Chapter 8

Continuous Systems

8.1 $c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{4000}{5}\right)^{1/2} = 28.2843 \text{ m/s}$

8.2 ρ = mass density x area = $7830 \left\{\frac{\pi}{4}(2 \times 10^{-3})^2\right\} = 24.5987 \times 10^{-3}$ kg/m

$P = 250$ N, $l = 2$ m

(b) $c = (P/\rho)^{1/2} = \left(\dfrac{250}{24.5987 \times 10^{-3}}\right)^{1/2} = 100.8124$ m/s

(a) $\omega_1 = \dfrac{c\pi}{l} = \dfrac{100.8124\ \pi}{2} = 158.3561$ rad/s $= 25.2031$ Hz

---

8.3 $\omega_1 = 3000\,(2\pi)$ rad/sec $= \dfrac{\pi c}{l} = \dfrac{\pi c}{2}$

$c = (6000\,\pi \times 2/\pi) = 12000$ m/s

$\omega_3 = 3\pi c/l = 3\,\omega_1 = 9000$ Hz

$c_{original} = (P/\rho)^{1/2} = 12000$ m/s

$c_{new} = (1.2\,P/\rho)^{1/2} = 1.0954\,(P/\rho)^{1/2} = 1.0954\ c_{original}$

$\therefore$ ω_1 and ω_3 are increased by 9.54%

8.4 $P = 30000$ N, $\rho = 2$ kg/m

$c = \left(\frac{P}{\rho}\right)^{1/2} = (30000/2)^{1/2} = 122.4745$ m/s

Time taken $= \dfrac{300}{122.4745} = 2.4495$ s

8.5 At $x = 0$: $$P\frac{\partial w}{\partial x} = m\frac{\partial^2 w}{\partial t^2} \qquad (E_1)$$

At $x = l$: $$P\frac{\partial w}{\partial x} = -k\,w \qquad (E_2)$$

General solution is

$$w(x,t) = W(x)\cdot T(t) = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) \qquad (E_3)$$

Equations (E_1) and (E_3) give:

$$\left\{P\left(-A\frac{\omega}{c}\sin\frac{\omega x}{c} + B\frac{\omega}{c}\cos\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) = -m\omega^2\left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)\right\}_{x=0}$$

i.e., $$A\,(m\omega^2) + B\left(\frac{P\omega}{c}\right) = 0 \qquad (E_4)$$

Eqs. (E_2) and (E_3) yield

$$P\left(-\frac{\omega}{c} A \sin\frac{\omega l}{c} + B\frac{\omega}{c}\cos\frac{\omega l}{c}\right) = -k\left(A\cos\frac{\omega l}{c} + B\sin\frac{\omega l}{c}\right)$$

i.e.,

$$A\left(-\frac{P\omega}{c}\sin\frac{\omega l}{c} + k\cos\frac{\omega l}{c}\right) + B\left(\frac{P\omega}{c}\cos\frac{\omega l}{c} + k\sin\frac{\omega l}{c}\right) = 0 \quad (E_5)$$

Eqs. (E_4) and (E_5) give the frequency equation:

$$\begin{vmatrix} (m\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c}\sin\frac{\omega l}{c} + k\cos\frac{\omega l}{c}\right) & \left(\frac{P\omega}{c}\cos\frac{\omega l}{c} + k\sin\frac{\omega l}{c}\right) \end{vmatrix} = 0$$

which, upon simplification, becomes

$$\tan\alpha = \left\{\frac{Pk - \left(\frac{P m c^2}{l^2}\right)\alpha^2}{\left(\frac{c^2 m k}{l}\right)\alpha + \left(\frac{P^2 c}{l}\right)\alpha}\right\} \quad (E_6)$$

8.6 $l = 2\,m$, $d = 0.5\,mm$, $\rho = 7800\,kg/m^3$, $\omega_n = \frac{n c \pi}{l}$, $c = \sqrt{\frac{P}{\rho}}$

(a) $\omega_1 = 1(2\pi)$ rad/sec $= \frac{\pi c}{l} = \frac{\pi}{2}\sqrt{\frac{P}{7800}}$

i.e., $(2\pi)^2 = \frac{\pi^2}{4}\left(\frac{P}{7800}\right)$

i.e., $P = 124{,}800$ N

(b) $\omega_1 = 5(2\pi)$ rad/sec $= \frac{\pi c}{l} = \frac{\pi}{2}\sqrt{\frac{P}{7800}}$

i.e., $100\pi^2 = \frac{\pi^2}{4(7800)} P$

i.e., $P = 3{,}120{,}000$ N

8.7 Let $x = 0$ be fixed and $x = l$ be connected to the pin which can move in a frictionless slot.

$$W(x) = A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}$$

$$w(0,t) = 0 \Rightarrow W(0) = 0 \Rightarrow A = 0$$

$$\frac{\partial w}{\partial x}(l,t) = 0 \Rightarrow \frac{dW}{dx}(l) = 0 \Rightarrow B\cos\frac{\omega l}{c} = 0 \Rightarrow \cos\frac{\omega l}{c} = 0$$

$$\therefore \frac{\omega l}{c} = (2n+1)\frac{\pi}{2}\ ; \quad n = 0, 1, 2, \ldots$$

or $$\omega_n = \frac{(2n+1)\pi c}{2l}\ ; \quad n = 0, 1, 2, \ldots$$

8.8 $$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left[C_n\cos\frac{nc\pi t}{l} + D_n\sin\frac{nc\pi t}{l}\right]$$

where

$$C_n = \frac{2}{l}\int_0^l w_0(x)\sin\frac{n\pi x}{l}\,dx, \quad D_n = \frac{2}{\pi c n}\int_0^l \dot{w}_0(x)\sin\frac{n\pi x}{l}\,dx$$

Since $w_0(x) = w(x,0) = 0$, $C_n = 0$

$$D_n = \frac{2}{\pi c n}\left[\int_0^{l/2} \frac{2ax}{l} \sin\frac{n\pi x}{l}\, dx + \int_{l/2}^{l} 2a\left(1-\frac{x}{l}\right) \sin\frac{n\pi x}{l}\, dx\right]$$

$$= \frac{8al}{\pi^3 c n^3} \sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}} \dfrac{8al}{\pi^3 n^3 c} & \text{if } n \text{ is odd} \end{cases}$$

$$w(x,t) = \frac{8al}{\pi^3 c} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3} \sin\frac{n\pi x}{l} \sin\frac{n\pi c t}{l}$$

8.9 Eq. (8.18) is $\quad c^2 \dfrac{d^2W}{dx^2} = a\, W(x)$

Multiply this equation by $W(x)$ and integrate from 0 to l:

$$c^2 \int_0^l W(x) \frac{d^2W(x)}{dx^2}\, dx = a \int_0^l [W(x)]^2\, dx$$

This shows that the sign of a will be same as that of the integral on the left side. Integration by parts gives

$$I = \int_0^l W(x) \cdot \frac{d^2W(x)}{dx^2}\, dx = W(l) \cdot \frac{dW}{dx}(l) - W(0)\frac{dW}{dx}(0) - \int_0^l \left[\frac{dW}{dx}(x)\right]^2 dx$$

Since the first two terms on the right side of this equation are zero for common boundary conditions, I and hence a will be negative.

Common boundary conditions (examples):

Fixed at ends: $\quad W(0) = W(l) = 0$

Free at ends: $\quad \dfrac{dW}{dx}(0) = \dfrac{dW}{dx}(l) = 0$

One end fixed and other end free: $\quad W(0) = \dfrac{dW}{dx}(l) = 0$

one end fixed and other end connected to a spring:

$$W(0) = 0; \quad \frac{dW}{dx}(l) = -k\, W(l)$$

8.10 $l = 2000$ m, $\rho = 8890$ kg/m^3, $\quad 0 \le \omega_1$ to $\omega_4 \le 20$ Hz

$$\omega_n = \frac{n c \pi}{l} = \frac{n\pi}{l}\sqrt{\frac{P}{\rho}}$$

For $\omega_4 \le 20(2\pi)$ rad/sec, $\quad \dfrac{4\pi}{l}\sqrt{\dfrac{P}{\rho}} \le 40\pi$

i.e., $\quad \sqrt{\dfrac{P}{8890}} \le \dfrac{40\pi(2000)}{4\pi}$

i.e., $\quad P \le 35\,560 \times 10^8$ N

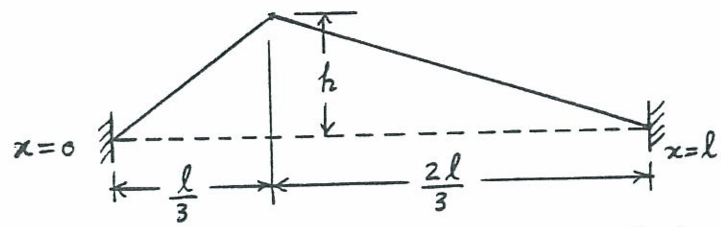
Let the permissible (yield) stress of the material be 300 MPa $= 3 \times 10^8$ N/m^2.

Assuming the diameter of cable as 0.1 m, area of cross-section is $\frac{\pi}{4}(0.1)^2 = 0.007854\ m^2$, and the permissible tension (with factor of safety of one) is

$$(0.007854)(3 \times 10^8) = 2,356,200 \text{ N}$$

$\therefore$ Initial tension $= 2.3562 \times 10^6$ N

8.11



Solution is given by Eq. (8.30):

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} \cos\frac{nc\pi t}{l} \qquad (E_1)$$

where

$$C_n = \frac{2}{l}\int_0^l w_0(x) \sin\frac{n\pi x}{l}\, dx \qquad (E_2)$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} 3hx/l & \text{for } 0 \le x \le l/3 \\ 3h(l-x)/2l & \text{for } l/3 \le x \le l \end{cases} \qquad (E_3)$$

Substitution of Eq. (E_3) into (E_2) gives

$$C_n = \frac{2}{l}\left\{\int_0^{l/3} \frac{3hx}{l} \sin\frac{n\pi x}{l}\, dx + \int_{l/3}^{l} \frac{3h(l-x)}{2l} \sin\frac{n\pi x}{l}\, dx\right\}$$

$$= \frac{6h}{l^2}\left\{\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - \frac{xl}{n\pi}\cos\frac{n\pi x}{l}\right\}_0^{l/3} - \frac{3h}{l}\left(\frac{l}{n\pi}\right)\left(\cos\frac{n\pi x}{l}\right)_{l/3}^{l}$$

$$- \frac{3h}{l^2}\left\{\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - \frac{xl}{n\pi}\cos\frac{n\pi x}{l}\right\}_{l/3}^{l}$$

$$= \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$= \frac{9h\sqrt{3}}{2\pi^2} \text{ for } n=1, \quad \frac{9h\sqrt{3}}{8\pi^2} \text{ for } n=2, \quad 0 \text{ for } n=3,$$

$$- \frac{9h\sqrt{3}}{32\pi^2} \text{ for } n=4, \quad -\frac{9h\sqrt{3}}{50\pi^2} \text{ for } n=5, \quad 0 \text{ for } n=6, \ldots$$

$$w(x,t) = \frac{9\sqrt{3}}{2\pi^2} h \sin\frac{\pi x}{l}\cos\frac{c\pi t}{l} + \frac{9\sqrt{3}}{8\pi^2} h \sin\frac{2\pi x}{l}\cos\frac{2c\pi t}{l}$$

$$- \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{l}\cos\frac{4c\pi t}{l} - \cdots$$

At $t=0$:

$$w(x,0) = \left\{ \frac{9\sqrt{3}}{2\pi^2} h \sin\frac{\pi x}{\ell} + \frac{9\sqrt{3}}{8\pi^2} h \sin\frac{2\pi x}{\ell} - \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{\ell} - \frac{9\sqrt{3}}{50\pi^2} h \sin\frac{5\pi x}{\ell} \right\}$$

At $t = \frac{\ell}{4c}$:

$$w\left(x, \frac{\ell}{4c}\right) = \frac{9\sqrt{3}}{2\pi^2} \frac{h}{\sqrt{2}} \sin\frac{\pi x}{\ell} - \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{\ell} + \frac{9\sqrt{3}}{50\pi^2} \frac{h}{\sqrt{2}} \sin\frac{5\pi x}{\ell}$$

At $t = \frac{\ell}{3c}$:

$$w\left(x, \frac{\ell}{3c}\right) = \frac{9\sqrt{3}\, h}{4\pi^2}\left\{ \sin\frac{\pi x}{\ell} - \frac{1}{4}\sin\frac{2\pi x}{\ell} + \frac{1}{16}\sin\frac{4\pi x}{\ell} - \frac{1}{25}\sin\frac{5\pi x}{\ell} \right\}$$

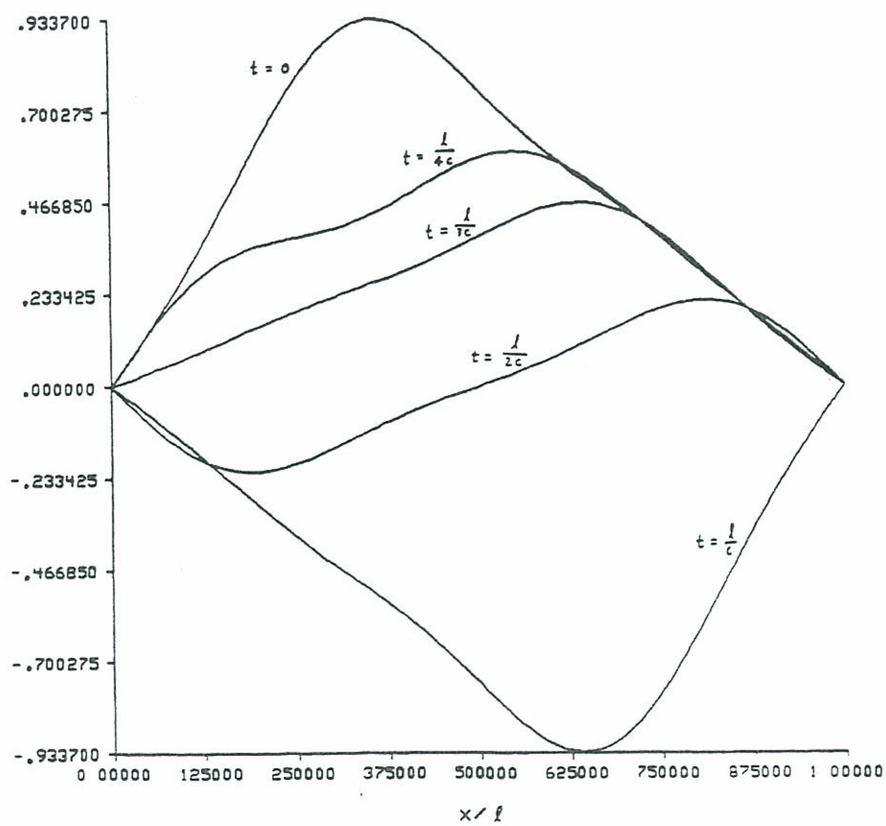
At $t = \frac{\ell}{2c}$:

$$w\left(x, \frac{\ell}{2c}\right) = -\frac{9\sqrt{3}\, h}{8\pi^2}\left\{ \sin\frac{2\pi x}{\ell} + \frac{1}{4}\sin\frac{4\pi x}{\ell} \right\}$$

At $t = \frac{\ell}{c}$:

$$w\left(x, \frac{\ell}{c}\right) = \frac{9\sqrt{3}\, h}{2\pi^2}\left\{ -\sin\frac{\pi x}{\ell} + \frac{1}{4}\sin\frac{2\pi x}{\ell} - \frac{1}{16}\sin\frac{4\pi x}{\ell} + \frac{1}{25}\sin\frac{5\pi x}{\ell} \right\}$$

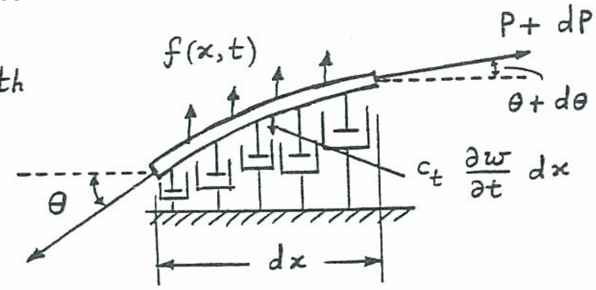
These deflection shapes are shown below:



8.12

c_t = viscous damping coefficient

= force / unit velocity / unit length



$$(P + dP)\sin(\theta + d\theta) + f(x,t)\,dx - c_t \frac{\partial w}{\partial t} dx - P \sin\theta = \rho\, dx\, \frac{\partial^2 w}{\partial t^2} \qquad \ldots (E_1)$$

with $dP = \frac{\partial P}{\partial x} dx$ and $\sin(\theta + d\theta) \approx \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx$

Eq. (E_1) can be rewritten as

$$\frac{\partial}{\partial x}\left(P \frac{\partial w}{\partial x}\right) + f(x,t) = \rho(x)\frac{\partial^2 w(x,t)}{\partial t^2} + c_t \frac{\partial w(x,t)}{\partial t}$$

8.13

Free vibration solution is given by

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left\{C_n \cos\frac{nc\pi t}{l} + D_n \sin\frac{nc\pi t}{l}\right\}$$

with $C_n = \frac{2}{l}\int_0^l w_0(x)\sin\frac{n\pi x}{l}\,dx$

and $D_n = \frac{2}{nc\pi}\int_0^l \dot{w}_0(x)\sin\frac{n\pi x}{l}\,dx$

Since $\dot{w}_0(x) = \frac{\partial w}{\partial t}(x, t=0) = 0$, $D_n = 0$.

For $w_0(x) = w_0 \sin\frac{\pi x}{l}$, $\quad C_n = \frac{2 w_0}{l}\int_0^l \sin\frac{\pi x}{l}\sin\frac{n\pi x}{l}\,dx$

Using the relations

$$\int \sin m_1 x \sin n_1 x\, dx = \frac{\sin(m_1 - n_1)x}{2(m_1 - n_1)} - \frac{\sin(m_1+n_1)x}{2(m_1+n_1)}\,; \quad m_1 \neq n_1$$

$$\int \sin^2 m_1 x\, dx = \frac{x}{2} - \frac{1}{4m_1}\sin 2m_1 x\,,$$

we get for $n = 1$,

$$C_1 = \frac{2w_0}{l}\int_0^l \sin^2\frac{\pi x}{l}\,dx = w_0$$

and for $n = 2, 3, \ldots$,

$$C_n = \frac{2w_0}{l}\left\{\frac{\sin\frac{\pi}{l}(n-1)x}{2\frac{\pi}{l}(n-1)} - \frac{\sin\frac{\pi}{l}(n+1)x}{2\frac{\pi}{l}(n+1)}\right\}_0^l = 0$$

$$\therefore\ w(x,t) = w_0 \sin\frac{\pi x}{l}\cos\frac{c\pi t}{l}$$

8.16

$$u(x,t) = \left(\underset{\sim}{A} \cos \frac{\omega x}{c} + \underset{\sim}{B} \sin \frac{\omega x}{c} \right)\left(C \cos \omega t + D \sin \omega t \right)$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{c}\left(- \underset{\sim}{A} \sin \frac{\omega x}{c} + \underset{\sim}{B} \cos \frac{\omega x}{c} \right)\left(C \cos \omega t + D \sin \omega t \right)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow \underset{\sim}{B} = 0$$

$$\frac{\partial u}{\partial x}(l,t) = 0 \Rightarrow -\frac{\omega}{c} \underset{\sim}{A} \sin \frac{\omega l}{c} \left(C \cos \omega t + D \sin \omega t \right) = 0$$

$$\Rightarrow \sin \frac{\omega l}{c} = 0 \quad ; \quad \frac{\omega_n l}{c} = n\pi$$

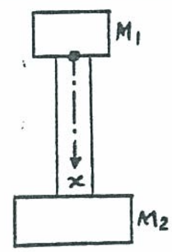
$$\omega_n = \frac{n \pi c}{l} = \frac{n\pi}{l}\sqrt{\frac{E}{\rho}}$$

$$u(x,t) = \sum_{n=1}^{\infty} \cos \frac{n\pi x}{l} \left[C_n \cos \frac{n\pi c t}{l} + D_n \sin \frac{n \pi c t}{l} \right]$$

where $C_n = \frac{2}{l} \int_0^l u_0(x) \cos \frac{n\pi x}{l} \, dx$

and $D_n = \frac{2}{n\pi c} \int_0^l \dot{u}_0(x) \cos \frac{n \pi x}{l} \, dx$

8.17



(a) $u(x,t) = \left(\underset{\sim}{A} \cos \frac{\omega x}{c} + \underset{\sim}{B} \sin \frac{\omega x}{c} \right)\left(C \cos \omega t + D \sin \omega t \right)$

At $x=0$: $M_1 \frac{\partial^2 u}{\partial t^2} = AE \frac{\partial u}{\partial x}$

$$-\omega^2 M_1 \underset{\sim}{A} = AE \frac{\omega}{c} \underset{\sim}{B} \Rightarrow \underset{\sim}{B} = -\left(\frac{\omega M_1 c}{AE} \right) \underset{\sim}{A}$$

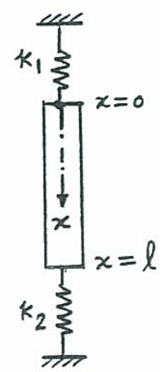
At $x = l$: $M_2 \frac{\partial^2 u}{\partial t^2} = -AE \frac{\partial u}{\partial x}$

$$-\omega^2 M_2 \left(\underset{\sim}{A} \cos \frac{\omega l}{c} + \underset{\sim}{B} \sin \frac{\omega l}{c} \right) = -AE \frac{\omega}{c}\left(-\underset{\sim}{A} \sin \frac{\omega l}{c} + \underset{\sim}{B} \cos \frac{\omega l}{c} \right)$$

i.e. $\underset{\sim}{A}\left(-\omega^2 M_2 \cos \frac{\omega l}{c} - AE \frac{\omega}{c} \sin \frac{\omega l}{c} \right) = \underset{\sim}{B}\left(-\frac{AE\omega}{c} \cos \frac{\omega l}{c} + \omega^2 M_2 \sin \frac{\omega l}{c} \right)$

i.e. $\omega^2 M_2 \cos \frac{\omega l}{c} + \frac{AE\omega}{c} \sin \frac{\omega l}{c} + \frac{\omega M_1 c}{AE}\left(-\omega^2 M_2 \sin \frac{\omega l}{c} + \frac{AE\omega}{c} \cos \frac{\omega l}{c} \right) = 0$

This is the frequency equation.



(b) $u(x,t) = \left(\underset{\sim}{A} \cos \frac{\omega x}{c} + \underset{\sim}{B} \sin \frac{\omega x}{c} \right)\left(C \cos \omega t + D \sin \omega t \right)$ --- (E_1)

At $x=0$, $k_1 u = AE \frac{\partial u}{\partial x} \Rightarrow \underset{\sim}{B} = \frac{k_1 c}{AE\omega} \underset{\sim}{A}$ --- (E_2)

At $x = l$, $k_2 u = -AE \frac{\partial u}{\partial x}$

$$\Rightarrow k_2 \left(\underset{\sim}{A} \cos \frac{\omega l}{c} + \underset{\sim}{B} \sin \frac{\omega l}{c} \right) = -AE \frac{\omega}{c}\left\{ -\underset{\sim}{A} \sin \frac{\omega l}{c} + \underset{\sim}{B} \cos \frac{\omega l}{c} \right\}$$ --- (E_3)

Substituting (E_2) into (E_3), we get

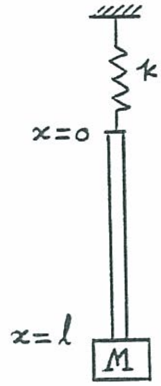
$$\underset{\sim}{A}\left[\left(k_2 - \frac{k_1 c}{AE\omega} \right) \cos \frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c} \right) \sin \frac{\omega l}{c} \right] = 0$$

Hence the frequency equation is given by

$$\left(k_2 - \frac{k_1 c}{AE\omega}\right)\cos\frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega l}{c} = 0$$

i.e., $$\tan\frac{\omega l}{c} = \left(\frac{k_1 c^2 - k_2 AE\omega c}{k_1 k_2 c^2 - A^2E^2\omega^2}\right)$$

(c) $$u(x,t) = \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) \quad \text{--- } (E_1)$$



At $x=0$, $\quad k\,u = AE\dfrac{\partial u}{\partial x} \;\Rightarrow\; \underset{\sim}{B} = \dfrac{kc}{AE\omega}\underset{\sim}{A} \quad \text{--- } (E_2)$

At $x=l$, $\quad AE\dfrac{\partial u}{\partial x} = -M\dfrac{\partial^2 u}{\partial t^2}$

$$\Rightarrow AE\left(-\underset{\sim}{A}\frac{\omega}{c}\sin\frac{\omega l}{c} + \underset{\sim}{B}\frac{\omega}{c}\cos\frac{\omega l}{c}\right) = M\left(\underset{\sim}{A}\cos\frac{\omega l}{c} + \underset{\sim}{B}\sin\frac{\omega l}{c}\right)\omega^2 \quad \text{--- } (E_3)$$

Substituting (E_2) into (E_3), we get

$$\frac{AE\omega}{c}\underset{\sim}{A}\left(-\sin\frac{\omega l}{c} + \frac{kc}{AE\omega}\cos\frac{\omega l}{c}\right) = M\omega^2\underset{\sim}{A}\left(\cos\frac{\omega l}{c} + \frac{kc}{AE\omega}\sin\frac{\omega l}{c}\right)$$

This gives the frequency equation

$$\tan\frac{\omega l}{c} = \left\{\frac{AE\omega c\,(k - M\omega^2)}{A^2E^2\omega^2 - M\omega^2 k c^2}\right\}$$

8.18 For a clamped-free bar in longitudinal vibration, fundamental frequency $= \dfrac{\pi c}{2l}$

If the bar is fixed at $x=0$ and attached to a mass M at $x=l$, its fundamental frequency $= \dfrac{\alpha_1 c}{l}$.

Here $\dfrac{\alpha_1 c}{l} = \dfrac{1}{2}\left(\dfrac{\pi c}{2l}\right) \;\Rightarrow\; \alpha_1 = \dfrac{\pi}{4} = 0.7854$

As an approximation, use linear relationship for the values in Table 8.1. When $\beta = 0.1$, $\alpha_1 = 0.3113$

when $\beta = 1.0$, $\alpha_1 = 0.8602$

This gives $\alpha_1(\beta) = a + b\beta \equiv 0.6099\,\beta + 0.2503$

For $\alpha_1 = 0.7854$, we get $\beta = \dfrac{0.7854 - 0.2503}{0.6099} = 0.8774 = \dfrac{m}{M}$

$\therefore$ Mass to be attached $= M \simeq \dfrac{m}{0.8774} = 1.1397\,m$.

8.19 Equation of motion for free vibration $\quad c^2\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} \quad (E_1)$

Assume separation of variables in Eq. (E_1): $\quad u(x,t) = U(x)\cdot T(t)$

so that $\quad c^2\dfrac{d^2U}{dx^2}\,T = U\dfrac{d^2T}{dt^2} \quad$ or $\quad c^2\dfrac{1}{U}\dfrac{d^2U}{dx^2} = \dfrac{1}{T}\dfrac{d^2T}{dt^2} = a = -\omega^2 \quad \text{--- } (E_2)$

i.e., $$c^2 \frac{d^2U(x)}{dx^2} + \omega^2 U(x) = 0 \tag{E3}$$

and $$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0 \tag{E4}$$

To show that the constant a on the right hand side of Eq. (E_2) is a negative quantity, multiply Eq. (E_3) by $U(x)$ and integrate w.r.t. x from 0 to l:

$$c^2 \int_0^l U(x) \frac{d^2U}{dx^2} dx = a \underbrace{\int_0^l \{U(x)\}^2 dx}_{\text{positive always}} \tag{E5}$$

Eq. (E_5) shows that a will have the same sign as the integral on the left side:

$$I = \int_0^l U(x) \frac{d^2U(x)}{dx^2} dx \tag{E6}$$

Integrate (E_6) by parts to get

$$I = U(l) \cdot \frac{dU(l)}{dx} - U(0) \cdot \frac{dU(0)}{dx} - \int_0^l \left\{ \frac{dU(x)}{dx} \right\}^2 dx \tag{E7}$$

Since $U(0) = 0$, the second term on the r.h.s. of (E_7) will be zero. Also,

$$AE \frac{dU(l)}{dx} = -k\, U(l) \tag{E8}$$

at $x = l$. Multiplication of (E_8) by $U(l)$ gives

$$U(l) \cdot \frac{dU}{dx}(l) = -k \{U(l)\}^2 < 0$$

This shows that $I < 0$ and hence $a < 0$.

In Eqs. (E_3) and (E_4), there exists an eigen (normal) function for each frequency (constant) ω. Let $U_m(x)$ and $U_n(x)$ denote the normal functions corresponding to the frequencies ω_m and ω_n, respectively. Eq. (E_3) gives

$$c^2 \frac{d^2U_m}{dx^2} + \omega_m^2 U_m = 0 \quad \text{--- } (E_9) \; ; \quad c^2 \frac{d^2U_n}{dx^2} + \omega_n^2 U_n = 0 \quad \text{--- } (E_{10})$$

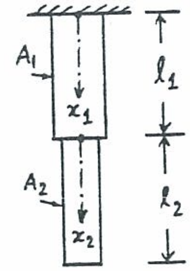
Multiply (E_9) by U_n, (E_{10}) by U_m and subtract the resulting equations one from the other to get

$$c^2 U_n \frac{d^2U_m}{dx^2} - c^2 U_m \frac{d^2U_n}{dx^2} + \left(\omega_m^2 - \omega_n^2\right) U_m U_n = 0 \tag{E11}$$

Integrate (E_{11}) with respect to x from 0 to l to get, after integration by parts, the desired orthogonality relation:

$$\int_0^l U_m(x)\, U_n(x)\, dx = 0 \quad , \quad \omega_m \neq \omega_n \tag{E12}$$

8.20 Set up two coordinates x_1 and x_2 as shown.



$$u_1(x_1,t) = \left(\underset{\sim}{A_1} \cos\frac{\omega x_1}{c_1} + \underset{\sim}{B_1} \sin\frac{\omega x_1}{c_1}\right)(C\cos\omega t + D\sin\omega t)$$

$$u_2(x_2,t) = \left(\underset{\sim}{A_2} \cos\frac{\omega x_2}{c_2} + \underset{\sim}{B_2} \sin\frac{\omega x_2}{c_2}\right)(C\cos\omega t + D\sin\omega t)$$

$$u_1(0,t) = 0 \quad\Rightarrow\quad \underset{\sim}{A_1} = 0$$

$$u_1(l_1,t) = u_2(0,t) \quad\Rightarrow\quad \underset{\sim}{B_1}\sin\frac{\omega l_1}{c_1} = \underset{\sim}{A_2}$$

$$A_1E_1\frac{\partial u_1}{\partial x_1}(l_1,t) = A_2E_2\frac{\partial u_2}{\partial x_2}(0,t) = \text{tensile force same in both areas}$$

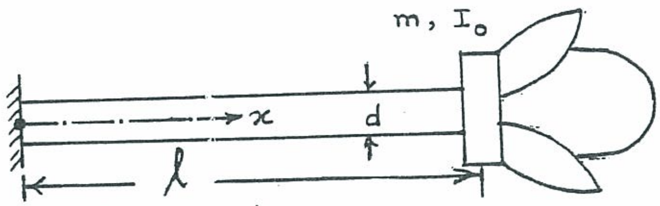
$$\text{i.e.}\quad A_1E_1\underset{\sim}{B_1}\frac{\omega}{c_1}\cos\frac{\omega l_1}{c_1} = A_2E_2\frac{\omega}{c_2}\underset{\sim}{B_2} \quad\Rightarrow\quad \underset{\sim}{B_2} = \frac{A_1E_1c_2}{A_2E_2c_1}\cos\frac{\omega l_1}{c_1}\cdot\underset{\sim}{B_1}$$

$$u_2(x_2,t) = \underset{\sim}{B_1}\left(\sin\frac{\omega l_1}{c_1}\cos\frac{\omega x_2}{c_2} + \frac{A_1E_1c_2}{A_2E_2c_1}\cos\frac{\omega l_1}{c_1}\sin\frac{\omega x_2}{c_2}\right)(C\cos\omega t + D\sin\omega t)$$

$$\frac{\partial u_2}{\partial x_2}(l_2,t) = 0 \quad\Rightarrow\quad \underset{\sim}{B_1}\frac{\omega}{c_2}\left\{-\sin\frac{\omega l_1}{c_1}\cdot\sin\frac{\omega l_2}{c_2} + \frac{A_1E_1c_2}{A_2E_2c_1}\cdot\cos\frac{\omega l_1}{c_1}\cdot\cos\frac{\omega l_2}{c_2}\right\} = 0$$

$\therefore$ Frequency equation is
$$\tan\frac{\omega l_1}{c_1}\cdot\tan\frac{\omega l_2}{c_2} = \frac{A_1E_1c_2}{A_2E_2c_1}$$

8.21 (a) Axial vibration:



$$\beta = \frac{m_0}{m} = \frac{\text{mass of rod}}{\text{end mass}}$$

Using $\rho = 76.5\ \text{kN/m}^3$ for steel, we find

$$m_0 = \frac{\pi d^2 \ell \rho}{4} = \frac{\pi}{4}\left(\frac{5}{100}\right)^2 (1)\left(\frac{76.5\,(10^3)}{9.81}\right) = 15.3117\ \text{kg}$$

$$\frac{m_0}{m} = \frac{15.3117}{100} = 0.1531$$

From Table 8.1, the value of α_1 for $\beta = 0.1531$ (using linear interpolation between values of $\beta = 0.1$ and $\beta = 1.0$) is:

$$\alpha_1 = 0.6099\,(0.1531) + 0.2503 = 0.3437$$

$$\omega_1 = \frac{\alpha_1 c}{\ell} = \frac{\alpha_1}{\ell}\sqrt{\frac{E}{\rho}} = \frac{0.3437}{1}\sqrt{\frac{207\,(10^9)\,(9.81)}{76500}} = 1770.7958\ \text{rad/sec}$$

(b) Torsional vibration:

In this case, we use the result of Example 8.6.

$$\beta = \frac{\tilde{J}_{rod}}{I_0} = \frac{J\rho\ell}{I_0}$$

where J = polar moment of inertia of the shaft:

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05)^4 = 2.4544 \,(10^{-8})\ \mathrm{m}^4$$

$$\rho = 76500/9.81\ \mathrm{kg/m^3} \ ; \ \ell = 1\ \mathrm{m}$$

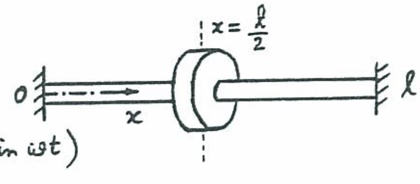
$$\tilde{J}_{rod} = (2.4544\,(10^{-8}))\, \frac{76500}{9.81}\,(1) = 1.9140\,(10^{-4})\ \mathrm{kg{-}m^2}$$

$$\beta = \frac{1.9140\,(10^{-4})}{10} = 1.9140\,(10^{-5})$$

Since β is close to zero, we have from Example 8.6,

$$\omega_1 \approx \frac{c}{\ell}\sqrt{\beta} = \sqrt{\frac{G\,\beta}{\rho}} = \sqrt{\frac{(80\,(10^9))\,(9.81)}{76500}} = 14.0126\ \mathrm{rad/sec}$$

8.22 Consider half the shaft and half the disc.



$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\theta(0,t) = 0 \quad \Rightarrow \quad A = 0$$

$$GJ\, \frac{\partial \theta}{\partial x}\left(\frac{\ell}{2}, t\right) = -\frac{J_0}{2}\, \frac{\partial^2 \theta}{\partial t^2}\left(\frac{\ell}{2}, t\right)$$

$$\Rightarrow \quad GJ \frac{\omega}{c} \cdot B \cos \frac{\omega \ell}{2c} = \frac{J_0\, \omega^2}{2} \cdot B \sin \frac{\omega \ell}{2c}$$

$$\Rightarrow \quad \tan \frac{\omega \ell}{2c} = \frac{2GJ}{J_0\, \omega c} = \frac{2c}{\omega \ell} \cdot \frac{GJ\ell}{J_0 c^2}$$

Frequency equation: $\quad \alpha \tan \alpha = \beta \quad$ where $\alpha = \frac{\omega \ell}{2c}$ and $\beta = \frac{GJ\ell}{J_0 c^2} = \left(\frac{J\rho \ell}{J_0}\right)$

If roots of this equation are given by α_n,

$\omega_n = \frac{2 \alpha_n c}{\ell}$ and

$$\theta(x,t) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \left(C_n \cos \omega_n t + D_n \sin \omega_n t\right) \qquad \text{---- } (E_1)$$

$$\theta(x,0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \,(C_n) = 0 \quad \Rightarrow \quad C_n = 0 \qquad \text{---- } (E_2)$$

$$\dot{\theta}(x,0) = \sum_{n=1}^{\infty} \sin \frac{\omega_n x}{c} \,(\omega_n D_n) = \frac{2\, \dot{\theta}_0\, x}{\ell} \qquad \text{---- } (E_3)$$

as $\dot{\theta}\Big|_{\frac{\ell}{2},\, t=0} = \dot{\theta}_0$ and hence $\dot{\theta}(x,0) = \frac{2\,\dot{\theta}_0\, x}{\ell}$

Multiply Eq. (E_3) by $\sin \frac{\omega_n x}{c}$ and integrate from 0 to $\frac{\pi c}{2\omega_n}$:

$$\omega_n D_n \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} \sin^2 \frac{\omega_n x}{c}\, dx = \frac{2\,\dot{\theta}_0}{\ell} \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} x \sin \frac{\omega_n x}{c}\, dx \qquad (E_4)$$

i.e., $$D_n = \frac{8\, c\, \dot{\theta}_0}{\pi\, \ell\, \omega_n^2} \qquad (E_5)$$

$$\therefore \quad \theta(x,t) = \sum_{n=1}^{\infty} \left(\frac{8\, c\, \theta_0}{\pi \ell\, \omega_n^2} \right) \sin \frac{\omega_n x}{c} \sin \omega_n t \qquad (E_6)$$

8.23

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\theta(0,t) = 0 \Rightarrow A = 0$$

$$\theta(\ell,t) = 0 \Rightarrow \sin \frac{\omega \ell}{c} = 0 \quad ; \quad \frac{\omega_n \ell}{c} = n\pi$$

$$\therefore \omega_n = \frac{n\pi c}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{G}{\rho}} \quad ; \quad n = 1,2,3,\dots$$

8.24

Boundary conditions:

At $x=0$, $\quad GJ \dfrac{\partial \theta}{\partial x}(0,t) = k_{t1}\, \theta(0,t) + c_{t1} \dfrac{\partial \theta}{\partial t}(0,t) + J_1 \dfrac{\partial^2 \theta}{\partial t^2}(0,t)$

At $x=\ell$, $\quad GJ \dfrac{\partial \theta}{\partial x}(\ell,t) = -k_{t2}\, \theta(\ell,t) - c_{t2} \dfrac{\partial \theta}{\partial t}(\ell,t) - J_2 \dfrac{\partial^2 \theta}{\partial t^2}(\ell,t)$

8.25

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$$

$$\theta(0,t) = 0 \Rightarrow A = 0$$

$$\frac{\partial \theta}{\partial x}(\ell,t) = 0 \Rightarrow \cos \frac{\omega \ell}{c} = 0 \quad ; \quad \frac{\omega_n \ell}{c} = \left(n - \tfrac{1}{2}\right)\pi$$

$$\therefore \omega_n = \left(n - \tfrac{1}{2}\right)\pi \sqrt{\frac{G}{\rho \ell^2}} \quad ; \quad n = 1,2,\dots$$

8.26

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right) \quad \text{----} \ (E_1)$$

at $x=0$: $\quad J_1 \dfrac{\partial^2 \theta}{\partial t^2} = GJ \dfrac{\partial \theta}{\partial x} \quad \text{----} \ (E_2)$

at $x=\ell$: $\quad -J_2 \dfrac{\partial^2 \theta}{\partial t^2} = GJ \dfrac{\partial \theta}{\partial x} \quad \text{----} \ (E_3)$

Eqs. (E_1) and (E_2) give $\quad (J_1 \omega^2) A + \left(\dfrac{GJ\omega}{c}\right) B = 0 \quad \text{----} \ (E_4)$

(E_1) and (E_3) give $\quad \left(J_2 \omega^2 \cos \dfrac{\omega \ell}{c} + \dfrac{GJ\omega}{c} \sin \dfrac{\omega \ell}{c}\right) A + \left(J_2 \omega^2 \sin \dfrac{\omega \ell}{c} - \dfrac{GJ\omega}{c} \cos \dfrac{\omega \ell}{c}\right) B = 0 \quad \text{----} \ (E_5)$

Setting the determinant of the coefficients of A and B in (E_4) and (E_5) equal to zero, we get the frequency equation

$$\begin{vmatrix} J_1 \omega^2 & \dfrac{GJ\omega}{c} \\ \left(J_2 \omega^2 \cos \dfrac{\omega \ell}{c} + \dfrac{GJ\omega}{c} \sin \dfrac{\omega \ell}{c}\right) & \left(J_2 \omega^2 \sin \dfrac{\omega \ell}{c} - \dfrac{GJ\omega}{c} \cos \dfrac{\omega \ell}{c}\right) \end{vmatrix} = 0$$

i.e.

$$J_1 J_2 \omega^2 \sin \frac{\omega \ell}{c} - (J_1 + J_2) \frac{GJ\omega}{c} \cos \frac{\omega \ell}{c} - \frac{G^2 J^2}{c^2} \sin \frac{\omega \ell}{c} = 0$$

8.27 Equation of motion: $c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$ ---- (E_1)

Let steady state vibration be $\theta(x,t) = X(x) . \cos \omega t$ ---- (E_2)

(E_1) and (E_2) give $\frac{d^2 X}{dx^2} + \frac{\omega^2}{c^2} X = 0$

Solution is $X(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$

$\theta(x=0, t) = 0 \Rightarrow X(x=0) = 0 \Rightarrow A = 0$

At free end, $GJ \frac{\partial \theta}{\partial x}(x=l, t) = M_{t0} \cos \omega t$

$$B \, GJ \frac{\omega}{c} \cos \frac{\omega l}{c} \cos \omega t = M_{t0} \cos \omega t$$

$$B = \frac{M_{t0} \, c}{GJ \omega} \sec \frac{\omega l}{c}$$

$$\therefore \quad \theta(x,t) = \frac{M_{t0} \, c}{GJ \omega} \sec \frac{\omega l}{c} \sin \frac{\omega x}{c} . \cos \omega t$$

8.28 From solution of problem 8.23, $\omega_n = \frac{n \pi}{l} \sqrt{\frac{G}{\rho}}$

$$\therefore \omega_1 = \frac{\pi}{l} \sqrt{\frac{G}{\rho}} = \frac{\pi}{2} \sqrt{\frac{0.8 \times 10^{11}}{7800}} = 5030.5861 \text{ rad/sec}$$

8.29 Let the angular displacement of the shaft be measured from the position occupied at the instant the shaft is stopped. Then the initial conditions are

$$\theta(x,0) = 0, \quad \frac{\partial \theta}{\partial t}(x,0) = \underset{\sim}{\omega} \qquad \text{---- } (E_1)$$

Angular displacement of shaft is

$$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right) \qquad \text{--- } (E_2)$$

As the shaft is supported at $x=0$ and free at $x = l$,

$$\theta(0,t) = 0, \quad \frac{\partial \theta}{\partial x}(l,t) = 0 \qquad \text{---- } (E_3)$$

Eqs. (E_2) and (E_3) give $A = 0$

$$B \frac{\omega}{c} \cos \frac{\omega l}{c} = 0 \quad \text{or} \quad \frac{\omega_n l}{c} = \frac{n\pi}{2}; \; n = 1,3,\ldots$$

$$\theta_n(x,t) = \sin \frac{n \pi x}{2l}\left(C_n \cos \omega_n t + D_n \sin \omega_n t\right) \qquad \text{---- } (E_4)$$

$$\theta(x,t) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l}\left(C_n \cos \omega_n t + D_n \sin \omega_n t\right) \qquad \text{---- } (E_5)$$

At $t = 0$, $$\theta(x,0) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n \pi x}{2l} \cdot C_n \qquad \text{---- } (E_6)$$

$$\frac{\partial \theta}{\partial t}(x,0) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \omega_n D_n = \underset{\sim}{\omega} \qquad \text{---- } (E_7)$$

Since $\theta(x,0) = 0$, $C_n = 0$. By multiplying both sides of (E_7) by $\sin \frac{n \pi x}{2l}$, integrating from 0 to l, and noting that

$$\int_0^l \sin^2 \frac{n\pi x}{2l}\, dx = \frac{l}{2} \quad \text{and} \quad \int_0^l \underset{\sim}{\omega} \sin \frac{n\pi x}{2l}\, dx = \underset{\sim}{\omega} \frac{2l}{n\pi},$$

we get $\quad \underset{\sim}{\omega} \frac{2l}{n\pi} = \frac{\omega_n l}{2} D_n = \frac{n\pi c}{4} D_n \Rightarrow D_n = \frac{8 l \underset{\sim}{\omega}}{n^2 \pi^2 c}$; $n = 1, 3, \ldots$

$$\theta(x,t) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \left(\frac{8 l \underset{\sim}{\omega}}{n^2 \pi^2 c} \right) \sin \omega_n t$$

8.30 $I = \frac{1}{12}(0.1)(0.3)^3 = 2.25 \times 10^{-4}\ m^4$

$A = 0.03\ m^2$, $l = 2\ m$, $E = 20.5 \times 10^{10}\ N/m^2$, $\rho = 7.83 \times 10^3\ kg/m^3$

Fig. 8.15 gives the values of $\beta_n l$:

$$\omega_n = (\beta_n l)^2 \sqrt{\frac{EI}{\rho A l^4}}$$

Here $\sqrt{\frac{EI}{\rho A l^4}} = \left\{ \frac{(20.5 \times 10^{10})(2.25 \times 10^{-4})}{7830 \times 0.03 \times 16} \right\}^{\frac{1}{2}} = 110.7814$

(a) For pinned- pinned beam :

$\beta_1 l = \pi$, $\omega_1 = \pi^2 (110.7814) = 1093.3737$ rad/sec

$\beta_2 l = 2\pi$, $\omega_2 = 4\omega_1 = 4373.4948$ rad/sec

$\beta_3 l = 3\pi$, $\omega_3 = 9\omega_1 = 9840.3634$ rad/sec

(b) For fixed - fixed beam :

$\beta_1 l = 4.730841$, $\quad \omega_1 = 2479.3826$ rad/sec

$\beta_2 l = 7.853205$, $\quad \omega_2 = 6832.2023$ rad/sec

$\beta_3 l = 10.995608$, $\quad \omega_3 = 13393.8474$ rad/sec

(c) For fixed - free beam :

$\beta_1 l = 1.875104$, $\quad \omega_1 = 389.5091$ rad/sec

$\beta_2 l = 4.694091$, $\quad \omega_2 = 2441.0117$ rad/sec

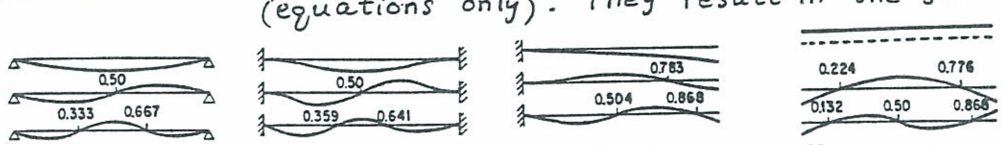
$\beta_3 l = 7.854757$, $\quad \omega_3 = 6834.9030$ rad/sec

(d) For free- free beam :

$\beta_1 l = 0$, $\omega_1 = 0$; $\beta_2 l = 4.730841$, $\quad \omega_2 = 2479.3826$ rad/sec

$\beta_3 l = 7.853205$, $\quad \omega_3 = 6832.2023$ rad/sec.

Mode shapes: The mode shapes are given in Fig. 8.15 (equations only). They result in the following.



Pinned- pinned   Fixed- fixed   Fixed- free   Free - free

8.31 Boundary conditions are:

At $x = 0$, fixed end $\Rightarrow$ $W(0) = 0$ --- (E_1), $\frac{dW}{dx}(0) = 0$ --- (E_2)

At $x = l$, free end $\Rightarrow$ $\frac{d^2W}{dx^2}(l) = 0$ --- (E_3), $\frac{d^3W}{dx^3}(0) = 0$ --- (E_4)

The deflection (normal) function is given by

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x \qquad (E_5)$$

from which

$$\frac{dW}{dx}(x) = \beta\left[-C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x\right] \qquad (E_6)$$

Eqs. (E_1) and (E_5) give $\quad C_1 + C_3 = 0 \qquad (E_7)$

Eqs. (E_2) and (E_6) yield $\quad \beta\,(C_2 + C_4) = 0 \qquad (E_8)$

Use of (E_7) and (E_8) in (E_5) leads to

$$W(x) = C_1(\cos\beta x - \cosh\beta x) + C_2(\sin\beta x - \sinh\beta x) \qquad (E_9)$$

Use of Eqs. (E_3), (E_4) and (E_9) yields

$$C_1(\cos\beta l + \cosh\beta l) + C_2(\sin\beta l + \sinh\beta l) = 0 \qquad (E_{10})$$

$$C_1(\sin\beta l - \sinh\beta l) - C_2(\cos\beta l + \cosh\beta l) = 0 \qquad (E_{11})$$

The frequency equation can be obtained by setting the coefficient matrix in (E_{10}) and (E_{11}) to zero as

$$\begin{vmatrix} \cos\beta l + \cosh\beta l & \sin\beta l + \sinh\beta l \\ \sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta l \end{vmatrix} = 0 \qquad (E_{12})$$

Upon simplification, Eq. (E_{12}) yields the frequency equation

$$\cos\beta l \cdot \cosh\beta l = -1 \qquad (E_{13})$$

First four roots of (E_{13}) are given by

$\beta_1 l = 1.875104$, $\beta_2 l = 4.694091$, $\beta_3 l = 7.854757$, $\beta_4 l = 10.995541$.

8.32 Boundary conditions are:

At $x=0$; $\quad EI \frac{d^3W}{dx^3} = 0$ --- (E_1), $\quad EI \frac{d^2W}{dx^2} = -k_t \frac{dW}{dx}$ --- (E_2)

At $x=l$; $\quad EI \frac{d^3W}{dx^3} = k\,W$ --- (E_3), $\quad EI \frac{d^2W}{dx^2} = 0$ --- (E_4)

Eq. (8.105) gives

$$\int_0^l W_i\, W_j\, dx = -\frac{c^2}{\omega_i^2 - \omega_j^2}\left[W_i W_j''' - W_j W_i''' + W_j' W_i'' - W_i' W_j''\right]_0^l \quad \text{--- } (E_5)$$

At $x=l$, Eq. (E_4) gives

$$W_i'' = 0 \text{ --- } (E_6), \qquad W_j'' = 0 \text{ --- } (E_7)$$

Eq. (E_3) gives

$$EI\, W_i''' = k\, W_i \text{ --- } (E_8), \qquad EI\, W_j''' = k\, W_j \text{ --- } (E_9)$$

Multiplying (E_8) by W_j, (E_9) by W_i, and subtracting one from the other, we get

$$EI\left(W_j''' W_i - W_i''' W_j\right) = k\left(W_j W_i - W_i W_j\right) = 0 \quad \text{--- } (E_{10})$$

This shows that the r.h.s. of (E_5) is zero at $x=l$.

At $x=0$, Eq. (E_1) gives $\quad W_i''' = 0$ --- (E_{11}), $\quad W_j''' = 0$ --- (E_{12})

Eq. (E_2) gives

$$EI\, W_i'' = -k_t\, W_i' \text{ --- } (E_{13}), \qquad EI\, W_j'' = -k_t\, W_j' \text{ --- } (E_{14})$$

Multiplying (E_{13}) by W_j', (E_{14}) by W_i', and subtracting one from the other, we get

$$EI\left(W_i'' W_j' - W_j'' W_i'\right) = -k_t\left(W_i' W_j' - W_j' W_i'\right) = 0 \quad \text{--- } (E_{15})$$

This shows that the r.h.s. of Eq. (E_5) is zero at $x=0$.

8.33 $w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = w(l,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = 0$; $\quad t \geq 0$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{d^2W}{dx^2}(x) = C_1\beta^2(-\cos\beta x + \cosh\beta x) + C_2\beta^2(-\cos\beta x - \cosh\beta x) + C_3\beta^2(-\sin\beta x + \sinh\beta x) + C_4\beta^2(-\sin\beta x - \sinh\beta x)$$

$W(0) = 0 \Rightarrow C_1 = 0$

$\frac{d^2W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$

$W(l) = 0 \Rightarrow C_3(\sin\beta l + \sinh\beta l) + C_4(\sin\beta l - \sinh\beta l) = 0$ --- (E_1)

$\frac{d^2W}{dx^2}(l) = 0 \Rightarrow C_3(-\sin\beta l + \sinh\beta l) - C_4(\sin\beta l + \sinh\beta l) = 0$ --- (E_2)

Setting the determinant of the coefficient matrix of C_3 and C_4 in (E_1) and (E_2) to zero, we get the frequency equation

$$-(\sin\beta l + \sinh\beta l)^2 + (\sin\beta l - \sinh\beta l)^2 = 0$$

or $\quad \sin\beta l \ \sinh\beta l = 0$

Since $\sinh\beta l \neq 0$, the frequency equation becomes

$$\sin\beta l = 0$$

$$\therefore \ \beta_n l = n\pi \ ; \quad \omega_n = \frac{n^2\pi^2}{l^2}\sqrt{\frac{EI}{\rho A}} \ ; \ n = 1, 2, \ldots$$

8.34 $w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(l,t) = \frac{\partial^3 w}{\partial x^3}(l,t) = 0 \ ; \ t \geq 0$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{d^2W}{dx^2}(x) = C_1\beta^2(-\cos\beta x + \cosh\beta x) + C_2\beta^2(-\cos\beta x - \cosh\beta x) + C_3\beta^2(-\sin\beta x + \sinh\beta x) + C_4\beta^2(-\sin\beta x - \sinh\beta x)$$

$$\frac{d^3W}{dx^3}(x) = C_1\beta^3(\sin\beta x + \sinh\beta x) + C_2\beta^3(\sin\beta x - \sinh\beta x) + C_3\beta^3(-\cos\beta x + \cosh\beta x) + C_4\beta^3(-\cos\beta x - \cosh\beta x)$$

$W(0) = 0 \ \Rightarrow \ C_1 = 0$

$\frac{d^2W}{dx^2}(0) = 0 \ \Rightarrow \ C_2 = 0$

$\frac{d^2W}{dx^2}(l) = 0 \Rightarrow C_3(-\sin\beta l + \sinh\beta l) - C_4(\sin\beta l + \sinh\beta l) = 0 \quad \text{----} (E_1)$

$\frac{d^3W}{dx^3}(l) = 0 \Rightarrow C_3(-\cos\beta l + \cosh\beta l) - C_4(\cos\beta l + \cosh\beta l) = 0 \quad \text{----} (E_2)$

Setting the determinant of coefficients of C_3 and C_4 in Eqs. (E_1) and (E_2) to zero, the frequency equation can be obtained:

$$-(-\sin\beta l + \sinh\beta l)(\cos\beta l + \cosh\beta l) + (\sin\beta l + \sinh\beta l)(-\cos\beta l + \cosh\beta l) = 0$$

or $\quad \tan\beta l - \tanh\beta l = 0$

8.35 For a simply supported beam, the first three natural frequencies are given by

$$\omega_1 = (\beta_1 l)^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}} = \pi^2 \left(EI/\rho A l^4\right)^{1/2}$$

$$\omega_2 = (\beta_2 l)^2 \left(EI/\rho A l^4\right)^{1/2} = 4\pi^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}}$$

$$\omega_3 = (\beta_3 l)^2 \left(EI/\rho A l^4\right)^{\frac{1}{2}} = 9\pi^2 \left(EI/\rho A l^4\right)^{1/2}$$

For steel, $E = 2.07 \times 10^{11}\ N/m^2$, $\rho = 7880\ kg/m^3$, $l = 1.0\ m$.

Setting $\omega_1 \geq 1500\ Hz = 9424.8\ rad/sec = \pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{(7880) A (1)} \right\}^{1/2}$

we get

$$(9424.8)^2 \geq \pi^4 \left(\frac{2.07 \times 10^{11}}{7880} \right) \left(\frac{I}{A} \right)$$

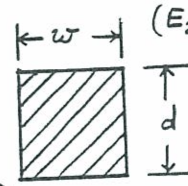
or

$$\frac{I}{A} \leq 0.03471 \qquad (E_1)$$

Setting $\omega_3 \leq 5000\ Hz = 31416.0\ rad/sec = 9\pi^2 \left\{ \frac{(2.07 \times 10^{11})\ I}{7880\ A\ (1)} \right\}^{\frac{1}{2}}$

we get

$$\frac{I}{A} \geq 0.0047617 \qquad (E_2)$$



Let the beam have a rectangular section

$$A = wd, \quad I = \frac{1}{12} w d^3, \quad \frac{I}{A} = \frac{d^2}{12}$$

Let $\frac{I}{A} = 0.005$ to satisfy inequalities (E_1) and (E_2).

Then $d^2 = 0.06$ or $d = 0.2449\ m$

Taking $w = 0.1\ m$ (w can have any value), we get

$A = wd = 0.02449\ m^2$ and $I = 1.2240 \times 10^{-4}\ m^4$.

8.36 From the solution of problem 8.31, we find for $\sin \beta l = 0$,

$$C_3 = C_4$$

Hence $W_n(x) = C_n \sin \beta_n x = C_n \sin \frac{n \pi x}{l}$

General free vibration solution is

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right)$$

When beam vibrates in the first mode,

$$w(x,t) = \sin \frac{\pi x}{l} \left(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \right)$$

Bending moment in the beam is

$$M(x,t) = EI \frac{\partial^2 w}{\partial x^2}(x,t) = -\frac{\pi^2 EI}{l^2} \sin \frac{\pi x}{l} \left(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \right)$$

$$= -\frac{\pi^2}{l^2} EI\ w(x,t)$$

If the amplitude of vibration is Y, the maximum bending moment is

$$|M_{max}| = \frac{\pi^2}{l^2} EI\ Y$$

For the given data,

$$|M_{max}| = \frac{\pi^2}{(1)^2} (20.5 \times 10^{10}) \left(\frac{10^3}{10^{12}} \right) \left(\frac{10}{1000} \right) = 20.2328\ N\text{-}m$$

8.37

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x$$

$$\frac{d^2 W}{dx^2}(x) = -C_1 \beta^2 \cos\beta x - C_2 \beta^2 \sin\beta x + C_3 \beta^2 \cosh\beta x + C_4 \beta^2 \sinh\beta x$$

$$\frac{d^3 W}{dx^3}(x) = C_1 \beta^3 \sin\beta x - C_2 \beta^3 \cos\beta x + C_3 \beta^3 \sinh\beta x + C_4 \beta^3 \cosh\beta x$$

$$EI\,\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow \quad C_1 = C_3 \qquad \text{--- } (E_1)$$

$$EI\,\frac{d^3 W}{dx^3}(0) = -\underset{\sim}{k}_1\, W(0) \Rightarrow \quad EI\beta^3(-C_2 + C_4) + \underset{\sim}{k}_1 (C_1 + C_3) = 0$$

$$C_2(-EI\beta^3) + C_3(+2\underset{\sim}{k}_1) + C_4(EI\beta^3) = 0 \qquad \text{--- } (E_2)$$

$$EI\,\frac{d^2 W}{dx^2}(l) = 0 \Rightarrow C_2(-\sin\beta l) + C_3(\cosh\beta l - \cos\beta l) + C_4(\sinh\beta l) = 0 \qquad \text{--- } (E_3)$$

$$EI\,\frac{d^3 W}{dx^3}(l) = +\underset{\sim}{k}_2\, W(l) \Rightarrow$$

$$C_2(-EI\beta^3\cos\beta l - \underset{\sim}{k}_2 \sin\beta l) + C_3(EI\beta^3 \sin\beta l + EI\beta^3 \sinh\beta l - \underset{\sim}{k}_2 \cos\beta l - \underset{\sim}{k}_2 \cosh\beta l) + C_4(EI\beta^3\cosh\beta l - \underset{\sim}{k}_2 \sinh\beta l) = 0 \qquad \text{--- } (E_4)$$

Setting the determinant of the coefficients of C_2, C_3 and C_4 in (E_2) to (E_4) to zero, we get the frequency equation:

$$\begin{vmatrix} -EI\beta^3 & +2\underset{\sim}{k}_1 & EI\beta^3 \\ -\sin\beta l & (\cosh\beta l - \cos\beta l) & \sinh\beta l \\ (-EI\beta^3\cos\beta l - \underset{\sim}{k}_2\sin\beta l) & (EI\beta^3\{\sin\beta l + \sinh\beta l\} - \underset{\sim}{k}_2\{\cos\beta l + \cosh\beta l\}) & (EI\beta^3\cosh\beta l - \underset{\sim}{k}_2 \sinh\beta l) \end{vmatrix} = 0$$

8.38

Because of symmetry, consider only half of beam to the left of M.

Boundary conditions are:

$$w(0,t) = 0, \quad EI\,\frac{\partial^2 w}{\partial x^2}(0,t) = 0$$

For symmetry, $\frac{\partial w}{\partial x} = 0$ at $x = \frac{l}{2}$.

$$EI\,\frac{\partial^3 w}{\partial x^3} = +\frac{M}{2}\,\frac{\partial^2 w}{\partial t^2} \quad \text{at } x = \frac{l}{2}$$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2 W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(x) = C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{dW}{dx}(x) = \beta\, C_3 \left(\cos\beta x + \cosh\beta x\right) + \beta C_4 \left(\cos\beta x - \cosh\beta x\right)$$

$$\frac{d^3W}{dx^3}(x) = \beta^3\, C_3 \left(-\cos\beta x + \cosh\beta x\right) - \beta^3 C_4 \left(\cos\beta x + \cosh\beta x\right)$$

$$\frac{dW}{dx}\left(\frac{l}{2}\right) = 0 \Rightarrow \quad C_3\left(\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right) + C_4\left(\cos\frac{\beta l}{2} - \cosh\frac{\beta l}{2}\right) = 0 \quad \text{----}\ (E_1)$$

$$EI\,\frac{d^3W}{dx^3}\left(\frac{l}{2}\right) = -\frac{M}{2}\,\omega^2\, W\left(\frac{l}{2}\right) \quad \Rightarrow$$

$$EI\,\beta^3\left[C_3\left(-\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right) - C_4\left(\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right)\right] =$$

$$-\frac{M}{2}\,\omega^2\left[C_3\left(\sin\frac{\beta l}{2} + \sinh\frac{\beta l}{2}\right) + C_4\left(\sin\frac{\beta l}{2} - \sinh\frac{\beta l}{2}\right)\right] \text{---}(E_2)$$

i.e.

$$R_1 \cos\frac{\beta l}{2} + R_2 \cosh\frac{\beta l}{2} = 0$$

$$R_1\left(-\cos\frac{\beta l}{2} + \lambda \sin\frac{\beta l}{2}\right) + R_2\left(\cosh\frac{\beta l}{2} + \lambda\sinh\frac{\beta l}{2}\right) = 0$$

where $R_1 = C_3 + C_4\,, \quad R_2 = C_3 - C_4\,, \quad \lambda = \dfrac{M\,\omega^2}{2EI\,\beta^3}$.

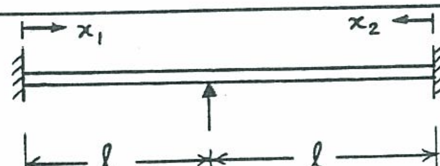
By setting the coefficient matrix of R_1 and R_2 equal to zero, we obtain the frequency equation:

$$\cos\frac{\beta l}{2}\left(\cosh\frac{\beta l}{2} + \lambda\sinh\frac{\beta l}{2}\right) + \cosh\frac{\beta l}{2}\left(\cos\frac{\beta l}{2} + \lambda\sin\frac{\beta l}{2}\right) = 0$$

or

$$\lambda\left(-\tanh\frac{\beta l}{2} + \tan\frac{\beta l}{2}\right) = 2$$

8.39 For convenience, set up two coordinates x_1 and x_2 as shown.



$$W_1(x_1) = C_1\cos\beta x_1 + C_2\sin\beta x_1 + C_3\cosh\beta x_1 + C_4\sinh\beta x_1$$

$$\frac{dW_1}{dx_1}(x_1) = -C_1\beta\sin\beta x_1 + C_2\beta\cos\beta x_1 + C_3\beta\sinh\beta x_1 + C_4\beta\cosh\beta x_1$$

$$W_1(0) = 0 \Rightarrow C_1 + C_3 = 0\ ; \quad C_1 = -C_3$$

$$\frac{dW_1}{dx_1}(0) = 0 \Rightarrow C_2 + C_4 = 0\ ; \quad C_2 = -C_4$$

$$W_1(x_1) = C_3\left(\cosh\beta x_1 - \cos\beta x_1\right) + C_4\left(\sinh\beta x_1 - \sin\beta x_1\right)$$

$$W_1(l) = 0 \Rightarrow \quad C_3 = -C_4\left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)$$

$$W_1(x_1) = C_4\left[\left(\sinh\beta x_1 - \sin\beta x_1\right) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)\left(\cosh\beta x_1 - \cos\beta x_1\right)\right] \quad \text{----}\ (E_1)$$

$$W_2(x_2) = C_1'\cos\beta x_2 + C_2'\sin\beta x_2 + C_3'\cosh\beta x_2 + C_4'\sinh\beta x_2$$

$$\frac{dW_2}{dx_2}(x_2) = -c_1' \beta \sin\beta x_2 + c_2' \beta \cos\beta x_2 + c_3' \beta \sinh\beta x_2 + c_4' \beta \cosh\beta x_2$$

$W_2(0) = 0 \Rightarrow c_1' = -c_3'$

$\frac{dW_2}{dx_2}(0) = 0 \Rightarrow c_2' = -c_4'$

$W_2(l) = 0 \Rightarrow c_3' = -c_4' \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)$

$$W_2(x_2) = c_4' \left[(\sinh\beta x_2 - \sin\beta x_2) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta x_2 - \cos\beta x_2)\right] \quad ---- (E_2)$$

$\frac{dW_1}{dx_1}(x_1 = l) = -\frac{dW_2}{dx_2}(x_2 = l)$ gives

$$c_4\left[(\cosh\beta l - \cos\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\sinh\beta l + \sin\beta l)\right]$$
$$+ c_4'\left[(\cosh\beta l - \cos\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\sinh\beta l + \sin\beta l)\right] = 0 \quad --- (E_3)$$

$\frac{d^2W_1}{dx_1^2}(x_1 = l) = -\frac{d^2W_2}{dx_2^2}(x_2 = l)$ gives

$$c_4\left[(\sinh\beta l + \sin\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta l + \cos\beta l)\right]$$
$$+ c_4'\left[(\sinh\beta l + \sin\beta l) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta l + \cos\beta l)\right] = 0 \quad --- (E_4)$$

Since the determinant of the coefficients of c_4 and c_4' in (E_3) and (E_4) is identically zero, we set the coefficients of c_4 and c_4' in (E_3) and (E_4) to zero. This leads to the frequency equations

$$\cos\beta l \cosh\beta l = 1$$

and $\tan\beta l = \tanh\beta l$

8.40 General free vibration solution of a simply supported beam is

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\,(A_n \cos\omega_n t + B_n \sin\omega_n t)$$

At $t=0$, $\frac{\partial w}{\partial t}(x,0) = 0$ and $EI\frac{\partial^4 w}{\partial x^4}(x,0) = f_0$

Thus $\sum_{n=1}^{\infty} B_n \omega_n \sin\frac{n\pi x}{l} = 0$ or $B_n = 0$

and $EI\sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{l}\right)^4 \sin\frac{n\pi x}{l} = f_0 \quad ---- (E_1)$

Multiplying Eq. (E_1) by $\sin \frac{n\pi x}{l}$ and integrating from 0 to l, we obtain

$$A_n = \frac{2 l^3}{EI\, n^4 \pi^4} \int_0^l f_0 \sin \frac{n\pi x}{l}\, dx = \frac{2 l^4 f_0}{EI\, n^5 \pi^5} (1 - \cos n\pi)$$

$$= \frac{4 f_0 l^4}{EI\, n^5 \pi^5} \quad ; \; n = 1, 3, 5, \ldots$$

$\therefore$ Solution is $\displaystyle w(x,t) = \frac{4 f_0 l^4}{\pi^5 EI} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^5} \sin \frac{n\pi x}{l} \cos \omega_n t$

8.41

Let $W(x) = \left(1 - \frac{x}{l}\right)^2$; $\frac{dW}{dx} = -\frac{2}{l}\left(1 - \frac{x}{l}\right)$; $\frac{d^2W}{dx^2} = \frac{2}{l^2}$

$$N = \int_0^l EI \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{EI_0}{l} \int_0^l x \left(\frac{4}{l^4}\right) dx = \frac{4 EI_0}{l^3}$$

$$D = \int_0^l \rho A \{W(x)\}^2 dx = \frac{\rho A_0}{l} \int_0^l x \left(1 - \frac{x}{l}\right)^4 dx = \frac{\rho A_0 l}{30}$$

$$\omega^2 = \frac{N}{D} \simeq \frac{4 EI_0 (30)}{l^3 (\rho A_0 l)} = \frac{120\, EI_0}{\rho A_0 l^4}$$

$$\omega \simeq \sqrt{120} \left(\frac{EI_0}{\rho A_0 l^4}\right)^{1/2}$$

8.42

(a) Equation of motion for a uniform beam is

$$EI \frac{\partial^4 w}{\partial x^4} = f(x,t) - \rho A \frac{\partial^2 w}{\partial t^2} \qquad \text{---- } (E_1)$$

Let $w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, T_n(t)$ where $W_n(x) = n^{th}$ normal mode satisfying the boundary conditions and the equation

$$EI \frac{d^4 W}{dx^4} = \omega_n^2 \rho A W_n \; ; \; n = 1, 2, \ldots \qquad \text{---- } (E_2)$$

Eq. (E_1) becomes for the assumed solution,

$$EI \sum_{n=1}^{\infty} \frac{d^4 W}{dx^4} \cdot T_n(t) = f(x,t) - \rho A \sum_{n=1}^{\infty} W_n(x) \cdot \frac{d^2 T_n}{dt^2} \qquad \text{---- } (E_3)$$

which, in view of (E_2), becomes

$$\sum_{n=1}^{\infty} \omega_n^2 W_n(x)\, T_n(t) = \frac{1}{\rho A} f(x,t) - \sum_{n=1}^{\infty} W_n(x) \frac{d^2 T_n}{dt^2} \qquad \text{---- } (E_4)$$

Multiplying Eq. (E_4) by $W_m(x)$ and integrating from 0 to l, we get

$$\frac{d^2 T_n}{dt^2} + \omega_n^2 T_n = \frac{2}{\rho A l} F_n(t) \qquad \text{---- } (E_5)$$

where

$$F_n(t) = \int_0^l f(x,t)\, W_n(x)\, dx \qquad \text{---- } (E_6)$$

Note that the orthogonality of normal modes, namely,

$$\int_0^l W_m(x)\, W_n(x)\, dx = \begin{cases} 0 & \text{if } m = n \\ l/2 & \text{if } m \neq n \end{cases}$$

is used in deriving (E_5). The solution of Eq. (E_5) can be written as

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^l F_n(\tau) \sin \omega_n (t-\tau)\, d\tau \qquad \text{---- } (E_7)$$

where A_n and B_n are constants to be determined from initial conditions. Thus the total solution can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A l \omega_n} \int_0^t F_n(\tau) \sin \omega_n (t-\tau)\, d\tau \right] W_n(x) \qquad \text{---- } (E_8)$$

(b) For a simply supported beam,

$$W_n(x) = \sin \frac{n\pi x}{l}$$

For the given loading, (E_6) becomes $F_n(t) = F_0 \sin \frac{n\pi a}{l} \sin \omega t$

Hence (E_7) becomes

$$T_n(t) = \frac{2 F_0 l^3}{E I \pi^4} \sin \frac{n\pi a}{l} \left[\frac{1}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \omega t - \frac{\left(\frac{\omega}{\omega_1}\right)}{n^2 \left\{ n^4 - \left(\frac{\omega}{\omega_1}\right)^2 \right\}} \sin n^2 \omega_1 t \right]$$

where $\omega_1 = \frac{\pi^2}{l^2} \sqrt{\frac{EI}{\rho A}}$ = fundamental frequency of the beam

Thus the forced response of the beam is given by

$$w(x,t) = \frac{2 F_0 l^3}{\pi^4 E I} \left[\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi a}{l}\right)}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \frac{n\pi x}{l} \right] \sin \omega t$$

$$- \frac{2 \left(\frac{\omega}{\omega_1}\right) F_0 l^3}{\pi^4 E I} \left[\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi a}{l}\right)}{n^2 \left(n^4 - \left\{\frac{\omega}{\omega_1}\right\}^2\right)} \sin \frac{n\pi x}{l} \sin n^2 \omega_1 t \right]$$

8.43 Derivation of Eq. (E_5):

If shear deformation is zero, $M = EI \frac{\partial^2 w}{\partial x^2}$

Eq. (8.131) $\Rightarrow$ $\quad \frac{\partial V}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} + f$

Eq. (8.132) $\Rightarrow$ $\quad \frac{\partial M}{\partial x} - V = \rho I \frac{\partial^2 \phi}{\partial t^2}$, $\quad \frac{\partial^2 M}{\partial x^2} - \frac{\partial V}{\partial x} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$

or $\quad \frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} - f = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$

$$\text{or} \quad EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - f = 0 \qquad \text{---- } (E.1)$$

Derivation of Eq. (E_6):

Eq. (E.1) becomes, for $f = 0$,

$$\frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - \frac{I}{A}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \qquad \text{---- (E.2)}$$

If $w(x,t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, Eq. (E.2) becomes

$$\alpha^2 \left(\frac{n\pi}{l}\right)^4 - \omega_n^2 - r^2 \left(\frac{n\pi}{l}\right)^2 \omega_n^2 = 0 \qquad \alpha^2 = \frac{EI}{\rho A}$$

i.e.

$$\omega_n^2 = \frac{\alpha^2 \left(\frac{n\pi}{l}\right)^4}{1 + r^2\left(\frac{n\pi}{l}\right)^2} = \frac{\alpha^2 n^4 \pi^4}{l^4 \left(1 + \frac{n^2\pi^2 r^2}{l^2}\right)} \qquad r^2 = \frac{I}{A}$$

8.44 Derivation of (E_7):

If rotary inertia is neglected,

$$M = EI \frac{\partial \phi}{\partial x} \qquad \text{(E.1)}$$

$$V = kAG\left(\phi - \frac{\partial w}{\partial x}\right) \qquad \text{(E.2)}$$

Eq. (8.131) $\Rightarrow$

$$-\frac{\partial V}{\partial x} + f = \rho A \frac{\partial^2 w}{\partial t^2} \qquad \text{(E.3)}$$

Eq. (8.132) $\Rightarrow$

$$\frac{\partial M}{\partial x} - V = 0 \qquad \text{(E.4)}$$

Using (E.1) and (E.2), (E.3) and (E.4) can be written as

$$-kAG\left(\frac{\partial \phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + f = \rho A \frac{\partial^2 w}{\partial t^2}$$

or

$$\frac{\partial \phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} + \frac{f}{kAG} - \frac{\rho A}{kAG}\frac{\partial^2 w}{\partial t^2} \qquad \text{(E.5)}$$

and

$$EI \frac{\partial^2 \phi}{\partial x^2} - kAG\left(\phi - \frac{\partial w}{\partial x}\right) = 0$$

or

$$EI \frac{\partial^2}{\partial x^2}\left(\frac{\partial \phi}{\partial x}\right) - kAG \frac{\partial \phi}{\partial x} + kAG \frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{(E.6)}$$

Substitution of (E.5) into (E.6) gives

$$EI \frac{\partial^4 w}{\partial x^4} + \frac{EI}{kAG}\frac{\partial^2 f}{\partial x^2} - \frac{\rho A EI}{kAG}\frac{\partial^4 w}{\partial x^2 \partial t^2} - f + \rho A \frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{(E.7)}$$

If $f = 0$, (E.7) becomes

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \frac{\rho EI}{kG}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \qquad \text{(E.8)}$$

Derivation of (E_8):

With $w(x,t) = C \sin \frac{n\pi x}{l} \cos \omega_n t$, (E.8) gives

$$\omega_n^2 = \frac{\alpha^2 n^4 \pi^4}{l^4 \left(1 + \frac{E}{kG}\cdot\frac{r^2 n^2 \pi^2}{l^2}\right)}$$

8.45 Multiply Eq. (8.83) by $W(x)$ and integrate from 0 to l:

$$\int_0^l W(x) \frac{d^4 W}{dx^4}\, dx = \frac{\omega^2}{c^2}\int_0^l \left[W(x)\right]^2 dx$$

This shows that the sign of ω^2 will be same as that of the left hand side term. Integrating the left hand side expression by parts, we get

$$\frac{d^3W}{dx^3}(x)\cdot W(x)\Big|_0^\ell - \frac{d^2W}{dx^2}(x)\cdot\frac{dW}{dx}(x)\Big|_0^\ell + \int_0^\ell \left[\frac{d^2W}{dx^2}(x)\right]^2 dx \qquad (E_1)$$

For common boundary conditions, the first two terms in (E_1) will be zero, and (E_1) will be positive. Hence ω^2 is positive.

Common boundary conditions:

Simply supported end: $W(x) = \frac{d^2W}{dx^2} = 0$

Fixed end: $W(x) = \frac{dW}{dx} = 0$

Free end: $\frac{d^2W}{dx^2}(x) = \frac{d^3W}{dx^3} = 0$

8.46 Equation of motion:

$$EI\,\frac{\partial^4 w}{\partial x^4} + \rho A\,\frac{\partial^2 w}{\partial t^2} = F_0 \sin\omega t \qquad (1)$$

For steady state response, we assume the particular solution as

$$w(x,t) = W(x)\sin\omega t \qquad (2)$$

Substitution of Eq. (2) into (1) gives

$$\frac{d^4W}{dx^4} - \frac{\omega^2}{c^2}W = \frac{F_0}{\rho A c^2} \qquad (3)$$

$$\text{where}\quad c^2 = \frac{EI}{\rho A} \qquad (4)$$

The complete solution of Eq. (3) can be written as

$$W(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x - \frac{F_0}{\rho A c^2} \qquad (5)$$

$$\text{where}\quad \beta^2 = \frac{\omega^2}{c^2} \qquad (6)$$

The boundary conditions for a simply supported beam are

$$W(x{=}0) = 0 \qquad (7)$$

$$\frac{d^2W}{dx^2}(x{=}0) = 0 \qquad (8)$$

$$W(x{=}\ell) = 0 \qquad (9)$$

$$\frac{d^2W}{dx^2}(x{=}\ell) = 0 \qquad (10)$$

Equations (7) and (5) give:

$$C_1 + C_3 - \frac{F_0}{\rho A c^2} = 0 \qquad (11)$$

Equations (8) and (5) yield:

$$-C_1\,\beta^2 + C_3\,\beta^2 = 0 \quad \text{or} \quad C_1 = C_3 \tag{12}$$

Eqs. (11) and (12) provide:

$$C_1 = C_3 = \frac{F_0}{2\,\rho\,A\,c^2} \tag{13}$$

Eqs. (9) and (5) give:

$$C_2\,\sin\beta\ell + C_4\,\sinh\beta\ell = -\frac{F_0}{\rho\,A\,c^2}\left(\frac{\cos\beta\ell + \cosh\beta\ell}{2} - 1\right) \tag{14}$$

Eqs. (10) and (5) yield:

$$-C_2\,\sin\beta\ell + C_4\,\sinh\beta\ell = \frac{F_0}{2\,\rho\,A\,c^2}\left(\cos\beta\ell - \cosh\beta\ell\right) \tag{15}$$

Solution of Eqs. (14) and (15) gives:

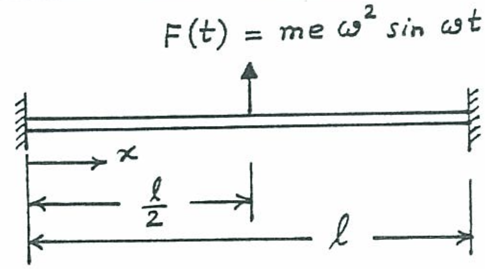
$$C_4 = \frac{F_0}{2\,\rho\,A\,c^2\,\sinh\beta\ell}\,(1-\cosh\beta\ell) = -\frac{F_0}{2\,\rho\,A\,c^2}\,\tanh\frac{\beta\ell}{2} \tag{16}$$

$$C_2 = -\frac{F_0}{2\,\rho\,A\,c^2\,\sin\beta\ell}\,(1-\cos\beta\ell) = \frac{F_0}{2\,\rho\,A\,c^2}\,\tan\frac{\beta\ell}{2} \tag{17}$$

Thus the complete solution becomes

$$w(x,t) = \frac{F_0}{2\,\rho\,A\,c^2}\,\left[\left(\cos\beta x + \cosh\beta x\right) + \tan\frac{\beta\ell}{2}\,\sin\beta x - \tanh\frac{\beta\ell}{2}\,\sinh\beta x - 2\right]\sin\omega t$$

8.47



$$\omega = \frac{1000}{60}\,(2\,\pi) = 104.72 \text{ rad/sec.}$$

$$F(t) = m\,e\,\omega^2\,\sin\omega t = (0.5)\,(104.72^2)\,\sin 104.72\,t = F_0\,\sin\omega t \tag{1}$$

where $F_0 = 5483.1392$ N ; $\omega = 104.72$ rad/sec

Steady state response of the beam can be expressed as:

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\,q_n(t) \tag{2}$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15):

$$W_n(x) = \sinh\beta_n x - \sin\beta_n x + \alpha_n\left(\cosh\beta_n x - \cos\beta_n x\right) \tag{3}$$

$$\text{and} \quad \alpha_n = \frac{\sinh \beta_n \ell - \sin \beta_n \ell}{\cos \beta_n \ell - \cosh \beta_n \ell} \tag{4}$$

The generalized force, $Q_n(t)$, can be expressed as:

$$Q_n(t) = \int_0^\ell f(x,t)\, W_n(x)\, dx = F_0\, W_n\left(\frac{\ell}{2}\right) \sin \omega t \tag{5}$$

The steady state values of the generalized coordinates, $q_n(t)$, are given by Eq. (8.117):

$$q_n(t) = \frac{1}{\rho A b\, \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0\, W_n\left(\frac{\ell}{2}\right)}{\rho A \ell\, \omega_n} \int_0^t \sin \omega \tau \sin \omega_n (t-\tau)\, d\tau \tag{6}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x)\, dx = \ell \tag{7}$$

The integral of Eq. (6) can be evaluated as:

$$\int_0^t \sin \omega \tau \left\{ \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right\} d\tau$$

$$= \sin \omega_n t \int_0^t \frac{1}{2} \sin 2 \omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \frac{1}{2} (1 - \cos 2 \omega_n \tau)\, d\tau$$

$$= \frac{\sin \omega_n t}{2\, \omega_n} - \frac{t \cos \omega_n t}{2} \tag{8}$$

Thus $q_n(t)$ can be expressed as

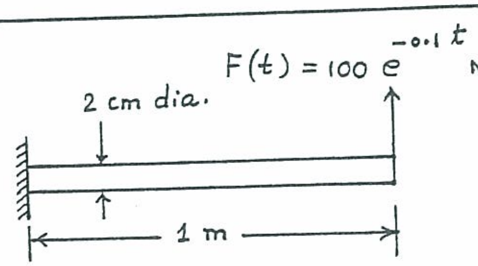
$$q_n(t) = \frac{F_0\, W_n\left(\frac{\ell}{2}\right)}{\rho A \ell\, \omega_n} \left\{ \frac{\sin \omega_n t}{2\, \omega_n} - \frac{t \cos \omega_n t}{2} \right\}$$

$$= \frac{18.2771\, W_n\left(\frac{\ell}{2}\right)}{\omega_n} \left\{ \sin \omega_n t - t \cos \omega_n t \right\} \tag{9}$$

The steady state response of Eq. (2) can be expressed as

$$w(x,t) = 18.2771 \sum_{n=1}^{\infty} \frac{W_n\left(\frac{\ell}{2}\right)}{\omega_n} W_n(x) \left(\sin \omega_n t - t \cos \omega_n t \right) \ \text{m} \tag{10}$$

8.48 Steady state displacement of the beam can be expressed as



$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, q_n(t) \tag{1}$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15)

$$W_n(x) = C_n \left[\sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \right] \tag{2}$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \tag{3}$$

and the generalized coordinates, $q_n(t)$, are given by

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 \, q_n(t) = \frac{1}{\rho A b} \, Q_n(t) \tag{5}$$

$$\text{where} \quad Q_n(t) = \int_0^\ell f(x,t) \, W_n(x) \, dx = F(t) \, W_n(\ell) \tag{6}$$

The steady state solution of Eq. (5) is given by

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{7}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x) \, dx = \ell \tag{8}$$

$$\int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau = 100 \, W_n(\ell) \int_0^t e^{-0.1 \tau} \sin \omega_n (t - \tau) \, d\tau$$

$$= -100 \, W_n(\ell) \, e^{-0.1 t} \int_{t-\tau=t}^{t-\tau=0} e^{0.1 (t-\tau)} \sin \omega_n (t - \tau) \, (-d\tau) \tag{9}$$

Using the formula

$$\int e^{a x} \sin b x \, dx = \frac{1}{a^2 + b^2} \, e^{a x} \left\{ a \sin b x - b \cos b x \right\} \tag{10}$$

Eq. (9) can be evaluated to obtain

$$q_n(t) = \frac{100 \, W_n(\ell)}{\rho A \ell \omega_n (\omega_n^2 + 0{,}01)} \left\{ \omega_n \, e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t \right\} \tag{11}$$

Using $\rho = 7500 \text{ kg/m}^3$, $\ell = 1$ m, $A = \frac{\pi}{4} (0.02^2) = 3.1416 \, (10^{-4}) \text{ m}^2$, Eq. (11) can be written as

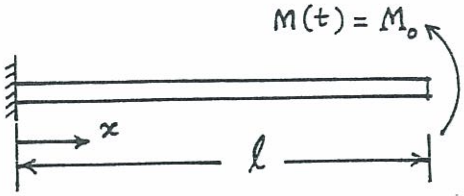
$$q_n(t) = \frac{42.4412 \, W_n(\ell)}{\omega_n (\omega_n^2 + 0.01)} \left\{ \omega_n \, e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t \right\} \tag{12}$$

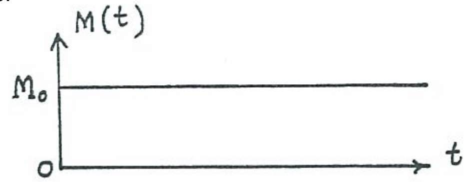
The total steady state solution can be found from Eq. (1).

8.49 The solution is assumed as

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) \, q_n(t) \tag{1}$$

where the normal modes of a cantilever beam are given by (Fig. 8.15):





$$W_n(x) = \sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \tag{2}$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \tag{3}$$

where $\beta_n \ell$ are given by the frequency equation:

$$\cos \beta_n \ell \cosh \beta_n \ell = -1 \tag{4}$$

The generalized force $Q_n(t)$ given by Eq. (8.115) becomes

$$Q_n(t) = M_0 \frac{dW_n}{dx} \Big|_{x=\ell} \tag{5}$$

where

$$\frac{dW_n}{dx} \Big|_{x=\ell} = \beta_n (\cos \beta_n \ell - \cosh \beta_n \ell) + \alpha_n \beta_n (\sin \beta_n \ell + \sinh \beta_n \ell) \tag{6}$$

The steady state response of the beam is given by Eq. (1) with

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau)\, d\tau$$

$$= \frac{1}{\rho A b \omega_n} M_0 \frac{dW_n}{dx} \Big|_{x=\ell} \int_0^t \sin \omega_n (t - \tau)\, d\tau \tag{7}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x)\, dx = \ell \tag{8}$$

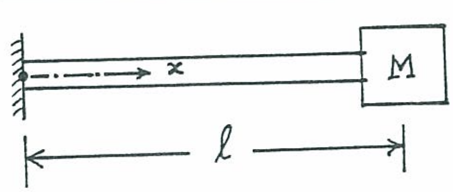
Noting that

$$\int_0^t \sin \omega_n (t - \tau)\, d\tau = \frac{1}{\omega_n} (1 - \cos \omega_n t) \tag{9}$$

Eq. (7) can be written as

$$q_n(t) = \frac{M_0}{\rho A \ell \omega_n^2} \frac{dW_n}{dx} \Big|_{x=\ell} (1 - \cos \omega_n t) \tag{10}$$

8.50



$$w(x,t) = W(x) \sin \omega t \tag{1}$$

$$\text{where} \quad W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \tag{2}$$

Boundary conditions:

$$w(0,\ t) = 0 \tag{3}$$

$$\frac{\partial w}{\partial x}(0,\ t) = 0 \tag{4}$$

$$\frac{\partial^2 w}{\partial x^2}(x = \ell,\ t) = 0 \tag{5}$$

$$EI\ \frac{\partial^3 w}{\partial x^3}(x = \ell,\ t) = m\ \frac{\partial^2 w}{\partial t^2}(x = \ell,\ t) \tag{6}$$

Eq. (2) gives:

$$\frac{dW}{dx} = -\beta\, C_1 \sin \beta x + \beta\, C_2 \cos \beta x + \beta\, C_3 \sinh \beta x + \beta\, C_4 \cosh \beta x \tag{7}$$

$$\frac{d^2W}{dx^2} = \beta^2 \left\{ - C_1 \cos \beta x - C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \right\} \tag{8}$$

$$\frac{d^3W}{dx^3} = \beta^3 \left\{ C_1 \sin \beta x - C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \right\} \tag{9}$$

Eqs. (2) and (3) give:

$$C_1 + C_3 = 0 \tag{10}$$

Eqs. (2) and (4) lead to:

$$C_2 + C_4 = 0 \tag{11}$$

Eqs. (2) and (5) yield:

$$- C_1 \cos \beta \ell - C_2 \sin \beta \ell + C_3 \cosh \beta \ell + C_4 \sinh \beta \ell = 0 \tag{12}$$

Eqs. (2) and (6) result in:

$$\begin{aligned} & C_1 (\sin \beta \ell + k \cos \beta \ell) + C_2 (- \cos \beta \ell + k \sin \beta \ell) \\ & + C_3 (\sinh \beta \ell + k \cosh \beta \ell) + C_4 (\cosh \beta \ell + k \sinh \beta \ell) = 0 \end{aligned} \tag{13}$$

where

$$k = \frac{m\, \omega^2}{E\, I\, \beta^3} \tag{14}$$

Eqs. (10) to (13) can be expressed in matrix form as

$$[A]\ \vec{C} = \vec{0} \tag{15}$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ - \cos \beta \ell & - \sin \beta \ell & \cosh \beta \ell & \sinh \beta \ell \\ (\sin \beta \ell & (- \cos \beta \ell & (\sinh \beta \ell & (\cosh \beta \ell \\ + k \cos \beta \ell) & + k \sin \beta \ell) & + k \cosh \beta \ell) & + k \sinh \beta \ell) \end{bmatrix} \tag{16}$$

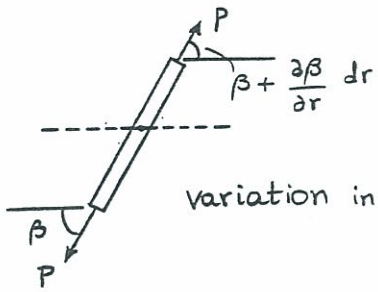
$$\vec{C} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \quad ; \quad \vec{0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

By setting the determinant of the coefficient matrix, [A], to zero, we obtain the frequency equation:

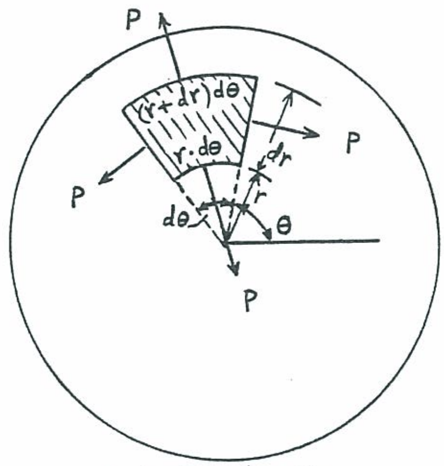
$$\Big| [A] \Big| = 0 \tag{17}$$

8.51

variation in θ-direction



variation in r-direction



Let P = tension

Forces in radial direction are $P\,r\,d\theta$ and $P(r+dr)\,d\theta$

Vertical component of radial forces

$$= P(r+dr)\,d\theta\left(\beta + \frac{\partial \beta}{\partial r} dr\right) - P\,r\,d\theta\,\beta = P\,r\left(\frac{\partial \beta}{\partial r} + \frac{1}{r}\beta\right) dr\,d\theta$$

$$= P\,r\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) dr\,d\theta \qquad \text{since } \beta = \frac{\partial w}{\partial r}$$

Forces in tangential direction are $P\,dr$ and $P\,dr$.

Vertical component of tangential forces

$$= P\,dr\left(\alpha + \frac{\partial \alpha}{\partial \theta} d\theta\right) - P\,dr\,\alpha = P\frac{\partial \alpha}{\partial \theta} dr\,d\theta$$

$$= P \cdot \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\,r\,dr\,d\theta \qquad \text{since } \alpha = \frac{1}{r}\frac{\partial w}{\partial \theta} \text{ and } \frac{\partial \alpha}{\partial \theta} = \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}$$

Equating the total vertical force to mass times acceleration, we get

$$P\,r\,dr\,d\theta\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] = \rho\frac{\partial^2 w}{\partial t^2}\,r\,dr\,d\theta$$

i.e. $$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{\rho}{P}\frac{\partial^2 w}{\partial t^2}$$

8.56 For harmonic motion, equation of motion becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = -\frac{\rho}{P}\omega^2 W$$

Let $W(r,\theta) = X(r)\cdot Y(\theta)$

$$\frac{r^2}{X}\left(\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \frac{\rho\omega^2}{P}X\right) = -\frac{1}{Y}\frac{d^2Y}{d\theta^2} = \alpha^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2Y}{d\theta^2} + \alpha^2 Y = 0 \qquad (E_1)$$

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \left(\frac{\rho\omega^2}{P} - \frac{\alpha^2}{r^2}\right)X = 0 \qquad (E_2)$$

Solution of (E_2) is $X(r) = C_1 J_m(\gamma\cdot r) + C_2 I_m(\gamma\cdot r)$; $m = 0,1,2,\ldots$

where J_m and I_m are Bessel functions of the first and second kinds, respectively, and $\gamma = \frac{\rho\omega^2}{P}$

Since $I_m(\gamma\cdot r) \to \infty$ when $r \to 0$, $C_2 = 0$ to keep $X(r)$ and hence $w(r,\theta,t)$ finite.

$$\therefore X(r) = C_1 J_m(\gamma\cdot r)$$

At $r = R$, $X(r) = 0$ i.e., $J_m(\gamma\cdot R) = 0$ $\qquad (E_3)$

Eq. (E_3) has several roots: $\gamma_1 R, \gamma_2 R, \ldots, \gamma_n R, \ldots$

$$\therefore \omega_{mn}^2 = \frac{\gamma_n P}{\rho}$$

8.57 (a) Equation of motion: $\rho\frac{\partial^2 w}{\partial t^2} = P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f \qquad (E_1)$

Let the forced response of the rectangular membrane be of the form

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (E_2)$$

where $T_{mn}(t)$ is to be determined. Eqs. (E_1) and (E_2) give

$$\pi^2 P\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)T_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$+\rho\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{d^2T_{mn}}{dt^2}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} - f = 0 \qquad (E_3)$$

Multiply (E_3) by $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ and integrate with respect to x from 0 to a and with respect to y from 0 to y to get

$$\frac{d^2T_{mn}}{dt^2}(t) + \frac{P\pi^2}{\rho}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)T_{mn}(t) = \frac{4}{ab\rho}F_{mn}(t);$$
$$m,n = 1,2,\ldots \qquad (E_4)$$

where

$$F_{mn}(t) = \int_0^a \int_0^b f(x,y,t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy \qquad (E_5)$$

Solution of (E_4) can be expressed as

$$T_{mn}(t) = A_{mn} \cos \pi \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\, t + B_{mn} \sin \pi \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\, t$$

$$+ \frac{4}{\pi \rho a b \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}} \int_0^t F_{mn}(\tau) \cdot \sin\left\{\pi \sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\,(t-\tau)\right\} d\tau \qquad (E_6)$$

where A_{mn} and B_{mn} are determined from the known initial conditions of the membrane. Thus the general solution is given by Eq. (E_2) with $T_{mn}(t)$ shown in the last equation, (E_6).

(b) For $f(x,y,t) = f_0$, (E_5) becomes

$$F_{mn}(t) = \int_0^a \int_0^b f_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$

$$= \frac{f_0\, ab}{\pi^2 mn}\left(1-\cos m\pi\right)\left(1-\cos n\pi\right)$$

$$= \begin{cases} 0 & \text{for } m \text{ or } n \text{ even} \\ \dfrac{4 f_0\, ab}{\pi^2 mn} & \text{for } m \text{ and } n \text{ odd} \end{cases}$$

Since the membrane is at rest initially, $A_{mn} = B_{mn} = 0$ and Eq. (E_6) gives

$$T_{mn}(t) = \frac{4}{\pi \rho a b \Delta} * \frac{4 f_0\, ab}{\pi^2 mn} \int_0^t \sin \pi\Delta (t-\tau) \cdot d\tau$$

$$= \frac{16 f_0}{\pi^3 mn \rho \Delta} \times \frac{(1 - \cos \pi\Delta t)}{\pi\Delta}$$

$$= \frac{16 f_0}{\pi^4 mn\rho\Delta^2}\left(1 - \cos\pi\Delta t\right) \qquad \text{for } m \text{ and } n \text{ odd}$$

where $\Delta = \left\{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\right\}^{1/2}$

Final solution is

$$w(x,y,t) = \frac{16 f_0}{\pi^4 P} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \cdot \left(1-\cos\pi\Delta t\right)$$

8.58 From example 8.11,

$$w(x,y,t) = X(x)\, Y(y)\, T(t) = \left(C_1 \cos\alpha x + C_2 \sin\alpha x\right)\left(C_3 \cos\beta y + C_4 \sin\beta y\right) * \left(A\cos\omega t + B\sin\omega t\right)$$

Boundary conditions:

$w(0,y,t) = w(a,y,t) = 0 \; ; \quad 0 \le y \le b$ and $t \ge 0$

$w(x,0,t) = w(x,b,t) = 0 \; ; \quad 0 \le x \le a$ and $t \ge 0$

i.e. $\quad X(0) = X(a) = 0 \quad , \quad Y(0) = Y(b) = 0$

i.e. $\quad C_1 = 0, \; C_3 = 0 \;\Rightarrow\; \sin \alpha a = 0, \; \sin \beta b = 0$

$$\alpha_m = \frac{m\pi}{a} \quad (m = 1,2,\dots), \quad \beta_n = \frac{n\pi}{b} \quad (n = 1,2,\dots)$$

$$\therefore \quad \omega_{mn}^2 = c^2 \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \quad ; \quad m,n = 1,2,\dots$$

and the corresponding displacement solution is

$$w_{mn}(x,y,t) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

General solution is

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

8.59 General solution

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right) \qquad (E_1)$$

If $\quad w(x,y,0) = w_0(x,y)$

and $\quad \frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$ $\Big\} ; \quad 0 \le x \le a, \; 0 \le y \le b \qquad (E_2)$

Eq. (E_1) gives

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x,y) \qquad (E_3)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x,y) \qquad (E_4)$$

These are double Fourier sine series expansions so that

$$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} w_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= \frac{4}{ab} \int_{x=0}^{a} w_0 \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} \, dx \int_{y=0}^{b} \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} \, dy \qquad (E_5)$$

$$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_{x=0}^{a} \int_{y=0}^{b} \dot{w}_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \qquad (E_6)$$

$= 0$ for given data

Using the relation $\int \sin \alpha t \sin \beta t \, dt = \frac{1}{2}\left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\}; \alpha \neq \beta$

Eq. (E_5) can be simplified as

$$A_{mn} = \frac{4 w_0}{ab} \left\{ \frac{1}{2}\left[\frac{\sin \frac{(m-1)\pi x}{a}}{\frac{(m-1)\pi}{a}} - \frac{\sin \frac{(m+1)\pi x}{a}}{\frac{(m+1)\pi}{a}} \right]_0^a \right\} \left\{ \frac{1}{2}\left[\frac{\sin \frac{(n-1)\pi y}{b}}{\frac{(n-1)\pi}{b}} - \frac{\sin \frac{(n+1)\pi y}{b}}{\frac{(n+1)\pi}{b}} \right]_0^b \right\} \tag{E_7}$$

$= 0$ for $m > 1$ and/or $n > 1$

For $m = 1$ and $n = 1$, Eq. (E_5) can be simplified as

$$A_{11} = \frac{4 w_0}{ab} \int_0^a \sin^2 \frac{\pi x}{a} \, dx \int_0^b \sin^2 \frac{\pi y}{b} \, dy = w_0 \tag{E_8}$$

$$\therefore \quad w(x,y,t) = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \omega_{11} t$$

with $\quad \omega_{11} = \left\{ c^2 \pi^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \right\}^{1/2}$

(from the solution of problem 8.52)

8.60 General solution:

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

If $w(x,y,0) = w_0(x,y)$ and $\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$ $\Big\}$; $\quad 0 \le x \le a, \quad 0 \le y \le b$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x,y)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x,y)$$

These are double Fourier sine series expansions of $w_0(x,y)$ and $\dot{w}_0(x,y)$.

$$A_{mn} = \frac{4}{ab} \int_0^a \int_0^b w_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$= 0$ for given data

$$B_{mn} = \frac{4}{ab\, \omega_{mn}} \int_0^a \int_0^b \dot{w}_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= \frac{4 \dot{w}_0}{ab\, \omega_{mn}} \int_0^a \sin \frac{m\pi x}{a} \sin \frac{\pi x}{a} \, dx \int_0^b \sin \frac{n\pi y}{b} \sin \frac{2\pi y}{b} \, dy \quad \text{---} (E_1)$$

using the relation

$$\int \sin \alpha t \cdot \sin \beta t \cdot dt = \frac{1}{2}\left\{ \frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta} \right\} \quad \text{for } \alpha \neq \beta,$$

Eq. (E_1) becomes

$$B_{mn} = \frac{4\dot{w}_0}{ab\,\omega_{mn}} \left\{ \frac{1}{2} \left(\frac{\sin\frac{(m-1)\pi x}{a}}{m-1} - \frac{\sin\frac{(m+1)\pi x}{a}}{m+1} \right)\Bigg|_0^a \right\} \times$$

$$\left\{ \frac{1}{2} \left(\frac{\sin\frac{(n-2)\pi y}{b}}{n-2} - \frac{\sin\frac{(n+2)\pi y}{b}}{n+2} \right)\Bigg|_0^b \right\}$$

$$= 0 \quad \text{for } m \neq 1 \text{ and } n \neq 2.$$

$$B_{12} = \frac{4\dot{w}_0}{ab\,\omega_{mn}} \left(\int_0^a \sin^2\frac{\pi x}{a}\,dx \right)\left(\int_0^b \sin^2\frac{2\pi y}{b}\,dy \right)$$

$$= \frac{4\dot{w}_0}{ab\,\omega_{mn}} \left\{ \left(\frac{x}{2} - \frac{\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}}{\left(\frac{2\pi}{a}\right)} \right)\Bigg/_0^a \right\} \left\{ \left(\frac{y}{2} - \frac{\sin\frac{2\pi y}{b}\cos\frac{2\pi y}{b}}{\left(\frac{4\pi}{b}\right)} \right)\Bigg/_0^b \right\}$$

$$= \frac{4\dot{w}_0}{ab\,\omega_{12}} \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \frac{\dot{w}_0}{\omega_{12}}$$

$$\therefore\ w(x,y,t) = \frac{\dot{w}_0}{\omega_{12}} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} \sin\omega_{12} t$$

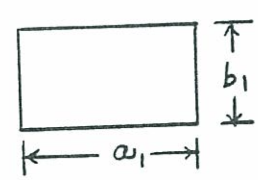
8.61 Fundamental natural frequency of transverse vibration is given by:

$$\omega_{11}^2 = c^2\pi^2\left(\frac{1}{a_1^2} + \frac{1}{b_1^2}\right) \quad \text{for rectangular membrane of sides } a_1 \text{ and } b_1 \quad \ldots (E_1)$$

with $\quad c^2 = P/\rho \quad \ldots (E_2)$

(a) For $a_1 = b_1 = a$, $\quad \omega_{11}^2 = c^2\pi^2\left(\frac{2}{a^2}\right) \quad \ldots (E_3)$

(c) For $a_1 = 2b_1$, $\quad$ area $= a_1 b_1 = a^2 = 2b_1^2$



$\therefore\ b_1 = a/\sqrt{2}, \quad a_1 = \sqrt{2}\,a$

$$\omega_{11}^2 = c^2\pi^2\left(\frac{1}{a_1^2} + \frac{1}{b_1^2}\right) = c^2\pi^2\left(\frac{5}{2a^2}\right) \quad \ldots (E_4)$$

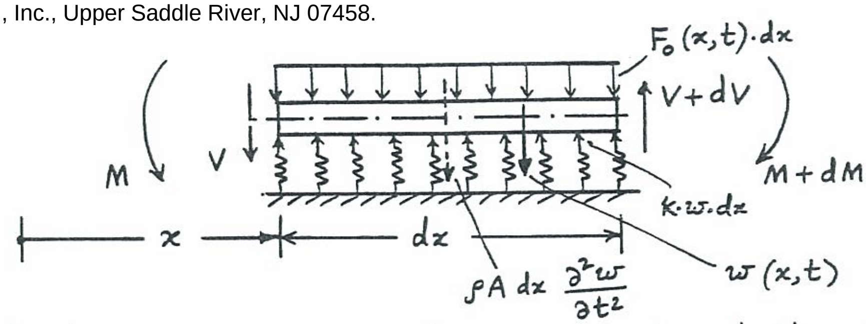
(b) From solution of problem 8.50, the fundamental natural frequency of a circular membrane of radius R is given by

$$\omega_{01}^2 = \frac{\gamma_1 P}{\rho} = c^2\gamma_1 \quad \text{where } \gamma_1 R = 2.404 = \text{first zero of Bessel function of the first kind.}$$

Since $\pi R^2 = a^2$, $R = a/\sqrt{\pi}$ and

$$\omega_{01}^2 = c^2\left(\frac{2.404}{R}\right) = c^2\left(\frac{4.2610}{a}\right) \quad \ldots (E_5)$$

8.62



(a) If load due to the car moves with a constant velocity v_0 in the x-direction,

$$F(x,t) = F(x - v_0 t) \tag{E1}$$

Equilibrium equations are

$$(V+dV) - V - F_0(x,t)\, dx + k w\, dx = -\rho A\, dx \frac{\partial^2 w}{\partial t^2}$$

or
$$\frac{\partial V}{\partial x} - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \tag{E2}$$

and
$$(M+dM) - M - (V+dV)\, dx + F(x,t)\, dx \cdot \frac{dx}{2} = 0$$

or
$$\frac{\partial M}{\partial x} - V = 0 \tag{E3}$$

Using (E_3), (E_2) can be rewritten as

$$\frac{\partial^2 M}{\partial x^2} - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \tag{E4}$$

But
$$M = EI \frac{\partial^2 w}{\partial x^2} \tag{E5}$$

Eqs. (E_4) and (E_5) lead to

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) - F_0(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \tag{E6}$$

For a uniform beam, (E_6) simplifies to, in view of (E_1),

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + k w = F_0(x - v_0 t) \tag{E7}$$

(b) Solution:

Defining
$$y = x - v_0 t \tag{E8}$$

Eq. (E_7) can be expressed as

$$EI \frac{d^4 w}{dy^4} + \rho A v_0^2 \frac{d^2 w}{dy^2} + k w = F_0(y) \tag{E9}$$

Let F_0 = concentrated load. Then the governing equation at all points of the beam, except at $y = 0$, is

$$EI \frac{d^4 w}{dy^4} + \rho A v_0^2 \frac{d^2 w}{dy^2} + k w = 0 \tag{E10}$$

Solution of (E_{10}) can be expressed as

$$w(y) = e^{\alpha y} \tag{E_{11}}$$

Substitution of (E_{11}) into (E_{10}) gives

$$EI\,\alpha^4 + \rho A\, v_0^2 + k = 0 \tag{E_{12}}$$

whose roots can be expressed as

$$\alpha_{1,2} = \pm\,(a + i\,b) \quad , \quad \alpha_{3,4} = \pm\,(a - i\,b) \tag{E_{13}}$$

with

$$\left.\begin{array}{l} a = \sqrt{1-c}\; d\ , \quad b = \sqrt{1+c}\; d\ , \quad c = \dfrac{v_0^2}{\sqrt{\dfrac{4EIk}{\rho^2 A^2}}} \\ \text{and} \quad d = \left(\dfrac{k}{4EI}\right)^{1/4} \end{array}\right\} \tag{E_{14}}$$

Thus the solution of (E_{10}) becomes

$$w(y) = A_1\, e^{\alpha_1 y} + A_2\, e^{\alpha_2 y} + A_3\, e^{\alpha_3 y} + A_4\, e^{\alpha_4 y} \tag{E_{15}}$$

where $A_i\ (i = 1, 2, 3, 4)$ are constants which can be determined from the following conditions:

$$\left.\begin{array}{l} w = 0 \quad \text{at} \quad y = \infty \\ \dfrac{d^2 w}{dy^2} = 0 \quad \text{at} \quad y = \infty \end{array}\right\} \text{deflection \& bending moment are zero at } y = \infty$$

$$\left.\dfrac{dw}{dy} = 0 \quad \text{at} \quad y = 0 \right\} \text{slope is zero under the load}$$

$$\left.\begin{array}{l} EI \left.\dfrac{d^3 w}{dy^3}\right|_{y=0^+} - EI \left.\dfrac{d^3 w}{dy^3}\right|_{y=0^-} = P \\ \text{i.e.,} \qquad EI\, \dfrac{d^2 w}{dy^2} = \dfrac{P}{2} \end{array}\right\} \text{shear force has discontinuity under the load}$$

8.63 $W(x) = \dfrac{c_0\, x^2}{24\, EI}\,(l - x)^2 = \dfrac{c_0}{24 EI}\,(x^2 l^2 + x^4 - 2 l x^3)$

$$\frac{dW}{dx} = \frac{c_0}{24EI}\,(2x l^2 + 4x^3 - 6 l x^2)$$

$$\frac{d^2W}{dx^2} = \frac{c_0}{24EI}\,(2l^2 + 12x^2 - 12 l x)$$

$$N = EI \int_0^l \left(\frac{d^2W}{dx^2}\right)^2 dx = 4EI\left(\frac{c_0}{24EI}\right)^2 \int_0^l (l^2 + 6x^2 - 6lx)^2\, dx$$

$$= \frac{c_0^2\, l^5}{720\; EI} \tag{E_1}$$

$$D = \rho A \int_0^l \left(W(x)\right)^2 dx = \rho A \left(\frac{c_0}{24EI}\right)^2 \int_0^l \left(x^2 l^2 + x^4 - 2 l x^3\right)^2 dx$$

$$= \frac{\rho A\, l^9\, c_0^2}{362\,880\; E^2 I^2} \tag{E_2}$$

$$\omega^2 = \frac{N}{D} = \left(\frac{c_o^2 \ell^5}{720 EI}\right)\left(\frac{362880\ E^2 I^2}{\rho A \ell^9 c_o^2}\right) = 504 \frac{EI}{\rho A \ell^4}$$

$$\therefore \quad \omega = 22.4499 \sqrt{\frac{EI}{\rho A \ell^4}}$$

This can be compared with the exact solution (Fig. 8.15):

$$\omega_{exact} = (4.730041)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 22.3733 \sqrt{\frac{EI}{\rho A \ell^4}}.$$

8.64

$$W(x) = c_o\left(1 - \cos\frac{2\pi x}{\ell}\right)$$

$$\frac{dW}{dx} = c_o \frac{2\pi}{\ell} \sin\frac{2\pi x}{\ell} \quad ; \quad \frac{d^2W}{dx^2} = c_o\left(\frac{2\pi}{\ell}\right)^2 \cos\frac{2\pi x}{\ell}$$

$$N = EI \int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = EI\, c_o^2 \left(\frac{2\pi}{\ell}\right)^4 \int_0^\ell \cos^2\frac{2\pi x}{\ell}\,.dx$$

$$= 8 EI\, c_o^2 \pi^4 / \ell^3$$

$$D = \rho A \int_0^\ell (W(x))^2 dx = \rho A\, c_o^2 \int_0^\ell \left(1 + \cos^2\frac{2\pi x}{\ell} - 2\cos\frac{2\pi x}{\ell}\right)dx$$

$$= 3 c_o^2 \rho A \ell / 2$$

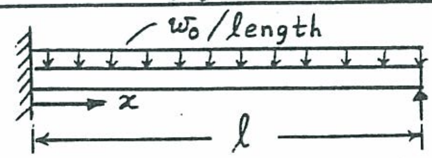
$$\omega^2 = \frac{N}{D} = \frac{16\pi^4}{3}\cdot\frac{EI}{\rho A \ell^4}$$

$$\therefore \omega = 22.7930 \sqrt{\frac{EI}{\rho A \ell^4}}$$. Compare this with

$$\omega_{exact} = (4.730041)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 22.3733 \sqrt{\frac{EI}{\rho A \ell^4}}.$$

8.65

From strength of materials, the static deflection curve is given by



$$W(x) = \frac{w_o x^2}{48EI}(\ell - x)(2x - 3\ell) = \frac{c_o}{48EI}(5\ell x^3 - 2x^4 - 3\ell^2 x^2)$$

$$\frac{dW}{dx} = \frac{c_o}{48EI}(15\ell x^2 - 8x^3 - 6\ell^2 x)$$

$$\frac{d^2W}{dx^2} = \frac{c_o}{8EI}(5\ell x - 4x^2 - \ell^2)$$

$$N = EI \int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{c_o^2 \ell^5}{320\ EI}$$

$$D = \rho A \int_0^l \left(W(x)\right)^2 dx = 1.3090 \times 10^{-5} \frac{c_0^2 \rho A l^9}{E^2 I^2}$$

$$\omega^2 = \frac{N}{D} = 238.7319 \frac{EI}{\rho A l^4}$$

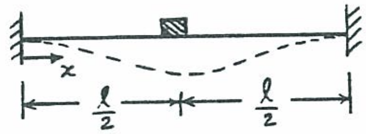
$$\therefore \quad \omega = 15.4510 \sqrt{\frac{EI}{\rho A l^4}}.$$

This can be compared with the exact value

$$\omega_{exact} = (3.926602)^2 \sqrt{\frac{EI}{\rho A l^4}} = 15.4182 \sqrt{\frac{EI}{\rho A l^4}}.$$

8.66

$$W(x) = \begin{cases} \dfrac{x^2}{48EI}(-4x+3l) \; ; & 0 \le x \le \dfrac{l}{2} \\ \dfrac{(l-x)^2}{48EI}(4x-l) \; ; & \dfrac{l}{2} \le x \le l \end{cases}$$



$$\frac{d^2W}{dx^2} = \begin{cases} \dfrac{1}{8EI}(-4x+l) \; ; & 0 \le x \le \dfrac{l}{2} \\ \dfrac{1}{8EI}(4x-3l) \; ; & \dfrac{l}{2} \le x \le l \end{cases}$$

$$T_{max} \text{ of beam} = \frac{\omega^2}{2} \int_0^l \rho A \left[W(x)\right]^2 dx$$

$$= \frac{\omega^2 \rho A}{2} \left(\frac{1}{48EI}\right)^2 \left[\int_0^{l/2} x^4 \left(16x^2 + 9l^2 - 24xl\right) dx + \int_{l/2}^{l} (l-x)^4 (4x-l)^2 \, dx \right]$$

$$= \frac{\rho A \omega^2}{4608 E^2 I^2} \left\{ \left[\frac{16x^7}{7} + \frac{9l^2x^5}{5} - \frac{24lx^6}{6} \right]_0^{l/2} + \left[\frac{16x^7}{7} - \frac{72lx^6}{6} + \frac{129l^2x^5}{5} - \frac{116l^3x^4}{4} + \frac{54l^4x^3}{3} - \frac{12l^5x^2}{2} + l^6 x \right]_{l/2}^{l} \right\}$$

$$= 5 \times 10^{-6} \left(\frac{\rho A \omega^2 l^7}{E^2 I^2} \right)$$

$$T_{max} \text{ of mass} = \frac{1}{2} m \, \dot{w}_{max}^2 \Big|_{x=\frac{l}{2}} = \frac{1}{2} m \omega^2 W^2\left(x = \frac{l}{2}\right)$$

$$= \frac{1}{2} m \omega^2 \left(\frac{l^3}{192EI}\right)^2 = 13.6 \times 10^{-6} \left(\frac{m\omega^2 l^6}{E^2 I^2}\right)$$

$$T_{max} \text{ total} = 5 \times 10^{-6} \frac{\rho A \omega^2 l^7}{E^2 I^2} + 13.6 \times 10^{-6} \frac{m\omega^2 l^6}{E^2 I^2}$$

$$V_{max} = \frac{1}{2}\int_0^l EI\left(\frac{d^2W}{dx^2}\right)^2 dx$$

$$= \frac{1}{128\,EI}\left[\int_0^{\frac{l}{2}} (16x^2 + l^2 - 8lx)\,dx + \int_{l/2}^{l} (16x^2 + 9l^2 - 24lx)\,dx\right]$$

$$= \frac{1}{128\,EI}\left[\left(\frac{16}{3}x^3 + l^2x - \frac{8lx^2}{2}\right)_0^{l/2} + \left(\frac{16}{3}x^3 + 9l^2x - \frac{24lx^2}{2}\right)_{l/2}^{l}\right]$$

$$= \frac{l^3}{384\,EI}$$

T_{max} total $= V_{max}$ gives

$$\omega^2 = \frac{l^3}{384\,EI}\cdot\frac{E^2I^2}{\left(5\times10^{-6}\rho A l^7 + 13.6\times10^{-6}\, m\, l^6\right)} = \frac{2590.7\,EI}{l^3(5\rho A l + 13.6\,m)}$$

$$\therefore\ \omega = 50.8987\sqrt{\frac{EI}{l^3(5M_b + 13.6\,m)}}$$ where $M_b = \rho A l$ = mass of beam

---

8.67 Let $W(x) = c_0\left(1 - \frac{x}{l}\right)^2 = c_0\left(1 + \frac{x^2}{l^2} - \frac{2x}{l}\right)$

$$\frac{d^2W}{dx^2} = 2c_0\left(\frac{1}{l^2}\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho A(x)\,[W(x)]^2\,dx = \frac{\rho\omega^2 A_0 c_0^2}{2l}\int_0^l x\left(1 + \frac{x^2}{l^2} - \frac{2x}{l}\right)^2 dx$$

$$= \frac{\rho\omega^2 A_0 c_0^2}{2l}\left[\frac{x^2}{2} + \frac{x^6}{6l^4} + \frac{4x^4}{4l^2} + \frac{2x^4}{4l^2} - \frac{4x^5}{5l^3} - \frac{4x^3}{3l}\right]_0^l$$

$$= \frac{\rho\omega^2 A_0 c_0^2\, l}{60}$$

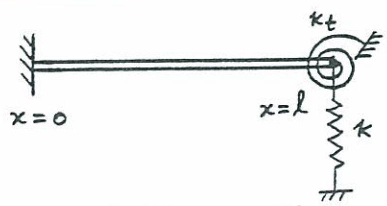
$$V_{max} = \frac{1}{2}\int_0^l EI(x)\left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{E}{2}\int_0^l \left(\frac{I_0 x}{l}\right)\left(\frac{4c_0^2}{l^4}\right) dx = \frac{EI_0 c_0^2}{l^3}$$

$T_{max} = V_{max}$ gives

$$\omega^2 = \frac{EI_0 c_0^2}{l^3}\cdot\frac{60}{\rho A_0 c_0^2 l} = \frac{60\,EI_0}{\rho A_0 l^4}$$

$$\omega = 7.7460\sqrt{\frac{EI_0}{\rho A_0 l^4}}$$

8.68 Let $W(x) = \frac{c_0}{24EI}\left(x^4 - 4lx^3 + 6l^2x^2\right)$



$$\frac{d^2W}{dx^2} = \frac{c_0}{2EI}\left(x^2 - 2lx + l^2\right)$$

$$V_{max} = \frac{EI}{2}\int_0^l \left(\frac{d^2W}{dx^2}\right)^2 dx + \frac{1}{2}k\,[W(x=l)]^2 + \frac{1}{2}k_t\left[\frac{dW}{dx}(x=l)\right]^2$$

Since $\frac{k}{2}\left[W(x=l)\right]^2 = \frac{k}{2}\, c_0^2\, \frac{l^8}{64 E^2 I^2}$,

$\frac{k_t}{2}\left[\frac{dW}{dx}(x=l)\right]^2 = \frac{k_t}{2}\, c_0^2\, \frac{l^6}{36 E^2 I^2}$,

and $\frac{1}{2} EI \int_0^l \left(\frac{d^2 W}{dx^2}\right)^2 dx = \frac{c_0^2\, l^5}{40\, EI}$ (from solution of problem 8.57)

$$V_{max} = \frac{c_0^2\, l^5}{40\, EI} + \frac{k\, c_0^2\, l^8}{128\, E^2 I^2} + \frac{k_t\, c_0^2\, l^6}{72\, E^2 I^2}$$

$$T_{max} = \frac{\omega^2 \rho A}{2} \int_0^l \left[W(x)\right]^2 dx = \omega^2 \rho A \left(\frac{13\, c_0^2\, l^9}{6480\, E^2 I^2}\right)$$ from problem 8.57

$V_{max} = T_{max}$ gives

$$\omega^2 = \left(12.4615\, \frac{EI}{\rho A l^4} + 3.8942\, \frac{k}{\rho A l} + 6.9231\, \frac{k_t}{\rho A l^4}\right)^{1/2}$$

8.69 $W(x) = c_1\left(1 - \cos\frac{2\pi x}{l}\right)$, $\quad \frac{d^2 W}{dx^2} = c_1\left(\frac{2\pi}{l}\right)^2 \cos\frac{2\pi x}{l}$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho A\left[W(x)\right]^2 dx = \frac{\omega^2 \rho A c_1^2}{2}\int_0^l \left(1 + \cos^2\frac{2\pi x}{l} - 2\cos\frac{2\pi x}{l}\right) dx$$

$$= \frac{\rho A \omega^2 c_1^2}{2}\left[x + \left(\frac{x}{2} + \frac{l}{8\pi}\sin\frac{4\pi x}{l}\right) + \left(\frac{l}{\pi}\sin\frac{2\pi x}{l}\right)\right]_0^l$$

$$= \frac{3}{4}\rho A \omega^2 c_1^2\, l$$

$$V_{max} = \frac{EI}{2}\int_0^l \left(\frac{d^2 W}{dx^2}\right)^2 dx = \frac{EI}{2}\int_0^l c_1^2 \left(\frac{2\pi}{l}\right)^4 \cos^2\frac{2\pi x}{l}\, dx$$

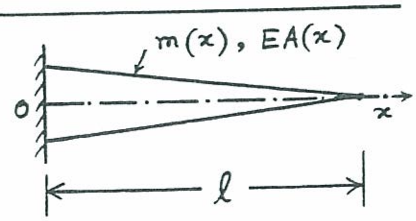
$$= \frac{8 EI\, c_1^2 \pi^4}{l^4}\left(\frac{x}{2} + \frac{l}{8\pi}\sin\frac{4\pi x}{l}\right)_0^l = \frac{4 EI\, c_1^2\, \pi^4}{l^3}$$

$T_{max} = V_{max}$ gives

$$\omega^2 = \frac{4 EI\, c_1^2\, \pi^4}{l^3}\cdot\frac{4}{3\rho A c_1^2\, l} = \frac{519.52\, EI}{\rho A l^4}$$

$$\omega = 22.7930\sqrt{\frac{EI}{\rho A l^4}}$$

8.70 $U(x) = c_1 \sin\frac{\pi x}{2l}$



$\frac{dU}{dx} = c_1\, \frac{\pi}{2l} \cos\frac{\pi x}{2l}$

$$V_{max} = \frac{1}{2}\int_0^l EA(x)\left(\frac{dU}{dx}\right)^2 dx$$

$$= \frac{1}{2}\int_0^l 2EA_0\left(1-\frac{x}{l}\right)c_1^2\left(\frac{\pi}{2l}\right)^2\cos^2\frac{\pi x}{2l}\cdot dx$$

$$= EA_0\, c_1^2\left(\frac{\pi}{2l}\right)^2\left[\int_0^l \cos^2\frac{\pi x}{2l}\,dx - \int_0^l \frac{x}{l}\cos^2\frac{\pi x}{2l}\,dx\right]$$

$$= EA_0\, c_1^2\left(\frac{\pi}{2l}\right)^2\left[\left(\frac{x}{2}+\frac{l}{2\pi}\sin\frac{\pi x}{l}\right)_0^l - \frac{1}{l}\left(\frac{x^2}{4}+\frac{x\sin\frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)}+\frac{\cos\frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)}\right)_0^l\right]$$

$$= \frac{EA_0\, c_1^2\pi^2}{16\, l}\left(1+\frac{4}{\pi^2}\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l m(x)\, U^2\, dx = \frac{\omega^2}{2}\int_0^l 2m_0\left(1-\frac{x}{l}\right)c_1^2\sin^2\frac{\pi x}{2l}\cdot dx$$

$$= \omega^2 m_0\, c_1^2\left[\int_0^l \sin^2\frac{\pi x}{2l}\cdot dx - \int_0^l \frac{x}{l}\sin^2\frac{\pi x}{2l}\,dx\right]$$

$$= \omega^2 m_0\, c_1^2\left[\left(\frac{x}{2}-\frac{l}{2\pi}\sin\frac{\pi x}{l}\right)_0^l - \frac{1}{l}\left(\frac{x^2}{4}-\frac{x\sin\frac{\pi x}{l}}{\left(\frac{2\pi}{l}\right)}-\frac{\cos\frac{\pi x}{l}}{\left(\frac{8\pi^2}{4l^2}\right)}\right)_0^l\right]$$

$$= \omega^2 m_0\, c_1^2\,\frac{l}{4}\left(1-\frac{4}{\pi^2}\right)$$

$V_{max} = T_{max}$ gives

$$\omega^2 = \frac{EA_0\pi^2}{16\, l}\;\frac{\left(1+\frac{4}{\pi^2}\right)4}{m_0\, l\left(1-\frac{4}{\pi^2}\right)} = 5.8303\,\frac{EA_0}{m_0\, l^2}$$

$$\therefore\ \omega_1 = 2.4146\sqrt{\frac{EA_0}{m_0\, l^2}}$$

8.71 We take the deflection curve satisfying the boundary conditions

$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$ as

$$w(x,y) = c_1\, xy\,(x-a)(y-b)$$

$$\frac{\partial w}{\partial x} = c_1\, y\,(x-a)(y-b) + c_1\, xy\,(y-b)$$

$$\frac{\partial w}{\partial y} = c_1\, x\,(x-a)(y-b) + c_1\, xy\,(x-a)$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = c_1^2 y^2(y-b)^2(x-a)^2 + c_1^2x^2y^2(y-b)^2 + c_1^2x^2(x-a)^2(y-b)^2 + c_1^2x^2y^2(x-a)^2$$

$$= c_1^2\big[2x^2y^4 + x^2y^2(2a^2+2b^2) + x^2y^3(-4b) + y^4(a^2) + y^2(a^2b^2) + y^3(-2ba^2) + xy^4(-2a) + xy^2(-2ab^2) + xy^3(4ab) + 2x^4y^2 + x^4(b^2) + x^4y(-2b) + x^2(a^2b^2) + x^2y(-2ba^2) + x^3y^2(-2a) + x^3(-2ab^2) + x^3y(4ab) + x^3y^2(-2a)\big]$$

$$\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dx\, dy = c_1^2 \left(\frac{a^3 b^5}{45} + \frac{a^5 b^3}{45} \right)$$

$$w^2(x,y) = c_1^2 \left[x^4 y^4 + a^2 x^2 y^4 + b^2 x^4 y^2 + a^2 b^2 x^2 y^2 - 2a\, x^3 y^4 - 2b\, x^4 y^3 + 2ab\, x^3 y^3 + 2ab\, x^3 y^3 - 2a^2 b\, x^2 y^3 - 2ab^2 x^3 y^2 \right]$$

$$\int_0^a \int_0^b w^2 \cdot dx\, dy = c_1^2\, a^5 b^5 / 900$$

$$V_{max} = \frac{P}{2} \frac{c_1^2}{45} (a^3 b^3)(a^2 + b^2)$$

$$T_{max} = \frac{\omega^2 \rho}{2} \cdot \frac{c_1^2\, a^5 b^5}{900}$$

$V_{max} = T_{max}$ gives $\quad \dfrac{\omega^2 \rho}{P} = 20 \left(\dfrac{1}{a^2} + \dfrac{1}{b^2} \right)$

$$\therefore\ \omega_1 = 4.4721 \sqrt{\frac{P}{\rho}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$$

Exact value of $\omega_1 = \pi \sqrt{\dfrac{P}{\rho}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)}$

8.72 For a shaft under torsion, the shear stress (τ) induced at a radius r from the center of the cross section is given by

$$\tau = \frac{M_t(x)\, r}{J} \tag{1}$$

where $M_t(x)$ = torque at section x. The potential energy of the shaft is given by the strain energy:

$$V = \frac{1}{2} \int \frac{1}{G} \tau^2\, dA\, dx \tag{2}$$

Using Eq. (8.61):

$$M_t(x) = GJ \frac{\partial \theta}{\partial x} \tag{3}$$

Eq. (2) can be expressed as

$$V = \frac{1}{2} \int_0^{\ell} GJ \left(\frac{\partial \theta}{\partial x} \right)^2 dx \tag{4}$$

The kinetic energy of the shaft can be written as

$$T = \frac{1}{2} \int_0^{\ell} \rho\, J \left(\frac{\partial \theta}{\partial t} \right)^2 dx \tag{5}$$

By assuming a harmonic variation of $\theta(x,t)$ as

$$\theta(x,\ t) = \Theta(x) \cos \omega t \tag{6}$$

and equating V_{max} to T_{max}, we obtain

$$\omega^2 = \frac{\int_0^\ell GJ \left(\frac{d\Theta}{dx}\right)^2 dx}{\int_0^\ell \rho\, J\, \Theta^2\, dx} \tag{7}$$

For the shaft shown in Fig. 8.41,

$$G = 80\ (10^9)\ \mathrm{N/m^2} \quad ; \quad \rho\, g = 76.5\ \mathrm{kN/m^3}$$

$$J = \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\,(0.05^4) = 61.3594\ (10^{-8})\ \mathrm{m^4}$$

Let $\Theta(x)$ be assumed to vary linearly on either side of the disc for simplicity:

$$\begin{aligned} \Theta(x) &= \frac{\theta_0\, x}{0.8} \quad ; \; 0 \le x \le 0.8 \\ &= \frac{\theta_0\,(1-x)}{0.2} \quad ; \; 0.8 \le x \le 1 \end{aligned} \tag{8}$$

where θ_0 is the angular displacement of the disc. Equation (8) satisfies the boundary conditions and gives:

$$\begin{aligned} \frac{d\Theta}{dx} &= \frac{\theta_0}{0.8} \quad ; \; 0 \le x \le 0.8 \\ &= -\frac{\theta_0}{0.2} \quad ; \; 0.8 \le x \le 1.0 \end{aligned} \tag{9}$$

$$\int_0^\ell GJ \left(\frac{d\Theta}{dx}\right)^2 dx = 306\ 797\ \theta_0^2 \quad ; \quad \int_0^\ell \rho\, J\, \Theta^2\, dx = 159.5384\ (10^{-5})\ \theta_0^2$$

Thus Eq. (7) gives the approximate natural frequency as:

$$\omega^2 \approx \frac{306797\ \theta_0^2}{159.5384\ (10^{-5})\ \theta_0^2} = 1923.0292\ (10^5)$$

$$\omega \approx 13{,}867.3328\ \mathrm{rad/sec}$$

For comparison, the torsional natural frequency of a fixed-fixed shaft with no intermediate disc, is given by (see Fig. 8.12):

$$\omega_1 = \frac{\pi\, c}{\ell} = \frac{\pi}{\ell}\sqrt{\frac{G}{\rho}} = \frac{\pi}{1.0}\sqrt{\frac{80\ (10^9)\ (9.81)}{76500}} = 10{,}062.3557\ \mathrm{rad/sec}$$

8.73 (a) $W(x) = c_1 x (l - x)$, Let $P =$ tension

$$V_{max} = \frac{1}{2}\int_0^l P\left(\frac{dW}{dx}\right)^2 dx = \frac{1}{2} c_1^2 P \int_0^l (l - 2x)^2 dx$$

$$= \frac{1}{2} P c_1^2 \left[l^2 x + \frac{4}{3} x^3 - 2 l x^2 \right]_0^l = \frac{1}{6} P c_1^2 l^3$$

$$T_{max} = \frac{\omega^2}{2} \int_0^l \rho [W]^2 dx = \frac{1}{2}\rho c_1^2 \omega^2 \int_0^l (x^2 l^2 + x^4 - 2 l x^3) dx$$

$$= \frac{1}{2}\rho c_1^2 \omega^2 \left[\frac{1}{3} x^3 l^2 + \frac{1}{5} x^5 - \frac{1}{2} l x^4 \right]_0^l = \frac{1}{60} \omega^2 \rho c_1^2 l^5$$

$V_{max} = T_{max}$ gives

$$\omega = \left(\frac{10 P}{\rho l^2}\right)^{1/2} = 3.1623 \sqrt{\frac{P}{\rho l^2}}$$

Exact value is $\omega_1 = \pi \sqrt{\frac{P}{\rho l^2}}$.

(b) $W(x) = c_1 x (l - x) + c_2 x^2 (l - x)^2$

$$\frac{dW}{dx} = c_1 (l - 2x) + c_2 (2 l^2 x + 4 x^3 - 6 l x^2)$$

$$V_{max} = \frac{1}{2}\int_0^l P\left(\frac{dW}{dx}\right)^2 dx = \frac{P}{2}\int_0^l \Big[c_1^2 (l^2 + 4 x^2 - 4 l x)$$

$$+ c_2^2 (4 l^4 x^2 + 16 x^6 + 36 l^2 x^4 + 16 l^2 x^4 - 48 l x^5 - 24 l^3 x^3)$$

$$+ 2 c_1 c_2 (2 l^3 x + 4 l x^3 - 6 l^2 x^2 - 4 l^2 x^2 - 8 x^4 + 12 l x^3) \Big] dx$$

$$= \frac{P}{2}\left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5 \right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^l \rho [W(x)]^2 dx = \frac{\rho\omega^2}{2}\int_0^l \Big[c_1^2 (l^2 x^2 + x^4 - 2 l x^3)$$

$$+ c_2^2 (l^4 x^4 + x^8 + 6 l^2 x^6 - 4 l x^7 - 4 l^3 x^5)$$

$$+ c_1 c_2 (2 l^3 x^3 + 6 l x^5 - 6 l^2 x^4 - 2 x^6) \Big] dx$$

$$= \frac{\rho\omega^2}{2}\left[\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7 \right]$$

$$X = P\left(\frac{1}{3} c_1^2 l^3 + \frac{2}{105} c_2^2 l^7 + \frac{2}{15} c_1 c_2 l^5 \right)$$

$$Y = \rho\left(\frac{1}{30} c_1^2 l^5 + \frac{1}{630} c_2^2 l^9 + \frac{1}{70} c_1 c_2 l^7 \right)$$

$$\frac{\partial X}{\partial c_1} = P\left(\frac{2}{3} c_1 l^3 + \frac{2}{15} c_2 l^5 \right), \quad \frac{\partial X}{\partial c_2} = P\left(\frac{4}{105} c_2 l^7 + \frac{2}{15} c_1 l^5 \right)$$

$$\frac{\partial Y}{\partial c_1} = \rho\left(\frac{1}{15} c_1 l^5 + \frac{1}{70} c_2 l^7 \right), \quad \frac{\partial Y}{\partial c_2} = \rho\left(\frac{1}{315} c_2 l^9 + \frac{1}{70} c_1 l^7 \right)$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0, \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \quad \text{give}$$

$$P\left(c_1 \tfrac{2}{3}l^3 + c_2 \tfrac{2}{15} l^5\right) - \omega^2 \rho \left(c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70}\right) = 0$$

$$P\left(c_2 \frac{4l^7}{105} + c_1 \frac{2l^5}{15}\right) - \omega^2 \rho \left(c_2 \frac{l^9}{315} + c_1 \frac{l^7}{70}\right) = 0$$

i.e., $$c_1\left(\tfrac{2}{3} P l^3 - \frac{\rho\omega^2 l^5}{15}\right) + c_2\left(\tfrac{2}{15} P l^5 - \frac{\rho \omega^2 l^7}{70}\right) = 0$$

$$c_1\left(\tfrac{2}{15} P l^5 - \frac{\rho\omega^2 l^7}{70}\right) + c_2\left(\tfrac{4}{105} P l^7 - \frac{\rho \omega^2 l^9}{315}\right) = 0$$

Setting the determinant of the coefficients of c_1 and c_2 to zero gives

$$\left(\tfrac{2}{3} P l^3 - \tfrac{1}{15}\rho\omega^2 l^5\right)\left(\tfrac{4}{105} P l^7 - \tfrac{1}{315}\rho\omega^2 l^9\right) - \left(\tfrac{2}{15} P l^5 - \tfrac{1}{70}\rho\omega^2 l^7\right)\left(\tfrac{2}{15} P l^5 - \tfrac{1}{70}\rho\omega^2 l^7\right) = 0$$

i.e., $$\omega^4 - 112\left(\frac{P}{\rho l^2}\right)\omega^2 + 1008\left(\frac{P}{\rho l^2}\right)^2 = 0$$

$$\omega^2 = 9.8697 \frac{P}{\rho l^2}, \quad 102.1302 \frac{P}{\rho l^2}$$

$$\therefore \quad \omega_1 = 3.1416\sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 10.1059\sqrt{\frac{P}{\rho l^2}}$$

First mode — Third mode

Exact solution is :

$$\omega_1 = 3.1416\sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 6.2832\sqrt{\frac{P}{\rho l^2}}, \quad \omega_3 = 9.4248\sqrt{\frac{P}{\rho l^2}}$$

8.74 (a)

$$U(x) = \frac{c_1}{l} x, \qquad \frac{dU}{dx} = \frac{c_1}{l}$$

$$V_{max} = \frac{1}{2}\int_0^l EA\left(\frac{dU}{dx}\right)^2 dx = \frac{1}{2}\int_0^l EA\left(\frac{c_1}{l}\right)^2 dx = \frac{EA c_1^2}{2l} \text{ for uniform beam}$$

$$T_{max} = \frac{\rho\omega^2}{2}\int_0^l A\, U^2 dx = \frac{\rho\omega^2}{2}\int_0^l A \frac{c_1^2}{l^2} x^2 dx = \frac{\omega^2 \rho A c_1^2 l}{6} \text{ for uniform beam}$$

$T_{max} = V_{max}$ gives $\omega^2 = \dfrac{3E}{\rho l^2}$

$$\therefore \omega_1 = 1.73205\sqrt{\frac{E}{\rho l^2}}$$

$$U(x) = \frac{c_1 x}{l} + \frac{c_2 x^2}{l^2}$$

$$\frac{dU}{dx} = \frac{c_1}{l} + \frac{2 c_2 x}{l^2}$$

(b) $$V_{max} = \frac{EA}{2} \int_0^l \left(\frac{dU}{dx} \right)^2 dx$$

$$= \frac{EA}{2} \left[\frac{c_1^2}{l} + \frac{4}{3} \frac{c_2^2}{l} + \frac{2 c_1 c_2}{l} \right]$$

$$T_{max} = \frac{\rho \omega^2}{2} \int_0^l A \cdot U^2 \cdot dx$$

$$= \frac{\rho A \omega^2}{2} \left[\frac{1}{3} c_1^2 l + \frac{1}{5} c_2^2 l + \frac{1}{2} c_1 c_2 l \right]$$

Eqs. (E_{11}) and (E_{12}) of Example 8.12 give

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \qquad (E.1)$$

where $$X = \frac{EA}{2l} \left[c_1^2 + \frac{4}{3} c_2^2 + 2 c_1 c_2 \right] \qquad (E.2)$$

and $$Y = \frac{\rho A l}{2} \left[\frac{1}{3} c_1^2 + \frac{1}{5} c_2^2 + \frac{1}{2} c_1 c_2 \right] \qquad (E.3)$$

Eqs. (E.1) to (E.3) give

$$c_1 \left(\frac{EA}{l} - \frac{1}{3} \omega^2 \rho A l \right) + c_2 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

$$c_2 \left(\frac{4}{3} \frac{EA}{l} - \frac{1}{5} \omega^2 \rho A l \right) + c_1 \left(\frac{EA}{l} - \frac{1}{4} \omega^2 \rho A l \right) = 0$$

Frequency equation is

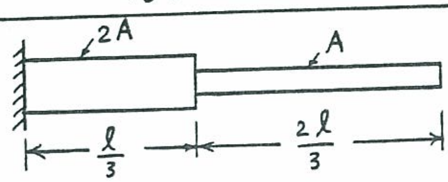
$$\begin{vmatrix} \left(1 - \frac{1}{3}\lambda\right) & \left(1 - \frac{1}{4}\lambda\right) \\ \left(1 - \frac{1}{4}\lambda\right) & \left(\frac{4}{3} - \frac{1}{5}\lambda\right) \end{vmatrix} = \frac{\lambda^2}{240} - \frac{13\lambda}{90} + \frac{1}{3} = 0$$

where $$\lambda = \frac{\omega^2 \rho l^2}{E}$$

$$\lambda = 2.48596, \quad 32.1807$$

$$\therefore \omega_1 = 1.57669 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 5.67280 \sqrt{\frac{E}{\rho l^2}}$$

8.75



$$U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3\pi x}{2l}$$

$$\frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3\pi x}{2l} \right)$$

$$V_{max} = \frac{EA}{2} \left(\frac{\pi}{2l} \right)^2 \left[\int_0^{l/3} 2 \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3\pi x}{2l} \right)^2 dx \right.$$

$$\left. + \int_{l/3}^{l} \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3\pi x}{2l} \right)^2 dx \right]$$

Using the relations $\int \cos^2 \frac{\pi x}{2l} dx = \frac{x}{2} + \frac{l}{2\pi} \sin \frac{\pi x}{l}$,

$$\int \cos^2 \frac{3\pi x}{2l}\, dx = \frac{x}{2} + \frac{l}{6\pi} \sin \frac{3\pi x}{l},$$

$$\int \cos \frac{\pi x}{2l} \cdot \cos \frac{3\pi x}{2l} \cdot dx = \frac{\sin \frac{\pi x}{l}}{(2\pi/l)} + \frac{\sin \frac{2\pi x}{l}}{(4\pi/l)},$$

$$V_{max} = \frac{EA\pi^2}{8l^2}\left[c_1^2\left(\frac{2}{3}l + \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2(6l) + c_1c_2\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$T_{max} = \frac{\rho\omega^2}{2}\left[\int_0^{\frac{l}{3}} 2A\left\{c_1^2 \sin^2\frac{\pi x}{2l} + c_2^2 \sin^2 \frac{3\pi x}{2l} + 2c_1c_2 \sin\frac{\pi x}{2l}\cdot\sin\frac{3\pi x}{2l}\right\}dx + \int_{l/3}^{l} A\left\{c_1^2 \sin^2\frac{\pi x}{2l} + c_2^2\sin^2\frac{3\pi x}{2l} + 2c_1c_2\sin\frac{\pi x}{2l}\sin\frac{3\pi x}{2l}\right\}dx\right]$$

But $\displaystyle\int \sin^2\frac{\pi x}{2l}\,dx = \frac{x}{2} - \frac{l}{2\pi}\sin\frac{\pi x}{l},$

$$\int \sin^2\frac{3\pi x}{2l}\,dx = \frac{x}{2} - \frac{l}{6\pi}\sin\frac{3\pi x}{l},$$

$$\int \sin\frac{\pi x}{2l}\cdot\sin\frac{3\pi x}{2l}\,dx = \frac{\sin\frac{\pi x}{l}}{(2\pi/l)} - \frac{\sin\frac{2\pi x}{l}}{(4\pi/l)}$$

$$T_{max} = \frac{\rho A\omega^2}{2}\left[c_1^2\left(\frac{2l}{3} - \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2\left(\frac{2l}{3}\right) + c_1c_2\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$X = \frac{EA\pi^2}{4l^2}\left[c_1^2\left(\frac{2l}{3} + \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2(6l) + c_1c_2\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$Y = \rho A\left[c_1^2\left(\frac{2l}{3} - \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2^2\left(\frac{2l}{3}\right) + c_1c_2\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$\frac{\partial X}{\partial c_1} = \frac{EA\pi^2}{4l^2}\left[2c_1\left(\frac{2l}{3} + \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$= \frac{EA\pi^2}{4l^2}\left[2c_1(0.8045l) + c_2(1.2405\, l)\right]$$

$$\frac{\partial X}{\partial c_2} = \frac{EA\pi^2}{4l^2}\left[c_2(12l) + c_1\left(\frac{3l}{\pi}\sin\frac{\pi}{3} + \frac{3l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$= \frac{EA\pi^2}{4l^2}\left[c_2(12l) + c_1(1.2405\, l)\right]$$

$$\frac{\partial Y}{\partial c_1} = \rho A\left[2c_1\left(\frac{2l}{3} - \frac{l}{2\pi}\sin\frac{\pi}{3}\right) + c_2\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$= \rho A\left[2c_1(0.5288\, l) + c_2(0.1378\, l)\right]$$

$$\frac{\partial Y}{\partial c_2} = \rho A\left[c_2\left(\frac{4l}{3}\right) + c_1\left(\frac{l}{\pi}\sin\frac{\pi}{3} - \frac{l}{2\pi}\sin\frac{2\pi}{3}\right)\right]$$

$$= \rho A\left[c_2(1.3333\, l) + c_1(0.1378\, l)\right]$$

$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0$ and $\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0$ give

$$c_1 \left(\frac{EA\pi^2}{l} 0.40225 - \rho A \omega^2 l \; 1.0576 \right) + c_2 \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l \; 0.1378 \right) = 0$$

$$c_1 \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l \; 0.1380 \right) + c_2 \left(\frac{EA\pi^2}{l} 3.0 - \rho A \omega^2 l \; 1.3333 \right) = 0$$

Frequency equation is

$$\left(0.40225 \frac{EA\pi^2}{l} - \rho A \omega^2 l \; 1.0576 \right) \left(\frac{EA\pi^2}{l} 3.0 - \rho A \omega^2 l \; 1.3333 \right) - \left(\frac{EA\pi^2}{l} 0.31012 - \rho A \omega^2 l \; 0.1380 \right)^2 = 0$$

i.e., $\omega^4 - 25.7090\, \omega^2 \left(\frac{E}{\rho l^2} \right) + 77.7692 \left(\frac{E}{\rho l^2} \right)^2 = 0$

$$\omega^2 = \left(\frac{25.7090 \pm 18.7010}{2} \right) \frac{E}{\rho l^2}$$

$$\therefore \omega_1 = 1.8719 \sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 4.7022 \sqrt{\frac{E}{\rho l^2}}$$

8.76 $U(x) = c_1 \sin \frac{\pi x}{2l} + c_2 \sin \frac{3\pi x}{2l}$, $\frac{dU}{dx} = \frac{\pi}{2l} \left(c_1 \cos \frac{\pi x}{2l} + 3 c_2 \cos \frac{3\pi x}{2l} \right)$

$$V_{max} = \frac{1}{2} \int_0^l EA(x) \left(\frac{dU}{dx} \right)^2 dx = \frac{1}{2} \int_0^l 2EA_0 \left(1 - \frac{x}{l}\right) \left(\frac{\pi}{2l} \right)^2 \left(c_1^2 \cos^2 \frac{\pi x}{2l} + 9 c_2^2 \cos^2 \frac{3\pi x}{2l} + 6 c_1 c_2 \cos \frac{\pi x}{2l} \cos \frac{3\pi x}{2l} \right) dx$$

$$= c_1^2 \frac{EA_0 \pi^2}{16 l} \left(1 + \frac{4}{\pi^2} \right) + c_2^2 \frac{9 EA_0 \pi^2}{16 l} \left(1 + \frac{4}{9\pi^2} \right) + c_1 c_2 \frac{3 EA_0 \pi^2}{2l}$$

$$T_{max} = \frac{\omega^2}{2} \int_0^l m(x) \cdot U^2 \cdot dx = m_0 \omega^2 \int_0^l \left(1 - \frac{x}{l}\right) \left(c_1^2 \sin^2 \frac{\pi x}{2l} + c_2^2 \sin^2 \frac{3\pi x}{2l} + 2 c_1 c_2 \sin \frac{\pi x}{2l} \sin \frac{3\pi x}{2l} \right) dx$$

$$= c_1^2 \frac{\omega^2 m_0 l}{4} \left(1 - \frac{4}{\pi^2} \right) + c_2^2 \frac{\omega^2 m_0 l}{4} \left(1 - \frac{4}{9\pi^2} \right) + c_1 c_2 \frac{2\omega^2 m_0 l}{\pi^2}$$

$$X = c_1^2 \frac{EA_0}{l} (0.86685) + c_2^2 \frac{EA_0}{l} (5.80165) + c_1 c_2 \frac{EA_0}{l} (1.5)$$

$$Y = c_1^2 m_0 l (0.14868) + c_2^2 m_0 l (0.23874) + c_1 c_2 m_0 l (0.20264)$$

$$\frac{\partial X}{\partial c_1} = c_1 \frac{EA_0}{l} (1.73370) + c_2 \frac{EA_0}{l} (1.5)$$

$$\frac{\partial X}{\partial c_2} = c_1 \frac{EA_0}{l}(1.5) + c_2 \frac{EA_0}{l}(11.60330)$$

$$\frac{\partial Y}{\partial c_1} = c_1 m_0 l (0.29736) + c_2 m_0 l (0.20264)$$

$$\frac{\partial Y}{\partial c_2} = c_1 m_0 l (0.20246) + c_2 m_0 l (0.47748)$$

$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0$ and $\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0$ give

$$c_1\left(\frac{EA_0}{l} 1.73370 - \omega^2 m_0 l\, 0.29736\right) + c_2\left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l\, 0.20264\right) = 0$$

$$c_1\left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l\, 0.20246\right) + c_2\left(\frac{EA_0}{l} 11.60330 - \omega^2 m_0 l\, 0.47748\right) = 0$$

Frequency equation is

$$\left(\frac{EA_0}{l} 1.7337 - \omega^2 m_0 l\, 0.29736\right)\left(\frac{EA_0}{l} 11.6033 - \omega^2 m_0 l\, 0.47748\right) - \left(\frac{EA_0}{l} 1.5 - \omega^2 m_0 l\, 0.20264\right)^2 = 0$$

or $$\omega^4 - 36.369\, \omega^2\left(\frac{EA_0}{m_0 l^2}\right) + 177.04292\left(\frac{EA_0}{m_0 l^2}\right)^2 = 0$$

$$\omega^2 = \left(\frac{36.369 \pm 24.7898}{2}\right)\frac{EA_0}{m_0 l^2}$$

$$\therefore\ \omega_1 = 2.4062\sqrt{\frac{EA_0}{m_0 l^2}}, \quad \omega_2 = 5.5299\sqrt{\frac{EA_0}{m_0 l^2}}$$

8.77 For a string, $V_{max} = \frac{1}{2}\int_0^l P\left(\frac{dW}{dx}\right)^2 \cdot dx$, $T_{max} = \frac{\omega^2}{2}\int_0^l \rho\, W^2 \cdot dx$

Here $$W(x) = c_1 x(l-x) + c_2 x^2 (l-x)^2$$

$$\frac{dW}{dx} = c_1 l - 2c_1 x + 2c_2 l^2 x - 6 c_2 l x^2 + 4 c_2 x^3$$

$$\int_0^l \left(\frac{dW}{dx}\right)^2 dx = c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5$$

$$\int_0^l W^2\, dx = c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70}$$

Equating T_{max} and V_{max}, we get $\omega^2 = \frac{X}{Y}$

where $$X = \frac{P}{2}\left[c_1^2 \frac{l^3}{3} + c_2^2 \frac{2}{105} l^7 + c_1 c_2 \frac{2}{15} l^5\right]$$

and $$Y = \frac{\rho}{2}\left[c_1^2 \frac{l^5}{30} + c_2^2 \frac{l^9}{630} + c_1 c_2 \frac{l^7}{70}\right]$$

$\frac{\partial(\omega^2)}{\partial c_1} = \frac{\partial(\omega^2)}{\partial c_2} = 0$ yield Eqs. (E_{11}) and (E_{12}) of Example 8.13.

$$\frac{\partial X}{\partial c_1} = \frac{P}{2}\left[c_1 \frac{2}{3} l^3 + c_2 \frac{2}{15} l^5\right], \quad \frac{\partial Y}{\partial c_1} = \frac{\rho}{2}\left[c_1 \frac{l^5}{15} + c_2 \frac{l^7}{70}\right]$$

$$\frac{\partial X}{\partial c_2} = \frac{P}{2}\left[c_1 \frac{2}{15} l^5 + c_2 \frac{4}{105} l^7\right], \quad \frac{\partial Y}{\partial c_2} = \frac{\rho}{2}\left[c_1 \frac{l^7}{70} + c_2 \frac{l^9}{315}\right]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \Rightarrow c_1\left(\frac{Pl^3}{3} - \frac{\rho\omega^2 l^5}{30}\right) + c_2\left(\frac{Pl^5}{15} - \frac{\rho\omega^2 l^7}{140}\right) = 0$$

$$\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \Rightarrow c_1\left(\frac{Pl^5}{15} - \frac{\rho\omega^2 l^7}{140}\right) + c_2\left(\frac{2Pl^7}{105} - \frac{\rho\omega^2 l^9}{630}\right) = 0$$

By setting the determinant of the coefficient matrix of c_1 and c_2 to zero, we get the frequency equation as

$$\begin{vmatrix} \left(\frac{1}{3} - \frac{\lambda}{30}\right) & \left(\frac{1}{15} - \frac{\lambda}{140}\right) \\ \left(\frac{1}{15} - \frac{\lambda}{140}\right) & \left(\frac{2}{105} - \frac{\lambda}{630}\right) \end{vmatrix} = \lambda^2 - 112\lambda + 1008 = 0$$

where $\lambda = \dfrac{\rho l^2 \omega^2}{P}$.

$$\lambda_1 = 9.8722, \quad \lambda_2 = 102.4144$$

$$\therefore \omega_1 = 3.142\sqrt{\frac{P}{\rho l^2}}, \quad \omega_2 = 10.12\sqrt{\frac{P}{\rho l^2}}$$

8.78

```
Results of Ex8_78
************************
1. beta = 0.01
>>program12
Roots of nonlinear equation

Data:
n     =    2
xs    =    5.000000e-001
xinc  =    1.000000e-003
nint  =   5000
iter  =    10000
eps   =    1.000000e-006


Roots
   3.144773e+000
   6.284776e+000

2. beta = 0.1
>>program12
Roots of nonlinear equation

Data:
n     =    2
xs    =    5.000000e-001
xinc  =    1.000000e-003
nint  =   5000
iter  =    10000
eps   =    1.000000e-006

Roots
   3.173097e+000
   6.299059e+000

3. beta = 1.0
>>program12
Roots of nonlinear equation

Data:
n     =    2
xs    =    5.000000e-001
xinc  =    1.000000e-003
nint  =   5000
iter  =    10000
eps   =    1.000000e-006


Roots
   8.603336e-001
   3.425618e+000
```

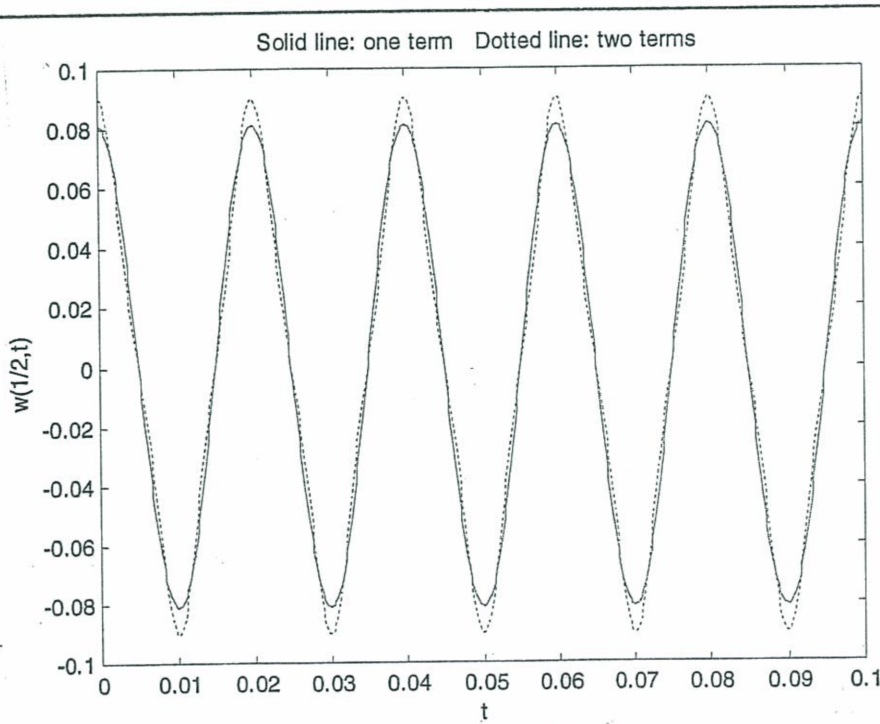
```
4. beta = 10.0
>>program12
Roots of nonlinear equation

Data:
n     =   2
xs    =   5.000000e-001
xinc  =   1.000000e-003
nint  =  5000
iter  =   10000
eps   =   1.000000e-006


Roots
  1.428870e+000
  7.228110e+000
```

```
5. beta = 100.0
>>program12
Roots of nonlinear equation

Data:
n     =   2
xs    =   5.000000e-001
xinc  =   1.000000e-003
nint  =  5000
iter  =   10000
eps   =   1.000000e-006


Roots
  1.555245e+000
  2.644501e+001
```

8.79

```
Results of  Ex8_79
********************
>>program12
Roots of nonlinear equation

Data:
n     =   5
xs    =   2.000000e+000
xinc  =   1.000000e-003
nint  =  50000
iter  =   100000
eps   =   1.000000e-006


Roots
  4.730041e+000
  7.853205e+000
  1.099561e+001
  1.413717e+001
  1.727876e+001
```

8.80



```
% Ex8_80.m
h = 0.1;
l = 1.0;
c = 100.0;
x = 1.0/2.0;
for i = 1: 501
   t(i) = (i-1)*0.1/500;
   w1(i) = ( 8*h/(pi^2) ) * ( sin(pi*x/l)*cos(pi*c*t(i)/l) );
   w2(i) = ( 8*h/(pi^2) ) * ( sin(pi*x/l)*cos(pi*c*t(i)/l)...
      - sin(3*pi*x/l)*cos(3*pi*c*t(i)/l)/9 );
end
plot(t,w1);
hold on;
plot(t, w2, ':');
xlabel('t');
ylabel('w(1/2,t)')
title('Solid line: one term   Dotted line: two terms');
```

8.81 From Example 8.7, the n^{th} mode shape is given by

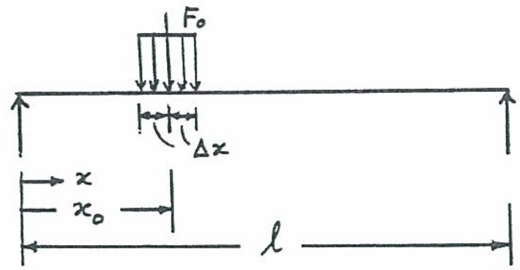
$$W_n(x) = C_{1n}\left\{(\cos\beta_n x - \cosh\beta_n x) - \left(\frac{\cos\beta_n l - \cosh\beta_n l}{\sin\beta_n l - \sinh\beta_n l}\right)(\sin\beta_n x - \sinh\beta_n x)\right\}$$

where $\beta_n l = 3.926602,\ 7.068583,\ 10.210176,\ 13.351768$ for $n = 1, 2, 3, 4$.
The program and results are given below.

```
C =================================================================
C
C PROBLEM 8.81
C
C =================================================================
      DIMENSION X(4),W(4,9)
      N=4
      M=9
      MD=10
C N       = NUMBER OF MODE SHAPES TO BE COMPUTED
C M       = NUMBER OF STATIONS IN THE BEAM AT WHICH VALUE OF DEFLECTION
C           IS REQUIRED
C MD      = NUMBGR OF SEGMENTS IN THE BEAM
C W(N,M)  = MATRIX CONTAINING THE MODE SHAPES IN ROWS
      DATA X/3.926602,7.068583,10.210176,13.351768/
      DO 10 I=1,N
      DO 20 J=1,M
      Y=X(I)*REAL(J)/REAL(MD)
      YY=X(I)
   20 W(I,J)=(COS(Y)-COSH(Y))-((COS(YY)-COSH(YY))/(SIN(YY)-SINH(YY)))*
     2 (SIN(Y)-SINH(Y))
   10 CONTINUE
      WRITE (86,30)
   30 FORMAT (//,2X,43H MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM,/)
      DO 40 I=1,N
   40 WRITE (86,50) I,(W(I,J),J=1,M)
   50 FORMAT (/,2X,6H MODE:,I4,4E15.6,/,(12X,4E15.6))
      STOP
      END
```

```
 MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM


 MODE:    1  -0.133996E+00  -0.455738E+00  -0.848519E+00  -0.120675E+01
             -0.144486E+01  -0.150550E+01  -0.136498E+01  -0.103456E+01
             -0.557241E+00

 MODE:    2  -0.382233E+00  -0.107449E+01  -0.149510E+01  -0.131923E+01
             -0.570351E+00   0.422676E+00   0.119881E+01   0.139349E+01
              0.917175E+00

 MODE:    3  -0.690370E+00  -0.147476E+01  -0.112212E+01   0.204393E+00
              0.130049E+01   0.114192E+01  -0.111633E+00  -0.126025E+01
             -0.120557E+01

 MODE:    4  -0.100204E+01  -0.141423E+01   0.927429E-01   0.139201E+01
              0.539917E+00  -0.114441E+01  -0.107568E+01   0.642578E+00
              0.137500E+01
```

8.82



Equation of motion is given by Eq. (8.77):

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \qquad (E_1)$$

Representation of load:

Load is $f(x)$ which is zero everywhere except over a length $2\Delta x$ at $x = x_0$, where it is equal to $(F_0/2\Delta x)$. The load $f(x)$ can be expanded into Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{l} \qquad (E_2)$$

where

$$f_n = \frac{2}{l} \int_0^{x_0-\Delta x} (0) \sin \frac{n\pi x}{l}\, dx + \frac{2}{l} \int_{x_0-\Delta x}^{x_0+\Delta x} \frac{F_0}{2\Delta x} \sin \frac{n\pi x}{l}\, dx$$

$$+ \frac{2}{l} \int_{x_0+\Delta x}^{l} (0) \sin \frac{n\pi x}{l}\, dx = \frac{F_0}{l\,\Delta x} \int_{x_0-\Delta x}^{x_0+\Delta x} \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2F_0}{l} \sin \frac{n\pi x_0}{l} \; \frac{\sin\left(\frac{n\pi \Delta x}{l}\right)}{\left(\frac{n\pi\Delta x}{l}\right)} \qquad (E_3)$$

As $\Delta x \to 0$, (E_3) becomes $\quad f_n = \frac{2F_0}{l} \sin \frac{n\pi x_0}{l} \qquad (E_4)$

$$\therefore f(x) = \frac{2F_0}{l} \sum_{n=1}^{\infty} \sin \frac{n\pi x_0}{l} \sin \frac{n\pi x}{l} \qquad (E_5)$$

Since F_0 is moving along the bridge with constant velocity v, at time t, F_0 will be at a distance $x_0 = vt$ from left support. Hence the load distribution of Eq. (E_5) can be rewritten as

$$f(x,t) = \frac{2F_0}{l} \left\{ \sin \frac{\pi v t}{l} \sin \frac{\pi x}{l} + \sin \frac{2\pi v t}{l} \sin \frac{2\pi x}{l} + \cdots \right\} \qquad (E_6)$$

Setting $\omega = \frac{\pi v}{l}$, (E_6) can be expressed as

$$f(x,t) = \frac{2F_0}{l} \left\{ \sin \frac{\pi x}{l} \sin \omega t + \sin \frac{2\pi x}{l} \sin 2\omega t + \cdots \right\} \qquad (E_7)$$

Equation of motion, (E_1), becomes

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{l} \sin n\omega t \qquad (E_8)$$

where $F_n = \dfrac{2 F_0}{l}$; $n = 1, 2, \ldots$

We can find the solution of Eq. (E_8) by superposing the solutions of the individual harmonic components. Thus, we need to find, for the n^{th} harmonic, the solution of the following equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_n \sin \frac{n\pi x}{l} \sin n\omega t \qquad (E_9)$$

The solution (particular integral) of Eq. (E_9) can be taken as

$$w(x,t) = w_n \sin \frac{n\pi x}{l} \sin n\omega t \qquad (E_{10})$$

Substitution of (E_{10}) into (E_9) gives

$$w_n = \frac{F_n\, l^4}{EI\,(n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \qquad (E_{11})$$

where $\omega_n = n^2 \pi^2 \sqrt{\dfrac{EI}{\rho A\, l^4}}$ $\qquad (E_{12})$

Adding the homogeneous solution to the particular integral of Eq. (E_{10}), the general solution can be expressed as

$$w(x,t) = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{l} \left\{ A_i \cos \omega_i t + B_i \sin \omega_i t \right\} + w_n \sin \frac{n\pi x}{l} \sin n\omega t \qquad (E_{13})$$

The initial conditions are given by

$$w(x,0) = 0 \quad , \quad \frac{\partial w}{\partial t}(x,0) = 0 \qquad (E_{14})$$

These initial conditions are satisfied if

$A_i = 0$ for all i, $\quad B_i = 0$ for all $i \neq n$

$$B_n = - w_n \left(\frac{n\omega}{\omega_n} \right)$$

$$w(x,t) = \frac{F_n\, l^4}{EI\,(n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{\omega_n} \sin \omega_n t \right\} \qquad (E_{15})$$

Since $\omega_n = n^2 \omega_1$, where ω_1 is the fundamental frequency of the bridge, Eq. (E_{15}) becomes,

$$w(x,t) = \frac{F_n \, l^4}{EI \, \pi^4 \left\{ n^4 - \left(\frac{n\omega}{\omega_1} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{n^2 \omega_1} \sin n^2 \omega_1 t \right\} \qquad (E_{16})$$

Thus the total response of the bridge can be expressed as

$$w(x,t) = \frac{2 \, F_0 \, l^3}{EI \, \pi^4} \left[\frac{\sin \frac{\pi x}{l}}{1 - \left(\frac{\omega}{\omega_1} \right)^2} \left\{ \sin \omega t - \left(\frac{\omega}{\omega_1} \right) \sin \omega_1 t \right\} \right.$$

$$+ \frac{\sin \frac{2\pi x}{l}}{2^4 - \left(\frac{2\omega}{\omega_1} \right)^2} \left\{ \sin 2\omega t - \left(\frac{\omega}{2\omega_1} \right) \sin 4 \omega_1 t \right\}$$

$$\left. + \frac{\sin \frac{3\pi x}{l}}{3^4 - \left(\frac{3\omega}{\omega_1} \right)^2} \left\{ \sin 3\omega t - \left(\frac{\omega}{3\omega_1} \right) \sin 9 \omega_1 t \right\} + \cdots \right] \qquad (E_{17})$$