Introduction to Neural Networks

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Module 9.1: Hopfield Memory Capacity

What We've Covered So Far…

- Learned about the Hopfield Network
  - Hebbian Learning
    - Recurrent neural networks
    - Matrix/vector representation of a recurrent network
    - Heat death
  - Excitation and Inhibition in Hopfield Networks via bi-polar values
  - Possibility of cycling in Hopfield networks in part because of inhibition and excitation.
- In this sub-module
  - We examine how many exemplars a Hopfield network can 'reasonably' store.

H-N Function Never Increases

- This means, the *H* value can only decrease and since there is a lower bound, it will eventually converge such that

$$f_h\left(\sum_j w_{ij}x_j\right) = \mathbf{x}$$

Cannot oscillate between solutions.

The Hopfield Outerproduct

Recall
$$w_{ij} = \begin{cases} \sum_{s=1}^{P} x_i^s x_j^s & i \neq j \\ 0 & i = j \end{cases}$$

$$\text{Let } \mathbf{x}^r = \left(x_1^r, x_2^r, \ldots, x_n^r\right)$$

This corresponds to the r[th] exemplar.

Memory Recall/Completion

Activity of the i^{th} neuron at the end of the first iteration.

$$a_i(1) = \sum_{\substack{j=1 \\ j \neq i}}^{n} w_{ij} x_j^r$$

Substituting in the expression for w_{ij}

$$a_i(1) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{s=1}^{P} x_i^s x_j^s x_j^r$$

Memory Recall/Completion

From previous slide:

$$a_i(1) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{s=1}^{P} x_i^s x_j^s x_j^r$$

Let's do some mathematical tricks and break up the double sum:

$$a_i(1) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[\underbrace{x_i^r x_j^r x_j^r}_{\substack{s=r^{\text{th}} \\ \text{term}}} + \sum_{\substack{s=1 \\ s \neq r}}^{P} x_i^s x_j^s x_j^r \right]$$

Memory Recall/Completion

Some further manipulations:

$$a_i(1) = \sum_{\substack{j=1 \\ j \neq i}}^{n} \left[\underbrace{x_i^r x_j^r x_j^r}_{\substack{s=r^{\text{th}} \\ \text{term}}} + \sum_{\substack{s=1 \\ s \neq r}}^{P} x_i^s x_j^s x_j^r \right]$$

$$= \sum_{\substack{j=1 \\ j \neq i}}^{n} x_i^r x_j^r x_j^r + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{\substack{s=1 \\ s \neq r}}^{P} x_i^s x_j^s x_j^r$$

Memory Recall/Completion

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$$a_i(1) = \sum_{\substack{j=1 \\ j \neq i}}^{n} x_i^r x_j^r x_j^r + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{\substack{s=1 \\ s \neq r}}^{P} x_i^s x_j^s x_j^r$$

$$= x_i^r \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j^r x_j^r + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{\substack{s=1 \\ s \neq r}}^{P} x_i^s x_j^s x_j^r$$

So …

$$a_i(1) = x_i^r (n-1) + N_i$$

Call this S_i

Memory Recall/Completion

$$a_i(1) = x_i^r \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j^r x_j^r + \sum_{\substack{j=1 \\ j \neq i}}^{n} \sum_{\substack{s=1 \\ s \neq r}}^{P} x_i^s x_j^s x_j^r$$

$$= x_i^r (n-1) + N_i$$

If the noise term is 0 and we put this into the hard-limiting function

$$f_h\left(a_i(1)\right) = f_h\left(x_i^r (n-1) + N_i\right)$$

$$= f_h\left(x_i^r (n-1)\right)$$

$$= x_i^r$$

When is the *Noise* term likely to be 0?

When two different exemplars are *statistically independent.*

What does that mean?

No significant correlations between two exemplars regarding corresponding vector elements.

Roughly, each set of corresponding vector elements has approximately a 50% chance of having the same value.

We can simply say therefore that the sum of the products of the corresponding elements have an expectation value of 0!

A = (1, 1, -1, 1, -1, 1)
B = (-1, 1, 1, -1, -1, 1)

-1 + 1 – 1 – 1 + 1 + 1 = 0

Memory Recall/Completion

What if N_i is not zero? Let's make some quick, hand-waving arguments:

Let's assume that the exemplars are "statistically independent". Therefore:

$$N_i \sim N(0,\sigma^2) \text{ that is } \mu(N_i) = 0$$

$$\text{and } \sigma^2(N_i) = (n-1)(P-1)$$

Recall that for a normally distributed random variable, 99.5% of all events lie within 3σ.

Memory Recall/Completion

We want to ensure that $|S_i| > |N_i|$

$$|S_i| = n - 1 \geq 3\sigma = 3\sqrt{(n-1)(P-1)}$$

$n - 1 \geq 3\sqrt{(n-1)(P-1)}$ and squaring both sides...

$$(n-1)^2 \geq 9(n-1)(P-1)$$

$$n - 1 \geq 9(P-1)$$

Or roughly $P \approx n/9$.