Chapter 11

Numerical Integration Methods in Vibration Analysis

11.1

$$\left.\frac{d^2x}{dt^2}\right|_i = \frac{\left.\frac{dx}{dt}\right|_{i+1} - \left.\frac{dx}{dt}\right|_i}{\Delta t} = \frac{\left(\frac{x_{i+2} - x_{i+1}}{\Delta t}\right) - \left(\frac{x_{i+1} - x_i}{\Delta t}\right)}{\Delta t} = \frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2}$$

$$\left.\frac{d^3x}{dt^3}\right|_i = \frac{\left.\frac{d^2x}{dt^2}\right|_{i+1} - \left.\frac{d^2x}{dt^2}\right|_i}{\Delta t} = \frac{\left(\frac{x_{i+3} - 2x_{i+2} + x_{i+1}}{(\Delta t)^2}\right) - \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2}\right)}{\Delta t}$$

$$= \frac{x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i}{(\Delta t)^3}$$

$$\left.\frac{d^4x}{dt^4}\right|_i = \frac{\left.\frac{d^3x}{dt^3}\right|_{i+1} - \left.\frac{d^3x}{dt^3}\right|_i}{\Delta t} = \frac{\left(\frac{x_{i+4} - 3x_{i+3} + 3x_{i+2} - x_{i+1}}{(\Delta t)^3}\right) - \left(\frac{x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i}{(\Delta t)^3}\right)}{\Delta t}$$

$$= \frac{x_{i+4} - 4x_{i+3} + 6x_{i+2} - 4x_{i+1} + x_i}{(\Delta t)^4}$$

11.2

$$\left.\frac{d^2x}{dt^2}\right|_i = \frac{\left.\frac{dx}{dt}\right|_i - \left.\frac{dx}{dt}\right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - x_{i-1}}{\Delta t}\right) - \left(\frac{x_{i-1} - x_{i-2}}{\Delta t}\right)}{\Delta t} = \frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2}$$

$$\left.\frac{d^3x}{dt^3}\right|_i = \frac{\left.\frac{d^2x}{dt^2}\right|_i - \left.\frac{d^2x}{dt^2}\right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2}\right) - \left(\frac{x_{i-1} - 2x_{i-2} + x_{i-3}}{(\Delta t)^2}\right)}{\Delta t}$$

$$= \frac{x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}}{(\Delta t)^3}$$

$$\left.\frac{d^4x}{dt^4}\right|_i = \frac{\left.\frac{d^3x}{dt^3}\right|_i - \left.\frac{d^3x}{dt^3}\right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}}{(\Delta t)^3}\right) - \left(\frac{x_{i-1} - 3x_{i-2} + 3x_{i-3} - x_{i-4}}{(\Delta t)^3}\right)}{\Delta t}$$

$$= \frac{x_i - 4x_{i-1} + 6x_{i-2} - 4x_{i-3} + x_{i-4}}{(\Delta t)^4}$$

11.3

$$\left.\frac{d^2x}{dt^2}\right|_i = \frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2}$$

$$\left.\frac{d^4x}{dt^4}\right|_i = \frac{\left.\frac{d^2x}{dt^2}\right|_{i+1} - 2\left.\frac{d^2x}{dt^2}\right|_i + \left.\frac{d^2x}{dt^2}\right|_{i-1}}{(\Delta t)^2}$$

$$= \left\{ \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2} \right) - 2\left(\frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2} \right) + \left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) \right\} \Big/ (\Delta t)^2$$

$$= \frac{x_{i+2} - 4x_{i+1} + 6x_i - 4x_{i-1} + x_{i-2}}{(\Delta t)^4}$$

11.4 Equation: $\ddot{x} + x = 0$

Central difference solution: $x_{i+1} = (\Delta t)^2 \left\{ \left(\frac{2}{(\Delta t)^2} - 1 \right) x_i - \frac{1}{(\Delta t)^2} x_{i-1} \right\}$

$$= (2 - \Delta t^2)\, x_i - x_{i-1} \qquad \text{---}(E_1)$$

For $x_0 = 0$ and $\dot{x}_0 = 1$, Eq. (11.9) gives $x_{-1} = -\Delta t$

(i) with $\Delta t = 1$, $x_{i+1} = x_i - x_{i-1}$ --- (E_2)

Repetitive application of (E_2), with $x_{-1} = -1$ and $x_0 = 0$, gives the results shown in the following table.

(ii) With $\Delta t = 0.5$, $x_{i+1} = 1.75\, x_i - x_{i-1}$ --- (E_3)

The results, for $x_{-1} = -0.5$ and $x_0 = 0$, are shown in the table.

comparison of solutions:

Time (t)	Value of $x(t)$ obtained with $\Delta t = 1$	$\Delta t = 0.5$	Exact value of $x(t)$ $x(t) = \sin t$
0	0	0	0
0.5	–	0.5	0.4794
1	1	0.8750	0.8415
1.5	–	1.0313	0.9975
2	1	0.9297	0.9093
2.5	–	0.5957	0.5985
3	0	0.1128	0.1411
3.5	–	−0.3983	−0.3508
4	−1	−0.8098	−0.7568
4.5	–	−1.0189	−0.9775
5	−1	−0.9733	−0.9589
5.5	–	−0.6843	−0.7055
6	0	−0.2242	−0.2794

11.5 At t_i, central difference formula gives

$$-\left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) + 0.1\, x_i = 0$$

i.e. $x_i = \dfrac{2x_{i-1} - x_{i-2}}{1 - 0.1(\Delta t)^2} = 1.1111\,(2x_{i-1} - x_{i-2})$ for $\Delta t = 1$ --- (E_1)

Since $\dot{x}_0 = \dfrac{x_0 - x_{-1}}{\Delta t} = 0$, $x_{-1} = x_0 = 1$

Eq. (E_1) gives the following results.

i	1	2	3	4	5	6	7	8	9	10
x_{i-1}	1	1.1111	1.3580	1.7832	2.2084	2.6336	3.0588	3.4840	3.9092	4.3344
x_{i-2}	1	1	1.1111	1.3580	1.7832	2.2084	2.6336	3.0588	3.4840	3.9092
x_i from (E_1)	1.1111	1.3580	1.7832	2.2084	2.6336	3.0588	3.4840	3.9092	4.3344	4.7596

11.6 For given data, Eq. (11.7) gives $x_{i+1} = 0.2(7x_i - 3x_{i-1})$ --- (E_1)

Eqs. (11.8) and (11.9) give $\ddot{x}_0 = -1$, $x_{-1} = -0.625$

$$\left(\frac{c}{2m}\right)^2 = \left(\frac{1}{2}\right)^2 < \left(\frac{k}{m} = 1\right) \Rightarrow \text{Underdamped case.}$$

Since $\tau_n = \frac{2\pi}{\omega_n} = 2\pi$ sec, we will consider the response for 15 steps.

$i+1$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	−0.625	0.375
2	0.375	0	0.525
3	0.525000E+00	0.375000E+00	0.510000E+00
4	0.510000E+00	0.525000E+00	0.399000E+00
5	0.399000E+00	0.510000E+00	0.252600E+00
6	0.252600E+00	0.399000E+00	0.114240E+00
7	0.114240E+00	0.252600E+00	0.837604E-02
8	0.837604E-02	0.114240E+00	-0.568176E-01
9	-0.568176E-01	0.837604E-02	-0.845702E-01
10	-0.845702E-01	-0.568176E-01	-0.843078E-01
11	-0.843078E-01	-0.845702E-01	-0.672887E-01
12	-0.672887E-01	-0.843078E-01	-0.436196E-01
13	-0.436196E-01	-0.672887E-01	-0.206942E-01
14	-0.206942E-01	-0.436196E-01	-0.280008E-02
15	-0.280008E-02	-0.206942E-01	0.849639E-02

11.7 $\left(\frac{c}{2m}\right)^2 = 1 = \left(\frac{k}{m} = 1\right) \Rightarrow$ critically damped case.

For the data, Eqs. (11.7) to (11.9) give

$$x_{i+1} = \frac{1}{6}(7x_i - 2x_{i-1}) \quad \text{--- } (E_1)$$

$\ddot{x}_0 = -2$, $x_{-1} = -0.75$. Response is given below.

$i+1$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	−0.75	0.25
2	0.25	0	0.2917
3	0.291667E+00	0.250000E+00	0.256944E+00
4	0.256944E+00	0.291667E+00	0.202546E+00
5	0.202546E+00	0.256944E+00	0.150656E+00
6	0.150656E+00	0.202546E+00	0.108250E+00
7	0.108250E+00	0.150656E+00	0.760728E-01
8	0.760728E-01	0.108250E+00	0.526683E-01
9	0.526683E-01	0.760728E-01	0.360888E-01

10	0.360888E-01	0.526683E-01	0.245475E-01
11	0.245475E-01	0.360888E-01	0.166091E-01
12	0.166091E-01	0.245475E-01	0.111948E-01
13	0.111948E-01	0.166091E-01	0.752425E-02
14	0.752425E-02	0.111948E-01	0.504668E-02
15	0.504668E-02	0.752425E-02	0.337971E-02

11.8 For given data, Eq. (11.7) gives $x_{i+1} = 0.875\, x_i$

This equation does not involve x_{i-1} and hence the response can't be found since $x_0 = 0$. Hence we change Δt to 0.4. This gives, from Eq. (11.7),

$$x_{i+1} = 0.08889\,(11.5\, x_i - 1.25\, x_{i-1}) \qquad \text{--- } (E_1)$$

Eqs. (11.8) and (11.9) give $\ddot{x}_0 = -4$, $x_{-1} = -0.72$

Results:

$i+1$	x_{i-1}	x_i	x_{i+1} from (E_1)
1	-0.720000E+00	0.000000E+00	0.800010E-01
2	0.000000E+00	0.800010E-01	0.817798E-01
3	0.800010E-01	0.817798E-01	0.747091E-01
4	0.817798E-01	0.747091E-01	0.672835E-01
5	0.747091E-01	0.672835E-01	0.604784E-01
6	0.672835E-01	0.604784E-01	0.543471E-01
7	0.604784E-01	0.543471E-01	0.488356E-01
8	0.543471E-01	0.488356E-01	0.438829E-01
9	0.488356E-01	0.438829E-01	0.394323E-01
10	0.438829E-01	0.394323E-01	0.354332E-01
11	0.394323E-01	0.354332E-01	0.318396E-01
12	0.354332E-01	0.318396E-01	0.286105E-01
13	0.318396E-01	0.286105E-01	0.257089E-01

11.9 $\omega_n = \sqrt{\frac{3000}{4}} = 27.3861$, $\tau_n = 2\pi/\omega_n = 0.22943$; $\Delta t = 0.05$

Eqs. (11.7) to (11.9) give

$$x_{i+1} = \frac{1}{1620}\,(200\, x_i - 1580\, x_{i-1} + F_i) \quad \text{where } F(t) = \begin{cases} 200 & ; 0 \le t \le 0.2 \\ -500t + 300 & ; 0.2 \le t \le 0.6 \end{cases} \qquad \text{--- } (E_1)$$

$\ddot{x}_0 = 50$, $x_{-1} = 0.0625$

Results:

$i+1$	t_i	$F_i = F(t_i)$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	0.200000E+03	0.000000E+00	0.625000E-01	0.625000E-01
2	0.05	0.200000E+03	0.625000E-01	0.000000E+00	0.131173E+00
3	0.1	0.200000E+03	0.131173E+00	0.625000E-01	0.786942E-01
4	0.15	0.200000E+03	0.786942E-01	0.131173E+00	0.523812E-02
5	0.2	0.200000E+03	0.523812E-02	0.786942E-01	0.473524E-01
6	0.25	0.175000E+03	0.473524E-01	0.523812E-02	0.108762E+00
7	0.3	0.150000E+03	0.108762E+00	0.473524E-01	0.598368E-01
8	0.35	0.125000E+03	0.598368E-01	0.108762E+00	-0.215287E-01
9	0.4	0.100000E+03	-0.215287E-01	0.598368E-01	0.711154E-03
10	0.45	0.750000E+02	0.711154E-03	-0.215287E-01	0.673812E-01
11	0.5	0.500000E+02	0.673812E-01	0.711154E-03	0.384892E-01
12	0.55	0.250000E+02	0.384892E-01	0.673812E-01	-0.455336E-01
13	0.6	-0.305176E-04	-0.455336E-01	0.384892E-01	-0.431603E-01

14	0.65	-0.250001E+02	-0.431603E-01	-0.455336E-01	0.236487E-01
15	0.7	-0.500001E+02	0.236487E-01	-0.431603E-01	0.141500E-01
16	0.75	-0.750001E+02	0.141500E-01	0.236487E-01	-0.676142E-01
17	0.8	-0.100000E+03	-0.676142E-01	0.141500E-01	-0.838765E-01
18	0.85	-0.125000E+03	-0.838765E-01	-0.676142E-01	-0.215709E-01
19	0.9	-0.150000E+03	-0.215709E-01	-0.838765E-01	-0.134502E-01
20	0.95	-0.175000E+03	-0.134502E-01	-0.215709E-01	-0.886470E-01
21	1.00	-0.200000E+03	-0.886470E-01	-0.134502E-01	-0.121283E+00

11.10 Equation: $m\ddot{x} + c\dot{x} + kx = F(t) = t$

Since $\tau_n = 2\pi$, Δt is selected as 0.5.

Eqs. (11.7) to (11.9) give
$$x_{i+1} = 0.2\,(7x_i - 3x_{i-1} + t_i) \quad \text{--- } (E_1)$$
$$\ddot{x}_0 = 0, \quad x_{-1} = 0$$

Exact solution (from Example 4.9):

For $\delta F = 1$, $k = 1$, $\omega_n = 1$, $\zeta = 0.5$, $\omega_d = 0.86603$, we find
$$x(t) = t - 1 + e^{-0.5t}\left(\cos 0.86603\,t - 0.57735 \sin 0.86603\,t\right) \quad \text{--- } (E_2)$$

Results:

$i+1$	t_i	x_i	x_{i-1}	x_{i+1}, (E_1)	x_{i+1}, (E_2)
1	0	0.000000E+00	0.000000E+00	0.000000E+00	0.182477E-01
2	0.5	0.000000E+00	0.000000E+00	0.100000E+00	0.126190E+00
3	1	0.100000E+00	0.000000E+00	0.340000E+00	0.364080E+00
4	1.5	0.340000E+00	0.100000E+00	0.716000E+00	0.731292E+00
5	2	0.716000E+00	0.340000E+00	0.119840E+01	0.120253E+01
6	2.5	0.119840E+01	0.716000E+00	0.174816E+01	0.174240E+01
7	3	0.174816E+01	0.119840E+01	0.232838E+01	0.231622E+01
8	3.5	0.232838E+01	0.174816E+01	0.291084E+01	0.289641E+01
9	4	0.291084E+01	0.232838E+01	0.347815E+01	0.346501E+01
10	4.5	0.347815E+01	0.291084E+01	0.402290E+01	0.401335E+01
11	5	0.402290E+01	0.347815E+01	0.454518E+01	0.454011E+01
12	5.5	0.454518E+01	0.402290E+01	0.504950E+01	0.504860E+01
13	6	0.504950E+01	0.454518E+01	0.554220E+01	0.554439E+01
14	6.5	0.554220E+01	0.504950E+01	0.602938E+01	0.603328E+01
15	7	0.602938E+01	0.554220E+01	0.651581E+01	0.652013E+01
16	7.5	0.651581E+01	0.602938E+01	0.700451E+01	0.700828E+01
17	8	0.700451E+01	0.651581E+01	0.749682E+01	0.749949E+01
18	8.5	0.749682E+01	0.700451E+01	0.799285E+01	0.799426E+01
19	9	0.799285E+01	0.749682E+01	0.849190E+01	0.849219E+01
20	9.5	0.849190E+01	0.799285E+01	0.899294E+01	0.899244E+01

11.11 Let $x_1 = x$, $x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$, $x_3 = \frac{dx_2}{dt} = \frac{d^2x}{dt^2}$, ..., $x_n = \frac{dx_{n-1}}{dt} = \frac{d^{n-1}x}{dt^{n-1}}$

Equation:
$$\frac{d^n x}{dt^n} = \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n}\frac{d^{n-1}x}{dt^{n-1}} - \cdots - \frac{a_1}{a_n}\frac{dx}{dt} \quad \text{--- } (E_1)$$
$$= \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n}x_n - \frac{a_{n-2}}{a_n}x_{n-1} - \cdots - \frac{a_1}{a_n}x_2 \quad \text{--- } (E_2)$$

(E_1) and (E_2) can be expressed as
$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}, t)$$

where $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ and $\vec{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{Bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n} x_n - \cdots - \frac{a_1}{a_n} x_2 \end{Bmatrix}$

11.12 The main program for problems (a) and (b) is given below.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C ======================================================================
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,1),XX(1),F(1),YI(1),YJ(1),YK(1),YL(1),
    2 UU(1)
      XX(1)=1.0
      NEQ=1
      NSTEP=40
      DT=0.1
      T=0.0
      WRITE (58,10)
10    FORMAT (//,3X,5H   I  ,10H   TIME(I),7X,5H X(1),/)
      DO 40 I=1,NSTEP
      CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (58,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
```

The subroutine FUN and output are given below.

For problem (a)

```
SUBROUTINE FUN (X,F,N,T)
DIMENSION X(N),F(N)
F(1)=X(1)-1.5*EXP(-0.5*T)
RETURN
END
```

I	TIME(I)	X(1)
1	0.1000	0.95122939E+00
2	0.2000	0.90483737E+00
3	0.3000	0.86070788E+00
4	0.4000	0.81873059E+00
5	0.5000	0.77880061E+00
6	0.6000	0.74081802E+00
7	0.7000	0.70468783E+00
8	0.8000	0.67031974E+00

For problem (b)

```
SUBROUTINE FUN (X,F,N,T)
DIMENSION X(N),F(N)
F(1)=-T*(X(1)**2)
RETURN
END
```

I	TIME(I)	X(1)
1	0.1000	0.99502486E+00
2	0.2000	0.98039210E+00
3	0.3000	0.95693767E+00
4	0.4000	0.92592573E+00
5	0.5000	0.88888866E+00
6	0.6000	0.84745741E+00
7	0.7000	0.80321264E+00
8	0.8000	0.75757557E+00

⋮			⋮		
31	3.1000	0.21224274E+00	31	3.1000	0.17226547E+00
32	3.2000	0.20189069E+00	32	3.2000	0.16339886E+00
33	3.3000	0.19204344E+00	33	3.3000	0.15515921E+00
34	3.4000	0.18267635E+00	34	3.4000	0.14749278E+00
35	3.5000	0.17376597E+00	35	3.5000	0.14035103E+00
36	3.6000	0.16529004E+00	36	3.6000	0.13368998E+00
37	3.7000	0.15722735E+00	37	3.7000	0.12746987E+00
38	3.8000	0.14955774E+00	38	3.8000	0.12165464E+00
39	3.9000	0.14226201E+00	39	3.9000	0.11621163E+00
40	4.0000	0.13532193E+00	40	4.0000	0.11111123E+00

11.13 The program and output are given below.

```
C ===================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK2
C
C ===================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),UU(2)
      XX(1)=0.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.31416/2
      T=0.0
      WRITE (57,10)
 10   FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK2 (T,DT,NEQ,XX,F,YI,YJ,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
 20   X(I,J)=XX(J)
      WRITE (57,30) I,TIME(I),(X(I,J),J=1,NEQ)
 30   FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
 40   CONTINUE
      STOP
      END
C ===================================================================
C SUBROUTINE RK2
C ===================================================================
      SUBROUTINE RK2 (T,DT,N,XX,F,XI,XJ,UU)
      DIMENSION XI(N),XJ(N),UU(N),XX(N),F(N)
      DO 10 I=1,N
 10   UU(I)=XX(I)
      CALL FUN (XX,F,N,T)
      DO 20 I=1,N
      XI(I)=F(I)*DT
 20   XX(I)=UU(I)+XI(I)
      T=T+DT
      CALL FUN (XX,F,N,T)
      DO 30 I=1,N
      XJ(I)=F(I)*DT
 30   XX(I)=UU(I)+(XI(I)+XJ(I))/2.0
      RETURN
      END
```

```
C ====================================================================
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK2
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C ====================================================================
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F0=1.0
      T0=3.1416
      XM=1.0
      XC=0.2
      XK=1.0
      FT=F0*(1.0-SIN(3.1416*T/(2.0*T0)))
      F(2)=(FT-XC*X(2)-XK*X(1))
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.1571	0.12337063E-01	0.14845039E+00
2	0.3142	0.46506263E-01	0.27647862E+00
3	0.4712	0.99086702E-01	0.38170516E+00
4	0.6283	0.16633771E+00	0.46245623E+00
5	0.7854	0.24431184E+00	0.51778144E+00
6	0.9425	0.32896915E+00	0.54745305E+00
7	1.0996	0.41628993E+00	0.55194682E+00
8	1.2566	0.50238299E+00	0.53240609E+00
9	1.4137	0.58358723E+00	0.49059010E+00
10	1.5708	0.65656364E+00	0.42880845E+00
⋮			
36	5.6549	-0.28463572E+00	0.51887971E+00
37	5.8120	-0.19237404E+00	0.65321267E+00
38	5.9690	-0.79548687E-01	0.77926701E+00
39	6.1261	0.52324414E-01	0.89440674E+00
40	6.2832	0.20133464E+00	0.99627036E+00

11.14 The computer program and output are given below.

```
C ====================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK3
C
C ====================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),UU(2)
      XX(1)=0.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.31416/2
      T=0.0
      WRITE (57,10)
 10   FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK3 (T,DT,NEQ,XX,F,YI,YJ,YK,UU)
      TIME(I)=T
```

```
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (57,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
C ======================================================================
C SUBROUTINE RK3
C ======================================================================
      SUBROUTINE RK3 (T,DT,N,XX,F,XI,XJ,XK,UU)
      DIMENSION XI(N),XJ(N),XK(N),UU(N),XX(N),F(N)
      DO 10 I=1,N
10    UU(I)=XX(I)
      CALL FUN (XX,F,N,T)
      DO 20 I=1,N
      XI(I)=F(I)*DT
20    XX(I)=UU(I)+XI(I)/2.0
      T=T+DT/2.0
      CALL FUN (XX,F,N,T)
      DO 30 I=1,N
      XJ(I)=F(I)*DT
30    XX(I)=UU(I)-XI(I)+2.0*XJ(I)
      CALL FUN (XX,F,N,T)
      DO 40 I=1,N
      XK(I)=F(I)*DT
40    XX(I)=UU(I)+(XI(I)+4.0*XJ(I)+XK(I))/6.0
      RETURN
      END
C ======================================================================
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK3
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C ======================================================================
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F0=1.0
      T0=3.1416
      XM=1.0
      XC=0.2
      XK=1.0
      FT=F0*(1.0-SIN(3.1416*T/(2.0*T0)))
      F(2)=(FT-XC*X(2)-XK*X(1))
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.0785	0.11884968E-01	0.14891791E+00
2	0.1571	0.46078302E-01	0.28356332E+00
3	0.2356	0.10011104E+00	0.40113181E+00
4	0.3142	0.17111324E+00	0.49932912E+00
5	0.3927	0.25589743E+00	0.57641083E+00
6	0.4712	0.35104716E+00	0.63120759E+00
7	0.5498	0.45300832E+00	0.66313553E+00
8	0.6283	0.55818105E+00	0.67219222E+00
9	0.7069	0.66300964E+00	0.65893871E+00
10	0.7854	0.76406908E+00	0.62446821E+00

36	2.8274	-0.56948423E+00	-0.28658101E+00
37	2.9060	-0.60655731E+00	-0.18535951E+00
38	2.9845	-0.62764353E+00	-0.83412744E-01
39	3.0631	-0.63280767E+00	0.17002784E-01
40	3.1416	-0.62246108E+00	0.11373823E+00

11.15 The main program, subroutine FUN and results are given.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK2
C
C ======================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),UU(2)
      XX(1)=5.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.01
      T=0.0
      WRITE (61,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(1),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK2 (T,DT,NEQ,XX,F,YI,YJ,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (61,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END

      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-1000.0*X(1)
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.0100	0.47500000E+01	-0.50000000E+02
2	0.0200	0.40124998E+01	-0.95000000E+02
3	0.0300	0.28618748E+01	-0.13037500E+03
4	0.0400	0.14150311E+01	-0.15247501E+03
5	0.0500	-0.18047059E+00	-0.15900157E+03
6	0.0600	-0.17614628E+01	-0.14924678E+03
7	0.0700	-0.31658576E+01	-0.12416982E+03
8	0.0800	-0.42492628E+01	-0.86302750E+02
9	0.0900	-0.48998270E+01	-0.39494984E+02
10	0.1000	-0.50497856E+01	0.11478039E+02
⋮			

```
36      0.3600     0.28330812E+01     0.13901421E+03
37      0.3700     0.40815692E+01     0.10373269E+03
38      0.3800     0.49148178E+01     0.57730366E+02
39      0.3900     0.52463808E+01     0.56956673E+01
40      0.4000     0.50410185E+01    -0.47052921E+02
```

11.16 The main program, subroutine FUN and results are given.

```
C ==================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK3
C
C ==================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),UU(2)
      XX(1)=5.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.01
      T=0.0
      WRITE (62,10)
10    FORMAT (//,3X,5H  I   ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK3 (T,DT,NEQ,XX,F,YI,YJ,YK,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (62,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-1000.0*X(1)
      RETURN
      END
```

```
--------------------------------------------------------------
  I     TIME(I)          X(1)                X(2)

  1     0.0050     0.47500000E+01    -0.49166668E+02
  2     0.0100     0.40290279E+01    -0.93416672E+02
  3     0.0150     0.29089794E+01    -0.12836461E+03
  4     0.0200     0.15012785E+01    -0.15055135E+03
  5     0.0250    -0.54207087E-01    -0.15778635E+03
  6     0.0300    -0.16030624E+01    -0.14936400E+03
  7     0.0350    -0.29916553E+01    -0.12613235E+03
  8     0.0400    -0.40823741E+01    -0.90407799E+02
  9     0.0450    -0.47672653E+01    -0.45744061E+02
 10     0.0500    -0.49787188E+01     0.34212494E+01
  :
 36     0.1800     0.18843936E+01     0.14399376E+03
 37     0.1850     0.32061126E+01     0.11826420E+03
 38     0.1900     0.42087383E+01     0.80824219E+02
 39     0.1950     0.47930727E+01     0.35397079E+02
 40     0.2000     0.49014902E+01    -0.13504654E+02
```

11.17 The main program, subroutine FUN and output are given.

```
C ====================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C ====================================================================
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),YL(2),
     2 UU(2)
      XX(1)=5.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.01
      T=0.0
      WRITE (59,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (59,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-1000.0*X(1)
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.0100	0.47520833E+01	-0.49166668E+02
2	0.0200	0.40329871E+01	-0.93457642E+02
3	0.0300	0.29140182E+01	-0.12848141E+03
4	0.0400	0.15061308E+01	-0.15076540E+03
5	0.0500	-0.51074505E-01	-0.15810023E+03
6	0.0600	-0.16031944E+01	-0.14975887E+03
7	0.0700	-0.29963315E+01	-0.12656857E+03
8	0.0800	-0.40923543E+01	-0.90828957E+02
9	0.0900	-0.47825933E+01	-0.46083870E+02
10	0.1000	-0.49986143E+01	0.32299538E+01
⋮			
36	0.3600	0.18898817E+01	0.14634213E+03
37	0.3700	0.32352061E+01	0.12050217E+03
38	0.3800	0.42597318E+01	0.82714409E+02
39	0.3900	0.48618784E+01	0.36725792E+02
40	0.4000	0.49819474E+01	-0.12903667E+02

11.18 The main program, subroutine EXTFUN and results are given.

```
C ========================================================================
C
C PROGRAM-
C MAIN PROGRAM WHICH CALLS CDIFF
C
C ========================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL CDIFF (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
     2 MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
      WRITE (53,10)
10    FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
      WRITE (53,20) N,NSTEP,DELT
20    FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
     2 E15.8,/)
      WRITE (53,30)
30    FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(1,2),2X,
     3 9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELT
40    WRITE (53,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),
     2 XDD(I,2)
50    FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C ========================================================================
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C ========================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
```

```
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=    2    NSTEP=   24    DELT= 0.24216267E+00
```

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.0000E+00	0.0000E+00	0.0000E+00	0.1466E+00	0.3077E-07	0.5000E+01
3	0.4843	0.1122E+00	0.2317E+00	0.1913E+01	0.5045E+00	0.1042E+01	0.3604E+01
4	0.7265	0.3601E+00	0.7436E+00	0.2314E+01	0.9859E+00	0.1733E+01	0.2105E+01
5	0.9687	0.6966E+00	0.1207E+01	0.1510E+01	0.1500E+01	0.2056E+01	0.5669E+00
6	1.2108	0.1036E+01	0.1396E+01	0.5531E-01	0.1961E+01	0.2013E+01	-.9224E+00
7	1.4530	0.1289E+01	0.1224E+01	-.1478E+01	0.2292E+01	0.1633E+01	-.2217E+01
8	1.6951	0.1388E+01	0.7260E+00	-.2635E+01	0.2438E+01	0.9851E+00	-.3136E+01
9	1.9373	0.1302E+01	0.2610E-01	-.3145E+01	0.2378E+01	0.1795E+00	-.3518E+01
10	2.1795	0.1045E+01	-.7088E+00	-.2925E+01	0.2127E+01	-.6433E+00	-.3277E+01
11	2.4216	0.6664E+00	-.1312E+01	-.2061E+01	0.1732E+01	-.1335E+01	-.2439E+01
12	2.6638	0.2433E+00	-.1655E+01	-.7648E+00	0.1270E+01	-.1769E+01	-.1146E+01
13	2.9060	-.1398E+00	-.1665E+01	0.6808E+00	0.8289E+00	-.1864E+01	0.3640E+00
14	3.1481	-.4070E+00	-.1343E+01	0.1979E+01	0.4940E+00	-.1602E+01	0.1803E+01
15	3.3903	-.5055E+00	-.7550E+00	0.2874E+01	0.3288E+00	-.1033E+01	0.2895E+01
16	3.6324	-.4166E+00	-.2001E-01	0.3196E+01	0.3645E+00	-.2673E+00	0.3427E+01
17	3.8746	-.1584E+00	0.7167E+00	0.2889E+01	0.5935E+00	0.5466E+00	0.3295E+01
18	4.1168	0.2183E+00	0.1311E+01	0.2019E+01	0.9707E+00	0.1252E+01	0.2527E+01
19	4.3589	0.6395E+00	0.1647E+01	0.7600E+00	0.1422E+01	0.1712E+01	0.1271E+01
20	4.6011	0.1023E+01	0.1662E+01	-.6399E+00	0.1861E+01	0.1838E+01	-.2264E+00
21	4.8433	0.1295E+01	0.1353E+01	-.1908E+01	0.2201E+01	0.1608E+01	-.1675E+01
22	5.0854	0.1403E+01	0.7832E+00	-.2800E+01	0.2378E+01	0.1067E+01	-.2793E+01
23	5.3276	0.1326E+01	0.6342E-01	-.3145E+01	0.2357E+01	0.3211E+00	-.3366E+01
24	5.5697	0.1080E+01	-.6657E+00	-.2877E+01	0.2143E+01	-.4839E+00	-.3283E+01
25	5.8119	0.7142E+00	-.1263E+01	-.2053E+01	0.1779E+01	-.1192E+01	-.2564E+01

11.19 The problem-dependent data to be used in the main program (which calls CDIFF), subroutine EXTFUN and output are given below.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=10.0*SIN(5.0*TIME)
      F(2)=0.0
      RETURN
      END
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=     2      NSTEP=    24      DELT= 0.24216267E+00

SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.5488E+00	0.1133E+01	0.9359E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0.4843	0.1291E+01	0.2666E+01	0.3301E+01	0.3219E-01	0.6645E-01	0.5488E+00
4	0.7265	0.1307E+01	0.1565E+01	-.1240E+02	0.1325E+00	0.2737E+00	0.1162E+01
5	0.9687	0.2964E+00	-.2054E+01	-.1749E+02	0.2784E+00	0.5084E+00	0.7765E+00
6	1.2108	-.9186E+00	-.4595E+01	-.3493E+01	0.3764E+00	0.5035E+00	-.8173E+00
7	1.4530	-.1279E+01	-.3252E+01	0.1458E+02	0.3322E+00	0.1110E+00	-.2424E+01
8	1.6951	-.6732E+00	0.5066E+00	0.1647E+02	0.1351E+00	-.4982E+00	-.2608E+01
9	1.9373	0.3324E-01	0.2709E+01	0.1722E+01	-.1332E+00	-.9609E+00	-.1214E+01
10	2.1795	0.1288E+00	0.1656E+01	-.1042E+02	-.3683E+00	-.1039E+01	0.5660E+00
11	2.4216	-.1236E+00	-.3238E+00	-.5933E+01	-.5094E+00	-.7768E+00	0.1602E+01
12	2.6638	0.8610E-02	-.2481E+00	0.6558E+01	-.5383E+00	-.3511E+00	0.1914E+01
13	2.9060	0.6165E+00	0.1528E+01	0.8111E+01	-.4404E+00	0.1424E+00	0.2162E+01
14	3.1481	0.9366E+00	0.1916E+01	-.4906E+01	-.2031E+00	0.6921E+00	0.2378E+01
15	3.3903	0.3481E+00	-.5540E+00	-.1549E+02	0.1368E+00	0.1192E+01	0.1749E+01
16	3.6324	-.7189E+00	-.3418E+01	-.8160E+01	0.4650E+00	0.1380E+01	-.1991E+00
17	3.8746	-.1185E+01	-.3166E+01	0.1024E+02	0.6420E+00	0.1043E+01	-.2579E+01
18	4.1168	-.5807E+00	0.2853E+00	0.1826E+02	0.5989E+00	0.2764E+00	-.3753E+01
19	4.3589	0.4129E+00	0.3300E+01	0.6635E+01	0.3812E+00	-.5385E+00	-.2976E+01
20	4.6011	0.8081E+00	0.2867E+01	-.1021E+02	0.9837E-01	-.1033E+01	-.1112E+01
21	4.8433	0.4651E+00	0.1077E+00	-.1259E+02	-.1602E+00	-.1118E+01	0.4146E+00
22	5.0854	0.1098E+00	-.1442E+01	-.2098E+00	-.3539E+00	-.9338E+00	0.1106E+01
23	5.3276	0.2596E+00	-.4244E+00	0.8612E+01	-.4582E+00	-.6152E+00	0.1525E+01
24	5.5697	0.5064E+00	0.8188E+00	0.1655E+01	-.4397E+00	-.1772E+00	0.2092E+01
25	5.8119	0.1089E+00	-.3112E+00	-.1099E+02	-.2885E+00	0.3504E+00	0.2265E+01

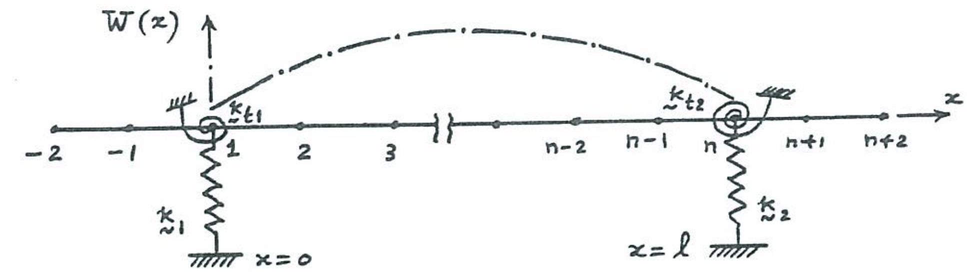
11.20 The problem-dependent data to be used in the main program (which calls CDIFF), subroutine EXTFUN and output are given below.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDD1(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XM1(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.25/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/2.0,0.0,0.0,1.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*TIME)
      RETURN
      END
```

```
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=    2      NSTEP=    24      DELT= 0.25000000E+00

SOLUTION:

STEP    TIME    X(I,1)    XD(I,1)    XDD(I,1)    X(I,2)    XD(I,2)    XDD(I,2)
```

1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7813E-01	0.0000E+00	0.2500E+01	0.1186E+01	0.2372E+01	0.1898E+02
3	0.5000	0.3720E+00	0.7440E+00	0.3452E+01	0.2834E+01	0.5668E+01	0.7381E+01
4	0.7500	0.9295E+00	0.1703E+01	0.4218E+01	0.3105E+01	0.3837E+01	-.2202E+02
5	1.0000	0.1663E+01	0.2582E+01	0.2816E+01	0.1517E+01	-.2633E+01	-.2974E+02
6	1.2500	0.2336E+01	0.2813E+01	-.9716E+00	-.2832E+00	-.6776E+01	-.3407E+01
7	1.5000	0.2709E+01	0.2092E+01	-.4790E+01	-.5484E+00	-.4131E+01	0.2456E+02
8	1.7500	0.2697E+01	0.7215E+00	-.6176E+01	0.4430E+00	0.1452E+01	0.2011E+02
9	2.0000	0.2362E+01	-.6938E+00	-.5147E+01	0.9807E+00	0.3058E+01	-.7259E+01
10	2.2500	0.1803E+01	-.1788E+01	-.3606E+01	0.3588E+00	-.1685E+00	-.1855E+02
11	2.5000	0.1084E+01	-.2557E+01	-.2549E+01	-.2105E+00	-.2382E+01	0.8438E+00
12	2.7500	0.3046E+00	-.2996E+01	-.9614E+00	0.5659E+00	0.4142E+00	0.2153E+02
13	3.0000	-.3400E+00	-.2847E+01	0.2152E+01	0.2052E+01	0.4524E+01	0.1135E+02
14	3.2500	-.6363E+00	-.1882E+01	0.5572E+01	0.2337E+01	0.3543E+01	-.1920E+02
15	3.5000	-.5110E+00	-.3421E+00	0.6746E+01	0.7394E+00	-.2625E+01	-.3013E+02
16	3.7500	-.8742E-01	0.1098E+01	0.4772E+01	-.1231E+01	-.7137E+01	-.5967E+01
17	4.0000	0.4318E+00	0.1886E+01	0.1531E+01	-.1764E+01	-.5007E+01	0.2301E+02
18	4.2500	0.9161E+00	0.2007E+01	-.5596E+00	-.9579E+00	0.5471E+00	0.2142E+02
19	4.5000	0.1325E+01	0.1786E+01	-.1206E+01	-.4066E+00	0.2715E+01	-.4080E+01
20	4.7500	0.1616E+01	0.1400E+01	-.1882E+01	-.8160E+00	0.2836E+00	-.1537E+02
21	5.0000	0.1710E+01	0.7696E+00	-.3165E+01	-.9849E+00	-.1157E+01	0.3850E+01
22	5.2500	0.1577E+01	-.7782E-01	-.3614E+01	0.4299E+00	0.2492E+01	0.2534E+02
23	5.5000	0.1332E+01	-.7549E+00	-.1802E+01	0.2808E+01	0.7586E+01	0.1542E+02
24	5.7500	0.1169E+01	-.8162E+00	0.1311E+01	0.4079E+01	0.7298E+01	-.1773E+02
25	6.0000	0.1198E+01	-.2684E+00	0.3071E+01	0.3241E+01	0.8649E+00	-.3374E+02

11.21



At $x=0$ (at node 1):

$$\frac{d}{dx}\left[EI\,\frac{d^2W(x)}{dx^2}\right] = EI\,\frac{d^3W}{dx^3} = -\,\underset{\sim}{k}_1\,W(x)$$

$$EI\,\frac{d^2W}{dx^2} = -\,\underset{\sim}{k}_{t1}\,\frac{dW(x)}{dx}$$

i.e. $$\frac{EI}{2h^3}\left(W_3 - 2W_2 + 2W_{-1} - W_{-2}\right) = -\,\underset{\sim}{k}_1 \cdot W_1$$

$$\frac{EI}{h^2}\left(W_2 - 2W_1 + W_{-1}\right) = -\,\underset{\sim}{k}_{t1}\cdot\frac{1}{2h}\left(W_2 - W_{-1}\right)$$

At $x=l$ (at node n):

$$\frac{d}{dx}\left[EI\,\frac{d^2W(x)}{dx^2}\right] = EI\,\frac{d^3W}{dx^3} = \underset{\sim}{k}_2\,W(x)$$

$$EI\,\frac{d^2W}{dx^2} = \underset{\sim}{k}_{t2}\,\frac{dW(x)}{dx}$$

i.e. $$\frac{EI}{2h^3}\left(W_{n+2} - 2W_{n+1} + 2W_{n-1} - W_{n-2}\right) = \underset{\sim}{k}_2 \cdot W_n$$

$$\frac{EI}{h^2}\left(W_{n+1} - 2W_n + W_{n-1}\right) = \underset{\sim}{k}_{t2}\cdot\frac{1}{2h}\left(W_{n+1} - W_{n-1}\right)$$

11.22 Given equations can be expressed as

$$\frac{d\vec{X}}{dt} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \frac{1}{2}(5 - 6x_1 + 2x_3) \\ x_4 \\ 20 \sin 5t + 2x_1 - 4x_3 \end{Bmatrix}$$

where

$$\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \text{original } x_1 \\ \text{original } \dot{x}_1 \\ \text{original } x_2 \\ \text{original } \dot{x}_2 \end{Bmatrix} \quad \text{and} \quad \vec{X}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

The main program which calls RK4, the subroutine FUN and the results are given.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C ======================================================================
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(25),X(25,4),XX(4),F(4),YI(4),YJ(4),YK(4),YL(4),
     2 UU(4)
      XX(1)=0.0
      XX(2)=0.0
      XX(3)=0.0
      XX(4)=0.0
      NEQ=4
      NSTEP=25
```

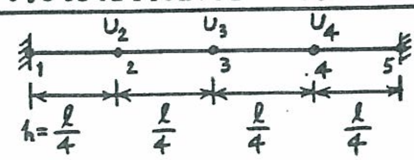
```
      DT=0.25
      T=0.0
      WRITE (60,10)
10    FORMAT (//,3X,5H   I   ,10H   TIME(I),7X,5H X(1),12X,5H X(2),
     2 12X,5H X(3),12X,5H X(4),/)
      DO 40 I=1,NSTEP
      CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (60,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,4(2X,E15.8))
40    CONTINUE
      STOP
      END
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=(5.0-6.0*X(1)+2.0*X(3))/2.0
      F(3)=X(4)
      F(4)=20.0*SIN(5.0*T)+2.0*X(1)-4.0*X(3)
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)	X(3)	X(4)
1	0.2500	0.76904297E-01	0.62070566E+00	0.24460433E+00	0.26932180E+01
2	0.5000	0.31453162E+00	0.13009652E+01	0.14515579E+01	0.64889636E+01
3	0.7500	0.74014312E+00	0.21077421E+01	0.29793470E+01	0.45641656E+01
4	1.0000	0.13591884E+01	0.27739763E+01	0.32607102E+01	-0.26126151E+01
5	1.2500	0.20711818E+01	0.27800636E+01	0.18916913E+01	-0.72935324E+01
6	1.5000	0.26687763E+01	0.18683126E+01	0.27935052E+00	-0.45067854E+01
7	1.7500	0.29524713E+01	0.35540020E+00	-0.78577667E-01	0.14555850E+01
8	2.0000	0.28473990E+01	-0.11584098E+01	0.60296357E+00	0.29186647E+01
9	2.2500	0.24058905E+01	-0.23112063E+01	0.90024042E+00	-0.96232796E+00
10	2.5000	0.17232550E+01	-0.30908332E+01	0.23876321E+00	-0.35336885E+01
11	2.7500	0.89089936E+00	-0.34831977E+01	-0.34061307E+00	-0.26938438E+00
12	3.0000	0.34346521E-01	-0.32377021E+01	0.25854278E+00	0.46597433E+01
13	3.2500	-0.65400219E+00	-0.21399481E+01	0.14909362E+01	0.39903281E+01
14	3.5000	-0.98161954E+00	-0.43759835E+00	0.17313157E+01	-0.24955246E+01
15	3.7500	-0.87797028E+00	0.11996664E+01	0.39478755E+00	-0.72439308E+01
16	4.0000	-0.43708453E+00	0.22119966E+01	-0.12451535E+01	-0.47495975E+01
17	4.2500	0.16884252E+00	0.25470126E+01	-0.16587874E+01	0.13506827E+01
18	4.5000	0.80284268E+00	0.24732747E+01	-0.92830122E+00	0.34720352E+01
19	4.7500	0.13844182E+01	0.21252785E+01	-0.39376926E+00	0.36745048E+00
20	5.0000	0.18346303E+01	0.14129710E+01	-0.64064902E+00	-0.15653036E+01
21	5.2500	0.20612290E+01	0.37833023E+00	-0.64773440E+00	0.23636723E+01
22	5.5000	0.20297880E+01	-0.56775379E+00	0.71066928E+00	0.81037455E+01
23	5.7500	0.18273093E+01	-0.94346517E+00	0.28840952E+01	0.79655704E+01
24	6.0000	0.16137744E+01	-0.69621664E+00	0.40854921E+01	0.10114455E+01
25	6.2500	0.14960963E+01	-0.25771955E+00	0.34342446E+01	-0.54408941E+01

11.23 Equation of motion

$$\frac{d^2 U}{dx^2} + \alpha^2 U = 0$$

At grid point i, this becomes



$$U_{i+1} - 2\,U_i + U_{i-1} + \alpha^2 h^2\, U_i = 0 \quad \text{or} \quad U_{i+1} - (2-\lambda)\, U_i + U_{i-1} = 0 \quad \text{--- } (E_1)$$

where $\lambda = \alpha^2 h^2 = \rho\, l^2 \omega^2 / (16 E)$.

Eq. (E_1), when applied to nodes 2, 3 and 4, gives

$$\left.\begin{aligned} U_3 - (2-\lambda)\, U_2 + U_1 &= 0 \\ U_4 - (2-\lambda)\, U_3 + U_2 &= 0 \\ U_5 - (2-\lambda)\, U_4 + U_3 &= 0 \end{aligned}\right\} \quad \text{--- } (E_2)$$

With boundary conditions $U_1 = U_5 = 0$, (E_2) becomes

$$\underset{3\times3}{[A]}\,\underset{3\times1}{\vec{U}} - \lambda\,\underset{3\times3}{[I]}\,\underset{3\times1}{\vec{U}} = \underset{3\times1}{\vec{0}} \quad \text{where} \quad [A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \ [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \vec{U} = \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

The solution of this eigenvalue problem is found using Program 9.F. The results are

$\lambda_1 = 0.585786$, $\quad \lambda_2 = 2.0$, $\quad \lambda_3 = 3.41421$

$\omega_1 = 3.06147\sqrt{\dfrac{E}{\rho l^2}}$, $\quad \omega_2 = 5.65685\sqrt{\dfrac{E}{\rho l^2}}$, $\quad \omega_3 = 7.39103\sqrt{\dfrac{E}{\rho l^2}}$

$$\vec{U}^{(1)} = \begin{Bmatrix} 0.5 \\ 0.707 \\ 0.5 \end{Bmatrix}, \quad \vec{U}^{(2)} = \begin{Bmatrix} 0.707 \\ 0 \\ -0.707 \end{Bmatrix}, \quad \vec{U}^{(3)} = \begin{Bmatrix} 0.5 \\ -0.707 \\ 0.5 \end{Bmatrix}$$

11.24 (i) Forced longitudinal vibration:

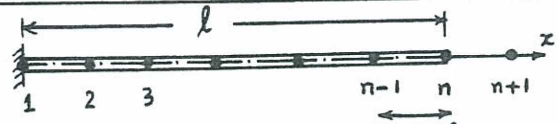


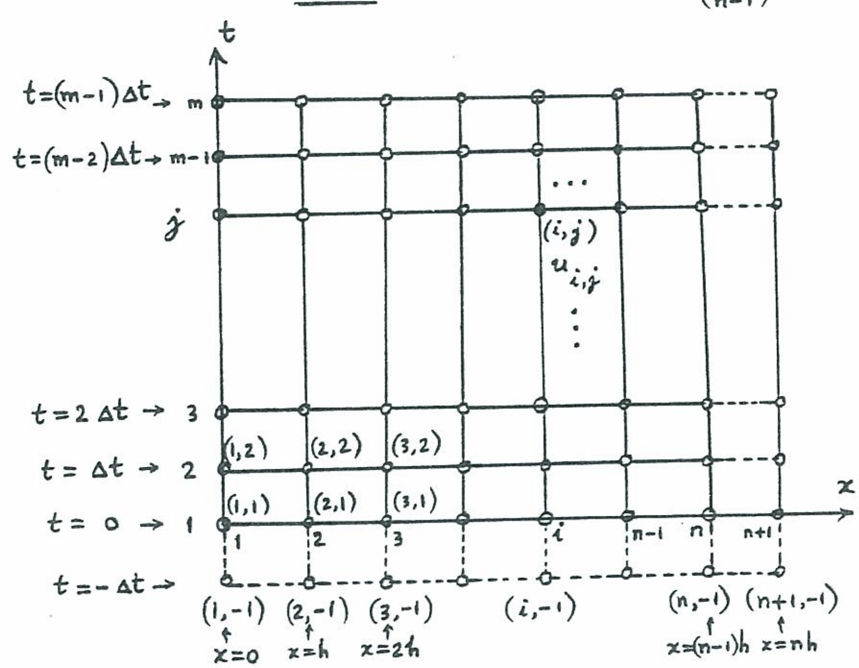
Fig. 1

Equation is

$$EA\,\frac{\partial^2 u}{\partial x^2} + f = \rho A\,\frac{\partial^2 u}{\partial t^2} \quad \text{--- } (E_1)$$

where $u = u(x,t)$.

Let the solution be required for $0 \le t \le T$. Set up the finite difference grid along x and t axes as shown in Fig. 2. Note that imaginary grid points are set up at $x = nh$ and $t = -\Delta t$ so that free boundary conditions can be applied at $x = (n-1)h$



$h = \dfrac{l}{n-1}$, $\quad \Delta t = \dfrac{T}{m-1}$

Fig. 2

and initial conditions can be applied at $t=0$.

Let $u_{i,j}$ = value of u at the grid point (i,j).

Eq. (E_1) can be approximated at grid point (i,j) as

$$EA\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right)+f_{i,j}=\rho A\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta t^2}\right) \quad ---(E_2)$$

where $f_{i,j} = f(x=x_i,\ t=t_j)$.

The procedure to set up the finite difference equations is:

(1). Apply (E_2) at grid points $(2,1),(3,1),\ldots,(n,1);(2,2),(3,2),\ldots,(n,2);\ldots;(2,m-1),(3,m-1),\ldots,(n,m-1)$. This gives $(n-1 \times m-1)$ equations in the unknowns associated with all the grid points shown in Fig. 2.

(2). Apply the boundary conditions.

At $x=0$, $u(x,t)=0 \Rightarrow u_{1,1}=u_{1,2}=\cdots=u_{1,m-1}=u_{1,m}=0$

At $x=l$, $\frac{\partial u}{\partial x}=0 \Rightarrow u_{n+1,j}=u_{n-1,j}$ for $j=1,2,\ldots,m$

(3). Apply the initial conditions.

At $t=0$, let $u(x,t)=U_0(x)$. Then $u_{i,1}=U_0(x=x_i)$ for $i=1,2,\ldots,n+1$

At $t=0$, let $\frac{\partial u}{\partial t}(x,t)=\dot{U}_0(x)$. Then $\frac{u_{i,2}-u_{i,-1}}{2\,\Delta t}=\dot{U}_0(x=x_i)$

i.e. $u_{i,-1}=u_{i,2}-2\,\Delta t\,\dot{U}_0(x_i)$ for $i=1,2,\ldots,n+1$.

(4). Express the resulting equations in matrix form. There will be as many linear equations as there are unknowns.

(ii) Free vibration:

Equation is $\frac{d^2U}{dx^2}+\alpha^2 U=0$ where $\alpha^2=\frac{\rho\omega^2}{E}$ $\quad ---(E_3)$

At node i, (E_3) becomes (Fig. 1)

$$U_{i+1}-2U_i+U_{i-1}+\lambda U_i=0 \quad \text{where } \lambda=h^2\alpha^2 \quad ---(E_4)$$

Applying (E_2) at nodes $2,3,\ldots,n$ gives

$$\left.\begin{aligned}
&U_3-(2-\lambda)U_2+U_1=0\\
&U_4-(2-\lambda)U_3+U_2=0\\
&\vdots\\
&U_n-(2-\lambda)U_{n-1}+U_{n-2}=0\\
&U_{n+1}-(2-\lambda)U_n+U_{n-1}=0
\end{aligned}\right\} \quad ---(E_5)$$

Boundary conditions are $U_1=0$ ($U=0$ at $x=0$) and $U_{n+1}=U_{n-1}$ $\left(\frac{dU}{dx}=0 \text{ at } x=l\right)$ $\quad ---(E_6)$

(E_5) and (E_6) give

$$U_3-(2-\lambda)U_2=0$$
$$U_4-(2-\lambda)U_3+U_2=0$$

$\vdots$

$$U_n - (2-\lambda) U_{n-1} + U_{n-2} = 0 \qquad \text{--- } (E_7)$$

$$-(2-\lambda) U_n + 2 U_{n-1} = 0$$

For $n = 4$, (E_7) can be expressed as

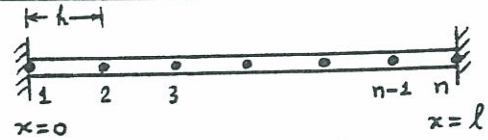
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{--- } (E_8)$$

Frequency equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 2 = 0$ --- (E_9)

Roots are $\lambda_1 = 0.267944$, $\lambda_2 = 2.0$, $\lambda_3 = 3.7320 50$

$\therefore\ \omega_1 = 1.55289 \sqrt{\dfrac{E}{\rho l^2}}$, $\omega_2 = 4.24264 \sqrt{\dfrac{E}{\rho l^2}}$, $\omega_3 = 5.79555 \sqrt{\dfrac{E}{\rho l^2}}$

11.25 Forced torsional vibration:



Equation $\quad GJ \dfrac{\partial^2\theta}{\partial x^2} + f = J_0 \dfrac{\partial^2\theta}{\partial t^2}$ --- (E_1)

Let the solution be required for $0 \le t \le T$. $\quad h = \dfrac{l}{n-1}$, $\Delta t = \dfrac{T}{m-1}$

Set up a finite difference grid along x (with n points) and along t (with m points) similar to Fig. 2 of problem 10.24.

Let $\theta_{i,j} = \theta(x = x_i, t = t_j)$ and $f_{i,j} = f(x = x_i, t = t_j)$

Eq. (E_1) at grid point (i, j) gives

$$GJ \left(\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{h^2} \right) + f_{i,j} = J_0 \left(\frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{\Delta t^2} \right) \qquad \text{--- } (E_2)$$

The procedure to set up the finite difference grid is:

(1). Apply Eq. (E_2) at grid points $(2,1), (3,1), \ldots, (n-1, 1); (2,2), (3,2), \ldots, (n-1, 2); \ldots; (2, m-1), (3, m-1), \ldots, (n-1, m-1)$. This gives $(n-2) \times (m-1)$ equations in the $(n) \times (m+1)$ unknowns.

(2). Apply boundary conditions:
At $x = 0$, $\theta(x,t) = 0$. Hence $\theta_{1,j} = 0$ for $j = 1, 2, \ldots, m$
At $x = l$, $\theta(x,t) = 0$. Hence $\theta_{n,j} = 0$ for $j = 1, 2, \ldots, m$

(3). Apply initial conditions.
At $t = 0$, let $\theta(x,t) = \Theta_0(x) \Rightarrow \theta_{i,1} = \Theta_0(x = x_i)$ for $i = 1, 2, \ldots, n$
At $t = 0$, let $\dfrac{\partial\theta}{\partial t}(x,t) = \dot{\Theta}_0(x) \Rightarrow \theta_{i,-1} = \Theta_{i,2} - 2\,\Delta t\ \dot{\Theta}_0(x_i)$ for $i = 1, 2, \ldots, n$.

(4) Express the resulting equations in matrix form. There will be $(n-2) \times m$ equations in $(n-2) \times m$ unknowns.

11.26 Applying Eq. (11.50) to mesh points 2, 3 and 4 gives

$$\left.\begin{aligned} W_4 - 4W_3 + (6-\lambda)W_2 - 4W_1 + W_{-1} &= 0 \\ W_5 - 4W_4 + (6-\lambda)W_3 - 4W_2 + W_1 &= 0 \\ W_6 - 4W_5 + (6-\lambda)W_4 - 4W_3 + W_2 &= 0 \end{aligned}\right\} \qquad (E_1)$$

Boundary conditions are $W = \frac{dW}{dx} = 0$ at $x=0$ and $x=\ell$

i.e. $W_1 = 0$, $W_{-1} = W_2$; $W_5 = 0$, $W_6 = W_4$

Eq. (E_1) becomes, after applying boundary conditions,

$$\begin{bmatrix} 7 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 7 \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E_2)$$

Solution (obtained using Program 9.F) is

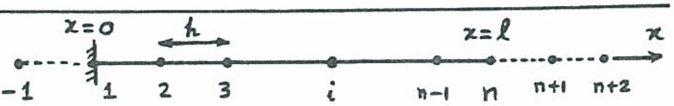
$\lambda_1 = 1.25544$, $\lambda_2 = 6.0$, $\lambda_3 = 12.7446$

$\omega_1 = 17.9274 \sqrt{\frac{EI}{\rho A \ell^4}}$, $\omega_2 = 39.1918 \sqrt{\frac{EI}{\rho A \ell^4}}$, $\omega_3 = 57.1193 \sqrt{\frac{EI}{\rho A \ell^4}}$

$$\vec{W}^{(1)} = \begin{Bmatrix} 0.4544 \\ 0.7662 \\ 0.4544 \end{Bmatrix}, \quad \vec{W}^{(2)} = \begin{Bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{Bmatrix}, \quad \vec{W}^{(3)} = \begin{Bmatrix} 0.5418 \\ -0.6426 \\ 0.5418 \end{Bmatrix}$$

11.27 Equation:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \qquad (E_1)$$



$h = \left(\frac{\ell}{n-1}\right)$, $\Delta t = \frac{T}{m-1}$

Let the solution be required for $0 \le t \le T$.

Set up a finite difference grid along x (grid points $-1, 1, 2, \ldots, n+2$) and along t (grid points $-1, 1, 2, \ldots, m$).

Imaginary grid points are located at $x = -h$, $x = \ell + h$ and $x = \ell + 2h$, and at $t = -\Delta t$.

Let $w_{i,j} = w(x = x_i,\ t = t_j)$ and $f_{i,j} = f(x = x_i,\ t = t_j)$.

Eq. (E_1) at grid point (i,j) gives

$$EI\left(\frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{h^4}\right) + \rho A\left(\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2}\right)$$
$$= f_{i,j} = f_0 \cos\{\omega(j-1)\Delta t\} \qquad (E_2)$$

The procedure to set up the finite difference equations is:

(1) Apply (E_2) at grid points $(2,1), (3,1), \ldots, (n,1)$; $(2,2), (3,2), \ldots, (n,2)$; $\ldots$; $(2, m-1), (3, m-1), \ldots, (n, m-1)$. This gives $(n-1) \times (m-1)$ equations in the unknowns associated with all grid points.

(2). Apply boundary conditions:

At $x=0$, $w = \frac{\partial w}{\partial x} = 0 \Rightarrow \quad w_{1,j} = 0, \quad w_{-1,j} = w_{2,j} \quad$ for $j = 1, 2, \ldots, m$

At $x = l$, $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \Rightarrow \quad w_{n+1,j} - 2\, w_{n,j} + w_{n-1} = 0,$

$$w_{n+2,j} - 2\, w_{n+1,j} + 2\, w_{n-1,j} - w_{n-2,j} = 0$$

$$\Rightarrow \quad w_{n+1,j} = 2\, w_{n,j} - w_{n-1,j}, \quad w_{n+2,j} = 4\, w_{n,j} - 4\, w_{n-1,j} + w_{n-2,j}$$

for $j = 1, 2, \ldots, m$

(3). Apply initial conditions:

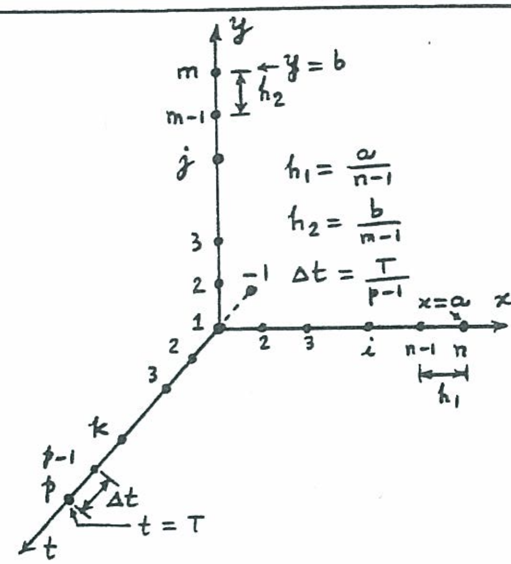
At $t=0$, let $w(x,t) = W_0(x) \Rightarrow w_{i,1} = W_0(x = x_i)$ for $i = 1, 2, \ldots, n$

At $t=0$, let $\frac{\partial w}{\partial t}(x,t) = \dot{W}_0(x) \Rightarrow w_{i,-1} = w_{i,2} - 2\,\Delta t\, \dot{W}_0(x_i)$ for $i = 1, 2, \ldots, n$

(4). Express the resulting equations in matrix form.

11.28 Equation:

$$P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f = \rho\, \frac{\partial^2 w}{\partial t^2} \quad \text{--- } (E_1)$$



Set up a finite difference grid with n, m and p points along x, y and t axes. Introduce an imaginary grid point at $t = -\Delta t$ to apply the initial conditions.

Let $w_{i,j,k} = w(x = x_i,\ y = y_j,\ t = t_k)$

and $f_{i,j,k} = f(x = x_i,\ y = y_j,\ t = t_k)$.

(E_1) gives at grid point (i,j,k):

$$P\left(\frac{w_{i+1,j,k} - 2\, w_{i,j,k} + w_{i-1,j,k}}{h_1^2}\right) + P\left(\frac{w_{i,j+1,k} - 2\, w_{i,j,k} + w_{i,j-1,k}}{h_2^2}\right)$$

$$+ f_{i,j,k} = \rho\left(\frac{w_{i,j,k+1} - 2\, w_{i,j,k} + w_{i,j,k-1}}{\Delta t^2}\right) \quad \text{--- } (E_2)$$

The following procedure is used to derive the final equations:

(1). Apply Eq. (E_2) at grid points (i,j,k) for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p-1$. This gives $n \times m \times (p-1)$ equations in $n \times m \times (p+1)$ unknowns.

(2). Apply boundary conditions:

At $x = 0$, $w(x,y,t) = 0 \Rightarrow w_{1,j,k} = 0$ for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p$

At $x = a$, $w(x,y,t) = 0 \Rightarrow w_{n,j,k} = 0$ for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p$

At $y=0$, $w(x,y,t)=0 \Rightarrow w_{i,1,k}=0$ for $i=1,2,\ldots,n$; $k=1,2,\ldots,p$

At $y=b$, $w(x,y,t)=0 \Rightarrow w_{i,m,k}=0$ for $i=1,2,\ldots,n$; $k=1,2,\ldots,p$

(3). Apply initial conditions:

At $t=0$, let $w(x,y,t)=W_0(x,y) \Rightarrow w_{i,j,1}=W_0(x_i,y_j)$ for $i=1,2,\ldots,n$; $j=1,2,\ldots,m$

At $t=0$, let $\frac{\partial w}{\partial t}(x,y,t)=\dot{W}_0(x,y) \Rightarrow w_{i,j,-1}=w_{i,j,2}-2\,\Delta t\,\dot{W}_0(x_i,y_j)$

for $i=1,2,\ldots,n$; $j=1,2,\ldots,m$

(4). Express the resulting equations in matrix form. There will be $(n-2)\times(m-2)\times p$ equations in the same number of unknowns.

11.29 Equations of motion:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k & -k \\ -k & k+k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\ddot{\vec{x}}+\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\dot{\vec{x}}+\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}\vec{x}=\begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \qquad (1)$$

Equations (1) can be rewritten as

$$\ddot{x}_1+2\dot{x}_1-2\dot{x}_2+6x_1-2x_2=0$$

$$2\ddot{x}_2-2\dot{x}_1+2\dot{x}_2-2x_1+8x_2=10$$

or

$$\ddot{x}_1=-2\dot{x}_1+2\dot{x}_2-6x_1+2x_2 \qquad (2)$$

$$\ddot{x}_2=\dot{x}_1-\dot{x}_2+x_1-4x_2+5 \qquad (3)$$

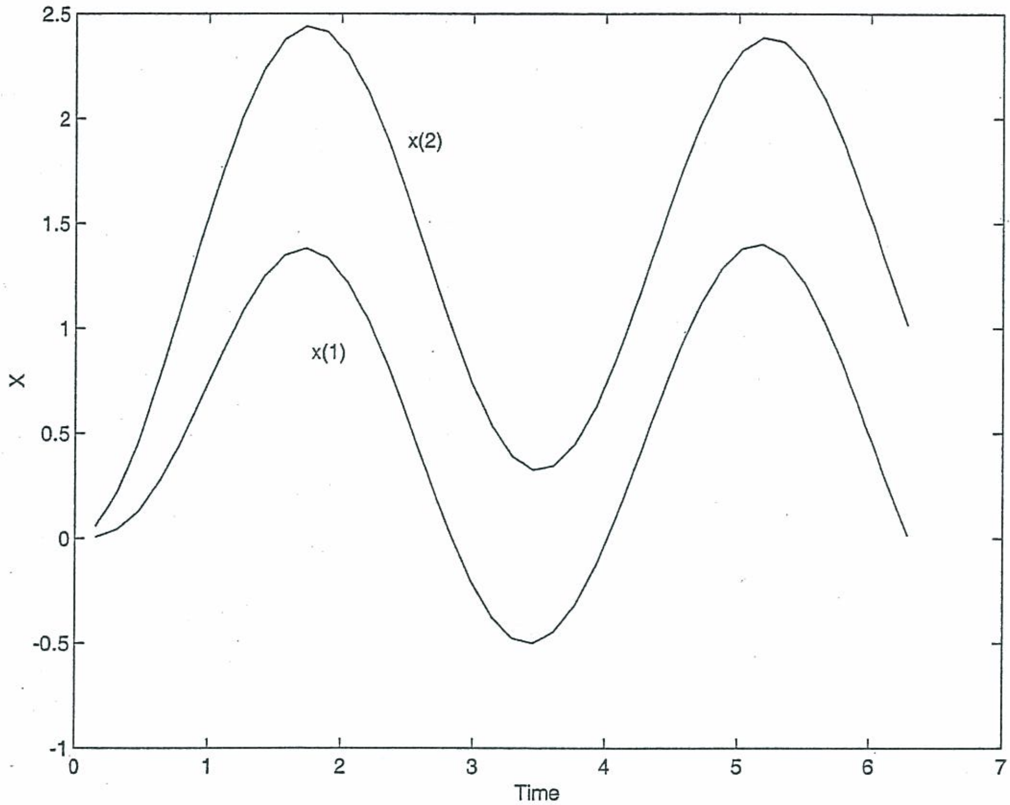
Using

$$\vec{y}=\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}=\begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} \quad \text{and} \quad \vec{Y}(t=0)=\vec{0}$$

Eqs. (2) and (3) can be expressed as

$$\frac{d\vec{Y}}{dt}=\begin{Bmatrix} y_2 \\ -2y_2+2y_4-6y_1+2y_3 \\ y_4 \\ y_2-y_4+y_1-4y_3+5 \end{Bmatrix}=\vec{f}(t)$$

Using $n = 4$, $xx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$, $dt = \frac{\pi}{20}$, Program 14.m can be used to find the solution.




```
Results of Ex11_29
***********************
>> program14
 I     Time(I)       x(1)        dx(1)        x(2)        dx(2)

 1 1.5708e-001 5.9523e-003 1.0944e-001 5.8328e-002 7.2086e-001
 2 3.1416e-001 4.2823e-002 3.7884e-001 2.1918e-001 1.3032e+000
 3 4.7124e-001 1.2857e-001 7.1703e-001 4.5992e-001 1.7366e+000
 4 6.2832e-001 2.6725e-001 1.0403e+000 7.5658e-001 2.0145e+000
 5 7.8540e-001 4.5102e-001 1.2817e+000 1.0845e+000 2.1343e+000
 6 9.4248e-001 6.6316e-001 1.3960e+000 1.4189e+000 2.0980e+000
 7 1.0996e+000 8.8165e-001 1.3607e+000 1.7358e+000 1.9137e+000
 8 1.2566e+000 1.0827e+000 1.1755e+000 2.0132e+000 1.5971e+000
 9 1.4137e+000 1.2440e+000 8.5854e-001 2.2318e+000 1.1714e+000
10 1.5708e+000 1.3472e+000 4.4240e-001 2.3770e+000 6.6715e-001
11 1.7279e+000 1.3799e+000 -3.0920e-002 2.4392e+000 1.2076e-001
12 1.8850e+000 1.3369e+000 -5.1500e-001 2.4149e+000 -4.2760e-001
13 2.0420e+000 1.2200e+000 -9.6373e-001 2.3069e+000 -9.3696e-001
14 2.1991e+000 1.0382e+000 -1.3356e+000 2.1246e+000 -1.3687e+000
15 2.3562e+000 8.0625e-001 -1.5969e+000 1.8828e+000 -1.6895e+000
16 2.5133e+000 5.4353e-001 -1.7248e+000 1.6009e+000 -1.8747e+000
17 2.6704e+000 2.7201e-001 -1.7082e+000 1.3017e+000 -1.9098e+000
18 2.8274e+000 1.4410e-002 -1.5489e+000 1.0090e+000 -1.7921e+000
19 2.9845e+000 -2.0779e-001 -1.2607e+000 7.4627e-001 -1.5312e+000
20 3.1416e+000 -3.7614e-001 -8.6809e-001 5.3442e-001 -1.1482e+000
21 3.2987e+000 -4.7674e-001 -4.0407e-001 3.9037e-001 -6.7391e-001
22 3.4558e+000 -5.0135e-001 9.2720e-002 3.2551e-001 -1.4669e-001
23 3.6128e+000 -4.4804e-001 5.8124e-001 3.4484e-001 3.9092e-001
```

```
   24 3.7699e+000 -3.2129e-001 1.0214e+000 4.4658e-001 8.9549e-001
   25 3.9270e+000 -1.3159e-001 1.3773e+000 6.2229e-001 1.3263e+000
   26 4.0841e+000 1.0545e-001 1.6202e+000 8.5755e-001 1.6487e+000
   27 4.2412e+000 3.7042e-001 1.7305e+000 1.1332e+000 1.8367e+000
   28 4.3982e+000 6.4167e-001 1.6997e+000 1.4267e+000 1.8755e+000
   29 4.5553e+000 8.9711e-001 1.5307e+000 1.7143e+000 1.7622e+000
   30 4.7124e+000 1.1160e+000 1.2378e+000 1.9728e+000 1.5063e+000
   31 4.8695e+000 1.2807e+000 8.4493e-001 2.1811e+000 1.1290e+000
   32 5.0265e+000 1.3779e+000 3.8448e-001 2.3227e+000 6.6103e-001
   33 5.1836e+000 1.3999e+000 -1.0601e-001 2.3860e+000 1.4069e-001
   34 5.3407e+000 1.3451e+000 -5.8663e-001 2.3663e+000 -3.8975e-001
   35 5.4978e+000 1.2182e+000 -1.0184e+000 2.2653e+000 -8.8723e-001
   36 5.6549e+000 1.0296e+000 -1.3664e+000 2.0915e+000 -1.3115e+000
   37 5.8119e+000 7.9480e-001 -1.6025e+000 1.8590e+000 -1.6283e+000
   38 5.9690e+000 5.3301e-001 -1.7080e+000 1.5869e+000 -1.8122e+000
   39 6.1261e+000 2.6552e-001 -1.6747e+000 1.2974e+000 -1.8486e+000
   40 6.2832e+000 1.4057e-002 -1.5055e+000 1.0141e+000 -1.7349e+000
%=======================================================================
%
% Program14.m
% Main program for calling the subroutine RK4
%
%=======================================================================
% Run "Program14" in MATLAB command window. Progrm14.m, rk4.m, and fun.m
% should be in the same folder,and set the Matalb path to this folder
% following 5 lines contain problem-dependent data
format long
xx=[0 0 0 0];
neq=4;
nstep=40;
dt=pi/20;
t=0;
%end of problem-dependent data
fprintf(' I    Time(I)     x(1)      dx(1)       x(2)        dx(2) \n\n');
for i=1:nstep
   [xx,f,t]=rk4(t,dt,neq,xx);
   time(i)=t;
   for j=1:neq
      x(i,j)=xx(j);
   end
   fprintf('%2.0f %8.4e %8.4e %8.4e %8.4e %8.4e\n',i,time(i),x(i,1:neq));
end
plot(time', x(1:40,1));
gtext('x(1)');
hold on;
plot(time', x(1:40,3));
gtext('x(2)');
xlabel('Time');
ylabel('X');
%=================================================================
%
% Function rk4.m
%
%=================================================================
function [xx,f,t]=rk4(t,dt,n,xx)
[xi]=fun(xx,n,t);
for i=1:n
   uu(i)=xx(i)+.5*dt*xi(i);
end
```

```
tn=t+0.5*dt;
[xj]=fun(uu,n,tn);
for i=1:n
   uu(i)=xx(i)+.5*dt*xj(i);
end
[xk]=fun(uu,n,tn);
for i=1:n
   uu(i)=xx(i)+dt*xk(i);
end
tn=t+dt;
[xl]=fun(uu,n,tn);
for i=1:n
   f(i)=xl(i);
   xx(i)=xx(i)+(xi(i)+2*xj(i)+2*xk(i)+xl(i))*dt/6;
end
t=t+dt;
%=====================================================================
%
% Function fun.m
%
%=====================================================================
function [f]=fun(x,n,t)
f(1)=x(2);
f(2)=-2*x(2) + 2*x(4) - 6*x(1) + 2*x(3);
f(3) = x(4);
f(4) = x(2) - x(4) + x(1) - 4*x(3) + 5;
```

11.30 Equations of motion: $\ddot{x}_1 + 6x_1 - 2x_2 = 10\sin 5t$

$$2\ddot{x}_2 - 2x_1 + 8x_2 = 0$$

or

$$\ddot{x}_1 = -6x_1 + 2x_2 + 10\sin 5t$$

$$\ddot{x}_2 = x_1 - 4x_2$$

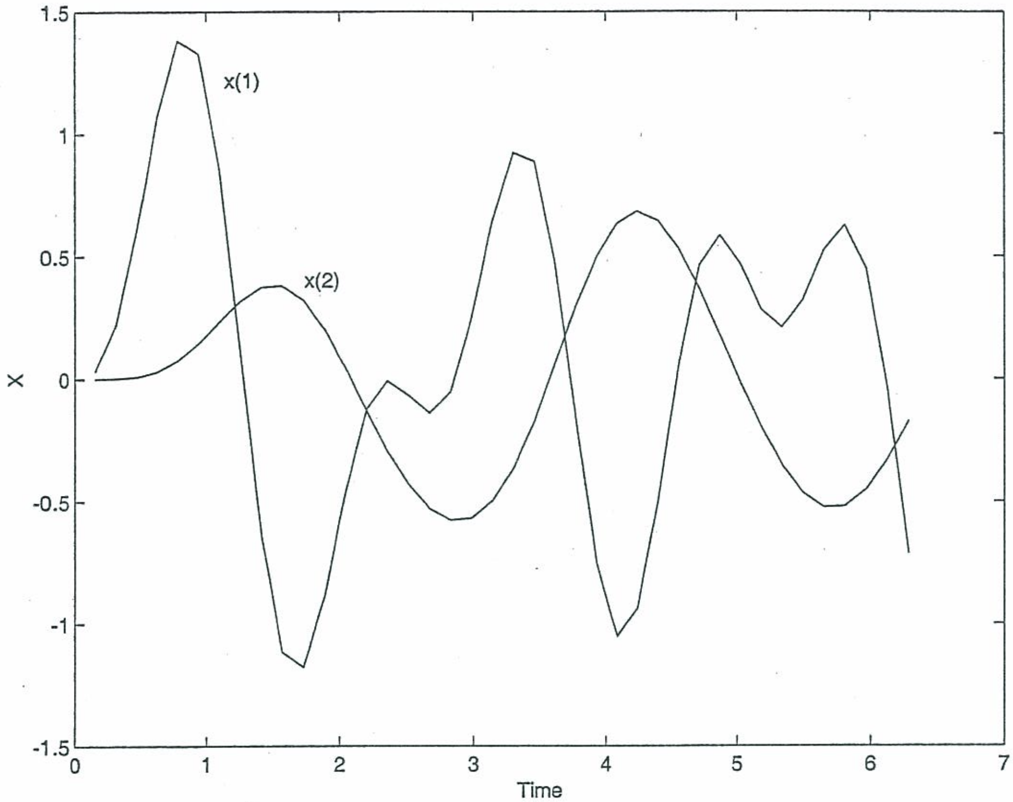
Problem to be solved is:

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -6y_1 + 2y_3 + 10\sin 5t \\ y_4 \\ y_1 - 4y_3 \end{Bmatrix} = \vec{f}$$

with

$$\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} \text{ and } \vec{Y}(0) = \vec{0}$$

Program14.m is used to solve these equations with $n = 4$, $xx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$ and $dt = \frac{\pi}{20}$ (40 steps used).



Results of Ex11_30

```
*********************************
>> program14
```

I	Time(I)	x(1)	dx(1)	x(2)	dx(2)
1	1.5708e-001	3.1474e-002	5.7845e-001	0.0000e+000	1.2360e-003
2	3.1416e-001	2.2179e-001	1.8901e+000	1.1243e-003	1.8174e-002
3	4.7124e-001	6.1079e-001	2.9264e+000	8.0316e-003	7.9050e-002
4	6.2832e-001	1.0743e+000	2.7210e+000	2.9302e-002	2.0111e-001
5	7.8540e-001	1.3821e+000	9.7131e-001	7.3678e-002	3.6666e-001
6	9.4248e-001	1.3282e+000	-1.7201e+000	1.4387e-001	5.1794e-001
7	1.0996e+000	8.5623e-001	-4.1349e+000	2.3154e-001	5.7695e-001
8	1.2566e+000	1.0705e-001	-5.0999e+000	3.1685e-001	4.8144e-001
9	1.4137e+000	-6.4233e-001	-4.1400e+000	3.7374e-001	2.1824e-001
10	1.5708e+000	-1.1164e+000	-1.7473e+000	3.7893e-001	-1.6442e-001
11	1.7279e+000	-1.1780e+000	8.9242e-001	3.2064e-001	-5.7311e-001
12	1.8850e+000	-8.8870e-001	2.5560e+000	2.0294e-001	-9.0595e-001
13	2.0420e+000	-4.5992e-001	2.6458e+000	4.3964e-002	-1.0910e+000
14	2.1991e+000	-1.2531e-001	1.4788e+000	-1.3077e-001	-1.1074e+000
15	2.3562e+000	-9.7785e-003	4.2490e-002	-2.9627e-001	-9.7979e-001
16	2.5133e+000	-7.0457e-002	-6.2931e-001	-4.3343e-001	-7.5366e-001
17	2.6704e+000	-1.4170e-001	-8.8991e-002	-5.2993e-001	-4.6650e-001
18	2.8274e+000	-5.4399e-002	1.2595e+000	-5.7769e-001	-1.3404e-001
19	2.9845e+000	2.4398e-001	2.4172e+000	-5.6974e-001	2.4201e-001
20	3.1416e+000	6.4251e-001	2.4157e+000	-4.9981e-001	6.5101e-001
21	3.2987e+000	9.2294e-001	9.3611e-001	-3.6547e-001	1.0521e+000
22	3.4558e+000	8.8747e-001	-1.4469e+000	-1.7347e-001	1.3711e+000
23	3.6128e+000	4.8295e-001	-3.5521e+000	5.6379e-002	1.5210e+000
24	3.7699e+000	-1.5268e-001	-4.2469e+000	2.9198e-001	1.4382e+000
25	3.9270e+000	-7.5178e-001	-3.0953e+000	4.9531e-001	1.1148e+000
26	4.0841e+000	-1.0537e+000	-6.2016e-001	6.3242e-001	6.0943e-001
27	4.2412e+000	-9.4005e-001	1.9747e+000	6.8277e-001	2.9645e-002

```
 28 4.3982e+000 -4.9282e-001 3.4645e+000 6.4407e-001 -5.0708e-001
 29 4.5553e+000 5.7387e-002 3.2671e+000 5.3052e-001 -9.1308e-001
 30 4.7124e+000 4.6138e-001 1.7307e+000 3.6614e-001 -1.1528e+000
 31 4.8695e+000 5.8458e-001 -1.1398e-001 1.7658e-001 -1.2380e+000
 32 5.0265e+000 4.6760e-001 -1.1837e+000 -1.6504e-002 -1.2032e+000
 33 5.1836e+000 2.8161e-001 -9.8207e-001 -1.9670e-001 -1.0768e+000
 34 5.3407e+000 2.0842e-001 1.2869e-001 -3.5051e-001 -8.6708e-001
 35 5.4978e+000 3.1891e-001 1.1786e+000 -4.6447e-001 -5.6937e-001
 36 5.6549e+000 5.2369e-001 1.2108e+000 -5.2494e-001 -1.8930e-001
 37 5.8119e+000 6.2652e-001 -9.9538e-002 -5.2141e-001 2.3595e-001
 38 5.9690e+000 4.4921e-001 -2.2015e+000 -4.5213e-001 6.3386e-001
 39 6.1261e+000 -4.6585e-002 -3.9510e+000 -3.2835e-001 9.1654e-001
 40 6.2832e+000 -7.1472e-001 -4.2611e+000 -1.7399e-001 1.0159e+000
%==================================================================
%
% Function fun.m
%
%==================================================================
function [f]=fun(x,n,t)
f(1)=x(2);
f(2)= - 6*x(1) + 2*x(3) + 10*sin(5*t);
f(3) = x(4);
f(4) = x(1) - 4*x(3);
```

11.31 Equations of motion:

$$[m]\ \ddot{\vec{x}} + [c]\ \dot{\vec{x}} + [k]\ \vec{x} = \vec{F}(t)$$

with $[m] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $[c] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[k] = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$,

$$\vec{F}(t) = \begin{Bmatrix} 5 \\ 20 \sin 5t \end{Bmatrix}$$

Solution using Program 15.m :

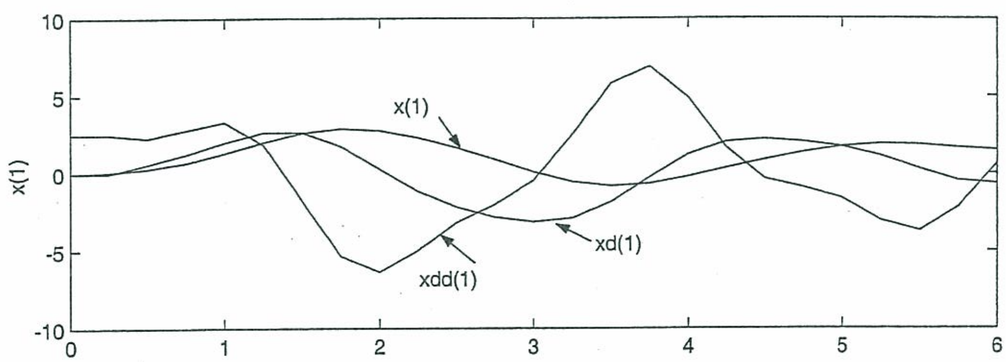
$n = 2$, $m = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

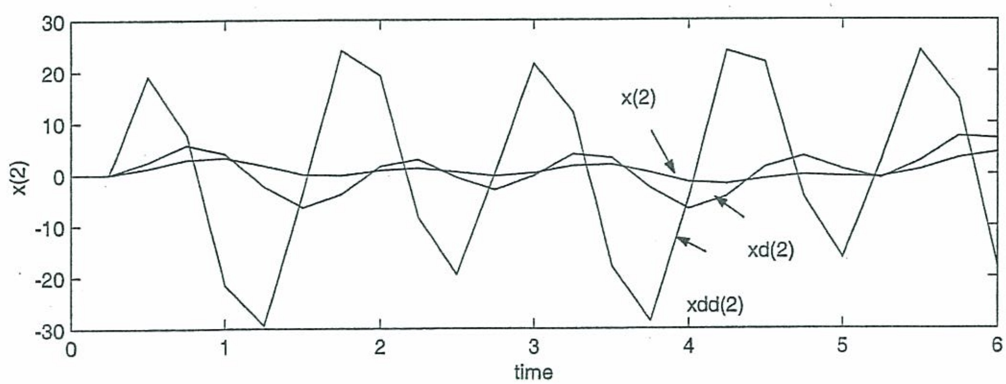
$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, nstep = 24, delt = 0.25

In subprogram: $\vec{F} = \begin{Bmatrix} 5 \\ 20 \sin 5t \end{Bmatrix}$

```
Results of Ex11_31
*******************************
>> program15
 Solution by central difference method

 Given data:

 n=  2      nstp= 24      delt=2.500000e-001

 Solution:

step time       x(i,1)      xd(i,1)       xdd(i,1)       x(i,2)          xd(
i,2)    xdd(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
 2 0.2500 7.8125e-002 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
```

```
 3 0.5000 2.9785e-001 5.9570e-001 2.2656e+000 1.1960e+000 2.3920e+000
1.9136e+001
 4 0.7500 6.9273e-001 1.2292e+000 2.8024e+000 2.8783e+000 5.7566e+000
7.7812e+000
 5 1.0000 1.2939e+000 1.9920e+000 3.3001e+000 3.2132e+000 4.0344e+000
-2.1559e+001
 6 1.2500 2.0095e+000 2.6335e+000 1.8316e+000 1.7079e+000 -2.3409e+000
 -2.9444e+001
 7 1.5000 2.6113e+000 2.6349e+000 -1.8206e+000 -1.4739e-002 -6.4559e+0
00 -3.4760e+000
 8 1.7500 2.8788e+000 1.7387e+000 -5.3487e+000 -2.3474e-001 -3.8852e+0
00 2.4042e+001
 9 2.0000 2.7482e+000 2.7373e-001 -6.3712e+000 7.4471e-001 1.5189e+000
 1.9191e+001
10 2.2500 2.3050e+000 -1.1476e+000 -4.9998e+000 1.2015e+000 2.8724e+00
0 -8.3629e+000
11 2.5000 1.6610e+000 -2.1743e+000 -3.2136e+000 4.3624e-001 -6.1695e-0
01 -1.9552e+001
12 2.7500 8.8908e-001 -2.8319e+000 -2.0468e+000 -3.1334e-001 -3.0296e+
000 2.5063e-001
.
.
.

0 2.1734e+001
20 4.7500 1.3287e+000 2.0191e+000 -9.4959e-001 -1.5950e-001 3.4773e+00
0 -4.4812e+000
21 5.0000 1.7010e+000 1.6947e+000 -1.6456e+000 -4.5222e-001 8.7311e-00
1 -1.6352e+001
22 5.2500 1.8823e+000 1.1071e+000 -3.0551e+000 -5.8471e-001 -8.5041e-0
01 2.5638e+000
23 5.5000 1.8304e+000 2.5880e-001 -3.7315e+000 7.8790e-001 2.4802e+000
 2.4081e+001
24 5.7500 1.6408e+000 -4.8304e-001 -2.2032e+000 3.0664e+000 7.3022e+00
0 1.4494e+001
25 6.0000 1.4914e+000 -6.7792e-001 6.4412e-001 4.2109e+000 6.8461e+000
 -1.8143e+001
```





11.32 Equations of motion:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}(t)$$

with

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ [c] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ [k] = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix},\ \vec{F}(t) = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

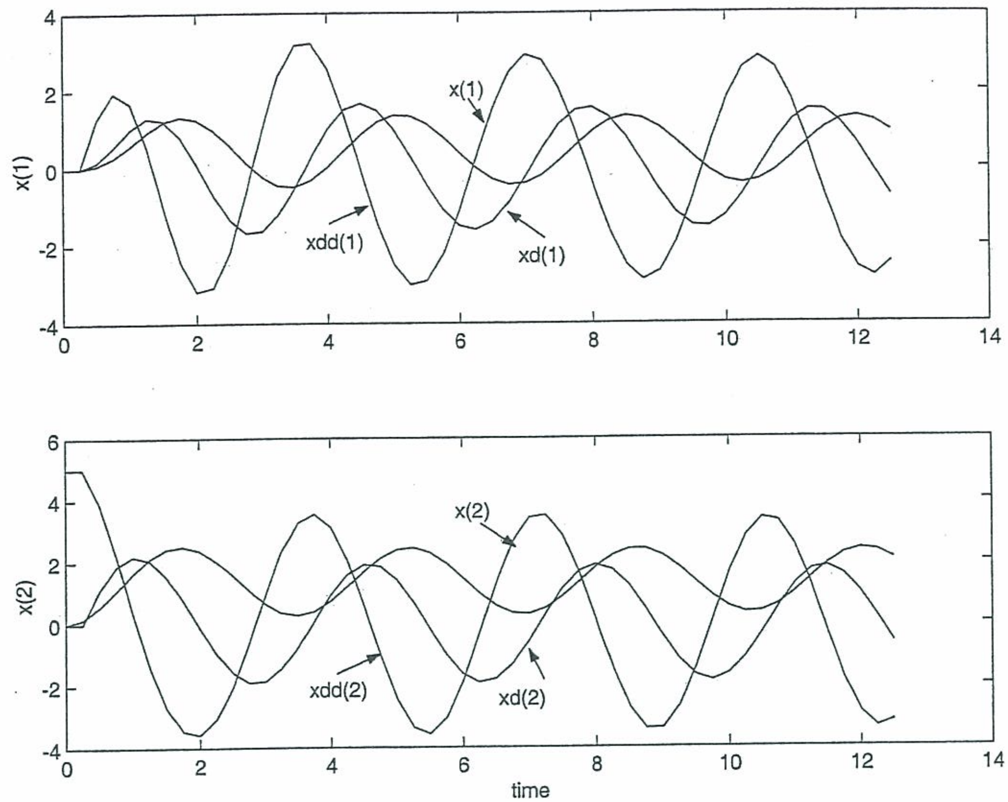
Solution using Program 15.m:

$n = 2$, $m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$,

$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, nstep = 50, delt $= \frac{\pi}{20}$.

In subprogram: $\vec{F} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$

```
Results of Ex11_32
******************************
>> program15
 Solution by central difference method

 Given data:

 n=  2     nstp= 50     delt=2.500000e-001

 Solution:

step time       x(i,1)      xd(i,1)       xdd(i,1)      x(i,2)          xd(
i,2)    xdd(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000
5.0000e+000
 2 0.2500 3.4694e-018 6.9389e-018 5.5511e-017 1.5625e-001 0.0000e+000
5.0000e+000
 3 0.5000 7.9153e-002 1.5831e-001 1.2664e+000 5.5613e-001 1.1123e+000
3.8980e+000
 4 0.7500 2.8025e-001 5.6050e-001 1.9511e+000 1.0934e+000 1.8742e+000
2.1978e+000
 5 1.0000 5.8633e-001 1.0144e+000 1.6798e+000 1.6506e+000 2.1889e+000
3.1952e-001
 6 1.2500 9.2647e-001 1.2924e+000 5.4495e-001 2.1205e+000 2.0542e+000
-1.3969e+000
 7 1.5000 1.2034e+000 1.2342e+000 -1.0110e+000 2.4211e+000 1.5410e+000
 -2.7089e+000
 8 1.7500 1.3294e+000 8.0596e-001 -2.4148e+000 2.5053e+000 7.6957e-001
 -3.4627e+000
   .
   .
   .

 45 11.0000 2.6959e-001 1.1967e+000 1.7533e+000 9.8753e-001 1.2500e+000
  2.5015e+000
 46 11.2500 6.5996e-001 1.4887e+000 5.8245e-001 1.4537e+000 1.7136e+000
  1.2070e+000
 47 11.5000 1.0064e+000 1.4736e+000 -7.0280e-001 1.8992e+000 1.8233e+00
 0 -3.2947e-001
 48 11.7500 1.2380e+000 1.1560e+000 -1.8379e+000 2.2327e+000 1.5582e+00
 0 -1.7914e+000
 49 12.0000 1.3076e+000 6.0249e-001 -2.5905e+000 2.3864e+000 9.7439e-00
 1 -2.8789e+000
 50 12.2500 1.2018e+000 -7.2283e-002 -2.8077e+000 2.3293e+000 1.9320e-0
 01 -3.3706e+000
 51 12.5000 9.4307e-001 -7.2916e-001 -2.4473e+000 2.0743e+000 -6.2413e-
 001 -3.1680e+000
```



(11.33) Equations of motion: $[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}(t)$

with $[m] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $[c] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[k] = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$,

$\vec{F}(t) = \begin{Bmatrix} 10 \sin 5t \\ 0 \end{Bmatrix}$

Solution using Program 16.m:

$n = 2$, $m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$,

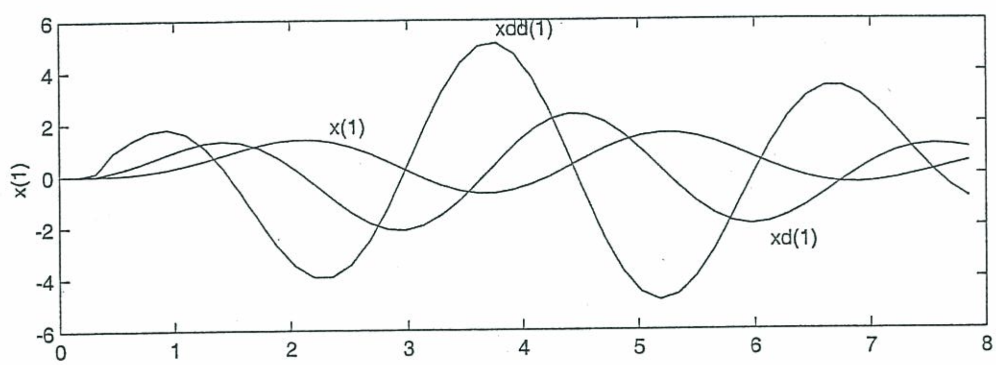
$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, nstep = 50, delt = $\frac{\pi}{20}$.

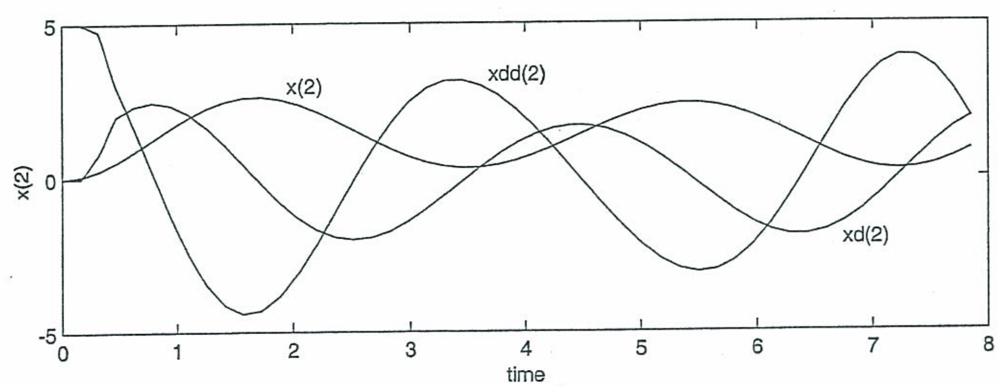
In subprogram: $\vec{F}(t) = \begin{Bmatrix} 10 \sin 5t \\ 0 \end{Bmatrix}$

```
Results of Ex11_33
********************************
>> program16
 Solution by Hobolt method

 Given data:

 n=  2     nstp= 50     delt=1.570796e-001

 Solution:

step time    x(i,1)   xd(i,1)      xdd(i,1)     x(i,2)      xd(i,2)     xdd
(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000
5.0000e+000
 2 0.1571 0.0000e+000 0.0000e+000 0.0000e+000 6.1685e-002 2.2087e-017
5.0000e+000
```

```
 3 0.3142 3.0440e-003 9.6895e-003 1.2337e-001 2.4065e-001 7.6602e-001
4.7533e+000
 4 0.4712 1.8912e-002 1.6259e-001 9.1608e-001 5.1478e-001 2.0011e+000
2.9598e+000
 5 0.6283 5.8032e-002 3.4519e-001 1.3650e+000 8.5661e-001 2.3334e+000
1.6316e+000
 6 0.7854 1.2964e-001 5.7894e-001 1.6914e+000 1.2346e+000 2.4551e+000
1.9106e-001
 7 0.9425 2.3963e-001 8.3483e-001 1.7936e+000 1.6157e+000 2.3650e+000
-1.2232e+000
```

```
45 6.9115 -3.7289e-001 1.7185e-001 3.0312e+000 3.9690e-001 -1.0620e+00
0 3.0395e+000
46 7.0686 -3.1591e-001 5.6154e-001 2.4307e+000 2.6762e-001 -5.5794e-00
1 3.6136e+000
47 7.2257 -2.0460e-001 8.5380e-001 1.6722e+000 2.2229e-001 5.0005e-003
 3.9062e+000
48 7.3827 -5.5446e-002 1.0351e+000 8.6621e-001 2.6677e-001 5.8147e-001
 3.8775e+000
49 7.5398 1.1405e-001 1.1066e+000 1.1476e-001 3.9952e-001 1.1229e+000
3.5160e+000
50 7.6969 2.8752e-001 1.0823e+000 -5.0214e-001 6.1148e-001 1.5822e+000
 2.8416e+000
51 7.8540 4.5143e-001 9.8440e-001 -9.3551e-001 8.8654e-001 1.9178e+000
 1.9053e+000
```





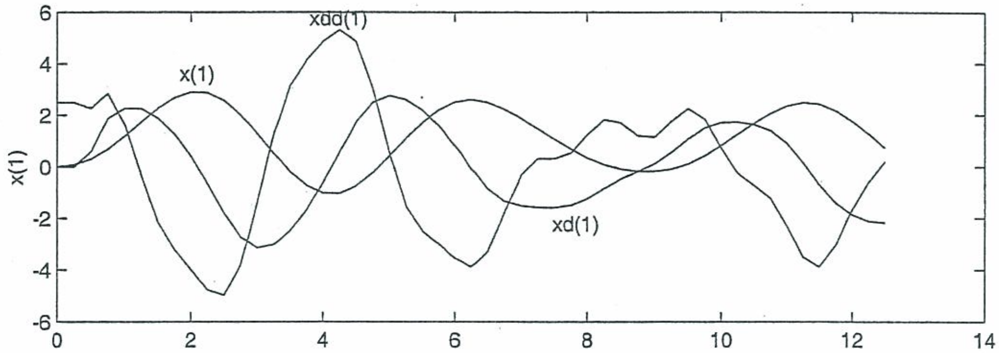
11.34 Equations of motion: Given in solution of Problem 11.20

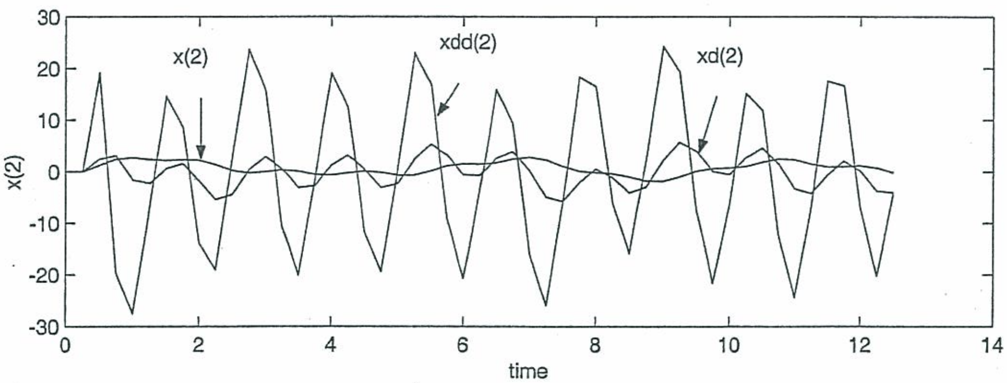
Solution using Program16.m:

$n = 2$, $m = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$,

$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, nstep = 50, delt = 0.25

In subprogram: $\vec{F} = \begin{Bmatrix} 5 \\ 20 \sin 5t \end{Bmatrix}$






```
Results of Ex11_34
************************
>> program16
 Solution by Hobolt method

 Given data:

 n=  2      nstp= 50      delt=2.500000e-001

 Solution:

step time    x(i,1)    xd(i,1)      xdd(i,1)     x(i,2)      xd(i,2)      xdd
(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
 2 0.2500 7.8125e-002 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
 3 0.5000 2.9785e-001 5.9570e-001 2.2656e+000 1.1960e+000 2.3920e+000
1.9136e+001
 4 0.7500 6.7731e-001 1.8615e+000 2.8459e+000 2.3779e+000 3.0857e+000
-1.9588e+001
 5 1.0000 1.1875e+000 2.2638e+000 1.6286e+000 2.6912e+000 -1.6232e+000
 -2.7568e+001
```

```
.
.
.
45 11.0000 2.3544e+000 9.1504e-001 -2.2782e+000 2.2850e+000 -3.3050e+0
00 -2.4426e+001
46 11.2500 2.4907e+000 1.5041e-001 -3.4958e+000 1.4763e+000 -4.2224e+0
00 -6.8088e+000
47 11.5000 2.4280e+000 -7.0701e-001 -3.8836e+000 9.0044e-001 -6.8674e-
001 1.7538e+001
48 11.7500 2.1705e+000 -1.4140e+000 -3.0502e+000 9.6130e-001 2.0554e+0
00 1.6650e+001
49 12.0000 1.7634e+000 -1.8673e+000 -1.6701e+000 1.1202e+000 1.1342e-0
01 -7.0501e+000
50 12.2500 1.2630e+000 -2.1135e+000 -5.9375e-001 6.9518e-001 -3.7771e+
000 -2.0254e+001
51 12.5000 7.2135e-001 -2.1793e+000 1.7567e-001 -1.6028e-001 -4.0782e+
000 -4.4321e+000
```

11.35 The problem-dependent data to be used in the main program that calls WILSON (given in Problem 11.49), subroutine EXTFUN and output are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
```

```
OLUTION BY WILSON METHOD

IVEN DATA:
= 2      NSTEP=   24      TH= 0.14000000E+01      DELTA= 0.24216267E+00

OLUTION:
```

TEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.1551E-01	0.1922E+00	0.1587E+01	0.1330E+00	0.1042E+01	0.3610E+01
3	0.4843	0.1136E+00	0.6390E+00	0.2103E+01	0.4767E+00	0.1735E+01	0.2112E+01
4	0.7265	0.3250E+00	0.1087E+01	0.1593E+01	0.9440E+00	0.2063E+01	0.5986E+00
5	0.9687	0.6233E+00	0.1328E+01	0.4052E+00	0.1447E+01	0.2036E+01	-.8224E+00
6	1.2108	0.9431E+00	0.1256E+01	-.9998E+00	0.1905E+01	0.1691E+01	-.2029E+01
7	1.4530	0.1206E+01	0.8697E+00	-.2195E+01	0.2246E+01	0.1095E+01	-.2892E+01
8	1.6951	0.1346E+01	0.2556E+00	-.2877E+01	0.2423E+01	0.3457E+00	-.3297E+01
9	1.9373	0.1323E+01	-.4452E+00	-.2910E+01	0.2411E+01	-.4384E+00	-.3179E+01
10	2.1795	0.1136E+01	-.1078E+01	-.2318E+01	0.2218E+01	-.1131E+01	-.2545E+01

11	2.4216	0.8171E+00	-.1511E+01	-.1252E+01	0.1879E+01	-.1620E+01	-.1489E+01
12	2.6638	0.4274E+00	-.1656E+01	0.5148E-01	0.1456E+01	-.1822E+01	-.1836E+00
~~13~~	~~2.9060~~	~~0.4036E-01~~	~~-.1489E+01~~	~~0.1330E+01~~	~~0.1023E+01~~	~~-.1705E+01~~	~~0.1149E+01~~
14	3.1481	-.2713E+00	-.1044E+01	0.2341E+01	0.6543E+00	-.1291E+01	0.2276E+01
15	3.3903	-.4500E+00	-.4093E+00	0.2903E+01	0.4155E+00	-.6520E+00	0.2998E+01
~~16~~	~~3.6324~~	~~-.4638E+00~~	~~0.2960E+00~~	~~0.2923E+01~~	~~0.3474E+00~~	~~0.9684E-01~~	~~0.3186E+01~~
17	3.8746	-.3114E+00	0.9418E+00	0.2410E+01	0.4606E+00	0.8226E+00	0.2808E+01
18	4.1168	-.2189E-01	0.1411E+01	0.1468E+01	0.7335E+00	0.1396E+01	0.1931E+01
~~19~~	~~4.3589~~	~~0.3512E+00~~	~~0.1622E+01~~	~~0.2714E+00~~	~~0.1116E+01~~	~~0.1717E+01~~	~~0.7159E+00~~
20	4.6011	0.7399E+00	0.1539E+01	-.9597E+00	0.1540E+01	0.1729E+01	-.6189E+00
21	4.8433	0.1074E+01	0.1180E+01	-.2005E+01	0.1929E+01	0.1432E+01	-.1834E+01
~~22~~	~~5.0854~~	~~0.1294E+01~~	~~0.6124E+00~~	~~-.2680E+01~~	~~0.2213E+01~~	~~0.8814E+00~~	~~-.2711E+01~~
23	5.3276	0.1362E+01	-.5939E-01	-.2868E+01	0.2343E+01	0.1781E+00	-.3097E+01
24	5.5697	0.1267E+01	-.7143E+00	-.2541E+01	0.2297E+01	-.5512E+00	-.2926E+01
~~25~~	~~5.8119~~	~~0.1027E+01~~	~~-.1235E+01~~	~~-.1763E+01~~	~~0.2085E+01~~	~~-.1176E+01~~	~~-.2233E+01~~

11.36 The problem-dependent data to be used in the main program that calls WILSON (given in Problem 11.49), subroutine EXTFUN, and the results are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=10.0*SIN(5.0*TIME)
      F(2)=0.0
      RETURN
      END
```

SOLUTION BY WILSON METHOD

GIVEN DATA:
N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.8209E-01	0.1017E+01	0.8399E+01	0.1461E-02	0.1810E-01	0.1495E+00
3	0.4843	0.5226E+00	0.2406E+01	0.3071E+01	0.1381E-01	0.9867E-01	0.5160E+00
4	0.7265	0.1062E+01	0.1503E+01	-.1053E+02	0.5495E-01	0.2498E+00	0.7325E+00
5	0.9687	0.1074E+01	-.1584E+01	-.1496E+02	0.1330E+00	0.3781E+00	0.3265E+00
6	1.2108	0.3655E+00	-.3797E+01	-.3306E+01	0.2246E+00	0.3401E+00	-.6398E+00
7	1.4530	-.5063E+00	-.2807E+01	0.1148E+02	0.2799E+00	0.8172E-01	-.1494E+01
8	1.6951	-.8380E+00	0.1135E+00	0.1263E+02	0.2548E+00	-.2930E+00	-.1600E+01
9	1.9373	-.5584E+00	0.1706E+01	0.5215E+00	0.1431E+00	-.6043E+00	-.9714E+00
10	2.1795	-.2170E+00	0.7537E+00	-.8390E+01	-.2359E-01	-.7389E+00	-.1402E+00
11	2.4216	-.2285E+00	-.6334E+00	-.3066E+01	-.2004E+00	-.6951E+00	0.5020E+00
12	2.6638	-.3632E+00	-.3125E-01	0.8040E+01	-.3488E+00	-.5089E+00	0.1036E+01

13	2.9060	-.1304E+00	0.1973E+01	0.8514E+01	-.4357E+00	-.1847E+00	0.1641E+01
14	3.1481	0.4742E+00	0.2513E+01	-.4054E+01	-.4280E+00	0.2662E+00	0.2083E+01
15	3.3903	0.8633E+00	0.2840E+00	-.1436E+02	-.3050E+00	0.7389E+00	0.1822E+01
16	3.6324	0.5643E+00	-.2534E+01	-.8916E+01	-.8396E-01	0.1040E+01	0.6659E+00
17	3.8746	-.1616E+00	-.2845E+01	0.6349E+01	0.1726E+00	0.1017E+01	-.8543E+00
18	4.1168	-.5959E+00	-.4607E+00	0.1334E+02	0.3836E+00	0.6825E+00	-.1911E+01
19	4.3589	-.4018E+00	0.1711E+01	0.4600E+01	0.4907E+00	0.1932E+00	-.2130E+01
20	4.6011	0.2833E-01	0.1349E+01	-.7598E+01	0.4783E+00	-.2815E+00	-.1791E+01
21	4.8433	0.1332E+00	-.4775E+00	-.7484E+01	0.3623E+00	-.6571E+00	-.1311E+01
22	5.0854	-.9143E-01	-.9221E+00	0.3812E+01	0.1706E+00	-.9025E+00	-.7150E+00
23	5.3276	-.1416E+00	0.7616E+00	0.1009E+02	-.5983E-01	-.9628E+00	0.2168E+00
24	5.5697	0.2564E+00	0.2184E+01	0.1656E+01	-.2753E+00	-.7698E+00	0.1377E+01
25	5.8119	0.7047E+00	0.9850E+00	-.1156E+02	-.4136E+00	-.3408E+00	0.2166E+01

11.37 The problem-dependent data to be used in the main program that calls WILSON (given in Problem 11.49), subroutine EXTFUN, and output are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XI(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.25/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/2.0,0.0,0.0,1.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*TIME)
      RETURN
      END
```

SOLUTION BY WILSON METHOD

GIVEN DATA:
N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.25000000E+00

SOLUTION:

STEP	TIME	X(1,1)	XD(1,1)	XDD(I,1)	X(1,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7846E-01	0.6290E+00	0.2532E+01	0.1849E+00	0.2219E+01	0.1775E+02
3	0.5000	0.3173E+00	0.1292E+01	0.2772E+01	0.1185E+01	0.5347E+01	0.7278E+01
4	0.7500	0.7246E+00	0.1956E+01	0.2539E+01	0.2473E+01	0.3847E+01	-.1928E+02
5	1.0000	0.1277E+01	0.2398E+01	0.9958E+00	0.2760E+01	-.1840E+01	-.2622E+02
6	1.2500	0.1880E+01	0.2322E+01	-.1598E+01	0.1722E+01	-.5495E+01	-.3023E+01
7	1.5000	0.2387E+01	0.1638E+01	-.3881E+01	0.5026E+00	-.3266E+01	0.2085E+02
8	1.7500	0.2666E+01	0.5579E+00	-.4757E+01	0.2826E+00	0.1285E+01	0.1556E+02
9	2.0000	0.2661E+01	-.5871E+00	-.4402E+01	0.8331E+00	0.2090E+01	-.9119E+01
10	2.2500	0.2384E+01	-.1595E+01	-.3661E+01	0.9796E+00	-.1282E+01	-.1786E+02
11	2.5000	0.1880E+01	-.2397E+01	-.2755E+01	0.3022E+00	-.3333E+01	0.1457E+01
12	2.7500	0.1212E+01	-.2882E+01	-.1122E+01	-.2795E+00	-.4970E+00	0.2123E+02
13	3.0000	0.4833E+00	-.2841E+01	0.1444E+01	0.1675E+00	0.3704E+01	0.1238E+02
14	3.2500	-.1555E+00	-.2164E+01	0.3975E+01	0.1194E+01	0.3364E+01	-.1510E+02
15	3.5000	-.5604E+00	-.1028E+01	0.5115E+01	0.1459E+01	-.1662E+01	-.2510E+02
16	3.7500	-.6636E+00	0.1787E+00	0.4536E+01	0.4757E+00	-.5336E+01	-.4289E+01

17	4.0000	-.4914E+00	0.1142E+01	0.3169E+01	-.7410E+00	-.3392E+01	0.1984E+02
18	4.2500	-.1194E+00	0.1784E+01	0.1965E+01	-.1005E+01	0.1134E+01	0.1637E+02
19	4.5000	0.3765E+00	0.2138E+01	0.8735E+00	-.4563E+00	0.2272E+01	-.7272E+01
20	4.7500	0.9225E+00	0.2166E+01	-.6563E+00	-.2094E+00	-.6720E+00	-.1628E+02
21	5.0000	0.1425E+01	0.1778E+01	-.2441E+01	-.6840E+00	-.2317E+01	0.3120E+01
22	5.2500	0.1783E+01	0.1046E+01	-.3417E+01	-.9452E+00	0.1110E+01	0.2429E+02
23	5.5000	0.1943E+01	0.2598E+00	-.2874E+01	0.1076E-01	0.6216E+01	0.1655E+02
24	5.7500	0.1934E+01	-.2762E+00	-.1414E+01	0.1780E+01	0.6725E+01	-.1248E+02
25	6.0000	0.1832E+01	-.4927E+00	-.3190E+00	0.2923E+01	0.1835E+01	-.2663E+02

11.38 The problem-dependent data to be used in the main program that calls NUMARK (given in Problem 11.50), subroutine EXTFUN, and output are given:

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
     2 0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
```

```
SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N=     2      NSTEP=    24      ALPHA= 0.16666667E+00    BETA= 0.50000000E+00
                                                         DELTA= 0.24216267E+00
SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.1776E-01	0.2200E+00	0.1817E+01	0.1335E+00	0.1048E+01	0.3656E+01
3	0.4843	0.1287E+00	0.7143E+00	0.2266E+01	0.4799E+00	0.1753E+01	0.2170E+01
4	0.7265	0.3613E+00	0.1178E+01	0.1568E+01	0.9532E+00	0.2093E+01	0.6342E+00
5	0.9687	0.6793E+00	0.1393E+01	0.2007E+00	0.1464E+01	0.2067E+01	-.8511E+00
6	1.2108	0.1008E+01	0.1259E+01	-.1303E+01	0.1927E+01	0.1704E+01	-.2145E+01
7	1.4530	0.1263E+01	0.7991E+00	-.2497E+01	0.2268E+01	0.1071E+01	-.3080E+01
8	1.6951	0.1377E+01	0.1221E+00	-.3094E+01	0.2433E+01	0.2741E+00	-.3505E+01
9	1.9373	0.1317E+01	-.6142E+00	-.2987E+01	0.2398E+01	-.5540E+00	-.3334E+01
10	2.1795	0.1088E+01	-.1246E+01	-.2231E+01	0.2173E+01	-.1270E+01	-.2581E+01
11	2.4216	0.7330E+00	-.1639E+01	-.1013E+01	0.1802E+01	-.1748E+01	-.1365E+01
12	2.6638	0.3203E+00	-.1712E+01	0.4055E+00	0.1353E+01	-.1901E+01	0.9807E-01
13	2.9060	-.6943E-01	-.1453E+01	0.1736E+01	0.9094E+00	-.1703E+01	0.1543E+01
14	3.1481	-.3607E+00	-.9133E+00	0.2721E+01	0.5536E+00	-.1189E+01	0.2700E+01
15	3.3903	-.4977E+00	-.1994E+00	0.3176E+01	0.3513E+00	-.4559E+00	0.3354E+01
16	3.6324	-.4544E+00	0.5510E+00	0.3022E+01	0.3395E+00	0.3593E+00	0.3379E+01
17	3.8746	-.2394E+00	0.1195E+01	0.2295E+01	0.5196E+00	0.1104E+01	0.2773E+01
18	4.1168	0.1059E+00	0.1610E+01	0.1138E+01	0.8573E+00	0.1639E+01	0.1647E+01
19	4.3589	0.5158E+00	0.1720E+01	-.2284E+00	0.1289E+01	0.1865E+01	0.2165E+00
20	4.6011	0.9129E+00	0.1506E+01	-.1545E+01	0.1732E+01	0.1740E+01	-.1251E+01
21	4.8433	0.1222E+01	0.1008E+01	-.2564E+01	0.2105E+01	0.1288E+01	-.2478E+01
22	5.0854	0.1386E+01	0.3231E+00	-.3096E+01	0.2337E+01	0.5967E+00	-.3235E+01
23	5.3276	0.1374E+01	-.4202E+00	-.3043E+01	0.2385E+01	-.2046E+00	-.3382E+01
24	5.5697	0.1189E+01	-.1081E+01	-.2419E+01	0.2241E+01	-.9643E+00	-.2893E+01
25	5.8119	0.8670E+00	-.1537E+01	-.1342E+01	0.1933E+01	-.1540E+01	-.1862E+01

11.39 The problem-dependent data to be used in the main program that calls NUMARK (given in Problem 11.50), the subroutine EXTFUN, and the output are given here:

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
     2 0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=10.0*SIN(5.0*TIME)
      F(2)=0.0
      RETURN
      END
```

SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N= 2 NSTEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00
DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.8642E-01	0.1071E+01	0.8842E+01	0.8129E-03	0.1007E-01	0.8317E-01
3	0.4843	0.5509E+00	0.2542E+01	0.3308E+01	0.9875E-02	0.8206E-01	0.5114E+00
4	0.7265	0.1120E+01	0.1570E+01	-.1134E+02	0.4878E-01	0.2560E+00	0.9251E+00
5	0.9687	0.1119E+01	-.1784E+01	-.1636E+02	0.1345E+00	0.4383E+00	0.5807E+00
6	1.2108	0.3303E+00	-.4220E+01	-.3762E+01	0.2457E+00	0.4296E+00	-.6523E+00
7	1.4530	-.6401E+00	-.3127E+01	0.1279E+02	0.3182E+00	0.1190E+00	-.1913E+01
8	1.6951	-.1003E+01	0.2052E+00	0.1473E+02	0.2886E+00	-.3739E+00	-.2158E+01
9	1.9373	-.6501E+00	0.2182E+01	0.1601E+01	0.1439E+00	-.7835E+00	-.1226E+01
10	2.1795	-.1786E+00	0.1284E+01	-.9018E+01	-.6887E-01	-.9202E+00	0.9693E-01
11	2.4216	-.8747E-01	-.3477E+00	-.4458E+01	-.2797E+00	-.7836E+00	0.1031E+01
12	2.6638	-.1894E+00	-.2747E-01	0.7103E+01	-.4342E+00	-.4713E+00	0.1547E+01
13	2.9060	0.2207E-01	0.1815E+01	0.8110E+01	-.4984E+00	-.3990E-01	0.2016E+01
14	3.1481	0.5744E+00	0.2232E+01	-.4664E+01	-.4456E+00	0.4895E+00	0.2357E+01
15	3.3903	0.8748E+00	-.1784E+00	-.1524E+02	-.2622E+00	0.1008E+01	0.1924E+01
16	3.6324	0.4459E+00	-.3111E+01	-.8975E+01	0.2292E-01	0.1284E+01	0.3542E+00
17	3.8746	-.4039E+00	-.3220E+01	0.8070E+01	0.3241E+00	0.1121E+01	-.1700E+01
18	4.1168	-.8682E+00	-.2884E+00	0.1614E+02	0.5329E+00	0.5516E+00	-.3000E+01
19	4.3589	-.5593E+00	0.2449E+01	0.6467E+01	0.5797E+00	-.1601E+00	-.2878E+01
20	4.6011	0.8159E-01	0.2258E+01	-.8046E+01	0.4672E+00	-.7250E+00	-.1787E+01
21	4.8433	0.3764E+00	0.1102E+00	-.9692E+01	0.2506E+00	-.1017E+01	-.6259E+00
22	5.0854	0.2284E+00	-.8793E+00	0.1519E+01	-.5534E-02	-.1063E+01	0.2506E+00
23	5.3276	0.1303E+00	0.3584E+00	0.8703E+01	-.2470E+00	-.8969E+00	0.1118E+01
24	5.5697	0.3961E+00	0.1522E+01	0.9086E+00	-.4220E+00	-.5091E+00	0.2084E+01
25	5.8119	0.6650E+00	0.1770E+00	-.1202E+02	-.4793E+00	0.5587E-01	0.2582E+01

11.40 The problem-dependent data to be used in the main program given in Problem 11.50 (which calls NUMARK), subroutine EXTFUN and results are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/
     2 2,24,25,0.16666667,0.5,0.25/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/2.0,0.0,0.0,1.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*TIME)
      RETURN
      END
```

SOLUTION BY NEWMARK METHOD

GIVEN DATA:

N= 2 NSTEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00

DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(1,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7769E-01	0.6198E+00	0.2458E+01	0.1914E+00	0.2296E+01	0.1837E+02
3	0.5000	0.3129E+00	0.1276E+01	0.2789E+01	0.1228E+01	0.5553E+01	0.7683E+01
4	0.7500	0.7202E+00	0.1988E+01	0.2905E+01	0.2565E+01	0.3982E+01	-.2025E+02
5	1.0000	0.1293E+01	0.2534E+01	0.1469E+01	0.2847E+01	-.2048E+01	-.2798E+02
6	1.2500	0.1940E+01	0.2517E+01	-.1606E+01	0.1715E+01	-.6001E+01	-.3641E+01
7	1.5000	0.2488E+01	0.1742E+01	-.4594E+01	0.3703E+00	-.3674E+01	0.2226E+02
8	1.7500	0.2769E+01	0.4543E+00	-.5706E+01	0.9920E-01	0.1312E+01	0.1763E+02
9	2.0000	0.2712E+01	-.8750E+00	-.4927E+01	0.7083E+00	0.2480E+01	-.8290E+01
10	2.2500	0.2353E+01	-.1940E+01	-.3596E+01	0.9630E+00	-.8686E+00	-.1850E+02
11	2.5000	0.1768E+01	-.2694E+01	-.2435E+01	0.3681E+00	-.3089E+01	0.7367E+00
12	2.7500	0.1035E+01	-.3095E+01	-.7734E+00	-.1674E+00	-.3396E+00	0.2126E+02
13	3.0000	0.2664E+00	-.2940E+01	0.2019E+01	0.3183E+00	0.3851E+01	0.1227E+02
14	3.2500	-.3745E+00	-.2064E+01	0.4988E+01	0.1364E+01	0.3319E+01	-.1652E+02
15	3.5000	-.7216E+00	-.6614E+00	0.6231E+01	0.1566E+01	-.2150E+01	-.2722E+02
16	3.7500	-.7048E+00	0.7455E+00	0.5024E+01	0.4094E+00	-.6182E+01	-.5035E+01
17	4.0000	-.3862E+00	0.1704E+01	0.2642E+01	-.1016E+01	-.4117E+01	0.2155E+02
18	4.2500	0.1030E+00	0.2133E+01	0.7952E+00	-.1396E+01	0.9885E+00	0.1929E+02
19	4.5000	0.6503E+00	0.2201E+01	-.2522E+00	-.8012E+00	0.2745E+01	-.5238E+01
20	4.7500	0.1181E+01	0.1991E+01	-.1429E+01	-.3879E+00	0.1236E+00	-.1573E+02
21	5.0000	0.1617E+01	0.1437E+01	-.3003E+01	-.6516E+00	-.1444E+01	0.3194E+01
22	5.2500	0.1874E+01	0.5849E+00	-.3813E+01	-.6909E+00	0.2016E+01	0.2449E+02
23	5.5000	0.1912E+01	-.2353E+00	-.2748E+01	0.4886E+00	0.7060E+01	0.1586E+02
24	5.7500	0.1792E+01	-.6351E+00	-.4500E+00	0.2425E+01	0.7132E+01	-.1527E+02
25	6.0000	0.1636E+01	-.5461E+00	0.1163E+01	0.3569E+01	0.1377E+01	-.3077E+02

11.41

Equation: $5\ddot{x} + 4\dot{x} + 3x = 6\sin t$

or $\ddot{x} = -0.8\dot{x} - 0.6x + 1.2\sin t$

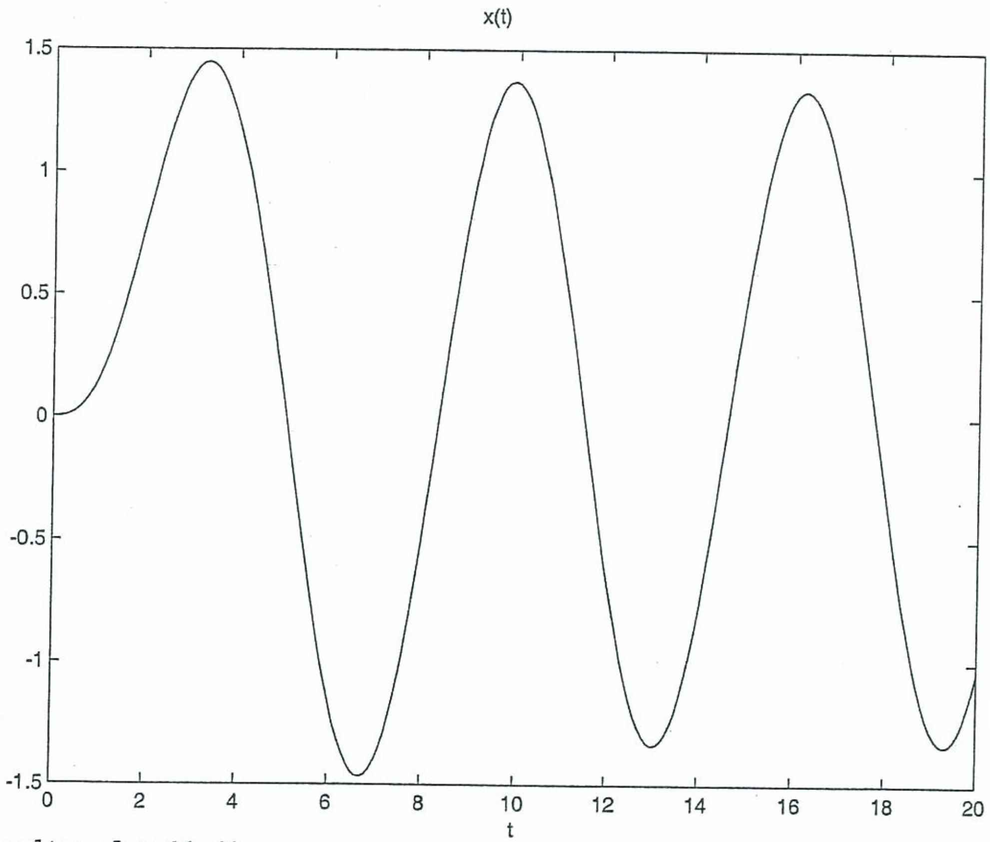
Differential equations to be solved:

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -0.8\,y_2 - 0.6\,y_1 + 1.2\sin t \end{Bmatrix}$$

with $\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ and $\vec{Y}(0) = \vec{0}$

```
% Ex11_41.m
% This program will use the function dfunc11_41.m, they should
% be in the same folder
tspan = [0: 0.1: 20];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc11_41', tspan, x0);
disp('      t          x(t)       xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc11_41.m
function f = dfunc11_41(t,x)
f = zeros(2,1);
f(1) = x(2);
f(2) = - 0.8 * x(2) - 0.6 * x(1) + 1.2*sin(t);
```




```
Results of Ex11_41
*************************
>> Ex11_41
       t          x(t)       xd(t)
         0           0           0
    0.1000      0.0002      0.0058
    0.2000      0.0015      0.0226
    0.3000      0.0051      0.0493
    0.4000      0.0117      0.0847
    0.5000      0.0222      0.1274
    0.6000      0.0374      0.1762
    0.7000      0.0576      0.2298
    0.8000      0.0835      0.2867
    0.9000      0.1151      0.3456
    1.0000      0.1526      0.4051
```

```
  .
  .
  .
19.1000   -1.3092   -0.2840
19.2000   -1.3310   -0.1518
19.3000   -1.3395   -0.0181
19.4000   -1.3346    0.1158
19.5000   -1.3163    0.2486
19.6000   -1.2849    0.3789
19.7000   -1.2406    0.5054
19.8000   -1.1839    0.6268
19.9000   -1.1154    0.7420
20.0000   -1.0358    0.8498
```

11.42 Equations:

$$2\ddot{x}_1 + 10x_1 - 5x_2 = F_1(t)$$

$$4\ddot{x}_2 - 5x_1 + 15x_2 = 0$$

or

$$\ddot{x}_1 = -5x_1 + 2.5x_2 + 0.5F_1(t)$$

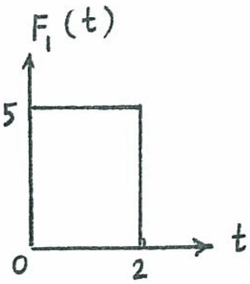
$$\ddot{x}_2 = 1.25x_1 - 3.75x_2$$

Equations in vector form:

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -5y_1 + 2.5y_3 + 0.5F_1(t) \\ y_4 \\ 1.25y_1 - 3.75y_3 \end{Bmatrix} \equiv \vec{f}(t)$$

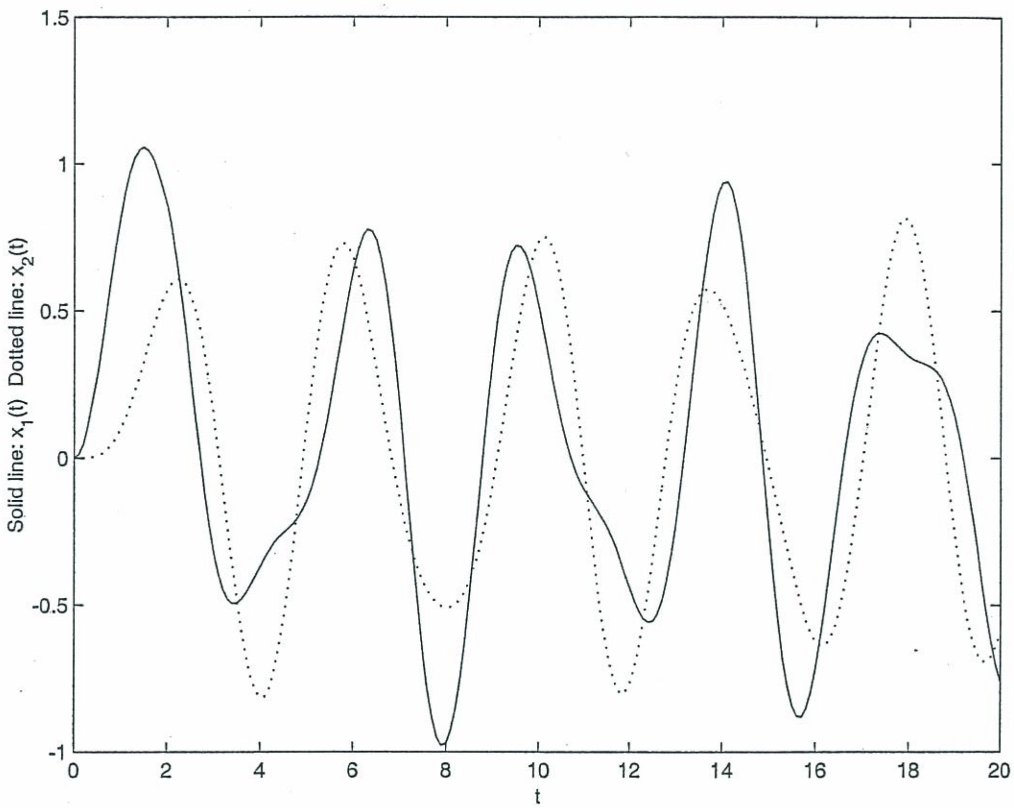
with

$$\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix}, \quad \vec{Y}(0) = \vec{0} \text{ and}$$




```
% Ex11_42.m
% This program will use the function dfunc11_42.m, they should
% be in the same folder
tspan = [0: 0.1: 20];
x0 = [0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc11_42', tspan, x0);
disp('      t        x1(t)      xd1(t)        x2(t)        xd2(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
ylabel('Solid line: x_1(t)  Dotted line: x_2(t)');
hold on;
plot(t,x(:,3), ':');
```

```
% dfunc11_42.m
function f = dfunc11_42(t,x)
F = 5*( stepfun(t,0) - stepfun(t,2.0) );
f = zeros(4,1);
f(1) = x(2);
f(2) = 2.5 * x(3) - 5 * x(1) + 0.5*F;
f(3) = x(4);
f(4) = 1.25 * x(1) - 3.75 * x(3);
```




```
Results of Ex11_42
************************
>> Ex11_42
```

t	x1(t)	xd1(t)	x2(t)	xd2(t)
0	0	0	0	0
0.1000	0.0124	0.2479	0.0000	0.0005
0.2000	0.0492	0.4835	0.0002	0.0041
0.3000	0.1084	0.6952	0.0010	0.0135
0.4000	0.1871	0.8725	0.0032	0.0311
0.5000	0.2814	1.0072	0.0076	0.0583
0.6000	0.3869	1.0932	0.0152	0.0958
0.7000	0.4984	1.1272	0.0271	0.1434
0.8000	0.6106	1.1086	0.0442	0.1995
0.9000	0.7184	1.0397	0.0672	0.2620
1.0000	0.8170	0.9256	0.0967	0.3277
19.3000	-0.1051	-0.9836	-0.5591	-0.7481
19.4000	-0.2073	-1.0540	-0.6238	-0.5451
19.5000	-0.3146	-1.0855	-0.6678	-0.3349
19.6000	-0.4229	-1.0711	-0.6907	-0.1256
19.7000	-0.5273	-1.0066	-0.6931	0.0750
19.8000	-0.6225	-0.8902	-0.6762	0.2603
19.9000	-0.7036	-0.7229	-0.6417	0.4248
20.0000	-0.7655	-0.5090	-0.5920	0.5644

11.46

```
C ===================================================================
C
C MAIN PROGRAM WHICH CALLS WILSON
C
C ===================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL WILSON (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,F1,F2,FT,R,LA,
     2 LB,S,ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,TH,N)
      PRINT 10
 10   FORMAT (//,26H SOLUTION BY WILSON METHOD,/)
      PRINT 20, N,NSTEP,TH,DELTA
 20   FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,4H TH=,
     2 E15.8,4X,7H DELTA=,E15.8,/)
      PRINT 30
 30   FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3 9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELTA
 40   PRINT 50, I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(I,2)
 50   FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C ===================================================================
C
C SUBROUTINE WILSON
C
C ===================================================================
      SUBROUTINE WILSON (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,F1,F2,FT,
     2 R,LA,LB,S,ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,TH,N)
      REAL M(N,N),MI(N,N),K(N,N)
      DIMENSION C(N,N),XI(N),XDI(N),XDDI(N),X(NSTEP1,N),XD(NSTEP1,N),
     2 XDD(NSTEP1,N),XT(N),F(N),F1(N),F2(N),FT(N),R(N),LA(N),LB(N,2),
     3 S(N),ZA(N),TK(N,N),XN1(N),XN2(N),XN3(N),XN4(N)
      DO 5 I=1,N
      DO 5 J=1,N
 5    MI(I,J)=M(I,J)
      CALL SIMUL (MI,ZA,N,0,LA,LB,S)
      CALL EXTFUN (F,0.0,N)
      DO 20 I=1,N
      R(I)=F(I)
```

```
      DO 10 J=1,N
10    R(I)=R(I)-C(I,J)*XDI(J)-K(I,J)*XI(J)
20    CONTINUE
      CALL XMULT (MI,R,XDDI,N)
      DO 25 J=1,N
      X(1,J)=XI(J)
      XD(1,J)=XDI(J)
25    XDD(1,J)=XDDI(J)
      A1=6.0/((TH*DELTA)**2)
      A2=3.0/(TH*DELTA)
      A3=2.0*A2
      A4=TH*DELTA/2.0
      A5=A1/TH
      A6=-A3/TH
      A7=1.0-(3.0/TH)
      A8=DELTA/2.0
      A9=(DELTA**2)/6.0
      DO 30 I=1,N
      DO 30 J=1,N
30    TK(I,J)=K(I,J)+A1*M(I,J)+A2*C(I,J)
      CALL SIMUL (TK,ZA,N,0,LA,LB,S)
      DO 100 II=2,NSTEP1
      DO 40 I=1,N
      XN1(I)=A1*X(II-1,I)+A3*XD(II-1,I)+2.0*XDD(II-1,I)
40    XN2(I)=A2*X(II-1,I)+2.0*XD(II-1,I)+A4*XDD(II-1,I)
      CALL XMULT (M,XN1,XN3,N)
      CALL XMULT (C,XN2,XN4,N)
      TIME1=REAL(II-2)*DELTA
      TIME2=REAL(II-1)*DELTA
      CALL EXTFUN (F1,TIME1,N)
      CALL EXTFUN (F2,TIME2,N)
      DO 45 J=1,N
45    FT(J)=F1(J)+TH*(F2(J)-F1(J))+XN3(J)+XN4(J)
      CALL XMULT (TK,FT,XT,N)
      DO 50 J=1,N
      XDD(II,J)=A5*(XT(J)-X(II-1,J))+A6*XD(II-1,J)+A7*XDD(II-1,J)
      XD(II,J)=XD(II-1,J)+A8*(XDD(II,J)+XDD(II-1,J))
50    X(II,J)=X(II-1,J)+DELTA*XD(II-1,J)+A9*(XDD(II,J)+2.0*XDD(II-1,J))
100   CONTINUE
      RETURN
      END
C ==================================================================
C
C SUBROUTINE EXTFUN
C
C ==================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
C ==================================================================
C
C SUBROUTINE XMULT
C
C ==================================================================
      SUBROUTINE XMULT (A,B,BB,N)
      DIMENSION A(N,N),B(N),BB(N)
```

```
      DO 10 I=1,N
      BB(I)=0.0
      DO 10 J=1,N
10    BB(I)=BB(I)+A(I,J)*B(J)
      RETURN
      END
```

SOLUTION BY WILSON METHOD

GIVEN DATA:

N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.3344E-02	0.4143E-01	0.3422E+00	0.1392E+00	0.1119E+01	0.4244E+01
3	0.4843	0.2893E-01	0.1926E+00	0.9065E+00	0.5201E+00	0.1967E+01	0.2756E+01
4	0.7265	0.1072E+00	0.4742E+00	0.1419E+01	0.1058E+01	0.2395E+01	0.7809E+00
5	0.9687	0.2649E+00	0.8337E+00	0.1550E+01	0.1641E+01	0.2336E+01	-.1267E+01
6	1.2108	0.5076E+00	0.1152E+01	0.1076E+01	0.2153E+01	0.1825E+01	-.2958E+01
7	1.4530	0.8074E+00	0.1281E+01	-.4223E-02	0.2498E+01	0.9863E+00	-.3964E+01
8	1.6951	0.1104E+01	0.1106E+01	-.1442E+01	0.2619E+01	0.6096E-02	-.4131E+01
9	1.9373	0.1316E+01	0.5916E+00	-.2807E+01	0.2506E+01	-.9176E+00	-.3497E+01
10	2.1795	0.1369E+01	-.1866E+00	-.3621E+01	0.2193E+01	-.1615E+01	-.2266E+01
11	2.4216	0.1218E+01	-.1052E+01	-.3526E+01	0.1750E+01	-.1979E+01	-.7362E+00
12	2.6638	0.8710E+00	-.1772E+01	-.2420E+01	0.1264E+01	-.1974E+01	0.7743E+00
13	2.9060	0.3897E+00	-.2127E+01	-.5128E+00	0.8208E+00	-.1638E+01	0.2002E+01
14	3.1481	-.1186E+00	-.1981E+01	0.1717E+01	0.4905E+00	-.1059E+01	0.2781E+01
15	3.3903	-.5290E+00	-.1330E+01	0.3658E+01	0.3183E+00	-.3521E+00	0.3056E+01
16	3.6324	-.7333E+00	-.3125E+00	0.4747E+01	0.3207E+00	0.3643E+00	0.2860E+01
17	3.8746	-.6708E+00	0.8244E+00	0.4643E+01	0.4872E+00	0.9876E+00	0.2287E+01
18	4.1168	-.3477E+00	0.1791E+01	0.3340E+01	0.7852E+00	0.1440E+01	0.1451E+01
19	4.3589	0.1627E+00	0.2337E+01	0.1170E+01	0.1167E+01	0.1672E+01	0.4655E+00
20	4.6011	0.7389E+00	0.2323E+01	-.1288E+01	0.1575E+01	0.1660E+01	-.5639E+00
21	4.8433	0.1243E+01	0.1757E+01	-.3380E+01	0.1951E+01	0.1406E+01	-.1535E+01
22	5.0854	0.1558E+01	0.7952E+00	-.4566E+01	0.2239E+01	0.9371E+00	-.2339E+01
23	5.3276	0.1617E+01	-.3112E+00	-.4571E+01	0.2392E+01	0.3075E+00	-.2860E+01
24	5.5697	0.1418E+01	-.1283E+01	-.3457E+01	0.2382E+01	-.4004E+00	-.2987E+01
25	5.8119	0.1024E+01	-.1894E+01	-.1588E+01	0.2201E+01	-.1082E+01	-.2639E+01

11.47

```
C ==================================================================
C
C MAIN PROGRAM WHICH CALLS NUMARK
C
C ==================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
     2 0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
```

```
      CALL NUMARK (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,R,LA,LB,S,ZA,
     2 TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,ALPHA,BETA,N)
      PRINT 10
10    FORMAT (//,27H SOLUTION BY NEWMARK METHOD,/)
      PRINT 20, N,NSTEP,ALPHA,BETA,DELTA
20    FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,7H ALPHA=,
     2 E15.8,2X,6H BETA=,E15.8,2X,7H DELTA=,E15.8,/)
      PRINT 30
30    FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3 9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELTA
40    PRINT 50, I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(I,2)
50    FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C ==================================================================
C
C SUBROUTINE NUMARK
C
C ==================================================================
      SUBROUTINE NUMARK (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,R,LA,LB,S,
     2 ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,ALPHA,BETA,N)
      REAL M(N,N),MI(N,N),K(N,N)
      DIMENSION C(N,N),XI(N),XDI(N),XDDI(N),X(NSTEP1,N),XD(NSTEP1,N),
     2 XDD(NSTEP1,N),XT(N),F(N),R(N),LA(N),LB(N,2),
     3 S(N),ZA(N),TK(N,N),XN1(N),XN2(N),XN3(N),XN4(N)
      DO 5 I=1,N
      DO 5 J=1,N
5     MI(I,J)=M(I,J)
      CALL SIMUL (MI,ZA,N,0,LA,LB,S)
      CALL EXTFUN (F,0.0,N)
      DO 20 I=1,N
      R(I)=F(I)
      DO 10 J=1,N
10    R(I)=R(I)-C(I,J)*XDI(J)-K(I,J)*XI(J)
20    CONTINUE
      CALL XMULT (MI,R,XDDI,N)
      DO 25 J=1,N
      X(1,J)=XI(J)
      XD(1,J)=XDI(J)
25    XDD(1,J)=XDDI(J)
      A1=1.0/(ALPHA*(DELTA**2))
      A2=1.0/(ALPHA*DELTA)
      A3=(1.0/(2.0*ALPHA))-1.0
      A4=(1.0-BETA)*DELTA
      A5=BETA*DELTA
      A6=BETA/(ALPHA*DELTA)
      A7=(BETA/ALPHA)-1.0
      A8=(A7-1.0)*DELTA/2.0
      DO 30 I=1,N
      DO 30 J=1,N
30    TK(I,J)=A1*M(I,J)+A6*C(I,J)+K(I,J)
      CALL SIMUL (TK,ZA,N,0,LA,LB,S)
      DO 100 II=2,NSTEP1
```

```
      TIME=REAL(II-1)*DELTA
      CALL EXTFUN (F,TIME,N)
      DO 40 J=1,N
      XN1(J)=A1*X(II-1,J)+A2*XD(II-1,J)+A3*XDD(II-1,J)
 40   XN2(J)=A6*X(II-1,J)+A7*XD(II-1,J)+A8*XDD(II-1,J)
      CALL XMULT (M,XN1,XN3,N)
      CALL XMULT (C,XN2,XN4,N)
      DO 45 J=1,N
 45   XN1(J)=F(J)+XN3(J)+XN4(J)
      CALL XMULT (TK,XN1,XT,N)
      DO 50 J=1,N
      X(II,J)=XT(J)
      XDD(II,J)=A1*(X(II,J)-X(II-1,J))-A2*XD(II-1,J)-A3*XDD(II-1,J)
 50   XD(II,J)=XD(II-1,J)+A4*XDD(II-1,J)+A5*XDD(II,J)
 100  CONTINUE
      RETURN
      END
C ==================================================================
C
C SUBROUTINE EXTFUN
C
C ==================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
C ==================================================================
C
C SUBROUTINE XMULT
C
C ==================================================================
      SUBROUTINE XMULT (A,B,BB,N)
      DIMENSION A(N,N),B(N),BB(N)
      DO 10 I=1,N
      BB(I)=0.0
      DO 10 J=1,N
 10   BB(I)=BB(I)+A(I,J)*B(J)
      RETURN
      END
```

```
SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N=    2     NSTEP=    24     ALPHA= 0.16666667E+00     BETA= 0.50000000E+00
                                                       DELTA= 0.24216267E+00
SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.2606E-02	0.3228E-01	0.2666E+00	0.1411E+00	0.1143E+01	0.4438E+01
3	0.4843	0.2461E-01	0.1757E+00	0.9181E+00	0.5329E+00	0.2030E+01	0.2893E+01
4	0.7265	0.1005E+00	0.4775E+00	0.1574E+01	0.1088E+01	0.2471E+01	0.7467E+00
5	0.9687	0.2644E+00	0.8845E+00	0.1788E+01	0.1687E+01	0.2382E+01	-.1483E+01

6	1.2108	0.5257E+00	0.1252E+01	0.1251E+01	0.2203E+01	0.1805E+01	-.3285E+01
7	1.4530	0.8530E+00	0.1398E+01	-.5063E-01	0.2534E+01	0.8884E+00	-.4281E+01
8	1.6951	0.1173E+01	0.1175E+01	-.1792E+01	0.2623E+01	-.1529E+00	-.4318E+01
9	1.9373	0.1389E+01	0.5460E+00	-.3400E+01	0.2467E+01	-.1097E+01	-.3481E+01
10	2.1795	0.1413E+01	-.3806E+00	-.4253E+01	0.2114E+01	-.1766E+01	-.2042E+01
11	2.4216	0.1200E+01	-.1369E+01	-.3914E+01	0.1643E+01	-.2058E+01	-.3706E+00
12	2.6638	0.7690E+00	-.2124E+01	-.2317E+01	0.1148E+01	-.1960E+01	0.1175E+01
13	2.9060	0.2111E+00	-.2384E+01	0.1724E+00	0.7195E+00	-.1536E+01	0.2333E+01
14	3.1481	-.3348E+00	-.2017E+01	0.2854E+01	0.4223E+00	-.8929E+00	0.2976E+01
15	3.3903	-.7196E+00	-.1078E+01	0.4907E+01	0.2946E+00	-.1570E+00	0.3102E+01
16	3.6324	-.8292E+00	0.2025E+00	0.5664E+01	0.3445E+00	0.5568E+00	0.2793E+01
17	3.8746	-.6221E+00	0.1475E+01	0.4843E+01	0.5550E+00	0.1156E+01	0.2158E+01
18	4.1168	-.1445E+00	0.2382E+01	0.2647E+01	0.8899E+00	0.1574E+01	0.1296E+01
19	4.3589	0.4811E+00	0.2667E+01	-.2884E+00	0.1299E+01	0.1766E+01	0.2842E+00
20	4.6011	0.1091E+01	0.2257E+01	-.3098E+01	0.1724E+01	0.1702E+01	-.8069E+00
21	4.8433	0.1529E+01	0.1281E+01	-.4966E+01	0.2103E+01	0.1377E+01	-.1882E+01
22	5.0854	0.1689E+01	0.2673E-01	-.5390E+01	0.2372E+01	0.8102E+00	-.2799E+01
23	5.3276	0.1548E+01	-.1150E+01	-.4326E+01	0.2480E+01	0.6281E-01	-.3374E+01
24	5.5697	0.1163E+01	-.1938E+01	-.2188E+01	0.2396E+01	-.7600E+00	-.3421E+01
25	5.8119	0.6543E+00	-.2166E+01	0.3099E+00	0.2118E+01	-.1515E+01	-.2817E+01