Chapter 13

Nonlinear Vibration

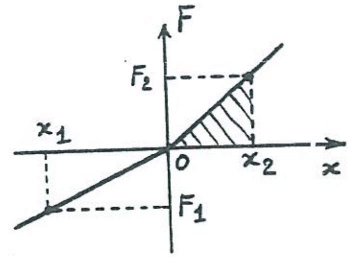
13.1 $\ddot{\theta}_1 + \omega_0^2 \theta_1 = f$, $\ddot{\theta}_2 + \omega_0^2 \theta_2 = \frac{\omega_0^2}{6} \theta_2^3$

Use $\theta = \theta_1 + \theta_2$ in Eq. (E_2):

$$\ddot{\theta}_1 + \ddot{\theta}_2 + \omega_0^2 \theta_1 + \omega_0^2 \theta_2 = f + \frac{\omega_0^2}{6} \theta_1^3 + \frac{\omega_0^2}{6} \theta_2^3 + \frac{\omega_0^2}{2} \theta_1^2 \theta_2 + \frac{\omega_0^2}{2} \theta_1 \theta_2^2$$

Left hand side is not equal to the right hand side.

Thus superposition principle is not valid.



13.4 T = kinetic energy at time zero $= \frac{1}{2} m (\dot{x}_0)^2$

Let x_2 = maximum displacement on right side.

V = potential energy in spring at displacement

$x_2 = \frac{1}{2} k_2 x_2^2$ ($\dot{x}$ is zero at x_2)

Since T = V, $\quad x_2 = \sqrt{\frac{m(\dot{x}_0)^2}{k_2}} = \sqrt{\frac{m}{k_2}}\ \dot{x}_0$

Let x_1 = maximum displacement to left side. $\quad V = \frac{1}{2} k_1 x_1^2$

T = V gives $\quad x_1 = \sqrt{\frac{m}{k_1} (\dot{x}_0)^2} = \sqrt{\frac{m}{k_1}}\ \dot{x}_0$

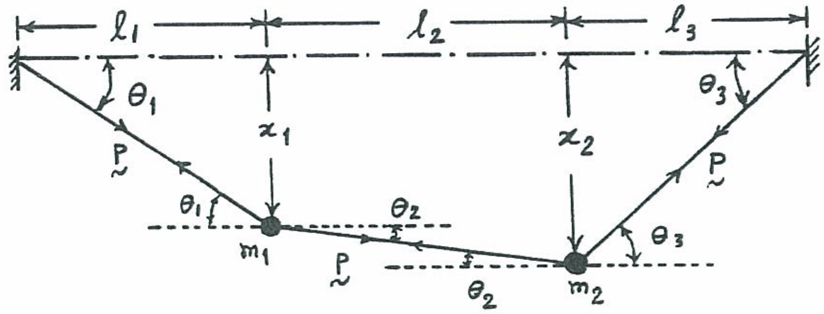
(a) Since $k_1 < k_2$, maximum deflection $= x_1 = \sqrt{\frac{m}{k_1}}\ \dot{x}_0$

(b) Period of vibration for a spring-mass system is $\tau_n = 2\pi \sqrt{\frac{m}{k}}$.

In the present case, $\tau_n =$ (time for m to go to $x = x_1$ from $x = 0$ and return to $x = 0$) + (time for m to go to $x = x_2$ from $x = 0$ and return to $x = 0$)

$$\therefore \ \tau_n = \pi \left(\sqrt{\frac{m}{k_1}} + \sqrt{\frac{m}{k_2}} \right)$$

13.7



Let

$\underset{\sim}{P}$ = tension in wire after displacement of masses

P = initial tension in wire

$\underset{\sim}{l}_i$ = length of i^{th} segment of wire after displacement of masses

x_i = transverse displacement of mass i $(i = 1, 2)$

$$\underset{\sim}{l}_1 = \sqrt{l_1^2 + x_1^2}, \qquad \underset{\sim}{l}_2 = \sqrt{l_2^2 + (x_2 - x_1)^2}, \qquad \underset{\sim}{l}_3 = \sqrt{l_3^2 + x_2^2}$$

$$\epsilon = \text{strain in wire} = \left(\frac{\underset{\sim}{l}_1 + \underset{\sim}{l}_2 + \underset{\sim}{l}_3 - l_1 - l_2 - l_3}{l_1 + l_2 + l_3} \right) \qquad (E_1)$$

New tension in wire $= \underset{\sim}{P} = (P + A E \epsilon)$

where A = cross-sectional area of wire and E = Young's modulus.

Equations of motion of masses:

$$m_1 \ddot{x}_1 + \underset{\sim}{P} \sin \theta_1 + \underset{\sim}{P} \sin \theta_2 = 0 \qquad (E_2)$$

$$m_2 \ddot{x}_2 + \underset{\sim}{P} \sin \theta_2 + \underset{\sim}{P} \sin \theta_3 = 0 \qquad (E_3)$$

i.e.,

$$m_1 \ddot{x}_1 + (P + AE\epsilon)\left(\frac{x_1}{\sqrt{l_1^2 + x_1^2}} + \frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} \right) = 0 \qquad (E_4)$$

$$m_2 \ddot{x}_2 + (P + AE\epsilon)\left(\frac{x_2 - x_1}{\sqrt{l_2^2 + (x_2 - x_1)^2}} + \frac{x_2}{\sqrt{l_3^2 + x_2^2}} \right) = 0 \qquad (E_5)$$

with

$$\frac{x_1}{\sqrt{l_1^2 + x_2^2}} = \frac{x_1}{l_1 \sqrt{1 + \left(\frac{x_2}{l_1}\right)^2}} = \frac{x_1}{l_1}\left[1 + \left(\frac{x_2}{l_1}\right)^2\right]^{-\frac{1}{2}} \approx \frac{x_1}{l_1}\left[1 - \frac{1}{2}\left(\frac{x_2}{l_1}\right)^2\right] \qquad (E_6)$$

$$\frac{x_2 - x_1}{\sqrt{\ell_2^2 + (x_2 - x_1)^2}} \approx \frac{x_2 - x_1}{\ell_2}\left[1 - \frac{1}{2}\left(\frac{x_2 - x_1}{\ell_2}\right)^2\right] \qquad (E_7)$$

and
$$\frac{x_2}{\sqrt{\ell_3^2 + x_2^2}} \approx \frac{x_2}{\ell_3}\left[1 - \frac{1}{2}\left(\frac{x_2}{\ell_3}\right)^2\right] \qquad (E_8)$$

similarly,
$$\underset{\sim}{\ell}_1 - \ell_1 = \sqrt{\ell_1^2 + x_1^2} - \ell_1 \approx \ell_1\left[1 + \frac{1}{2}\left(\frac{x}{\ell}\right)^2\right] - \ell_1 \approx \frac{1}{2}\,\frac{x_1^2}{\ell_1}$$

$$\underset{\sim}{\ell}_2 - \ell_2 = \sqrt{\ell_2^2 + (x_2 - x_1)^2} - \ell_2 \approx \frac{1}{2}\,\frac{(x_2 - x_1)^2}{\ell_2}$$

$$\underset{\sim}{\ell}_3 - \ell_3 \approx \frac{1}{2}\,\frac{x_2^2}{\ell_3}$$

and hence
$$\epsilon = \frac{\left\{\frac{x_1^2}{\ell_1} + \frac{(x_2 - x_1)^2}{\ell_2} + \frac{x_3^2}{\ell_3}\right\}}{2(\ell_1 + \ell_2 + \ell_3)} \qquad (E_9)$$

Substitution of Eqs. (E_6) to (E_9) into Eqs. (E_4) and (E_5) yields the nonlinear equations of motion of the masses m_1 and m_2.

13.8 Using x and θ as the coordinates, the kinetic and potential energies of the system can be expressed as

$$T = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 \qquad (1)$$

where $J_0 = m\,(\ell + x)^2$ and

$$V = \frac{1}{2}\, k\, (x + \delta_{st})^2 - m\, g\, (\ell + x) \cos\theta \qquad (2)$$

where $\delta_{st} = \frac{mg}{k}$. Equations (1) and (2) give

$$\frac{\partial T}{\partial \dot{x}} = m\,\dot{x} \quad ; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\,\ddot{x}$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0\,\dot{\theta} \quad ; \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \dot{J}_0\,\dot{\theta} + J_0\,\ddot{\theta} = 2\, m\, (\ell + x)\,\dot{x}\,\dot{\theta} + J_0\,\ddot{\theta}$$

$$\frac{\partial T}{\partial x} = m\,(\ell + x)\,\dot{\theta}^2 \quad ; \quad \frac{\partial T}{\partial \theta} = 0$$

$$\frac{\partial V}{\partial x} = k\,(x + \delta_{st}) - m\, g \cos\theta \quad ; \quad \frac{\partial V}{\partial \theta} = m\, g\, (\ell + x) \sin\theta$$

The equations of motion can be derived using Lagrange's equations, Eq. (6.44), as:

$$m\,\ddot{x} - m\,(\ell + x)\,\dot{\theta}^2 + k\, x + m\, g - m\, g \cos\theta = 0 \qquad (3)$$

$$m\,(\ell + x)^2\,\ddot{\theta} + 2\, m\,(\ell + x)\,\dot{x}\,\dot{\theta} + m\, g\,(\ell + x)\sin\theta = 0 \qquad (4)$$

Using $\sin\theta \approx \theta$, $\cos\theta \approx 1$, and neglecting nonlinear terms involving $x^2\,\ddot{\theta}$, $\dot{\theta}^2$, $x\,\theta$, and $\dot{x}\,\dot{\theta}$, Eqs. (3) and (4) can be reduced (linearized) to obtain:

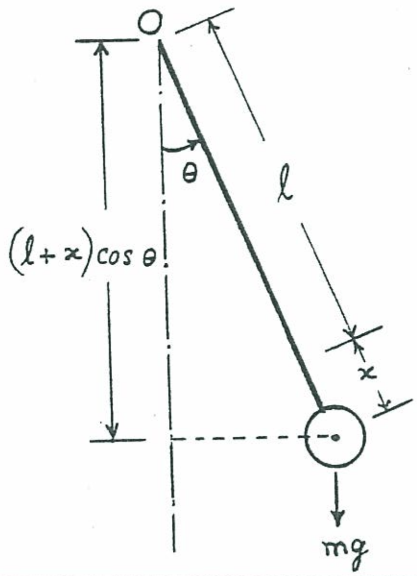
$$m\,\ddot{x} + k\,x = 0 \tag{5}$$

$$m\,\ell^2\,\ddot{\theta} + m\,g\,\ell\,\theta = 0 \tag{6}$$

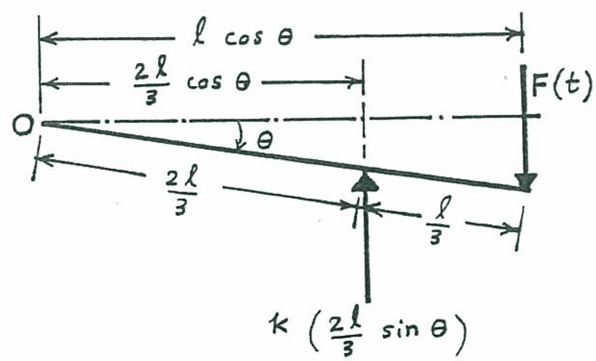
Equations (5) and (6) correspond to the natural frequencies:

$$\omega_{n1} = \sqrt{\frac{k}{m}} \tag{7}$$

$$\omega_{n2} = \sqrt{\frac{g}{\ell}} \tag{8}$$



13.9 $J_0 = \frac{1}{3}\,m\,\ell^2$



$$\sum M_0 = J_0\,\ddot{\theta}$$

$$\text{or}\quad J_0\,\ddot{\theta} = -\,k\left(\frac{2\,\ell}{3}\sin\theta\right)\left(\frac{2\,\ell}{3}\cos\theta\right) + (\ell\cos\theta)\,F(t)$$

$$\text{or}\quad \left(\frac{1}{3}\,m\,\ell^2\right)\ddot{\theta} + \frac{4}{9}\,k\,\ell^2\sin\theta\cos\theta = \ell\cos\theta\,F(t) \tag{1}$$

Eq. (1) can be approximated using

$$\sin\theta = \theta - \frac{\theta^3}{6} \quad \text{and} \quad \cos\theta = 1 - \frac{\theta^2}{2}$$

which yields

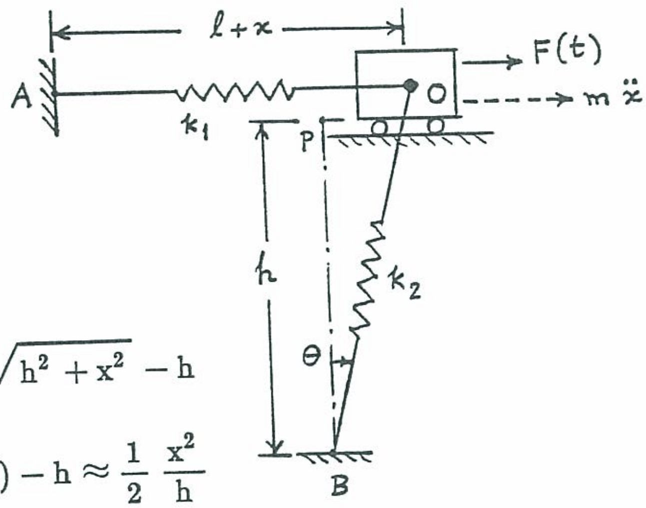
$$\left(\frac{1}{3}\,m\,\ell^2\right)\ddot{\theta} + \frac{4}{9}\,k\,\ell^2\,(\theta - \frac{\theta^3}{6})\,(1 - \frac{\theta^2}{2}) = \ell\,(1 - \frac{\theta^2}{2})\,F(t)$$

$$\text{or}\quad \frac{1}{3}\,m\,\ell^2\,\ddot{\theta} + \frac{4}{9}\,k\,\ell^2\,(\theta - \frac{2}{3}\,\theta^3) = (\ell - \frac{\ell\,\theta^2}{2})\,F(t) \qquad (2)$$

Neglecting the nonlinear term, Eq. (2) can be reduced to

$$\frac{1}{3}\,m\,\ell^2\,\ddot{\theta} + \frac{4}{9}\,k\,\ell^2\,\theta = \ell\,F(t) \qquad (3)$$

13.10



Extension of k_2:

$$OB - PB = \sqrt{BP^2 + PO^2} - PB = \sqrt{h^2 + x^2} - h$$

$$= h\,(1 + \frac{x^2}{h^2})^{\frac{1}{2}} - h \approx h\,(1 + \frac{1}{2}\,\frac{x^2}{h^2}) - h \approx \frac{1}{2}\,\frac{x^2}{h}$$

$$\sin\theta = \frac{x}{BO} = \frac{x}{h + \frac{1}{2}\,\frac{x^2}{h}}$$

$$\sum F = m\,\ddot{x}$$

$$\text{or}\quad m\,\ddot{x} = -\,k_1\,x - \left(\frac{1}{2}\,\frac{x^2}{h}\right)k_2\,\sin\theta + F(t)$$

$$\text{or}\quad m\,\ddot{x} + k_1\,x + \left(\frac{\frac{1}{2}\,\frac{x^3\,k_2}{h}}{h + \frac{1}{2}\,\frac{x^2}{h}}\right) = F(t) \qquad (1)$$

Note that

$$\frac{x^3\,k_2}{2\,h\,(h + \frac{1}{2}\,\frac{x^2}{h})} \approx \frac{x^3\,k_2}{2\,h^2} - \frac{x^5\,k_2}{4\,h^4} \qquad (2)$$

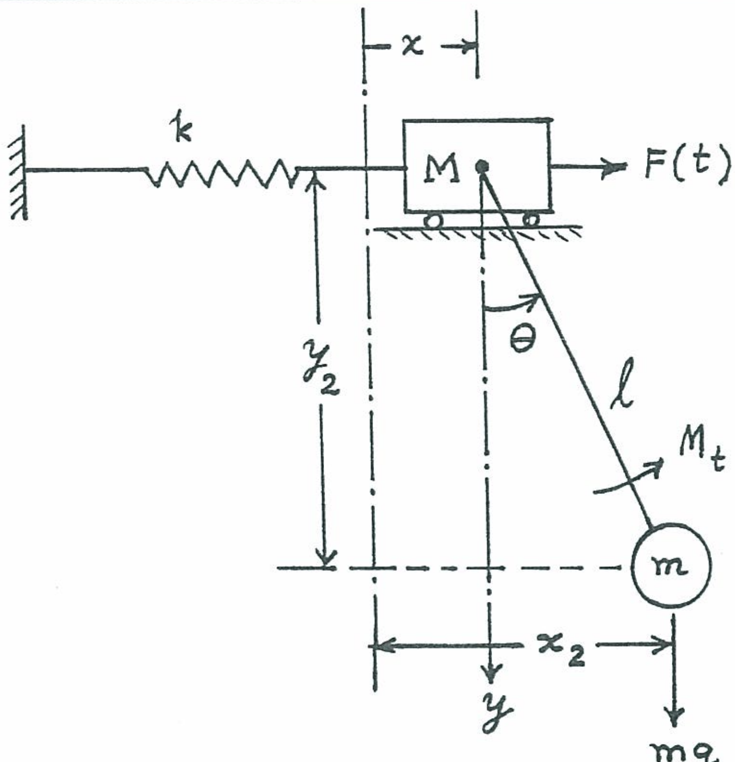
Substituting Eq. (2) into (1), we obtain

$$m\,\ddot{x} + k_1\,x + \frac{x^3\,k_2}{2\,h^2} - \frac{x^5\,k_2}{4\,h^4} = F(t) \qquad (3)$$

Neglecting the terms involving x^5, Eq. (3) can be reduced to

$$m\,\ddot{x} + k_1\,x + \frac{k_2}{2\,h^2}\,x^3 = F(t) \tag{4}$$

13.11



$x_2 = x + \ell\sin\theta$; $\dot{x}_2 = \dot{x} + \ell\,\dot{\theta}\cos\theta$

$y_2 = \ell\cos\theta$; $\dot{y}_2 = -\ell\,\dot{\theta}\sin\theta$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + \ell\,\dot{\theta}\cos\theta)^2 + (-\ell\,\dot{\theta}\sin\theta)^2\right]$$

$$= \frac{1}{2}(M+m)\,\dot{x}^2 + \frac{1}{2}m\,\ell^2\,\dot{\theta}^2 + m\,\ell\,\dot{x}\,\dot{\theta}\cos\theta$$

$$V = \frac{1}{2}k\,x^2 + m\,g\,\ell\,(1-\cos\theta)$$

$Q_x = F(t)$; $Q_\theta = M_t(t)$

Equations of motion:

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta}\cos\theta - m\,\ell\,\dot{\theta}^2\sin\theta + k\,x = F(t) \tag{1}$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x}\cos\theta - m\,\ell\,\dot{x}\,\dot{\theta}\sin\theta + m\,g\,\ell\sin\theta = M_t(t) \tag{2}$$

Using the approximations

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \;\; ; \;\; \sin\theta \approx \theta - \frac{\theta^3}{6}$$

Eqs. (1) and (2) can be expressed as

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta} - \frac{1}{2}\,m\,\ell\,\theta^2\,\ddot{\theta} - m\,\ell\,\theta\,\dot{\theta}^2 + \frac{1}{6}\,m\,\ell\,\theta^3\,\dot{\theta}^2 + k\,x = F(t) \qquad (3)$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x} - \frac{1}{2}\,m\,\ell\,\theta^2\,\ddot{x} - m\,\ell\,\theta\,\dot{\theta}\,\dot{x} + \frac{1}{6}\,m\,\ell\,\theta^3\,\dot{\theta}\,\dot{x}$$
$$+ m\,g\,\ell\,\theta - \frac{1}{6}\,m\,g\,\ell\,\theta^3 = M_t(t) \qquad (4)$$

By neglecting the nonlinear terms, the linearized equations of motion can be written as

$$(M+m)\,\ddot{x} + m\,\ell\,\ddot{\theta} + k\,x = F(t) \qquad (5)$$

$$m\,\ell^2\,\ddot{\theta} + m\,\ell\,\ddot{x} + m\,g\,\ell\,\theta = M_t(t) \qquad (6)$$

13.12 Eq. (13.1) is $\quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \quad --- \; (E_1)$

But $\frac{d^2\theta}{dt^2} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\,\frac{d\theta}{dt} = \dot{\theta}\,\frac{d\dot{\theta}}{d\theta}$

Eq. (E_1) can be rewritten as $\quad \dot{\theta}\,\frac{d\dot{\theta}}{d\theta} + \frac{g}{\ell}\sin\theta = 0$

Integrating this, we get $\quad \frac{(\dot{\theta})^2}{2} - \frac{g}{\ell}\cos\theta = c \quad --- \; (E_2)$

At $t=0$, $\dot{\theta}=0$ and $\theta=\theta_0 \Rightarrow c = -(g/\ell)\cos\theta_0$

Eq. (E_2) gives $\left(\frac{d\theta}{dt}\right)^2 = (\dot{\theta})^2 = 2\left(c + \frac{g}{\ell}\cos\theta\right)$

$$dt = \frac{d\theta}{\sqrt{2\left(c + \frac{g}{\ell}\cos\theta\right)}} = \frac{d\theta}{\sqrt{2\,\frac{g}{\ell}\left(\cos\theta - \cos\theta_0\right)}}$$

$$t = \int \frac{d\theta}{\sqrt{2\,\frac{g}{\ell}\left(\cos\theta - \cos\theta_0\right)}} \quad --- \; (E_3)$$

But $\cos\theta - \cos\theta_0 = \left(1 - 2\sin^2\frac{\theta}{2}\right) - \left(1 - 2\sin^2\frac{\theta_0}{2}\right)$

$$= 2\sin^2\frac{\theta_0}{2}\left(1 - \frac{\sin^2\frac{\theta}{2}}{\sin^2\frac{\theta_0}{2}}\right) \quad --- \; (E_4)$$

If $\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}} = \frac{\sin\frac{\theta}{2}}{k} = \sin\phi \quad$ where $\quad k = \sin\frac{\theta_0}{2}$,

differentiation gives

$$\frac{1}{k}\cos\frac{\theta}{2}\cdot\frac{d\theta}{2} = \cos\phi\cdot d\phi \Rightarrow d\theta = \frac{2k\cos\phi\,d\phi}{\cos\frac{\theta}{2}} \quad --(E_5)$$

Eqs. (E_3) to (E_5) give

$$t = \int \left(\frac{2k \cos\phi \; d\phi}{\cos \frac{\theta}{2}} \right) \Big/ \left(2 \sqrt{\frac{g}{l}} \sin \frac{\theta_0}{2} \cos\phi \right) = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\cos \frac{\theta}{2}}$$

$$= \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\theta}{2}}} = \sqrt{\frac{l}{g}} \int \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

$$\tau = \text{period} = 4 \sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} \qquad \text{where } k = \sin \frac{\theta_0}{2}.$$

The integral in τ is the complete elliptic integral of the first kind. Hence the numerical value of τ corresponding to any θ_0 can be obtained from the elliptic integral tables.

13.13 Equation of motion : $\ddot{\theta} + \frac{g}{l} \sin\theta = 0$

or $\frac{d}{d\theta}\left(\dot{\theta}^2\right) + \frac{2g}{l} \sin\theta = 0$

Let $\theta = \theta_0$ at $\frac{d\theta}{dt} = 0$. $\int_{\dot{\theta}=0}^{\dot{\theta}} \frac{d}{d\theta}\left(\dot{\theta}^2\right) d\theta + \int_{\theta=\theta_0}^{\theta} \frac{2g}{l} \sin\theta \, d\theta = 0$

i.e., $(\dot{\theta})^2 + \frac{2g}{l} \left(-\cos\theta\right)_{\theta=\theta_0}^{\theta} = 0$

i.e., $(\dot{\theta})^2 = \frac{2g}{l} \left(\cos\theta - \cos\theta_0\right)$

i.e., $\frac{d\theta}{dt} = \pm \sqrt{\frac{2g}{l}} \sqrt{\cos\theta - \cos\theta_0}$

i.e., $dt = \pm \sqrt{\frac{l}{2g}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$

Time taken by pendulum to reach vertical position can be found as

$$t = \int_0^t dt = \sqrt{\frac{l}{2g}} \int_{\theta=\theta_0}^{0} \frac{d\theta}{\left(\cos\theta - \cos\theta_0\right)^{\frac{1}{2}}} \qquad (E_1)$$

Using the relation $\cos\theta = 1 - 2 \sin^2 \frac{\theta}{2}$,

Eq. (E_1) can be rewritten as

$$t = \frac{1}{2} \sqrt{\frac{l}{g}} \int_{\theta_0}^{0} \frac{d\theta}{\sqrt{\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2}}} \qquad (E_2)$$

Let $a = \sin \frac{\theta_0}{2}$ and $\sin \frac{\theta}{2} = a \cdot \sin\phi$ so that when $\theta = 0$, $\phi = 0$ and when $\theta = \theta_0$, $\phi = \frac{\pi}{2}$.

$$\cos\frac{\theta}{2}\cdot\frac{d\theta}{2} = a\cos\phi\cdot d\phi \quad ; \qquad d\theta = \left(\frac{2a\cos\phi}{\cos\frac{\theta}{2}}\right) d\phi$$

$$\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2} = a^2 - a^2\sin^2\phi = a^2\cos^2\phi$$

Eq. (E_2) becomes

$$t = \pm\frac{1}{2}\sqrt{\frac{l}{g}}\int_{\pi/2}^{0}\frac{2a\cos\phi\, d\phi}{\cos\frac{\theta}{2}\cdot a\cos\phi} = \pm\sqrt{\frac{l}{g}}\int_{\pi/2}^{0}\frac{1}{\sqrt{1-\cos^2\frac{\theta}{2}}}\,d\phi$$

$$= \sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-a^2\sin^2\phi}} = \sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\cdot\sin^2\phi}}$$

$$= \sqrt{\frac{l}{g}}\cdot F\left(a,\frac{\pi}{2}\right) \qquad (E_3)$$

where F is called the incomplete elliptic integral of the first kind. Here $\sqrt{\frac{l}{g}} = \sqrt{\frac{30}{386.4}} = 0.2786$

and $a = \sin\frac{\theta_0}{2} = \sin 40^\circ = 0.6428$.

From CRC Standard Mathematical Tables, we find for

$F\left(a,\frac{\pi}{2}\right) = F\left(\sin\frac{\theta_0}{2},\frac{\pi}{2}\right)$ with $\theta_0 = 80^\circ$, $F = 1.7868$

$\therefore\ t = 0.2786\,(1.7868) = 0.4978$ second.

Alternatively,

$l = 30''$, $\theta_0 = 80^\circ = 1.3963$ rad and $g = 386.4$ in/sec^2.

Time taken by pendulum to reach vertical position (one-quarter of the time period) can be found from Eq. (E_{15}) in solution of problem 13.11 as

$$t = \frac{\tau}{4} = \frac{1}{\omega_0\left(1-\frac{\theta_0^2}{12}\right)}\ F\left(a,\frac{\pi}{2}\right) \qquad (E_4)$$

Here $\omega_0 = \sqrt{g/l} = \sqrt{386.4/30} = 3.5889$, $\theta_0 = 1.3963$ rad,

$$a^2 = \frac{\theta_0^2}{12\left(1-\frac{1}{12}\theta_0^2\right)} = \frac{1.9496}{12\,(0.8375)} = 0.1940 = (0.4405)^2$$

$$= \left(\sin^{-1} 26.1358^\circ\right)^2$$

$F\left(a,\frac{\pi}{2}\right) \simeq 1.6490$ from CRC Standard Mathematical Tables.

$$t \simeq \frac{1.6490}{(3.5889)(0.8375)} = 0.5486 \text{ sec}.$$

Note that this result is only approximate due to the approximation $\sin\theta \simeq \theta - \frac{1}{6}\theta^3$ used in the solution of problem 13.14.

13.14

$$\ddot{\theta} + \omega_0^2 \left(\theta - \frac{1}{6}\theta^3\right) = 0 \qquad (E_1)$$

This equation is similar to Eq. (13.9) with

$$x = \theta, \quad a = \omega_0, \quad F(x) = F(\theta) = \theta - \frac{1}{6}\theta^3 .$$

Eq. (E_1) can be rewritten as

$$\frac{d}{d\theta}\left(\dot{\theta}^2\right) + 2\,\omega_0^2\left(\theta - \frac{1}{6}\theta^3\right) = 0 \qquad (E_2)$$

which upon integration gives

$$\dot{\theta}^2 = 2\,\omega_0^2 \int_{\theta}^{\theta_0} F(\eta)\cdot d\eta = 2\,\omega_0^2 \int_{\theta}^{\theta_0} \left(\eta - \frac{1}{6}\eta^3\right)\cdot d\eta$$

$$= 2\,\omega_0^2 \left(\frac{1}{2}\eta^2 - \frac{1}{24}\eta^4\right)_{\theta}^{\theta_0} = \omega_0^2\left(\theta_0^2 - \frac{1}{12}\theta_0^4 - \theta^2 + \frac{1}{12}\theta^4\right) \qquad (E_3)$$

$$= \omega_0^2\left(\theta_0^2 - \theta^2\right)\left\{1 - \frac{1}{12}\left(\theta_0^2 + \theta^2\right)\right\} \qquad (E_4)$$

Since the maximum value of θ is θ_0, we assume

$$\theta(t) = \theta_0 \sin\beta \qquad (E_5)$$

Thus

$$\theta_0^2 - \theta^2 = \theta_0^2 - \theta_0^2 \sin^2\beta = \theta_0^2 \cos^2\beta \qquad (E_6)$$

$$\theta_0^2 + \theta^2 = \theta_0^2\left(1 + \sin^2\beta\right) \qquad (E_7)$$

and

$$\dot{\theta} = A_0 \cos\beta \, \frac{d\beta}{dt} \qquad (E_8)$$

Substitution of Eqs. (E_6) to (E_8) into (E_4) gives

$$\theta_0^2 \cos^2\beta \left(\frac{d\beta}{dt}\right)^2 = \omega_0^2\, \theta_0^2 \cos^2\beta \left\{1 - \frac{1}{12}\theta_0^2\left(1 + \sin^2\beta\right)\right\}$$

i.e.,

$$\left(\frac{d\beta}{dt}\right)^2 = \omega_0^2\left(1 - \frac{1}{12}\theta_0^2\right)\left\{1 - \frac{\theta_0^2 \sin^2\beta}{12\left(1 - \frac{1}{12}\theta_0^2\right)}\right\} \qquad (E_9)$$

Defining

$$a^2 = \frac{\theta_0^2}{12\left(1 - \frac{1}{12}\theta_0^2\right)} \qquad (E_{10})$$

Eq. (E_9) can be used to express (taking positive root):

$$\frac{d\beta}{dt} = \omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}\left(1 - a^2 \sin^2\beta\right)^{\frac{1}{2}} \qquad (E_{11})$$

i.e.,

$$\omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} dt = \int \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{12})$$

Integration of (E_{12}) yields

$$\omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}\left(t - t_0\right) = \int_{\beta_0}^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} \qquad (E_{13})$$

Using the initial conditions $\beta_0 = 0$ at $t_0 = 0$, Eq. (E_{13}) can be reduced to

$$\omega_0 \left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}} \cdot t = \int_0^{\beta} \frac{d\beta}{\sqrt{1 - a^2 \sin^2\beta}} = F(a,\beta) \qquad (E_{14})$$

where $F(a,\beta)$ is an incomplete elliptic integral of the first kind. Using $\beta = \frac{\pi}{2}$ when $\theta = \theta_0$ and $\beta = 0$ when $\theta = 0$, we get for one-quarter period,

$$\frac{\tau}{4} = t = \frac{1}{\omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{15})$$

Thus the time period of the pendulum is given by

$$\tau = \frac{4}{\omega_0\left(1 - \frac{1}{12}\theta_0^2\right)^{\frac{1}{2}}} \cdot F\left(a, \frac{\pi}{2}\right) \qquad (E_{16})$$

13.15 Equation: $E[x] = \ddot{x} + \omega_0^2 x - \frac{1}{6}\omega_0^2 x^3 = 0 \qquad (E_1)$

Assumed solution: $\tilde{x}(t) = A_0 \sin\omega t + A_3 \sin 3\omega t \qquad (E_2)$

$$\ddot{\tilde{x}}(t) = -A_0\omega^2 \sin\omega t - A_3 (3\omega)^2 \sin 3\omega t$$

$$\tilde{x}^3 = A_0^3 \sin^3\omega t + A_3^3 \sin^3 3\omega t \quad \text{(neglecting cross product terms)}$$

$$= \sin\omega t\left(\frac{3A_0^3}{4}\right) + \sin 3\omega t\left(-\frac{1}{4}A_0^3 + \frac{3}{4}A_3^3\right) + \sin 9\omega t\left(-\frac{1}{4}A_3^3\right)$$

This gives

$$\tilde{E} = E[\tilde{x}] = -A_0\omega^2 \sin\omega t - A_3(3\omega)^2 \sin 3\omega t + A_0\omega_0^2 \sin\omega t$$
$$+ A_3\omega_0^2 \sin 3\omega t - \frac{1}{8}\omega_0^2 A_0^3 \sin\omega t$$
$$+ \frac{1}{24}\omega_0^2\left(A_0^3 - 3A_3^3\right)\sin 3\omega t + \frac{1}{24}\omega_0^2 A_3^3 \sin 9\omega t$$

$$= B_1 \sin\omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t \qquad (E_3)$$

Where

$$B_1 = -A_0\omega^2 + A_0\omega_0^2 - \frac{1}{8}A_0^3\omega_0^2$$

$$B_2 = -9A_3\omega^2 + A_3\omega_0^2 + \frac{1}{24}A_0^3\omega_0^2 - \frac{1}{8}A_3^3\omega_0^2$$

$$B_3 = \frac{1}{24}A_3^3\omega_0^2$$

In Ritz-Galerkin method,

$$\int_0^{\tau} \tilde{E}\,\frac{\partial\tilde{E}}{\partial A_0}\,dt = 0 \qquad (E_4)$$

$$\int_0^{\tau} \tilde{E}\,\frac{\partial\tilde{E}}{\partial A_3}\,dt = 0 \qquad (E_5)$$

i.e., $\int_0^\tau \left(B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t\right)\left(B_4 \sin \omega t + B_5 \sin 3\omega t\right) dt = 0$

with $B_4 = -\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2$ and $B_5 = \frac{1}{8} A_0^2 \omega_0^2$.

i.e., $\int_0^\tau \left(B_1 B_4 \sin^2 \omega t + B_2 B_4 \sin \omega t \sin 3\omega t + B_3 B_4 \sin \omega t \sin 9\omega t\right.$

$\left. + B_1 B_5 \sin \omega t \sin 3\omega t + B_2 B_5 \sin^2 3\omega t + B_3 B_5 \sin 3\omega t \sin 9\omega t\right) dt = 0$

i.e., $$B_1 B_4 + B_2 B_5 = 0 \qquad (E_6)$$

and Eq. (E_5) leads to

$\int_0^\tau \left(B_1 \sin \omega t + B_2 \sin 3\omega t + B_3 \sin 9\omega t\right)\left(B_6 \sin 3\omega t + B_7 \sin 9\omega t\right) dt = 0$

with $B_6 = -9\omega^2 + \omega_0^2 - \frac{3}{8} A_3^2 \omega_0^2$ and $B_7 = \frac{1}{8} A_3^2 \omega_0^2$.

i.e. $$B_2 B_6 + B_3 B_7 = 0 \qquad (E_7)$$

Eqs. (E_6) and (E_7) can be rewritten as

$$\left(-A_0 \omega^2 + A_0 \omega_0^2 - \frac{1}{8} A_0^3 \omega_0^2\right)\left(-\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2\right)$$
$$+ \left(-A_3 \cdot 9\omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2\right)\left(\frac{1}{8} A_0^2 \omega_0^2\right) = 0 \qquad (E_8)$$

and

$$\left(-A_3 \cdot 9\omega^2 + A_3 \omega_0^2 + \frac{1}{24} A_0^3 \omega_0^2 - \frac{1}{8} A_3^3 \omega_0^2\right)\left(-9\omega^2 + \omega_0^2 - \frac{3}{8} A_3^2 \omega_0^2\right)$$
$$+ \left(\frac{1}{24} A_3^3 \omega_0^2\right)\left(\frac{1}{8} A_3^2 \omega_0^2\right) = 0 \qquad (E_9)$$

The solution of Eqs. (E_8) and (E_9) gives the values of A_0 and A_3.

Note that (E_8) and (E_9) are two simultaneous algebraic equations in A_0 and A_3 for which general closed-form solution is not possible.

Particular case :

If $A_3 = 0$, Eq. (E) yields

$$A_0 \left(-\omega^2 + \omega_0^2 - \frac{1}{8} A_0^2 \omega_0^2\right)\left(-\omega^2 + \omega_0^2 - \frac{3}{8} A_0^2 \omega_0^2\right) + \frac{1}{192} A_0^5 \omega_0^4 = 0$$

i.e., $$A_0 \left\{\omega^4 + \omega^2\left(-2\omega_0^2 + \frac{1}{2} A_0^2 \omega_0^2\right) + \omega_0^4 \left(1 - \frac{1}{2} A_0^2 + \frac{5}{96} A_0^4\right)\right\} = 0$$

since $A_0 \neq 0$, we get

$$\omega^4 - \omega^2\left(2\omega_0^2 - \frac{1}{2} A_0^2 \omega_0^2\right) + \omega_0^4 \left(1 - \frac{1}{2} A_0^2 + \frac{5}{96} A_0^4\right) = 0 \qquad (E_{10})$$

Which can be seen to be identical to Eq. (E.6) of Example 13.1.

13.16 Equation: $\ddot{x} + \omega_0^2 x + \alpha x^3 = 0 \qquad (E_1)$

we assume

$$x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) \qquad (E_2)$$

and

$$\omega^2 = \omega_0^2 + \alpha\, \omega_1(A_0) + \alpha^2\, \omega_2(A_0) \qquad (E_3)$$

or

$$\omega_0^2 = \omega^2 - \alpha\, \omega_1(A_0) - \alpha^2\, \omega_2(A_0) \qquad (E_4)$$

where A_0 = amplitude and ω = true fundamental frequency.

Substitution of (E_2) and (E_4) into (E_1) gives

$$\ddot{x}_0 + \alpha \ddot{x}_1 + \alpha^2 \ddot{x}_2 + \omega^2 x_0 - \alpha \omega_1 x_0 - \alpha^2 \omega_2 x_0 + \omega^2 \alpha x_1$$
$$- \alpha^2 \omega_1 x_1 - \alpha^3 \omega_2 x_1 + \alpha^2 \omega^2 x_2 - \alpha^3 \omega_1 x_2 - \alpha^4 \omega_2 x_2 + \alpha x_0^3$$
$$+ \alpha^4 x_1^3 + \alpha^7 x_2^3 + 3\alpha^2 x_0^2 x_1 + 3\alpha^3 x_0 x_1^2 + 3\alpha^3 x_0^2 x_2$$
$$+ 3\alpha^5 x_0 x_2^2 + 3\alpha^5 x_1^2 x_2 + 3\alpha^6 x_1 x_2^2 + 6\alpha^4 x_0 x_1 x_2 = 0 \qquad (E_5)$$

Neglecting terms involving $\alpha^3, \alpha^4, \ldots$, (E_5) can be rewritten as

$$\alpha^0 (\ddot{x}_0 + \omega^2 x_0) + \alpha^1 (\ddot{x}_1 - \omega_1 x_0 + \omega^2 x_1 + x_0^3)$$
$$+ \alpha^2 (\ddot{x}_2 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2 + 3 x_0^2 x_1) = 0 \qquad (E_6)$$

Setting the coefficient of α^0 to zero in (E_6), we get

$$\ddot{x}_0 + \omega^2 x_0 = 0 \qquad (E_7)$$

whose solution is

$$x_0(t) = D_1 \cos \omega t + D_2 \sin \omega t \qquad (E_8)$$

and hence $\quad \dot{x}_0(t) = -D_1 \omega \sin \omega t + D_2 \omega \cos \omega t \qquad (E_9)$

Let the initial conditions of the system be

$$x_0(t=0) = A_0 \quad \text{and} \quad \dot{x}_0(t=0) = 0 \qquad (E_{10})$$

Eqs. (E_8) to (E_{10}) yield $D_1 = A_0$ and $D_2 = 0$ so that

$$x_0(t) = A_0 \cos \omega t \qquad (E_{11})$$

Thus, if $\alpha = 0$, the solution becomes

$$x_0(t) = A_0 \cos \omega t, \qquad \omega = \omega_0 \qquad (E_{12})$$

By setting the coefficient of α^1 to zero in (E_6), we get

$$\ddot{x}_1 + \omega^2 x_1 = \omega_1 x_0 - x_0^3 \qquad (E_{13})$$

Substituting Eq. (E_{11}) for x_0, (E_{13}) leads to

$$\ddot{x}_1 + \omega^2 x_1 = \left(\omega_1 A_0 - \frac{3}{4} A_0^3\right) \cos \omega t - \frac{1}{4} A_0^3 \cos 3\omega t \qquad (E_{14})$$

The complete solution of (E_{14}) can be expressed as (can be verified by substitution):

$$x_1(t) = D_1 \cos \omega t + D_2 \sin \omega t + \left(\omega_1 A_0 - \frac{3}{4} A_0^3\right) \frac{1}{2} \omega t \cdot \sin \omega t$$
$$+ \frac{1}{32} \cdot \frac{1}{\omega^2} \cdot A_0^3 \cos 3\omega t \qquad (E_{15})$$

Note that the third term on the r·h·s· of (E_{15}) is a secular term which can be eliminated by setting

$$\omega_1 A_0 - \frac{3}{4} A_0^3 = 0 \quad \text{i.e.,} \quad A_0 = 0 \quad \text{or} \quad \omega_1 = \frac{3}{4} A_0^2 \qquad (E_{16})$$

Since $A_0 \neq 0$, we have $\quad \omega_1 = \frac{3}{4} A_0^2 \qquad (E_{17})$

Using the initial conditions $x_1(t=0) = \dot{x}_1(t=0) = 0$ in (E_{15}), we get

$$D_1 = -\frac{1}{32\,\omega^2} A_0^3 , \quad D_2 = 0 \qquad (E_{18})$$

Hence the first-order correction is given by

$$x_1(t) = -\frac{A_0^3}{32\omega^2} (\cos \omega t - \cos 3\omega t) \;;\quad \omega_1 = \frac{3}{4} A_0^2 \qquad (E_{19})$$

Finally, by setting the coefficient of α^2 in (E_6) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \omega_2 x_0 + x_1 \omega_1 - 3 x_0^2 x_1 \qquad (E_{20})$$

Substituting for x_0, x_1 and ω_1, Eq. (E_{20}) leads to

$$\ddot{x}_2 + \omega^2 x_2 = \left(\omega_2 A_0 - \frac{3}{128} \frac{A_0^5}{\omega^2}\right) \cos \omega t + \left(\frac{3}{128} \frac{A_0^5}{\omega^2}\right) \cos 3\omega t$$
$$+ \frac{3}{32} \frac{A_0^5}{\omega^2} \left(\cos^3 \omega t - \cos^2 \omega t \cos 3\omega t \right) \qquad (E_{21})$$

Here $\quad \cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t \qquad (E_{22})$

and $\quad \cos^2 \omega t \cdot \cos 3\omega t = \frac{1}{4} \cos \omega t + \frac{1}{2} \cos 3\omega t + \frac{1}{4} \cos 5\omega t \qquad (E_{23})$

Eqs. (E_{21}) to (E_{23}) lead to

$$\ddot{x}_2 + \omega^2 x_2 = \left(\omega_2 A_0 + \frac{3}{128} \frac{A_0^5}{\omega^2}\right) \cos \omega t - \frac{3}{128} \frac{A_0^5}{\omega^2} \cos 5\omega t \qquad (E_{24})$$

Since the first term on the r·h·s· of (E_{24}) leads to the secular term, we eliminate it by setting

$$\omega_2 A_0 + \frac{3}{128} \frac{A_0^5}{\omega^2} = 0 \quad \text{or} \quad \omega_2 = -\frac{3}{128} \frac{A_0^4}{\omega^2} \qquad (E_{25})$$

With this, the total solution of (E_{24}), after applying the initial conditions $x_2(t=0)=0$ and $\dot{x}_2(t=0)=0$, can be obtained as

$$x_2(t) = -\frac{A_0^5}{1024\,\omega^4}\left(\cos\omega t - \cos 5\omega t\right); \quad \omega_2 = -\frac{3}{128}\frac{A_0^4}{\omega^2} \qquad (E_{26})$$

Thus the total solution becomes

$$x(t) = A_0\cos\omega t - \frac{1}{32}\frac{A_0^3\alpha}{\omega^2}\left(\cos\omega t - \cos 3\omega t\right) - \frac{1}{1024}\cdot\frac{A_0^5\alpha^2}{\omega^4}\left(\cos\omega t - \cos 5\omega t\right)$$

and

$$\omega^2 = \omega_0^2 + \frac{3}{4}\cdot A_0^2\cdot\alpha - \frac{3}{128}\cdot\frac{A_0^4}{\omega^2}\cdot\alpha^2 \qquad (E_{27})$$

13.17 Equation: $\ddot{x} + c\dot{x} + k_1 x + k_2 x^3 = a_1\cos 3\omega t - a_2\sin 3\omega t \qquad (E_1)$

Assume

$$x(t) = A\cos\omega t + B\cos 3\omega t + C\sin 3\omega t \qquad (E_2)$$

so that

$$\dot{x}(t) = -\omega A\sin\omega t - 3\omega B\sin 3\omega t + 3\omega C\cos 3\omega t \qquad (E_3)$$

$$\ddot{x}(t) = -\omega^2 A\cos\omega t - 9\omega^2 B\cos 3\omega t - 9\omega^2 C\sin 3\omega t \qquad (E_4)$$

$$\begin{aligned}
x^3 =\ & A^3\cos^3\omega t + B^3\cos^3 3\omega t + C^3\sin^3 3\omega t \\
& + 3A^2B\cos^2\omega t\cos 3\omega t + 3AB^2\cos\omega t\cos^2 3\omega t \\
& + 3A^2C\cos^2\omega t\sin 3\omega t + 3AC^2\cos\omega t\sin^2 3\omega t \\
& + 3B^2C\cos^2 3\omega t\sin 3\omega t + 3BC^2\cos 3\omega t\sin^2 3\omega t \\
& + 6ABC\cos\omega t\cos 3\omega t\sin 3\omega t \\
=\ & \cos\omega t\left(\frac{3}{4}A^3 + \frac{3}{4}A^2B + \frac{3}{2}AB^2 + \frac{3}{2}AC^2\right) + \sin\omega t\left(\frac{3}{4}A^2C\right) \\
& + \cos 3\omega t\left(\frac{1}{4}A^3 + \frac{3}{4}B^3 + \frac{3}{2}A^2B + \frac{3}{2}BC^2\right) \\
& + \sin 3\omega t\left(\frac{3}{4}C^3 + \frac{3}{2}A^2C + \frac{3}{2}B^2C - \frac{3}{4}B^2C - \frac{3}{4}BC^2\right) \\
& + \cos 9\omega t\left(\frac{1}{4}B^3\right) + \sin 9\omega t\left(-\frac{1}{4}C^3 + \frac{3}{4}B^2C - \frac{3}{4}BC^2\right) \\
& + \cos 5\omega t\left(\frac{3}{4}A^2B + \frac{3}{4}AB^2 - \frac{3}{4}AC^2\right) \\
& + \cos 7\omega t\left(\frac{3}{4}AB^2 - \frac{3}{4}AC^2\right) + \sin 5\omega t\left(\frac{3}{4}A^2C + \frac{3}{2}ABC\right) \\
& + \sin 7\omega t\left(\frac{3}{2}ABC\right)
\end{aligned} \qquad (E_5)$$

For convenience, we use the notation

$\theta = \omega t$ so that $\dot{x} = \frac{dx}{dt} = \frac{dx}{d\theta}\frac{d\theta}{dt} = \omega \frac{dx}{d\theta} = \omega x'$

and $\ddot{x} = \frac{d}{dt}(\omega x') = \omega^2 x''$

With this, Eqs. (E_1) to (E_5) become

$$\omega^2 x'' + c\omega x' + k_1 x + k_2 x^3 = a_1 \cos 3\theta - a_2 \sin 3\theta \tag{E6}$$

$$x(\theta) = A \cos\theta + B\cos 3\theta + C \sin 3\theta \tag{E7}$$

$$x'(\theta) = \dot{x}/\omega = -A\sin\theta - 3B\sin 3\theta + 3C\cos 3\theta \tag{E8}$$

$$x''(\theta) = \ddot{x}/\omega^2 = -A\cos\theta - 9B\cos 3\theta - 9C\sin 3\theta \tag{E9}$$

$$x^3(\theta) = \left(\frac{3}{4}A^3 + \frac{3}{4}A^2 B + \frac{3}{2}AB^2 + \frac{3}{2}AC^2\right)\cos\theta + \left(\frac{3}{4}A^2 C\right)\sin\theta + \cdots + \left(\frac{3}{2}ABC\right)\sin 7\theta \tag{E10}$$

Terms containing $\cos 5\theta$, $\sin 5\theta$, $\cos 7\theta$, $\sin 7\theta$, $\cos 9\theta$ and $\sin 9\theta$ in Eq. (E_{10}) can be neglected for the present analysis. Then the substitution of Eqs. (E_7) to (E_{10}) into (E_6) and equating the coefficients of $\cos\theta$, $\sin\theta$, $\cos 3\theta$ and $\sin 3\theta$ on both sides gives:

$$(k_1 - \omega^2) + \frac{3}{4}k_2(A^2 + AB + 2B^2 + 2C^2) = 0 \tag{E11}$$

$$-c\omega + \frac{3}{4}k_2 AC = 0 \tag{E12}$$

$$(k_1 - 9\omega^2)B + 3c\omega C + \frac{1}{4}k_2(A^3 + 6A^2B + 3B^3 + 3BC^2) = a_1 \tag{E13}$$

$$(k_1 - 9\omega^2)C - 3c\omega B + \frac{3}{4}k_2 C(C^2 + 2A^2 + B^2) = -a_2 \tag{E14}$$

Eqs. (E_{11}) to (E_{14}) represent four simultaneous nonlinear algebraic equations in A, B, C and ω. For any specific case, these equations should yield non-zero real values for A, B and C for the existence of subharmonic oscillations of order 3.

<u>More specifically, for A:</u>

Eqs. (E_{11}) and (E_{12}) can be solved to find

$$\omega^2 = k_1 + \frac{3}{4}k_2(A^2 + AB + 2B^2 + 2C^2) \tag{E15}$$

$$C = \frac{4c\omega}{3k_2 A} \tag{E16}$$

Eq. (E_{15}) can be solved to obtain

$$A = -\frac{B}{2} \pm \frac{1}{2}\left\{\frac{16}{3k_2}(\omega^2 - k_1) - 7B^2 - 8C^2\right\}^{\frac{1}{2}} \qquad (E_{17})$$

This equation gives the condition for the existence of subharmonic of order 3 (nonzero A) as

$$\frac{16}{3k_2}(\omega^2 - k_1) - 7B^2 - 8C^2 > 0$$

i.e.,

$$\omega^2 > k_1 + \frac{3k_2}{16}(7B^2 + 8C^2) \qquad (E_{18})$$

13.18 Equation:

$$\ddot{x} + c\dot{x} + k_1 x + k_2 x^2 = a \cos 2\omega t \qquad (E_1)$$

As in the solution of problem 13.17, we introduce

$$\theta = \omega t, \quad \dot{x} = \omega x' \quad \text{and} \quad \ddot{x} = \omega^2 x'' \quad \text{with} \quad x' = \frac{dx}{d\theta}.$$

Eq. (E_1) becomes

$$\omega^2 x'' + c\omega x' + k_1 x + k_2 x^2 = a \cos 2\theta \qquad (E_2)$$

The solution of (E_2) must contain terms involving $\cos\theta$ and/or $\sin\theta$ in order to include subharmonics of order 2. If the nonlinear term $k_2 x^2$ is assumed to be small with $k_2 > 0$, we can express c and a as $c = \epsilon k_2^2$ and $a = a_0 k_2$. Thus (E_2) becomes

$$\omega^2 x'' + \epsilon\omega k_2^2 x' + k_1 x + k_2 x^2 = k_2 a_0 \cos 2\theta \qquad (E_3)$$

Using perturbation method, we assume

$$x(t) = x_0(t) + k_2 x_1(t) + k_2^2 x_2(t) + \cdots \qquad (E_4)$$

$$\omega = \omega_0 + k_2\omega_1 + k_2^2\omega_2 + \cdots \qquad (E_5)$$

Substituting (E_4) and (E_5) into (E_3), we get

$$(\omega_0 + k_2\omega_1 + k_2^2\omega_2)^2(x_0'' + k_2 x_1'' + k_2^2 x_2'') + \epsilon k_2^2(\omega_0 + k_2\omega_1 + k_2^2\omega_2)(x_0' + k_2 x_1' + k_2^2 x_2') + k_1(x_0 + k_2 x_1 + k_2^2 x_2) + k_2(x_0 + k_2 x_1 + k_2^2 x_2)^2 = k_2 a_0 \cos 2\theta$$

i.e.,

$$k_2^0(x_0''\omega_0^2 + k_1 x_0) + k_2^1(x_1''\omega_0^2 + 2\omega_0\omega_1 x_0'' + k_1 x_1 + x_0^2 - a_0\cos 2\theta) + k_2^2(x_2''\omega_0^2 + x_0''\omega_1^2 + 2\omega_0\omega_1 x_1'' + 2\omega_0\omega_2 x_0'' + \epsilon\omega_0 x_0' + k_1 x_2 + 2x_0 x_1) + k_2^3(\cdots) + \cdots = 0 \qquad (E_6)$$

By setting the coefficients of various powers of k_2 equal to zero in (E_6), we get the following.

coefficient of k_2^0:

$$x_0'' \omega_0^2 + k_1 x_0 = 0 \tag{E7}$$

Solution of (E7) is

$$x_0(t) = A_1 \cos\theta + A_2 \sin\theta \tag{E8}$$

and

$$\omega_0^2 = k_1 \tag{E9}$$

coefficient of k_2^1:

$$x_1'' \omega_0^2 + k_1 x_1 = -2\omega_0 \omega_1 x_0'' - x_0^2 + a_0 \cos 2\theta \tag{E10}$$

Substituting (E8) and (E9), Eq. (E10) becomes

$$x_1'' \omega_0^2 + k_1 x_1 = -2\omega_0\omega_1 \left(-A_1 \cos\theta - A_2 \sin\theta\right) - A_1^2 \cos^2\theta - A_2^2 \sin^2\theta - A_1 A_2 \sin 2\theta + a_0 \cos 2\theta$$

or,

$$x_1'' + x_1 = \left(\frac{2\omega_1 A_1}{\omega_0}\right)\cos\theta + \left(\frac{2\omega_1 A_2}{\omega_0}\right)\sin\theta + \frac{1}{k_1}\left(-\frac{A_1^2}{2} + \frac{A_2^2}{2} + a_0\right)\cos 2\theta - \left(\frac{A_1 A_2}{k_1}\right)\sin 2\theta - \left(\frac{A_1^2 + A_2^2}{2k_1}\right) \tag{E11}$$

Since the first two terms on the r.h.s. of (E11) lead to secular terms, we should have

$$\omega_1 = 0 \tag{E12}$$

to avoid those terms. Thus (E11) reduces to

$$x_1'' + x_1 = \frac{1}{k_1}\left(-\frac{A_1^2}{2} + \frac{A_2^2}{2} + a_0\right)\cos 2\theta - \frac{A_1 A_2}{k_1}\sin 2\theta - \frac{A_1^2 + A_2^2}{2k_1} \tag{E13}$$

The particular integral of (E13) can be written as

$$x_1(\theta) = -\left(\frac{A_1^2 + A_2^2}{2k_1}\right) + A_3 \cos 2\theta + A_4 \sin 2\theta \tag{E14}$$

where A_3 and A_4 can be found by substituting (E14) into (E13):

$$A_3 = \frac{1}{3k_1}\left(\frac{A_1^2 - A_2^2}{2} - a_0\right), \quad A_4 = \frac{A_1 A_2}{3k_1} \tag{E15}$$

coefficient of k_2^2:

$$\omega_0^2 x_2'' + k_1 x_2 = -\omega_1^2 x_0'' - 2\omega_0\omega_1 x_1'' - 2\omega_0\omega_2 x_0'' - \epsilon\omega_0 x_0' - 2x_0 x_1$$

or,

$$x_2'' + x_2 = -2\frac{\omega_2}{\omega_0}x_0'' - \frac{\epsilon}{\omega_0}x_0' - \frac{2}{k_1}x_0 x_1 \tag{E16}$$

since $\omega_1 = 0$ and $\omega_0^2 = k_1$. By substituting (E8) and (E14) on the r.h.s. of (E16), we get

$$x_2'' + x_2 = \left(\frac{2\omega_2}{\omega_0}A_1 - \frac{\epsilon}{\omega_0}A_2 + \frac{A_1}{k_1^2}(A_1^2 + A_2^2) - \frac{A_1 A_3}{k_1} - \frac{A_2 A_4}{k_1}\right)\cos\theta$$
$$+ \left(\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{k_1^2}(A_1^2 + A_2^2) + \frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1}\right)\sin\theta$$

$$+ \left(-\frac{A_1 A_3}{k_1} + \frac{A_2 A_4}{k_1} \right) \cos 3\theta + \left(-\frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1} \right) \sin 3\theta \qquad (E_{17})$$

To avoid secular terms, the coefficients of $\cos\theta$ and $\sin\theta$ in (E_{17}) must be zero. This gives

$$\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon A_2}{\omega_0} + \frac{A_1}{k_1^2}(A_1^2 + A_2^2) - \frac{A_1 A_3}{k_1} - \frac{A_2 A_4}{k_1} = 0 \qquad (E_{18})$$

$$\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{k_1^2}(A_1^2 + A_2^2) + \frac{A_2 A_3}{k_1} - \frac{A_1 A_4}{k_1} = 0 \qquad (E_{19})$$

Thus (E_{17}) reduces to

$$x_2'' + x_2 = (-A_1 A_3 + A_2 A_4)\frac{1}{k_1}\cos 3\theta - (A_2 A_3 + A_1 A_4)\frac{1}{k_1}\sin 3\theta \qquad (E_{20})$$

Substituting (E_{15}) into (E_{18}) and (E_{19}), we get

$$\frac{2\omega_2 A_1}{\omega_0} - \frac{\epsilon A_2}{\omega_0} + \frac{A_1}{6k_1^2}(6A_1^2 + 6A_2^2 - A_1^2 + A_2^2 - 2A_2^2) + \frac{A_1 a_0}{3k_1^2} = 0 \qquad (E_{21})$$

$$\frac{2\omega_2 A_2}{\omega_0} + \frac{\epsilon A_1}{\omega_0} + \frac{A_2}{6k_1^2}(6A_1^2 + 6A_2^2 + A_1^2 - A_2^2 - 2A_1^2) - \frac{A_2 a_0}{3k_1^2} = 0 \qquad (E_{22})$$

Dividing (E_{21}) and (E_{22}) by A_1 and A_2, respectively, we get

$$\frac{2\omega_2}{\omega_0} - \frac{\epsilon}{\omega_0}\frac{A_2}{A_1} + \frac{5}{6k_1^2}(A_1^2 + A_2^2) + \frac{a_0}{3k_1^2} = 0 \qquad (E_{23})$$

$$\frac{2\omega_2}{\omega_0} + \frac{\epsilon}{\omega_0}\frac{A_1}{A_2} + \frac{5}{6k_1^2}(A_1^2 + A_2^2) - \frac{a_0}{3k_1^2} = 0 \qquad (E_{24})$$

Addition and subtraction of (E_{23}) and (E_{24}) give

$$\omega_2 = \frac{\epsilon}{4}\left(\frac{A_2}{A_1} - \frac{A_1}{A_2}\right) - \frac{5\omega_0}{12k_1^2}(A_1^2 + A_2^2) \qquad (E_{25})$$

and
$$r + \frac{1}{r} = p \qquad (E_{26})$$

where $r = A_2/A_1$ and
$$p = \left(\frac{2a_0}{3\epsilon k_1 \omega_0}\right) \qquad (E_{27})$$

Solution of (E_{26}) is:
$$r = \frac{A_2}{A_1} = \frac{p \pm \sqrt{p^2 - 4}}{2} \qquad (E_{28})$$

Thus ω_2 becomes (Eq. (E_{25})):

$$\omega_2 = \pm \frac{(a_0^2 - 9\epsilon^2 k_1^3)^{1/2}}{6k_1\omega_0} - \frac{5(A_1^2 + A_2^2)}{12k_1\omega_0} \qquad (E_{29})$$

The solution of Eq. (E_{20}) can be determined as

$$x_2(\theta) = \left(\frac{A_1 A_3 - A_2 A_4}{8k_1}\right)\cos 3\theta + \left(\frac{A_2 A_3 + A_1 A_4}{8k_1}\right)\sin 3\theta \qquad (E_{30})$$

Thus Eqs. (E_4) and (E_5) become

$$x(t) = x_0(t) + k_2 x_1(t) + k_2^2 x_2(t) \;;\quad \omega = \omega_0 + k_2\omega_1 + k_2^2\omega_2$$

along with Eqs. (E_8), (E_9), (E_{12}), (E_{14}), (E_{29}) and (E_{30}).

13.19 Eq. (13.82) can be rewritten as

$$(\omega^2)^3 - \omega_0^2 (\omega^2)^2 - \frac{3}{4}\alpha A^2 (\omega^2)^2 + \frac{3}{32}\alpha F A (\omega^2) - \frac{3}{128}\alpha F^2 = 0 \qquad (E_1)$$

Differentiation of (E_1) gives

$$d\omega^2 \left(3\omega^4 - 2\omega_0^2 \omega^2 - \frac{3}{2}\alpha A^2 \omega^2 + \frac{3}{32}\alpha F A\right)$$
$$+ dA \left(-\frac{3}{2}\alpha \omega^4 A + \frac{3}{32}\alpha F \omega^2\right) = 0 \qquad (E_2)$$

Setting $\frac{d\omega^2}{dA} = 0$, Eq. (E_2) gives $\qquad A = \frac{1}{16}\frac{F}{\omega^2} \qquad (E_3)$

With this Eq. (E_3), Eq. (E_1) becomes

$$\omega^6 - \omega_0^2 \omega^4 = \frac{3}{128}\alpha F^2 + \frac{3}{1024}\alpha F^2 - \frac{3}{512}\alpha F^2 = \frac{21}{1024}\alpha F^2 \qquad (E_4)$$

Dividing (E_4) by ω_0^6, we get

$$\left(\frac{\omega}{\omega_0}\right)^6 - \left(\frac{\omega}{\omega_0}\right)^4 = \frac{21}{1024}\frac{\alpha F^2}{\omega_0^6} \qquad (E_5)$$

Letting $\frac{\omega}{\omega_0} = 1 + \delta$ with $\delta << 1$, Eq. (E_5) can be expressed as

$$(1+\delta)^6 - (1+\delta)^4 = \frac{21}{1024}\frac{\alpha F^2}{\omega_0^6} \qquad (E_6)$$

Since $\quad (1+\delta)^6 \approx 1 + 6\delta \quad$ and $\quad (1+\delta)^4 \approx 1 + 4\delta$,

Eq. (E_6) becomes

$$2\delta = \frac{21}{1024}\frac{\alpha F^2}{\omega_0^6}$$

or

$$\omega_{min} = \omega_0 + \frac{21}{2048}\frac{\alpha F^2}{\omega_0^5} \qquad (E_7)$$

13.20 Derivation of Eq. (13.113 b):

Using $a_0 = \frac{1}{4}$ and $y_0 = \sin\frac{t}{2}$, Eq. (13.103) gives

$$\ddot{y}_1 + a_0 y_1 = -a_1 y_0 - y_0 \cos t$$

or

$$\ddot{y}_1 + \frac{1}{4} y_1 = \left(-a_1 + \frac{1}{2}\right)\sin\frac{t}{2} - \frac{1}{2}\sin\frac{3t}{2} \qquad (E_1)$$

Homogeneous solution of (E_1) is:

$$y_1(t) = A_1 \cos\frac{t}{2} + A_2 \sin\frac{t}{2} \qquad (E_2)$$

where A_1 and A_2 are constants of integration. To avoid secular term, the coefficient of $\sin\frac{t}{2}$ must be set equal to zero in (E_1). This gives $a_1 = \frac{1}{2}$ and

$$\ddot{y}_1 + \frac{1}{4} y_1 = -\frac{1}{2}\sin\frac{3t}{2} \qquad (E_3)$$

Using the particular solution

$$y_1(t) = A_2 \sin\frac{3t}{2} \qquad (E_4)$$

in (E_3), we get $A_2 = \frac{1}{4}$ and $y_1(t) = \frac{1}{4} \sin \frac{3t}{2}$ $\quad (E_5)$

With $a_0 = \frac{1}{4}$, $a_1 = \frac{1}{2}$, $y_0 = \sin \frac{t}{2}$ and $y_1 = \frac{1}{4} \cdot \sin \frac{3t}{2}$, Eq. (13.104) becomes

$$\ddot{y}_2 + \frac{1}{4} y_2 = \left(-a_2 - \frac{1}{8}\right) \sin \frac{t}{2} - \frac{1}{8} \sin \frac{3t}{2} - \frac{1}{8} \sin \frac{5t}{2} \qquad (E_6)$$

Again the coefficient of $\sin \frac{t}{2}$ in (E_6) must be set equal to zero to avoid the secular term. This gives $a_2 = -\frac{1}{8}$. Thus Eq. (13.100) becomes

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 = \frac{1}{4} + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} \qquad (13.113b)$$

<u>Derivation of Eq. (13.116 b):</u>

With $a_0 = 1$ and $y_0 = \sin t$, Eq. (13.103) becomes

$$\ddot{y}_1 + y_1 = -a_1 y_0 - y_0 \cos t = -a_1 \sin t - \frac{1}{2} \sin 2t \qquad (E_7)$$

Coefficient of $\sin t$ in (E_7) must be zero to avoid the secular term. This gives $a_1 = 0$ and hence (E_7) reduces to

$$\ddot{y}_1 + y_1 = -\frac{1}{2} \sin 2t \qquad (E_8)$$

By substituting the particular solution

$$y_1(t) = A_2 \sin 2t \qquad (E_9)$$

into (E_8), we get $A_2 = \frac{1}{6}$. Using $a_0 = 1$, $y_0 = \sin t$, $a_1 = 0$ and $y_1 = \frac{1}{6} \sin 2t$, Eq. (13.104) becomes

$$\ddot{y}_2 + a_0 y_2 = -a_2 y_0 - a_1 y_1 - y_1 \cos t$$

$$= \left(-a_2 - \frac{1}{12}\right) \sin t - \frac{1}{12} \sin 3t \qquad (E_{10})$$

To avoid secular terms, the coefficient of $\sin t$ on the r.h.s. of Eq. (E_{10}) must be zero. This gives $a_2 = -\frac{1}{12}$. Thus Eq. (13.100) becomes

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 = 1 - \frac{1}{12} \epsilon^2 \qquad (13.116b)$$

13.21 $\ddot{x} + 0.4 \dot{x} + 0.8 x = 0$. This is in the form $\ddot{x} + 2 \zeta \omega_n \dot{x} + \omega_n^2 x = 0$ with $\omega_n = \sqrt{0.8} = 0.89443$ and $\zeta = 0.22361$.

$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.87178$.

Solution is $x(t) = e^{-\zeta \omega_n t} \left(A_1 \cos \omega_d t + A_2 \sin \omega_d t\right)$

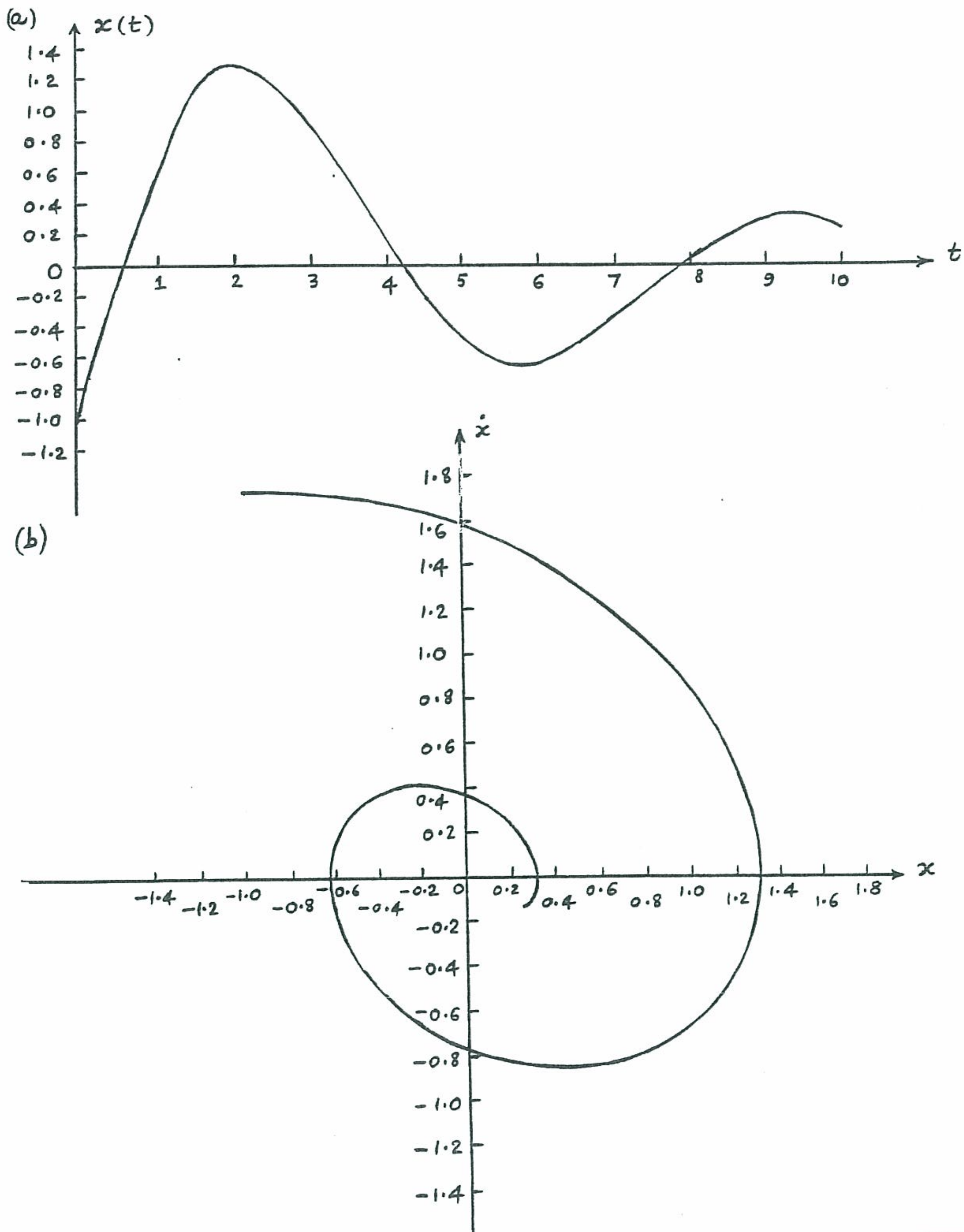
$$\dot{x}(t) = e^{-\zeta \omega_n t} \left[\left(-\zeta \omega_n A_1 + \omega_d A_2\right) \cos \omega_d t - \left(\omega_d A_1 + \zeta \omega_n A_2\right) \sin \omega_d t \right]$$

Using $x(0) = A_1 = -1$ and $\dot{x}(0) = -\zeta \omega_n A_1 + \omega_d A_2 = 2$, we obtain

$A_1 = -1$ and $A_2 = (2 + \zeta \omega_n A)/\omega_d = 1.77058$

$$\therefore\ x(t) = e^{-0.2t}\left(-\cos 0.87178t + 1.77058 \sin 0.87178t\right)$$

and $$\dot{x}(t) = e^{-0.2t}\left(1.74356 \cos 0.87178t + 0.51766 \sin 0.87178t\right)$$



13.22 $\ddot{x} + 0.1\,(x^2-1)\,\dot{x} + x = 0$ or $\ddot{x} = -\left[0.1\,(x^2-1)\,\dot{x} + x\right]$

Let $x = x_1$. $\dot{x}_1 = x_2 = f_1(x_1, x_2)$

$$\dot{x}_2 = -\left[0.1\,(x_1^2-1)\,x_2 + x_1\right] = f_2(x_1, x_2)$$

For equilibrium, from Eq. (13.129),

$$f_1 = 0 \Rightarrow x_2 = 0\ ;\quad f_2 = 0 \Rightarrow x_1 = 0$$

Eq. (13.132) becomes

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{Bmatrix} x_1\\ x_2\end{Bmatrix}$$

where

$$a_{11} = \left.\frac{\partial f_1}{\partial x_1}\right|_{(0,0)} = 0,\qquad a_{12} = \left.\frac{\partial f_1}{\partial x_2}\right|_{(0,0)} = 1,$$

$$a_{21} = \left.\frac{\partial f_2}{\partial x_1}\right|_{(0,0)} = -\left.\left(0.2\,x_1 x_2 + 1\right)\right|_{(0,0)} = -1,$$

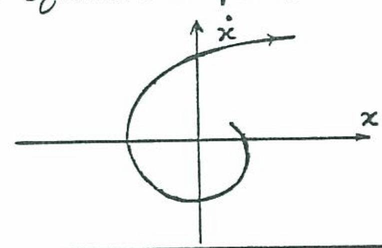
$$a_{22} = \left.\frac{\partial f_2}{\partial x_2}\right|_{(0,0)} = -\left.\left[0.1\,(x_1^2-1)\right]\right|_{(0,0)} = 0.1$$

From Eq. (13.135), we find $p = 0.1$, $q = 1$

$\lambda_1, \lambda_2 = \frac{1}{2}\left(0.1 \pm \sqrt{0.01-4}\right)$ = complex with positive real parts

Since $p > 0$, the system is unstable at the equilibrium point $(x, \dot{x}) = (0, 0)$.

Hence the phase-plane trajectory in the neighborhood of the equilibrium position appears as shown in the figure.



13.23 $\ddot{x} + 0.4\,\dot{x} + 0.8\,x = 0$ Initial conditions:

Let $x = x$, $\dot{x} = y$, $\dot{y} = -0.4\,y - 0.8\,x$ $\quad x(0) = 2,\ y(0) = 1$

$$\frac{dy}{dx} = \frac{-0.4\,y - 0.8\,x}{y} = c = \text{slope of the trajectory}$$

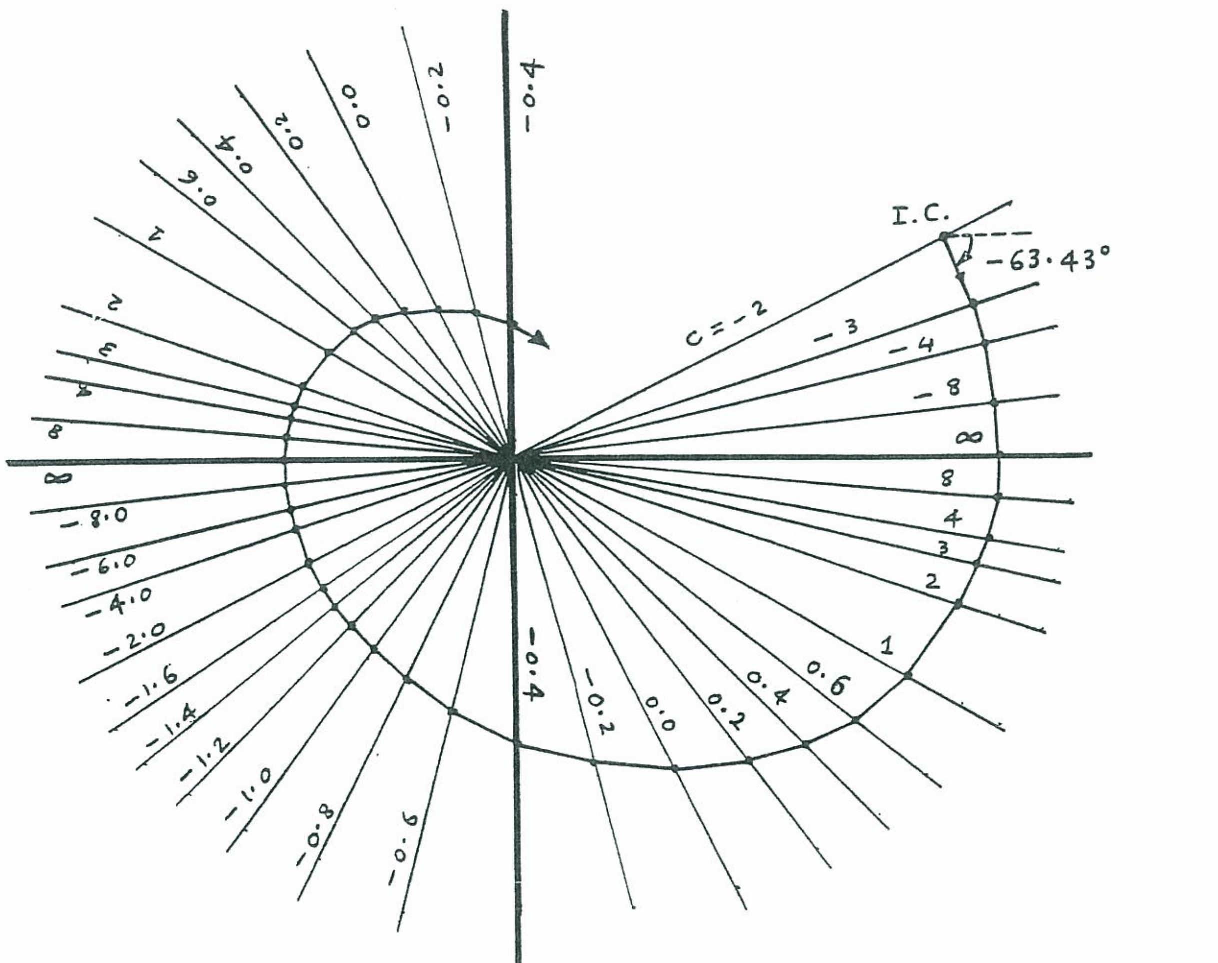
Equation of isocline is $-0.4\,y - 0.8\,x = c\,y$ or $\dfrac{y}{x} = \dfrac{-0.8}{c+0.4}$

This equation is plotted for different values of c. Then small line segments are drawn and slopes are used for extrapolation.

c	-2	-3	-4	-8	∞	8	4
value of y/x	0.5	0.3077	0.2222	0.1053	0	-0.0952	-0.1818
Angle	26.5651°	17.10°	12.53°	6.01°	0°	$-5.44°$	$-10.30°$
Slope of trajectory	$-63.43°$	$-71.56°$	$-75.96°$	$-82.87°$	$-90°$	82.87°	75.96°

3	2	1	0.6	0.4	0.2	0	−0.2	−0.4
−0.2353	−0.3333	−0.5714	−0.8	−1.0	−1.3333	−2.0	−4.0	−∞
−13.24°	−18.43°	−29.74°	−38.65°	−45°	−53.13°	−63.43°	−75.96°	−90°
71.56°	63.43°	45°	30.96°	21.80°	11.31°	0°	−11.31°	−21.80°

−0.6	−0.8	−1.0	−1.2	−1.6	−2.0	−4	−8	∞
4.0	2.0	1.3333	1.0	0.6667	0.5	0.2222	0.1053	0
75.96°	63.43°	53.13°	45°	33.69°	26.57°	12.53°	6.01°	0°
−30.96°	−38.66°	−45.00°	−50.19°	−57.99°	−63.43°	−75.96°	−82.87°	−90°



13.24 Equation of motion: $\ddot{x} + 0.1\,\dot{x} + x = 5$ (E₁)

Letting $\dfrac{dx}{dt} = y, \quad \dfrac{dy}{dt} = 5 - x - 0.1\,y$ (E₂)

we obtain
$$\frac{dy}{dx} = \frac{5 - x - 0.1\,y}{y} \qquad (E_3)$$

Instead of integrating (E_3) directly, we find the solution of (E_1) using the procedures of chapter 4.

Noting that $m = 1$, $c = 0.1$, $k = 1$, $F_0 = 5$, we find

$$\omega_n = \sqrt{\frac{k}{m}} = 1 \quad \text{and} \quad \zeta = \frac{c}{2\sqrt{km}} = \frac{0.1}{2(1)} = 0.05.$$

Hence the solution of (E_1), with the initial conditions $x(0) = \dot{x}(0) = 0$, is given by [see Example 4.9]:

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)\right]$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2} = 1\sqrt{1-0.05^2} = 0.9987$

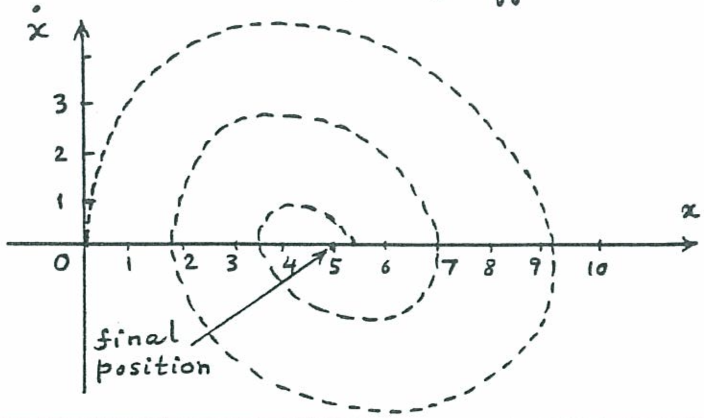
and $\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = \tan^{-1}\left(\frac{0.05}{0.9987}\right) = 2.8681°$

$$\therefore\ x(t) = 5\left[1 - 1.0013\, e^{-0.05t} \cos(0.9987t - 2.8681°)\right] \qquad (E_4)$$

Velocity is given by (differentiating Eq. (E_4)):

$$\dot{x}(t) = 0.2503\, e^{-0.05t} \cos(0.9987t - 2.8681°) + 5.0\, e^{-0.05t} \sin(0.9987t - 2.8681°) \qquad (E_5)$$

The trajectory in the phase-plane ($x - \dot{x}$ plane) appears as shown below.



13.25 Equation of motion: $\ddot{x} + f\frac{\dot{x}}{|\dot{x}|} + \omega_n^2 x = 0$ $\qquad (E_1)$

i.e. $\ddot{x} + \omega_n^2 (x + a) = 0$ for $\dot{x} > 0$ $\qquad (E_2)$

and $\ddot{x} + \omega_n^2 (x - a) = 0$ for $\dot{x} < 0$ $\qquad (E_3)$

where $a = f/\omega_n^2$ $\qquad (E_4)$

Multiplying by $2\dot{x}$ and integrating, (E_2) and (E_3) yield

$$\dot{x}^2 + \omega_n^2 (x + a)^2 = R_j^2 \quad \text{for } \dot{x} > 0 \qquad (E_5)$$

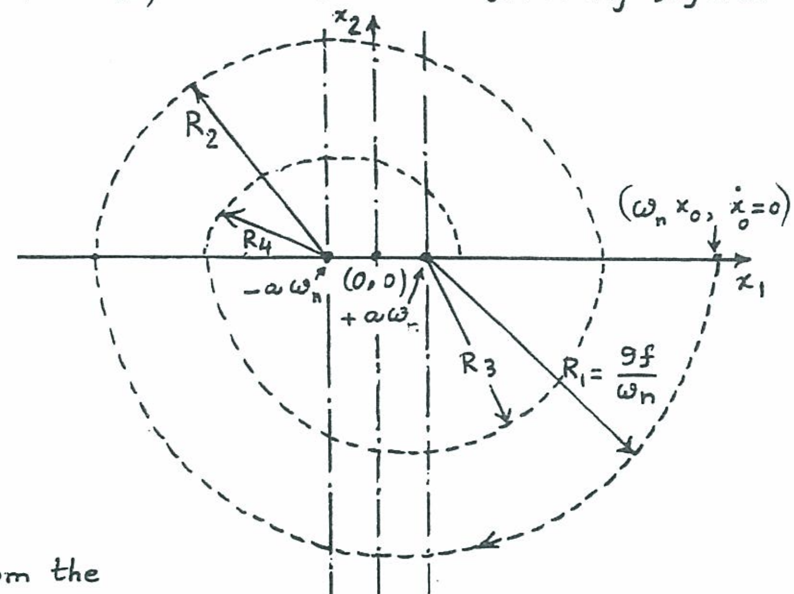
$$\dot{x}^2 + \omega_n^2 (x - a)^2 = R_{j+1}^2 \quad \text{for } \dot{x} < 0 \qquad (E_6)$$

where R_j^2 and R_{j+1}^2 are integration constants which are to be computed at each switching of sign of $\dot{x}$.

We can plot the trajectories of a representative point whose coordinates are

$$x_1 = \omega_n x \quad , \quad x_2 = \dot{x} \tag{E_7}$$

Eqs. (E_5) and (E_6) show that the trajectory is made of semicircles whose centers are located at $x = -a$ (or $x_1 = -a\omega_n$) and $x = +a$ (or $x_1 = +a\omega_n$) as shown in the following figure.



R_1 can be obtained from the initial conditions using Eq. (E_6) as:

$$R_1^2 = 0^2 + \omega_n^2 \left(\frac{10f}{\omega_n^2} - \frac{f}{\omega_n^2} \right)^2 = \left(\frac{9f}{\omega_n} \right)^2 \quad ; \quad R_1 = \frac{9f}{\omega_n} \tag{E_8}$$

Notice that the radii of the circles $R_1, R_2, \ldots$ decrease according to the relation

$$R_j = R_{j-1} - 2a\omega_n \quad ; \quad j = 1, 2, \ldots$$

and the system will stop when

$$R_k \le 2a\omega_n$$

Here $R_1 = \frac{9f}{\omega_n}$, $R_2 = R_1 - \frac{2f}{\omega_n} = \frac{7f}{\omega_n}$, $R_3 = R_2 - \frac{2f}{\omega_n} = \frac{5f}{\omega_n}$,

$R_4 = R_3 - \frac{2f}{\omega_n} = \frac{3f}{\omega_n}$, $R_5 = R_4 - \frac{2f}{\omega_n} = \frac{f}{\omega_n}$,

and the motion stops at this point (after five half-cycles) since $R_6 < 2a\omega_n = \frac{2f}{\omega_n}$.

13.26 $\ddot{\theta} + c\dot{\theta} + \sin\theta = 0$ or $\ddot{\theta} = -c\dot{\theta} - \sin\theta$

Let $x = \theta$ and $y = \frac{dx}{dt} = \dot{\theta}$

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -cy - \sin x \tag{E$_1$}$$

Equilibrium or critical point (where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$) of this system is $(x=0, y=0)$. Linearization of Eqs. (E$_1$) about the equilibrium point (origin) leads to

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -cy - x$$

or
$$\begin{Bmatrix} dx/dt \\ dy/dt \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{E$_2$}$$

The eigenvalues of this system are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

i.e. $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda - c \end{vmatrix} = 0$ i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $$\lambda_{1,2} = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - 1} \tag{E$_3$}$$

If $c = 0$: $p = 0;\ q = 1;\ \lambda_{1,2} = \pm\sqrt{-1}$
The origin will be a center.

If $0 < c < 2$: $p > 0;\ q > 0;\ \lambda_{1,2} =$ complex conjugates
The origin will be a stable focal point (spiral point).

If $c = 2$: $p > 0;\ q > 0;\ \lambda_{1,2} =$ negative and equal.
The origin will be a stable nodal point.

If $c > 2$: $p > 0;\ q > 0;$ If $\lambda_{1,2} =$ negative real, the origin will be a stable nodal point.

If $-2 < c < 0$: $p < 0;\ q > 0;\ \lambda_{1,2} =$ complex conjugates.
The origin will be an unstable focal point (spiral point).

13.27 Equation of motion: $$\ddot{\theta} + 0.5\dot{\theta} + \sin\theta = 0.8 \tag{E$_1$}$$

Let $x = \theta$ and $y = dx/dt$

$$\therefore \frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\sin x - 0.5y + 0.8 \tag{E$_2$}$$

$$\frac{dy}{dx} = \frac{-\sin x - 0.5\,y + 0.8}{y} \qquad (E_3)$$

At $(x = \sin^{-1} 0.8,\ y = 0)$, $\frac{dy}{dx} = \frac{0}{0}$ and hence it will be an equilibrium point. To investigate the nature of singularity, we rewrite Eqs. (E_2) in linearized form as

$$\left.\begin{aligned} \frac{dx}{dt} &= (0)x + (1)y \\ \frac{dy}{dt} &= (0)x - 0.5\,y \end{aligned}\right\} \qquad (E_4)$$

Thus the eigenvalues of the system are given by

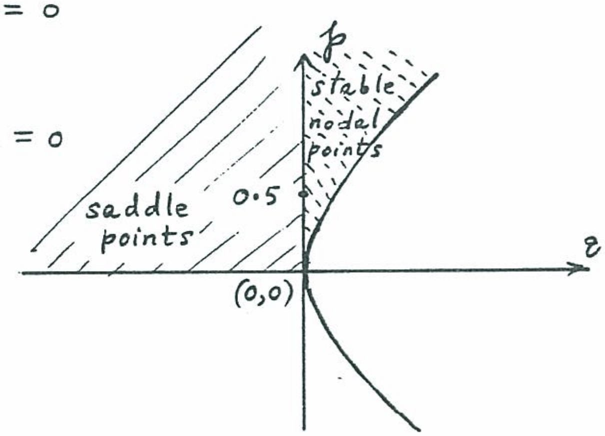
$$\left| \begin{bmatrix} 0 & 1 \\ 0 & -0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & 1 \\ 0 & -0.5-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + 0.5\lambda \equiv \lambda^2 + p\lambda + q = 0$

$\therefore\ \lambda_1 = 0,\ \ \lambda_2 = \text{negative}$

Here p = positive ; $q = 0$, $\lambda_1 = 0$ and λ_2 = negative.

Thus the equilibrium point falls on the border of saddle points and stable nodal points as shown in the adjacent figure.



13.28

$$\frac{dx}{dt} = (0)\,x + (1)\,y \qquad (E_1)$$

$$\frac{dy}{dt} = -1.x - c\,y + (0.1)\,x^3 \qquad (E_2)$$

Eqs. (E_1) and (E_2) are zero at $(x = 0,\ y = 0)$. Hence the origin $(0,0)$ will be equilibrium point (singularity). The eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + \lambda c + 1 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $$\lambda_{1,2} = \left\{ \frac{-c \pm \sqrt{c^2 - 4}}{2} \right\}$$

For $c > 0$ and $c < 2$:

$p > 0$, $q > 0$ and $\lambda_{1,2}$ = complex conjugates.

Hence the origin will be a stable focus (or spiral point).

For $c \geq 2$:

$p > 0$, $q > 0$; $\lambda_{1,2}$ = negative real.

Hence the origin will be a stable nodal point.

13.29

$$\ddot{x} - \alpha \dot{x} (1 - x^2) + x = 0 \quad , \quad \alpha > 0 \qquad (E_1)$$

This equation can be rewritten as

$$\left. \begin{aligned} dx/dt &= y \\ dy/dt &= -x + y \alpha (1 - x^2) \end{aligned} \right\} \qquad (E_2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-x + y \alpha (1 - x^2)}{y} \qquad (E_3)$$

Thus the system has singularity at $(x = 0, y = 0)$. Near the origin, the nature of singularity can be investigated by considering the linearized equations:

$$\left. \begin{aligned} dx/dt &= (0)\, x + (1)\, y \\ dy/dt &= (-1)\, x + (\alpha) y \end{aligned} \right\} \qquad (E_4)$$

The eigenvalues corresponding to Eqs. (E_4) are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -1 & \alpha - \lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 - \alpha \lambda + 1 \equiv \lambda^2 + p \lambda + q = 0$

i.e., $\lambda_{1,2} = \dfrac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$

Here $p = -\alpha$, $q = 1$ and $\lambda_{1,2}$ = complex conjugates for $\alpha > 0$ and α = small. Hence the origin will be an unstable focus (spiral point).

13.30

$$\ddot{x} + \omega_n^2 \, x \, (1 + k^2 x^2) = 0 \qquad (E_1)$$

Let $dx/dt = y$, $\quad dy/dt = -\omega_n^2 x - \omega_n^2 k^2 x^3 \qquad (E_2)$

Singularity is at $(x = 0, y = 0)$. Eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \begin{vmatrix} -\lambda & 1 \\ -\omega_n^2 & -\lambda \end{vmatrix} = 0$$

i.e., $\lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$

i.e., $\lambda_{1,2} = \pm i\,\omega_n$

Here $p = 0$; $q = \omega_n^2 = $ positive ; $\lambda_{1,2} = $ imaginary (conjugates)

$\therefore$ Equilibrium point is a center.

13.31 $\ddot{x} + \omega_n^2 x\,(1 - k^2 x^2) = 0$ (E_1)

with $dx/dt = y$, $dy/dt = -\omega_n^2 x + \omega_n^2 k^2 x^3$ (E_2)

Singularity at $(x = 0,\ y = 0)$.

Linearized equations about origin are

$$dx/dt = y, \qquad dy/dt = -\omega_n^2 x \qquad (E_3)$$

Eigenvalues are given by

$$\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$$

i.e., $\lambda_{1,2} = \pm i\,\omega_n$

$\therefore$ $p = 0$; $q = $ positive $= \omega_n^2$; $\lambda_{1,2} = $ conjugates (imaginary)

Hence equilibrium point is a center.

13.32 $\ddot{\theta} + \omega_n^2 \sin\theta = 0$ (E_1)

Using $x = \theta$, Eq. (E_1) can be expressed as

$$dx/dt = y, \qquad dy/dt = -\omega_n^2 \sin x \qquad (E_2)$$

Singularity is at $(x = 0,\ y = 0)$.

Linearized equations about origin yield

$$dx/dt = y, \qquad dy/dt = -\omega_n^2 x$$

Eigenvalues are given by $\left| \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$

i.e., $\lambda^2 + \omega_n^2 \equiv \lambda^2 + p\lambda + q = 0$; $\lambda_{1,2} = \pm i\,\omega_n$

Here $p = 0$; $q = \omega_n^2 = $ positive ; $\lambda_{1,2} = $ imaginary (conjugates)

$\therefore$ Equilibrium point is a center.

13.33 (a) $\dot{x} = x - y$

$\dot{y} = x + 3y$

Assuming the solution as $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} e^{\lambda t}$,

we get the following eigenvalue problem

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_1)$$

The eigenvalues are given by

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$$

i.e., $\lambda_{1,2} = \dfrac{4 \pm \sqrt{16-4}}{2} = 2, 2$

Both eigenvalues are same.

Using $\lambda_1 = 2$ in the first equation of (E_1), we get

$Y = -X$.

Letting $X = 1$ arbitrarily, we get $\begin{Bmatrix} X \\ Y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ for $\lambda_1 = 2$ $\qquad (E_2)$

Since the substitution of $\lambda_2 = 2$ gives the same eigenvector as in (E_2), we seek a second solution in the form $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} t\, e^{2t} \qquad (E_3)$

Substitution of (E_3) into the original equations gives

$$\begin{Bmatrix} X \\ Y \end{Bmatrix} e^{2t} + 2 \begin{Bmatrix} X \\ Y \end{Bmatrix} t\, e^{2t} - \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} t\, e^{2t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_4)$$

It can be seen that the only way (E_4) can be satisfied for all values of t is to have $\begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$. This implies that solution of the form of Eq. (E_3) can not exist. Hence we seek a solution of the form

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} t\, e^{2t} \qquad (E_5)$$

where $\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}$ and $\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix}$ are to be determined. Substitution of (E_5) into original equations gives

$$2 \begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} t\, e^{2t} + \left(\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} + 2 \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} \right) e^{2t} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \left(\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} t\, e^{2t} + \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} e^{2t} \right) \qquad (E_6)$$

By equating the coefficients of $t\, e^{2t}$ and e^{2t} on both sides of (E_6), we get

$$2 \begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} \qquad (E_7)$$

and $\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} + 2 \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}$ (E_8)

It can be seen that (E_7) will be satisfied if

$$\begin{Bmatrix} X_2 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \text{eigenvector corresponding to } \lambda_1 = 2.$$

Eqs. (E_8) can be rewritten in scalar form as

$$\left.\begin{aligned} X_2 + 2X_1 &= X_1 - Y_1 \\ Y_2 + 2Y_1 &= X_1 + 3Y_1 \end{aligned}\right\} \quad \text{or,} \quad \left.\begin{aligned} X_1 + Y_1 &= -X_2 \\ -X_1 - Y_1 &= -Y_2 \end{aligned}\right\} \qquad (E_9)$$

Thus for $X_2 = 1$ and $Y_2 = -1$, (E_9) give $X_1 = c$ = any constant and $Y_1 = -1 - c$.

$$\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix} = \begin{Bmatrix} c \\ -1-c \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} + c \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \qquad (E_{10})$$

$\therefore$ The solution of (E_5) becomes

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} + c \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t\, e^{2t} \qquad (E_{11})$$

We can ignore the middle term on the r.h.s. of (E_{11}) since it is same as $\begin{Bmatrix} X_1 \\ Y_1 \end{Bmatrix}^{(1)}$. Thus the second solution of original equations is

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t\, e^{2t} \qquad (E_{12})$$

(b) $$\left.\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 4x + y \end{aligned}\right\} \qquad (E_1)$$

Assuming the solution as $\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} e^{\lambda t}$ (E_2)

Eqs. (E_1) give

$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_3)$$

Eigenvalues are given by $\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$ (E_4)

i.e., $\lambda_1 = -1$, $\lambda_2 = 3$

substitution of $\lambda_1 = -1$ in (E_3) gives $\left.\begin{aligned} 2X^{(1)} + Y^{(1)} &= 0 \\ 4X^{(1)} + 2Y^{(1)} &= 0 \end{aligned}\right\}$

or $Y^{(1)} = -2X^{(1)}$

choosing $X^{(1)} = 1$, arbitrarily, the first eigenvector becomes

$$\begin{Bmatrix} X \\ Y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ -2 \end{Bmatrix}$$

Next, by substituting $\lambda_2 = 3$ in (E_3), we get $-2X^{(2)} + Y^{(2)} = 0$ or $Y^{(2)} = 2X^{(2)}$. By choosing $X^{(2)} = 1$ arbitrarily, the second eigenvector becomes $\begin{Bmatrix} X \\ Y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$.

13.34 $\left.\begin{array}{l}\dot{x} = x - 2y \\ \dot{y} = 4x - 5y\end{array}\right\}, \qquad \dfrac{dy}{dx} = \dfrac{4x - 5y}{x - 2y} \qquad (E_1)$

Eigenvalues are defined by

$$\left[\begin{bmatrix} 1 & -2 \\ 4 & -5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_2)$$

This gives $\begin{vmatrix} 1-\lambda & -2 \\ 4 & -5-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = 0$

$\lambda_1 = -3, \quad \lambda_2 = -1.$

Eigenvectors: For $\lambda_1 = -3$, Eq. (E_2) gives $4X = 2Y$

For $\lambda_2 = -1$, Eq. (E_2) gives $2X = 2Y$

$$\vec{X}^{(1)} = \begin{Bmatrix} X \\ Y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \quad \text{and} \quad \vec{X}^{(2)} = \begin{Bmatrix} X \\ Y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Thus the solutions are: $\vec{x}^{(1)} = \begin{Bmatrix} X \\ Y \end{Bmatrix}^{(1)} e^{\lambda_1 t}, \quad \vec{x}^{(2)} = \begin{Bmatrix} X \\ Y \end{Bmatrix}^{(2)} e^{\lambda_2 t}$

General solution is

$$\vec{x} = \begin{Bmatrix} x(t) \\ y(t) \end{Bmatrix} = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) = c_1 \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} e^{-3t} + c_2 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} e^{-t} \qquad (E_3)$$

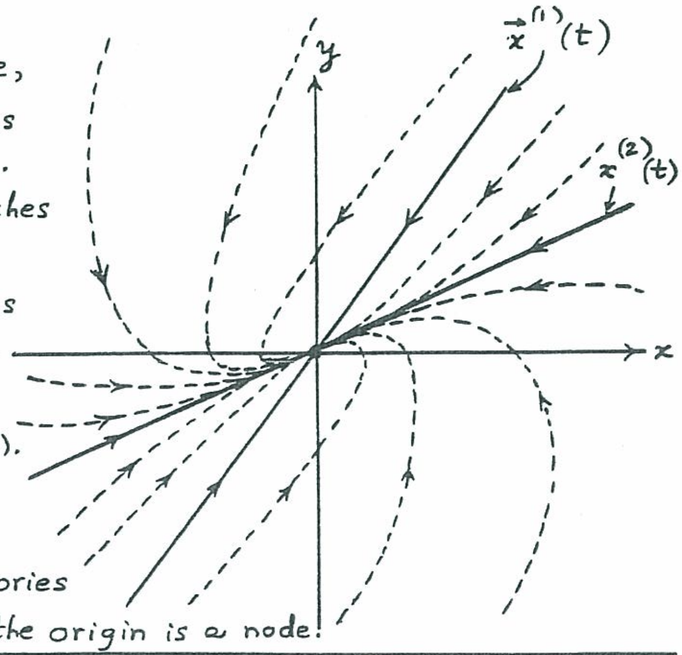
where c_1 and c_2 are arbitrary constants.

The solution can be represented in the xy plane for various values of c_1 and c_2.

First, we consider $c_2 = 0$ so that

$x = c_1 e^{-3t}, \quad y = 2c_1 e^{-3t}$

Eliminating t between these, we find that the solution lies on the straight line $y = 2x$. The solution $x^{(1)}(t)$ approaches origin along the line $y = 2x$. Similarly, $x^{(2)}(t)$ approaches origin along the line $y = x$. As $t \to \infty$, $x^{(1)}(t)$ is negligible compared to $x^{(2)}(t)$. Hence $\vec{x}(t)$ of (E_3) approaches origin tangential to line $y = x$. The trajectories are shown in the figure. Here the origin is a node.



13.35 $\dot{x} = x - y, \quad \dot{y} = x + 3y$ (E_1)

$$\frac{dy}{dx} = \frac{x + 3y}{x - y} \qquad (E_2)$$

The eigenvalues and eigenvectors of this system were found in Problem 13.33(a)

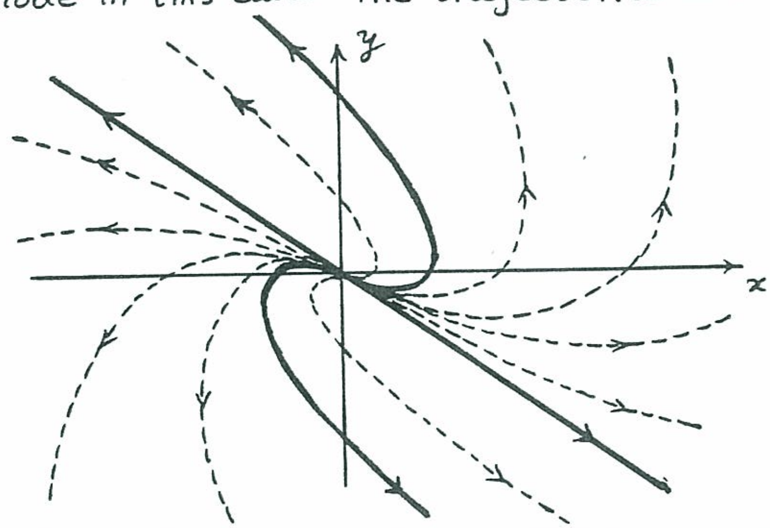
$\lambda_1 = 2, \ \lambda_2 = 2, \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$ = only one independent eigenvector

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t}, \quad \vec{x}^{(2)}(t) = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} + \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} \qquad (E_3)$$

The general solution of (E_1) is

$$\vec{x}(t) = c_1 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{2t} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} t e^{2t} + c_2 \begin{Bmatrix} 0 \\ -1 \end{Bmatrix} e^{2t} \qquad (E_4)$$

To draw the trajectories given by (E_4), we first observe the following behavior. As $t \to \infty$, $\vec{x}(t)$ becomes unbounded and as $t \to -\infty$, $\vec{x}(t) \to \vec{0}$. Also, as $t \to -\infty$, the solutions approach the origin $\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ tangentially to the line $y = -x$ (which denotes the eigenvector, $\vec{X}^{(1)}$). Further, as $t \to \infty$, the trajectories will lie asymptotic to the line $y = -x$. The origin will be a node in this case. The trajectories are shown below.



13.36 $\dot{x} = 2x + y, \quad \dot{y} = -3x - 2y$ (E_1)

$$\frac{dy}{dx} = \frac{-3x - 2y}{2x + y} \qquad (E_2)$$

Eigenvalue problem is: $\left[\begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ (E_3)

Eigenvalues are given by $\begin{vmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0$

$$\lambda_1 = -1, \quad \lambda_2 = 1 \qquad (E_4)$$

Eigenvectors: For $\lambda_1 = -1$; $(2-\lambda_1) X + Y = 0 \Rightarrow Y = -3X$

For $\lambda_2 = 1$; $(2-\lambda_2) X + Y = 0 \Rightarrow Y = -X$

$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ -3 \end{Bmatrix}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

Hence the general solution is: $\vec{x}(t) = c_1 \begin{Bmatrix} 1 \\ -3 \end{Bmatrix} e^{-t} + c_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} e^{t}$ $\quad (E_5)$

where c_1 and c_2 are arbitrary constants.

To plot the trajectories, first consider the case of $c_1 = 0$.

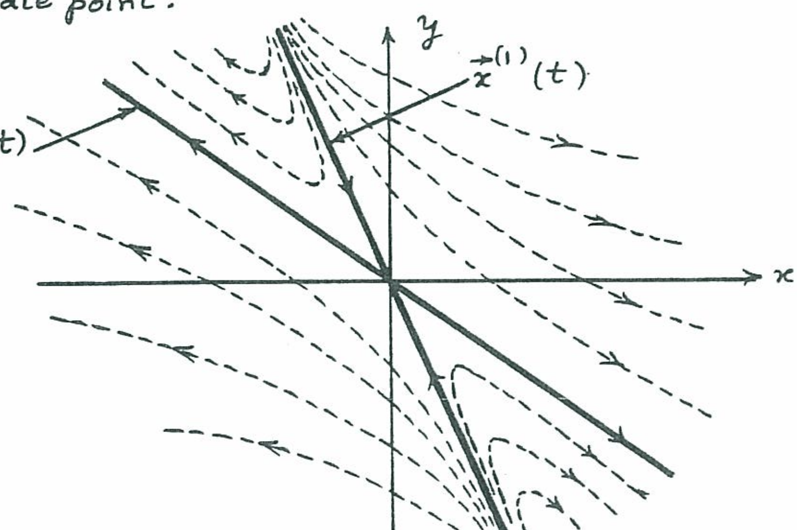
$\vec{x}(t) = c_2 \vec{x}^{(2)}(t)$ i.e., $x(t) = c_2 e^{t}$, $y(t) = -c_2 e^{t}$ $\quad (E_6)$

Eliminating t from (E_6), we get $y = -x$. This line denotes the eigenvector $\vec{X}^{(2)}$. If we consider the trajectory of a representative point, the particle moves away from origin as t increases. Next, we consider $c_2 = 0$. Eq. (E_5) gives

$x(t) = c_1 e^{-t}$, $y(t) = -3 c_1 e^{-t}$ $\quad (E_7)$

This leads to the equation $y = -3x$, which represents the eigenvector, $\vec{X}^{(1)}$. A representative point in this case goes to origin as $t \to \infty$.

In the general solution of (E_5), the term $c_2 \vec{x}^{(2)}(t)$ dominates for large values of t. Thus the solutions with $c_2 \neq 0$ will be asymptotic to the line $y = -3x$ as $t \to \infty$. Similarly, all solutions with $c_1 \neq 0$ will be asymptotic to the line $y = -3x$ as $t \to -\infty$. The trajectories are shown below, with origin denoting a saddle point.



13.37 Van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0$, $\alpha > 0$ $\quad (E_1)$

Assume $x(t) = x_0(t) + \alpha\, x_1(t) + \alpha^2 x_2(t)$ $\quad (E_2)$

$\omega_0^2 = 1 = \omega^2 - \alpha\, \omega_1 - \alpha^2 \omega_2$ $\quad (E_3)$

where $\omega_0^2 = 1 =$ coefficient of x in Eq. (E_1).
Substitution of (E_2) and (E_3) into (E_1) gives

$$\alpha^0 \left[\ddot{x}_0 + \omega^2 x_0 \right] + \alpha^1 \left[\ddot{x}_1 - \dot{x}_0 + \dot{x}_0 x_0^2 - \omega_1 x_0 + \omega^2 x_1 \right]$$
$$+ \alpha^2 \left[\ddot{x}_2 - \dot{x}_1 + \dot{x}_1 x_0^2 + 2 x_0 \dot{x}_0 x_1 - \omega_2 x_0 - \omega_1 x_1 + \omega^2 x_2 \right]$$
$$+ \alpha^3 [\cdots] + \cdots = 0 \tag{E_4}$$

Setting coefficient of α^0 in (E_4) to zero, we obtain

$$\ddot{x}_0 + \omega^2 x_0 = 0 \quad , \quad \text{i.e.,} \quad x_0(t) = A_1 \cos \omega t + A_2 \sin \omega t \tag{E_5}$$

Assuming the initial conditions $x(0) = A$ and $\dot{x}(0) = 0$, we get $A_1 = A$ and $A_2 = 0$. Thus (E_5) reduces to

$$x_0(t) = A \cos \omega t \tag{E_6}$$

Setting coefficient of α^1 to zero, in Eq. (E_4),

$$\ddot{x}_1 + \omega^2 x_1 = \dot{x}_0 - \dot{x}_0 x_0^2 + \omega_1 x_0$$
$$= -A\omega \sin \omega t + A^3 \omega \sin \omega t . \cos^2 \omega t + \omega_1 A \cos \omega t$$
$$= \left(-A\omega + \frac{1}{4} A^3 \omega \right) \sin \omega t + \omega_1 A \cos \omega t + \frac{A^3 \omega}{4} \sin 3\omega t \tag{E_7}$$

The coefficients of $\sin \omega t$ and $\cos \omega t$ must be zero in Eq. (E_7) to avoid secular terms. This gives

$$A = \pm 2 \; , \quad \omega_1 = 0 \tag{E_8}$$

Thus the particular solution of (E_7) can be expressed as

$$x_1(t) = A_3 \sin 3\omega t + A_4 \cos 3\omega t \tag{E_9}$$

Substitution of (E_8) and (E_9) into E_7 gives

$$A_3 = \frac{1}{32} \frac{A^3}{\omega} \quad \text{and} \quad A_4 = 0 \tag{E_{10}}$$

Thus

$$x_1(t) = \frac{1}{32} \frac{A^3}{\omega} \sin 3\omega t \tag{E_{11}}$$

Finally, setting coefficient of α^2 in (E_4) to zero, we get

$$\ddot{x}_2 + \omega^2 x_2 = \dot{x}_1 - \dot{x}_1 x_0^2 - 2 x_0 \dot{x}_0 x_1 + \omega_2 x_0 + \omega_1 x_1 \tag{E_{12}}$$

Substitution of (E_{11}), (E_6) and (E_8) into (E_{12}) leads to

$$\ddot{x}_2 + \omega^2 x_2 = \frac{3}{32} A^3 \cos 3\omega t - \left(\frac{3}{32} A^3 \cos 3\omega t \right) A^2 \cos^2 \omega t$$
$$- 2 (A \cos \omega t)(-A\omega \sin \omega t) \left(\frac{A^3}{32\omega} \sin 3\omega t \right) + \omega_2 A \cos \omega t$$
$$= \left(-\frac{3}{128} A^5 + \frac{1}{64} A^5 + A\omega_2 \right) \cos \omega t + \left(\frac{3}{32} A^3 - \frac{3}{64} A^5 \right) \cos 3\omega t$$
$$+ \left(-\frac{3}{128} A^5 - \frac{1}{64} A^5 \right) \cos 5\omega t \tag{E_{13}}$$

To avoid secular terms, the coefficient of $\cos \omega t$ in (E_{13}) must be zero. This gives $\omega_2 = \frac{1}{128} A^4$ $\qquad (E_{14})$

With this, and using $A = 2$, Eq. (E_{13}) reduces to

$$\ddot{x}_2 + \omega^2 x_2 = -\frac{3}{4} \cos 3\omega t - \frac{5}{4} \cos 5\omega t \qquad (E_{15})$$

Assuming $\quad x_2(t) = A_5 \cos 3\omega t + A_6 \cos 5\omega t \qquad (E_{16})$

we find, from Eq. (E_{15}),

$$A_5 = \frac{3}{32} \cdot \frac{1}{\omega^2}, \qquad A_6 = \frac{5}{96} \cdot \frac{1}{\omega^2} \qquad (E_{17})$$

$$\therefore\ x_2(t) = \frac{3}{32\omega^2} \cos 3\omega t + \frac{5}{96\omega^2} \cos 5\omega t \qquad (E_{18})$$

Thus the complete solution, Eqs. (E_2) and (E_3), become

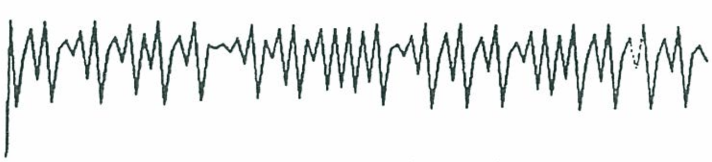
$$x(t) = 2 \cos \omega t + \frac{\alpha}{4\omega} \sin 3\omega t + \frac{3\alpha^2}{32\omega^2} \cos 3\omega t + \frac{5\alpha^2}{96\omega^2} \cos 5\omega t$$

and $\quad \omega^2 = 1 + \frac{\alpha^2}{8}$.

13.38 With k = 3.25, the sequence of values generated oscillate between two distinct values. With k = 3.5, the sequence of values generated oscillate between four distinct values. With k = 3.75, the sequence of values generated do not show any apparent periodicity (i.e., exhibit chaotic behavior). The resulting plots are shown below.

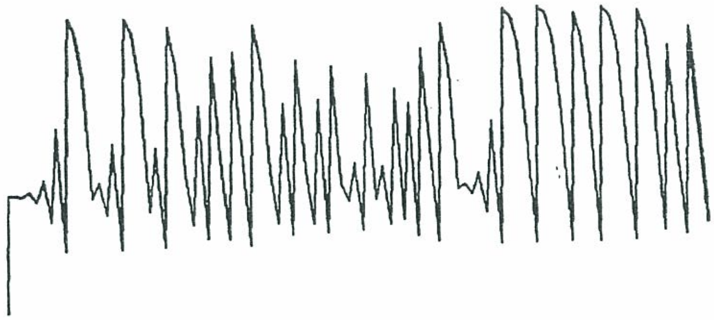
k = 3.25 (2 distinct values)

k = 3.50 (4 distinct values)

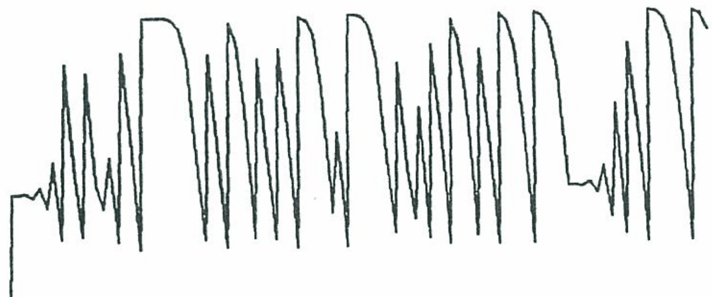


k = 3.75 (chaotic)

13.39 By taking two different initial values of x_1 which apparently do not differ much from one another, completely different sequences of values are generated. The sequences are shown plotted in the following figure.



$x_1 = 1.002$



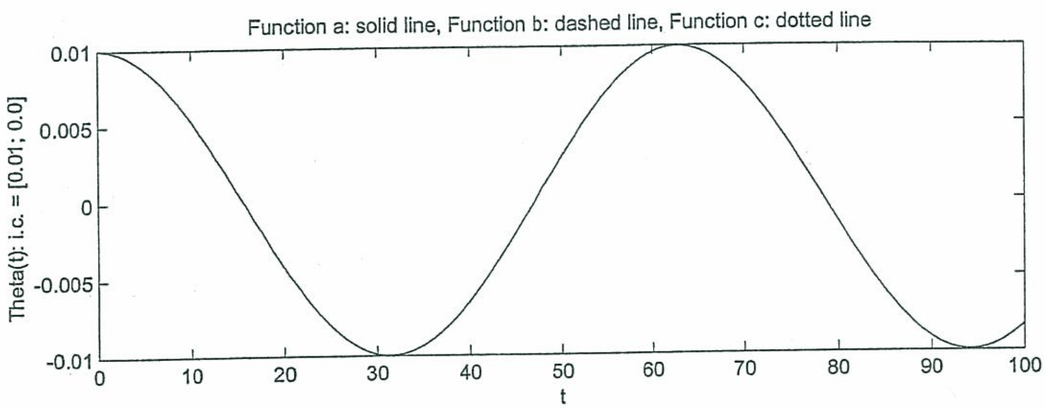
$x_1 = 1.003$

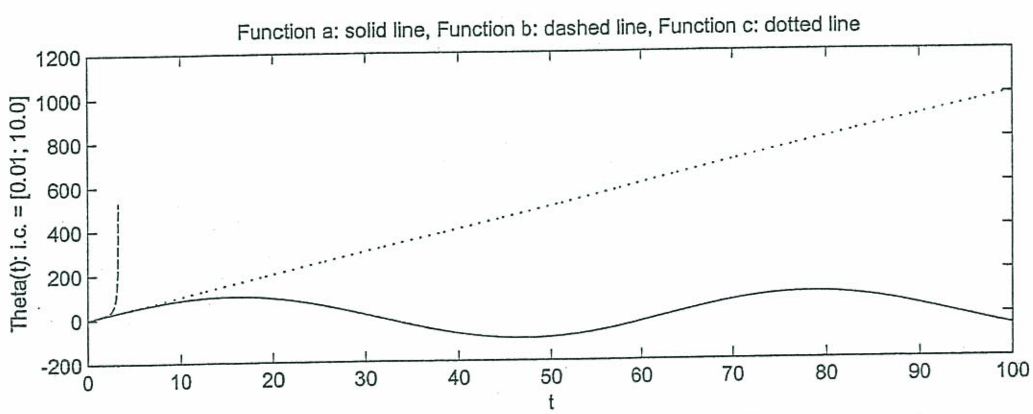
13.40

13.41

```
% Ex13_40_41.m
% This program will use the function dfunc1_a.m, dfunc1_b.m and dfun1_c.m
% they should be in the same folder, two different initial conditions
tspan = [0: 0.1: 100];
x0 = [0.01; 0.0];
x0_1 = [0.01; 10.0];
[t,xa] = ode23('dfunc1_a', tspan, x0);
[t,xb] = ode23('dfunc1_b', tspan, x0);
[t,xc] = ode23('dfunc1_c', tspan, x0);
[t1,xa1] = ode23('dfunc1_a', tspan, x0_1);
[t2,xb1] = ode23('dfunc1_b', tspan, x0_1);
[t3,xc1] = ode23('dfunc1_c', tspan, x0_1);
subplot(211);
plot(t,xa(:,1));
ylabel('Theta(t): i.c. = [0.01; 0.0]');
xlabel('t');
title...
('Function a: solid line, Function b: dashed line, Function c: dotted line');
hold on;
plot(t,xb(:,1),'--');
hold on;
plot(t,xc(:,1),':');
subplot(212);
hold off;
plot(t1,xa1(:,1));
ylabel('Theta(t): i.c. = [0.01; 10.0]');
xlabel('t');
title...
('Function a: solid line, Function b: dashed line, Function c: dotted line');
hold on;
plot(t2,xb1(:,1),'--');
hold on;
plot(t3,xc1(:,1),':');
```

```
% dfunc1_a.m
function f = dfunc1_a(t,x);
w0 = 0.1;
f = zeros(2,1);
f(1) = x(2);
f(2) = -w0^2 * x(1);

% dfunc1_b.m
function f = dfunc1_b(t,x);
w0 = 0.1;
f = zeros(2,1);
f(1) = x(2);
f(2) = w0^2 * ((x(1))^3) / 6.0 - w0^2 * x(1);

% dfunc1_c.m
function f = dfunc1_c(t,x);
w0 = 0.1;
f = zeros(2,1);
f(1) = x(2);
f(2) = -w0^2 * sin(x(1));
```

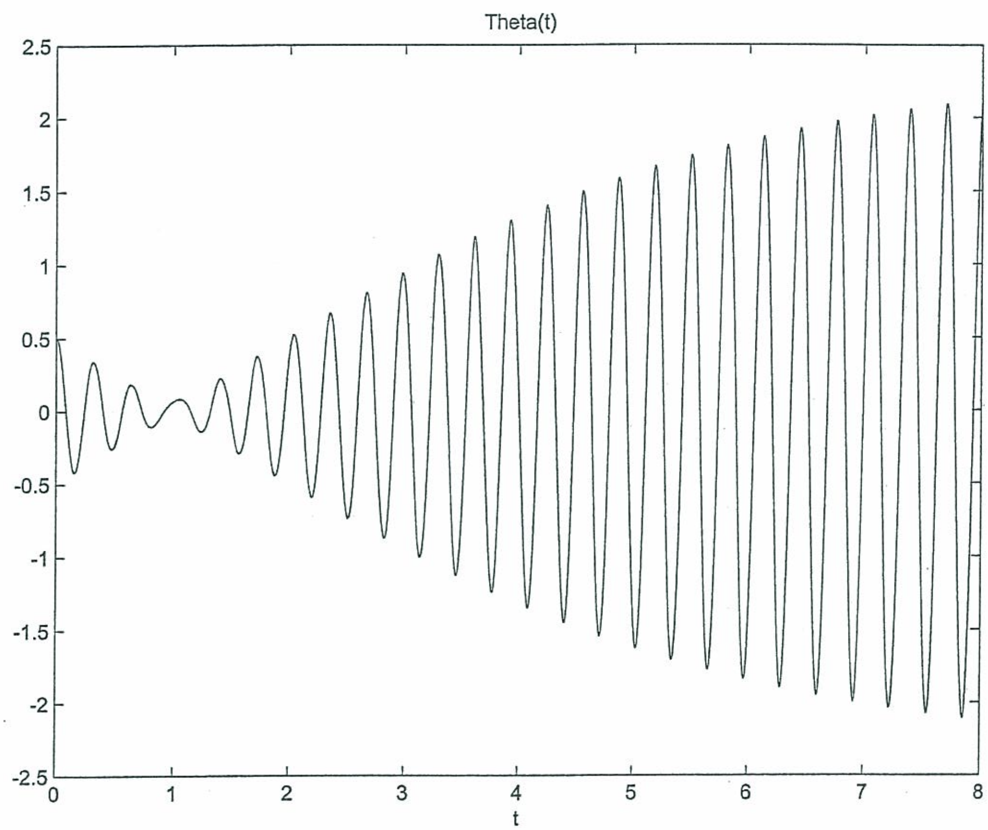




13.42

```
% Ex13_42.m
% This program will use the function dfunc13_42.m,
% they should be in the same folder
tspan = [0: 0.001: 8];
x0 = [0.5; 1.0];
[t,x] = ode23('dfunc13_42', tspan, x0);
plot(t,x(:,1));
title('Theta(t)');
xlabel('t');
```

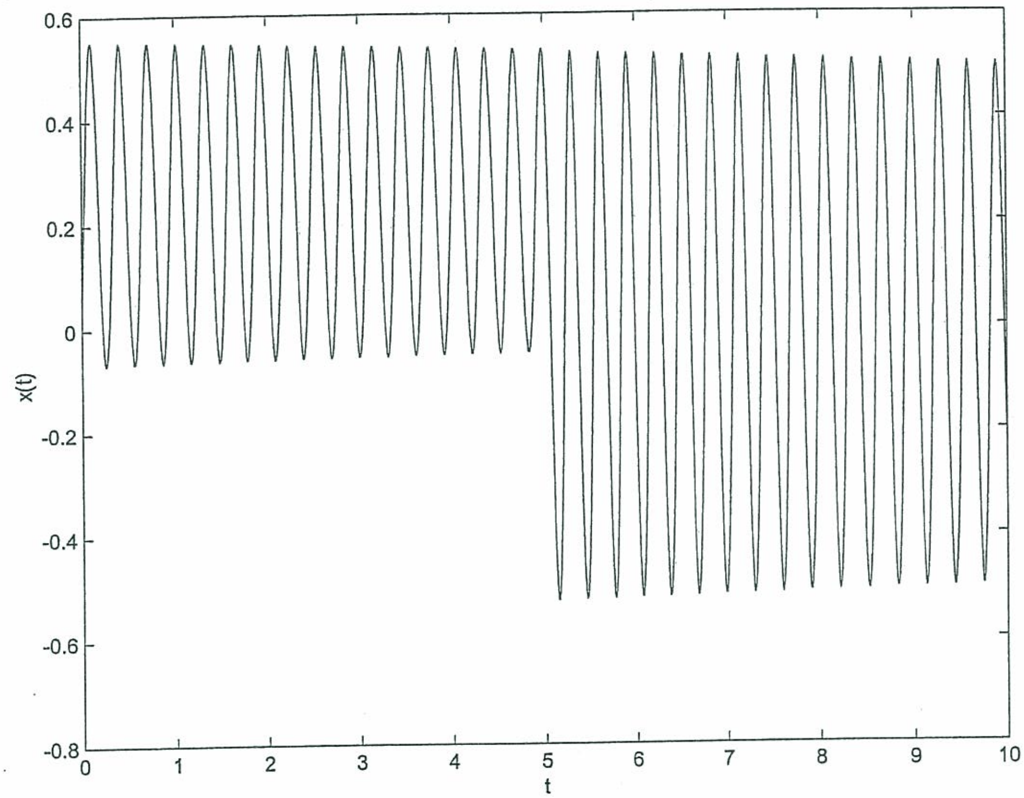
```
% dfunc13_42.m
function f = dfunc13_42(t,x);
f0 = 200;
m = 10;
c = 0.1;
k = 4000;
w = 20;
f = zeros(2,1);
f(1) = x(2);
f(2) = f0*sin(w*t)/m - c* x(2)^2 * sign(x(2))/m - k*x(1)/m;
```



13.43

```
% Ex13_43.m
% This program will use the function dfunc13_43.m ,
% they should be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.05; 5.0];
[t,x] = ode23('dfunc13_43', tspan, x0);
plot(t,x(:,1));
ylabel('x(t)');
xlabel('t');
```

```
% dfunc13_43.m
function f = dfunc13_43(t,x)
f0 = 1000;
m = 10;
k1 = 4000;
k2 = 1000;
FF = f0* ( stepfun(t, 0.0)-stepfun(t, 5.0) );
f = zeros(2,1);
f(1) = x(2);
f(2) = FF/m - k1*x(1)/m - k2*x(1)^3/m;
```

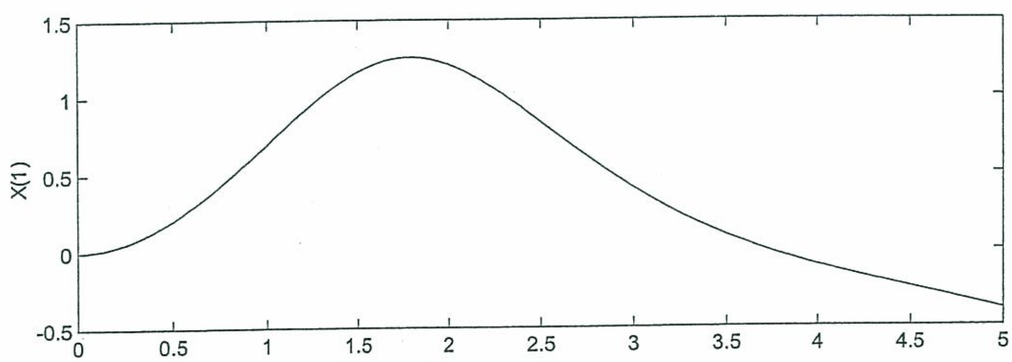


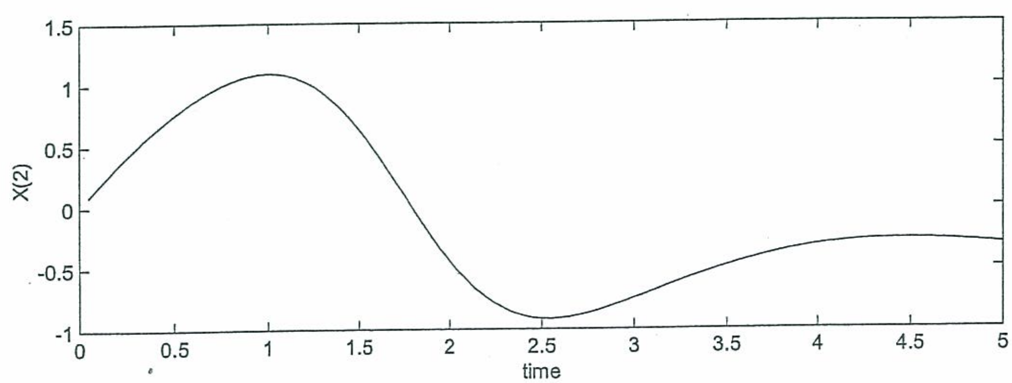
13.44

```
Results of Ex13_44
*****************
» program18
Solution of nonlinear vibration problem
by fourth order Runge-kutta method

Data:
ym = 1.000000e+000
yc = 5.000000e-001
yk = 1.00000000e+000
yks = 1.20000000e+000

Results
```

i	time(i)	x(i,1)	x(i,2)
1	5.000000e-002	2.230824e-003	8.884132e-002
2	1.000000e-001	8.843311e-003	1.752342e-001
3	1.500000e-001	1.971028e-002	2.589902e-001
4	2.000000e-001	3.469540e-002	3.399312e-001
5	2.500000e-001	5.365366e-002	4.178885e-001
6	3.000000e-001	7.643182e-002	4.927007e-001

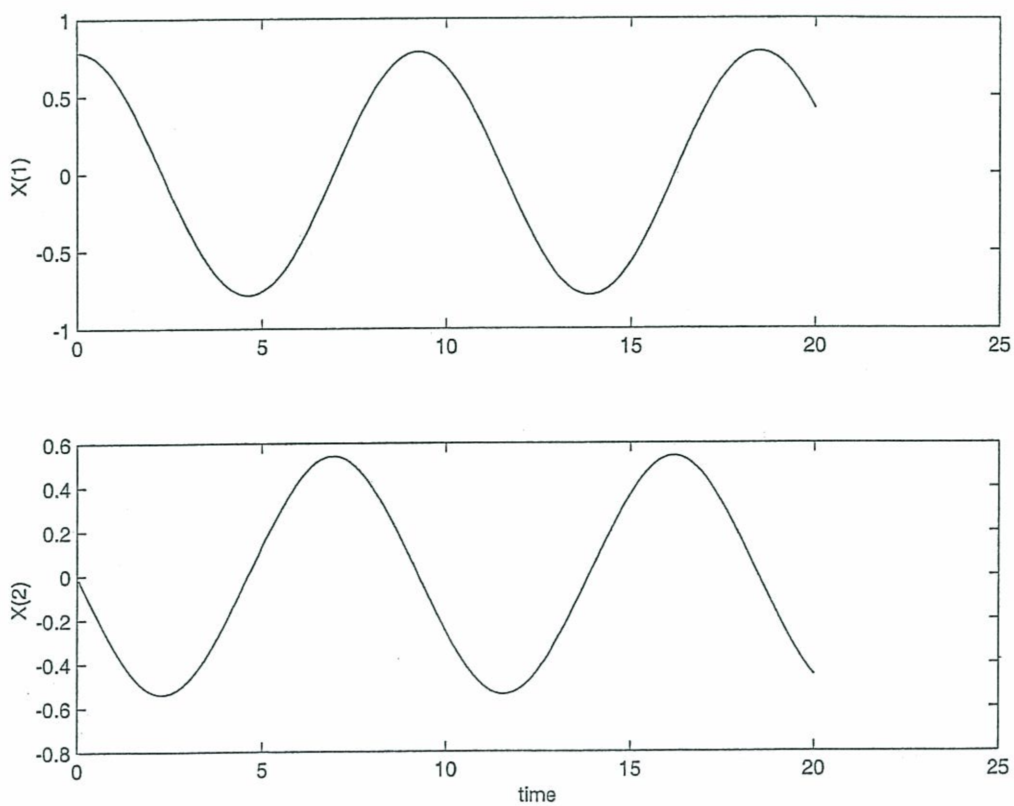
7	3.500000e-001	1.028687e-001	5.642099e-001
8	4.000000e-001	1.327951e-001	6.322582e-001
9	4.500000e-001	1.660341e-001	6.966810e-001
10	5.000000e-001	2.023999e-001	7.573016e-001
11	5.500000e-001	2.416976e-001	8.139235e-001
12	6.000000e-001	2.837219e-001	8.663231e-001
13	6.500000e-001	3.282553e-001	9.142430e-001
14	7.000000e-001	3.750665e-001	9.573844e-001
15	7.500000e-001	4.239083e-001	9.954025e-001
⋮			
81	4.050000e+000	-1.175452e-001	-3.112257e-001
82	4.100000e+000	-1.328833e-001	-3.025009e-001
83	4.150000e+000	-1.478152e-001	-2.949702e-001
84	4.200000e+000	-1.623999e-001	-2.886102e-001
85	4.250000e+000	-1.766953e-001	-2.833937e-001
86	4.300000e+000	-1.907578e-001	-2.792902e-001
87	4.350000e+000	-2.046423e-001	-2.762657e-001
88	4.400000e+000	-2.184017e-001	-2.742829e-001
89	4.450000e+000	-2.320873e-001	-2.733010e-001
90	4.500000e+000	-2.457478e-001	-2.732763e-001
91	4.550000e+000	-2.594301e-001	-2.741616e-001
92	4.600000e+000	-2.731783e-001	-2.759064e-001
93	4.650000e+000	-2.870341e-001	-2.784572e-001
94	4.700000e+000	-3.010365e-001	-2.817569e-001
95	4.750000e+000	-3.152213e-001	-2.857454e-001
96	4.800000e+000	-3.296215e-001	-2.903593e-001
97	4.850000e+000	-3.442666e-001	-2.955318e-001
98	4.900000e+000	-3.591828e-001	-3.011930e-001
99	4.950000e+000	-3.743928e-001	-3.072695e-001
100	5.000000e+000	-3.899154e-001	-3.136849e-001





13.45

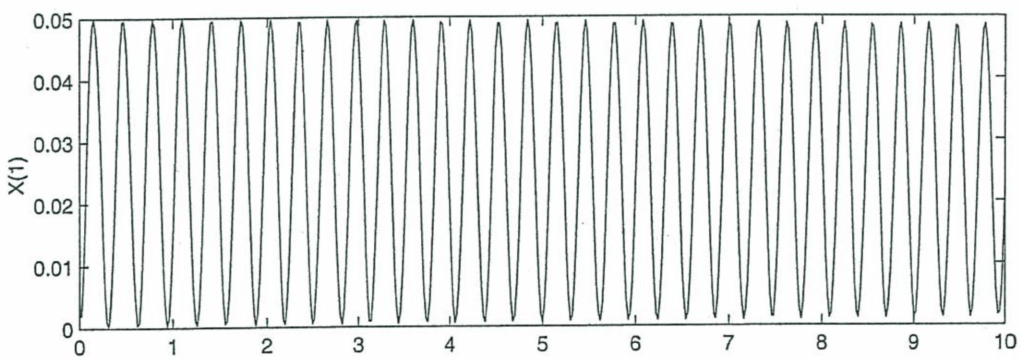
```
% Results of Ex13_45
***************************
>> program18
Solution of nonlinear vibration problem
by fourth order Runge-kutta method

Data:
ym = 1.000000e+000
yc = 0.000000e+000
yk = 5.00000000e-001
yks = -8.33333333e-002

Results

   i        time(i)          x(i,1)          x(i,2)

   1    5.000000e-002     7.849578e-001    -1.761378e-002
   6    3.000000e-001     7.695847e-001    -1.051476e-001
  11    5.500000e-001     7.325772e-001    -1.903594e-001
  16    8.000000e-001     6.747684e-001    -2.712057e-001
  21    1.050000e+000     5.975204e-001    -3.454950e-001
  26    1.300000e+000     5.027601e-001    -4.109125e-001
  31    1.550000e+000     3.929984e-001    -4.651246e-001
  36    1.800000e+000     2.713140e-001    -5.059555e-001
  41    2.050000e+000     1.412872e-001    -5.316073e-001
  46    2.300000e+000     6.877705e-003    -5.408763e-001
  51    2.550000e+000    -1.277459e-001    -5.333127e-001
   .
   .
   .
 371    1.855000e+001     7.849876e-001    -1.700658e-002
 376    1.880000e+001     7.697655e-001    -1.045496e-001
 381    1.905000e+001     7.329049e-001    -1.897843e-001
 386    1.930000e+001     6.752355e-001    -2.706680e-001
 391    1.955000e+001     5.981157e-001    -3.450103e-001
 396    1.980000e+001     5.034682e-001    -4.104971e-001
```

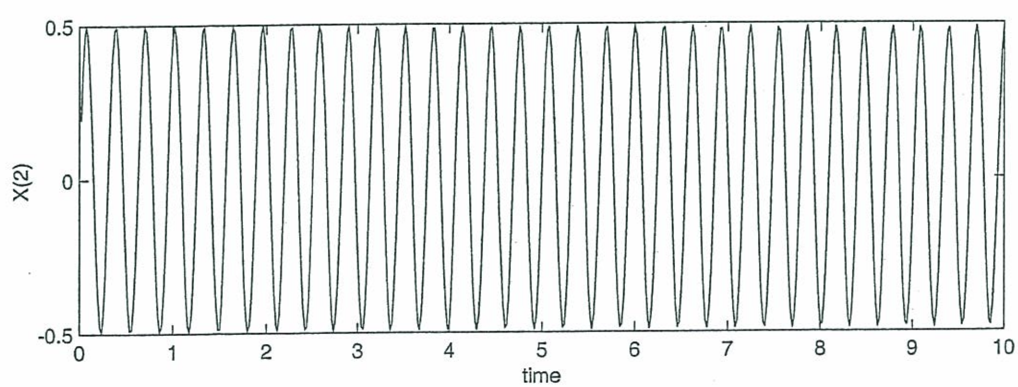


13.46

```
Results of EX13_46
*******************
>> program18
Solution of nonlinear vibration problem
by fourth order Runge-kutta method

Data:
yc = 1.000000e+001
yk = 8.00000000e+005
yks = 6.00000000e+003

Results
```

i	time(i)	x(i,1)	x(i,2)
1	2.000000e-002	1.973319e-003	1.946639e-001
6	1.200000e-001	4.341183e-002	3.375669e-001
11	2.200000e-001	3.266651e-002	-4.754773e-001
16	3.200000e-001	2.145722e-004	5.888314e-002
21	4.200000e-001	3.800441e-002	4.259926e-001
26	5.200000e-001	3.888710e-002	-4.145664e-001
31	6.200000e-001	3.953964e-004	-7.910512e-002
36	7.200000e-001	3.170805e-002	4.804600e-001
41	8.200000e-001	4.394467e-002	-3.234886e-001
.			
.			
.			
461	9.220000e+000	3.397870e-002	-4.512661e-001
466	9.320000e+000	1.308378e-003	4.378072e-002
471	9.420000e+000	3.776651e-002	4.106881e-001
476	9.520000e+000	3.692116e-002	-4.211170e-001
481	9.620000e+000	1.278773e-003	-2.297464e-002
486	9.720000e+000	3.492314e-002	4.419094e-001
491	9.820000e+000	3.953430e-002	-3.843744e-001
496	9.920000e+000	1.673648e-003	-8.686608e-002





```
%==========================================================================
%
% Program18.m
% Main program for solving a nonlinear vibration problem using the
% subroutine RK4
%
%==========================================================================
% Run "Program18" in MATLAB command window. Progrm18.m, fun.m, and
% rk4.m should be in the same folder,and set the Matalb path to this folder
% following 8 lines contain problem-dependent data
xx=[0 0];
n=2;
nstep=500;
dt=0.02;
yc=10;
yk=8e5;
yks=6000;
%end of problem-dependent data
t=0;
fprintf('Solution of nonlinear vibration problem\n');
fprintf('by fourth order Runge-kutta method\n\n');
fprintf('Data:\n');
fprintf('yc = %8.6e\n',yc);
fprintf('yk = %10.8e\n',yk);
fprintf('yks = %10.8e\n\n',yks);
fprintf('Results\n\n');
fprintf('   i         time(i)             x(i,1)           x(i,2)\n\n');
for i=1:nstep
   [xx,f,t]=rk4(t,dt,n,xx);
   time(i)=t;
   for j=1:n
      x(i,j)=xx(j);
   end
end
for i=1:5:nstep
   fprintf('%3.0f    %8.6e       %8.6e     %8.6e\n',i,time(i),x(i,1),x(i,2))
end
subplot(211);
plot(time,x(1:nstep,1));
ylabel('X(1)');
subplot(212);
plot(time,x(1:nstep,2));
xlabel('time');
ylabel('X(2)');
%==========================================================================
%
% Function rk4.m
%
%==========================================================================
function [xx,f,t]=rk4(t,dt,n,xx)
[xi]=fun(xx,n,t);
for i=1:n
   uu(i)=xx(i)+.5*dt*xi(i);
end
tn=t+0.5*dt;
[xj]=fun(uu,n,tn);
```

```
for i=1:n
   uu(i)=xx(i)+.5*dt*xj(i);
end
[xk]=fun(uu,n,tn);
for i=1:n
   uu(i)=xx(i)+dt*xk(i);
end
tn=t+dt;
[xl]=fun(uu,n,tn);
for i=1:n
   f(i)=xl(i);
   xx(i)=xx(i)+(xi(i)+2*xj(i)+2*xk(i)+xl(i))*dt/6;
end
t=t+dt;
%=====================================================================
%
% Function fun.m
%
%=====================================================================
function [f]=fun(x,n,t)
ym=(2000 - 10*t)*( stepfun(t,0) - stepfun(t,100) );
yc=10;
yk=8e5;
yks=6000;
f(1)=x(2);
f(2)=10*2000*( stepfun(t,0) - stepfun(t,100) )/ym...
    - (yc/ym)*x(2) - (20/ym)* x(2)^2 - (yk/ym)*x(1) - (yks/ym)*(x(1)^3);
```

13.47 For $\ddot{x} + a^2 F(x) = 0$,
$$\tau = \frac{2\sqrt{2}}{a} \int_0^{x_0} \frac{d\xi}{\left\{ \int_\xi^{x_0} F(\eta) \cdot d\eta \right\}^{1/2}} \qquad (E.1)$$

Here $\ddot{\theta} + \frac{g}{l}\left(\theta - \frac{\theta^3}{6}\right) = 0$; $\frac{g}{l} = 0.5 \Rightarrow a = \sqrt{0.5}$; $F(\theta) = \theta - \frac{\theta^3}{6}$;
$\theta_0 = 0.7854$ rad.

Simpson's rule is used for integration; note that one integral is embedded inside a second integral in (E.1). According to Simpson's rule,
$$\int_a^b f(x) \cdot dx = \frac{h}{3}\left\{ f(a) + 2 \sum_{i=1}^{\left(\frac{N}{2}-1\right)} f_{2i+1} + 4 \sum_{i=1}^{\left(\frac{N}{2}\right)} f_{2i} + f(b) \right\}$$

where $h = \frac{b-a}{N}$, N = even number of equal intervals and
$f_i = f\left(x = x_i = a + \{i-1\}\left\{\frac{b-a}{N}\right\}\right)$.

The program listing and the result are given below.

```
C ======================================================================
C
C PROBLEM 13.47
C
C ======================================================================
      AA=0.0
      BB=0.7854
      NN=40
      HH=(BB-AA)/FLOAT(NN)
      CALL FFINT (AA,FA)
      BB1=BB-(BB-AA)/100.0
      CALL FFINT (BB1,FB)
      CC=BB-HH
      CALL FFINT (CC,FC)
      TAU=HH*(FA+FB+4.0*FC)/3.0
      KK=NN/2
      KKM1=KK-1
      DO 10 JJ=1,KKM1
      XX=AA+HH*FLOAT(2*JJ-1)
      CALL FFINT (XX,FX)
      XXHH=XX+HH
      CALL FFINT (XXHH,FXH)
      TAU=TAU+HH*(4.0*FX+2.0*FXH)/3.0
 10   CONTINUE
      TAU=4.0*TAU
      PRINT 20, TAU
 20   FORMAT (2X,E15.8)
      STOP
      END
```

```
C ======================================================================
C SUBROUTINE FFINT
C ======================================================================
      SUBROUTINE FFINT (X,Y)
      A=X
      B=0.7854
      N=40
      CALL SIM (A,B,N,FINT)
      Y=1.0/SQRT(FINT)
      RETURN
      END
C ======================================================================
C SUBROUTINE SIM
C ======================================================================
      SUBROUTINE SIM (A,B,N,FINT)
      H=(B-A)/FLOAT(N)
      FINT=H*(F(A)+F(B)+4.0*F(B-H))/3.0
      K=N/2
      KM1=K-1
      DO 10 J=1,KM1
      X=A+H*FLOAT(2*J-1)
      FINT=FINT+H*(4.0*F(X)+2.0*F(X+H))/3.0
 10   CONTINUE
      RETURN
      END
C ======================================================================
C FUNCTION F
C ======================================================================
      FUNCTION F(X)
      F=X-(X**3)/6.0
      RETURN
      END
```

RESULT: $\tau = 8.7627392$

This can be compared with the value of τ of a linear pendulum which is equal to $\frac{2\pi}{\omega} = 2\pi / \sqrt{\frac{g}{l}} = 2\pi / \sqrt{0.5} = 8.8857867.$

13.50 van der Pol's equation: $\ddot{x} - \alpha(1-x^2)\dot{x} + x = 0$ $\quad (E_1)$

i.e.,

$$\begin{Bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{Bmatrix} = \begin{Bmatrix} y \\ \alpha(1-x^2)y - x \end{Bmatrix} \quad (E_2)$$

(a) Equation for the phase plane trajectory is

$$\frac{dy}{dx} = \frac{\alpha(1-x^2)y - x}{y} \quad (E_3)$$

The trajectories corresponding to $\alpha = 0.1$, $\alpha = 1.0$ and $\alpha = 10$ are shown in Figs. F_1, F_2 and F_3, respectively. It can be seen, from these figures, that irrespective of the initial point, the trajectories converge to closed curves. These closed curves denote periodic motions with constant amplitude. In addition, as the value of α increases, the closed curve deviates more and more from a circle.

(b) If (E_1) is solved numerically, the variation of the solution $x(t)$ can be plotted against the t-axis. The result will be as shown in Figs. F_4, F_5 and F_6 for $\alpha = 0.1$, 1.0 and 10, respectively. It can be seen from these figures, that as the value of α is increased, the variation of x with respect to t differs more and more from harmonic motion.

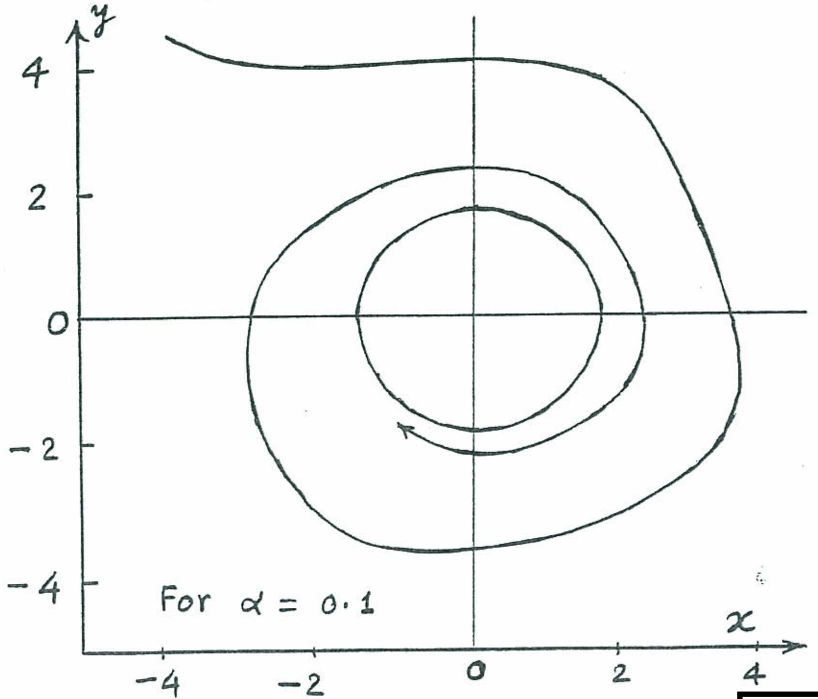


Fig. F_1

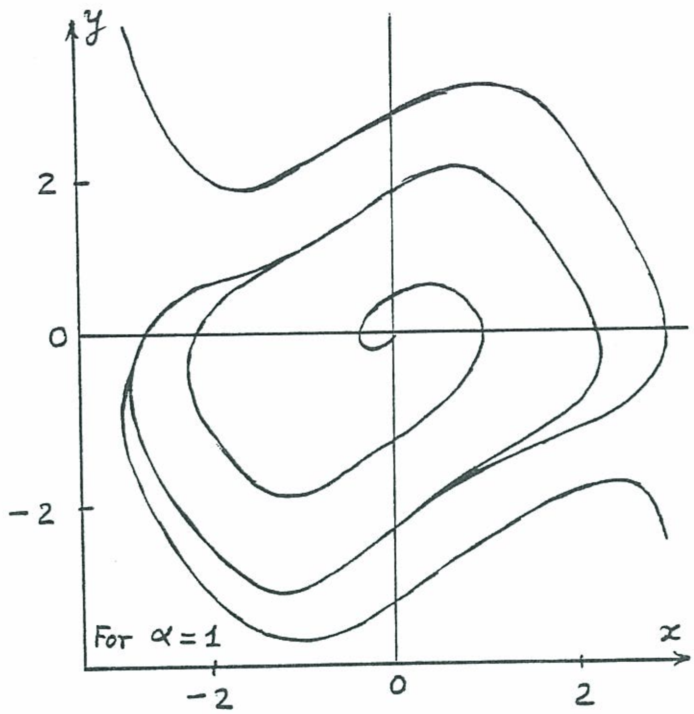


Fig. F2

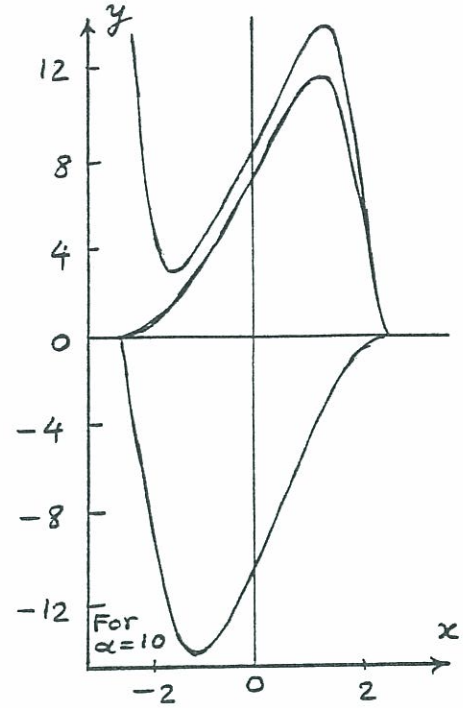


Fig. F3

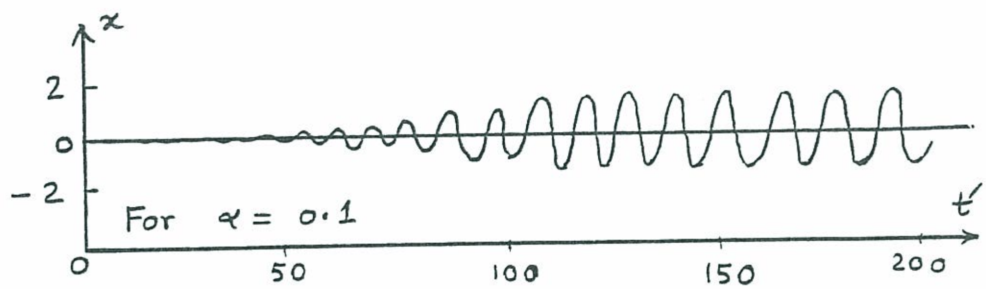


Fig. F4

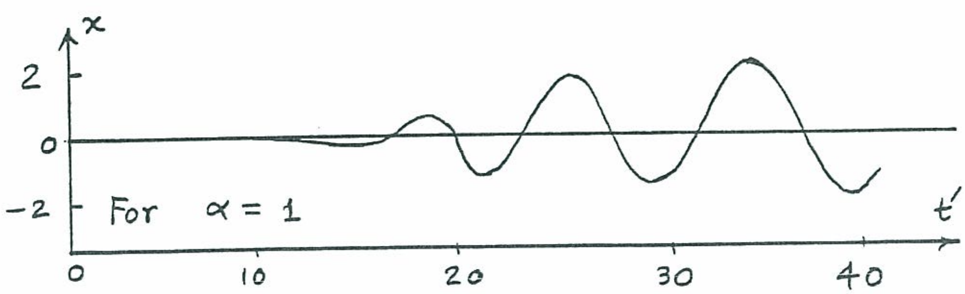


Fig. F5

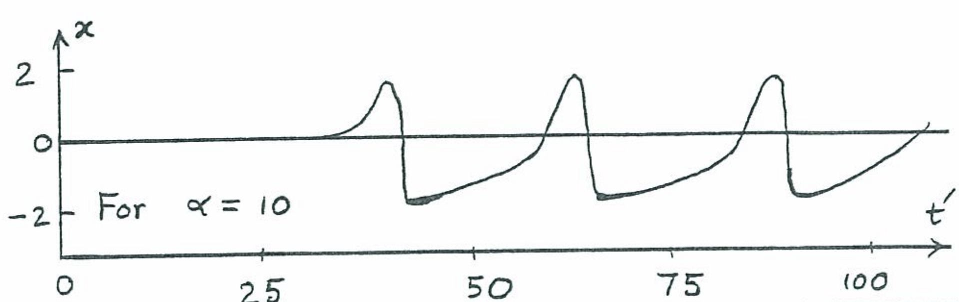


Fig. F6

13.51 Equations of motion:

$$m\ddot{x} + k_{11}(x - l_1\theta) + k_{12}(x - l_1\theta)^3 + k_{21}(x + l_2\theta) + k_{22}(x + l_2\theta)^3 = 0$$

$$J_0\ddot{\theta} - k_{11}(x - l_1\theta)l_1 - k_{12}(x - l_1\theta)^3 l_1 + k_{21}(x + l_2\theta)l_2 + k_{22}(x + l_2\theta)^3 l_2 = 0$$

Using $x_1 = x$, $x_2 = \theta$, $x_3 = \frac{dx}{dt} = \dot{x}_1$ and $x_4 = \frac{d\theta}{dt} = \dot{x}_2$, the equations of motion can be expressed as

$$\dot{\vec{X}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix} = \vec{F} \equiv \begin{Bmatrix} x_3 \\ x_4 \\ -\frac{1}{m}\left[k_{11}(x_1 - l_1x_2) + k_{12}(x_1 - l_1x_2)^3 + k_{21}(x_1 + l_2x_2) + k_{22}(x_1 + l_2x_2)^3\right] \\ \frac{1}{J_0}\left[k_{11}(x_1 - l_1x_2)l_1 + k_{12}(x_1 - l_1x_2)^3 l_1 - k_{21}(x_1 + l_2x_2)l_2 - k_{22}(x_1 + l_2x_2)^3 l_2\right] \end{Bmatrix}$$

Data:

$m = 1000$ kg, $J_0 = 2500$ kg-m^2, $l_1 = 1$ m, $l_2 = 1.5$ m,

$k_{11} = 40000$, $k_{12} = 10000$, $k_{21} = 50000$, $k_{22} = 5000$.

Runge-Kutta method can be used to solve the equations of motion using the initial conditions $x(0) = \dot{x}(0) = \theta(0) = \dot{\theta}(0) = 0$.