A SIMPLE PROOF OF THE CHAIN RULE

1 Let's Start at the Very Beginning

Let's look at the basic definition of the derivative where we'll emphasize certain, specific patterns that will help us navigate through a more complicated-looking derivative. First, the derivative a function $g(x)$ should be quite familiar:

$$\frac{dg}{dx} = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \tag{1}$$

which tells us what the relative change is of the function value g compared to changes in the value of x. In other words, we want to know how fast g changes if we slightly change the value of its argument by a small amount h. Notice that the increment h is thus *added to the argument of the function* which is why we refer to this as the derivative of g *with respect to* x — because x is the variable we are *perturbing* with h. The value of h simply represents the difference between $x + h$ and x —the "run" and $g(x+h)-g(x)$ is the "rise" and their ratio corresponds to the slope and in the limit corresponds to the derivative. When we add the increment h directly to the argument of the function, as is the case here, we can write the derivative as $dg(x)/dx$ or simply dg/dx — again, the derivative of g *with respect to* x which tells us the proportional change in g when we change x by adding h to x. Also notice that the increment h in the numerator of (1) is the very same increment that's in the denominator of (1). This is necessary so that we can correctly compute the rise over the run. If the h in the numerator was different than the one in the denominator, we wouldn't actually be calculating the slope would we.

2 What About a Function of a Function?

Now let's look at a compound function: $f(g(x))$. In this case, as before, the argument or variable of the function g is x, but the argument or variable of the function f is g! So changing x changes g and changes in g change the value of f. So if we want to define the derivative of f *with respect to* x, we have to add the increment h to the variable x and so by the definition of the derivative and using the same general form as in (1),

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \tag{2}$$

So this is the derivative of f with respect to x based on the definition of a derivative. The problem with (2) however is that it's hard to actually use and work with. This is due to several issues. For one thing, the argument of the function f is the value of g and *not* the value x. Remember, that in (1) we added the increment h directly to the argument of that function. In this case, we're not exactly doing that. Instead of adding the increment h to the argument of f, we're adding h to the argument of g!

It might be helpful therefore if we could somehow modify the numerator of (2) so that the increment h is added to the argument of f as this is the function we're taking the derivative of and since the argument of f is g, we want to add the increment h to g and not to x. Thus, we need to somehow change the expression thusly:

$$f(g(x+h)) \longrightarrow f(g(x)+h)$$

or, ignoring the argument x since it is the argument of g (think of g as the argument or variable for f), we want to somehow establish a conversion

$$f(g(x+h)) \longrightarrow f(g+h)$$

in the numerator of (2). In this way, we would be adding the increment h directly to the argument of f (which is g) and not to the argument of the function g (which is x)! Do you see the difference? So the question is: is there a way to change $f(g(x+h))$ to look something more like $f(g(x)+h)$ or $f(g+h)$ in a legitimate way? To do this requires that we find some way of relating or converting $g(x+h)$ to $g(x)+h$ or something similar to it. Do you see a way to connect these two expressions? (*hint:* look at equation (1)).

From (1) we see quite directly (after doing just a little algebra) that

$$g(x+h) = g(x) + h\frac{dg}{dx} \tag{3}$$

Now we're getting somewhere. Note also that in (3) as $h \to 0$ so does hdg/dx which means the increment goes to 0 just like it's supposed to in the definition of a derivative like in (1). So let's simplify some notation and write $h\frac{dg}{dx}$ as $\hat{h}$ so that (3) looks like

$$g(x+h) = g(x) + \hat{h} \tag{4}$$

which looks much cleaner and simpler. Now, we can substitute (4) for $g(x+h)$ in equation (2) which now looks like

$$\frac{df}{dx} = \lim_{h\to 0} \frac{f(g(x)+\hat{h}) - f(g(x))}{h} \tag{5}$$

or, ignoring the argument x of g to emphasize that it is g that is the argument of f, (5) can be rewritten as

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(g + \hat{h}) - f(g)}{h} \tag{6}$$

Now we see very clearly that g is the variable or argument of f and that we've added a type of increment $\hat{h}$ directly to the argument of f just like in (1). Thus, the right-hand side of (6) is beginning to look a lot like df/dg!

The only problem in (6) is that the increment $\hat{h}$ in the numerator of (6) is not the same as the increment h in the denominator of (6). These two increments must be the same for there to be a proper evaluation of the slope and to correspond to the definition of a derivative as in (1). How can we get these two increments to be the same?

Well, if we multiply the the quantity in (6) by $h/\hat{h}$ then the h's would cancel as in

$$\frac{df}{dx} = \lim_{h \to 0} \left[\frac{f(g + \hat{h}) - f(g)}{\cancel{h}} \right] \cdot \frac{\cancel{h}}{\hat{h}}$$

and we'd end up with an $\hat{h}$ in both the numerator and denominator and that really would be df/dg! But we can't just willy-nilly multiply the the right-hand side of (6) by $h/\hat{h}$ just to make it look like we want it to look — doing that could change its value. The only multiplication that we can do to (6) *without changing its value* is to multiply (somehow) by a factor that is *always* 1! So instead of just multiplying (6) by $h/\hat{h}$, we're going to multiply by

$$\frac{h}{\hat{h}} \cdot \frac{\hat{h}}{h}$$

which is just another way of writing 1. Thus, (6) becomes

$$\frac{df}{dx} = \lim_{h \to 0} \left[\frac{f(g + \hat{h}) - f(g)}{h} \right] \cdot \frac{h}{\hat{h}} \cdot \frac{\hat{h}}{h} \tag{7}$$

and the value is still the same. Notice that we can now cancel the h's thusly:

$$\frac{df}{dx} = \lim_{h \to 0} \left[\frac{f(g + \hat{h}) - f(g)}{\cancel{h}} \right] \cdot \frac{\cancel{h}}{\hat{h}} \cdot \frac{\hat{h}}{h} \tag{8}$$

and then moving the $\hat{h}$ in the denominator into the denominator of the bracketed quantity we get

$$\frac{df}{dx} = \lim_{h \to 0} \left[\frac{f(g + \hat{h}) - f(g)}{\hat{h}} \right] \cdot \frac{\hat{h}}{h} \tag{9}$$

and now we see that the quantity in brackets is actually $\frac{df}{dg}$ because g is the argument of f to which the increment $\hat{h}$ has been added, the increments in the numerator and denominator of the bracketed quantity are the same, and as $h \to 0$ so does $\hat{h}$. Thus, everything in the brackets corresponds to the definition of a derivative of the function f with respect to g!

So what about the fraction $\hat{h}/h$ on the right of the brackets in (9)? Well, recall from (3) and (4) that $\hat{h} = h\frac{dg}{dx}$ therefore

$$\frac{\hat{h}}{h} = \frac{h(dg/dx)}{h} = \frac{dg}{dx} \tag{10}$$

because the h's cancel and so (9) becomes

$$\frac{df}{dx} = \lim_{h \to 0} \left[\frac{f(g + \hat{h}) - f(g)}{\hat{h}} \right] \cdot \frac{dg}{dx} \tag{11}$$

or

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} \tag{12}$$

3 Final Thoughts

In our quest to understand the chain rule, we first had to use symbols and abstractions to clearly state what we meant by a derivative of some function with respect to some variable. Such a quantity represents the rate of change of a function's value as its argument is changed and we were able to write (1) to symbolize this concept. Then using good 'ole algebra and our knowledge of limits, we manipulated the symbols, all the while adhering to fundamental rules of mathematics to show a *chain* of reasoning and logic to arrive at the simple rule that you will use a lot — the chain rule — a multiplication of rates of change where one variable affects another. Remember, a chain is only as strong as its weakest link, so learn this rule well!

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