Figure 3.4: String perturbation theory as described in terms of vertex operators, the information of which is encoded at the vertex insertion points on the world-sheet[2].

### 3.3.1 String Perturbation Theory in Type IIB

Multiple different formulations of superstring scattering amplitudes exist. In this report we shall give the expressions for genus $h$ contribution to the amplitudes and give references to the detailed derivations.

We consider four graviton scattering in Type IIB superstring theory. Each graviton has momenta $k_i$ and polarisation tensors $\varepsilon_i^\mu(k)\tilde{\varepsilon}_i^\mu(k)$ where $k_i^2 = k_i \cdot \varepsilon_i = k_i \cdot \tilde{\varepsilon}_i$. Momentum conservation implies $s+t+u=0$. Closed type IIB superstring perturbation theory has the on-shell four graviton scattering amplitude [1]

$$A(\varepsilon_i; k_i) = \kappa_{10}^2 R^4 \sum_{h=0}^{\infty} g_s^{2h-2} A^h(s_{ij}) \tag{3.22}$$

where $s_{ij} = -\frac{\alpha'}{4}(k_i + k_j)$ and $R$ is the on-shell linearised Riemann tensor. $R^4$ is shared by amplitudes at all loop order because of the space-time supersymmetry of the Type II superstring. $A^{(h)}$ is a Lorentz scalar function that depends only on $s, t,$ and $u$. $k_i$ and $\varepsilon_i$ represent the momenta and the polarisation tensors of the four gravitons. We have the following explicit formulas for the tree-level, one-loop, and two-loop amplitudes that have been derived from first principles [10] [11] [12] [13].

$$A^{(0)}(s_{ij}) = \frac{1}{stu} \frac{\Gamma(1-s)\Gamma(1-t)\Gamma(1-u)}{\Gamma(1+s)\Gamma(1+t)\Gamma(1+u)} \tag{3.23}$$

$$A^{(1)}(s_{ij}) = \frac{\pi}{16} \int_{\mathcal{M}_\infty} \frac{|d\tau|^2}{\Im(\tau)^2} \mathcal{B}^{(1)}(s,t,u;\tau) \tag{3.24}$$

$$A^{(2)}(s_{ij}) = \frac{\pi}{64} \int_{\mathcal{M}_\in} \frac{|d\Omega|^2}{\det(\Im(\tau))^3} \mathcal{B}^{(2)}(s,t,u|\Omega) \tag{3.25}$$

Where $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ are Lorentz scalar function given by [1]

$$\mathcal{B}^{(1)}(s,t,u|\tau) = \int_{\Sigma^4} \prod_{i=1}^{4} \frac{d^2 z_i}{\Im(\tau)} \exp \sum_{i<j} s_{ij} G(z_i, z_j|\tau) \tag{3.26}$$

$$\mathcal{B}^{(2)}(s,t,u|\Omega) = \int_{\Sigma^4} \frac{|\dagger|^2}{(\det(\Im(\Omega)))^2} \exp \sum_{i<j} s_{ij} \mathcal{G}(z_i, z_j|\Omega) \tag{3.27}$$

where $z_i$ are the vertex insertion points on $\Sigma$, $\tau$ is the modulus used as a local complex coordinate for the genus one moduli space $\mathcal{M}_{\infty}$, $\Omega$ is the period matrix used as a set of local coordinates for the genus two moduli space $\mathcal{M}_{\in}$, $\dagger$ is a holomorphic $(1,0)$ form in each one of the vertex points $z_i$

$$\dagger = (t-u)\Delta(1,2)\Delta(3,4) + (s-t)\Delta(1,3)\Delta(4,2) + (u-s)\Delta(1,4)\Delta(2,3) \quad (3.28)$$

where $\Delta$ is a holomorphic $(1,0)$ form in its entries given by

$$\Delta(i,j) = \omega_1(z_i)\omega_2(z_j) - \omega_2(z_i)\omega_1(z_j) \quad (3.29)$$

$\omega_I(z)$ are the canonically normalised holomorphic Abelian differentials on the Riemann surface of genus 2.

## 3.4 Low-Energy Expansion of Type IIB

To compute the low-energy expansion of superstring amplitudes we use a different notation for $s_{ij}$ called the mandelstram invariants. This time we do not factor out the constant $-\frac{\alpha'}{4}$ to make evident the dependence of $\alpha'$. We do this for the reader's benefit, to facilitate greater understanding of the literature. We consider the mandelstram invariants given by

$$s = -2k_1 \cdot k_2, \quad t = -2k_1 \cdot k_4, \quad u = -2k_1 \cdot k_3 \quad (3.30)$$

We again consider the simplest amplitude in closed type IIB superstring theory. The maximal supersymmetry of type IIB allows us to factor out a value of the linearised curvature tensor with $R^4 \sim k_\mu k_v \epsilon_{\rho\sigma}$ as

$$A^{(4)}(\epsilon_r, k_r; \tau) = R^4 T^{(4)}(s,t,u;\tau) \quad (3.31)$$

Where $T$ is a scalar function of the mandelstram invariants and the moduli. The modulus, the complex coupling constant given by

$$\tau = \tau_1 + i\tau_2, \qquad \tau_2 = \frac{1}{g_s} = e^{-\phi} \quad (3.32)$$

where $g_s$ is the string coupling constant. $T$ contains the dynamics and is a symmetric function of $s,t,u$. Momentum conservation and masslessness imply $s+t+u=0$. Any symmetric polynomial in $s,t,u$ is then a polynomial in

$$\sigma_2 = s^2 + t^2 + u^2 \quad \text{and} \quad \sigma_3 = s^3 + t^3 + u^3. \quad (3.33)$$

We now wish to calculate the low energy expansion of the individual tree-level, one-loop, two-loop, and three-loop terms of the perturbative expansion. We motivate the derivations and give references to the detailed derivations in the literature. It turns out we can express them in terms of multiple zeta values.

### 3.4.1 Tree-level Amplitude

The first order term in the tree-level term (Virasoro Amplitude) is given by

$$A_0^{(4)}(\epsilon_r, k_r) = g_s^{-2} R^4 T_0^{(4)}(s,t,u) \tag{3.34}$$

Where $T_0$ is the genus 0 part of the function $T$ given by (for the superstring)

$$T_0^{(4)} = \frac{1}{stu}\frac{\Gamma(1-\alpha' s)\Gamma(1-\alpha' t)\Gamma(1-\alpha' u)}{\Gamma(1+\alpha' s)\Gamma(1+\alpha' t)\Gamma(1+\alpha' u)} \tag{3.35}$$

we can expand this to arbitrary order in $\alpha'$ using properties of the gamma function as [14]

$$\frac{3}{\sigma_3}\exp\left(\sum_{n=1}^{\infty}\frac{2\zeta(2n+1)}{2n+1}{\alpha'}^{2n+1}\sigma_{2n+1}\right) \tag{3.36}$$

where $\sigma_n = s^n + t^n + u^n$. Expanding to the first few orders are:

$$\begin{aligned} &\frac{3}{\sigma_3} + 2\zeta(3){\alpha'}^3 + \zeta(5){\alpha'}^5\sigma_2 + 2\frac{\zeta(3)^2}{3}{\alpha'}^6\sigma_3 + \frac{\zeta(7)}{2}{\alpha'}^7\sigma_2^2 \\ &+ \frac{2\zeta(3)\zeta(5)}{3}{\alpha'}^8\sigma_2\sigma_3 + \frac{\zeta(9)}{4}{\alpha'}^8\sigma_2^3\frac{2}{27}(2\zeta(3)^2+\zeta(9)){\alpha'}^9\sigma_3^2 + ... \end{aligned} \tag{3.37}$$

Where we used $s+t+u=0$. The first term corresponds to tree-level supergravity (the Einstein Hilbert term) [15], the second term to $R^4$ term, the third term to $D^4R^4$, the fourth term to $D^6R^4$, the fifth term to $D^8R^4$, the sixth term to $D^{10}R^4$, and both the seventh and eighth term to $D^{12}R^4$.

The second term $R^4$ is called the first order correction (stringy correction) to classical supergravity. Note this is if we are trying to write an effective action involving the curvature, so for the four point function we're studying we only get 4 powers of the curvature with higher powers of derivatives multiplying it to get the translation of higher powers of $s$, $t$, and $u$.

We notice the coefficients in this series are rational numbers multiplying odd zeta values. For N-particle scattering please view the appendix. [5]

### 3.4.2 Genus-one Amplitude

The genus-one level amplitude is much more difficult to expand.

$$A_1^{(4)}(\epsilon_r, k_r) = \frac{\pi}{16}R^4\int_{\mathbb{M}_{\mathbb{K}}}\frac{d\tau^2}{y^2}\mathcal{B}_1(s,t,u;\tau) \tag{3.38}$$

where we integrate over the fundamental domain of the modular group for some modular function $\mathbb{B}_1$ and $\tau = x+iy$ is the complex structure. $\mathbb{B}_1$ is the standard integral over the positions of the four particles over the torus.

$$\mathcal{B}_1(s,t,u;\tau) = \frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4}d^2z\exp -\frac{\alpha'}{2}\sum_{i<j}k_i\cdot k_j G(z_i,z_j) \tag{3.39}$$

[5] If we generalise to n-particle scattering we get multiple zeta values.

Where $G(z_i, z_j)$ is the Green function, and $k_i \cdot k_j$ is the contracted momentum.

We want to calculate the low energy expansion. This is quite tricky. We have to:

1. Bring down powers of the momenta $s, t$, and $u$
2. Integrate products of Greens functions over the torus
3. Integrate over the fundamental domain of the modular group.

Expanding in a power series of momenta gives (with $\alpha' = 4$)

$$\frac{1}{w!}\frac{1}{y^4}\int_{\Sigma^4}\prod_{i=1}^{4} d^2 z_i (\sum_{0<i<j\leq 4} s_i j G(z_i - z_j))^w = \sum_i \sigma_2^{p_i}\sigma_3^{q_i} j^{(p_i,q_i)}(\tau) \tag{3.40}$$

Where we rearranged the terms to get a power series of $\sigma_2$ and $\sigma_3$ with coefficients that are functions of $\tau$. Integrating over $\tau$ we get

$$A_1^{(4)} = \frac{\pi}{3}(1 + 0\sigma_2 + \frac{\zeta(3)}{3}\sigma_3 + 0\sigma_2^2 + \frac{116\zeta(5)}{5}\sigma_2\sigma_3 ...)R^4 \tag{3.41}$$

where as in the tree level amplitude each term is associated with the $R^4$, $D^4R^4$, $D^6R^4$, $D^8R^4$, $D^{10}R^4$.

### 3.4.3 Higher genus amplitudes

For genus two we have

$$A_2^{(4)} = g_s^2(\frac{4}{3}\zeta(4)\sigma_2 R^4 + 4\zeta(4)\sigma_3 R^4 + ...) \tag{3.42}$$

where the terms correspond to the $D^4R^4$, and $D^6R^4$ term respectively.

For genus three we have

$$A_3^{(4)} = g_s^4(\frac{4}{27}\zeta(6)\sigma_3 + ...)R^4 \tag{3.43}$$

where the first term corresponds to the $D^6R^4$ term.

## 3.5 Summary

In this chapter we have investigated Type IIB superstring theory and it's interaction. We studies four-graviton scattering and gave expressions for the tree-level, one-loop, and two-loop amplitudes. We then took the low energy limit of each of these giving us expressions in terms of multiple zeta values. We now combine them to get the low-energy effective Lagrangian up to $(\alpha')^6$ order

$$\begin{aligned} L_S \propto & R_S + (\alpha')^3\left(2\zeta(3) + 4\zeta(2)g_s^2 + \cdots\right)R_S^4 + (\alpha')^5\left(\zeta(5) + \cdots\right)D^4R_S^4 \\ & + (\alpha')^6\left(\frac{2}{3}\zeta(3)^2 + \frac{4}{3}\zeta(2)\zeta(3)g_s^2 + \cdots\right)D^6R_S^4 + \mathcal{O}((\alpha')^7) \end{aligned} \tag{3.44}$$

Which in the Einstein frame gives us

$$
\begin{aligned}
L \propto R_E &+ (\alpha')^3 \left(2\zeta(3) g_s^{-\frac{3}{2}} + 4\zeta(2) g_s^{\frac{1}{2}} + \cdots\right) R_E^4 \\
&+ (\alpha')^5 \left(\zeta(5) g_s^{-\frac{5}{2}} + \cdots\right) D^4 R_E^4 \\
&+ (\alpha')^6 \left(\frac{2}{3}\zeta(3)^2 g_s^{-3} + \frac{4}{3}\zeta(2)\zeta(3) g_s^{-1} + \cdots\right) D^6 R_E^4 + \mathcal{O}((\alpha')^7)
\end{aligned}
\tag{3.45}
$$

Where we can write the terms in brackets as coefficients parameterised by the axion-dilaton field $\tau$. We also give the Lagrangian as a function of the action:

$$
S = \int d\mu_G \left(\varepsilon_0(\tau) R_E^4 + \varepsilon_4(\tau) D^4 R_E^4 + \varepsilon_6(\tau) D^6 R_E^4 + \varepsilon_8(\tau) D^8 R_E^4 + O((\alpha')^8)\right) \tag{3.46}
$$

We get the low energy effective theory action order by order in $\alpha'$ with corrections from the four-graviton amplitudes of the form [14]

$$
\varepsilon_{2k}(\tau) D^{2w} R^4 \tag{3.47}
$$

with $D$ the covariant derivative on the target space time $M$ and $R^4$ (not to be confused with the Ricci curvature scalar) is a contraction of two $t_8$ (rank 8) tensors and four Riemann curvature tensors. See [16] for exact forms of the higher derivative contributions. Note there is no reason the dependence of the functions on $g_s$ be analytic. The non analytic terms in $g_s$ are the non perturbative effects and appear in the functions through "instanton" contributions.

# Chapter 4

# Modular Constraints in the Low-Energy Expansion

Let us first get some intuition of why modular forms, and more generically automorphic forms should appear in string theory [2]. In physics, charges of physical states are restricted to lie in the lattice rather than continuous spaces after Dirac quantisation. Certain discrete symmetries called dualities preserve the lattice. We have the following in these certain real physical systems:

1. For a discrete symmetry, all observable quantities must necessarily be given by functions that are invariant under that symmetry.
2. Dynamics of other symmetries impose differential equations on the observables
3. There are well-defined perturbative regimes in the theory

These general physical concepts actually satisfy the conditions that define automorphic forms:

1. Invariant under the action of a discrete subgroup $\Gamma \subset G : f(\gamma \cdot g) = f(g)$ for all $\gamma \in \Gamma$ where G is a real lie group (in our case $\Gamma$ is $SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R})$).
2. Satisfy eigenvalue differential equations under the action of the ring of G-invariant differential operators.
3. Have well-behaved growth conditions.

Therefore the physical concepts of quanitsation, discrete symmetries, and perturbations being well defined exactly give the conditions for automorphic forms (and specifically modular forms in type IIB superstring theory) to show up!

We first show that the coefficients of the low energy expansion described in the previous chapter are modular invariant functions, and then with supersymmetric constraints and results from superstring perturbation theory give their closed form expressions as functions of non-holomorphic Eisenstein series and then as functions of multiple zeta values. Together with the Fourier expansion of the Eisenstein series allows us to define some "Non-re-normalisation theorems".

This Chapter mainly follows Chapter 14 of the D'hoker Kaidi notes [1] however we make an effort to cite to the original papers. We also cite other resources when used.

## 4.1 Coefficients of the Low-Energy Expansion

Supergravity in ten dimensions has non-renormalisable UV divergences starting at one-loop. It's UV completion is Type IIB superstring theory. Supergravity is therefore the effective low energy field theory that describes the mass-less states of Type IIB superstring theory, to leading order in the $\alpha'$ expansion.

We can however extend beyond the leading order contributions with higher order contributions in $\alpha'$. These contributions are due to the massive string states with mass of order $\frac{1}{\alpha'}$ and are known as effective interactions. These string-induced effective interactions provide systematic corrections to supergravity in the expansion of $\alpha'$.

As $SL(2,\mathbb{Z})$ is a symmetry of Type IIB superstring theory, it must also be a symmetry of the low energy effective action of the theory.

In the Type IIB superstring theory, the effective interactions arise at order $(\alpha')^3$, higher order effective interactions are symbolically represented by $D^{2w}R^4$ where $D$ is the covariant derivative and $R$ is the Riemann tensor. The low energy effective action is given by

$$\int d\mu_G \left(\varepsilon_0(\tau)R_E^4 + \varepsilon_4(\tau)D^4R_E^4 + \varepsilon_6(\tau)D^6R_E^4 + \varepsilon_8(\tau)D^8R_E^4 + O((\alpha')^8)\right) \quad (4.1)$$

where $\tau$ is the vacuum expectation value of the axion-dilaton field described above and $R_E$ is the Riemann tensor associated to the Einstein frame metric $G_E$. The coefficients $\varepsilon_{2w}(\tau)$ are real-valued scalar function of $\tau$. Since the action is $SL(2,\mathbb{Z})$ invariant and (in the Einstein frame) $G_E$ and $R_E$ are modular invariants also, we have that the coefficients must be invariant under $SL(2,\mathbb{Z})$!

$$\varepsilon_{2w}\left(\frac{a\tau+b}{c\tau+d}\right) = \varepsilon_{2w}(\tau) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \quad (4.2)$$

Therefore, they are non-holomorphic modular functions from chapter 2!

Converting the action to the string frame $G_{\mu v}$ by $G_{\mu v} = e^{\frac{\Phi}{2}} G_{E\mu v}$ we get

$$\sqrt{G_E}\varepsilon_{2w}(\tau)D_E^{2w}R^4{}_E = e^{\frac{(k-1)\Phi}{2}}\sqrt{G}\varepsilon_{2w}(\tau)D^{2w}R^4 \quad (4.3)$$

with $D$ and $R$ in the string frame. We now conjecture that the coefficients are combinations of Eisenstein series. If we express combinations of the non-holomorphic Eisenstein series with effective interactions $D_E^{2w}R_E^4$ in the Einstein frame such that the leading perturbative contribution in the string frame is tree level we get

$$\frac{1}{2}\pi^{3/2}\sqrt{G_E}E_{\frac{3}{2}}(\tau)R_E^4 = e^{-2\Phi}\zeta(3)R^4 + \frac{\pi^2}{3}R^4 + \cdots \tag{4.4}$$

and

$$\frac{1}{2}\pi^{5/2}\sqrt{G_E}E_{\frac{5}{2}}(\tau)D_E^4R_E^4 = e^{-2\Phi}\zeta(5)D^4R^4 + \frac{2\pi^4}{135}e^{2\Phi}D^4R^4 + \cdots \tag{4.5}$$

Matching the tree level, genus one, and genus two perturbative results in terms of their zeta value representations. This suggests the coefficients $\varepsilon_{2w}(\tau)$ are given by non-holomorphic Eisenstein series which we will prove with supersymmetric constraints

## 4.2 Supersymmetric Constraints

We now use the structure and algebra of supersymmetry to impose constraints on the coefficients in the low-energy expansion. The constraints from supersymmetry then leads to first order differential equations on moduli space relating automorphic coefficients in the low energy expansion. We can therefore determine moduli dependent coefficients, at least for low dimension interactions. We follow the derivation from Chapter 12 of the D'hoker Kaidi notes [1]. The detailed derivations are given in [17] [18] [19] [20]. The method is as follows:

1. Find the supersymmetric completion to the "derivative correction term" ($R^4, D^4R^4, D^6R^4$ etc.)
2. Show $SL(2,\mathbb{Z})$ symmetry of that part of the action gives the "specific term" coefficients as non-holomorphic modular forms.
3. By defining a supersymmetric transformation in powers of $\alpha'$ we consider the terms in $S^{(0)}$ that "mix" with each other but not with other terms. This allows us to place differential constraints on the "specfic term" coefficients.
4. Closure of the supersymmetry algebra, combined with the differential constraints then gives us a Laplace eigenvalue equation for the "specfic term" coefficient.
5. Using the covariant derivative gives a Laplace eigenvalue equation for the coefficients of the low energy expansion we are looking for.

Following this method and combining with the know results in the asymptotic expansion gives us that the coefficients are functions of non-holomorphic Eisenstein series and generalised Eisenstein series.

### $R^4$ term

Let's first calculate the coefficient of $R^4$ which is $\varepsilon_0(\tau)$. First define $M$ as

$$M = e^{i\phi}\sqrt{-2i\tau_2}\left(F_1^3 - \tau F_2^3\right) \tag{4.6}$$

$M$ is $SL(2,\mathbb{Z})$ invariant as the complex 2-form potential $C_2$ and its field strength $F_3$ are both invariant. Define $\hat{M}$ as the "super-covariant combination"

$$\hat{M}_{c\mu\nu\rho} = M_{\mu\nu\rho} - 3\hat{\psi}_{[\mu}\gamma_{\nu\rho]}\lambda - 6i\hat{\psi}^*_{[\mu}\gamma_\nu\psi_{\rho]} \tag{4.7}$$

where $\psi$ and $\lambda$ are the gravitino and dilatino fields. $\hat{M}$ under supersymmetry transformation does not contain derivatives of the transformation parameter. The supersymmetric completion to the $R^4$ term is given by

$$\begin{aligned} S^{(3)} = \int d\mu_G(&\varepsilon_0^{(12,-12)}\lambda_{16} + \varepsilon_0^{(11,-11)}M\lambda^{14} + \cdots + \varepsilon_0^{(0,0)}R^4 + \cdots + \\ &\varepsilon_0^{(-12,12)}(\lambda^*)^{16}) \end{aligned} \tag{4.8}$$

with $\lambda_{16}$ and $\hat{M}_\lambda^{14}$ are given by

$$\lambda^{16} = \frac{1}{16!}\epsilon_{a_1\ldots a_{16}}\lambda^{a_1}\ldots\lambda^{a_{16}} \tag{4.9}$$

and

$$\hat{M}\lambda^{14} = \frac{1}{14!}\hat{M}_{\mu\nu\rho}(\gamma^{\mu\nu\rho}\gamma^0)_{a_{15}a_{16}}\epsilon_{a_1\ldots a_{16}}\lambda^{a_1}\ldots\lambda^{a_{14}} \tag{4.10}$$

The $SL(2,\mathbb{Z})$ symmetry of the action $S^{(3)}$ implies the coefficient functions $\varepsilon_0^{(w,-w)}(\tau)$ are non-holomorphic modular forms with holomorphic and anti-holomorphic weights $w$ and $-w$ respectively. By supersymmetry they must also satisfy

$$\varepsilon_0^{(w,-w)}(\tau) = D_{w-1}\ldots D_0\varepsilon_0(\tau) \tag{4.11}$$

where $D_w$ is the covariant derivative given by

$$D_w = i\tau_2\left(\frac{\partial}{\partial\tau} - i\frac{w}{2\tau_2}\right) \tag{4.12}$$

which maps modular forms of weight $(w, w')$ to weight $(w+1, w'-1)$.

$$S = \frac{1}{(\alpha')^4}\left(S^{(0)} + (\alpha')^3S^{(3)} + (\alpha')^4S^{(4)} + (\alpha')^5S^{(5)} + \cdots\right) \tag{4.13}$$

Define a supersymmetric transformation $\delta$ in powers of $\alpha'$

$$\delta = \delta^{(0)} + (\alpha')^3\delta^{(3)} + (\alpha')^4\delta^{(4)} + (\alpha')^5\delta^{(5)} + \cdots \tag{4.14}$$

As the action has supersymmetry we therefore get

$$\delta S^{(0)} = \delta^{(0)}S^{(3)} + \delta^{(3)}S^{(0)} = \delta^{(0)}S^{(5)}\delta^{(5)}S^{(0)} = \cdots = 0 \tag{4.15}$$

on the equations of motion. Consider now the first two terms in the action

$$L^{(3)} = \varepsilon_0^{(12,-12)} \lambda^{16} + \varepsilon_0^{(11,-11)} \hat{M} \lambda^{14} \tag{4.16}$$

The terms are related by supersymmetric transformations that don't mix with any of the other terms at this order. Defining the supersymmetry parameter by $\epsilon$ and only keeping the terms proportional to $\lambda^{16}\psi_\mu^* \epsilon$ gives

$$\delta^{(0)} L^{(3)}\big|_{\lambda^{16}\psi_\mu^*\epsilon} = -8i \left( \varepsilon_0^{(12,-12)}(\tau) + 108 D_{11} \varepsilon_0^{(11,-11)}(\tau) \right) \tag{4.17}$$

The $\delta^{(3)}$ variation of $S^{(0)}$ doesn't contain any of those terms implying

$$\delta^{(0)} L^{(3)}\big|_{\lambda^{16}\psi_\mu^*\epsilon} = 0 \tag{4.18}$$

which itself implies our first supersymmetric constraint

$$D_{11} \varepsilon_0^{(11,-11)}(\tau) = -\frac{1}{108} \varepsilon_0^{(12,-12)}(\tau) \tag{4.19}$$

Now consider the term in the supersymmetric variation proportional to $\lambda^{16}\lambda^*\epsilon^*$. This results in the second supersymmetric constraint

$$\bar{D}_{-12} \varepsilon_0^{(12,-12)}(\tau) + 3240 \varepsilon_0^{(11,-11)}(\tau) - 90 g(\tau,\bar{\tau}) = 0 \tag{4.20}$$

where $g(\tau,\bar{\tau})$ is some function. Closure of the supersymmetric algebra on $\lambda^*$ gives us our third supersymmetric constraint

$$32 D_{11} g(\tau,\bar{\tau}) = \varepsilon_0^{(12,-12)}(\tau) \tag{4.21}$$

Combining the three supersymmetric constraints gives us a Laplace eigenvalue equation for $\varepsilon_0^{(12,-12)}$

$$\Delta \varepsilon_0^{(12,-12)}(\tau) = \left( -\frac{132}{4} + \frac{3}{4} \right) \varepsilon_0^{(12,-12)}(\tau) \tag{4.22}$$

as for weight $(12,-12)$ modular functions, the Laplacian acting upon it is given by $\Delta = 4 D_{11} \bar{D}_{-12}$. Translating $\varepsilon_0^{(12,-12)}(\tau)$ to $\varepsilon_0(\tau)$ using the covariant derivative finally give us the Laplace eigenvalue equation for $\varepsilon_0(\tau)$

$$\Delta \varepsilon_0(\tau) = \frac{3}{4} \varepsilon_0(\tau) \tag{4.23}$$

Combining this with the asymptotic expansion from superstring perturbation theory gives us that $\varepsilon_0(\tau)$ is the non-holomorphic Eisenstein series $E_{\frac{3}{2}}(\tau)$.

### $D^4R^4$ **Term**

Analogous with the previous derivation we first find the terms in $S^{(0)}$ that mix with each other but not with other terms, under a subset of supersymmetry transformations. Choosing

$$L^{(5)} = \lambda^{16}\bar{M}^4\varepsilon_4^{(14,-14)} + \lambda^{15}\gamma^\mu\psi_\mu^*\bar{M}^4\varepsilon_4^{(13,-13)} + \lambda^{16}\bar{M}_2\bar{M}_{\rho_1\rho_2\rho_3}\bar{M}^{\rho_1\rho_2\rho_3}\tilde{\varepsilon}_4^{(13,-13)} \tag{4.24}$$

Modular invariance implies $\varepsilon_4^{(w,-w)}$ are modular forms and $\varepsilon_4^{(0,0)} = \varepsilon_4$. Checking invariance under supersymmetry, the only term proportional to $\lambda^{16}\psi_\mu^*\epsilon$ is from $\delta^{(0)}L^{(5)}$. As this vanishes we get

$$2D_{13}\varepsilon_4^{(13,-13)} - 11\varepsilon_4^{(14,-14)} = 0 \tag{4.25}$$

the only term proportional to $\lambda^{16}\lambda^*\epsilon^*$ is from the $\delta^{(0)}$ variation of $L^{(5)}$ and the $\delta^{(5)}$ variation of $S^{(0)}$. As this vanishes we get

$$2\bar{D}_{-14}\varepsilon_4^{(14,-14)} + 15\varepsilon_4^{(13,-13)} - \frac{9i}{16}\tilde{\varepsilon}_4^{(13,-13)} - 1080ig_1 - \frac{3}{4}ig_2 = 0 \tag{4.26}$$

Closure of the supersymmetry algebra on $\lambda^*$ gives the following constraints

$$-192iD_{13}g_1 = \varepsilon_4^{(14,-14)} \tag{4.27}$$

$$-108g_1 = \tilde{\varepsilon}_4^{(13,-13)} \tag{4.28}$$

$$ig_2 + 191ig_1 = \frac{1}{2}\varepsilon_4^{(13,-13)} \tag{4.29}$$

Combining all the constraints give us a Laplace eigenvalue equation for $\varepsilon_4^{(14,-14)}$

$$\Delta\varepsilon_4^{(14,-14)} = \left(-\frac{182}{4} + \frac{15}{4}\right)\varepsilon_4^{(14,-14)} \tag{4.30}$$

Translating $\varepsilon_4^{(14,-14)}(\tau)$ to $\varepsilon_4(\tau)$ using the covariant derivative finally gives us the Laplace eigenvalue equation for $\varepsilon_4(\tau)$

$$\Delta\varepsilon_4(\tau) = \frac{15}{4}\varepsilon_4(\tau) \tag{4.31}$$

which together with the asymptotic expansion gives $\varepsilon_4 = kE_{\frac{5}{2}}$ where $k$ is a constant

### $D^6R^4$ **term**

Repeating the analysis from the first two terms we now get the in homogeneous Laplace eigenvalue equation

$$(\Delta - 12)\varepsilon_6(\tau) = -6\pi^3(E_{\frac{3}{2}}(\tau))^2 \tag{4.32}$$

Substituting in the fourier series of $E_{\frac{3}{2}}$ and only keeping the Laurent polynomial terms we get

$$\begin{aligned}(\Delta - 12)\varepsilon_6^{(0)}(\tau) = &-6\left(2\zeta(3)\tau_2^{\frac{3}{2}} + \frac{2}{3}\pi^2\tau_2^{\frac{-1}{2}}\right)^2 \\ &+ 64\pi^2\tau_2 \sum_{N\neq 0} |(N)|^{-2}\sigma_2 N^2 |(N)|^2 K_1(2\pi\tau_2|N|)^2\end{aligned} \tag{4.33}$$

We have that $\varepsilon_6^{(0)}$ must be a polynomial in $\tau_2$. The specific powers are $\tau_2^3, \tau_2, \tau_2^{-1}$ and the powers solving the equation $(\Delta - 12)f = 0$ which are $\tau_2^{-3}$ and $\tau_2^4$, the latter term we reject since $\tau_2^3$ is the only singular term. The term $\tau_2^{-3}$ corresponds to the three loop contribution. We substitute the following ansatz into the inhomgeneous Laplace eigenvalue equation.

$$\varepsilon_6^{(0)}(\tau) = a_1\tau_2^3 + a_2\tau^2 + a_3\tau_2^{-1} + a_4\tau_2^{-3} \tag{4.34}$$

Which finally gives us

$$\varepsilon_6^{(0)}(\tau) = 4\zeta(3)\tau_2^3 + \frac{4}{3}\pi^2\zeta(3)\tau^2 + \frac{4\pi}{15}\tau_2^{-1} + a_4\tau_2^{-3} \tag{4.35}$$

Which matches the values from string perturbation theory. the term $a_4$ can be derived from first principles and is given by $a_4 = \frac{8\pi^6}{8505}$.

## 4.3 Eisenstein Series in String Theory

We can now interpret the non-holomorphic Eisenstein series in terms of string theory. Given $\tau = \tau_1 + i\tau_2$ where $\tau_2 = e^{-\phi} = \frac{1}{g_s}$ given by the inverse of the coupling constant, we get for the Fourier series

$$E_s(\tau) = 2\sum_{k=0}^{\infty} \mathcal{F}_k(\tau_2) cos(2\pi i k\tau_1) \tag{4.36}$$

where the zero mode $k = 0$ is given by

$$\mathcal{F}_0(s;\tau_2) = {\tau_2}^s + \frac{\sqrt{\pi}\Gamma\left(s - \frac{1}{2}\right)\zeta(2s-1)}{\zeta(2s)\Gamma(s)}{\tau_2}^{1-s} \tag{4.37}$$

As $\tau_2$ is the inverse of the coupling constant we interpret the zero mode as perturbative terms because its in terms of powers of the coupling constant $g_s$.

The non-zero modes $k > 0$ are given by

$$\mathcal{F}_k(s;\tau_2) = \frac{2\pi^s}{\zeta(2s)\Gamma(s)}|k|^{s-\frac{1}{2}}\sigma_{2s-1}(k){\tau_2}^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|k|\tau_2) \tag{4.38}$$

with measure $\sigma_n(k) = \sum_{p|k} p^n$. If we look asymptotically at weak coupling which is just as $g_s \to 0$ or as $\tau_2 \to \infty$ the Bessel functions $K$ decrease exponentially. This decrease together with the phase is characteristic of instantons or non-perturbative terms given by:

$$\frac{\pi^{s-\frac{1}{2}}}{\zeta(2s)\Gamma(s)}|k|^{s-1}\sigma_{2s-1}(k)e^{-2\pi|k|\tau_2}(1 + O({\tau_2}^{-\gamma})) \tag{4.39}$$

### 4.3.1 Non-renormalisation theorems

We now have the following Laplace equations for the coefficient functions with the corresponding solutions

1. For the $R^4$ term: $\Delta\varepsilon_0(\tau) = \frac{3}{4}\varepsilon_0(\tau)$ with $\varepsilon_0(\tau) = E_{\frac{3}{2}}(\tau)$.
2. For the $D^4R^4$ term: $\Delta\varepsilon_4(\tau) = \frac{15}{4}\varepsilon_4(\tau)$ with $\varepsilon_4(\tau) = kE_{\frac{5}{2}}(\tau)$ where $k$ is a constant.
3. For the $D^6R^4$ term: $(\Delta - 12)\varepsilon_6(\tau) = -6\pi^3(E_{\frac{3}{2}}(\tau))^2$ with "zero mode" $\varepsilon_6^{(0)}(\tau) = 4\zeta(3)\tau_2^3 + \frac{4}{3}\pi^2\zeta(3)\tau^2 + \frac{4\pi}{15}\tau_2^{-1} + \frac{8\pi^6}{8505}\tau_2^{-3}$.

#### $R^4$ term

Returning to the deviations from Einsteins equations, the first term is the $R^4$ term and we have in the Einstein frame (with re-scaled constant outlined in Chapter 2)

$$\varepsilon_0(\tau)R^4 = 2\zeta(3)E_{\frac{3}{2}}(\tau)R^4 \tag{4.40}$$

where $\Delta_s\varepsilon_0(\tau) = \frac{3}{4}\varepsilon_0(\tau)$. We get in the string frame:

$$g_s^{-\frac{1}{2}}\varepsilon_0(\tau) = 2\zeta(3)g_s^{-\frac{1}{2}}E_{\frac{3}{2}}(\tau) \sim 2\zeta(3)g_s^{-2} + 4\zeta(2)g_S^0 + \text{D-instantons} \tag{4.41}$$

This expression only has two power behaved terms, where the latter terms are the tree-level, genus one, and exponential $e^{\frac{-c}{g_s}}$ terms respectively. There are an infinite set of exponentially small terms. This shows there are no contributions in string theory beyond one-loop in string perturbation theory that contribute to scattering amplitude of order $R^4$. The coefficients are rational multiples of zeta values which agree with explicit calculations in string perturbation theory which are not at all obvious at first.

#### $D^4R^4$ term

For the term in $D^4R^4$ we have that its coefficients are proportional to the Eisenstein series:

$$\varepsilon_4(\tau)D^4R^4 = 2\zeta(3)E_{\frac{3}{2}}(\tau)D^4R^4 \tag{4.42}$$

This gives

$$g_s^{\frac{1}{2}}\varepsilon_4(\tau) = \zeta(5)g_s^{\frac{1}{2}}E_{\frac{5}{2}} \sim \zeta(5)g_s^{-2} + \frac{4}{3}\zeta(4)g_s^2 + \text{D-instantons} \tag{4.43}$$

The two perturbative terms in this case are tree-level and genus-two respectively. This agrees with the calculation in string perturbation theory and therefore there are no contributions to this term beyond two loops.

$D^6R^4$ **term**

For the term $D^6R^4$ have that its coefficients are proportional to the generalised Eisenstein series of Chapter 2, that instead of satisfying the Laplace eigenvalue equation satisfies the inhomogeneous Laplace equation:

$$(\Delta_\tau - 12)\varepsilon_6(\tau) = -(2\zeta(3)E_{\frac{3}{2}}(\tau))^2 \tag{4.44}$$

which is the square of the coefficient of $R^4$. Recall its zero mode is given by

$$\begin{aligned}\hat{f}_0(\tau_2)&\\ &= \frac{2}{3}\zeta(3)^2\tau_2^3 + \frac{4}{3}\zeta(2)\zeta(3)\tau_2 + \frac{8}{5}\zeta(2)^2\tau_2^{-1} + \frac{4}{27}\zeta(6)\tau_2^{-3} + O(e^{-4\pi\tau_2}),\end{aligned} \tag{4.45}$$

We can see there are four perturbative terms given representing genus 0, 1, 2, and 3 respectively and a non-perturbative term. The perturbative contributions agree with explicit string theory calculations.

## 4.4 Conclusions

In summary we have used S-Duality and supersymmetric arguments to obtain Laplace equations with the following solutions and non-renormalisation theorems:

1. The $R^4$ coefficient is $\varepsilon_0(\tau) = E_{\frac{3}{2}}(\tau)$ in which the term has non-renormalisation beyond one-loop.
2. The $D^4R^4$ coefficient with $\varepsilon_4(\tau) = E_{\frac{5}{2}}(\tau)$ in which the term has non-renormalisation beyond two-loops.
3. The $D^6R^4$ coefficient with $\varepsilon_6(\tau)$ is not an Eisenstein series in which the term has non-renormalisation beyond three-loops.

Where $E_s(\tau)$ is the non-holomorphic Eisenstein series that has two power-behaved terms $\tau_2^s$, $\tau_2^{1-s}$. All these power-behaved terms agree with explicit perturbative string calculations outlined in Chapter 3 and 5. There are instanton corrections to all of these. Note this has been done in 10 dimensions with corresponding $SL(2,\mathbb{Z})$ duality, which can be generalised with toroidal compactifications. The corresponding Eisenstein series associated with the higher rank duality group generalise to maximal parabolic Langland Eisenstein series.

# Chapter 5

# Modular Graph Functions in String Perturbation Theory

In the last chapter, we focused on the modular invariant coefficients on the spacetime of the low energy expansion of four graviton closed string scattering in type IIB superstring theory.

Now we study $SL(2,\mathbb{Z})$ invariant functions on the world-sheet. We outline how this type of modularity is different from the previous one we considered related to the S-duality group, which was a space-time concept where Type IIB in 10 dimension has $SL(2,\mathbb{Z})$ symmetry which acts on the space-time, not on the world sheet. This gives us to 2 types of modular invariant functions we study.

1. World sheet: On the torus world-sheet we get modular graph functions where $\tau$ is the world sheet torus modulus.

2. Space-time: Coefficient functions for the low-energy expansion where $\tau$ is the axion-dilaton i.e $\frac{1}{g_s}$

So we can think of $\tau$ as the modulus of the torus or the inverse string coupling. Both obviously satisfy

$$(\Delta - s(s-1))f(\tau) = S(\tau) \tag{5.1}$$

Where $\Delta = \tau_2^2(\partial_{\tau_1}^2 + \partial_{\tau_2}^2)$ is the $SL(2,\mathbb{Z})$ invariant Laplacian, $\tau = \tau_1 + i\tau_2$, and $S(\tau)$ is some modular invariant source term. However, the origin of this differential equation is completely different for the two types of modular invariant functions.

1. From the world sheet point of view we act on the lattice sum (a modular graph function) with the Laplacian, leading to the differential equation

2. From the space-time point of view, the differential equation comes from supersymmetric constraints

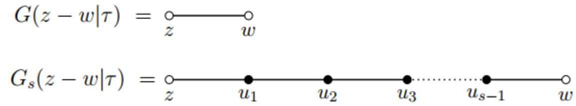


Figure 5.1: The Greens functions with their associated graph. The points $u_1, \cdots, u_{s-1}$ are integrated over $\Sigma$. The white dots are points in $\Sigma$ that are not integrated over. [1]

For example for the modular graph function:

$$(\Delta-6)C_{3,1,1}(\tau) = \frac{172}{5}\pi^{-5}\zeta(10)E_5(\tau) - 16\pi^{-5}\zeta(4)\zeta(6)E_2(\tau)E_3(\tau) + \frac{\zeta(5)}{10} \tag{5.2}$$

whereas for the low energy expansion coefficient we had

$$(\Delta - 12)\varepsilon_6(\tau) = -6\pi^3(E_{\frac{3}{2}}(\tau))^2 \tag{5.3}$$

This section closely follows Chapter 9 and 12 from the D'hoker Kaidi notes [1].

## 5.1 Modular Graph Functions and Feynman Diagrams

Modular graph functions map graphs to complex valued $SL(2,\mathbb{Z})$ invariant functions on the Poincare upper half-plane $\mathbb{H}$. Consider the scalar Green function $G(z-w|\tau)$ on the torus $\Sigma = \mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is the lattice with modulus $\tau \in \mathbb{H}$ and $z, w \in \Sigma$. It can be shown that the Green function can be expressed in terms of a double Fourier series in terms of a sum over the lattice $\Lambda \backslash 0$

$$G(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2}{\pi|m+n\tau|^2} e^{2\pi i(nx-my})) \tag{5.4}$$

where $z = x + \tau y \in \mathbb{R}/\mathbb{Z}$. We can construct modular invariant functions by concatenating the Green function

$$G_s(z|\tau) = \int_\Sigma \frac{d^2u}{\tau_2} G(z-u|\tau)G_{s-1}(u|\tau) \tag{5.5}$$

where $G_1 = G$ and $s$ is some positive integer. We associate a (Feynman) graph to $G$ and $G_s$ shown in in Figure 5.1. We get a sum of modular functions because as we take powers of the Greens functions we get Feynman diagrams on the torus. We can Fourier transform the Greens function to obtain

$$G_s(z|\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{\pi^s|m+n\tau|^{2s}} e^{2\pi i(nx-my)} \tag{5.6}$$

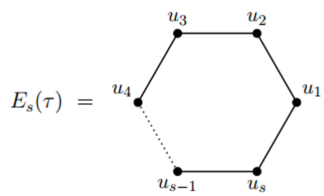


Figure 5.2: Closing the open chain in Figure 5.1 we get a "one-loop" modular graph function, which via its Greens function definition is the non-holomorphic Eisenstein series. [1]

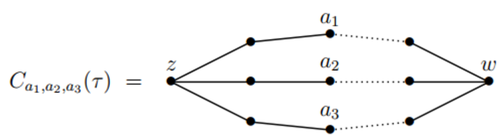


Figure 5.3: A connected two-loop graph can be visualized as having two key vertices, each with three connections (trivalent vertices), and it can also have any number of simpler vertices with two connections (bivalent vertices). These vertices are linked by three paths (or edges) that form a sort of chain or series connecting the two trivalent vertices located at positions z and w. The lengths of these three connecting paths are determined by three positive numbers, $a_1$, $a_2$, and $a_3$. These numbers represent how many bivalent vertices are placed on each of the three edges [1]

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### 5.1.1 One-loop Modular Graph Functions

We can get a one-loop modular graph by setting $z = 0$ (where $s \geq 2$) which closes the open chain. This is shown in Figure 5.2. We have that $G_s(0, \tau) = E_s(\tau)$ which is the non-holomorphic Eisenstein series. We have that the non-holomorphic Eisenstein series is associated with simple one-loop graphs.

### 5.1.2 Two-loop Modular Graph Functions

Consider the two-loop modular graph function in Figure 5.3 We define the modular graph function associated to the two-loop graphs as

$$C_{a_1,a_2,a_2}(\tau) = \sum_{\Sigma} \frac{d^2 z}{\tau_2} \int_{\Sigma} \frac{d^2 w}{\tau_2} G_{a_1}(z - w|\tau) G_{a_2}(z - w|\tau) G_{a_3}(z - w|\tau) \quad (5.7)$$

Due to translation invariance we have

$$C_{a_1,a_2,a_3}(\tau) = \int_{\Sigma} \frac{d^2 z}{\tau_2} G_{a_1}(z|\tau) G_{a_2}(z|\tau) G_{a_3}(z|\tau) \quad (5.8)$$

which we can Fourier transform to

$$C_{a_1,a_2,a_3}(\tau) = \sum_{\substack{m_r,n_r\in\mathbb{Z}\\ r=1,2,3}} \delta\left(\sum_{r=1}^{3} m_r\right)\delta\left(\sum_{r=1}^{3} n_r\right)\prod_{r=1}^{3}\left(\frac{\tau_2}{\pi|m_r+n_r\tau|^2}\right)^{a_r} \tag{5.9}$$

The sums are absolutely convergent [1]. As the non-holomorphic Eisenstein series are the solution to a differential equation and have an expansion at the cusp, implies some facts about two-loop modular graph functions.

Two-loop modular graph functions $C_{a_1,a_2,a_3}$ of weight $w = a_1 + a_2 + a_3$ obey a system of differential equations of uniform weight $w$:

$$\begin{aligned} 2\Delta C_{a_1,a_2,a_3} =& 2a_1a_2C_{a_1+1,a_2-1,a_3} + a_1a_2C_{a_1+1,a_2+1,a_3-2} - 4a_1a_2C_{a_1+1,a_2,a_3-1} \\ &+ a_1(a_1-1)C_{a_1,a_2,a_3} + 5 \text{ permutations of } (a_1,a_2,a_3) \end{aligned} \tag{5.10}$$

where $\Delta = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$. "When $a_1, a_2, a_3 \in \mathbb{N}$, the system of differential equations for a given weight truncates to a finite-dimensional linear system of inhomogeneous equations where the inhomogeneous part consists of a linear combination of $E_w$ and $E_{w_1}E_{w_2}$ with $w = w_1 + w_2$ and $w_1, w_2 \geq 2$, as a consequence of the following relations" [1]

$$\begin{aligned} C_{w_1,w_2,0} &= E_{w_1}E_{w_2} - E_w \\ C_{w_1+1,w_2,-1} &= E_{w_1}E_{w_2} + E_{w_1+1}E_{w_2-1} \end{aligned} \tag{5.11}$$

"The eigenvalues of the linear system are of the form $s(s-1)$ where $1 \leq s \leq w-2$ and $w - s \in 2\mathbb{N}$ and have multiplicity $[(s + 2)/3]$." [1]

For example up to weight 5 we have [1]

$$\begin{aligned} \Delta C_{1,1,1} &= 6E_3 \\ (\Delta-2)C_{2,1,1} &= 9E_4 - E_2^2 \\ (\Delta-6)C_{3,1,1} - 3C_{2,2,1} &= 16E_5 - 4E_2E_3 \\ \Delta C_{2,2,1} &= 8E_5 \end{aligned} \tag{5.12}$$

The first and last expressions imply that $C_{1,1,1} - E_3$ and $5C_{2,2,1} - 2E_5$ are constant. We can determine their value by looking at the expansion at the cusp.

As $\tau \to i\infty$ we get that the expansion of $C_{a_1,a_2,a_3}(\tau)$ is given by:

$$\begin{aligned} C_{a_1,a_2,a_3}(\tau) =& c_w(-4\pi\tau_2)^w + \frac{c_{2-w}}{(4\pi\tau_2)^{w-2}} \\ &+ \sum_{k=1}^{w-1}\frac{c_{w-2k-1}\zeta(2k+1)}{(4\pi\tau_2)^{2k+1-w}} + O(e^{-2\pi\tau_2}) \end{aligned} \tag{5.13}$$

with rational $c_w$ and $c_{w-2k-1}$ and $c_{2-w}$ given by

$$c_{2-w} = \sum_{k=1}^{w-2}\gamma_k\zeta(2k+1)\zeta(2w-2k-3) \tag{5.14}$$

with integer $\gamma_k$.

For every odd weight, there exists a unique linear combination of the differential identities mentioned earlier. This combination is found in the subspace where the eigenvalue is zero, and its inhomogeneous component does not include bi-linear terms from the Eisenstein series. This characteristic can be confirmed through the equations for $C_{1,1,1}$ and $C_{2,2,1}$ as listed in Equation 5.12. This property is consistent across all odd weights.

Moreover, each of these identities can be integrated to yield an additive constant. This constant can be precisely determined by examining the asymptotic behavior of the functions as they approach the cusp. We get

$$\begin{aligned} C_{1,1,1} &= E_3 + \zeta(3) \\ C_{2,2,1} &= \frac{2}{5}E_5 + \frac{\zeta(5)}{30} \\ C_{3,3,1} + C_{3,2,2} &= \frac{3}{7}E_7 + \frac{\zeta(7)}{252} \\ 9C_{4,4,1} + 18C_{4,3,2} + 4C_{3,3,3} &= 4E_7 + \frac{\zeta(9)}{240} \end{aligned} \tag{5.15}$$

## 5.2 Genus-one amplitude

Recall in Chapter 3 we gave for four graviton scattering the genus-one amplitude $A^{(1)}$ by integrating $\mathcal{B}^{(1)}(s,t,u|\tau)$ over the genus-one moduli space $\mathcal{M}_1$ as so

$$A^{(1)}(s_{ij}) = \frac{\pi}{16}\int_{\mathcal{M}_\infty} \frac{|d\tau|^2}{\Im(\tau)^2}\mathcal{B}^{(1)}(s,t,u;\tau) \tag{5.16}$$

Where the Lorentz scalar modular function $\mathcal{B}^{(1)}(s,t,u|\tau)$ was given by integrating over four copies of the torus $\Sigma$ with modulus $\tau$ given by[1]

$$\mathcal{B}^{(1)}(s,t,u|\tau) = \int_{\Sigma^4}\prod_{i=1}^{4}\frac{d^2 z_i}{\Im(\tau)}\exp\sum_{i<j} s_{ij}G(z_i,z_j|\tau) \tag{5.17}$$

where $z_i$ are the vertex insertion points on $\Sigma$, $\tau$ is the modulus used as a local complex coordinate for the genus one moduli space $\mathcal{M}_\infty$.

As $\mathcal{B}^{(1)}(s,t,u|\tau)$ is a symmetric function in $s,t,u$ we can expand it in powers of $\sigma_2$ and $\sigma_3$

$$\mathcal{B}^{(1)}(s,t,u|\tau) = \sum_{p,q=0}^{\infty}\mathcal{B}^{(1)}_{p,q}(\tau)\frac{\sigma_2^p\sigma_3^q}{p!q!} \tag{5.18}$$

We have that each coefficient $\mathcal{B}^{(1)}_{p,q}(\tau)$ must be a modular function of $\tau$ also. Expanding $\mathcal{B}^{(1)}(s,t,u|\tau)$ via the exponential in its integrand in powers of its argument to $w$ order given by $w = 2p+3q$ we get

$$\sum_{\substack{p,q\geq 0\\ 2p+3q=w}}\mathcal{B}^{(1)}_{p,q}(\tau)\frac{\sigma_2^p\sigma_3^q}{p!q!} = \frac{1}{w!}\prod_{i=1}^{4}\int_{\Sigma}\frac{d^2 z_i}{\Im(\tau)}\left(\sum_{i<j} s_{ij}G(z_i - z_j|\tau)\right)^w \tag{5.19}$$

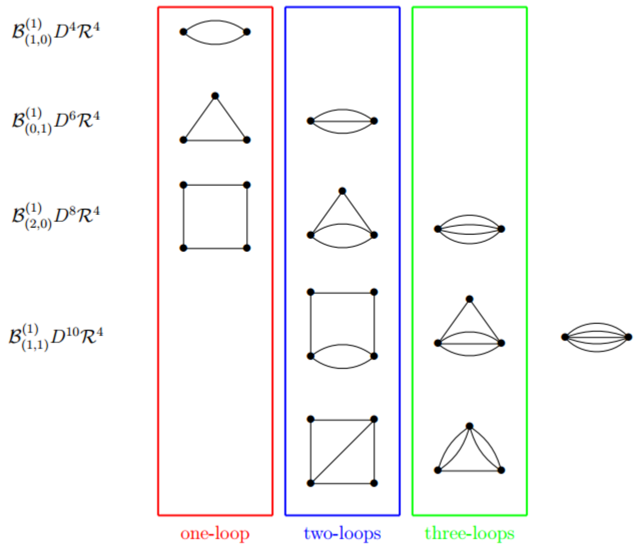


Figure 5.4: Modular graph functions contributing to the genus-one four graviton scattering amplitude up to weight $w = 5$ included. [1]

Expanding in $w$ we can represent it graphically in terms of modular graph functions Expanding the exponential and using the modular graph function identities (5.15) we get [1]

$$\begin{aligned}
\mathcal{B}^{(1)}_{(0,0)} &= 1, \\
\mathcal{B}^{(1)}_{(1,0)} &= E_2, \\
\mathcal{B}^{(1)}_{(0,1)} &= \frac{5}{3}E_3 + \frac{1}{3}\zeta(3), \\
\mathcal{B}^{(1)}_{(2,0)} &= 2C_{2,1,1} + E_2^2 - E_4, \\
\mathcal{B}^{(1)}_{(1,1)} &= \frac{7}{3}C_{3,1,1} + \frac{5}{3}E_2E_3 + \frac{1}{3}\zeta(3)E_3 - \frac{34}{15}E_5 + \frac{1}{5}\zeta(5),
\end{aligned} \tag{5.20}$$

## 5.3 Genus-two amplitude

Similarly in Chapter 2 we gave the genus-two amplitude

$$A^{(2)}(s_{ij}) = \frac{\pi}{64}\int_{\mathcal{M}_\in} \frac{|d\Omega|^2}{\det(\Im(\tau))^3}\mathcal{B}^{(2)}(s,t,u|\Omega) \tag{5.21}$$

Where $\mathcal{B}^{(2)}$ is the modular Lorentz scalar function given by [1]

$$\mathcal{B}^{(2)}(s,t,u|\Omega) = \int_{\Sigma^4} \frac{|\dagger|^2}{(\det(\Im(\Omega)))^2} \exp\sum_{i<j} s_{ij}\mathcal{G}(z_i,z_j|\Omega) \tag{5.22}$$

where $\Omega$ is the period matrix used as a set of local coordinates for the genus two moduli space $\mathcal{M}_{\in}$, $\dagger$ is a holomorphic $(1,0)$ form in each one of the vertex points $z_i$. The function $\mathcal{G}(z_i, z_j|\Omega)$ is the scalar Arakelov Green function.

As before $\mathcal{B}^{(2)}(s,t,u|\Omega)$ is symmetric so we can expand it as

$$\mathcal{B}^{(2)}(s,t,u|\Omega) = \sum_{p,q=0}^{\infty} \mathcal{B}^{(2)}_{p,q}(\Omega)\frac{\sigma_2^p \sigma_3^q}{p!q!} \tag{5.23}$$

The function $\mathcal{B}^{(2)}(s,t,u|\Omega)$ is invariant under the so called "genus-two" modular group $Sp(2,\mathbb{Z})$. All this means for us is that each coefficient $\mathcal{B}^{(2)}_{p,q}(\Omega)$ is a modular function of $\Omega$. Again expanding the exponential in the integrand of $\mathcal{B}^{(2)}(s,t,u|\Omega)$ in powers of its argument to a given order $w = 2p+3q-2$ we get

$$\sum_{\substack{p,q\geq 0 \\ 2p+3q=w+2}} B^{(2)}_{p,q}(\Omega)\frac{\sigma_2^p \sigma_3^q}{p!q!} = \frac{1}{w!}\int_{\Sigma^4} \frac{|\dagger|^2}{(\det(\Im(\Omega)))^2}\left(\sum_{i<j} s_{ij}\mathcal{G}(z_i, z_j|\Omega)\right)^w \tag{5.24}$$

We can now make some statements on the low-energy expansion. As $\dagger$ is linear in $s,t,u$ we have that the genus-two contribution to the $R^4$ term must vanish i.e. $\mathcal{B}^{(2)}_{(0,0)} = 0$. We have that the $D^2R^4$ contribution vanishes due to momentum conservation $(s+t+u=0)$ and the leading low energy contribution is the $D^4R^4$ interaction. This follows from the zeroth order expansion term of the exponential term for the coefficients of $\mathcal{B}^{(1)}_{(p,q)}$.

## 5.4 Conclusions

As we have deduced the contributions at any given weight, we now have to integrate them over the torus giving the genus-one expansion coefficients:

$$A_1^{(4)} = \frac{\pi}{3}(1 + 0\sigma_2 + \frac{\zeta(3)}{3}\sigma_3 + 0\sigma_2^2 + \frac{116\zeta(5)}{5}\sigma_2\sigma_3 ...)R^4 \tag{5.25}$$

Where the terms correspond to the $R^4$, $D^4R^4$, $D^6R^4$, $D^8R^4$, $D^{10}R^4$ respectively.

We have limited knowledge however for other genus's. For example for genus two the expansion coefficients are known up to this order [1] [3]:

$$A_2^{(4)} = g_s^2(\frac{4}{3}\zeta(4)\sigma_2 R^4 + 4\zeta(4)\sigma_3 R^4 + ...) \tag{5.26}$$

At genus three we know the leading low energy behaviour [1] [3]:

$$A_3^{(4)} = g_s^4(\frac{4}{27}\zeta(6)\sigma_3 + ...)R^4 \tag{5.27}$$

At higher orders we know no explicit expressions.

# Chapter 6

# Conclusions and Further Work

We have shown that there exist beautiful and elucidating relations between string theory and modular forms. Particularly of note are the fascinating non-renormalisation theorems we have been able to prove with the power of modular and supersymmetric constraints in string theory.

In the final chapter we described the "two types" of modularity that appears in Type IIB Superstring theory:

1. On the world-sheet in superstring perturbation theory
2. In the coefficients of the low-energy expansion

A natural question to ask is what is the relationship between these two forms of modularity? Indeed the generalised Eisenstein series arises in both of these context. Firstly in the coefficients of effective action of the low energy expansion, and secondly an infinite class of objects that contains all two-loop modular graph functions. Recent work has shown a Poincare series approach that unifies both classes of generalised Eisenstein series [21]. Defining the seed function as

$$v(a,b,r,s|\gamma\cdot z)=\sum_{m\neq 0}\sigma_a(m)|m|^{b-\frac{1}{2}}y^{r+\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|m|\tau_2)e^{2\pi i m\tau_1} \tag{6.1}$$

then gives us the Poincare series

$$\mathcal{Y}(a,b,r,s|z)=\sum_{\gamma\in B(\mathbb{Z})\backslash SL(2,\mathbb{Z})}v(a,b,r,s|\gamma\cdot z) \tag{6.2}$$

One can then find the asymptotic expansion in the $\tau_2\to 0$ limit leading to interesting connections to the non-trivial zeroes of the Riemann zeta function. As we can take the generalised Eisenstein series that corresponds to the two-loop modular graph functions, the limit corresponds to a degeneration of the world-sheet torus. For the generalised Eisenstein series that corresponds to the coefficients of the action of the low energy expansion we get this limit is given by $g_s \to \infty$. The fact that the Riemann hypothesis is linked to the strong coupling regime of string theory through modular forms is extremely elucidating and suggests a deeper connection between theoretical physics and number theory.

Many more open questions remain. Two of immense interest are if we can go to higher orders in the low energy expansion and to what extent does supersymmetry and duality determine the low energy expansion?

String theory is UV finite at all orders and the supergravity Feynman diagrams are combined into a single diagram for each genus. This leads us to the following question: can we determine the properties of supergravity Feynman diagrams by a suitable limit of string theory diagrams? This could answer if the gravity field theory has UV divergences.

We have to mention that due to the holographic correspondence we have a duality between the S-Duality of the type IIB superstring and the Montonen-Olive $SL(2,\mathbb{Z})$ of $N = 4$ super-symmetric Yang-Mills theory [3]. This is achieved by studying the flat space limit Low energy expansion of type IIB superstring amplitudes in Ad$S_5$ X $S^5$ through its holographic correspondence to $SL(2,\mathbb{Z})$ covariance of correlation functions of $N = 4$ super-symmetric Yang-Mills theory. Much recent work has been done in this area and a great resource is the SageX review [3].

In summary, this thesis has endeavored to bridge distinct yet interrelated domains—mathematics and physics—through the lens of modular forms and string theory. By delving into the modular structures and their compelling implications in string theory, we have illuminated aspects of string amplitudes, gravitational dynamics, and beyond. The exploration of modular forms not only enriches our theoretical arsenal but also deepens our understanding of the universe's most fundamental aspects, pointing towards a unified framework where mathematics does not merely describe but actively informs the underlying principles of physics. As we continue to uncover these intricate connections, the path forward beckons with promises of further discoveries and insights, potentially leading to revolutionary advancements in both fields. This thesis, therefore, stands not only as a testament to what has been achieved but also as a beacon, guiding future inquiries into the profound interplay between the abstract world of numbers and the tangible realm of physical reality.

# Chapter 7

# Appendix

## 7.1 Zeta Values

Zeta values can be described as special values of poly-logarithms $Li_a(z)$:

$$Li_a(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^a} \qquad \zeta(a) = Li_a(1) \tag{7.1}$$

For the even zeta values we get $\zeta(2n) = c_n \pi^{2n}$. The odd values $\zeta(2n+1)$ are thought to be transcendental although this is still up to conjecture.

### Multiple-Zeta Values

Multiple zeta values can be described as special values of multiple polylogarithms:

$$\begin{aligned} Li_{a_1,\ldots,a_r}(z_1,...,z_r) &= \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} (\frac{z_l}{k_l})^{a_l}, \\ \zeta(a_1,...a_r) = Li_{a_1,\ldots,a_r}(1,...,1) &= \sum_{0<k_1<\ldots<k_r} \prod_{l=1}^{r} (k_l)^{-a_l} \end{aligned} \tag{7.2}$$

with weight $w = \sum_{l=1}^{r} a_l$ and depth $r$.

Multiple zeta values are interesting because they are numbers with certain algebraic properties. The properties are derivatives of multiple polylogarithms. For example, the first irreducible case is at $w = 8$:

$$350\zeta(3,5) = 875\zeta(6,2) + 240\zeta(2)^4 - 1400\zeta(3)\zeta(5) \tag{7.3}$$

These multiple zeta values arise in N-particle tree amplitudes. For the open string we have for $N > 4$ the coefficients of the higher derivative interactions of order $\alpha'^n$ are multiple zeta values of weight n.

For the closed string we have for $N > 4$ the coefficients are so called "single-valued multiple zeta values of weight n" which are special values of single-valued multiple polylogarithms without monodromies.

We get for single valued multiple zeta-values the following properties

$$\zeta_{\rm sv}(2n) = 0, \qquad \zeta_{\rm sv}(2n+1) = 2\zeta(2n+1) \tag{7.4}$$

Where the second relation is for odd zeta's only.

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