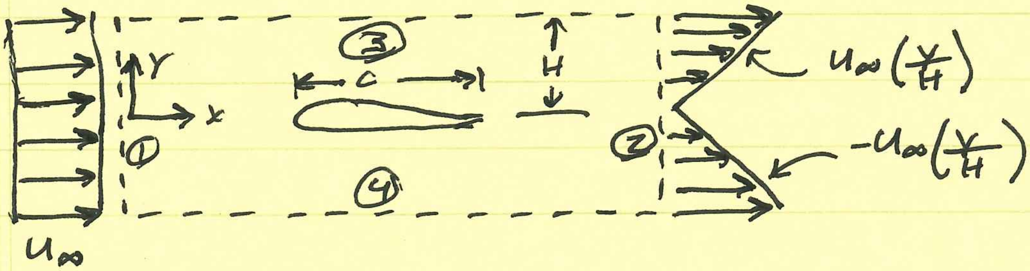
Part of Problem 1:



Assume: incompressible, steady, 2-D.

Q: What is the total Q across ③ and ④

Q ← volume flow rate

Use the conservation of mass equation (continuity):

$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall}}_{\text{steady}}^{0} + \int_{cs} \rho(\vec{V}\cdot\hat{n})\,dA = 0$$

$$\rho \int_{cs} (\vec{V}\cdot\hat{n})\,dA = 0 \quad \text{(incompressible really const. } \rho\text{)}$$

$$\int_{cs} \vec{V}\cdot\hat{n}\,dA = 0$$

$$\int_{①} \vec{V}\cdot\hat{n}\,dA + \int_{②} \vec{V}\cdot\hat{n}\,dA + \underbrace{\int_{③} \vec{V}\cdot\hat{n}\,dA + \int_{④} \vec{V}\cdot\hat{n}\,dA}_{Q_{34}} = 0$$

$$Q_{34} = -\int_{①} \vec{V}\cdot\hat{n}\,dA - \int_{②} \vec{V}\cdot\hat{n}\,dA$$

$$= -(-u_\infty)\int_{-H}^{H} dA - u_\infty \int_0^H \frac{y}{H}\,dA - u_\infty \int_{-H}^{0} \frac{-y}{H}\,dA$$

$dA = w\,dy$

↑ unit width → $w = 1$

$$Q_{34} = 2u_\infty H - \frac{u_\infty}{H}\frac{y^2}{2}\Big|_0^H + \frac{u_\infty}{H}\frac{y^2}{2}\Big|_{-H}^{0}$$

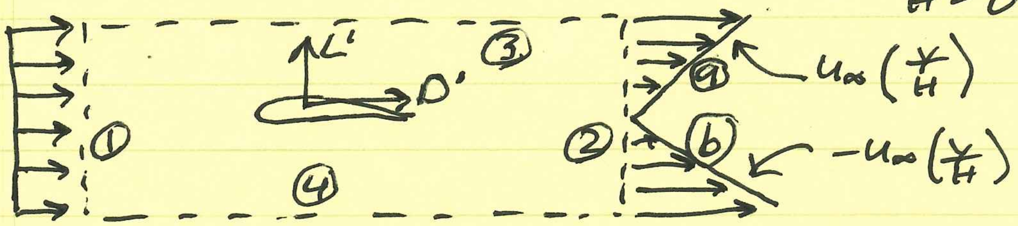
$$Q_{34} = 2u_\infty H - \frac{u_\infty H}{2} - \frac{u_\infty H}{2}$$

$$\boxed{Q_{34} = u_\infty H} \text{ (per unit width)}$$

Problem 1:

Same setup as Problem 5

$H = 0.025c$



Assume: Incompressible, steady, 2-D, pressure is constant on CS.
Inviscid

Q: What is the drag coefficient?

$$c_d = \frac{D'}{\frac{1}{2}\rho_\infty u_\infty^2 c}$$

<u>Momentum equation:</u>

$$\underbrace{\cancelto{0}{\frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall}}_{\text{steady}} + \int_{CS} \vec{V}\rho(\vec{V}\cdot\hat{n})\,dA = \underbrace{\cancelto{0}{\int_{CS} -p\hat{n}\,dA}}_{\text{const. } P \text{ on CS}} + \underbrace{\cancelto{0}{\vec{F}_{visc}}}_{\text{Inviscid}} + \vec{F}_{ext}$$

$\vec{F}_{ext} = -D'\hat{i} - L'\hat{j}$ ← ~~Flows~~ The fluid acts <u>on</u> the <u>airfoil</u> to produce lift and drag.
The airfoil acts on the fluid with equal and opposite force by Newton's third law.
i.e. The force acting on the fluid is $= -D'\hat{i} - L'\hat{j}$

On ③ and ④ we assume: $\vec{V} = u_\infty\hat{i} + v(x)\hat{j}$

↑ small flow perturbations to the $\hat{i}$ component of velocity have little effect on the boundary.

X-momentum equation:

$$\int_{①} u_\infty \rho_\infty (-u_\infty)\, dy + \int_{②a} \left(u_\infty \frac{y}{H}\right) \rho_\infty \left(u_\infty \frac{y}{H}\right) dy \quad (0 \le y \le H)$$

$$+ \int_{②b} \left(-u_\infty \frac{y}{H}\right) \rho_\infty \left(-u_\infty \frac{y}{H}\right) dy \quad (-H \le y \le 0)$$

$$+ \int_{③+④} u_\infty \rho_\infty (\vec{V}\cdot\hat{n})\, dy$$

$$\int_{-H}^{H} ① : \quad -\rho_\infty u_\infty^2 (2H)$$

$$\int_{0}^{H} ②a : \quad \frac{\rho_\infty u_\infty^2}{H^2} \frac{y^3}{3}\Big|_0^H = \frac{\rho_\infty u_\infty^2 H}{3}$$

$$\int_{-H}^{0} ②b : \quad \frac{\rho_\infty u_\infty^2}{H^2} \frac{y^3}{3}\Big|_{-H}^{0} = \frac{\rho_\infty u_\infty^2 H}{3}$$

$$= u_\infty \rho_\infty \underbrace{\int_{③+4} (\vec{V}\cdot\hat{n})\, dy}_{Q_{34} \text{ from Problem 5!}}$$

$$= u_\infty \rho_\infty (u_\infty H) = \rho_\infty u_\infty^2 H$$

Put these together with the RHS

$$-D' = -\rho_\infty u_\infty^2 (2H) + \frac{\rho_\infty u_\infty^2 H}{3} + \frac{\rho_\infty u_\infty^2 H}{3} + \rho_\infty u_\infty^2 H$$

$$-D' = -\frac{\rho_\infty H u_\infty^2}{3} \Rightarrow \frac{D'}{\frac{1}{2}\rho_\infty u_\infty^2 c} = \frac{2H}{3c} \qquad H = 0.025c$$

$$c_d = +\frac{2}{3}(0.025)$$

$$\boxed{c_d = 0.0167}$$

Problem 2 (3.7.1 Kuethe)

Derive: $\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P = \rho\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V}\right] = \rho\vec{g} - \vec{\nabla}P$

From: $\frac{d}{dt}\int_{\hat{R}} \rho\vec{V}\, d\hat{R} + \int_{\hat{S}} \rho\vec{V}(\vec{V}\cdot\hat{n})\, d\hat{S} = -\vec{F}_e - \int_{\hat{S}} P\hat{n}\, d\hat{S} + \int_{\hat{R}} \rho\vec{g}\, d\hat{R}$

Neglecting external forces $\rightarrow \vec{F}_e = 0$

Divergence theorem: $\int_{\hat{S}} \hat{n}\cdot\vec{G}\, d\hat{S} = \int_{\hat{R}} \vec{\nabla}\cdot\vec{G}\, d\hat{R}$

Momentum flux term: $\int_{\hat{S}} \rho\vec{V}(\vec{V}\cdot\hat{n})\, dS = \int_{\hat{S}} (\hat{n}\cdot\rho\vec{V})\vec{V}\, d\hat{S}$

Applying divergence $\rightarrow \int_{\hat{S}} (\hat{n}\cdot\rho\vec{V})\vec{V}\, d\hat{S} = \int_{\hat{R}} \vec{\nabla}\cdot(\rho\vec{V}\vec{V})\, d\hat{R}$

Expanding $\rightarrow \int_{\hat{R}} \vec{\nabla}\cdot(\rho\vec{V}\vec{V})\, d\hat{R} = \int_{\hat{R}} (\vec{\nabla}\cdot\rho\vec{V})\vec{V}\, d\hat{R} + \int_{\hat{R}} \rho(\vec{V}\cdot\vec{\nabla})\vec{V}\, d\hat{R}$

Pressure term: $\int_{\hat{S}} \hat{n}P\, d\hat{S} = \int_{\hat{R}} \vec{\nabla}P\, d\hat{R}$

easy to see using index notation.

Collecting terms into a single integral:

$$\int_{\hat{R}} \left[\frac{\partial}{\partial t}(\rho\vec{V}) + (\vec{\nabla}\cdot\rho\vec{V})\vec{V} + \rho(\vec{V}\cdot\vec{\nabla})\vec{V} + \vec{\nabla}P - \rho\vec{g}\right] d\hat{R} = 0$$

Integrand must be zero since $\hat{R}$ is an arbitrary volume:

$$\frac{\partial}{\partial t}(\rho\vec{V}) + (\vec{\nabla}\cdot\rho\vec{V})\vec{V} + \rho(\vec{V}\cdot\vec{\nabla})\vec{V} + \vec{\nabla}P - \rho\vec{g} = 0$$

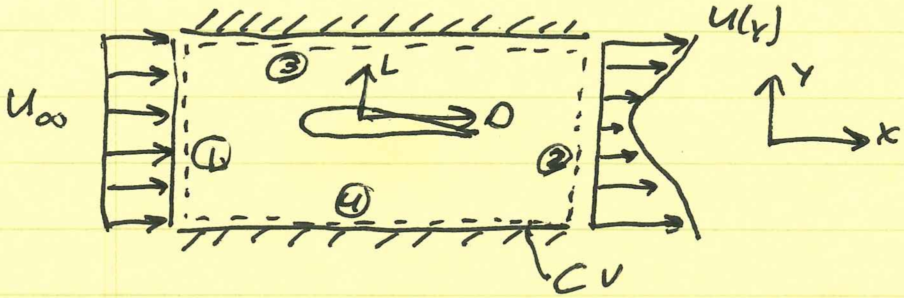
Expanding terms: $\rho\frac{\partial\vec{V}}{\partial t} + \underbrace{\vec{V}\frac{\partial\rho}{\partial t} + (\vec{\nabla}\cdot\rho\vec{V})\vec{V}}_{\left[\frac{\partial\rho}{\partial t} + (\vec{\nabla}\cdot\rho\vec{V})\right]\vec{V}} + \rho(\vec{V}\cdot\vec{\nabla})\vec{V} + \vec{\nabla}P - \rho\vec{g} = 0$

continuity equation $= 0$

Now: $\rho\frac{\partial\vec{V}}{\partial t} + \rho(\vec{V}\cdot\vec{\nabla})\vec{V} = \rho\vec{g} - \vec{\nabla}P$

and $\rightarrow$ $\boxed{\rho\frac{D\vec{V}}{Dt} = \rho\vec{g} - \vec{\nabla}P}$

Problem 3:



To solve this problem we need to consider a control volume, CV. We can write down the momentum equation:

$$\frac{\partial}{\partial t}\int_{CV} \vec{v}\,\rho\, d\forall + \int_{CS} \vec{V}\rho(\vec{V}\cdot\hat{n})\,dA = \sum\left(\begin{array}{c}\text{Forces acting}\\ \underline{\text{on}}\text{ the }\underline{\text{fluid}}\end{array}\right)$$

* Remember the forces on the right hand side (RHS) are forces acting <u>on</u> the <u>fluid</u>. These forces are equal and opposite to the forces acting <u>on</u> the <u>airfoil</u>!

The forces are:

$$\sum(\text{Forces}) = \int_{CS} -P\hat{n}\,dA + \vec{F}_{viscous} + \vec{F}_{external}$$

$$\vec{F}_{external} = -D\hat{\imath} - L\hat{\jmath}$$ ← equal and opposite to L and D. Acting on the fluid.

Since lift is in the $\hat{\jmath}$ direction we only need to consider the $\hat{\jmath}$ component of the momentum equation.

$$\underbrace{\cancel{\frac{\partial}{\partial t}\int_{CV}(v_y\hat{\jmath})\,\rho\, d\forall}}_{\text{Steady}}^{\,0} + \int_{CS}(v_y\hat{\jmath})\,\rho\,(\vec{V}\cdot\hat{n})\,dA = \int_{CS} -P\hat{n}_y\hat{\jmath}\,dA + \cancel{F_{y,visc}\hat{\jmath}}^{\,0} \;\; -L\hat{\jmath}$$

Only significant viscous forces on top and bottom wall in $\hat{\imath}$ direction

$$\int_{①} (\cancel{V_y \hat{j}})^{0} \rho (\vec{V}\cdot\hat{n})\, dA + \int_{②} (\cancel{V_y \hat{j}})^{0} \rho (\vec{V}\cdot\hat{n})\, dA$$

zero on ①

② far enough downsteam that V_y on ② is zero

$$+ \int_{③} (\cancel{V_y \hat{j}}) \rho (\cancel{\vec{V}\cdot\hat{n}})^{0}\, dA + \int_{④} (\cancel{V_y \hat{j}})^{0} \rho (\cancel{\vec{V}\cdot\hat{n}})^{0}\, dA = 0$$

zero on ③ | no mass flux through a wall

Same as ③

$\Rightarrow$

$$0 = \int_{CS} -P \hat{j}\, dA - L\hat{j}$$

pressure distribution on lower wall

$$L = \int_{④} P_L\, dA - \int_{③} P_u\, dA$$

pressure on upper wall

$$\boxed{L = \int_A (P_L - P_u)\, dA}$$

* Assumptions:

1.) Inviscid on <u>entrance</u> and <u>exit</u>

2.) Steady flow

3.) Neglecting gravity

4.) CV is long enough for $V_y = 0$ on ③

Problem 4

$\psi = u_\infty r \sin\theta + \frac{\sigma}{2\pi}\theta \Rightarrow V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = u_\infty \cos\theta + \frac{\sigma}{2\pi r}$

$\frac{\sigma}{2\pi u_\infty} \equiv 1$

$V_\theta = -\frac{\partial\psi}{\partial r} = -u_\infty \sin\theta$

Stagnation occurs when $V_r = V_\theta = 0$

Stag pt: $(r,\theta) = \left(\frac{\sigma}{2\pi u_\infty}, \pi\right)$

Stagnation streamline occurs for the constant: $\psi = \frac{\sigma}{2}$

So, $\frac{\sigma}{2} = u_\infty r \sin\theta + \frac{\sigma}{2\pi}\theta$

$$\left(\frac{2\pi}{\sigma}\right)\left(\frac{\sigma}{2}\right) = u_\infty r \sin\theta\left(\frac{2\pi}{\sigma}\right) + \left(\frac{2\pi}{\sigma}\right)\frac{\sigma}{2\pi}\theta$$

$$\pi = \underbrace{\left(\frac{2\pi u_\infty}{\sigma}\right)}_{1} r\sin\theta + \theta$$

$$\pi = r\sin\theta + \theta$$

$$\boxed{r = \frac{\pi - \theta}{\sin\theta}}$$

give a range of θ and solve for r

not valid for $\theta = 0, \pi$

If $\theta = 0$: $\pi = r\sin(0) + (0) \rightarrow \boxed{r = \text{undefined!}}$

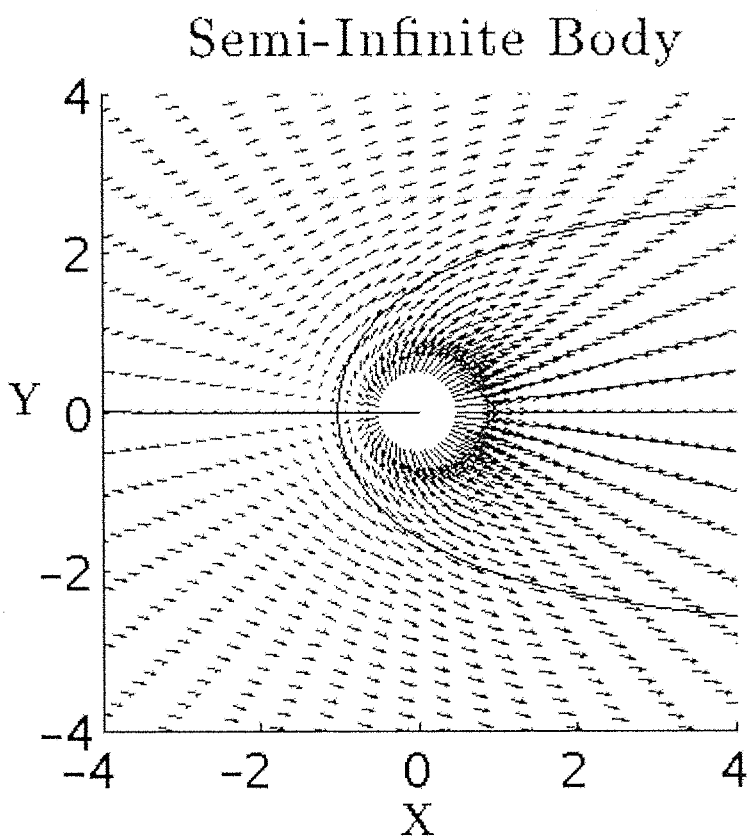
If $\theta = \pi$: $\pi = r\,\cancel{\sin(\pi)}^{0} + (\pi)$

$\pi = \pi \longrightarrow$ $\boxed{r \text{ can be any value from } (0 \le r \le \infty)}$

$\boxed{\text{Plot!}}$

```
clear
close all
clc

% Create r and theta vectors
r = linspace(1/2,10,50)';
theta = linspace(0,2*pi,51)';

% Create a grid of (r,theta) combinations
[R, Theta] = meshgrid(r,theta);

% Calculate V_r/Uinf and V_theta/Uinf
V_r_Uinf = cos(Theta) + 1./R;
V_theta_Uinf = -sin(Theta);

% Convert polar (r,theta) to Cartesian (x,y)
X = R.*cos(Theta);
Y = R.*sin(Theta);

% Find the matrix size of R and use for indices
[rows, cols] = size(R);

% Step through each (r,theta) combination
for i = 1:rows
    for j = 1:cols

        % Calculate the Cartesian velocity components using a
        % transformation matrix, Q:
        %
        % Q = [cos(theta) -sin(theta)
        %      sin(theta)  cos(theta)]
        %
        % then,
        %
        % [U; V] = Q*[V_r; V_theta]

        U(i,j) = V_r_Uinf(i,j)*cos(Theta(i,j)) - V_theta_Uinf(i,j)*sin(Theta(i,j));
        V(i,j) = V_r_Uinf(i,j)*sin(Theta(i,j)) + V_theta_Uinf(i,j)*cos(Theta(i,j));
    end
end

% Streamline calculations
theta_top = linspace(0.01,pi-0.01,1000)';
theta_bot = linspace(pi+0.01,2*pi - 0.01,1000)';
r_top = (pi - theta_top)./sin(theta_top);
r_bot = (pi - theta_bot)./sin(theta_bot);
r_pi = linspace(0,10,1000)';

% Convert to Cartesian coordinates
[x_top,y_top] = pol2cart(theta_top,r_top);
[x_bot,y_bot] = pol2cart(theta_bot,r_bot);
[x_pi,y_pi] = pol2cart(pi*ones(1000,1),r_pi);


% Plot the velocity field and streamlines
```

```
figure
set(gcf,'DefaultAxesfontsize',24,'DefaultAxesfontname','TimesNewRoman','DefaultAxesGridLineSty
le','-.')

hold on
    quiver(X,Y,U,V)
    plot(x_top,y_top,'-k','LineWidth',1.5)
    plot(x_bot,y_bot,'-k','LineWidth',1.5)
    plot(x_pi,y_pi,'-k','LineWidth',1.5)
hold off
axis equal
axis([-4 4 -4 4])

title('Semi-Infinite Body','Interpreter','Latex','FontName','TimesNewRoman','FontSize',28)
xlabel('X','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24)
ylabel('Y','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24,'Rotation',0)

print('-depsc', '-r600','Semi_Inf_Body.eps');
```



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Problem 5:

$$\psi = -\frac{\mu}{2\pi}\frac{\sin\theta}{r} \qquad V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \Rightarrow -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2}$$

$$V_\theta = -\frac{\partial\psi}{\partial r} \Rightarrow -\frac{\mu}{2\pi}\frac{\sin\theta}{r^2}$$

$$① \; V_r = \frac{\partial\phi}{\partial r} \qquad V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \; ②$$

$$① \; -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2} = \frac{\partial\phi}{\partial r} \Rightarrow \phi = \int -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2}\,dr + C$$

$$\phi = -\frac{\mu}{2\pi}\left(\frac{1}{-1}\right)\frac{\cos\theta}{r} + f(\theta) + C$$

$$\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r} + f(\theta) + C \quad ③$$

$$\cancel{② \; -\frac{\mu}{2\pi}\frac{\sin\theta}{r^2} = \frac{\partial\phi}{\partial r} \Rightarrow \psi =}$$

$$② \; -\frac{\mu}{2\pi}\frac{\sin\theta}{r^2} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \Rightarrow \phi = \int -\frac{\mu}{2\pi}\frac{\sin\theta}{r}\,d\theta + C$$

$$\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r} + g(r) + C \quad ④$$

Compare ③ to ④:

$f(\theta) = g(r) = 0$

C is an arbitrary constant that we can set to zero.

$$\boxed{\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r}}$$

Plot!

Equipotential lines $\rightarrow$ $\boxed{r = \frac{\mu}{2\pi C}\cos\theta}$

Contents


```
clear
close all
clc

% Create r and theta vectors
r = linspace(1/2,10,50)';
theta = linspace(0,2*pi,51)';

% Create a grid of (r,theta) combinations
[R, Theta] = meshgrid(r,theta);

% Calculate V_r and V_theta
V_r = -cos(Theta)./R.^2;
V_theta = -sin(Theta)./R.^2;
V_r_neg = cos(Theta)./R.^2;
V_theta_neg = sin(Theta)./R.^2;

% Convert polar (r,theta) to Cartesian (x,y)
X = R.*cos(Theta);
Y = R.*sin(Theta);

% Find the matrix size of R and use for indices
[rows, cols] = size(R);

% Step through each (r,theta) combination
for i = 1:rows
    for j = 1:cols

        % Calculate the Cartesian velocity components using a
        % transformation matrix, Q:
        %
        % Q = [cos(theta) -sin(theta)
        %      sin(theta)  cos(theta)]
        %
        % then,
        %
        % [U; V] = Q*[V_r; V_theta]

        U(i,j) = V_r(i,j)*cos(Theta(i,j)) - V_theta(i,j)*sin(Theta(i,j));
        V(i,j) = V_r(i,j)*sin(Theta(i,j)) + V_theta(i,j)*cos(Theta(i,j));
        U_neg(i,j) = V_r_neg(i,j)*cos(Theta(i,j)) - V_theta_neg(i,j)*sin(Theta(i,j));
        V_neg(i,j) = V_r_neg(i,j)*sin(Theta(i,j)) + V_theta_neg(i,j)*cos(Theta(i,j));
    end
end

% Equipotential line calculations
theta = linspace(0,2*pi,1000)';
C_1 = 1/2;
C_2 = 1;
C_3 = 2;
```

```
C_4 = -1/2;
C_5 = -1;
C_6 = -2;


r_1 = 1/C_1*cos(theta);
r_2 = 1/C_2*cos(theta);
r_3 = 1/C_3*cos(theta);
r_4 = 1/C_4*cos(theta);
r_5 = 1/C_5*cos(theta);
r_6 = 1/C_6*cos(theta);


% Convert to Cartesian coordinates
[x_1,y_1] = pol2cart(theta,r_1);
[x_2,y_2] = pol2cart(theta,r_2);
[x_3,y_3] = pol2cart(theta,r_3);
[x_4,y_4] = pol2cart(theta,r_4);
[x_5,y_5] = pol2cart(theta,r_5);
[x_6,y_6] = pol2cart(theta,r_6);
```

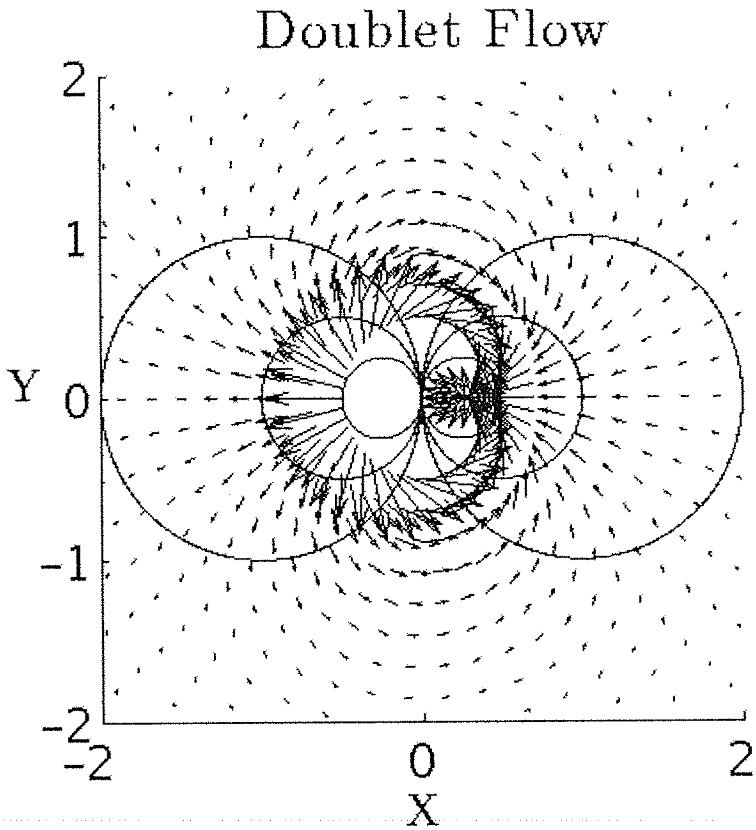
Plot the velocity field and equipotential lines: positive mu.

```
figure
set(gcf,'DefaultAxesfontsize',24,'DefaultAxesfontname','TimesNewRoman','DefaultAxesGridLineSty
le','-.')
num = 1;

hold on
    plot(x_1,y_1,'-k','LineWidth',num)
    plot(x_2,y_2,'-k','LineWidth',num)
    plot(x_3,y_3,'-k','LineWidth',num)
    plot(x_4,y_4,'-k','LineWidth',num)
    plot(x_5,y_5,'-k','LineWidth',num)
    plot(x_6,y_6,'-k','LineWidth',num)
    quiver(X,Y,U,V,'k')
hold off
axis equal
axis([-2 2 -2 2])

title('Doublet Flow','Interpreter','Latex','FontName','TimesNewRoman','FontSize',28)
xlabel('X','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24)
ylabel('Y','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24,'Rotation',0)

print('-depsc', '-r600','Doub_Flow_Pos.eps');
```



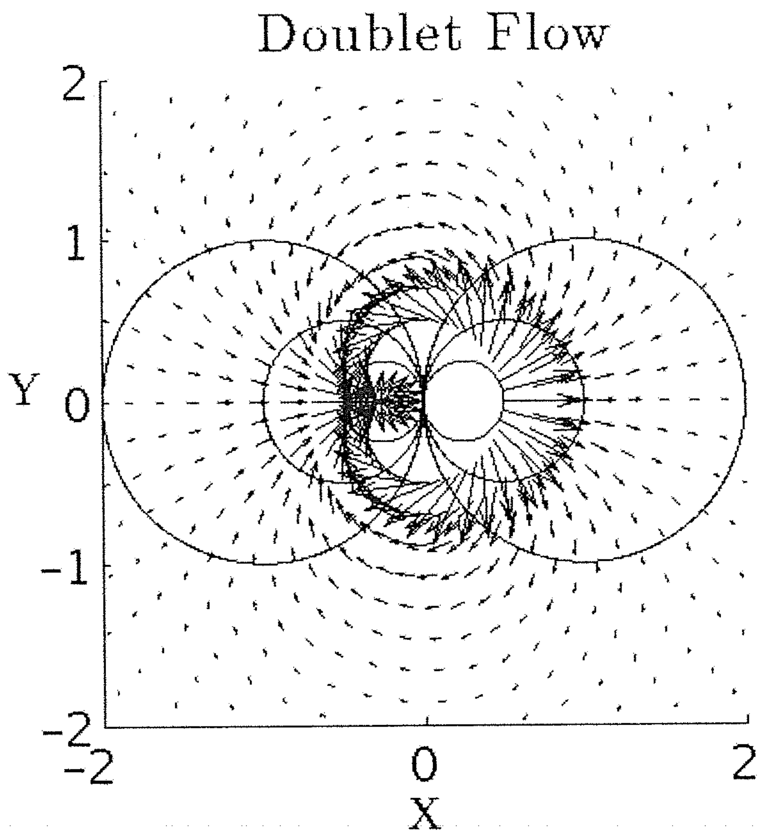
Plot the velocity field and equipotential lines: negative mu.

```
figure
set(gcf,'DefaultAxesfontsize',24,'DefaultAxesfontname','TimesNewRoman','DefaultAxesGridLineSty
le','-.')
num = 1;

hold on
    plot(x_1,y_1,'-k','LineWidth',num)
    plot(x_2,y_2,'-k','LineWidth',num)
    plot(x_3,y_3,'-k','LineWidth',num)
    plot(x_4,y_4,'-k','LineWidth',num)
    plot(x_5,y_5,'-k','LineWidth',num)
    plot(x_6,y_6,'-k','LineWidth',num)
    quiver(X,Y,U_neg,V_neg,'k')
hold off
axis equal
axis([-2 2 -2 2])

title('Doublet Flow','Interpreter','Latex','FontName','TimesNewRoman','FontSize',28)
xlabel('X','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24)
ylabel('Y','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24,'Rotation',0)

print('-depsc', '-r600','Doub_Flow_Neg.eps');
```



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