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Introduction to Neural Networks

Johns Hopkins University
Engineering for Professionals Program
605-447/625-438
Dr. Mark Fleischer

Module 11.4.1: RBM Mathematics, Insights and
Getting Your Head Around It!

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What We've Covered So Far

- Probabilistic foundations of RBMs.
  - *Energy/Consensus* associated with a visible and hidden pair of vectors.
  - Probability of a node's states

Goal

- Raise the probability that a visible vector with be faithfully reconstructed when a 'hidden' vector is presented to the visible layer.

Question:

How do we train an RBM so that 'reconstructions' are likely to create a reasonable facsimile of the original data?

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Weights → Energies → Probabilities

- Each possible joint configuration of the visible and hidden units has an energy
  - The energy is determined by the weights and biases (as in a Hopfield net).
- The energy of a joint configuration of the visible and hidden units determines its probability:

$$p(\mathbf{v},\mathbf{h}) \propto e^{-E(\mathbf{v},\mathbf{h})}$$

- The probability of a configuration over the visible units is found by summing the probabilities of all the joint configurations that contain it.

From Hinton 2007 (modified)

Using energies to define probabilities

- The probability of a joint configuration over both visible and hidden units depends on the energy of that joint configuration compared with the energy of all other joint configurations.
- The probability of a configuration of the visible units is the sum of the probabilities of all the joint configurations that contain it.

$$p(\mathbf{v},\mathbf{h}) = \frac{e^{-E(\mathbf{v},\mathbf{h})}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

partition function

$$p(\mathbf{v}) = \frac{\sum_{g} e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

From Hinton 2007 (modified)

How Do We Train an RBM?

$$p(\mathbf{v}) = \frac{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}} \rightarrow \ln p(\mathbf{v}) = \ln\left(\frac{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}\right)$$

$$= \ln \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)} - \ln \sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}$$

$$\frac{\partial \ln p(\mathbf{v})}{\partial w_{ij}} = \underbrace{\frac{\partial}{\partial w_{ij}} \ln \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}}_{\mathbf{A}} - \underbrace{\frac{\partial}{\partial w_{ij}} \ln \sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}_{\mathbf{B}}$$

Looking at Term A:

$$\frac{\partial}{\partial w_{ij}} \ln \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)} = \frac{1}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} \cdot \frac{\partial}{\partial w_{ij}} \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}$$

$$= \frac{1}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} \cdot \sum_g \frac{\partial e^{-E(\mathbf{v},\mathbf{h}^g)}}{\partial w_{ij}}$$

$$= \frac{1}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} \cdot \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)} \cdot \frac{\partial\left(-E(\mathbf{v},\mathbf{h}^g)\right)}{\partial w_{ij}}$$

Looking at Term A:

Recall that $E(\mathbf{v},\mathbf{h}) = -\sum_{i,j} v_i h_j w_{ij}$

$$\frac{\partial}{\partial w_{ij}} \ln \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)} = \frac{1}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} \cdot \sum_g e^{-E(\mathbf{v},\mathbf{h}^g)} \cdot \frac{\partial\left(-E(\mathbf{v},\mathbf{h}^g)\right)}{\partial w_{ij}}$$

$$= \frac{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)} v_i h_j^g}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} = \sum_g p(\mathbf{h}^g \middle| \mathbf{v}) v_i h_j^g = \langle v_i \cdot h_j \rangle_{\mathbf{v}}$$

Where does that conditional probability come from?

$$\Pr\left\{A \middle| B\right\} = \frac{\Pr\left\{A \cap B\right\}}{\Pr\left\{B\right\}}$$

$$p(\mathbf{h}^g \middle| \mathbf{v}) = \frac{p(\mathbf{v},\mathbf{h}^g)}{p(\mathbf{v})} = \frac{\frac{1}{Z} e^{-E(\mathbf{v},\mathbf{h}^g)}}{\frac{1}{Z}\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} = \frac{e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}}$$

$$\frac{e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}} = \frac{p\left(\mathbf{v},\mathbf{h}^g\right)}{p\left(\mathbf{v}\right)} = p(\mathbf{h}^g \middle| \mathbf{v})$$

Looking at Term B:

$$\frac{\partial}{\partial w_{ij}} \ln \sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)} = \frac{1}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}} \cdot \frac{\partial}{\partial w_{ij}} \sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}$$

$$= \frac{1}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}} \cdot \sum_{u,g} \frac{\partial e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}{\partial w_{ij}}$$

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Looking at Term B:

$$\frac{\partial}{\partial w_{ij}} \ln \sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)} = \frac{\sum_{u,g} \frac{\partial e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}{\partial w_{ij}}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

$$= \frac{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)} \frac{\partial\left(-E(\mathbf{v}^u,\mathbf{h}^g)\right)}{\partial w_{ij}}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

$$= \frac{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)} v_i^u h_j^g}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}} = \sum_{u,g} p(\mathbf{v}^u,\mathbf{h}^g) v_i^u h_j^g = \langle v_i \cdot h_j \rangle_{\mathbf{vh}}$$

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Basis of *Contrastive Divergence*

$$\frac{\partial \ln p(\mathbf{v})}{\partial w_{ij}} = \frac{\partial}{\partial w_{ij}} \ln \sum_{g} e^{-E(\mathbf{v},\mathbf{h}^g)} - \frac{\partial}{\partial w_{ij}} \ln \sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}$$
$$= \langle v_i \cdot h_j \rangle_{\mathbf{v}} - \langle v_i \cdot h_j \rangle_{\mathbf{vh}}$$

Using Contrastive Divergence

- Use the derivative to perform *stochastic* gradient ascent!

$$\frac{\partial \ln p(\mathbf{v})}{\partial w_{ij}} \propto \Delta w_{ij} = \eta\left(\langle v_i \cdot h_j \rangle_{\mathbf{v}} - \langle v_i \cdot h_j \rangle_{\mathbf{vh}}\right)$$

But why?

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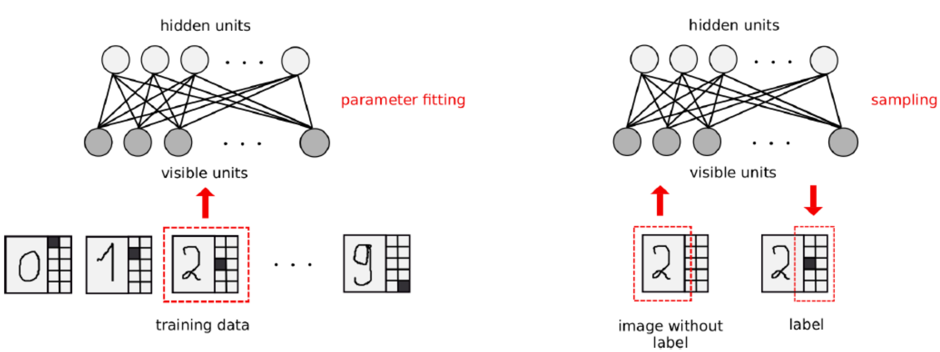
Some Motivation

learning with labels **classification**



From Fischer, Asja, "Training Restricted Boltzmann Machines", Doctoral Thesis, Faculty of Science, University of Copenhagen, 2014 at p.19

Our Goal

- Strengthen the probability of reconstructed visible vectors so that they correspond to their probability of occurring in a training set of data.

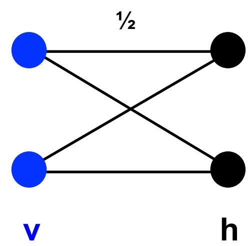
The Energy/Probability Relationships

$$E(\mathbf{v},\mathbf{h}) = -\sum_{i,j} w_{ij} v_i h_j - \sum_i a_i v_i - \sum_j b_j h_j$$

$$\Pr(\mathbf{v},\mathbf{h}) = \frac{e^{-E(\mathbf{v},\mathbf{h})}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

$$\Pr(\mathbf{v}) = \frac{\sum_g e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

A Numerical Example



We are going to increase the probability of the vector v = [0, 1].

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So What is the Generative Model?

v	h	Joint Probability
0 0	0 0	
0 0	0 1	
0 0	1 0	
0 0	1 1	
0 1	0 0	
0 1	0 1	
0 1	1 0	
0 1	1 1	

v	h	Joint Probability
1 0	0 0	
1 0	0 1	
1 0	1 0	
1 0	1 1	
1 1	0 0	
1 1	0 1	
1 1	1 0	
1 1	1 1	

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So What is the Generative Model?

	v1	v2	h1	h2	E	e^{-E}	Probability
1	0	0	0	0	0	1	0.031390208
2	0	0	0	1	0	1	0.031390208
3	0	0	1	0	0	1	0.031390208
4	0	0	1	1	0	1	0.031390208
5	0	1	0	0	0	1	0.031390208
6	0	1	0	1	-0.5	1.648721271	0.051753703
7	0	1	1	0	-0.5	1.648721271	0.051753703
8	0	1	1	1	-1	2.718281828	0.085327431
9	1	0	0	0	0	1	0.031390208
10	1	0	0	1	-0.5	1.648721271	0.051753703
11	1	0	1	0	-0.5	1.648721271	0.051753703
12	1	0	1	1	-1	2.718281828	0.085327431
13	1	1	0	0	0	1	0.031390208
14	1	1	0	1	-1	2.718281828	0.085327431
15	1	1	1	0	-1	2.718281828	0.085327431
16	1	1	1	1	-2	7.389056099	0.231944006
						31.8570685	1

All energy and probability values are based solely on the weights and biases!

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What Does This Tell Us?

- The different configurations will occur with the indicated probability.
- How?
  - Present (activate) a initial visible vector onto the visible nodes (set their states).
  - Stochastically assign states to the hidden vector nodes.
  - Let the hidden vector nodes stochastically influence the assignment of states to the visible nodes, and so on.

If we do this back and forth for a very large number of cycles and count the occurances of the different vectors (configurations), they will occur with the frequency from the preceding table!

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What is the Frequency of Occurrence of Visible Vector [0,1]?

From the table, we can calculate the marginal probability.

	v1	v2	h1	h2	E	e^-E	Probability
5	0	1	0	0	0	1	0.031390208
6	0	1	0	1	-0.5	1.648721271	0.051753703
7	0	1	1	0	-0.5	1.648721271	0.051753703
8	0	1	1	1	-1	2.718281828	0.085327431

$$p(\mathbf{v}) = \frac{\sum_{g} e^{-E(\mathbf{v},\mathbf{h}^g)}}{\sum_{u,g} e^{-E(\mathbf{v}^u,\mathbf{h}^g)}}$$

This sums to about 0.22022505.

All based on the energy values and Boltzmann distribution function.

Let's See How Stochastic Update Functions are consistent with the table values.

- Thus, given a visible vector, the hidden vectors are assigned states with certain probabilities based on the activity function.
- Remember?

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What are the probabilities of h given v?

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}\right\} = \frac{1}{1+e^{-S_{h_1}/T}}$$

If S = 0, then this probability is 1/2.

Can you determine the first row of the table?

$$\Pr\left\{\mathbf{v},\mathbf{h}\right\} = \Pr\left\{\mathbf{h} \middle| \mathbf{v}\right\} \times \Pr\left\{\mathbf{v}\right\}$$

What are the probabilities of h given v?

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}\right\} = \frac{1}{1+e^{-S_{h_1}/T}}$$

Let's assume for now that **v** = [0, 0]

So,
$$S_{h_1} = \sum_i w_i v_i + \theta_i$$
$$= \tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 0 + 0 = 0$$

The Generative Model

Since $S_{h1} = 0$, then

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}\right\} = \frac{1}{1+e^{-S_{h_1}/T}} = \frac{1}{1+1} = \frac{1}{2}$$

But for the first row of the table, we want to know the probability of $h_1 = 0$ (not 1). Also, we want to determine the probability of the **vector h given the vector $\mathbf{v_1}$**

Probability of **h**, given that v_1 = [0, 0], is ¼.

This is because h_1 is independent of h_2.

So What is the Generative Model?

	v	h	Joint Probability
P_{v1}	0 0	0 0	P_{v1} / 4
	0 0	0 1	P_{v1} / 4
	0 0	1 0	P_{v1} / 4
	0 0	1 1	P_{v1} / 4
P_{v2}	0 1	0 0	
	0 1	0 1	
	0 1	1 0	
	0 1	1 1	

	v	h	Joint Probability
P_{v3}	1 0	0 0	
	1 0	0 1	
	1 0	1 0	
	1 0	1 1	
P_{v4}	1 1	0 0	
	1 1	0 1	
	1 1	1 0	
	1 1	1 1	

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What are the probabilities of h given v?

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}_2\right\} = \frac{1}{1+e^{-S_{h_1}/T}}$$

Now, $\mathbf{v}$ = [0, 1]

So, again, $S_{h_1} = \sum_i w_i v_i + \theta_i$

$$= \tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 1 = \tfrac{1}{2}$$

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What are the probabilities of h given v?

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}_2\right\} = \frac{1}{1+e^{-1/2}} = 0.622459331$$

So, given that **v** = [0, 1], Pr { **h** = [0,0] | $\mathbf{v}_2$ } = 0.3775 × 0.3775 = 0.1425.

So What is the Generative Model?

	v	h	Joint Probability
P_{v1}	0 0	0 0	P_{v1} / 4
	0 0	0 1	P_{v1} / 4
	0 0	1 0	P_{v1} / 4
	0 0	1 1	P_{v1} / 4
P_{v2}	0 1	0 0	P_{v2} • 0.1425
	0 1	0 1	
	0 1	1 0	
	0 1	1 1	

	v	h	Joint Probability
P_{v3}	1 0	0 0	
	1 0	0 1	
	1 0	1 0	
	1 0	1 1	
P_{v4}	1 1	0 0	
	1 1	0 1	
	1 1	1 0	
	1 1	1 1	

What are the probabilities of h given v?

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}_2\right\} = \frac{1}{1+e^{-1/2}} = 0.622459331$$

So, given that **v** = [0, 1], Pr { **h = [0,1] | v$_2$** } = 0.3775 × 0.6224 = 0.2350.

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So What is the Generative Model?

	v	h	Joint Probability
P_{v1}	0 0	0 0	P_{v1} / 4
	0 0	0 1	P_{v1} / 4
	0 0	1 0	P_{v1} / 4
	0 0	1 1	P_{v1} / 4
P_{v2}	0 1	0 0	P_{v2} • 0.1425
	0 1	0 1	P_{v2} • 0.2350
	0 1	1 0	P_{v2} • 0.2350
	0 1	1 1	

	v	h	Joint Probability
P_{v3}	1 0	0 0	
	1 0	0 1	
	1 0	1 0	
	1 0	1 1	
P_{v4}	1 1	0 0	
	1 1	0 1	
	1 1	1 0	
	1 1	1 1	

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What are the probabilities of h given v?

$$\Pr\left\{h_1 = 1 \middle| \mathbf{v}_2\right\} = \frac{1}{1+e^{-1/2}} = 0.622459331$$

So, given that **v** = [0, 1], Pr { **h = [1,1] | v$_2$** } = 0.6224 × 0.6224 = 0.3874.

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So What is the Generative Model?

	v	h	Joint Probability
P_{v1}	0 0	0 0	P_{v1} / 4
	0 0	0 1	P_{v1} / 4
	0 0	1 0	P_{v1} / 4
	0 0	1 1	P_{v1} / 4
P_{v2}	0 1	0 0	P_{v2} • 0.1425
	0 1	0 1	P_{v2} • 0.2350
	0 1	1 0	P_{v2} • 0.2350
	0 1	1 1	P_{v2} • 0.3874

	v	h	Joint Probability
P_{v3}	1 0	0 0	
	1 0	0 1	
	1 0	1 0	
	1 0	1 1	
P_{v4}	1 1	0 0	
	1 1	0 1	
	1 1	1 0	
	1 1	1 1	

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So What is the Generative Model?

	v	h	Joint Probability
P_{v1}	0 0	0 0	$P_{v1} / 4$
	0 0	0 1	$P_{v1} / 4$
	0 0	1 0	$P_{v1} / 4$
	0 0	1 1	$P_{v1} / 4$
P_{v2}	0 1	0 0	$P_{v2} \bullet 0.1425$
	0 1	0 1	$P_{v2} \bullet 0.2350$
	0 1	1 0	$P_{v2} \bullet 0.2350$
	0 1	1 1	$P_{v2} \bullet 0.3874$

	v	h	Joint Probability
P_{v3}	1 0	0 0	$P_{v2} \bullet 0.1425$
	1 0	0 1	$P_{v2} \bullet 0.2350$
	1 0	1 0	$P_{v2} \bullet 0.2350$
	1 0	1 1	$P_{v2} \bullet 0.3874$
P_{v4}	1 1	0 0	
	1 1	0 1	
	1 1	1 0	
	1 1	1 1	

What are the probabilities of h given v?

$$S_{v_4} = \tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 1 = 1$$

$$\Pr\left\{ h_1 = 1 \middle| \mathbf{v}_4 \right\} = \frac{1}{1 + e^{-1}} = 0.7310$$

So, given that **v** = [1, 1], Pr { **h =** [0,0] | $\mathbf{v_2}$ } = 0.2689 × 0.2689 = 0.0723.

So What is the Generative Model?

	v	h	Joint Probability
P_{v1}	0 0	0 0	P_{v1} / 4
	0 0	0 1	P_{v1} / 4
	0 0	1 0	P_{v1} / 4
	0 0	1 1	P_{v1} / 4
P_{v2}	0 1	0 0	P_{v2} • 0.1425
	0 1	0 1	P_{v2} • 0.2350
	0 1	1 0	P_{v2} • 0.2350
	0 1	1 1	P_{v2} • 0.3874

	v	h	Joint Probability
P_{v3}	1 0	0 0	P_{v3} • 0.1425
	1 0	0 1	P_{v3} • 0.2350
	1 0	1 0	P_{v3} • 0.2350
	1 0	1 1	P_{v3} • 0.3874
P_{v4}	1 1	0 0	P_{v4} • 0.0723
	1 1	0 1	P_{v4} • 0.1966
	1 1	1 0	P_{v4} • 0.1966
	1 1	1 1	P_{v4} • 0.5344

So, based on stochastic updating…

- What is the probability of v_2?
- Let's look at the 5th row of the preceding slide.

P_{v2} • 0.142536957 = 0.031390208

From the spreadsheet
For the probability of the
configuration [0,1], [0, 0]

Solving for P_{v2} = 0.220225047!

As expected, stochastic updating is consistent with the energy/ probability functions defined earlier ---- that was the basis of stochastic updating afterall!

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Let's Increase the Probability that v = [0, 1] occurs

$$\Delta w_{ij} = \eta\left(\langle v_i \cdot h_j \rangle_{\mathbf{v}} - \langle v_i \cdot h_j \rangle_{\mathbf{vh}}\right)$$

	v1	v2	h1	h2	E	e^-E	Probability
5	0	1	0	0	0	1	0.031390208
6	0	1	0	1	-0.5	1.648721271	0.051753703
7	0	1	1	0	-0.5	1.648721271	0.051753703
8	0	1	1	1	-1	2.718281828	0.085327431

Recall that the first term is ... $= \sum_g p(\mathbf{h}^g \mid \mathbf{v}) v_i h_j^g = \langle v_i \cdot h_j \rangle_{\mathbf{v}}$

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Let's Do Some Calculations

$$\langle v_i \cdot h_j \rangle_{\mathbf{v}} = \sum_g p(\mathbf{h}^g | \mathbf{v}) v_i h_j^g = \sum_g \left(\frac{p(\mathbf{h}^g, \mathbf{v})}{p(\mathbf{v})} \right) v_i h_j^g$$

5	0	1	0	0	0	1	0.031390208
6	0	1	0	1	-0.5	1.648721271	0.051753703
7	0	1	1	0	-0.5	1.648721271	0.051753703
8	0	1	1	1	-1	2.718281828	0.085327431

So for Δw_{11}, this term is:

$$\langle v_1 \cdot h_1 \rangle_{\mathbf{v}} = \left(\frac{0.0313}{0.2202} \right) \bullet 0 \bullet 0 + \left(\frac{0.0517}{0.2202} \right) \bullet 0 \bullet 0 + \left(\frac{0.0517}{0.2202} \right) \bullet 0 \bullet 1 + \left(\frac{0.0853}{0.2202} \right) \bullet 0 \bullet 1 = 0$$

Now for the Second Term

Recall that $\sum_{u,g} p(\mathbf{v}^u, \mathbf{h}^g) v_i^u h_j^g = \langle v_i \cdot h_j \rangle_{\mathbf{vh}}$

	v1	v2	h1	h2	E	e^{-E}	Probability
1	0	0	0	0	0	1	0.031390208
2	0	0	0	1	0	1	0.031390208
3	0	0	1	0	0	1	0.031390208
4	0	0	1	1	0	1	0.031390208
5	0	1	0	0	0	1	0.031390208
6	0	1	0	1	-0.5	1.648721271	0.051753703
7	0	1	1	0	-0.5	1.648721271	0.051753703
8	0	1	1	1	-1	2.718281828	0.085327431
9	1	0	0	0	0	1	0.031390208
10	1	0	0	1	-0.5	1.648721271	0.051753703
11	1	0	1	0	-0.5	1.648721271	0.051753703
12	1	0	1	1	-1	2.718281828	0.085327431
13	1	1	0	0	0	1	0.031390208
14	1	1	0	1	-1	2.718281828	0.085327431
15	1	1	1	0	-1	2.718281828	0.085327431
16	1	1	1	1	-2	7.389056099	0.231944006
						31.8570685	1

The Second Term for v_1 h_1

So, $\sum_{u,g} p(\mathbf{v}^u, \mathbf{h}^g) v_1^u h_1^g = \langle v_1 \cdot h_1 \rangle_{\mathbf{vh}} = 0.45435257$

$$\Delta w_{11} = \eta\left(\langle v_1 \cdot h_1 \rangle_{\mathbf{v}} - \langle v_1 \cdot h_1 \rangle_{\mathbf{vh}}\right)$$
$$= 0.1\left(0 - 0.45435257\right)$$
$$= -0.045435257$$

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Doing the Same Calculations for all the other weights, we get …

$$\langle v_1 \cdot h_1 \rangle_{\mathrm{v}} = \left(\frac{0.0313}{0.2202}\right) \bullet 0 \bullet 0 + \left(\frac{0.0517}{0.2202}\right) \bullet 0 \bullet 0 + \left(\frac{0.0517}{0.2202}\right) \bullet 0 \bullet 0 + \left(\frac{0.0853}{0.2202}\right) \bullet 0 \bullet 0 = 0$$

$$\langle v_1 \cdot h_2 \rangle_{\mathrm{v}} = \left(\frac{0.0313}{0.2202}\right) \bullet 0 \bullet 0 + \left(\frac{0.0517}{0.2202}\right) \bullet 0 \bullet 1 + \left(\frac{0.0517}{0.2202}\right) \bullet 0 \bullet 0 + \left(\frac{0.0853}{0.2202}\right) \bullet 0 \bullet 1 = 0$$

$$\langle v_2 \cdot h_1 \rangle_{\mathrm{v}} = \left(\frac{0.0313}{0.2202}\right) \bullet 1 \bullet 0 + \left(\frac{0.0517}{0.2202}\right) \bullet 1 \bullet 0 + \left(\frac{0.0517}{0.2202}\right) \bullet 1 \bullet 1 + \left(\frac{0.0853}{0.2202}\right) \bullet 1 \bullet 1 = 0.622459331$$

$$\langle v_2 \cdot h_2 \rangle_{\mathrm{v}} = \left(\frac{0.0313}{0.2202}\right) \bullet 1 \bullet 0 + \left(\frac{0.0517}{0.2202}\right) \bullet 1 \bullet 1 + \left(\frac{0.0517}{0.2202}\right) \bullet 1 \bullet 0 + \left(\frac{0.0853}{0.2202}\right) \bullet 1 \bullet 1 = 0.622459331$$

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Now for all the Second Terms

$$\langle v_1 \cdot h_1 \rangle_{\mathbf{vh}} = 0.0517 + 0.0853 + 0.0853 + .2319 = 0.454352573$$
$$\langle v_1 \cdot h_2 \rangle_{\mathbf{vh}} = 0.0517 + 0.0853 + 0.0853 + .2319 = 0.454352573$$
$$\langle v_2 \cdot h_1 \rangle_{\mathbf{vh}} = 0.0517 + 0.0853 + 0.0853 + .2319 = 0.454352573$$
$$\langle v_2 \cdot h_2 \rangle_{\mathbf{vh}} = 0.0517 + 0.0853 + 0.0853 + .2319 = 0.454352573$$

Updated Weights

$$\Delta w_{11} = \eta\left(\langle v_1 \cdot h_1 \rangle_{\mathbf{v}} - \langle v_1 \cdot h_1 \rangle_{\mathbf{vh}}\right) = 0.1\left(0 - 0.45435257\right) = -0.045435257$$
$$\Delta w_{12} = 0.1\left(0 - 0.45435257\right) = -0.045435257$$
$$\Delta w_{21} = 0.1\left(0.622459331 - 0.45435257\right) = 0.016810676$$
$$\Delta w_{22} = 0.1\left(0.622459331 - 0.45435257\right) = 0.016810676$$

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The Updated Configurations

	v1	v2	h1	h2	E	e^-E	Probability
1	0	0	0	0	0	1	0.032197018
2	0	0	0	1	0	1	0.032197018
3	0	0	1	0	0	1	0.032197018
4	0	0	1	1	0	1	0.032197018
5	0	1	0	0	0	1	0.032197018
6	0	1	0	1	-0.516810676	1.676671664	0.053983827
7	0	1	1	0	-0.516810676	1.676671664	0.053983827
8	0	1	1	1	-1.033621352	2.811227869	0.090513154
9	1	0	0	0	0	1	0.032197018
10	1	0	0	1	-0.454564743	1.575487492	0.050725999
11	1	0	1	0	-0.454564743	1.575487492	0.050725999
12	1	0	1	1	-0.909129486	2.482160837	0.079918176
13	1	1	0	0	0	1	0.032197018
14	1	1	0	1	-0.971375419	2.641575235	0.085050845
15	1	1	1	0	-0.971375419	2.641575235	0.085050845
16	1	1	1	1	-1.942750838	6.97791972	0.224668205
						31.05877721	1

So now the total probability Pr{v=[0,1]} = 0.230677826

If eta = 1, the probability goes up to 0.332961042!

Recall, it was 0.22022505

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Hinton's Approximation in vector form

1. Take a training sample v, compute the probabilities of the hidden units and sample a hidden activation vector h from this probability distribution.
2. Compute the outer product of **v** and **h** and call this the positive gradient.
3. From **h**, sample a reconstruction **v'** of the visible units, then resample the hidden activations **h'** from this. (Gibbs sampling step)
4. Compute the outer product of **v'** and **h'** and call this the negative gradient.
5. Let the update to the weight matrix *W* be the positive gradient minus the negative gradient, times some learning rate: $\Delta W = \epsilon (\mathbf{v}\mathbf{h}^T - \mathbf{v}'\mathbf{h}'^T)$.
6. Update the biases *a* and *b* analogously: $\Delta a = \epsilon(\mathbf{v} - \mathbf{v}')$, $\Delta b = \epsilon (\mathbf{h} - \mathbf{h}')$.

From Wikipedia.

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Summary

- Showed the derivative of the log probability with respect to weights
- This can serve as the basis of a gradient ascent method for increasing the probability of the reconstructed vector v.