Engineering Vibrations & Systems

Module 8: Two Degree of Freedom Systems

ME 242
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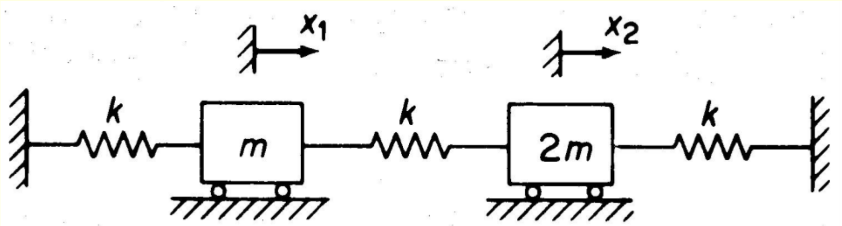
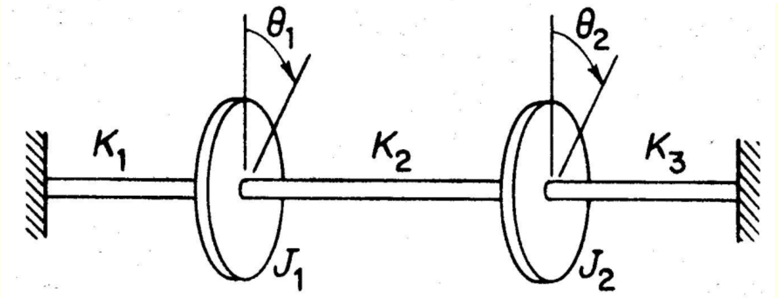
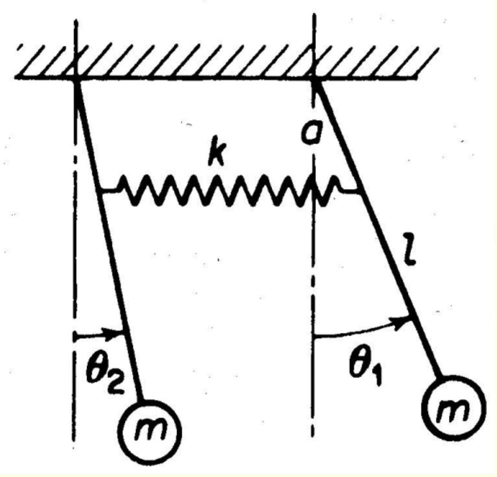
Module 3

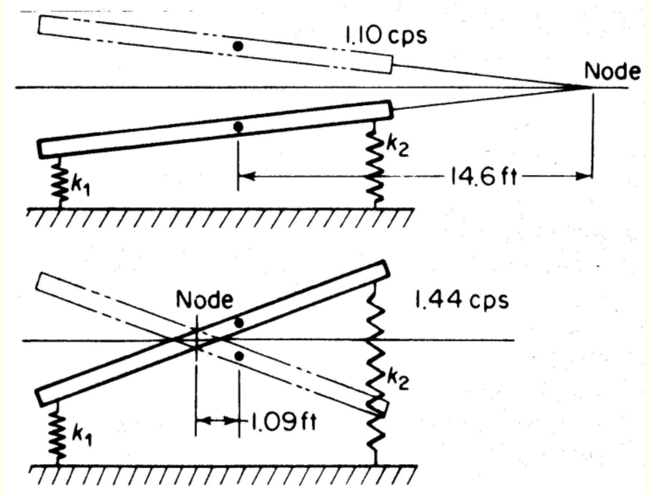
1. Two Degree of Freedom Systems

Examples of Two Degree of Freedom Systems









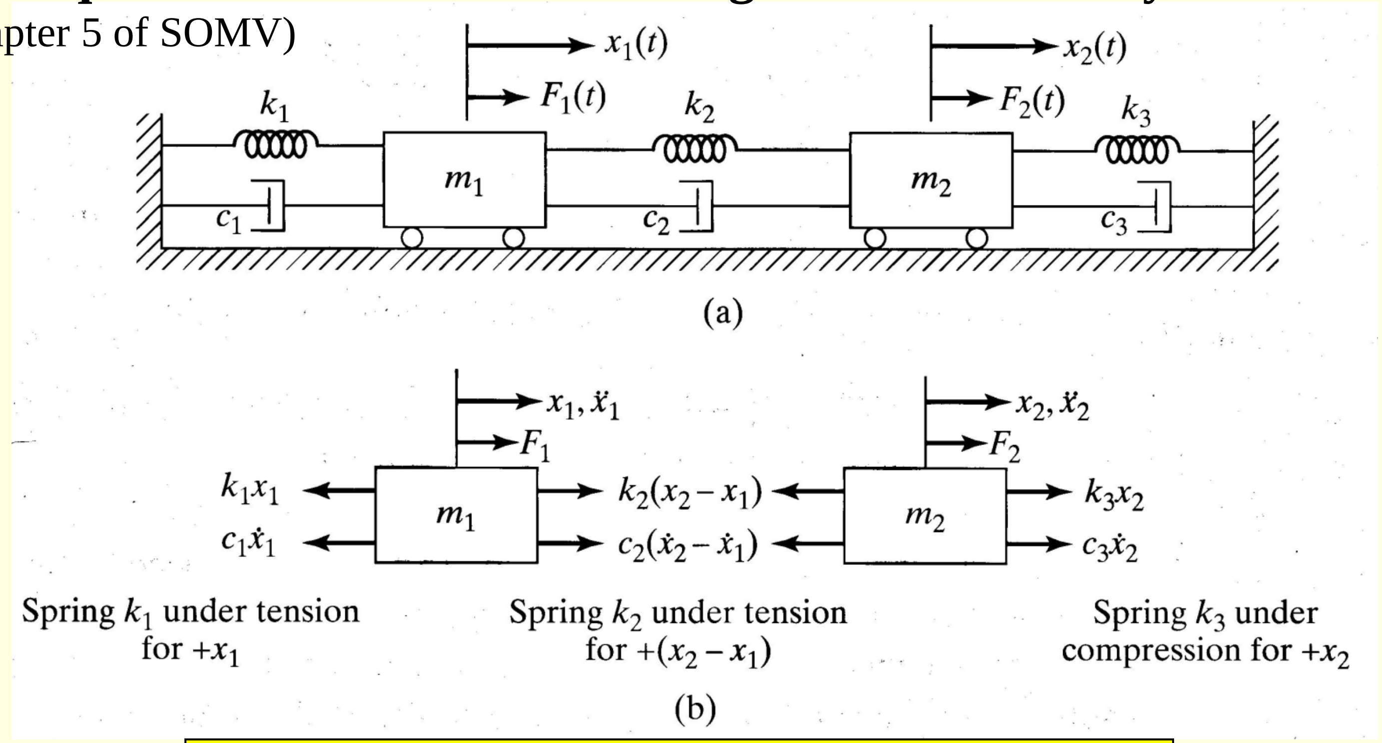
1. Each frequency corresponds to one mode of vibration
2. Number of degrees of freedom of system

   = (# of masses) * (# of possible motions per mass)

1. Two Degree of Freedom Systems

1.1 Equations of Motion -- Two Degree of Freedom Systems

(Chapter 5 of SOMV)



$$\left.\begin{aligned} m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1+(k_1+k_2)x_1-c_2\dot{x}_2-k_2x_2=F_1 \\ m_2\ddot{x}_2+(c_2+c_3)\dot{x}_2+(k_2+k_3)x_2-c_2\dot{x}_1-k_2x_1=F_2 \end{aligned}\right\} \qquad [1a]$$

1. Two Degree of Freedom Systems

1.1 Equations of Motion -- Two Degree of Freedom Systems

Let: $$\underline{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix};\ \underline{F}(t)=\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

Then $$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\ddot{\underline{x}}+\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2+c_3 \end{bmatrix}\dot{\underline{x}}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\underline{x}=\underline{F} \qquad [1b]$$

$$[m]\ddot{\underline{x}}+[c]\dot{\underline{x}}+[k]\underline{x}=\underline{F} \qquad [1c]$$

Observations:

1. $$\left.\begin{array}{l} [\mathrm{m}] \text{ inertia matrix} \\ [\mathrm{c}] \text{ damping matrix} \\ [\mathrm{k}] \text{ stiffness matrix} \end{array}\right\} \text{ for 2 Dof systems, size of matrices is 2x2}$$

2. Symmetric matrices: $[\mathrm{m}]=[\mathrm{m}]^T$

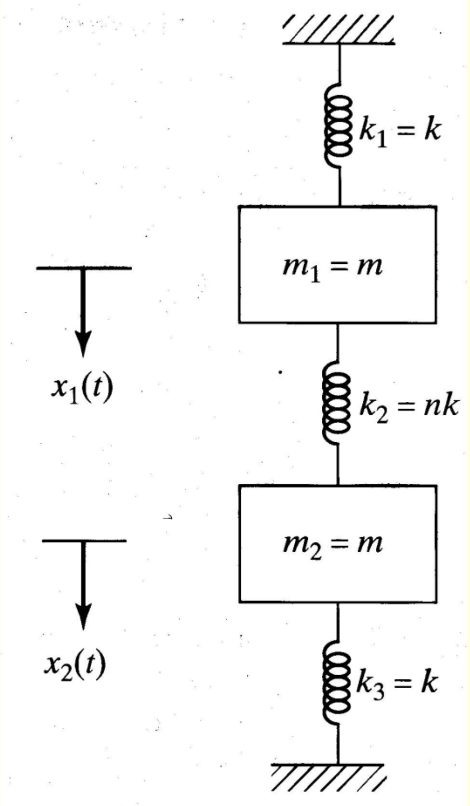
   $[c]=[c]^T$

   $[k]=[k]^T$

3. Equations are coupled: whatever motion of x_1 is, affects x_2 which then affects x_1 and so on. Equation becomes uncoupled when $c_2=k_2=0$ but this results in two separate systems, not physically connected.

1. Two Degree of Freedom Systems

1.2 Undamped System Response (Free Vibration)



$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 + k_2 x_1 + (k_2 + k_3)x_2 = 0 \qquad [2a]$$

Let:
$$x_1 = X_1 \cos(\omega t + \phi)$$
$$x_2 = X_2 \cos(\omega t + \phi) \qquad [2b]$$

[2b] into [2a]:
$$\left\{[-m_1\omega^2 + (k_1 + k_2)]X_1 - k_2 X_2\right\}\cos(\omega t + \phi) = 0$$
$$\left\{-k_2 X_1 + [-m_2\omega^2 + (k_2 + k_3)]X_2\right\}\cos(\omega t + \phi) = 0 \qquad [2c]$$

$$\therefore [-m_1\omega^2 + (k_1 + k_2)]X_1 - k_2 X_2 = 0$$
$$-k_2 X_1 + [-m_2\omega^2 + (k_2 + k_3)]X_2 = 0 \qquad [2d]$$

Simultaneous equations in X_1 and X_2.

Trivial solution: $X_1 = X_2 = 0$. → NO VIBRATIONS!

1. Two Degree of Freedom Systems

1.2 Undamped System Response (Continue)

For nontrivial solution:
$$\det\begin{bmatrix} -m_1\omega^2+(k_1+k_2) & -k_2 \\ -k_2 & -m_2\omega^2+(k_2+k_3) \end{bmatrix}=0 \qquad [2e]$$

i.e., $$\left(m_1m_2\right)\omega^4-[(k_1+k_2)m_2+(k_2+k_3)m_1]\omega^2+\left[(k_1+k_2)(k_2+k_3)-k_2{}^2\right]=0 \qquad [2f]$$

characteristic equation

Natural frequencies ω_1, ω_2 :

$$\omega_1{}^2,\omega_2{}^2=\frac{1}{2}\left[\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right]\mp\frac{1}{2}\sqrt{\left[\frac{(k_1+k_2)m_2+(k_2+k_3)m_1}{m_1m_2}\right]^2-4\left[\frac{(k_1+k_2)(k_2+k_3)-k_2{}^2}{m_1m_2}\right]} \qquad [2g]$$

For each frequency (eigenvalue), we get a set of X_1 and X_2 (eigenvectors):

e.g., for frequency ω_1 → $X_1{}^{(1)}, X_2{}^{(1)}$
for frequency ω_2 → $X_1{}^{(2)}, X_2{}^{(2)}$

Look at the ratios:
$$\left.\begin{aligned} r_1 &= X_2{}^{(1)}/X_1{}^{(1)}=\frac{-m_1\omega^2+(k_1+k_2)}{k_2}=\frac{k_2}{-m_2\omega_1{}^2+(k_2+k_3)} \\ r_2 &= X_2{}^{(2)}/X_1{}^{(2)}=\frac{-m_1\omega_2{}^2+(k_1+k_2)}{k_2}=\frac{k_2}{-m_2\omega_2{}^2+(k_2+k_3)} \end{aligned}\right\} \qquad [2h]$$

1. Two Degree of Freedom Systems

1.2 Undamped System Response (Continue)

Normal modes corresponding to $\omega_1{}^2$, $\omega_2{}^2$:

$$\underline{X}^{(1)} = \begin{bmatrix} X_1{}^{(1)} \\ X_2{}^{(1)} \end{bmatrix} = \begin{Bmatrix} X_1{}^{(1)} \\ r_1 X_2{}^{(1)} \end{Bmatrix}$$

and

$$\underline{X}^{(2)} = \begin{bmatrix} X_1{}^{(2)} \\ X_2{}^{(2)} \end{bmatrix} = \begin{Bmatrix} X_1{}^{(2)} \\ r_2 X_2{}^{(2)} \end{Bmatrix} \qquad [2i]$$

$\underline{X}^{(1)}, \underline{X}^{(2)}$ are **MODAL VECTORS** denoting the normal modes of the system vibration. (eigenvectors)

Solution:

$$\underline{x}^{(1)} = \begin{bmatrix} x_1{}^{(1)}(t) \\ x_2{}^{(1)}(t) \end{bmatrix} = \begin{Bmatrix} X_1{}^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1{}^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} \quad \text{First Mode}$$

$$\underline{x}^{(2)} = \begin{bmatrix} x_1{}^{(2)}(t) \\ x_2{}^{(2)}(t) \end{bmatrix} = \begin{Bmatrix} X_1{}^{(2)} \cos(\omega_2 t + \phi_2) \\ r_2 X_1{}^{(2)} \cos(\omega_2 t + \phi_2) \end{Bmatrix} \quad \text{Second Mode}$$

(Section 13.4 of SOMV)

where: $X_1{}^{(1)}, X_1{}^{(2)}, \phi_1, \phi_2$ are determined by i.c.

1. Two Degree of Freedom Systems

1.2 Undamped System Response (Continue)

i.c. :

$$x_1(t=0)=x_1(0)\;\dot{x}_1(t=0)=\dot{x}_1(0)$$

$$x_2(t=0)=x_2(0)\;\dot{x}_2(t=0)=\dot{x}_2(0)$$

Then the arbitrary constants are:

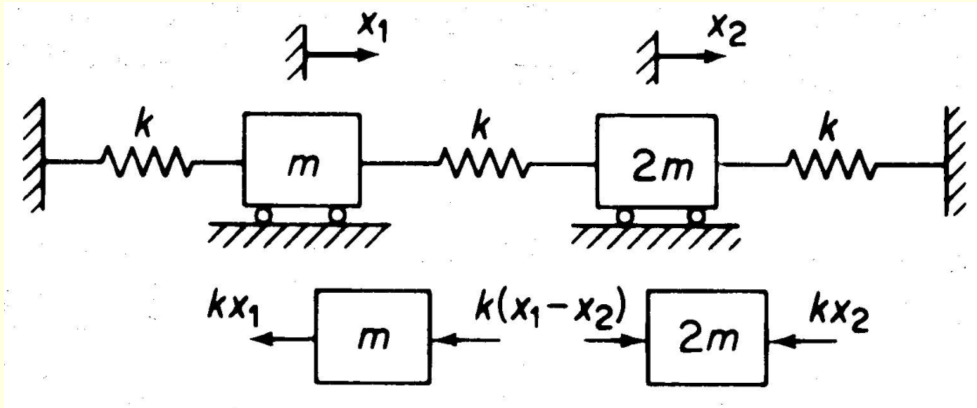
$$\left.\begin{aligned} X_1^{(1)} &= \frac{1}{(r_2-r_1)}\left\{\left[r_2x_1(0)-x_2(0)\right]^2+\frac{\left[-r_2\dot{x}_1(0)+\dot{x}_2(0)\right]^2}{\omega_1^{\,2}}\right\}^{1/2} \\ X_1^{(2)} &= \frac{1}{(r_2-r_1)}\left\{\left[-r_2x_1(0)+x_2(0)\right]^2+\frac{\left[r_1\dot{x}_1(0)-\dot{x}_2(0)\right]^2}{\omega_2^{\,2}}\right\}^{1/2} \end{aligned}\right\} \quad [3a]$$

$$\left.\begin{aligned} \phi_1 &= \tan^{-1}\left[\frac{-r_2\dot{x}_1(0)+\dot{x}_2(0)}{\omega_1\left[r_2x_1(0)-x_2(0)\right]}\right] \\ \phi_2 &= \tan^{-1}\left[\frac{r_1\dot{x}_1(0)-\dot{x}_2(0)}{\omega_2\left[-r_1x_1(0)+x_2(0)\right]}\right] \end{aligned}\right\} \quad [3b]$$

1. Two Degree of Freedom Systems

1.3 Examples (Chapters 5 & 13 of SOMV)

Example 1a (Translational System)



Step #1:

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$

$$2m\ddot{x}_2 = k(x_1 - x_2) - kx_2$$

Step #2:

Let:

1. Two Degree of Freedom Systems

Example 1a (Continue)

Step #5:

Amplitude ratios: $$r_{1,2} = \frac{A_2}{A_1} = \frac{k}{2k - 2\omega_i^2 m} = \frac{2k - \omega_i^2 m}{k} \qquad i = 1,2$$

For:

$$\omega_1^2 = 0.634\frac{k}{m}, r_1 = \left(\frac{A_2}{A_1}\right)^{(1)} = \frac{2k - \omega_1^2 m}{k} = \frac{2 - 0.0634}{1} = 1.368$$

For:

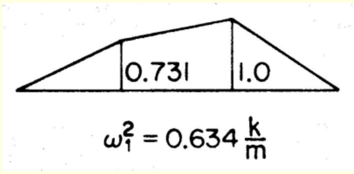
$$\omega_2^2 = 2.366\frac{k}{m}, r_2 = \left(\frac{A_2}{A_1}\right)^{(2)} = \frac{2k - \omega_2^2 m}{k} = \frac{2 - 2.366}{1} = -0.3663$$

Step #6:

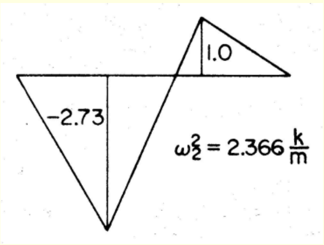
The two modes are:

$$\underline{X}^{(1)} = \begin{bmatrix} 1.000 \\ 1.368 \end{bmatrix} \qquad \underline{X}^{(2)} = \begin{bmatrix} 1.000 \\ -0.366 \end{bmatrix}$$

masses are moving in phase:



masses are moving out of phase:



First Mode:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^{(1)} = A_1 \begin{bmatrix} 1.000 \\ 1.368 \end{bmatrix} \sin(\omega_1 t + \phi_1)$$

Second Mode:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}^{(2)} = A_1 \begin{bmatrix} 1.000 \\ -0.366 \end{bmatrix} \sin(\omega_2 t + \phi_2)$$

1. Two Degree of Freedom Systems

Example 1b

Given i.c.: $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2.0 \\ 4.0 \end{bmatrix}$ and $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ determine free vibration of system.

Step #1:

Solution is the sum of the two modes:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1.000 \\ 1.368 \end{bmatrix} \sin(\omega_1 t + \phi_1) + c_2 \begin{bmatrix} 1.000 \\ -0.368 \end{bmatrix} \cos(\omega_2 t + \phi_2)$$

At t = 0:

$$\left.\begin{aligned} \begin{bmatrix} 2.0 \\ 4.0 \end{bmatrix} &= c_1 \begin{bmatrix} 1.000 \\ 1.368 \end{bmatrix} \sin\phi_1 + c_2 \begin{bmatrix} 1.000 \\ -0.366 \end{bmatrix} \sin\phi_2 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= \omega_1 c_1 \begin{bmatrix} 1.000 \\ 1.368 \end{bmatrix} \cos\phi_1 + \omega_2 c_2 \begin{bmatrix} 1.000 \\ -0.366 \end{bmatrix} \cos\phi_2 \end{aligned}\right\}$$

4 Eqns, 4 Unknowns: c_1, ϕ_1, c_2, ϕ_2

Step #2:

(Eqn. 2 / 0.366) + Eqn.1: (to eliminate $\sin\phi_2$): $12.929 = c_1(4.738)\sin\phi_1$

- (Eqn. 2 / 1.368) + Eqn.1: (to eliminate $\sin\phi_1$): $-0.928 = c_2(1.2675)\sin\phi_2$

(Eqn. 4 / 0.366) + Eqn.3: (to eliminate $\cos\phi_2$): $0 = \omega_1 c_1(4.738)\cos\phi_1 \rightarrow \cos\phi_1 = 0 \rightarrow \phi_1 = 90^o$

-(Eqn. 4 / 1.368) + Eqn.3: (to eliminate $\cos\phi_1$): $0 = \omega_2 c_2(1.2675)\cos\phi_2 \rightarrow \cos\phi_2 = 0 \rightarrow \phi_2 = 90^o$

1. Two Degree of Freedom Systems

Example 1b (Continue)

Back into Eqn.1: $c_1 = 2.7288$

Back into Eqn.2: $c_2 = -0.732$

Step #3:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = 2.7288 \begin{bmatrix} 1.000 \\ 1.368 \end{bmatrix} \sin\left(\omega_1 t + 90^o\right) + \left(-0.732\right) \begin{bmatrix} 1.000 \\ -0.366 \end{bmatrix} \sin\left(\omega_2 t + 90^o\right)$$

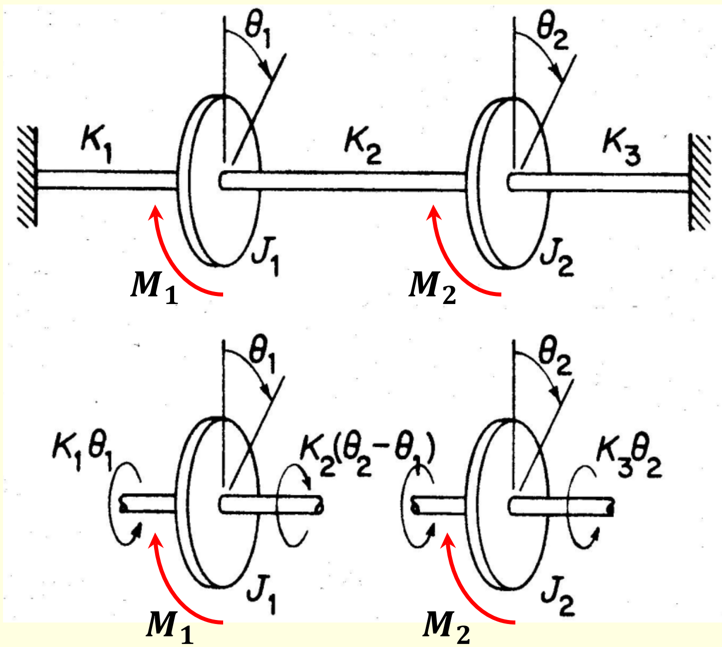
$$= \begin{bmatrix} 2.729 \\ 3.733 \end{bmatrix} \cos \omega_1 t + \begin{bmatrix} -0.732 \\ 0.268 \end{bmatrix} \cos \omega_2 t$$

Step #4: larger than

$\therefore$ For i.c. specified, most of the response is due to $X_1^{(1)}$ the first mode. This should be expected since the initial displacement of $\begin{bmatrix} 2.0 \\ 4.0 \end{bmatrix}$ is somewhat closer to the first mode than the second.

1. Two Degree of Freedom Systems

Example 2 (Torsional System)



From the free-body diagram of the 2 Disks:

$$J_1\ddot{\theta}_1 = -k_1\theta_1 + k_2\left(\theta_2 - \theta_1\right) + M_1$$
$$J_2\ddot{\theta}_2 = -k_2\left(\theta_2 - \theta_1\right) - k_3\theta_2 + M_2$$

Rewriting:

$$J_1\ddot{\theta}_1 + (k_1 + k_2)\theta_1 - k_2\theta_2 = M_1$$
$$J_2\ddot{\theta}_2 - k_2\theta_1 + (k_2 + k_3)\theta_2 = M_2$$

or

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & (k_2+k_3) \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad [4b]$$

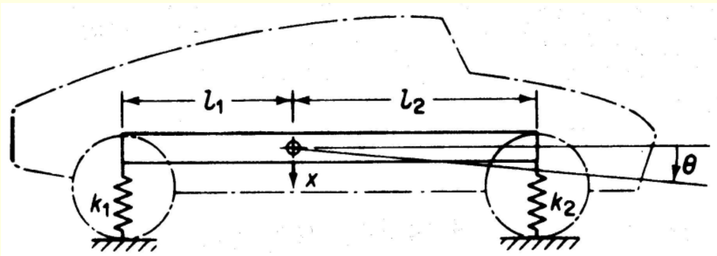
Compare [4b] with [1b] or [2a]. The torsional spring constants K_1, K_1 and K_1 have different units from the linear spring constants in Eqs [1] and [2] which have units of N/m. Torsional spring constants have units of Nm/rad.

$$K = \frac{GI_p}{l}$$

where G = shear modulus (N/m^2)
I_p = polar moment of inertia of shaft (m^4)
l = length of shaft (m)

1. Two Degree of Freedom Systems

Example 3 (Static & Dynamic Coupling) Study Section 13.4.1 of SD for derivations



Determine the normal mode of vibration of the automobile modeled by a 2-DOF system with the following values:

$W = 3220$ lb	$l_1 = 4.5$ ft	$k_1 = 2400$ lb/ft
$J_c = \dfrac{W}{g} r^2$	$l_2 = 5.5$ ft	$k_2 = 2600$ lb/ft
$r = 4$ ft	$l = 10$ ft	

Eqn. of Motion: (Come up with these on your own):

$$m\ddot{x} + k_1\left(x - l_1\theta\right) + k_2\left(x + l_2\theta\right) = 0$$

$$J_c\ddot{\theta} - k_1\left(x - l_1\theta\right)l_1 + k_2\left(x + l_2\theta\right)l_2 = 0$$

Assuming harmonic motion (as in Eqn. [2b]), we get 2 eqns similar to Eqn. [2d] resulting in:

$$\det\begin{bmatrix} (k_1 + k_2 - \omega^2 m) & -\left(k_1 l_1 - k_2 l_2\right) \\ -\left(k_1 l_1 - k_2 l_2\right) & \left(k_1 l_1^{\,2} + k_2 l_2^{\,2} - \omega^2 J_c\right) \end{bmatrix} = 0$$ compare to Eq. [2e]

Expanding this determinant and solving, we get 2 natural frequencies:

$$\omega_1 = 6.90 rad/s = 1.10 cps \rightarrow \left(Mode\#1 - bounce\right)$$

$$\omega_2 = 9.06 rad/s = 1.44 cps \rightarrow \left(Mode\#2 - pitch\right)$$

1. Two Degree of Freedom Systems

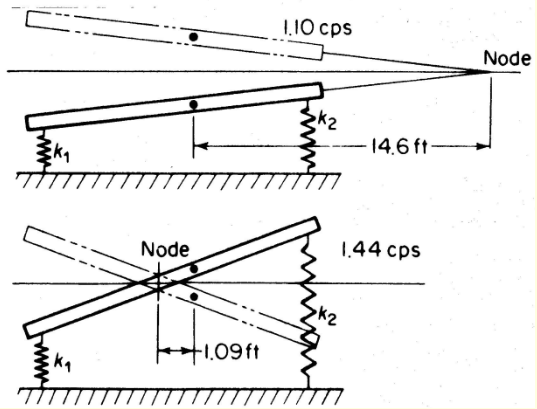
Example 3 (Continue)

Amplitude ratios for the two frequencies:

$$\left(\frac{x}{\theta}\right)_{\omega_1} = -14.6\, ft/rad = -3.06 in/\deg$$

$$\left(\frac{x}{\theta}\right)_{\omega_2} = 1.09\, ft/rad = 0.288 in/\deg$$

Mode Shapes for the two frequencies:



First mode: ω_1= 6.90 rad/s --- largely verticle bounce.
Second mode: ω_2= 9.06 rad/s --- mostly rotation. Therefore, following "quick and dirty" method works. Assume modes are uncoupled into two 1-DOF systems.

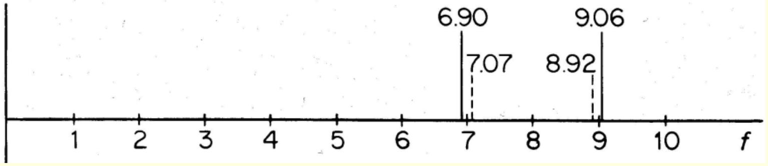
Then (for bounce):

$$\tilde{\omega}_1 = \sqrt{\frac{vertical\ stiffness}{translational\ mass}} = \sqrt{\frac{2400+2600}{3220/32.2}} = \sqrt{\frac{5000}{100}} = 7.07 rad/s$$

and for pitch:

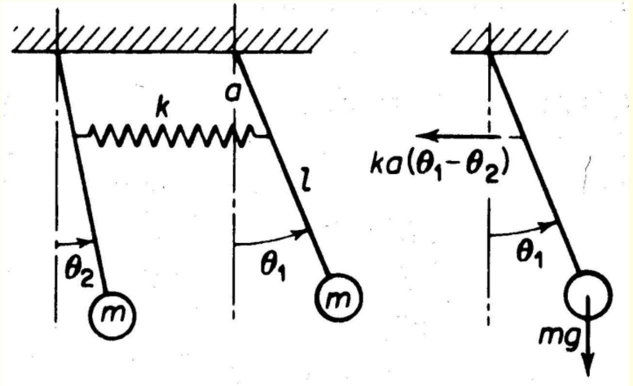
$$\tilde{\omega}_2 = \sqrt{\frac{Rotational\ stiffness}{Rotational\ Mmt.ofIn.}} = \sqrt{\frac{2400\times(4.5)^2 + 2600\times(5.5)^2}{J_c}} = \sqrt{\frac{127250}{1600}} = 8.92 rad/s$$

Comparing $\widetilde{\omega_1}$ and $\widetilde{\omega_2}$ to ω_1 and ω_2 respectively, we see that the uncoupled approximate frequencies are inside the range of the coupled frequencies as shown in the graph:



1. Two Degree of Freedom Systems

Example 4 (Coupled Pendulum)



Eqns. of Motion:

For Mass 1: $m_1 l_1^2 \ddot{\theta}_1 = -m_1 g l_1 \theta_1 + ka^2(\theta_2 - \theta_1)$

For Mass 2: $m_2 l_2^2 \ddot{\theta}_2 = -m_2 g l_2 \theta_2 - ka^2(\theta_2 - \theta_1)$

Rewriting:

$$\begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & m_1 l_1^2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (ka^2 + m_1 g l_1) & -ka^2 \\ -ka^2 & (ka^2 + m_2 g l_2) \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Simplification: $m_1 = m_2 = m$

$l_1 = l_2 = l$

Then:

$$ml^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} (ka^2 + mgl) & -ka^2 \\ -ka^2 & (ka^2 + mgl) \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assuming: $\theta_1 = A_1 \cos \omega t$

$\theta_2 = A_2 \cos \omega t$

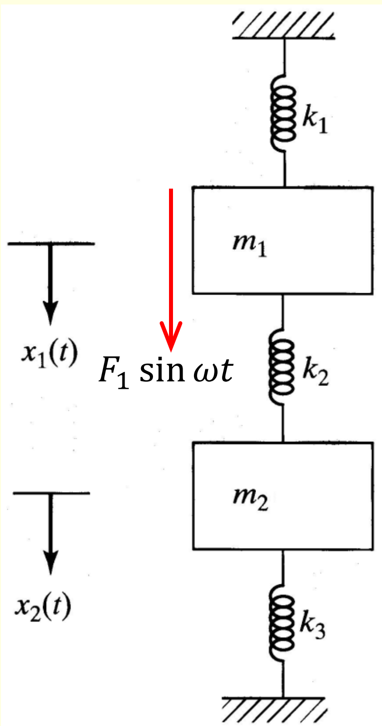
$\Rightarrow$ $\omega_1 = \sqrt{g/l}$ and $\omega_2 = \sqrt{\dfrac{g}{l} + 2\dfrac{k}{m}\dfrac{a^2}{l^2}}$

Normal modes: (in phase) (out of phase)

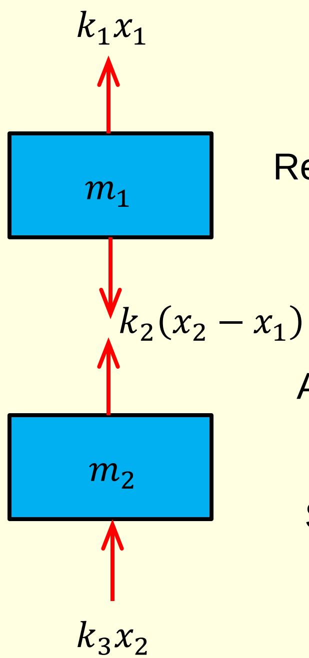
Amplitude ratios: $\left(\dfrac{A_1}{A_2}\right)^{(1)} = 1.0, \left(\dfrac{A_1}{A_2}\right)^{(2)} = -1.0$

2. Forced Vibration of Two DoF Systems

2.1 Equations of Motion – Forced Harmonic Vibration (Chapter 5 of SOMV)



Free Body Diagram



$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 \left(x_2 - x_1 \right) + F_1 \sin \omega t$$

$$m_2 \ddot{x}_2 = -k_2 \left(x_2 - x_1 \right) - k_3 x_2$$

Rewriting:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} (k_1 + k_2) & -k_2 \\ -k_2 & (k_2 + k_3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \sin \omega t \qquad [5a]$$

Assume: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sin \omega t$

Substituting into Diff. Eqn.:

$$\underbrace{\begin{bmatrix} (k_1 + k_2) - m_1 \omega^2 & -k_2 \\ -k_2 & (k_2 + k_3) - m_2 \omega^2 \end{bmatrix}}_{[Z(\omega)] \text{ impedance matrix}} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} \qquad [5b]$$

2. Forced Vibration of Two DoF Systems

2.1 Equations of Motion – Forced Harmonic Vibration (Continue)

$$\therefore \left[Z(\omega)\right]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

Solving:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \left[Z(\omega)\right]^{-1}\begin{bmatrix} F_1 \\ 0 \end{bmatrix} = \frac{adj\left[Z(\omega)\right]\begin{bmatrix} F_1 \\ 0 \end{bmatrix}}{\det\left|Z(\omega)\right|} \qquad [5c]$$

Refering to [5b], determinant $|Z(\omega)|$ can be written as:

$$\det\left[Z(\omega)\right] = m_1 m_2\left(\omega_1{}^2 - \omega^2\right)\left(\omega_2{}^2 - \omega^2\right)$$ where ω_1, ω_2 are normal mode frequencies

so that:

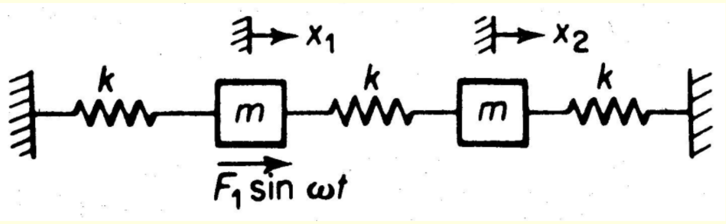
$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{\det\left|Z(\omega)\right|}\begin{bmatrix} (k_2+k_3) - m_2\omega^2 & -k_2 \\ -k_2 & (k_1+k_2) - m_1\omega^2 \end{bmatrix}\begin{bmatrix} F_1 \\ 0 \end{bmatrix} \qquad [5d]$$

$$X_1 = \frac{\left[(k_2+k_3) - m_2\omega^2\right]F_1}{m_1 m_2\left(\omega_1{}^2 - \omega^2\right)\left(\omega_2{}^2 - \omega^2\right)}$$

$$X_2 = \frac{\left[+k_2\right]F_1}{m_1 m_2\left(\omega_1{}^2 - \omega^2\right)\left(\omega_2{}^2 - \omega^2\right)} \qquad [5e]$$

2. Forced Vibration of Two DoF Systems

Example 5



Refering to the derivation above, we have:

$$m_1 = m_2 = m$$

$$k_1 = k_2 = k_3 = k$$

$$\therefore X_1 = \frac{\left[2k - m_2\omega^2\right]F_1}{m^2\left(\omega_1{}^2 - \omega^2\right)\left(\omega_2{}^2 - \omega^2\right)} \qquad [5f]$$

$$X_2 = \frac{kF_1}{m^2\left(\omega_1{}^2 - \omega^2\right)\left(\omega_2{}^2 - \omega^2\right)}$$

What are: $\omega_1{}^2$ and $\omega_1{}^2$?

Take the determinant $|Z(\omega)|$ in [5b]:

$$\det\left|Z(\omega)\right| = \left(2k - m\omega^2\right)^2 - k^2 = 0$$

or

$$\left(2\frac{k}{m} - \omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = 0 \qquad [5g]$$

2. Forced Vibration of Two DoF Systems

Example 5 (Continue)

Applying to [5g]: $A^2 - B^2 = (A+B)(A-B)$

$$\left(2\frac{k}{m} - \omega^2 + \frac{k}{m}\right)\left(2\frac{k}{m} - \omega^2 - \frac{k}{m}\right) = 0$$

so that:

$$\therefore \omega_1{}^2 = \frac{k}{m}$$

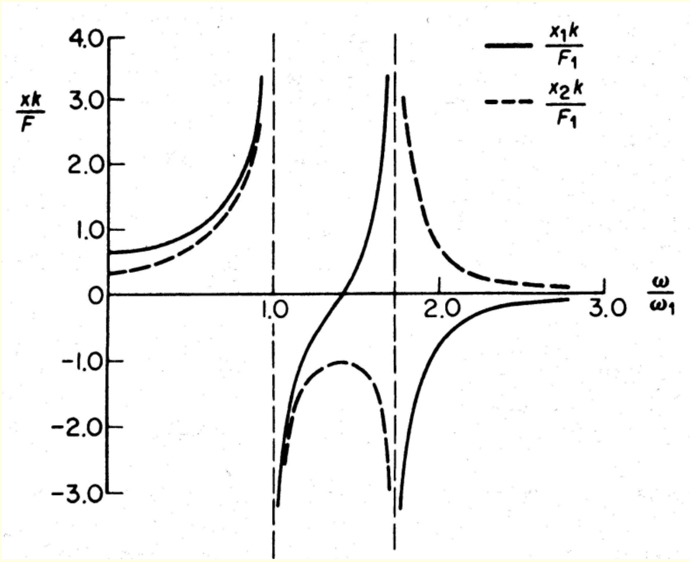
$$\omega_2{}^2 = \frac{3k}{m} \qquad [5h]$$

Therefore:

$$x_1{}^{(1)}(t) = X_1 \sin \omega_1 t \; ; \qquad x_1{}^{(2)}(t) = X_1 \sin \omega_2 t$$

$$x_2{}^{(1)}(t) = X_2 \sin \omega_1 t \; ; \qquad x_2{}^{(2)}(t) = X_2 \sin \omega_2 t$$

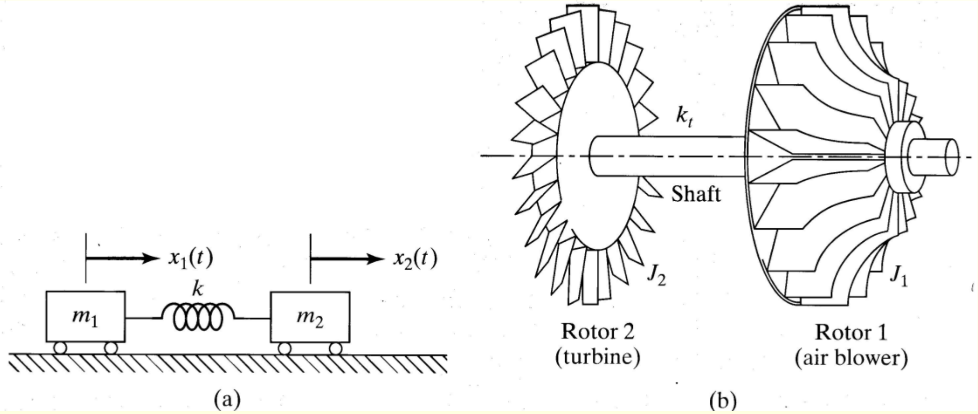
where X_1 and X_2 are given by [5f] after [5h] is applied to it.



3. Semi-definite Systems

(Degenerate/Unrestrained/Unconstrained) Systems

3.1 Equations of Motion



Semidefinite systems are also called unrestrained or degenerate systems. Two examples are shown, (a) in translation such as railway cars, and (b) in rotation such as the rotors in a turbo-charger.

Eqns. of Motion for (a):

For Mass 1: $m_1\ddot{x}_1 + k(x_1 - x_2) = 0$

For Mass 2: $m_2\ddot{x}_2 + k(x_2 - x_1) = 0$

Assume: $x_j(t) = X_j \cos\omega t \qquad j = 1,2$

then:

$$(-m_1\omega^2 + k)X_1 - kX_2 = 0$$
$$-kX_1 + (-m_2\omega^2 + k)X_2 = 0 \qquad [6a]$$

Eqns. of Motion for (b):

$$J_1\ddot{\theta}_1 + k\left(\theta_1 - \theta_2\right) = 0$$
$$J_2\ddot{\theta}_2 + k\left(\theta_2 - \theta_1\right) = 0$$

Assume: $\theta_j\left(t\right) = X_j \sin\omega t$

then:

$$\left(-J_1\omega^2 + k\right)X_1 - kX_2 = 0$$
$$-kX_1 + \left(-J_2\omega^2 + k\right)X_2 = 0 \qquad [6a]$$

3. Semi-definite Systems

3.1 Equations of Motion (Continue)

Characteristic Eqn.: $$\det\begin{vmatrix}(-J_1\omega^2+k) & -k \\ -k & (-J_2\omega^2+k)\end{vmatrix}=0$$

$$\Rightarrow \left(-J_1\omega^2+k\right)\left(-J_2\omega^2+k\right)-k^2=0$$

$$\omega^2\left[J_1J_2\omega^2-\left(J_1+J_2\right)k\right]+k^2-k^2=0$$

$$\left[J_1J_2\omega^2-\left(J_1+J_2\right)k\right]\omega^2=0$$

$$\left.\begin{aligned}\therefore\ \omega_1{}^2&=0\\ \omega_2{}^2&=\frac{k\left(J_1+J_2\right)}{J_1J_2}\end{aligned}\right\}\Rightarrow\begin{aligned}\omega_1&=0\\ \omega_2&=\sqrt{\frac{k\left(J_1+J_2\right)}{J_1J_2}}\end{aligned}$$

means constant velocity or zero velocity of entire shaft and rotors (Mode # 1)

means rotors move out-of-phase to each other.

To satisfy [6a]:

$$\frac{X_2}{X_1}=\frac{\left(-J_1\omega_2{}^2+k\right)}{k}=\frac{k}{\left(-J_2\omega_2{}^2+k\right)}$$

$$\therefore\frac{X_2}{X_1}=\frac{-J_1\dfrac{k\left(J_1+J_2\right)}{J_1J_2}+k}{k}=-\frac{J_1}{J_2}$$

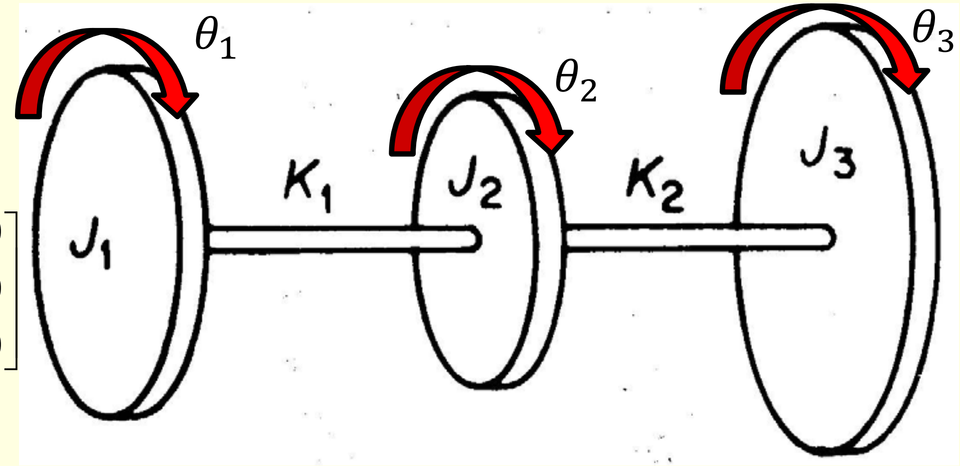
3. Semi-definite Systems

Example 6

Three mass torsional system are unrestrained to rotate freely in bearings.

Eqn. of Motion:

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix}+\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



Simplify: $J_1 = J_2 = J_3 = J$

$k_1 = k_2 = k$

Also, let $\lambda = \omega^2 J / k$

Assume: $\theta_j(t) = X_j \sin \omega t \quad j = 1, 2, 3$

Then:

$$\begin{bmatrix} -J_1\omega^2 X_1 + k_1 X_1 - k_1 X_2 \\ -J_2\omega^2 X_2 - k_1 X_1 + (k_1 + k_2) X_2 + k_2 X_3 \\ -J_3\omega^2 X_3 - k_2 X_2 + k_2 X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Or

$$\begin{bmatrix} \frac{-J\omega^2}{k} X_1 + X_1 - X_2 \\ \frac{-J\omega^2}{k} X_2 - X_1 + 2X_2 - X_3 \\ \frac{-J\omega^2}{k} X_3 - X_2 + X_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

3. Semi-definite Systems

For non-trivial solutions

$$\det\begin{vmatrix} (1-\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{vmatrix} = 0$$

Or

$$\lambda(1-\lambda)(\lambda-3)=0$$

Eigenvalues for the system are: $\lambda_1 = 0$

$$\lambda_2 = 1$$

$$\lambda_3 = 3$$

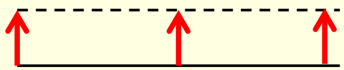
The corresponding eigenvectors are obtained by substituting each of the λ's into Eqn. of motion:

$$\begin{bmatrix} (1-\lambda) & -1 & 0 \\ -1 & (2-\lambda) & -1 \\ 0 & -1 & (1-\lambda) \end{bmatrix}\begin{bmatrix}\theta_1 \\ \theta_2 \\ \theta_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(i) When $\lambda_1 = 0$

$$\theta_1 = \theta_2 = \theta_3$$

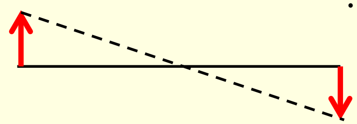
$\therefore$ normal mode (eigenvector) is $v_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$



(ii) When $\lambda_2 = 1$

$$\theta_2 = 0 \rightarrow \theta_3 = 1 \rightarrow \theta_1 = -1$$

$$\therefore v_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$$



(iii) When $\lambda_3 = 3$

Using 3[rd] eqn. : $\theta_3 = 1 \rightarrow \theta_2 = -2$

Using 2[nd] eqn. : $\theta_1 = 1$

$$v_3 = \begin{bmatrix}1\\-2\\1\end{bmatrix}$$