Chapter 6

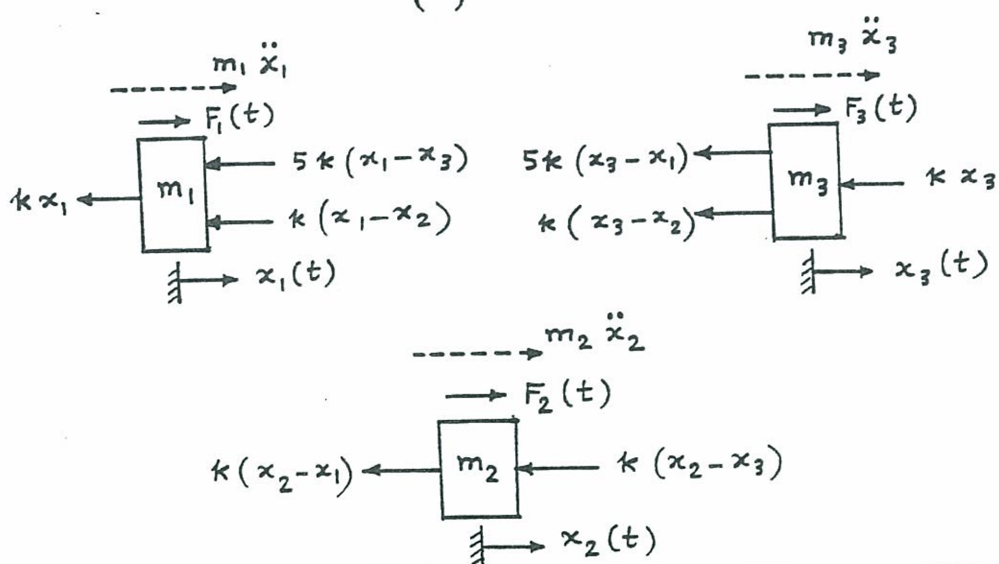
Multidegree of Freedom Systems

6.1 Equations of motion:

$$m_1\,\ddot{x}_1 = -\,k\,x_1 - 5\,k\,(x_1 - x_3) - k\,(x_1 - x_2) + F_1(t)$$
$$m_2\,\ddot{x}_2 = -\,k\,(x_2 - x_1) - k\,(x_2 - x_3) + F_2(t)$$
$$m_3\,\ddot{x}_3 = -\,5\,k\,(x_3 - x_1) - k\,(x_3 - x_2) - k\,x_3 + F_3(t)$$

or

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix}$$



6.2 Equations of motion:

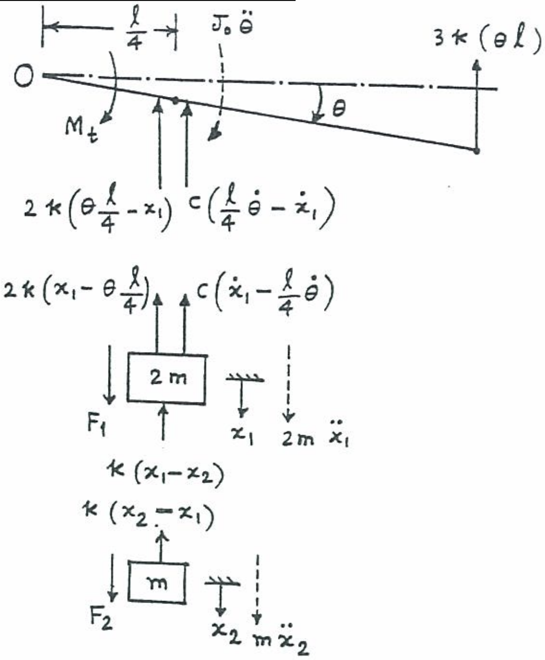
$$J_0\,\ddot{\theta} = -\,2\,k\left(\frac{\ell}{4}\,\theta - x_1\right)\frac{\ell}{4} - c\left(\frac{\ell}{4}\,\dot{\theta} - \dot{x}_1\right)\frac{\ell}{4} - 3\,k\,(\theta\,\ell)\,\ell + M_t$$
$$2\,m\,\ddot{x}_1 = -\,2\,k\left(x_1 - \frac{\ell}{4}\,\theta\right) - c\left(\dot{x}_1 - \frac{\ell}{4}\,\dot{\theta}\right) - k\,(x_1 - x_2) + F_1$$
$$m\,\ddot{x}_2 = -\,k\,(x_2 - x_1) + F_2$$

$$\text{where}\quad J_0 = \frac{1}{3}\,(2\,m)\,\ell^2 = \frac{2}{3}\,m\,\ell^2$$

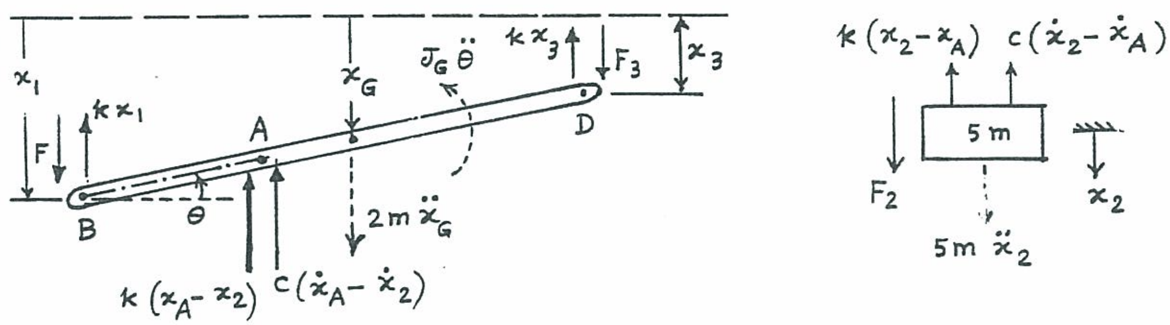
These equations can be stated in matrix form as:

$$\begin{bmatrix} \frac{2}{3}\,m\,\ell^2 & 0 & 0 \\ 0 & 2\,m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{c\,\ell^2}{16} & -\frac{c\,\ell}{4} & 0 \\ -\frac{c\,\ell}{4} & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} \frac{25\,k\,\ell^2}{8} & -\frac{k\,\ell}{2} & 0 \\ -\frac{k\,\ell}{2} & 3\,k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \theta \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} M_t \\ F_1 \\ F_2 \end{Bmatrix}$$



6.3



Equation of motion for rotation about B:

$$J_G\,\ddot{\theta} - 2\,m\,\ddot{x}_G\left(\frac{5\,\ell}{2}\right) = k\,x_3\,(5\,\ell) - F_3\,(5\,\ell) + k\,(x_A - x_2)\,(2\,\ell) + c\,(\dot{x}_A - \dot{x}_2)\,(2\,\ell) \tag{1}$$

Equation of motion for rotation about D:

$$J_G\,\ddot{\theta} + 2\,m\,\ddot{x}_G\left(\frac{5\,\ell}{2}\right) = -\,k\,x_1\,(5\,\ell) + F_1\,(5\,\ell) - k\,(x_A - x_2)\,(3\,\ell) - c\,(\dot{x}_A - \dot{x}_2)\,(3\,\ell) \tag{2}$$

Equation of motion of mass 5m in vertical direction:

$$5\,m\,\ddot{x}_2 = -\,k\,(x_2 - x_A) - c\,(\dot{x}_2 - \dot{x}_A) + F_2 \tag{3}$$

Noting that

$$J_G = \frac{1}{12}\,(2\,m)\,(5\,\ell)^2 = \frac{25}{6}\,m\,\ell^2 \quad ; \quad \theta = \frac{x_1 - x_3}{5\,\ell} \quad ; \quad x_G = \frac{x_1 + x_3}{2}$$

$$\text{and} \quad x_A = x_1 - 2\,\ell\,\theta = \frac{3}{5}\,x_1 + \frac{2}{5}\,x_3$$

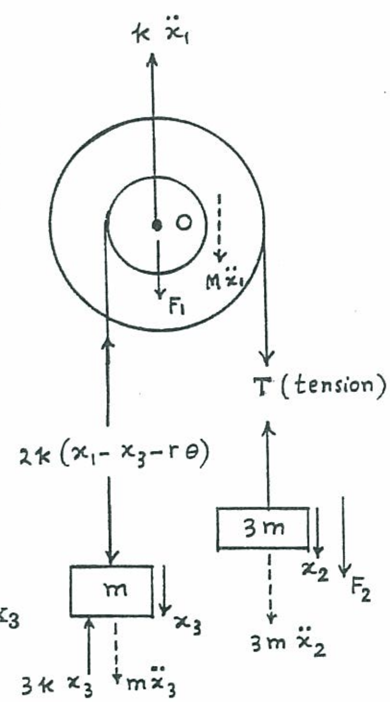
Eqs. (1) to (3) can be rewritten as:

$$\frac{m}{3}\ddot{x}_1 + \frac{2}{3} m \ddot{x}_3 + \frac{29}{25} k x_3 + \frac{6}{25} k x_1 - \frac{2}{5} k x_2 + \frac{6}{25} c \dot{x}_1 + \frac{4}{25} c \dot{x}_3 - \frac{2}{5} c \dot{x}_2 = F_3 \tag{4}$$

$$\frac{2}{3} m \ddot{x}_1 + \frac{m}{3}\ddot{x}_3 + \frac{34}{25} k x_1 - \frac{3}{5} k x_2 + \frac{6}{25} k x_3 + \frac{9}{25} c \dot{x}_1 - \frac{3}{5} c \dot{x}_2 + \frac{6}{25} c \dot{x}_3 = F_1 \tag{5}$$

$$5 m \ddot{x}_2 - \frac{3}{5} k x_1 + k x_2 - \frac{2}{5} k x_3 - \frac{3}{5} c \dot{x}_1 + c \dot{x}_2 - \frac{2}{5} c \dot{x}_3 = F_2 \tag{6}$$

6.4 Equations of motion:



$$\text{Mass M:} \quad M\ddot{x}_1 = -k x_1 + T - 2k(x_1 - x_3 - r\theta) + F_1 \tag{1}$$

$$\text{Mass m:} \quad m\ddot{x}_3 = -3k x_3 + 2k(x_1 - x_3 - r\theta) + F_3 \tag{2}$$

$$\text{Mass 3m:} \quad 3m\ddot{x}_2 = F_2 - T \tag{3}$$

Rotation of pulley:

$$J_0\ddot{\theta} = T(3r) + r(2k)(x_1 - x_3 - r\theta) \tag{4}$$

Noting that

$$\theta = \frac{x_2 - x_1}{3r}$$

and

$$x_1 - x_3 - r\theta = x_1 - x_3 - r\left(\frac{x_2 - x_1}{3r}\right) = \frac{4}{3}x_1 - \frac{1}{3}x_2 - x_3$$

Eq. (4) can be used to find the tension T as:

$$T = \left(\frac{J_0}{9r^2}\right)(\ddot{x}_2 - \ddot{x}_1) - \frac{8}{9} k x_1 + \frac{2}{9} k x_2 + \frac{2}{3} k x_3 \tag{5}$$

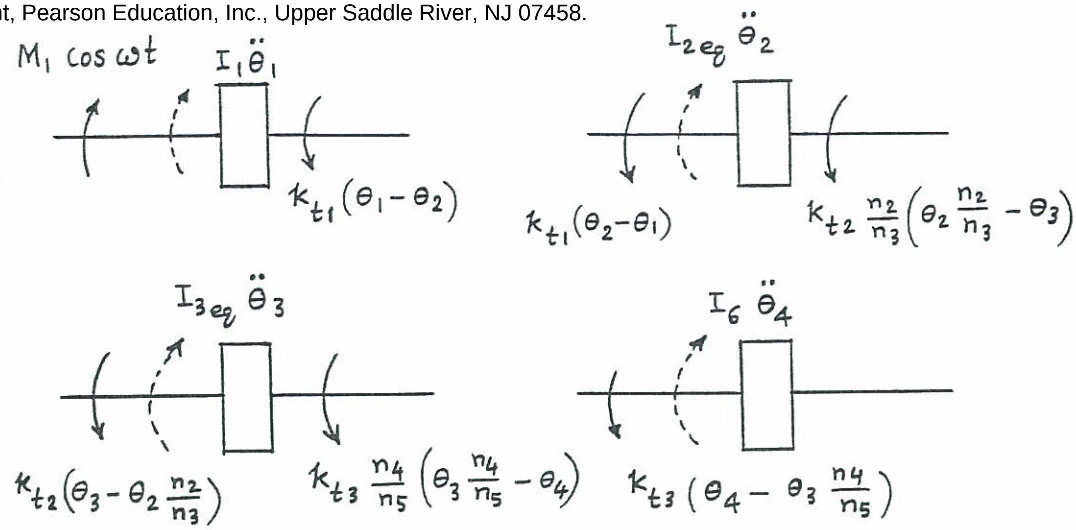
Using the expression of T, Eqs. (1) to (3) can be rewritten as

$$\left(M + \frac{J_0}{9r^2}\right)\ddot{x}_1 - \frac{J_0}{9r^2}\ddot{x}_2 + \frac{41}{9} k x_1 - \frac{8}{9} k x_2 - \frac{8}{3} k x_3 = F_1(t) \tag{6}$$

$$-\frac{J_0}{9r^2}\ddot{x}_1 + \left(3m + \frac{J_0}{9r^2}\right)\ddot{x}_2 - \frac{8}{9} k x_1 + \frac{2}{9} k x_2 + \frac{2}{3} k x_3 K = F_2(t) \tag{7}$$

$$m\ddot{x}_3 - \frac{8}{3} k x_1 + \frac{2}{3} k x_2 + 5 k x_3 = F_3(t) \tag{8}$$

6.5



$$I_2\text{eq} = I_2 + I_3 \left(\frac{n_2}{n_3}\right)^2 \;;\; I_3\text{eq} = I_4 + I_5 \left(\frac{n_4}{n_5}\right)^2$$

Equations of motion:

$$I_1 \ddot{\theta}_1 + k_{t1} (\theta_1 - \theta_2) = M_1 \cos \omega t$$

$$\left(I_2 + I_3 \frac{n_2^2}{n_3^2}\right) \ddot{\theta}_2 + k_{t1} (\theta_2 - \theta_1) + k_{t2} \frac{n_2}{n_3} (\theta_2 \frac{n_2}{n_3} - \theta_3) = 0$$

$$\left(I_4 + I_5 \frac{n_4^2}{n_5^2}\right) \ddot{\theta}_3 + k_{t2} (\theta_3 - \theta_2 \frac{n_2}{n_3}) + k_{t3} \frac{n_4}{n_5} (\theta_3 \frac{n_4}{n_5} - \theta_4) = 0$$

$$I_6 \ddot{\theta}_4 + k_{t3} (\theta_4 - \theta_3 \frac{n_4}{n_5}) = 0$$

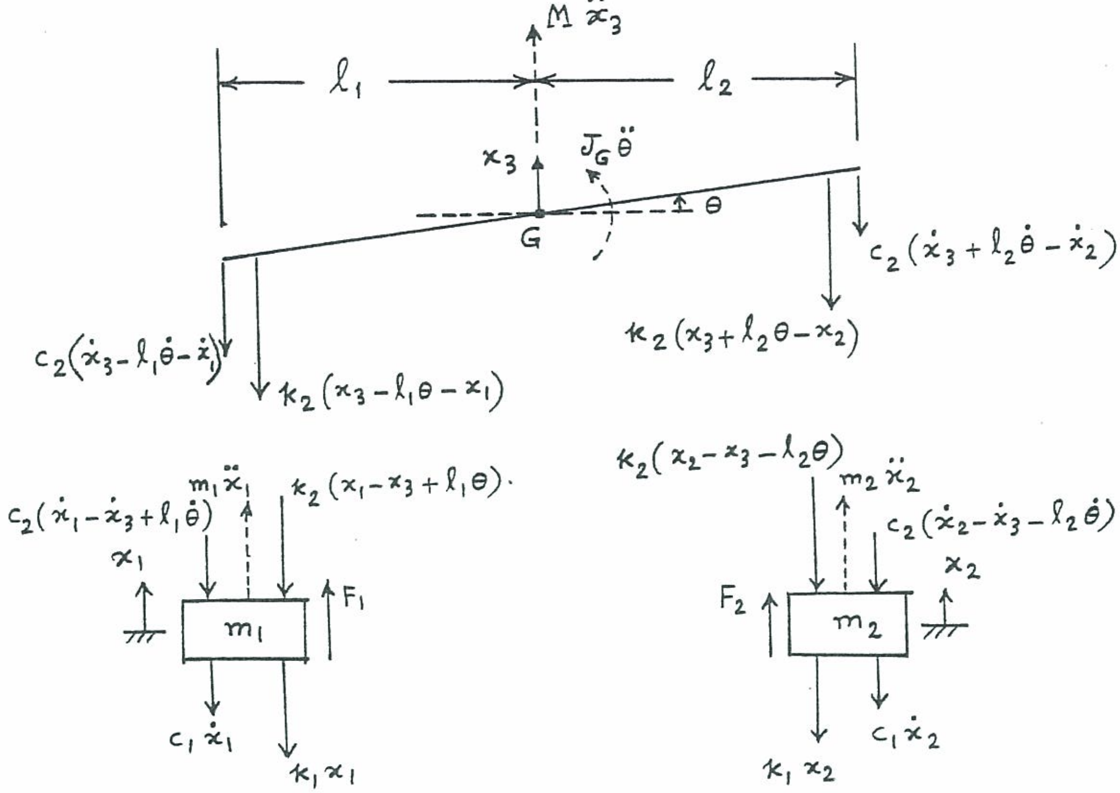
6.6

$$M \ddot{x}_3 = - c_2 (\dot{x}_3 - \ell_1 \dot{\theta} - \dot{x}_1) - k_2 (x_3 - \ell_1 \theta - x_1)$$
$$- c_2 (\dot{x}_3 + \ell_1 \dot{\theta} - \dot{x}_2) - k_2 (x_3 + \ell_2 \theta - x_2)$$

$$J_G \ddot{\theta} = c_2 (\dot{x}_3 - \ell_1 \dot{\theta} - \dot{x}_1) \ell_1 + k_2 (x_3 - \ell_1 \theta - x_1) \ell_1$$
$$- c_2 (\dot{x}_3 + \ell_2 \dot{\theta} - \dot{x}_2) \ell_2 - k_2 (x_3 + \ell_2 \theta - x_2) \ell_2 \qquad (2)$$
$$m_1 \ddot{x}_1 = - c_2 (\dot{x}_1 - \dot{x}_3 + \ell_1 \dot{\theta}) - k_2 (x_1 - x_3 + \ell_1 \theta)$$
$$- c_1 \dot{x}_1 - k_1 x_1 + F_1 \qquad (3)$$
$$m_2 \ddot{x}_2 = - c_2 (\dot{x}_2 - \dot{x}_3 - \ell_2 \dot{\theta}) - k_2 (x_2 - x_3 - \ell_2 \theta)$$
$$- k_1 x_2 - c_1 \dot{x}_2 + F_2 \qquad (4)$$

Eqs. (1) to (4) can be rewritten as

$$M_3 \dot{x}_3 + 2 c_2 \dot{x}_3 - c_2 \dot{x}_1 - c_2 \dot{x}_2 + 2 k_2 x_3$$
$$+ \theta (k_2 \ell_2 - k_2 \ell_1) - k_2 x_1 - k_2 x_2 = 0 \qquad (5)$$

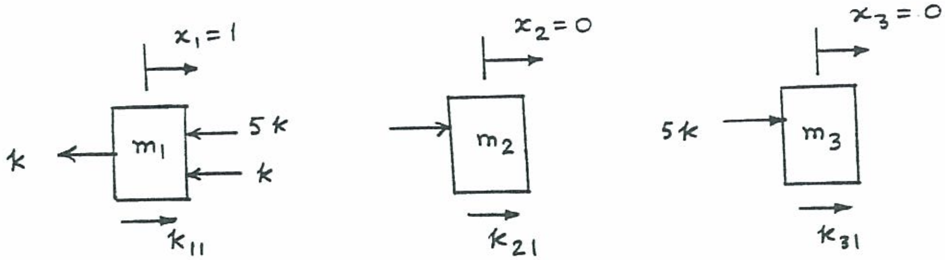


$$J_G\,\ddot{\theta} + (c_2\,\ell_1^2 + c_2\,\ell_2^2)\,\dot{\theta} + (-\,c_2\,\ell_1 + c_2\,\ell_2)\,\dot{x}_3 + c_2\,\ell_1\,\dot{x}_1 - c_2\,\ell_2\,\dot{x}_2$$
$$+ (-\,k_2\,\ell_1 + k_2\,\ell_2)\,x_3 + k_2\,\ell_1\,x_1 - k_2\,\ell_2\,x_2 + (k_2\,\ell_1^2 + k_2\,\ell_2^2)\,\theta = 0 \qquad (6)$$

$$m_1\,\ddot{x}_1 + (c_1 + c_2)\,\dot{x}_1 - c_2\,\dot{x}_3 + c_2\,\ell_1\,\dot{\theta} + (k_1 + k_2)\,x_1$$
$$-\,k_2\,x_3 + k_2\,\ell_1\,\theta = F_1 \qquad (7)$$

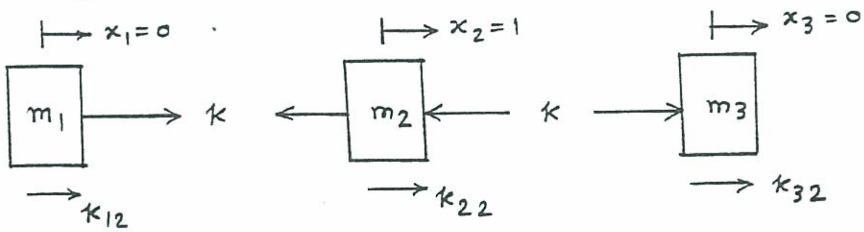
$$m_2\,\ddot{x}_2 + (c_1 + c_2)\,\dot{x}_2 - c_2\,\dot{x}_3 - c_2\,\ell_2\,\dot{\theta}$$
$$+ (k_1 + k_2)\,x_2 - k_2\,x_3 - k_2\,\ell_2\,\theta = F_2 \qquad (8)$$

6.12



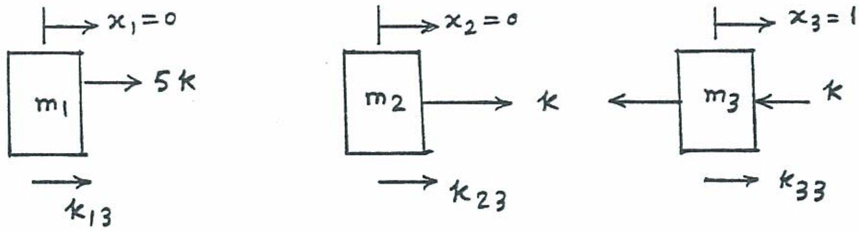
(i) Set $x_1 = 1, x_2 = x_3 = 0$:

Equilibrium of forces:
$k_{11} - k - 5k - k = 0$ or $k_{11} = 7k$; $k_{21} = -k$; $k_{31} = -5k$



(ii) Set $x_2 = 1$, $x_1 = x_3 = 0$:

Equilibrium of forces:
$k_{12} = -k$; $k_{22} - k - k = 0$ or $k_{22} = 2k$; $k_{32} = -k$

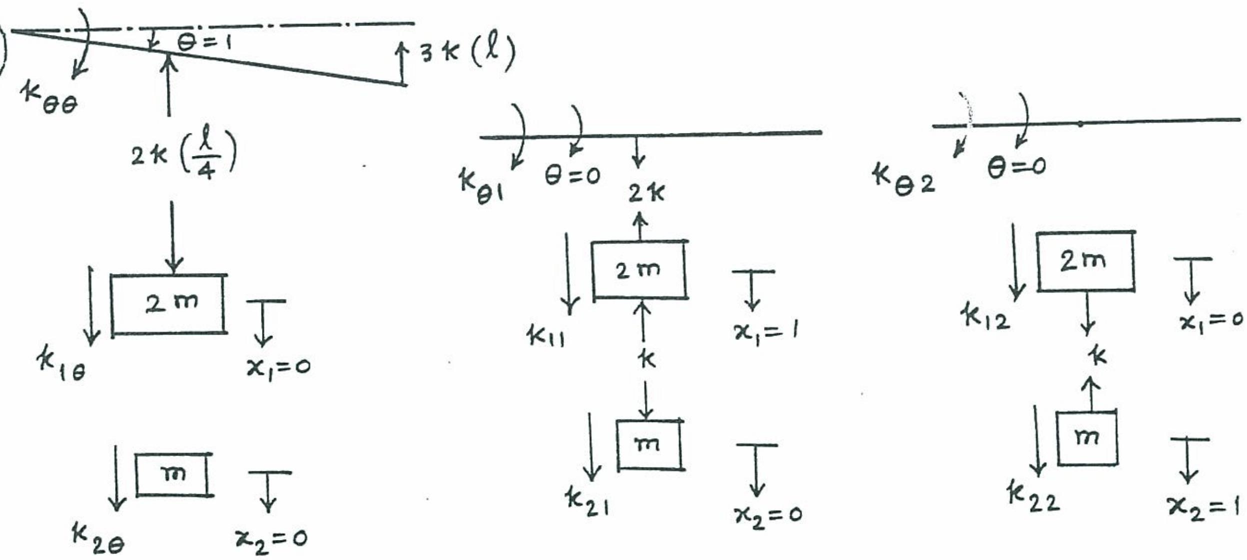


(iii) Set $x_3 = 1$, $x_1 = x_2 = 0$:

Equilibrium of forces:
$k_{13} = -5k$; $k_{23} = -k$; $k_{33} = 7k$

$$\therefore \quad [k] = \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix}$$

6.13



(i) Give $\theta = 1$, $x_1 = x_2 = 0$:

Equilibrium equations:

$$k_{\theta\theta} = 3\,k\,\ell^2 + 2\,k\,(\ell/4)^2 = \frac{25}{8}\,k\,\ell^2 \quad ; \quad k_{1\theta} = -\frac{k\,\ell}{2} \quad ; \quad k_{2\theta} = 0$$

(ii) Give $x_1 = 1$, $\theta = x_2 = 0$:

Equilibrium equations:

$$k_{\theta 1} = -2\,k\,(\ell/4) = -\frac{k\,\ell}{2} \quad ; \quad k_{11} = 2\,k + k = 3\,k \quad ; \quad k_{21} = -k$$

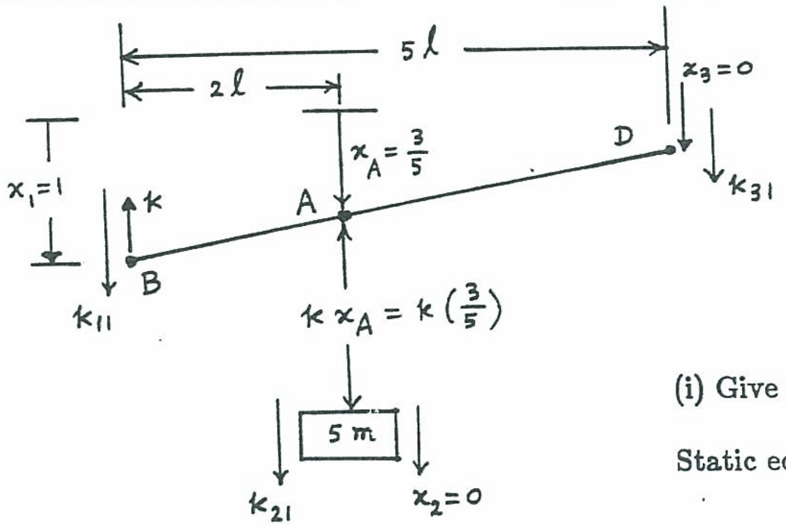
(iii) Give $x_2 = 1$, $\theta = x_1 = 0$:

Equilibrium equations:

$$k_{\theta 2} = 0 \;\; ; \;\; k_{12} = -k \;\; ; \;\; k_{22} = k$$

$$\therefore \;\; [k] = \begin{bmatrix} \dfrac{25\,k\,\ell^2}{8} & -\dfrac{k\,\ell}{2} & 0 \\ -\dfrac{k\,\ell}{2} & 3\,k & -k \\ 0 & -k & k \end{bmatrix}$$

6.14



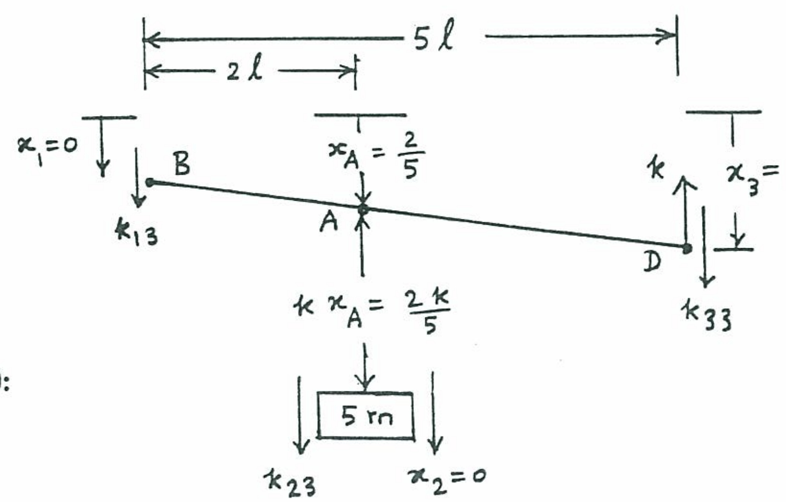
(i) Give $x_1 = 1$, $x_2 = x_3 = 0$:

Static equilibrium equations:

$$\sum M_B = 0 \;\text{ or }\; k_{31}\,(5\,\ell) - \frac{3}{5}\,k\,(2\,\ell) = 0 \;\text{ or }\; k_{31} = \frac{6}{25}\,k$$

$$\sum M_D = 0 \;\text{ or }\; k_{11}\,(5\,\ell) - k\,(5\,\ell) - \frac{3\,k}{5}\,(3\,\ell) = 0 \;\text{ or }\; k_{11} = \frac{34}{25}\,k$$

$$\sum F = 0 \text{ at mass 5m } \;\text{ or }\; k_{21} + \frac{3}{5}\,k = 0 \;\text{ or }\; k_{21} = -\frac{3}{5}\,k$$



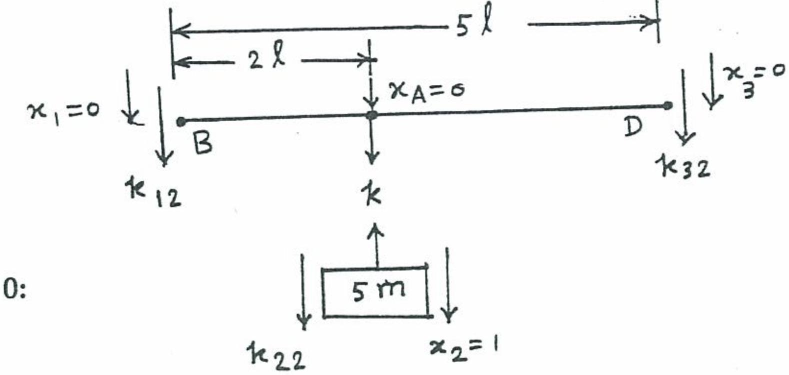
(ii) Give $x_3 = 1$, $x_1 = x_2 = 0$:

Static equilibrium equations:

$$\sum M_B = 0 \;\text{ or }\; k_{33}\,(5\,\ell) - \frac{2\,k}{5}\,(2\,\ell) - k\,(5\,\ell) = 0 \;\text{ or }\; k_{33} = \frac{29}{25}\,k$$

$$\sum M_D = 0 \text{ or } k_{13}\,(5\,\ell) - \frac{2\,k}{5}\,(3\,\ell) = 0 \text{ or } k_{13} = \frac{6}{25}\,k$$

$$\sum F = 0 \text{ at mass 5m or } k_{23} + \frac{2}{5}\,k = 0 \text{ or } k_{23} = -\frac{2}{5}\,k$$



(iii) Give $x_3 = 1, x_1 = x_2 = 0$:

Static equilibrium equations:

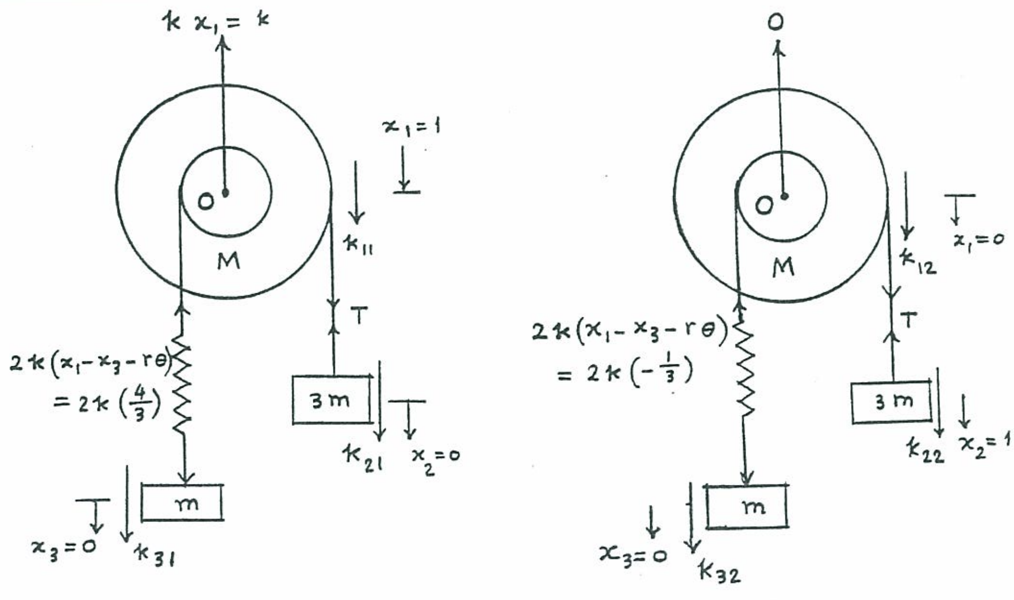
$$\sum M_B = 0 \text{ or } k_{32}\,(5\,\ell) + k\,(2\,\ell) = 0 \text{ or } k_{32} = -\frac{2}{5}\,k$$

$$\sum M_D = 0 \text{ or } k_{12}\,(5\,\ell) + k\,(3\,\ell) = 0 \text{ or } k_{12} = -\frac{3}{5}\,k$$

$$\sum F = 0 \text{ at mass 5m or } k_{22} - k = 0 \text{ or } k_{22} = k$$

$$\therefore \quad [k] = k\begin{bmatrix} \frac{34}{25} & -\frac{3}{5} & \frac{6}{25} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \\ \frac{6}{25} & -\frac{2}{5} & \frac{29}{25} \end{bmatrix}$$

6.15



$$\text{Here } \theta = \frac{x_2 - x_1}{3\,r}\,; \; x_1 - x_3 - r\,\theta = \frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3$$

(i) Give $x_1 = 1$, $x_2 = x_3 = 0$:

$$\sum F = 0 \text{ for mass M:} \quad k_{11} - k + T - \frac{8\,k}{3} = 0 \tag{1}$$

$$\sum F = 0 \text{ for mass m:} \quad k_{31} + \frac{8\,k}{3} = 0 \tag{2}$$

$$\sum F = 0 \text{ for mass 3m:} \quad k_{21} = T \tag{3}$$

$$\sum M_0 = 0 \text{ for pulley:} \quad T\,(3\,r) + 2\,k\left(\frac{4}{3}\right) r = 0 \text{ or } T = -\frac{8\,k}{9} \tag{4}$$

Eqs. (1) to (4) yield:

$$k_{21} = -\frac{8\,k}{9}, \quad k_{31} = -\frac{8\,k}{3}, \quad k_{11} = \frac{41}{9}\,k$$

(ii) Give $x_2 = 1$, $x_1 = x_3 = 0$:

$$\sum F = 0 \text{ for mass M:} \quad k_{12} + T + \frac{2\,k}{3} = 0 \tag{5}$$

$$\sum F = 0 \text{ for mass 3m:} \quad k_{22} - T = 0 \tag{6}$$

$$\sum F = 0 \text{ for mass m:} \quad k_{32} - \frac{2\,k}{3} = 0 \tag{7}$$

$$\sum M_0 = 0 \text{ for pulley:} \quad T\,(3\,r) + r\,(-\frac{2\,k}{3}) = 0 \tag{8}$$

Solution of Eqs. (5) to (8) yields:

$$k_{22} = T = \frac{2\,k}{9}, \quad k_{32} = \frac{2\,k}{3}, \quad k_{12} = -\frac{8\,k}{9}$$

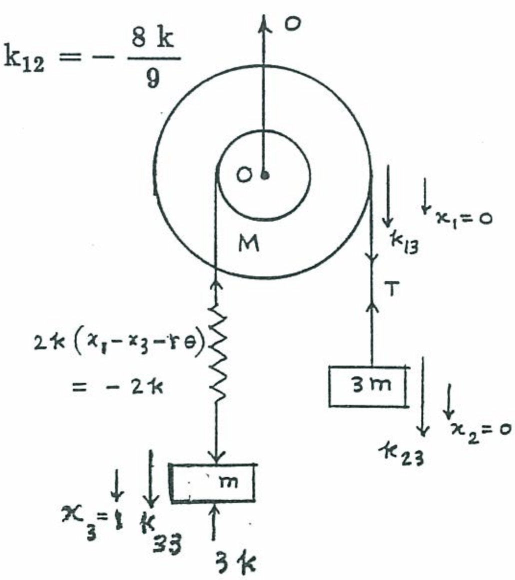
(iii) Give $x_3 = 1$, $x_1 = x_2 = 0$:

$$\sum F = 0 \text{ for mass M:} \quad k_{13} + T - (-2\,k) = 0 \tag{9}$$

$$\sum F = 0 \text{ for mass 3m:} \quad k_{23} - T = 0 \tag{10}$$

$$\sum F = 0 \text{ for mass m:} \quad k_{33} - 2\,k - 3\,k = 0 \tag{11}$$

$$\sum M_0 = 0 \text{ for pulley:} \quad T\,(3\,r) + r\,(-2\,k) = 0 \tag{12}$$

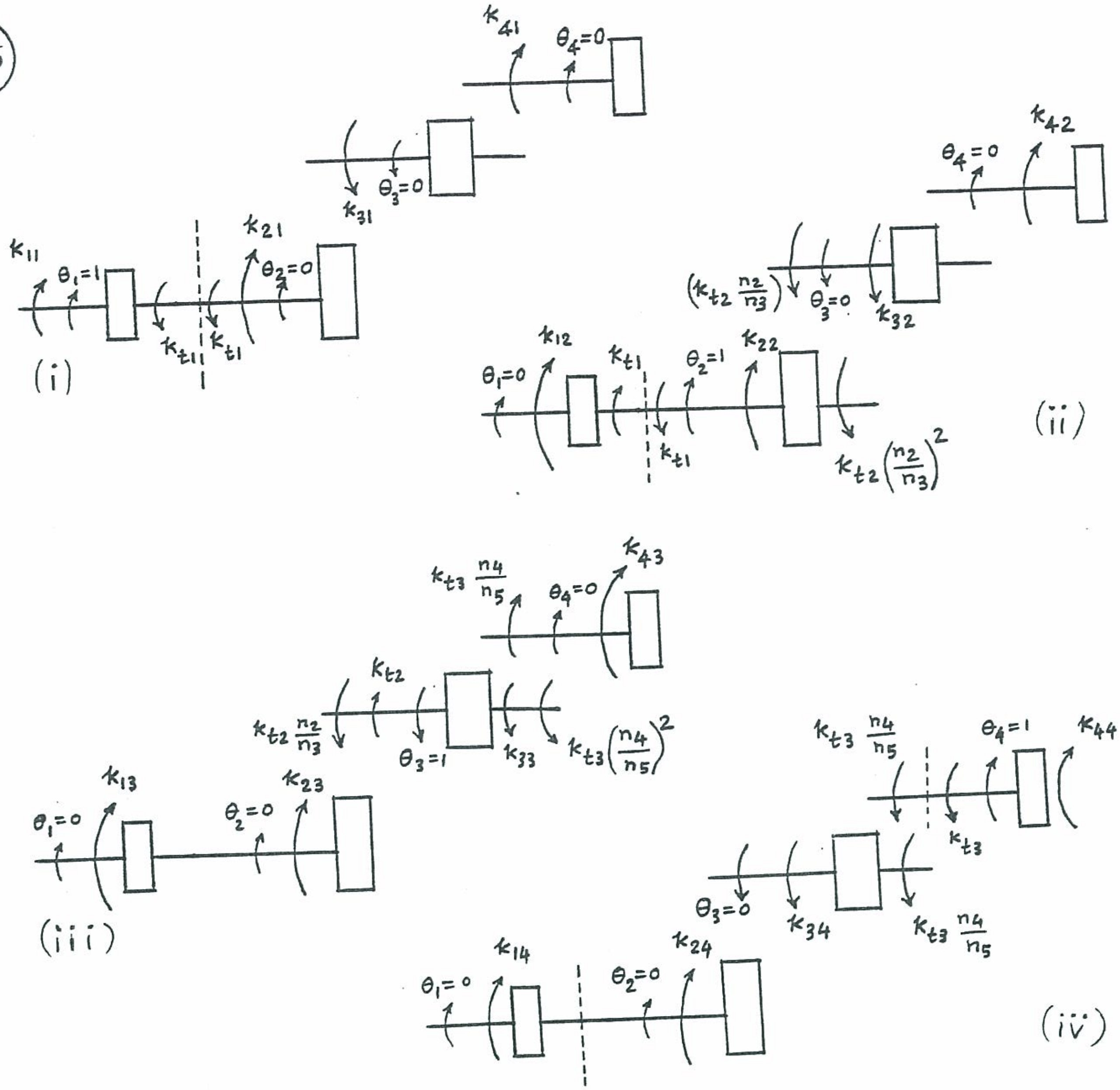


Solution of Eqs. (9) to (12) gives:

$$k_{33} = 5\,k, \quad k_{23} = \frac{2\,k}{3}, \quad k_{13} = -\frac{8\,k}{3}$$

$$\therefore \quad [k] = k\begin{bmatrix} \frac{41}{9} & -\frac{8}{9} & -\frac{8}{3} \\ -\frac{8}{9} & \frac{2}{9} & \frac{2}{3} \\ -\frac{8}{3} & \frac{2}{3} & 5 \end{bmatrix}$$

6.16



(i) Set $\theta_1 = 1$, $\theta_2 = \theta_3 = \theta_4 = 0$:

Equilibrium equations give:

$$k_{11} = k_{t1},\ \ k_{21} = -k_{t1},\ \ k_{31} = k_{41} = 0$$

(ii) Set $\theta_2 = 1$, $\theta_1 = \theta_3 = \theta_4 = 0$:

Equilibrium equations yield:

$$k_{12} = -k_{t1},\ \ k_{22} = k_{t1} + k_{t2}\left(\frac{n_2}{n_3}\right)^2 \ \ ,\ \ k_{32} = -k_{t2}\left(\frac{n_2}{n_3}\right),\ \ k_{42} = 0$$

(iii) Set $\theta_3 = 1$, $\theta_1 = \theta_2 = \theta_4 = 0$:

Equilibrium equations provide:

$$k_{13} = 0,\ \ k_{23} = -k_{t2}\left(\frac{n_2}{n_3}\right),\ \ k_{33} = k_{t2} + k_{t3}\left(\frac{n_4}{n_5}\right)^2,\ \ k_{43} = -k_{t3}\left(\frac{n_4}{n_5}\right)$$

(iv) Set $\theta_4 = 1$, $\theta_1 = \theta_2 = \theta_3 = 0$:

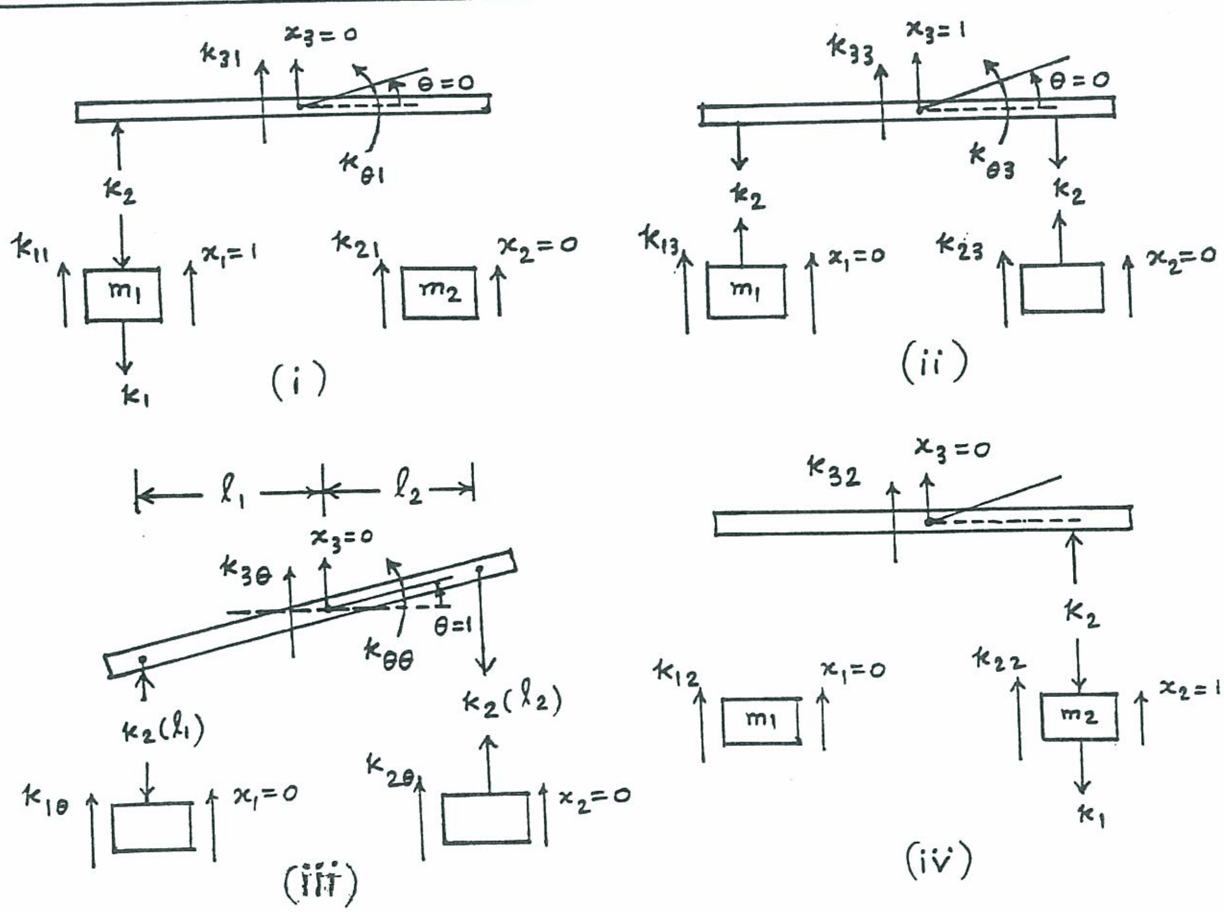
Equilibrium equations give:

$$k_{14} = k_{24} = 0,\ \ k_{34} = -k_{t3}\left(\frac{n_4}{n_5}\right),\ \ k_{44} = k_{t3}$$

Thus the stiffness matrix is:

$$[k] = \begin{bmatrix} k_{t1} & -k_{t1} & 0 & 0 \\ -k_{t1} & k_{t1} + k_{t2}\left(\frac{n_2}{n_3}\right)^2 & -k_{t2}\left(\frac{n_2}{n_3}\right) & 0 \\ 0 & -k_{t2}\left(\frac{n_2}{n_3}\right) & k_{t2} + k_{t3}\left(\frac{n_4}{n_5}\right)^2 & -k_{t3}\left(\frac{n_4}{n_5}\right) \\ 0 & 0 & -k_{t3}\left(\frac{n_4}{n_5}\right) & k_{t3} \end{bmatrix}$$

6.17



(i) Set $x_1 = 1$, $x_2 = x_3 = \theta = 0$:

Equilibrium equations:

$$k_{11} = k_1 + k_2,\ \ k_{31} = -k_2,\ \ k_{\theta 1} = k_2\,\ell_1,\ \ k_{21} = 0$$

(ii) Set $x_3 = 1$, $x_1 = x_2 = \theta = 0$:

Equilibrium equations:

$$k_{33} = 2\,k_2,\ \ k_{13} = -k_2,\ \ k_{23} = -k_2,\ \ k_{\theta 3} = -k_2\,\ell_1 + k_2\,\ell_2$$

(iii) Set $\theta = 1$, $x_1 = x_2 = x_3 = 0$:

Equilibrium equations:

$$k_{\theta\theta} = k_2\,(\ell_1^2 + \ell_2^2),\ \ k_{\theta 3} = k_2\,(\ell_2 - \ell_1)\ ,\ \ k_{1\theta} = k_2\,\ell_1,\ \ k_{2\theta} = -k_2\,\ell_2$$

(iv) Set $x_2 = 1$, $x_1 = x_3 = \theta = 0$:

Equilibrium equations:

$$k_{12} = 0,\ \ k_{32} = -k_2,\ \ k_{\theta 2} = -k_2\,\ell_2,\ \ k_{22} = k_1 + k_2$$

$$\therefore\ \ [k] = \begin{bmatrix} (k_1 + k_2) & 0 & -k_2 & k_2\,\ell_1 \\ 0 & (k_1 + k_2) & -k_2 & -k_2\,\ell_2 \\ -k_2 & -k_2 & 2\,k_2 & k_2\,(\ell_2 - \ell_1) \\ k_2\,\ell_1 & -k_2\,\ell_2 & k_2\,(\ell_2 - \ell_1) & k_2(\ell_1^2 + \ell_2^2) \end{bmatrix}$$

6.18 (i) Give $F_x = 1$, $M_\theta = 0$:

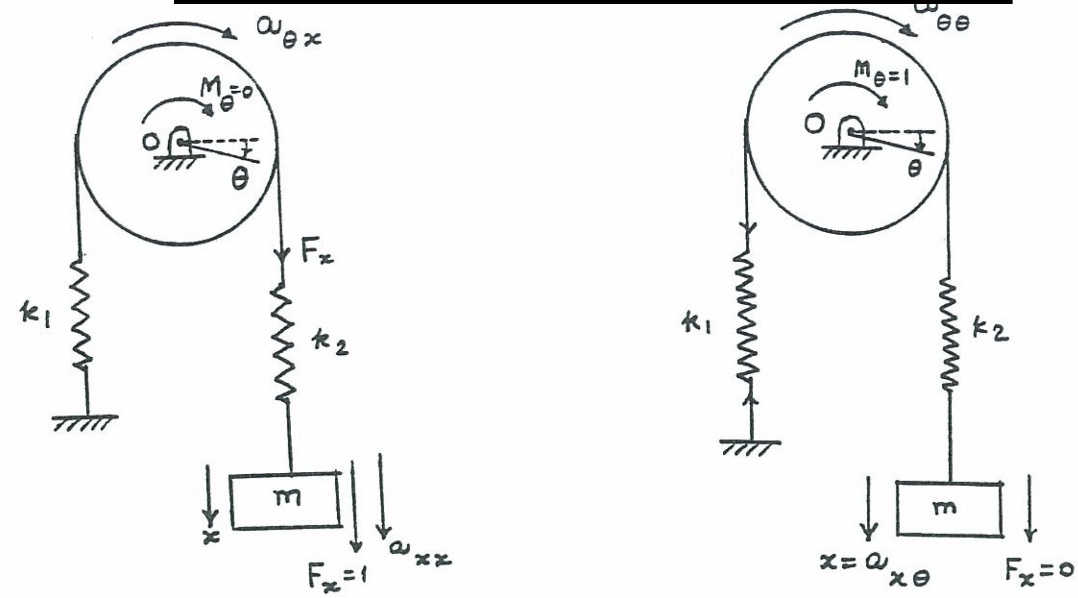
Same force of $F_x = 1$ is induced everywhere along the rope. Since k_1 and k_2 are in series,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$a_{xx} = \frac{\text{force}}{k_{eq}} = \text{deflection at m due } F_x \text{ of } 1 = \frac{k_1 + k_2}{k_1\,k_2}$$

Linear deflection of k_1 under $F_x = 1$ is $\dfrac{1}{k_1}$, angular deflection θ due to linear displacement of $\dfrac{1}{k_1} = a_{\theta x}$.

$$a_{\theta x} = \text{linear deflection of } k_1 = a_{\theta x}\,r = \frac{1}{k_1}\ \ ;\ \ a_{\theta x} = \frac{1}{k_1\,r}$$



(ii) Give $M_\theta = 1$, $F_x = 0$:

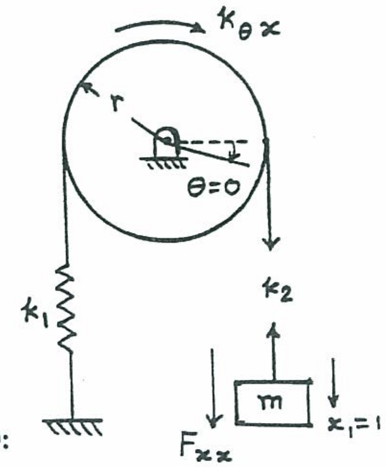
Extension of $k_1 = a_{\theta\theta}$ r, spring force $= k_1$ ($a_{\theta\theta}$ r).

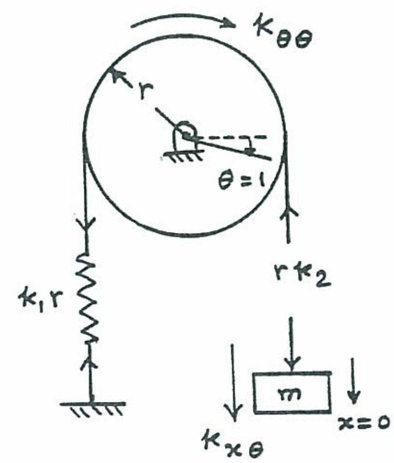
$$\sum M_\theta = 0 \text{ or } M_\theta = \left(k_1\ a_{\theta\theta}\ r\right) r = 1 \text{ or } a_{\theta\theta} = \frac{1}{k_1\ r^2}$$

Displacement of $m = a_{x\theta} = a_{\theta\theta}\ r = \dfrac{1}{k_1\ r}$

$$\therefore\ [a] = \begin{bmatrix} \dfrac{k_1+k_2}{k_1\ k_2} & \dfrac{1}{k_1\ r} \\ \dfrac{1}{k_1\ r} & \dfrac{1}{k_1\ r^2} \end{bmatrix}$$

6.19





(i) Give $x = 1$, $\theta = 0$:

Equilibrium equations give:

$$k_{xx} - k_2 = 0 \text{ or } k_{xx} = k_2 \quad ; \quad k_{\theta x} + k_2\ r = 0 \text{ or } k_{\theta x} = -\,k_2\ r$$

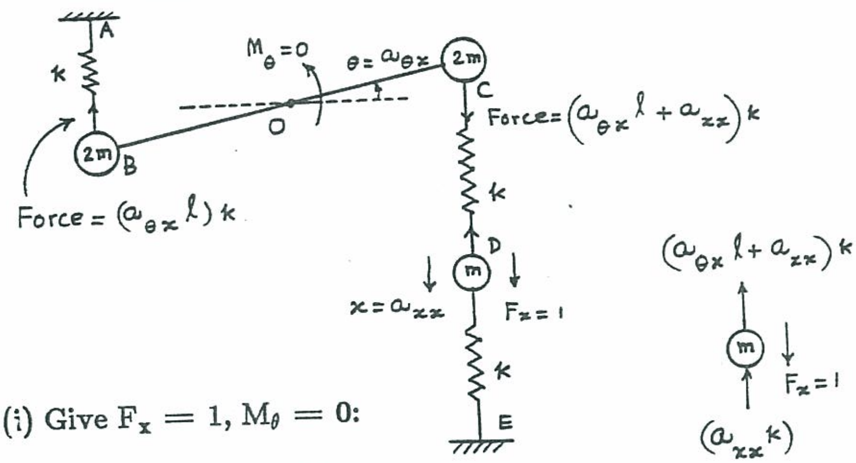
(ii) Give $\theta = 1$, $x = 0$:

Equilibrium equations yield:

$$k_{x\theta} + r\,k_2 = 0 \;\text{ or }\; k_{x\theta} = -k_2\,r$$
$$k_{\theta\theta} - r^2\,k_2 - r^2\,k_1 = 0 \;\text{ or }\; k_{\theta\theta} = r^2\,(k_1 + k_2)$$

$$\therefore\;[k] = \begin{bmatrix} k_2 & -k_2\,r \\ -k_2\,r & (k_1 + k_2)\,r^2 \end{bmatrix}$$

6.20



(i) Give $F_x = 1$, $M_\theta = 0$:

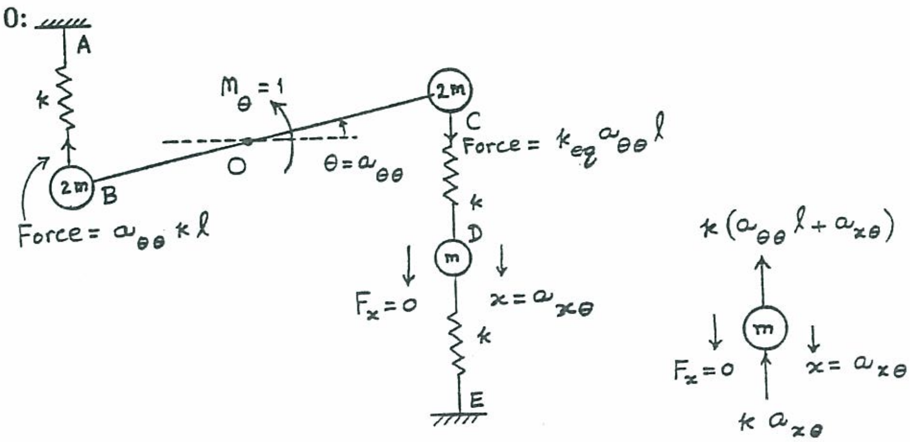
Extension of spring AB $= a_{\theta x}\,\ell$.
Total extension of spring CD $= (a_{\theta x}\,\ell + a_{xx})$.
Compression of spring DE $= a_{xx}$.

$$\sum F = 0 \;\text{ or }\; a_{xx}\,k + a_{\theta x}\,\ell\,k + a_{xx}\,k = 1 \;\text{ or }\; 2\,a_{xx}\,k + a_{\theta x}\,k\,\ell = 1 \qquad (1)$$
$$\sum M_0 = 0 \;\text{ or }\; a_{\theta x}\,k\,\ell + (a_{\theta x}\,k\,\ell + a_{xx}\,k) = 0 \;\text{ or }\; 2\,a_{\theta x}\,k\,\ell + a_{xx}\,k = 0$$

Solution of Eqs. (1) and (2):

$$a_{xx} = \frac{2}{3\,k}\;;\; a_{\theta x} = -\frac{1}{3\,k\,\ell}$$

(ii) Give $M_\theta = 1$, $F_x = 0$:



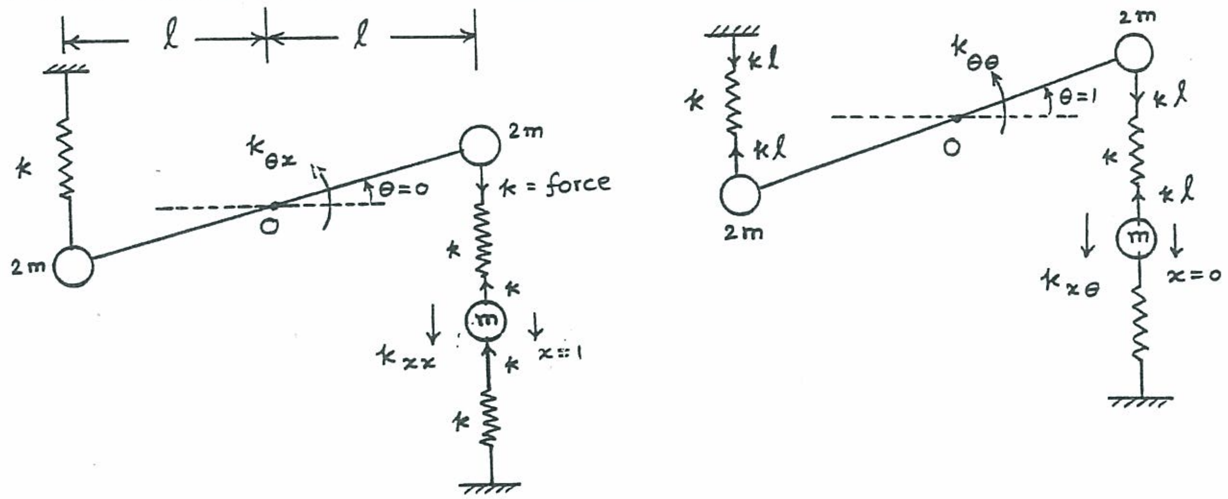
Extension of spring AB $= a_{\theta\theta}\,\ell$.
Total extension of springs CD and DE $= a_{\theta\theta}\,\ell$.
k_{eq} of series springs CD and DE $= \frac{k}{2}$.

$$\sum M_0 = 0 \text{ or } (a_{\theta\theta}\, k\, \ell)\, \ell + (k_{eq}\, a_{\theta\theta}\, \ell)\, \ell = 1 \text{ or } a_{\theta\theta} = \frac{2}{3\, k\, \ell^2} \qquad (3)$$

$$\sum F = 0 \text{ or } k\, a_{x\theta} + k\, a_{\theta\theta}\, \ell + k\, a_{x\theta} = 0 \text{ or } a_{x\theta} = -\frac{1}{3\, k\, \ell} \qquad (4)$$

$$\therefore\ [a] = \begin{bmatrix} \frac{2}{3\,k} & -\frac{1}{3\,k\,\ell} \\ -\frac{1}{3\,k\,\ell} & \frac{2}{3\,k\,\ell^2} \end{bmatrix}$$

6.21



(i) Give $x = 1$, $\theta = 0$:

$$\sum F = 0 \text{ or } k_{xx} - k - k = 0 \text{ or } k_{xx} = 2\,k$$
$$\sum M_0 = 0 \text{ or } k_{\theta x} - k\,\ell = 0 \text{ or } k_{\theta x} = k\,\ell$$

(ii) Give $\theta = 1$, $x = 0$:

$$\sum F = 0 \text{ or } k_{x\theta} - k\,\ell = 0 \text{ or } k_{x\theta} = k\,\ell$$
$$\sum M_0 = 0 \text{ or } k_{\theta\theta} - k\,\ell\,(\ell) - k\,\ell\,(\ell) = 0 \text{ or } k_{\theta\theta} = 2\,k\,\ell^2$$

$$\therefore\ [k] = \begin{bmatrix} 2\,k & k\,\ell \\ k\,\ell & 2\,k\,\ell^2 \end{bmatrix}$$

6.22 Kinetic energy of the system can be expressed as:

$$T = \frac{1}{2}\,(2m)\,(\ell\,\dot\theta)^2 + \frac{1}{2}\,(2m)\,(\ell\,\dot\theta^2) + \frac{1}{2}\,m\,\dot x^2 = \frac{1}{2}\,(4m)\,(\ell\,\dot\theta)^2 + \frac{1}{2}\,m\,\dot x^2$$

which can be expressed in matrix form as

$$T = \frac{1}{2}(\dot{x} \quad \dot{\theta})\,[m]\begin{Bmatrix}\dot{x}\\ \dot{\theta}\end{Bmatrix} \qquad \text{where} \quad [m] = \begin{bmatrix} m & 0 \\ 0 & 4\,m\,\ell^2 \end{bmatrix}$$

6.23 Flexibility influence coefficients:

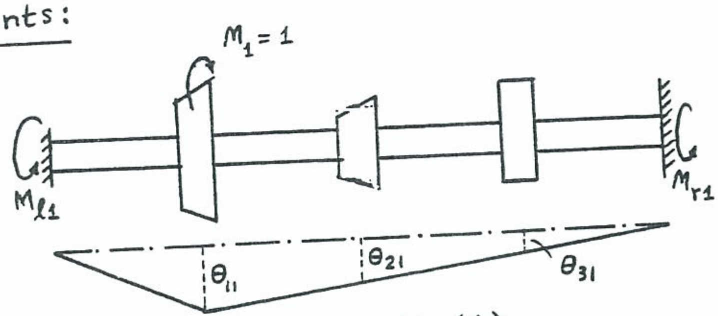


Fig. (A)

Spring constants of different sections of the shaft (k_i) are

$$k_i = \frac{(GJ)_i}{\ell_i}\ ; \ i = 1,2,3,4$$

where $(GJ)_i$ = torsional rigidity, J_i = polar moment of inertia, and ℓ_i = length of section i of shaft.

Consider disc 1. shaft to the left of disc 1 has a spring constant of k_1 while the shaft to the right side of disc 1 has an equivalent spring constant of $\quad k_{e1} = \dfrac{1}{\sum_{i=2}^{4}\left(\frac{1}{k_i}\right)}$

If we apply unit torque to disc 1 ($M_1 = 1$) as shown in Fig. (A), reactive torques at left and right ends of the shaft are

$$M_{r1} = k_{e1}\,\theta_{11}\ , \quad M_{\ell 1} = k_1\,\theta_{11}$$

Since $\quad M_{\ell 1} + M_{r1} = M_1 = 1$, we get

$$\theta_{11} = a_{11} = \left\{\sum_{i=2}^{4}\left(\frac{1}{k_i}\right)\right\} \Big/ \left\{k_1 \cdot \sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}$$

$$M_{\ell 1} = k_1\,\theta_{11} = \sum_{i=2}^{4}\left(\frac{1}{k_i}\right) \Big/ \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\ ; \quad M_{r1} = \frac{\theta_{11}}{\sum_{i=2}^{4}\left(\frac{1}{k_i}\right)} = \frac{1}{k_1 \cdot \sum_{i=1}^{4}\left(\frac{1}{k_i}\right)}$$

Also,
$$\theta_{31} = a_{31} = \frac{M_{r1}}{k_4} = \frac{1}{k_1 k_4 \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

$$\theta_{21} = a_{21} = M_{r1} \Big/ \left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} = \frac{1}{k_1 \cdot \left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} \cdot \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

Consider disc 2. Shaft to the left side of disc 2 has an equivalent spring constant of k_{e2} and the shaft to its right side has an equivalent spring constant of k_{e3} with

$$k_{e2} = \frac{1}{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)} , \qquad k_{e3} = \frac{1}{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)}$$

If we apply a unit torque to disc 2 ($M_2 = 1$), reactive torques at the left and right ends of shaft are

$$M_{\ell 2} = \theta_{22}\, k_{e2} , \qquad M_{r2} = \theta_{22}\, k_{e3} \quad \text{with} \quad M_2 = M_{\ell 2} + M_{r2} = 1$$

Hence

$$\theta_{22} = a_{22} = \left\{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)\right\}\cdot\left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} \Big/ \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}$$

$$\theta_{32} = a_{32} = \frac{M_{r2}}{k_4} = \frac{\theta_{22}}{k_{e3}\cdot k_4} = \left\{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)\right\} \Big/ \left[k_4 \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\right]$$

$$\theta_{12} = a_{12} = \frac{M_{\ell 2}}{k_1} = \frac{\theta_{22}}{k_{e2}\cdot k_1} = \left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} \Big/ \left[k_1 \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\right]$$

Consider disc 3. Apply unit torque to disc 3 ($M_3 = 1$) to obtain

$$M_{\ell 3} = k_{e4}\,\theta_{33} , \quad M_{r3} = k_4\,\theta_{33} , \quad M_{r3} + M_{\ell 3} = M_3 = 1$$

where $k_{e4} = \dfrac{1}{\sum_{i=1}^{3}\left(\frac{1}{k_i}\right)}$

Hence

$$\theta_{33} = a_{33} = \left\{\sum_{i=1}^{3}\left(\frac{1}{k_i}\right)\right\} \Big/ \left[k_4 \cdot \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\right]$$

$$\theta_{13} = a_{13} = \frac{M_{\ell 3}}{k_1} = \frac{1}{k_1 k_4 \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

$$\theta_{23} = a_{23} = \frac{M_{\ell 3}}{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)} = \frac{1}{k_4 \left\{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)\right\}\left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

Flexibility matrix is
$$[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Note: In this problem, it is much easier to derive the stiffness influence coefficients, k_{ij}, compared to a_{ij}. Hence it is advisable to find [k] first and then find [a] by inverting the matrix [k].

Stiffness influence coefficients:

Let angular displacement of disc 1 be unity ($\theta_1 = 1$) and of discs 2 and 3 be zero as shown in Fig. (B). If the torque applied to disc 1 is M_{11} and the reactions are $M_{\ell 1}$ and M_{21}, we have

$M_{\ell 1} = k_1\,\theta_{11} = k_1$, $\quad M_{21} = k_2\,\theta_{11} = k_2$

$M_{11} = M_{\ell_1} + M_{21} \equiv k_{11} = k_1 + k_2$

$k_{21} = -M_{21} = -k_2$ ($\because$ reactive torque is opposite to M_{11} in direction)

$k_{31} = M_{31} = 0$ ($\because$ disc 2 is fixed, no reactive torque is felt at disc 3)

Let displacement of disc 2 = 1 and displacements of discs 1 and 3 be zero.

$k_{22} = k_2 + k_3$, $\quad k_{12} = -k_2$, $\quad k_{32} = -k_3$

Similarly we can obtain

$k_{33} = k_3 + k_4$, $\quad k_{13} = 0$, $\quad k_{23} = -k_3$

stiffness matrix is $\quad [k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$

Equations of motion of the system:

$$[m]\,\ddot{\vec{\theta}} + [k]\,\vec{\theta} = \vec{M}_t$$

where $\quad [m] = \begin{bmatrix} J_{d1} & 0 & 0 \\ 0 & J_{d2} & 0 \\ 0 & 0 & J_{d3} \end{bmatrix}$ = matrix of mass moments of inertia of the discs

$\vec{\theta} = \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{Bmatrix}$ and $\vec{M}_t = \begin{Bmatrix} M_{t1}(t) \\ M_{t2}(t) \\ M_{t3}(t) \end{Bmatrix}$ = vector of external torques applied to discs $= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

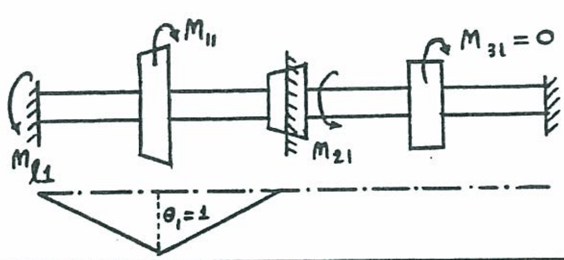
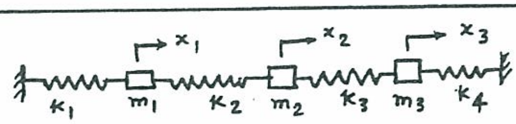


Fig. (B)

6.24 Stiffness influence coefficients:



Let $x_1 = 1$, $x_2 = x_3 = 0$.

Forces required at 1, 2, 3 are

$F_1 = k_1 + k_2 = k_{11}$; $\quad F_2 = -k_2 = k_{21}$; $\quad F_3 = 0 = k_{31}$

Let $x_2 = 1$, $x_1 = x_3 = 0$.

Forces required at 1, 2, 3 are

$F_1 = -k_2 = k_{12}$; $\quad F_2 = k_2 + k_3 = k_{22}$; $\quad F_3 = -k_3 = k_{32}$

Let $x_3 = 1$, $x_1 = x_2 = 0$.

Forces required at 1, 2, 3 are

$F_1 = 0 = k_{13}$; $\quad F_2 = -k_3 = k_{23}$; $\quad F_3 = k_3 + k_4 = k_{33}$

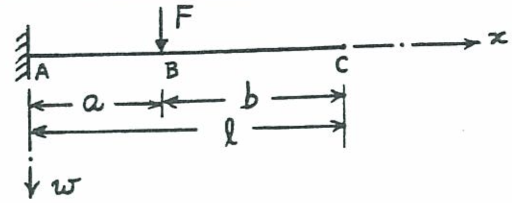
$$\therefore \quad [k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & (k_3+k_4) \end{bmatrix}$$

Flexibility influence coefficients:

Procedure and results are similar to those of problem 6.1.

Equations of motion: $[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}$

6.25 From strength of materials, the deflection of the cantilever beam shown is given by



$$w(x)\Big|_{\text{in } AB} = \frac{Fx^2}{6EI}(-x+3a) \quad --- (E_1)$$

$$w(x)\Big|_{\text{in } BC} = \frac{Fa^2}{6EI}(-a+3x) \quad --- (E_2)$$

Apply $F_1 = 1,\ F_2 = F_3 = 0$: $\quad a_{11} = (F=1,\ x=l,\ a=l \text{ in } (E_1)) = l^3/(3EI)$

$a_{21} = (F=1,\ x=2l,\ a=l \text{ in } (E_2)) = 5l^3/(6EI)$

$a_{31} = (F=1,\ x=3l,\ a=l \text{ in } (E_2)) = 4l^3/(3EI)$

Similarly apply $F_2 = 1,\ F_1 = F_3 = 0$ to get a_{22}, a_{32}, a_{12} and $F_3 = 1,\ F_1 = F_2 = 0$ to get a_{33}, a_{13}, a_{23}. Result is

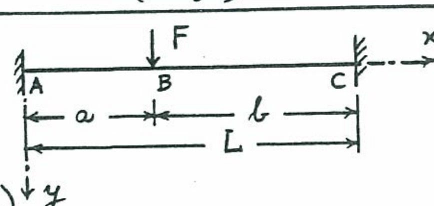
$$[a] = \frac{l^3}{EI}\begin{bmatrix} 1/3 & 5/6 & 4/3 \\ 5/6 & 8/3 & 14/3 \\ 4/3 & 14/3 & 9 \end{bmatrix}$$

Equations of motion:

$$[m]\ddot{\vec{w}} + [k]\vec{w} = \vec{0} \quad \text{or} \quad [a][m]\ddot{\vec{w}} + \vec{w} = \vec{0}$$

with $[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$ and $\vec{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$.

6.26 Deflection of a fixed-fixed beam is



$$y(x)\Big|_{\text{in } AB} = \frac{Fb^2x^2}{6EIL^3}\left\{-x(3a+b)+3aL\right\} --- (E_1)$$

$$y(x)\Big|_{\text{in } BC} = \frac{Fa^2(L-x)^2}{6EIL^3}\left\{-(L-x)(3b+a)+3bL\right\} --- (E_2)$$

Apply $F_1 = 1,\ F_2 = F_3 = 0$:

$a_{11} = (F=1,\ a=l,\ b=3l,\ x=l,\ L=4l \text{ in } (E_1)) = 9l^3/(64EI)$

$a_{21} = (F=1,\ a=l,\ b=3l,\ x=2l,\ L=4l \text{ in } (E_2)) = l^3/(6EI)$

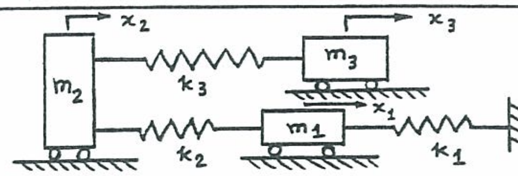
$a_{31} = (F=1,\ a=l,\ b=3l,\ x=3l,\ L=4l \text{ in } (E_2)) = 13l^3/(192EI)$

Similarly apply $F_2 = 1,\ F_1 = F_3 = 0$ to get a_{22}, a_{32}, a_{12} and

$F_3 = 1$, $F_1 = F_2 = 0$ to get a_{33}, a_{13}, a_{23}. Result is

$$[a] = \frac{l^3}{EI}\begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}$$

6.27 Flexibility matrix:



a_{11} = deflection of m_1 for a unit load on $m_1 = 1/k_1$

m_2 and m_3 get same displacement (as rigid body motion) as there are no other forces or constraints.

$a_{21} = a_{31} = \frac{1}{k_1}$. If we apply unit load to m_2, equivalent stiffness is given by $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$. $a_{22} = \frac{1}{k_{eq}} = \frac{k_1 + k_2}{k_1 k_2}$.

Mass m_3 follows deflection of m_2. $a_{32} = a_{22}$.

If we apply unit load to m_3, equivalent stiffness of springs is given by $\frac{1}{k_{eq}} = \frac{1}{k_3} + \frac{1}{k_1} + \frac{1}{k_2}$. $a_{33} = \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$.

Stiffness matrix:

Let $x_1 = 1$, $x_2 = x_3 = 0$. Forces required at 1, 2, 3 are

$F_1 = k_1 + k_2 = k_{11}$, $F_2 = -k_2 = k_{21}$, $F_3 = 0 = k_{31}$.

Let $x_2 = 1$, $x_1 = x_3 = 0$. Forces required at 1,2,3 are

$F_1 = -k_2 = k_{12}$, $F_2 = k_2 + k_3 = k_{22}$, $F_3 = -k_3 = k_{23}$.

Let $x_3 = 1$, $x_1 = x_2 = 0$. Forces required at 1, 2, 3 are

$F_1 = 0 = k_{13}$, $F_2 = -k_3 = k_{23}$, $F_3 = k_3 = k_{33}$.

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}; \quad [a] = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) \\ 1/k_1 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) & \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right) \end{bmatrix}$$

6.28 Assume small deflections; hence tension in spring (P) remains constant.

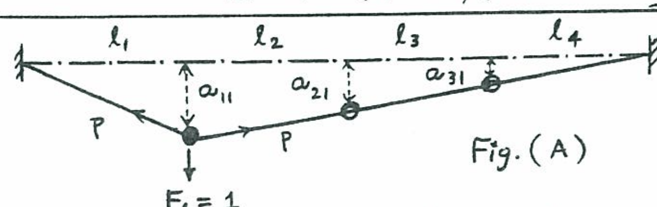


Fig. (A)

Let $F_1 = 1$, $F_2 = F_3 = 0$ as shown in Fig. (A).

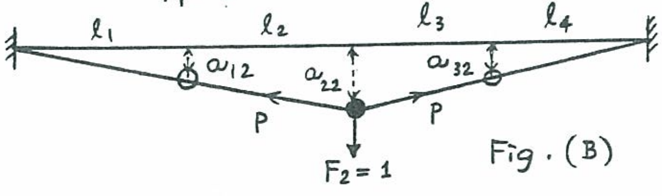


Fig. (B)

Vertical force balance gives

$$F_1 = 1 = \left(\frac{a_{11}}{l_1}\right)P + \left(\frac{a_{11}}{l_2 + l_3 + l_4}\right)P$$

$$a_{11} = \frac{1}{P\left(\frac{1}{l_1} + \frac{1}{l_2 + l_3 + l_4}\right)}$$

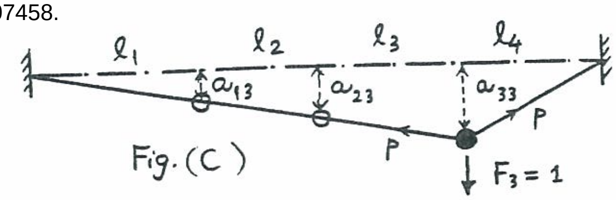


Fig. (C)

From relations of triangles,

$$\frac{a_{11}}{l_2 + l_3 + l_4} = \frac{a_{21}}{l_3 + l_4} \quad \text{and} \quad \frac{a_{11}}{l_2 + l_3 + l_4} = \frac{a_{31}}{l_4}$$

$$a_{21} = \left(\frac{l_3 + l_4}{l_2 + l_3 + l_4}\right) a_{11}, \quad a_{31} = \left(\frac{l_4}{l_2 + l_3 + l_4}\right) a_{11}$$

When $F_2 = 1$, $F_1 = F_3 = 0$, vertical force balance gives (Fig. (B))

$$F_2 = 1 = \left(\frac{a_{22}}{l_1 + l_2}\right) P + \left(\frac{a_{22}}{l_3 + l_4}\right) P \quad \Rightarrow a_{22} = \frac{1}{P\left\{\frac{1}{l_1 + l_2} + \frac{1}{l_3 + l_4}\right\}}$$

From triangle relations $\quad a_{12} = \left(\frac{l_1}{l_1 + l_2}\right) a_{22}, \quad a_{32} = \left(\frac{l_4}{l_3 + l_4}\right) a_{22}$

When $F_3 = 1$, $F_1 = F_2 = 0$ (Fig. (C)), vertical force balance gives

$$F_3 = 1 = \left(\frac{a_{33}}{l_1 + l_2 + l_3}\right) P + \left(\frac{a_{33}}{l_4}\right) P \; ; \quad a_{33} = \frac{1}{P\left(\frac{1}{l_1 + l_2 + l_3} + \frac{1}{l_4}\right)}$$

From triangle relations, $\quad a_{23} = \left(\frac{l_1 + l_2}{l_1 + l_2 + l_3}\right) a_{33}, \quad a_{13} = \left(\frac{l_1}{l_1 + l_2 + l_3}\right) a_{33}$

Equations of motion: $\quad [a][m] \ddot{\vec{w}} + \vec{w} = \vec{0}$

with $\quad [m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \vec{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}.$

6.29 Let x_1, x_2 and x_3 denote the displacements of top, middle and bottom masses. Equations of motion are

$$m \ddot{x}_1 = -k x_1 - k(x_1 - x_2) - 3k(x_1 - x_3)$$

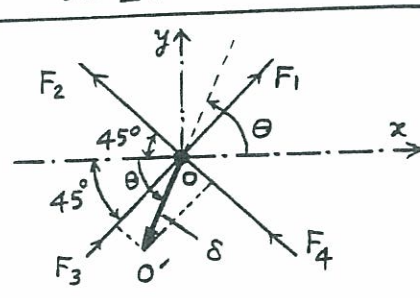
$$2m \ddot{x}_2 = -2k x_2 - k(x_2 - x_1) - k(x_2 - x_3)$$

$$m \ddot{x}_3 = -k x_3 - 3k(x_3 - x_1) - k(x_3 - x_2)$$

i.e. $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -k & -3k \\ -k & 4k & -k \\ -3k & -k & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

6.30 Let o move to o′ with δ small.

Then
$$F_1 = k \delta \cos(\theta - 45^\circ)$$
$$F_2 = k \delta \cos(135^\circ - \theta)$$
$$F_3 = k \delta \cos(\theta - 45^\circ)$$
$$F_4 = k \delta \cos(135^\circ - \theta)$$



Force along δ is:

$$F = F_1 \cos(\theta - 45^\circ) + F_2 \cos(135^\circ - \theta) + F_3 \cos(\theta - 45^\circ) + F_4 \cos(135^\circ - \theta)$$

$$= 2k\delta \left[\cos^2(\theta - 45^\circ) + \cos^2(135^\circ - \theta) \right]$$

$$= 2k\delta$$

$\therefore$ Stiffness influence coefficient of junction point in arbitrary direction $= F/\delta = 2k$

6.31 Stiffness matrix is given by Eq. (6.6):

$$[k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & \cdots & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & \cdots & 0 \\ 0 & -k_3 & (k_3+k_4) & -k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & (k_n+k_{n+1}) \end{bmatrix}$$

All elements except those on the three diagonals are zero. Hence $[k]$ is a band matrix. In fact, it is a tri-diagonal matrix.

6.32 We use the expression of kinetic energy to derive the mass matrix. Let the generalized coordinates be x_1, x_2 and x_3. The kinetic energy of the system is.

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

This can be expressed in matrix form as

$$T = \frac{1}{2} (\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3) [m] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}$$

with the mass matrix given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

6.33 We use the expression of kinetic energy to derive the mass matrix. Using the generalized coordinates θ, x_1 and x_2, the kinetic energy of the system can be expressed as:

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

where $J_0 = J_G + (2m) (\frac{\ell}{2})^2 = \frac{2}{3} m \ell^2$. T can be expressed in matrix form as:

$$T = \frac{1}{2}\left(\dot\theta \quad \dot x_1 \quad \dot x_2\right)[m]\begin{Bmatrix}\dot\theta\\ \dot x_1\\ \dot x_2\end{Bmatrix} \quad \text{with the mass matrix} \quad [m] = \begin{bmatrix}\frac{2}{3}\,m\,\ell^2 & 0 & 0\\ 0 & 2\,m & 0\\ 0 & 0 & m\end{bmatrix}$$

6.34 We derive the mass matrix from the kinetic energy expression. Using the coordinates x_1, x_2 and x_3, the kinetic energy of the system can be expressed as (see figure in the solution of Problem 6.3):

$$T = \frac{1}{2}\,J_G\,\dot\theta^2 + \frac{1}{2}\,(2\,m)\,\dot x_G^2 + \frac{1}{2}\,(5\,m)\,\dot x_2^2 \tag{1}$$

$$\text{Using} \quad J_G = \frac{1}{12}\,(2m)\,(5\,\ell)^2 = \frac{25}{6}\,m\,\ell^2 \tag{2}$$

$$x_G = \frac{x_1 + x_3}{2}, \quad \theta = \frac{x_1 - x_3}{5\,\ell} \tag{3}$$

Eq. (1) can be rewritten as:

$$T = \frac{1}{2}\left(\frac{25}{6}\,m\,\ell^2\right)\left(\frac{\dot x_1 - \dot x_3}{5\,\ell}\right)^2 + \frac{1}{2}\,(2m)\left(\frac{\dot x_1 + \dot x_3}{2}\right)^2 + \frac{1}{2}\,(5m)\,\dot x_2^2$$

$$= \frac{1}{2}\,\frac{2\,m}{3}\,\dot x_1^2 + \frac{1}{2}\,\frac{2\,m}{3}\,\dot x_3^2 + \frac{1}{2}\,\frac{1}{3}\,m\,(2\,\dot x_1\,\dot x_3) + \frac{1}{2}\,(5m)\,\dot x_2^2 \tag{4}$$

Equation (4) can be expressed in matrix form as

$$T = \frac{1}{2}\left(\dot x_1 \quad \dot x_2 \quad \dot x_3\right)[m]\begin{Bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\end{Bmatrix} \quad \text{with the mass matrix} \quad [m] = m\begin{bmatrix}\frac{2}{3} & 0 & \frac{1}{3}\\ 0 & 5 & 0\\ \frac{1}{3} & 0 & \frac{2}{3}\end{bmatrix}$$

6.35 The kinetic energy of the system can be expressed, in terms of the coordinates x_1, x_2 and x_3 as:

$$T = \frac{1}{2}\,M\,\dot x_1^2 + \frac{1}{2}\,J_0\,\dot\theta^2 + \frac{1}{2}\,(3m)\,\dot x_2^2 + \frac{1}{2}\,m\,\dot x_3^2 \tag{1}$$

Using the relation $\theta = \dfrac{x_2 - x_1}{3\,r}$, Eq. (1) can be rewritten as

$$T = \frac{1}{2}\left(M + \frac{J_0}{9\,r^2}\right)\dot x_1^2 + \frac{1}{2}\left(\frac{J_0}{9\,r^2} + 3\,m\right)\dot x_2^2 - \frac{1}{2}\,\frac{J_0}{9\,r^2}\,(2\,\dot x_1\,\dot x_2) + \frac{1}{2}\,m\,\dot x_3^2 \tag{2}$$

By expressing T in matrix form as

$$T = \frac{1}{2}\left(\dot{x}_1 \quad \dot{x}_2 \quad \dot{x}_3\right)[m]\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{Bmatrix}$$

the mass matrix

$$[m] = \begin{bmatrix}\left(M + \frac{J_0}{9\,r^2}\right) & -\frac{J_0}{9\,r^2} & 0\\ -\frac{J_0}{9\,r^2} & \left(\frac{J_0}{9\,r^2} + 3\,m\right) & 0\\ 0 & 0 & m\end{bmatrix}$$

6.36 The kinetic energy of the system can be expressed as

$$T = \frac{1}{2}\,I_1\,\dot{\theta}_1^2 + \frac{1}{2}\,I_2\,\dot{\theta}_2^2 + \frac{1}{2}\,I_3\left(\dot{\theta}_2\,\frac{n_2}{n_3}\right)^2 + \frac{1}{2}\,I_4\,\dot{\theta}_3^2 + \frac{1}{2}\,I_5\left(\dot{\theta}_3\,\frac{n_4}{n_5}\right)^2 + \frac{1}{2}\,I_6\,\dot{\theta}_4^2$$

This can be expressed in matrix form as

$$T = \frac{1}{2}\left(\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\theta}_3 \quad \dot{\theta}_4\right)[m]\begin{Bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\end{Bmatrix}$$

and the mass matrix can be identified as

$$[m] = \begin{bmatrix}I_1 & 0 & 0 & 0\\ 0 & \left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right) & 0 & 0\\ 0 & 0 & \left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right) & 0\\ 0 & 0 & 0 & I_6\end{bmatrix}$$

6.37

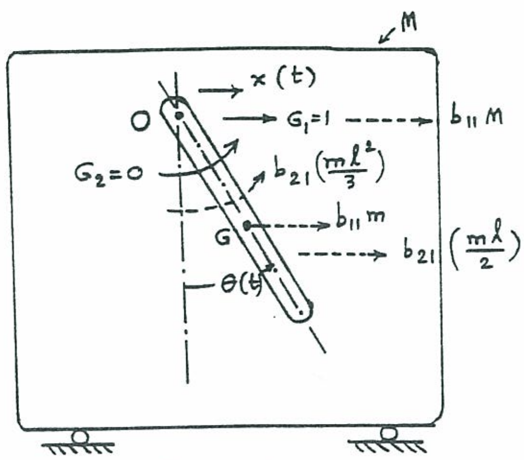


Fig. 1

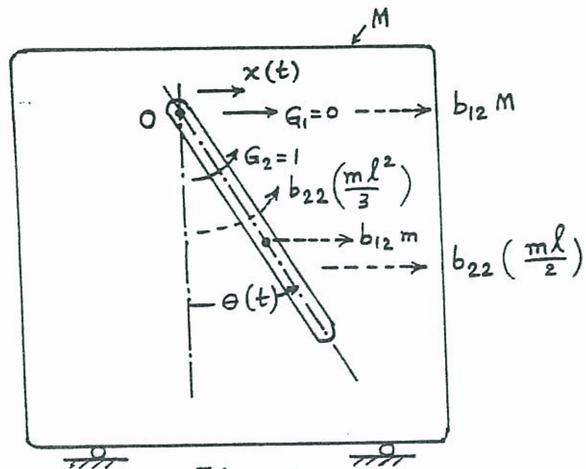


Fig. 2

Let x(t) and θ(t) indicate the coordinates to define the linear and angular positions of the trailer (M) and the pendulum (m), respectively. To derive the inverse inertia influence coefficients, first we apply a unit linear impulse at point o (along x(t)) and write the impulse-momentum relations as (see Fig. 1):

Linear impulse-momentum relation along x:

$$\dot{x} = 1 = b_{11}\ (M + m) + b_{21}\ (\frac{m\,\ell}{2}) \tag{1}$$

Angular impulse-momentum relation along θ:

$$\dot{\theta} = 0 = b_{11}\ (\frac{m\,\ell}{2}) + b_{21}\ (\frac{m\,\ell^2}{3}) \tag{2}$$

Solution of Eqs. (1) and (2) gives:

$$b_{11} = \frac{4}{4\,M + m}\ ;\ b_{21} = -\frac{6}{4\,M\,\ell + m\,\ell} \tag{3}$$

Next we apply a unit angular impulse at point o (along θ(t)) and write the impulse-momentum relations as (see Fig. 2):

Linear impulse-momentum relation along x:

$$\dot{x} = 0 = b_{12}\ (M + m) + b_{22}\ (\frac{m\,\ell}{2}) \tag{4}$$

Angular impulse-momentum about O along θ:

$$\dot{\theta} = 1 = b_{12}\ (\frac{m\,\ell}{2}) + b_{22}\ (\frac{m\,\ell^2}{3}) \tag{5}$$

Solution of Eqs. (4) and (5) gives:

$$b_{12} = -\frac{6}{4\,M\,\ell + m\,\ell}\ ;\ b_{22} = \frac{12\,(M + m)}{4\,M\,m\,\ell^2 + m^2\,\ell^2} \tag{6}$$

Thus the inverse mass matrix is given by

$$[b] = [m]^{-1} = \begin{bmatrix} \left(\frac{4}{4\,M + m}\right) & -\left(\frac{6}{4\,M\,\ell + m\,\ell}\right) \\ -\left(\frac{6}{4\,M\,\ell + m\,\ell}\right) & \left(\frac{12\,(M + m)}{4\,M\,m\,\ell^2 + m^2\,\ell^2}\right) \end{bmatrix} \tag{7}$$

6.38 The shear building can be modeled as shown below :

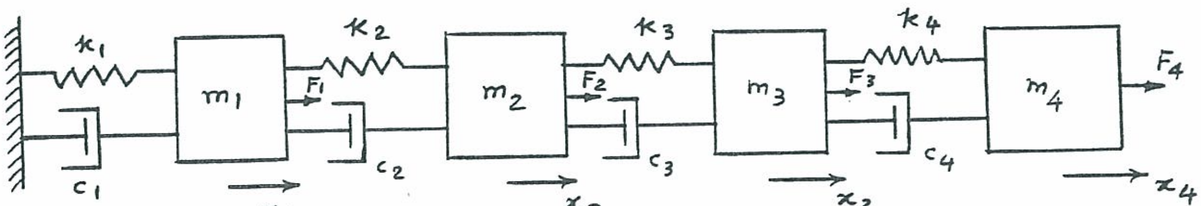


FIG. (A)

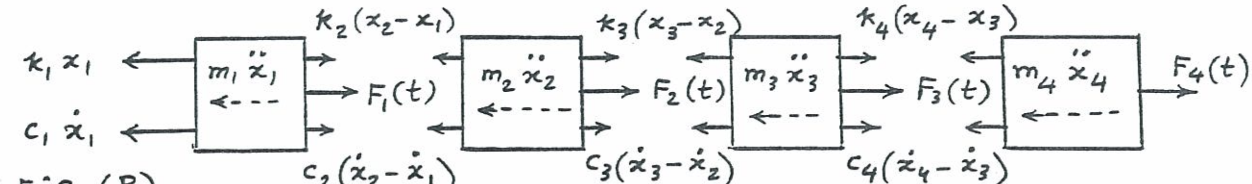


FIG. (B)

(a)
Equations of motion (from free body diagrams in Fig. B):

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + k_1 x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_1(t)$$
$$m_2 \ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) - c_3(\dot{x}_3 - \dot{x}_2) - k_3(x_3 - x_2) = F_2(t)$$
$$m_3 \ddot{x}_3 + c_3(\dot{x}_3 - \dot{x}_2) + k_3(x_3 - x_2) - c_4(\dot{x}_4 - \dot{x}_3) - k_4(x_4 - x_3) = F_3(t)$$
$$m_4 \ddot{x}_4 + c_4(\dot{x}_4 - \dot{x}_3) + k_4(x_4 - x_3) = F_4(t) \qquad \text{-----}(E_1)$$

(b) Lagrange's equations are

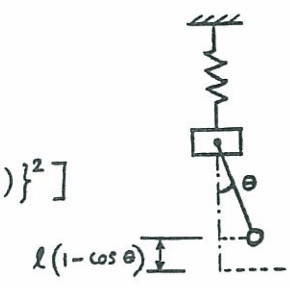
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j \qquad \text{-----}(E_2)$$

where T = kinetic energy, V = potential energy, R = Rayleigh's dissipation function, $Q_j = j^{th}$ generalized force and $q_j = j^{th}$ generalized coordinate:

$$T = \frac{1}{2}\left\{ m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + m_3 \dot{x}_3^2 + m_4 \dot{x}_4^2 \right\}$$
$$V = \frac{1}{2}\left\{ k_1 x_1^2 + k_2(x_2 - x_1)^2 + k_3(x_3 - x_2)^2 + k_4(x_4 - x_3)^2 \right\}$$
$$R = \frac{1}{2}\left\{ c_1 \dot{x}_1^2 + c_2(\dot{x}_2 - \dot{x}_1)^2 + c_3(\dot{x}_3 - \dot{x}_2)^2 + c_4(\dot{x}_4 - \dot{x}_3)^2 \right\}$$
$$Q_j = F_j \quad ; \quad j = 1, 2, 3, 4$$

Using $q_j = x_j$; $j = 1, 2, 3, 4$, the application of Eqs. (E_2) yields the equations of motion given in (E_1).

6.39 Coordinates of the bob are $(x + l \cos\theta,\ l \sin\theta)$



T = kinetic energy = kinetic energy of slider + kinetic energy of bob

$$= \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + \frac{1}{2} m \left[\left\{\frac{d}{dt}(x + l\cos\theta)\right\}^2 + \left\{\frac{d}{dt}(l\sin\theta)\right\}^2\right]$$
$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 (\sin^2\theta + \cos^2\theta) - \frac{1}{2} m (2 \dot{x}\, l \sin\theta\ \dot{\theta})$$
$$= m\dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 - m \dot{x} \dot{\theta} l \sin\theta \simeq m\dot{x}^2 + \frac{1}{2} m l^2 \dot{\theta}^2 \text{ for small } \theta.$$

V = potential energy = potential energy of spring + potential energy of bob

$= \frac{1}{2} k x^2 + mg\ell (1 - \cos\theta)$

(Note: Potential energy of slider need not be considered if $x=0$ corresponds to static equilibrium position)

Since $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, $\quad V = \frac{1}{2} k x^2 + \frac{1}{2} mg\ell\,\theta^2$

As there are no external forces, Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \; ; \quad j = 1, 2$$

Here $q_1 = x$ and $q_2 = \theta$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = 2m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = 2m\ddot{x}, \quad \frac{\partial V}{\partial x} = kx$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = m\ell^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta}, \quad \frac{\partial V}{\partial \theta} = mg\ell\theta$$

Lagrange's equations become

$$2m\ddot{x} + kx = 0 \; ; \quad m\ell^2\ddot{\theta} + mg\ell\theta = 0 \text{ or } \ell\ddot{\theta} + g\theta = 0$$

6.40 (1) With x_1 and x_2 as generalized coordinates:

Since $\quad x_1 = x - \ell_1\theta$ and $x_2 = x + \ell_2\theta$,

$$x = \left(\frac{x_1\ell_2 + x_2\ell_1}{\ell_1 + \ell_2}\right) \quad \text{and} \quad \theta = \left(\frac{x_2 - x_1}{\ell_1 + \ell_2}\right)$$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} J_0\dot{\theta}^2 = \frac{1}{2} m\left(\frac{\dot{x}_1\ell_2 + \dot{x}_2\ell_1}{\ell_1 + \ell_2}\right)^2 + \frac{1}{2} J_0\left(\frac{\dot{x}_2 - \dot{x}_1}{\ell_1 + \ell_2}\right)^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = \frac{m}{2(\ell_1+\ell_2)^2}\left(2\ell_2^2\dot{x}_1 + 2\ell_1\ell_2\dot{x}_2\right) + \frac{J_0}{2(\ell_1+\ell_2)^2}\left(2\dot{x}_1 - 2\dot{x}_2\right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{m}{(\ell_1+\ell_2)^2}\left(\ell_2^2\ddot{x}_1 + \ell_1\ell_2\ddot{x}_2\right) + \frac{J_0}{(\ell_1+\ell_2)^2}\left(\ddot{x}_1 - \ddot{x}_2\right)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = \frac{m}{2(\ell_1+\ell_2)^2}\left(2\ell_1^2\dot{x}_2 + 2\ell_1\ell_2\dot{x}_1\right) + \frac{J_0}{2(\ell_1+\ell_2)^2}\left(2\dot{x}_2 - 2\dot{x}_1\right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{m}{(\ell_1+\ell_2)^2}\left(\ell_1^2\ddot{x}_2 + \ell_1\ell_2\ddot{x}_1\right) + \frac{J_0}{(\ell_1+\ell_2)^2}\left(\ddot{x}_2 - \ddot{x}_1\right)$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k x_2^2$$

$$\frac{\partial V}{\partial x_1} = k_1 x_1, \quad \frac{\partial V}{\partial x_2} = k_2 x_2$$

Lagrange's equations, Eq. (6.41), give

$$\frac{m}{(\ell_1+\ell_2)^2}\left(\ell_2^2\ddot{x}_1 + \ell_1\ell_2\ddot{x}_2\right) + \frac{J_0}{(\ell_1+\ell_2)^2}\left(\ddot{x}_1 - \ddot{x}_2\right) + k_1 x_1 = 0$$

$$\frac{m}{(l_1+l_2)^2}\left(l_1^2\ddot{x}_2 + l_1 l_2 \ddot{x}_1\right) + \frac{J_0}{(l_1+l_2)^2}\left(\ddot{x}_2 - \ddot{x}_1\right) + k_2 x_2 = 0$$

i.e.

$$\ddot{x}_1\left(m l_2^2 + J_0\right) + \ddot{x}_2\left(m l_1 l_2 - J_0\right) + x_1 (l_1+l_2)^2 k_1 = 0$$

$$\ddot{x}_1\left(m l_1 l_2 - J_0\right) + \ddot{x}_2\left(m l_1^2 + J_0\right) + x_2 (l_1+l_2)^2 k_2 = 0$$

(2) With x and θ as generalized coordinates:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$V = \frac{1}{2} k_1 (x - l_1\theta)^2 + \frac{1}{2} k_2 (x + l_2\theta)^2$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}, \quad \frac{\partial V}{\partial x} = k_1(x - l_1\theta) + k_2(x + l_2\theta)$$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = J_0\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = J_0\ddot{\theta}, \quad \frac{\partial V}{\partial \theta} = -k_1 l_1 (x - l_1\theta) + k_2 l_2 (x + l_2\theta)$$

Lagrange's equations give

$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0$$

$$J_0\ddot{\theta} - k_1 l_1 (x - l_1\theta) + k_2 l_2 (x + l_2\theta) = 0$$

6.41

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (x_3 - x_2)^2 + \frac{1}{2} k_4 x_3^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1, \quad \frac{\partial V}{\partial x_1} = k_1 x_1 - k_2(x_2 - x_1)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = m_2\dot{x}_2, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2, \quad \frac{\partial V}{\partial x_2} = k_2(x_2 - x_1) - k_3(x_3 - x_2)$$

$$\frac{\partial T}{\partial x_3} = 0, \quad \frac{\partial T}{\partial \dot{x}_3} = m_3\dot{x}_3, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = m_3\ddot{x}_3, \quad \frac{\partial V}{\partial x_3} = k_3(x_3 - x_2) + k_4 x_3$$

Lagrange's equations give

$$m_1\ddot{x}_1 + x_1(k_1 + k_2) - k_2 x_2 = 0$$

$$m_2\ddot{x}_2 - k_2 x_1 + x_2(k_2 + k_3) - k_3 x_3 = 0$$

$$m_3\ddot{x}_3 - k_3 x_2 + x_3(k_3 + k_4) = 0$$

6.42

Using θ_1, θ_2 and θ_3 as generalized coordinates, we find

$$T \simeq \frac{1}{2} m_1 (l_1\dot{\theta}_1)^2 + \frac{1}{2} m_2 (l_1\dot{\theta}_1 + l_2\dot{\theta}_2)^2 + \frac{1}{3} m_3 (l_1\dot{\theta}_1 + l_2\dot{\theta}_2 + l_3\dot{\theta}_3)^2 \quad ---(E_1)$$

Let reference position correspond to $\theta_1 = \theta_2 = \theta_3 = 0$.

Vertical movement of m_1 is

$$l_1(1 - \cos\theta_1) \simeq l_1\left\{1 - \left(1 - \frac{\theta_1^2}{2}\right)\right\} = \frac{1}{2} l_1 \theta_1^2$$

vertical movement of $m_2 = \frac{1}{2} l_1 \theta_1^2 + \frac{1}{2} l_2 \theta_2^2$

vertical movement of $m_3 = \frac{1}{2} l_1 \theta_1^2 + \frac{1}{2} l_2 \theta_2^2 + \frac{1}{2} l_3 \theta_3^2$

$$V = \frac{m_1 g l_1 \theta_1^2}{2} + \frac{m_2 g}{2}\left(l_1\theta_1^2 + l_2\theta_2^2\right) + \frac{m_3 g}{2}\left(l_1\theta_1^2 + l_2\theta_2^2 + l_3\theta_3^2\right) \text{---} (E_2)$$

$$\frac{\partial T}{\partial \theta_1} = 0, \quad \frac{\partial T}{\partial \dot\theta_1} = m_1 l_1^2 \dot\theta_1 + m_2 l_1 (l_1\dot\theta_1 + l_2\dot\theta_2) + m_3 l_1 (l_1\dot\theta_1 + l_2\dot\theta_2 + l_3\dot\theta_3),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_1}\right) = (m_1+m_2+m_3) l_1^2 \ddot\theta_1 + (m_2+m_3) l_1 l_2 \ddot\theta_2 + m_3 l_1 l_3 \ddot\theta_3,$$

$$\frac{\partial V}{\partial \theta_1} = (m_1+m_2+m_3) g l_1 \theta_1$$

$$\frac{\partial T}{\partial \theta_2} = 0, \quad \frac{\partial T}{\partial \dot\theta_2} = m_2 l_2 (l_1\dot\theta_1 + l_2\dot\theta_2) + m_3 l_2 (l_1\dot\theta_1 + l_2\dot\theta_2 + l_3\dot\theta_3),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_2}\right) = (m_2+m_3) l_1 l_2 \ddot\theta_1 + (m_2+m_3) l_2^2 \ddot\theta_2 + m_3 l_2 l_3 \ddot\theta_3,$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2 \theta_2 + m_3 g l_2 \theta_2 = (m_2+m_3) g l_2 \theta_2$$

$$\frac{\partial T}{\partial \theta_3} = 0, \quad \frac{\partial T}{\partial \dot\theta_3} = m_3 l_3 (l_1\dot\theta_1 + l_2\dot\theta_2 + l_3\dot\theta_3),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta_3}\right) = m_3 l_3 (l_1\ddot\theta_1 + l_2\ddot\theta_2 + l_3\ddot\theta_3), \quad \frac{\partial V}{\partial \theta_3} = m_3 g l_3 \theta_3$$

Lagrange's equations give the equations of motion

$$\begin{bmatrix} (m_1+m_2+m_3) l_1^2 & (m_2+m_3) l_1 l_2 & m_3 l_1 l_3 \\ (m_2+l_3) l_1 l_2 & (m_2+m_3) l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2 \end{bmatrix} \begin{Bmatrix} \ddot\theta_1 \\ \ddot\theta_2 \\ \ddot\theta_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} (m_1+m_2+m_3) g l_1 & 0 & 0 \\ 0 & (m_2+m_3) g l_2 & 0 \\ 0 & 0 & m_3 g l_3 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{---} (E_3)$$

6.43 (a) Let generalized coordinates be $q_1 = x$ and $q_2 = \theta$

Displacement of mass M is $x + l\theta$.

Kinetic energy of airplane is $T = \frac{1}{2} M_0 \dot x^2 + 2\left\{\frac{1}{2} M (\dot x + l\dot\theta)^2\right\}$

Potential energy is $V = 2\left(\frac{1}{2} k_t \theta^2\right)$

Lagrange's equations are $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i; \quad i = 1, 2$

Here $\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot x} = M_0 \dot x + 2M(\dot x + l\dot\theta), \quad \frac{\partial V}{\partial x} = 0$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = 2 M \ell \left(\dot{x} + \ell \dot{\theta} \right), \quad \frac{\partial V}{\partial \theta} = 2 k_t \theta, \qquad Q_1 = Q_2 = 0$$

Hence Lagrange's equations become

$$\left.\begin{aligned} M_0 \ddot{x} + 2 M (\ddot{x} + \ell \ddot{\theta}) &= 0 \\ 2 M \ell (\ddot{x} + \ell \ddot{\theta}) + 2 k_t \theta &= 0 \end{aligned}\right\} \qquad (E_1)$$

(b) If $x(t) = X \cos(\omega t + \phi)$ and $\theta(t) = \Theta \cos(\omega t + \phi)$, the equations of motion become

$$\left.\begin{aligned} (-M_0 \omega^2 - 2 M \omega^2) X - 2 M \ell \omega^2 \Theta &= 0 \\ -2 M \ell \omega^2 X - 2 M \ell^2 \omega^2 \Theta + 2 k_t \Theta &= 0 \end{aligned}\right\} \qquad (E_2)$$

which yields the frequency equation:

$$\begin{vmatrix} M_0 \omega^2 + 2 M \omega^2 & 2 M \ell \omega^2 \\ 2 M \ell \omega^2 & 2 M \ell^2 \omega^2 - 2 k_t \end{vmatrix} = 0$$

i.e., $$\omega^2 \left[2 M_0 M \ell^2 \omega^2 - 2 M_0 k_t - 4 M k_t \right] = 0$$

i.e., $$\omega^2 = 0 \; ; \qquad \omega^2 = \left(\frac{2 M_0 k_t + 4 M k_t}{2 M_0 M \ell^2} \right)$$

Mode shapes: Eq. (E_2) gives

$$\frac{\Theta}{X} = \frac{2 M \ell \omega^2}{-2 M \ell^2 \omega^2 + 2 k_t}$$

For $\omega_1 = 0$; $$\left. \frac{\Theta}{X} \right|_{\omega_1} = \frac{0}{2 k_t} = 0 \quad \Rightarrow \Theta = 0$$

This corresponds to rigid body translation in x (vertical) direction.

For ω_2; $$\left. \frac{\Theta}{X} \right|_{\omega_2} = \frac{2 M \ell \left(\dfrac{2 M_0 k_t + 4 M k_t}{2 M_0 M \ell^2} \right)}{-2 M \ell^2 \left(\dfrac{2 M_0 k_t + 4 M k_t}{2 M_0 M \ell^2} \right) + 2 k_t}$$

$$= - \left(\frac{M_0}{2 M \ell} + \frac{1}{\ell} \right)$$

(c) For $\omega_2 > 4 \pi$ rad/sec,

$$\left(\frac{2 M_0 k_t + 4 M k_t}{2 M M_0 \ell^2} \right) > 16 \pi^2 \qquad (E_3)$$

When $M_0 = 1000$ kg, $M = 500$ kg and $l = 6$ m, inequality (E_3) becomes

$$\frac{2000\ k_t + 2000\ k_t}{(1\times10^6)(36)} > 16\pi^2$$

i.e., $k_t > 1.4212\times10^6$ N-m/rad.

6.44 Generalized coordinates: x_1, x_2, x_3:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k (x_3 - x_2)^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} (5k) (x_3 - x_1)^2$$

$Q_i = F_i$; $i = 1, 2, 3$

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i \ ; \ i = 1, 2, 3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = m_i \ddot{x}_i \ ; \ i = 1, 2, 3$$

$$\frac{\partial T}{\partial x_i} = 0 \ ; \ i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_1} = 7\,k\,x_1 - k\,x_2 - 5\,k\,x_3$$

$$\frac{\partial V}{\partial x_2} = -k\,x_1 + 2\,k\,x_2 - k\,x_3$$

$$\frac{\partial V}{\partial x_3} = -5\,k\,x_1 - k\,x_2 + 7\,k\,x_3$$

Lagrange's equations yield the equations of motion:

$$m_1 \ddot{x}_1 + 7\,k\,x_1 - k\,x_2 - 5\,k\,x_3 = F_1(t)$$
$$m_2 \ddot{x}_2 - k\,x_1 + 2\,k\,x_2 - k\,x_3 = F_2(t)$$
$$m_3 \ddot{x}_3 - 5\,k\,x_1 - k\,x_2 + 7\,k\,x_3 = F_3(t)$$

6.45 Generalized coordinates: θ, x_1, x_2:

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

where $J_0 = \frac{1}{3}(2m)\,\ell^2 = \frac{2}{3} m\,\ell^2$.

$$V = \frac{1}{2} (2k) (x_1 - \frac{\ell}{4}\theta)^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} (3k) (\theta\,\ell)^2$$

$$R = \frac{1}{2} c (\dot{x}_1 - \frac{\ell}{4}\dot{\theta})^2$$

$Q_\theta = M_t(t)$; $Q_1 = F_1(t)$; $Q_2 = F_2(t)$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0\,\dot{\theta} \ ; \ \frac{\partial T}{\partial \dot{x}_1} = 2\,m\,\dot{x}_1 \ ; \ \frac{\partial T}{\partial \dot{x}_2} = m\,\dot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) = J_0\,\ddot\theta \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot x_1}\right) = 2\,m\,\ddot x_1 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot x_2}\right) = m\,\ddot x_2$$

$$\frac{\partial T}{\partial \theta} = 0 \;;\; \frac{\partial T}{\partial x_i} = 0 \;;\; i = 1,\, 2$$

$$\frac{\partial R}{\partial \dot\theta} = -\,c\,\frac{\ell}{4}\,(\dot x_1 - \dot\theta\,\frac{\ell}{4}) \;;\; \frac{\partial R}{\partial \dot x_1} = c\,(\dot x_1 - \dot\theta\,\frac{\ell}{4}) \;;\; \frac{\partial R}{\partial \dot x_2} = 0$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{25}{8}\,k\,\ell^2\right)\theta - \frac{1}{2}\,k\,\ell\,x_1$$

$$\frac{\partial V}{\partial x_1} = -\,\frac{1}{2}\,k\,\ell\,\theta + 3\,k\,x_1 - k\,x_2$$

$$\frac{\partial V}{\partial x_2} = -\,k\,x_1 + k\,x_2$$

Lagrange's equations, Eqs. (6.118), yield the equations of motion as:

$$\frac{2}{3}\,m\,\ell^2\,\ddot\theta + \frac{1}{16}\,c\,\ell^2\,\dot\theta - \frac{1}{4}\,c\,\ell\,\dot x_1 + \frac{25}{8}\,k\,\ell^2\,\theta - \frac{1}{2}\,k\,\ell\,x_1 = M_t(t)$$
$$2\,m\,\ddot x_1 - \frac{1}{4}\,c\,\ell\,\dot\theta + c\,\dot x_1 - \frac{1}{2}\,k\,\ell\,\theta + 3\,k\,x_1 - k\,x_2 = F_2(t)$$
$$m\,\ddot x_2 - k\,x_1 + k\,x_2 = F_2(t)$$

6.46 Generalized coordinates: x_i ; $i = 1,\, 2,\, 3$:

$$T = \frac{1}{2}\,J_G\,\dot\theta^2 + \frac{1}{2}\,(2m)\,\dot x_G^2 + \frac{1}{2}\,(5m)\,\dot x_2^2$$

$$= \frac{1}{2}\left(\frac{25}{6}\,m\,\ell^2\right)\left(\frac{\dot x_1 - \dot x_3}{5\,\ell}\right)^2 + \frac{1}{2}\,(2m)\left(\frac{\dot x_1 + \dot x_3}{2}\right)^2 + \frac{1}{2}\,(5m)\,\dot x_2^2$$

where the subscript G denotes the mass center of the bar with

$$J_G = \frac{1}{12}\,(2m)\,(5\,\ell)^2 = \frac{25}{6}\,m\,\ell^2 \;;\; \theta = \frac{x_1 - x_3}{5\,\ell} \;;\; x_G = \frac{x_1 + x_3}{2}$$

$$V = \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,k\,x_3^2 + \frac{1}{2}\,k\,(x_A - x_2)^2$$
$$= \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,k\,x_3^2 + \frac{1}{2}\,k\,(\frac{3\,x_1 + 2\,x_3}{5} - x_2)^2$$

$$\text{since}\quad x_A = x_1 - \frac{2}{5}\,(x_1 - x_3) = \frac{3\,x_1 + 2\,x_3}{5}$$

$$R = \frac{1}{2}\,c\,(\dot x_A - \dot x_2)^2 = \frac{1}{2}\,c\left(\frac{3\,\dot x_1 + 2\,\dot x_3}{5} - \dot x_2\right)^2$$

$$Q_i(t) = F_i(t) \;\; ; \;\; i = 1,\, 2,\, 3$$

$$\frac{\partial T}{\partial \dot{x}_1} = \left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{5\,\ell}\right)\left(\frac{\dot{x}_1 - \dot{x}_3}{5\,\ell}\right) + (2m)\,\frac{1}{2}\left(\frac{\dot{x}_1 + \dot{x}_3}{2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{25\,\ell^2}\right)(\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2}\,(\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial \dot{x}_2} = (5m)\,\dot{x}_2 \;\; ; \;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = (5m)\,\ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = -\left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{5\,\ell}\right)\left(\frac{\dot{x}_1 - \dot{x}_3}{5\,\ell}\right) + (2m)\,\frac{1}{2}\left(\frac{\dot{x}_1 + \dot{x}_3}{2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = -\left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{25\,\ell^2}\right)(\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2}\,(\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial x_i} = 0 \;\; ; \;\; i = 1,\, 2,\, 3$$

$$\frac{\partial R}{\partial \dot{x}_1} = \frac{3}{5}\, c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right) \qquad \frac{\partial V}{\partial x_1} = k\, x_1 + \frac{3}{5}\, k\left(\frac{3\, x_1 + 2\, x_3}{5} - x_2\right)$$

$$\frac{\partial R}{\partial \dot{x}_3} = \frac{2}{5}\, c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right) \qquad \frac{\partial V}{\partial x_2} = -\, k\left(\frac{3\, x_1 + 2\, x_3}{5} - x_2\right)$$

$$\frac{\partial R}{\partial \dot{x}_2} = -\, c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right) \qquad \frac{\partial V}{\partial x_3} = k\, x_3 + \frac{2}{5}\, k\left(\frac{3\, x_1 + 2\, x_3}{5} - x_2\right)$$

Application of Lagrange's equations gives the equations of motion as:

$$\frac{2}{3}\, m\, \ddot{x}_1 + \frac{1}{3}\, m\, \ddot{x}_3 + \frac{9}{25}\, c\, \dot{x}_1 - \frac{3}{5}\, c\, \dot{x}_2 + \frac{6}{25}\, c\, \dot{x}_3 + \frac{34}{25}\, k\, x_1 - \frac{3}{5}\, k\, x_2 + \frac{6}{25}\, k\, x_3 = F_1(t)$$
$$5\, m\, \ddot{x}_2 - \frac{3}{5}\, c\, \dot{x}_1 + c\, \dot{x}_2 - \frac{2}{5}\, c\, \dot{x}_3 - \frac{3}{5}\, k\, x_1 + k\, x_2 - \frac{2}{5}\, k\, x_3 = F_2(t)$$
$$\frac{1}{3}\, m\, \ddot{x}_1 + \frac{2}{3}\, m\, \ddot{x}_3 + \frac{6}{25}\, c\, \dot{x}_1 - \frac{2}{5}\, c\, \dot{x}_2 + \frac{4}{25}\, c\, \dot{x}_3 + \frac{6}{25}\, k\, x_1 - \frac{2}{5}\, k\, x_2 + \frac{29}{25}\, k\, x_3 = F_3(t)$$

6.47

$$T = \frac{1}{2}\, M\, \dot{x}_1^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 + \frac{1}{2}\,(3m)\, \dot{x}_2^2 + \frac{1}{2}\, m\, \dot{x}_3^2$$
$$V = \frac{1}{2}\, k\, x_1^2 + \frac{1}{2}\,(2k)\,(x_1 - x_3 - r\,\theta)^2 + \frac{1}{2}\,(3k)\, x_3^2$$
$$Q_i = F_i \;\; ; \;\; i = 1,\, 2,\, 3$$

Noting that $\theta = \left(\frac{x_2 - x_1}{3\, r}\right)$, we can express

$$\frac{\partial T}{\partial \dot{x}_1} = M\,\dot{x}_1 + J_0 \left(-\frac{1}{3\,r}\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{3\,r}\right) \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = M\,\ddot{x}_1 - \frac{J_0}{3\,r}\left(\frac{\ddot{x}_2 - \ddot{x}_1}{3\,r}\right)$$

$$\frac{\partial T}{\partial \dot{x}_2} = J_0 \left(\frac{1}{3\,r}\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{3\,r}\right) + 3\,m\,\dot{x}_2 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{J_0}{9\,r^2}\,(\ddot{x}_2 - \ddot{x}_1) + 3\,m\,\ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = m\,\dot{x}_3 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = m\,\ddot{x}_3$$

$$\frac{\partial V}{\partial x_1} = k\,x_1 + (2k)\,\frac{4}{3}\,(\frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3)$$

$$\frac{\partial V}{\partial x_2} = -\,\frac{1}{3}\,(2k)\,(\frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3)$$

$$\frac{\partial V}{\partial x_3} = -\,(2k)\,(\frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3) + (3k)\,x_3$$

Application of Lagrange's equations give the equations of motion:

$$\left(M + \frac{J_0}{9\,r^2}\right)\ddot{x}_1 - \frac{J_0}{9\,r^2}\,\ddot{x}_2 + \frac{41}{9}\,k\,x_1 - \frac{8}{9}\,k\,x_2 - \frac{8}{3}\,k\,x_3 = F_1(t)$$
$$-\,\frac{J_0}{9\,r^2}\,\ddot{x}_1 + \left(3\,m + \frac{J_0}{9\,r^2}\right)\ddot{x}_2 - \frac{8}{9}\,k\,x_1 + \frac{2}{9}\,k\,x_2 + \frac{2}{3}\,k\,x_3 = F_2(t)$$
$$m\,\ddot{x}_3 - \frac{8}{3}\,k\,x_1 + \frac{2}{3}\,k\,x_2 + 5\,k\,x_3 = F_3(t)$$

6.48 Generalized coordinates: θ_i ; $i = 1, 2, 3, 4$:

$$T = \frac{1}{2}\,I_1\,\dot{\theta}_1^2 + \frac{1}{2}\left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\dot{\theta}_2^2 + \frac{1}{2}\left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\dot{\theta}_3^2 + \frac{1}{2}\,I_6\,\dot{\theta}_4^2$$

$$V = \frac{1}{2}\,k_{t1}\,(\theta_2 - \theta_1)^2 + \frac{1}{2}\,k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right)^2 + \frac{1}{2}\,k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)^2$$

$$Q_1 = M_1\,\cos\omega\,t$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = I_1\,\dot{\theta}_1 \;;\; \frac{\partial T}{\partial \dot{\theta}_2} = \left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = \left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\dot{\theta}_3 \;;\; \frac{\partial T}{\partial \dot{\theta}_4} = I_6\,\dot{\theta}_4$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = I_1\,\ddot{\theta}_1 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = \left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\ddot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = \left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\ddot{\theta}_3 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_4}\right) = I_6\,\ddot{\theta}_4$$

$$\frac{\partial V}{\partial \theta_1} = -k_{t1}\,(\theta_2 - \theta_1)$$

$$\frac{\partial V}{\partial \theta_2} = k_{t1}\,(\theta_2 - \theta_1) - k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right)\frac{n_2}{n_3}$$

$$\frac{\partial V}{\partial \theta_3} = k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right) - k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)\frac{n_4}{n_5}$$

$$\frac{\partial V}{\partial \theta_4} = k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)$$

Equations of motion (from Lagrange's equations):

$$I_1\,\ddot{\theta}_1 - k_{t1}\,(\theta_2 - \theta_1) = M_1\,\cos\omega t$$

$$\left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\ddot{\theta}_2 + k_{t1}\,(\theta_2 - \theta_1) - k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right)\frac{n_2}{n_3} = 0$$

$$\left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\ddot{\theta}_3 + k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right) - k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)\frac{n_4}{n_5} = 0$$

$$I_6\,\ddot{\theta}_4 + k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right) = 0$$

6.49 Equations of motion for the system of Fig. 6.8(a) are:

$$m_1\ddot{x}_1 + k_1 x_1 + k_2(x_1 - x_2) = 0 \qquad \text{--- } (E_1)$$

$$m_2\ddot{x}_2 - k_2(x_1 - x_2) + k_3(x_2 - x_3) = 0 \qquad \text{--- } (E_2)$$

$$m_3\ddot{x}_3 - k_3(x_2 - x_3) = 0 \qquad \text{--- } (E_3)$$

Let $q_1 = x_1$, $q_2 = x_2 - x_1$ and $q_3 = x_3 - x_2$. Eqs. (E_1) to (E_3) become

$$m_1\ddot{x}_1 + k_1 q_1 - k_2 q_2 = 0 \;\Rightarrow\; \ddot{x}_1 + \frac{k_1}{m_1} q_1 - \frac{k_2}{m_1} q_2 = 0 \qquad \text{--- } (E_4)$$

$$m_2\ddot{x}_2 + k_2 q_2 - k_3 q_3 = 0 \;\Rightarrow\; \ddot{x}_2 + \frac{k_2}{m_2} q_2 - \frac{k_3}{m_2} q_3 = 0 \qquad \text{--- } (E_5)$$

$$m_3\ddot{x}_3 + k_3 q_3 = 0 \;\Rightarrow\; \ddot{x}_3 + \frac{k_3}{m_3} q_3 = 0 \qquad \text{--- } (E_6)$$

(E_4) minus (E_5) gives $\quad (\ddot{x}_2 - \ddot{x}_1) - \frac{k_1}{m_1} q_1 + \left(\frac{k_2}{m_2} + \frac{k_2}{m_1}\right) q_2 - \frac{k_3}{m_2} q_3 = 0 \quad \text{--- } (E_7)$

(E_5) minus (E_6) gives $\quad (\ddot{x}_3 - \ddot{x}_2) - \frac{k_2}{m_2} q_2 + q_3\left(\frac{k_3}{m_3} + \frac{k_3}{m_2}\right) = 0 \quad \text{--- } (E_8)$

(E_4), (E_7) and (E_8) can be expressed as

$$\left.\begin{aligned}
&\ddot{q}_1 + \frac{k_1}{m_1} q_1 - \frac{k_2}{m_1} q_2 = 0\\
&\ddot{q}_2 - \frac{k_1}{m_1} q_1 + \left(\frac{k_2}{m_2} + \frac{k_2}{m_1}\right) q_2 - \frac{k_3}{m_2} q_3 = 0\\
&\ddot{q}_3 - \frac{k_2}{m_2} q_2 + \left(\frac{k_3}{m_3} + \frac{k_3}{m_2}\right) q_3 = 0
\end{aligned}\right\} \qquad \text{--- } (E_9)$$

For $k_i = k$ and $m_i = m$ $(i = 1, 2, 3)$, Eqs. (E_9) reduce to

$$\left.\begin{aligned} &\ddot{q}_1 + \frac{k}{m} q_1 - \frac{k}{m} q_2 = 0 \\ &\ddot{q}_2 - \frac{k}{m} q_1 + 2\frac{k}{m} q_2 - \frac{k}{m} q_3 = 0 \\ &\ddot{q}_3 - \frac{k}{m} q_1 + 2\frac{k}{m} q_3 = 0 \end{aligned}\right\} \quad --- (E_{10})$$

For $q_i(t) = Q_i \cos(\omega t + \phi)$; $i = 1, 2, 3$, (E_{10}) give

$$\begin{bmatrix} \left(-\omega^2 + \frac{k}{m}\right) & -\frac{k}{m} & 0 \\ -\frac{k}{m} & \left(-\omega^2 + 2\frac{k}{m}\right) & -\frac{k}{m} \\ 0 & -\frac{k}{m} & \left(-\omega^2 + 2\frac{k}{m}\right) \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad --- (E_{11})$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & -\omega^2 + 2\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & -\omega^2 + 2\frac{k}{m} \end{vmatrix} = \omega^6 - 5\omega^4 \frac{k}{m} + 6\omega^2 \frac{k^2}{m^2} - \frac{k^3}{m^3} = 0$$

i.e. $\alpha^3 - 5\alpha^2 + 6\alpha - 1 = 0$ where $\alpha = \frac{\omega^2 m}{k}$.

Roots of this equation give

$\alpha_1 = 0.19806$, $\omega_1 = 0.44504 \sqrt{\frac{k}{m}}$

$\alpha_2 = 1.5553$, $\omega_2 = 1.2471 \sqrt{\frac{k}{m}}$

$\alpha_3 = 3.2490$, $\omega_3 = 1.8025 \sqrt{\frac{k}{m}}$

It can be seen that the eigenvalues are same in both problems.

6.50 Equations of motion (from problem 6.24):

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2, 3$, we get

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 + k_4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 + k_4 \end{vmatrix} = 0$$

i.e. $$(-\omega^2 m_1 + k_1 + k_2)\left\{(-\omega^2 m_2 + k_2 + k_3)(-\omega^2 m_3 + k_3 + k_4) - k_3^2\right\} + k_2\left\{-k_2(-\omega^2 m_3 + k_3 + k_4)\right\} = 0$$

i.e. $$\omega^6 (m_1 m_2 m_3) - \omega^4 \left[m_1 m_2 (k_3 + k_4) + m_2 m_3 (k_1 + k_2) + m_1 m_3 (k_2 + k_3) \right] + \omega^2 \left[m_1 (k_2 + k_3)(k_3 + k_4) + m_2 (k_1 + k_2)(k_3 + k_4) + m_3 (k_1 + k_2)(k_2 + k_3) - m_1 k_3^2 - m_3 k_2^2 \right] - \left[(k_1 + k_2)(k_2 + k_3)(k_3 + k_4) + (k_1 + k_2) k_3^2 + (k_3 + k_4) k_2^2 \right] = 0$$

6.51 Equations of motion:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_1)$$

For harmonic motion, we get

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--} (E_2)$$

This gives the frequency equation (for $k_1 = k$, $k_2 = 2k$, $k_3 = 3k$, $m_1 = m$, $m_2 = 2m$ and $m_3 = 3m$)

$$\begin{vmatrix} -\omega^2 m + 3k & -2k & 0 \\ -2k & -2\omega^2 m + 5k & -3k \\ 0 & -3k & -3\omega^2 m + 3k \end{vmatrix} = 0 \quad \text{---} (E_3)$$

i.e. $$(-\omega^2 m + 3k)\left[(-2\omega^2 m + 5k)(-3\omega^2 m + 3k) - 9k^2 \right] + 2k\left[-2k(-3\omega^2 m + 3k) \right] = 0$$

i.e. $$2\alpha^3 - 13\alpha^2 + 19\alpha - 2 = 0 \quad \text{where } \alpha = \frac{\omega^2 m}{k} \quad \text{---} (E_4)$$

This gives the roots

$$\alpha_1 = 0.113992, \quad \alpha_2 = 2.00002, \quad \alpha_3 = 4.38600$$

$$\omega_1 = 0.337627 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.414221 \sqrt{\frac{k}{m}}, \quad \omega_3 = 2.094278 \sqrt{\frac{k}{m}}$$

<u>Mode shape in jth mode:</u>

Eq. (E_2) gives $$\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-\omega_j^2 m_1 + k_1 + k_2}{k_2} = \frac{-\omega_j^2 m + 3k}{2k}$$

$$\frac{X_3^{(j)}}{X_2^{(j)}} = \frac{k_3}{-\omega_j^2 m_3 + k_3} = \frac{3k}{-3\omega_j^2 m + 3k}$$

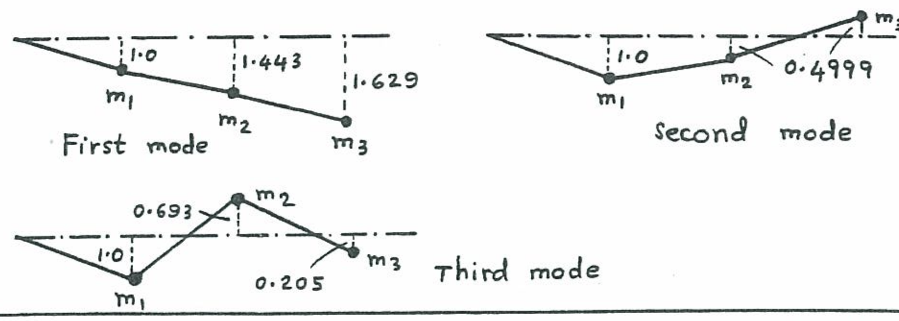
$$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{X_3^{(j)}}{X_2^{(j)}} \cdot \frac{X_2^{(j)}}{X_1^{(j)}} = \frac{3(-\omega_j^2 m + 3k)}{2(-3\omega_j^2 m + 3k)}$$

$$\vec{X}^{(j)} = \begin{Bmatrix} X_1^{(j)} \\ X_2^{(j)} \\ X_3^{(j)} \end{Bmatrix} = X_1^{(j)} \begin{Bmatrix} 1 \\ (-\omega_j^2 m + 3k)/(2k) \\ 3(-\omega_j^2 m + 3k)/[2(-3\omega_j^2 m + 3k)] \end{Bmatrix} \quad \text{--- } (E_5)$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.443004 \\ 1.628659 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.49999 \\ -0.49998 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -0.693 \\ 0.204666 \end{Bmatrix}$$

Mode shapes:



6.52 When $k_1 = 3k$, $k_2 = k_3 = k$, $m_1 = 3m$ and $m_2 = m_3 = m$, Eq. (E_2) of problem 6.46 gives the frequency equation

$$\begin{vmatrix} -3m\omega^2 + 4k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + k \end{vmatrix} = 0$$

i.e. $(-3m\omega^2 + 4k)\left[(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2\right] + k\left[-k(-m\omega^2 + k)\right] = 0$

i.e. $$3\alpha^3 - 13\alpha^2 + 14\alpha - 3 = 0 \quad \text{--- } (E_1)$$

where $\alpha = m\omega^2/k$. Roots of (E_1) are

$\alpha_1 = 0.284515$, $\quad \alpha_2 = 1.26053$, $\quad \alpha_3 = 2.78829$

$\omega_1 = 0.533399\sqrt{\frac{k}{m}}$, $\quad \omega_2 = 1.122733\sqrt{\frac{k}{m}}$, $\quad \omega_3 = 1.669817\sqrt{\frac{k}{m}}$

Eq. (E_5) of problem 6.51 gives the jth mode shape as

$$\vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-3m\omega_j^2 + 4k)/k \\ (-3m\omega_j^2 + 4k)/(-m\omega_j^2 + k) \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 3.146455 \\ 4.397653 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.21841 \\ -0.83833 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -4.36487 \\ 2.44081 \end{Bmatrix}$$

Orthogonality of normal modes:

$$[m] = m \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{X}^{(1)^T}[m]\vec{X}^{(1)} = 1 \Rightarrow (1.0 \quad 3.146455 \quad 4.397653)\begin{Bmatrix}3.0\\3.146455\\4.397653\end{Bmatrix} m\, X_1^{(1)^2} = 1$$

$$\Rightarrow 32.239531\; m\, X_1^{(1)^2} = 1 \quad \text{or} \quad X_1^{(1)} = \frac{1}{\sqrt{m}}(0.176119)$$

$$\vec{X}^{(1)} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.176119\\0.554151\\0.774510\end{Bmatrix}$$

$$\vec{X}^{(2)^T}[m]\vec{X}^{(2)} = 1 \Rightarrow (1.0 \quad 0.21841 \quad -0.83833)\begin{Bmatrix}3.0\\0.21841\\-0.83833\end{Bmatrix} m\, X_1^{(2)^2} = 1$$

$$\Rightarrow 3.7505\; m\, X_1^{(2)^2} = 1 \quad \text{or} \quad X_1^{(2)} = \frac{1}{\sqrt{m}}(0.516363)$$

$$\vec{X}^{(2)} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.516363\\0.112779\\-0.432883\end{Bmatrix}$$

$$\vec{X}^{(3)}[m]\vec{X}^{(3)} = 1 \Rightarrow (1.0 \quad -4.36487 \quad 2.44081)\begin{Bmatrix}3.0\\-4.36487\\2.44081\end{Bmatrix} m\, X_1^{(3)^2} = 1$$

$$\Rightarrow 28.00964\; m\, X_1^{(3)^2} = 1 \quad \text{or} \quad X_1^{(3)} = \frac{1}{\sqrt{m}}(0.18895)$$

$$\vec{X}^{(3)} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.18895\\-0.824742\\0.461191\end{Bmatrix}$$

It can be verified that

$$\vec{X}^{(1)^T}[m]\vec{X}^{(2)} = (0.176119 \quad 0.554151 \quad 0.774510)\begin{Bmatrix}1.549089\\0.112779\\-0.432883\end{Bmatrix} = 0$$

$$\vec{X}^{(1)^T}[m]\vec{X}^{(3)} = (0.176119 \quad 0.554151 \quad 0.774510)\begin{Bmatrix}0.56685\\-0.824742\\0.461191\end{Bmatrix} = 0$$

$$\vec{X}^{(2)^T}[m]\vec{X}^{(3)} = (0.516363 \quad 0.112779 \quad -0.432883)\begin{Bmatrix}0.56685\\-0.824742\\0.461191\end{Bmatrix} = 0$$

6.53 For $l_1 = 0.2$ m, $l_2 = 0.3$ m, $l_3 = 0.4$ m, $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg, Eq. (E_3) of problem 6.42 gives the equations of motion

$$\begin{bmatrix}6(0.04) & 5(0.06) & 3(0.08)\\5(0.06) & 5(0.09) & 3(0.12)\\3(0.08) & 3(0.12) & 3(0.16)\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\\ddot{\theta}_2\\\ddot{\theta}_3\end{Bmatrix} + \begin{bmatrix}6(9.81)(0.2) & 0 & 0\\0 & 5(9.81)(0.3) & 0\\0 & 0 & 3(9.81)(0.4)\end{bmatrix}\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad \text{---} (E_1)$$

For harmonic motion, (E_1) becomes

$$-\omega^2\begin{bmatrix}0.24 & 0.30 & 0.24\\0.30 & 0.45 & 0.36\\0.24 & 0.36 & 0.48\end{bmatrix}\begin{Bmatrix}\Theta_1\\\Theta_2\\\Theta_3\end{Bmatrix} + \begin{bmatrix}11.772 & 0 & 0\\0 & 14.715 & 0\\0 & 0 & 11.772\end{bmatrix}\begin{Bmatrix}\Theta_1\\\Theta_2\\\Theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad \text{---} (E_2)$$

This gives the frequency equation

$$\begin{vmatrix} \omega^2(0.24)-11.772 & \omega^2(0.30) & \omega^2(0.24) \\ \omega^2(0.30) & \omega^2(0.45)-14.715 & \omega^2(0.36) \\ \omega^2(0.24) & \omega^2(0.36) & \omega^2(0.48)-11.772 \end{vmatrix} = 0$$

i.e. $(0.24\omega^2 - 11.772)\left[(0.45\,\omega^2 - 14.715)(0.48\omega^2 - 11.772) - (0.36)^2\omega^4\right]$

$- 0.3\,\omega^2\left[0.3\,\omega^2(0.48\omega^2 - 11.772) - (0.24)(0.36)\,\omega^4\right]$

$+ 0.24\,\omega^2\left[(0.30)(0.36)\,\omega^4 - 0.24\,\omega^2(0.45\omega^2 - 14.715)\right] = 0$

i.e. $\omega^6 - 600.8625\,\omega^4 + 54132.806\,\omega^2 - 590047.6 = 0$

Roots of this equation are

$\omega_1^2 = 12.6335$, $\omega_1 = 3.554364$ rad/s

$\omega_2^2 = 94.6116$, $\omega_2 = 9.726849$ rad/s

$\omega_3^2 = 493.619$, $\omega_3 = 22.217538$ rad/s.

6.54 (a) By replacing l by $\frac{l}{4}$ in problem 6.26, we obtain

$$[a] = \frac{l^3}{64EI}\begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}$$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};\ [D] = [a][m] = \frac{ml^3}{EI}\begin{bmatrix} 0.0021975 & 0.0026042 & 0.0010579 \\ 0.0026042 & 0.0052083 & 0.0026042 \\ 0.0010579 & 0.0026042 & 0.0021973 \end{bmatrix}$$

Frequency equation is $\left|[D] - \lambda[I]\right| = 0$ where $\lambda = \frac{1}{\omega^2}$

i.e.
$$\begin{vmatrix} 0.0021973-\alpha & 0.0026042 & 0.0010579 \\ 0.0026042 & 0.0052083-\alpha & 0.0026042 \\ 0.0010579 & 0.0026042 & 0.0021973-\alpha \end{vmatrix} = 0 \quad ---(E_1)$$

where $\alpha = \frac{EI}{ml^3\lambda} = \frac{EI}{ml^3\omega^2}$. Eq. (E_1) gives

$(0.0021973-\alpha)\left[(0.0052083-\alpha)(0.0021973-\alpha) - (0.0026042)^2\right]$

$-(0.0026042)\left[0.0026042(0.0021973-\alpha) - (0.0010579)(0.0026042)\right]$

$+(0.0010579)\left[(0.0026042)^2 - (0.0010579)(0.0052083-\alpha)\right] = 0$

i.e. $\alpha^3 - 0.96029\times10^{-2}\,\alpha^2 + 0.1303355\times10^{-4}\,\alpha - 0.0038623\times10^{-6} = 0$

Roots are :

$\alpha_1 = 0.000421453$, $\omega_1 = 48.71082\sqrt{EI/(ml^3)}$

$\alpha_2 = 0.00113955$, $\omega_2 = 29.62329\sqrt{EI/(ml^3)}$

$\alpha_3 = 0.00804192$, $\omega_3 = 11.15116 \sqrt{EI/(m\,l^3)}$

(b) $m = 10$ kg, $l = 0.5$ m, $E = 2.07 \times 10^{11}$ N/m^2,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64}\left(\frac{2.5}{100}\right)^4 = 1.9175 \times 10^{-8} \text{ m}^4$$

$$\sqrt{\frac{EI}{m l^3}} = \sqrt{\frac{(2.07 \times 10^{11})(1.9175 \times 10^{-8})}{10\,(0.5)^3}} = 56.3505$$

$$\omega_3 = 48.71082\,(56.3505) = 2744.8791 \text{ rad/sec}$$

$$\omega_2 = 29.62329\,(56.3505) = 1669.2872 \text{ rad/sec}$$

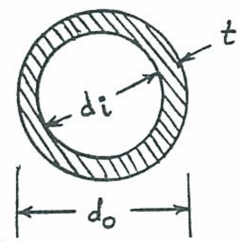
$$\omega_1 = 11.15116\,(56.3505) = 628.3734 \text{ rad/sec}$$

(c) In order to have the same natural frequencies, we need to have the same value of I.

(i) For solid circular cross-section of diameter 2.5 cm,

$$I = 1.9175 \times 10^{-8} \text{ m}^4$$

(ii) For hollow circular section:



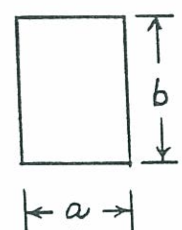
let $d_o = 5t$

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(625 - 81)\,t^4$$

$$= 26.7036\,t^4 = 1.9175 \times 10^{-8}$$

$t = 0.5177$ cm, $d_o = 2.5885$ cm, $d_i = 1.5531$ cm.

(iii) For solid rectangular section:

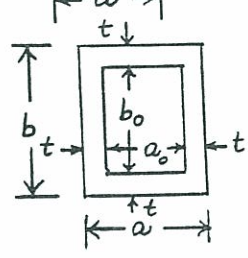


Let $b = 2a$.

$$I = \frac{1}{12}(a)\,b^3 = \frac{2}{3}a^4 = 1.9175 \times 10^{-8}$$

$a = 1.3023$ cm , $b = 2.6046$ cm.

(iv) For hollow rectangular section:



Let $b = 5t$, $b_o = 3t$

$a = 2.5t$, $a_o = 0.5t$

$$I = \frac{1}{12}\left[a b^3 - a_o b_o^3\right]$$

$$= \frac{1}{12}\left[2.5(125)\,t^4 - (0.5)(27)\,t^4\right]$$

$$= 24.9167\,t^4 = 1.9175 \times 10^{-8} \text{ m}^4$$

$t = 0.5267$ cm, $a = 1.3167$ cm, $b = 2.6335$ cm,

$a_o = 0.2634$ cm, $b_o = 1.5801$ cm.

Weights:

Weights are proportional to cross-sectional areas.

(i) For solid circular section:

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (2.5)^2 = 4.90875 \text{ cm}^2$$

(ii) For hollow circular section:

$$A = \frac{\pi}{4} (d_o^2 - d_i^2) = \frac{\pi}{4} \left[2.5885^2 - 1.5531^2 \right] = 3.3680 \text{ cm}^2$$

(iii) For solid rectangular section:

$$A = ab = (1.3023)(2.6046) = 3.3920 \text{ cm}^2$$

(iv) For hollow rectangular section:

$$A = ab - a_o b_o = (1.3167)(2.6335) - (0.2634)(1.5801) = 3.0513 \text{ cm}^2$$

$\therefore$ Least weight beam will have a hollow rectangular section.

6.55

$$\begin{vmatrix} \lambda - 5 & -3 & -2 \\ -3 & \lambda - 6 & -4 \\ -1 & -2 & \lambda - 6 \end{vmatrix} = 0$$

i.e. $(\lambda-5)\left[(\lambda-6)^2 - (-2)(-4)\right] - (-3)\left[-3(\lambda-6) - (-1)(-4)\right]$

$+ (-2)\left[-3(-2) - (-1)(\lambda-6)\right] = 0$

i.e. $\lambda^3 - 17\lambda^2 + 77\lambda - 98 = 0$

Roots give: $\lambda_1 = 2.21398$, $\lambda_2 = 4.16929$, $\lambda_3 = 10.6168$

6.56 From problem 6.24, for $k_i = k$; $i = 1,2,3,4$,

$$[k] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion:

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

These become, for harmonic motion,

$$\begin{bmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- } (E_1)$$

Frequency equation:

$$(-m\omega^2 + 2k)\left[(-m\omega^2 + 2k)^2 - k^2\right] + k\left[-k(-m\omega^2 + 2k)\right] = 0$$

i.e. $(-\alpha + 2)(\alpha^2 - 4\alpha + 2) = 0$ where $\alpha = \frac{m\omega^2}{k}$.

This gives $\alpha_1 = 2-\sqrt{2} = 0.585786$, $\alpha_2 = 2$, $\alpha_3 = 3.414214$

$$\omega_1 = 0.765367\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.414214\sqrt{\frac{k}{m}}, \quad \omega_3 = 1.847759\sqrt{\frac{k}{m}}$$

(E_1) gives $X_2^{(j)} = \left(\frac{-m\,\omega_j^2 + 2k}{k}\right) X_1^{(j)}$

$$-k\,X_1^{(j)} + (-m\,\omega_j^2 + 2k)\,X_2^{(j)} - k\,X_3^{(j)} = 0$$

or $\left[-k + (-m\,\omega_j^2 + 2k)^2 \cdot \frac{1}{k}\right] X_1^{(j)} - k\,X_3^{(j)} = 0$

or $X_3^{(j)} = \left\{\frac{(-m\,\omega_j^2 + 2k)^2 - k^2}{k^2}\right\} X_1^{(j)}$

$$j^{th} \text{ mode} = \vec{X}^{(j)} = \begin{Bmatrix} X_1^{(j)} \\ X_2^{(j)} \\ X_3^{(j)} \end{Bmatrix} = X_1^{(j)} \begin{Bmatrix} 1 \\ (-m\,\omega_j^2 + 2k)/k \\ [(-m\,\omega_j^2 + 2k)^2 - k^2]/k^2 \end{Bmatrix}$$

Mode shapes are:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.414214 \\ 1 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} X_1^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1.414214 \\ 1 \end{Bmatrix} X_1^{(3)}$$

6.57 From problem 6.24,

$$[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}, \quad [m] = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Equations of motion for harmonic motion:

$$\begin{bmatrix} -2m\,\omega^2 + 2k & -k & 0 \\ -k & -3m\,\omega^2 + 2k & -k \\ 0 & -k & -2m\,\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_1)$$

Frequency equation:

$$(-2m\,\omega^2 + 2k)\left[(-3m\,\omega^2 + 2k)(-2m\,\omega^2 + 2k) - k^2\right] + k\left[-k\,(-2m\,\omega^2 + 2k)\right] = 0$$

i.e. $(-m\omega^2 + k)\left[3m^2\,\omega^4 - 5\,k m\,\omega^2 + k^2\right] = 0$

i.e. $(-\alpha + 1)\,(3\alpha^2 - 5\alpha + 1) = 0$ where $\alpha = \frac{\omega^2 m}{k}$

$\therefore\ \alpha_1 = 0.232408$, $\omega_1 = 0.482087\sqrt{\frac{k}{m}}$

$\alpha_2 = 1.0$, $\omega_2 = \sqrt{\frac{k}{m}}$

$\alpha_3 = 1.434258$, $\omega_3 = 1.197605\sqrt{\frac{k}{m}}$

(E_1) gives $\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-2m\,\omega_j^2 + 2k}{k}$

$$(-3m\,\omega_j^2 + 2k)\, X_2^{(j)} - k\, X_3^{(j)} = k\, X_1^{(j)}$$

or $$(-3m\,\omega_j^2 + 2k)\left(\frac{-2m\,\omega_j^2 + 2k}{k}\right) X_1^{(j)} - k\, X_1^{(j)} = k\, X_3^{(j)}$$

or $$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{(-3m\,\omega_j^2 + 2k)(-2m\,\omega_j^2 + 2k) - k^2}{k^2}$$

$$j^{th} \text{ mode} = \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-2m\omega_j^2 + 2k)/k \\ \{(-3m\,\omega_j^2 + 2k)(-2m\,\omega_j^2 + 2k) - k^2\}/k^2 \end{Bmatrix}$$

This leads to:

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1 \\ 1.535184 \\ 1 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} -1 \\ -0.868516 \\ 1 \end{Bmatrix}$$

6.58 For $l_i = l$ and $m_i = m$ $(i = 1,2,3)$, problem 6.42 gives

$$\begin{bmatrix} 3m\,l^2 & 2m\,l^2 & m\,l^2 \\ 2m\,l^2 & 2m\,l^2 & m\,l^2 \\ m\,l^2 & m\,l^2 & m\,l^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 3mg\,l & 0 & 0 \\ 0 & 2mg\,l & 0 \\ 0 & 0 & mg\,l \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For harmonic motion,

$$\begin{bmatrix} -3\,l\,\omega^2 + 3g & -2\,l\,\omega^2 & -l\,\omega^2 \\ -2\,l\,\omega^2 & -2\,l\,\omega^2 + 2g & -l\,\omega^2 \\ -l\,\omega^2 & -l\,\omega^2 & -l\,\omega^2 + g \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Dividing throughout by $-g$ and defining $\alpha = \dfrac{\omega^2 l}{g}$, this gives

$$\begin{bmatrix} 3\alpha - 3 & 2\alpha & \alpha \\ 2\alpha & 2\alpha - 2 & \alpha \\ \alpha & \alpha & \alpha - 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{---- } (E_1)$$

Frequency equation:

$$(3\alpha - 3)\left[(2\alpha - 2)(\alpha - 1) - \alpha^2\right] - 2\alpha\left[2\alpha(\alpha - 1) - \alpha^2\right] + \alpha\left[2\alpha^2 - \alpha(2\alpha - 2)\right] = 0$$

i.e. $$\alpha^3 - 9\alpha^2 + 18\alpha - 6 = 0$$

Roots are:

$$\alpha_1 = 0.415764, \quad \omega_1 = 0.644798\sqrt{\frac{g}{l}}$$
$$\alpha_2 = 2.29431, \quad \omega_2 = 1.514698\sqrt{g/l}$$
$$\alpha_3 = 6.28995, \quad \omega_3 = 2.507977\sqrt{g/l}$$

Mode shapes: (E_1) gives $$\Theta_2^{(j)} = \left\{\frac{-2\alpha_j^2 + 6\alpha_j - 3}{\alpha_j(\alpha_j - 2)}\right\} \Theta_1^{(j)}, \quad \Theta_3^{(j)} = \left(\frac{\alpha_j - 3}{\alpha_j - 2}\right) \Theta_1^{(j)}$$

$$j^{th} \text{ mode} = \vec{\Theta}^{(j)} = \Theta_1^{(j)} \begin{Bmatrix} 1 \\ (-2\alpha_j^2 + 6\alpha_j - 3)/(\alpha_j^2 - 2\alpha_j) \\ (\alpha_j - 3)/(\alpha_j - 2) \end{Bmatrix}$$

Hence

$$\vec{\Theta}^{(1)} = \Theta_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix}, \quad \vec{\Theta}^{(2)} = \Theta_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix}, \quad \vec{\Theta}^{(3)} = \Theta_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

6.59 From problem 6.27 we find, for $m_1 = m_3 = m$, $m_2 = 2m$, $k_1 = k_2 = k$ and $k_3 = 2k$,

$$[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}, \quad [m] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Equations of motion for harmonic motion

$$\begin{bmatrix} -\omega^2 m + 2k & -k & 0 \\ -k & -2m\omega^2 + 3k & -2k \\ 0 & -2k & -\omega^2 m + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{---} (E_1)$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 m + 2k & -k & 0 \\ -k & -2m\omega^2 + 3k & -2k \\ 0 & -2k & -\omega^2 m + 2k \end{vmatrix} = 0$$

or $\quad (-\alpha + 2)(2\alpha^2 - 7\alpha + 1) = 0 \quad$ with $\quad \alpha = \dfrac{m\omega^2}{k}$

$$\therefore \quad \alpha_1 = 0.149219, \quad \omega_1 = 0.386289\sqrt{k/m}$$
$$\alpha_2 = 2.0, \quad \omega_2 = 1.414214\sqrt{k/m}$$
$$\alpha_3 = 3.350781, \quad \omega_3 = 1.830514\sqrt{k/m}$$

Eq. (E_1) gives, for ω_j,

$$\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-\omega_j^2 m + 2k}{k}$$

$$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{(-2m\omega_j^2 + 3k)(-m\omega_j^2 + 2k) - k^2}{2k^2}$$

$$j^{th} \text{ mode} = \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-\omega_j^2 m + 2k)/k \\ \{(-2m\omega_j^2 + 3k)(-m\omega_j^2 + 2k) - k^2\}/(2k^2) \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.850781 \\ 2.0 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.0 \\ -0.5 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.350781 \\ 2.0 \end{Bmatrix}$$

6.60 For $k_1 = 3k$, $k_2 = k_3 = k$, $m_1 = 4m$, $m_2 = 2m$ and $m_3 = m$, Eq. (E_2) of problem 6.51 gives

$$\begin{bmatrix} -4m\omega^2+4k & -k & 0 \\ -k & -2m\omega^2+2k & -k \\ 0 & -k & -m\omega^2+k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad --- (E_1)$$

Frequency equation:

$$(-4m\omega^2+4k)\left\{(-2m\omega^2+2k)(-m\omega^2+k) - k^2\right\} + k\left\{-k(-m\omega^2+k)\right\} = 0$$

i.e. $\quad (-\alpha+1)(8\alpha^2 - 16\alpha + 3) = 0 \quad$ with $\alpha = \dfrac{m\omega^2}{k}$

$\therefore \quad \alpha_1 = 0.209431 \quad , \quad \omega_1 = 0.457636\sqrt{k/m}$

$\alpha_2 = 1.0 \quad , \quad \omega_2 = \sqrt{k/m}$

$\alpha_3 = 1.790569 \quad , \quad \omega_3 = 1.338121\sqrt{k/m}$

Eq. (E_1) gives

$$X_2^{(j)} = \left\{\frac{-4m\omega_j^2 + 4k}{k}\right\} X_1^{(j)}$$

$$X_3^{(j)} = \left[-1 + (-2m\omega_j^2 + 2k)\left(\frac{-4m\omega_j^2+4k}{k^2}\right)\right] X_1^{(j)}$$

$$j^{th} \text{ mode} = \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-4m\omega_j^2 + 4k)/k \\ \left\{(-2m\omega_j^2+2k)(-4m\omega_j^2+4k) - k^2\right\}/k^2 \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 3.162276 \\ 4.0 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.0 \\ -1.0 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -3.162276 \\ 4.0 \end{Bmatrix}$$

6.61 For $m_1 = 2m$, $m_2 = m$, $m_3 = 3m$ and $l_i = l$ for all i, problem 6.28 gives

$$a_{11} = \frac{1}{P\left(\frac{1}{l} + \frac{1}{3l}\right)} = \frac{3l}{4P}, \quad a_{21} = \frac{2}{3}a_{11} = \frac{1}{2}\frac{l}{P}, \quad a_{31} = \frac{1}{3}a_{11} = \frac{1}{4}\frac{l}{P}$$

$$a_{22} = \frac{1}{P\left(\frac{1}{2l} + \frac{1}{2l}\right)} = \frac{l}{P}, \quad a_{32} = \frac{1}{2}a_{22} = \frac{1}{2}\frac{l}{P}, \quad a_{33} = \frac{1}{P\left(\frac{1}{3l} + \frac{1}{l}\right)} = \frac{3l}{4P}$$

$$[a] = \frac{l}{4P}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad [m] = m\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [a][m] = \frac{lm}{4P}\begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 9 \end{bmatrix}$$

Equations of motion:

$$[a][m]\ddot{\vec{x}} + [I]\vec{x} = \vec{0}$$

Frequency equation: $\quad \left| -[a][m]\omega^2 + [I] \right| = 0$

i.e.

$$\left| -\frac{\omega^2 \ell m}{4P}\begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

with $\alpha = \omega^2 \ell m/(4P)$, this equation becomes

$$\begin{vmatrix} 6\alpha - 1 & 2\alpha & 3\alpha \\ 4\alpha & 4\alpha - 1 & 6\alpha \\ 2\alpha & 2\alpha & 9\alpha - 1 \end{vmatrix} = 96\alpha^3 - 88\alpha^2 + 19\alpha - 1 = 0$$

Roots are:

$\alpha_1 = 0.079126$, $\omega_1 = 0.562587\sqrt{\frac{P}{\ell m}}$

$\alpha_2 = 0.209671$, $\omega_2 = 0.915797\sqrt{\frac{P}{\ell m}}$

$\alpha_3 = 0.627872$, $\omega_3 = 1.584767\sqrt{\frac{P}{\ell m}}$

6.62 For $(GJ)_i = GJ$, $J_{di} = J_0$ and $\ell_i = \ell$ for all i, problem 6.23 gives

$$[k] = \frac{GJ}{\ell}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [J_d] = J_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion for harmonic oscillation:

$$\begin{bmatrix} -\omega^2 J_0 + 2\frac{GJ}{\ell} & -\frac{GJ}{\ell} & 0 \\ -\frac{GJ}{\ell} & -\omega^2 J_0 + 2\frac{GJ}{\ell} & -\frac{GJ}{\ell} \\ 0 & -\frac{GJ}{\ell} & -\omega^2 J_0 + 2\frac{GJ}{\ell} \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_1)$$

Dividing throughout by GJ/ℓ and defining $\alpha = \frac{\omega^2 J_0 \ell}{GJ}$, (E_1) gives

$$\begin{bmatrix} -\alpha + 2 & -1 & 0 \\ -1 & -\alpha + 2 & -1 \\ 0 & -1 & -\alpha + 2 \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_2)$$

Frequency equation:

$$\begin{vmatrix} -\alpha + 2 & -1 & 0 \\ -1 & -\alpha + 2 & -1 \\ 0 & -1 & -\alpha + 2 \end{vmatrix} = (-\alpha + 2)(\alpha^2 - 4\alpha + 2) = 0$$

Roots are:

$\alpha_1 = 2 - \sqrt{2} = 0.585786$, $\alpha_2 = 2$, $\alpha_3 = 2 + \sqrt{2} = 3.414214$

$\omega_1 = 0.765367\sqrt{\frac{GJ}{\ell J_0}}$, $\omega_2 = 1.414214\sqrt{\frac{GJ}{\ell J_0}}$, $\omega_3 = 1.847759\sqrt{\frac{GJ}{\ell J_0}}$

Noting that Eq. (E_1) is similar to Eq. (E_1) of problem 6.56, we can use the same modeshapes:

$$\vec{\Theta}^{(1)} = \Theta_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.414214 \\ 1.0 \end{Bmatrix}, \quad \vec{\Theta}^{(2)} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{\Theta}^{(3)} = \Theta_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.414214 \\ 1.0 \end{Bmatrix}$$

6.63 Equations of motion

$$\frac{\rho A l}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \frac{2AE}{l} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_1)$$

This gives, for harmonic motion,

$$\begin{bmatrix} \left(-\frac{\omega^2 \rho A l}{4} + \frac{2AE}{l}\right) & -\frac{2AE}{l} & 0 \\ -\frac{2AE}{l} & \left(-\frac{2\omega^2 \rho A l}{4} + \frac{4AE}{l}\right) & -\frac{2AE}{l} \\ 0 & -\frac{2AE}{l} & \left(-\frac{\omega^2 \rho A l}{4} + \frac{2AE}{l}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_2)$$

Dividing throughout by $\frac{2AE}{l}$ and defining $\alpha = \frac{\omega^2 \rho A l}{8AE}$,

$$\begin{bmatrix} -\alpha+1 & -1 & 0 \\ -1 & -2\alpha+2 & -1 \\ 0 & -1 & -\alpha+1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_3)$$

Frequency equation:

$$\begin{vmatrix} -\alpha+1 & -1 & 0 \\ -1 & -2\alpha+2 & -1 \\ 0 & -1 & -\alpha+1 \end{vmatrix} = 2\alpha(-\alpha+1)(\alpha-2) = 0$$

$$\therefore \quad \alpha_1 = 0, \quad \omega_1 = 0$$

$$\alpha_2 = 1, \quad \omega_2 = \sqrt{\frac{8AE}{\rho A l^2}}$$

$$\alpha_3 = 2, \quad \omega_3 = \sqrt{\frac{16AE}{\rho A l^2}}$$

Principal modes:

Eq. (E_3) gives

$$X_2^{(j)} = (-\alpha_j + 1)\, X_1^{(j)}$$

$$X_3^{(j)} = \left[-1 + (-2\alpha_j+2)(-\alpha_j+1)\right] X_1^{(j)}$$

j^{th} mode: $$\vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-\alpha_j+1) \\ 2(-\alpha_j+1)^2 - 1 \end{Bmatrix}$$

Thus

$$\vec{x}^{(1)} = x_1^{(1)} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \vec{x}^{(2)} = x_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{x}^{(3)} = x_1^{(3)} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

6.64 For orthonormalization, $\vec{x}^{(i)T} [m] \vec{x}^{(i)} = 1 \; ; \; i = 1, 2, 3$

Let new $\vec{x}^{(1)} = a_1 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$, $\vec{x}^{(2)} = a_2 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ and $\vec{x}^{(3)} = a_3 \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$

$$\vec{x}^{(1)T}[m]\,\vec{x}^{(1)} = a_1^2 \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = 4 a_1^2 = 1 \;\Rightarrow\; a_1 = \frac{1}{2}$$

$$\vec{x}^{(2)T}[m]\,\vec{x}^{(2)} = a_2^2 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 4 a_2^2 = 1 \;\Rightarrow\; a_2 = \frac{1}{2}$$

$$\vec{x}^{(3)T}[m]\,\vec{x}^{(3)} = a_3^2 \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} = 6 a_3^2 = 1 \;\Rightarrow\; a_3 = \frac{1}{\sqrt{6}}$$

$[m]$- orthonormal modal matrix $= [X] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & \sqrt{2/3} \\ 1 & 1 & \sqrt{8/3} \end{bmatrix}$

6.65 Stiffness matrix:

Let $x_1 = 1$, $x_2 = x_3 = 0$. $\quad F_1 = 2+1+1 = 4 = k_{11}$, $F_2 = -1 = k_{21}$, $F_3 = -1 = k_{31}$

Let $x_2 = 1$, $x_1 = x_3 = 0$. $\quad F_2 = 1+ \; = 2 = k_{22}$, $F_1 = -1 = k_{12}$, $F_3 = 0 = k_{32}$

Let $x_3 = 1$, $x_1 = x_2 = 0$. $\quad F_3 = 1+1 = 2 = k_{33}$, $F_1 = -1 = k_{13}$, $F_2 = 0 = k_{23}$

$$\therefore [k] = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Mass matrix:

$$[m] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

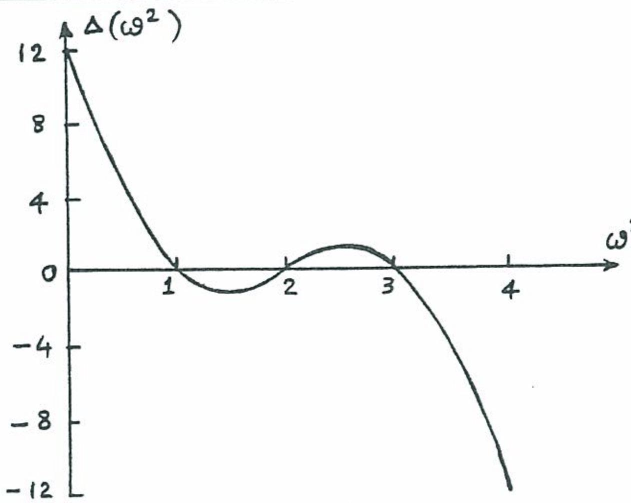
(a) Characteristic polynomial:

$$\begin{vmatrix} (-2\omega^2+4) & -1 & -1 \\ -1 & (-\omega^2+2) & 0 \\ -1 & 0 & (-\omega^2+2) \end{vmatrix} = 0$$

i.e. $2(-\omega^2+1)(-\omega^2+2)(-\omega^2+3) = 0$

$$\therefore \Delta(\omega^2) = 2(-\omega^2+1)(-\omega^2+2)(-\omega^2+3)$$

(b) Plot of $\Delta(\omega^2)$:



(c) Roots of equation:

$$\left.\begin{aligned} \omega_1^2 &= 1 \\ \omega_2^2 &= 2 \\ \omega_3^2 &= 3 \end{aligned}\right\} \text{from the graph.}$$

6.66 (a)

$$\vec{X}^{(1)} = \begin{Bmatrix} 0.2754946 \\ 0.3994672 \\ 0.4490562 \end{Bmatrix}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 0.6916979 \\ 0.2974301 \\ -0.3389320 \end{Bmatrix}, \quad [m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = (0.2754946 \quad 0.3994672 \quad 0.4490562) \begin{Bmatrix} 0.6916979 \\ 0.5948602 \\ -1.0167960 \end{Bmatrix} \simeq 0$$

Let $\vec{X}^{(3)} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$

Then $\vec{X}^{(3)T} [m] \vec{X}^{(3)} = 1, \quad \vec{X}^{(1)T} [m] \vec{X}^{(3)} = 0, \quad \vec{X}^{(2)T} [m] \vec{X}^{(3)} = 0$

These relations give

$$a_1^2 + 2 a_2^2 + 3 a_3^2 = 1 \qquad \text{--- } (E_1)$$

$$0.2754946 \; a_1 + 0.7989344 \; a_2 + 1.3471686 \; a_3 = 0 \qquad \text{--- } (E_2)$$

$$0.6916979 \; a_1 + 0.5948602 \; a_2 - 1.0167960 \; a_3 = 0 \qquad \text{--- } (E_3)$$

From (E_2) and (E_3),

$$a_1 = -2.9000002 \; a_2 - 4.89 \; a_3 = -0.86 \; a_2 + 1.4700001 \; a_3$$

or $\quad a_2 = -3.1176468 \; a_3 \qquad \text{--- } (E_4)$

and $\quad a_1 = -2.9000002 (-3.1176468 \; a_3) - 4.89 \; a_3 = 4.1511763 \; a_3 \qquad \text{--- } (E_5)$

(E_1), (E_4) and (E_5) give

$$a_3^2 \left(17.232265 + 9.7197216 + 1 \right) = 1 \;\Rightarrow\; a_3 = \pm 0.1891445$$

Hence $\quad a_2 = \mp 0.5896857, \quad a_1 = \pm 0.7851722$

$$\therefore \quad \vec{X}^{(3)} = \begin{Bmatrix} 0.7851722 \\ -0.5896857 \\ 0.1891445 \end{Bmatrix}$$

(b) $\omega_i^2 = \vec{X}^{(i)T} [k] \vec{X}^{(i)} \quad ; \quad [k] = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$$\omega_1^2 = (0.2754946 \quad 0.3994672 \quad 0.4490562) \begin{Bmatrix} 0.05509889 \\ 2.8926938 \\ 2.6943374 \end{Bmatrix} = 2.3806248$$

$$\omega_2^2 = (0.6916979 \quad 0.2974301 \quad -0.3389320) \begin{Bmatrix} 2.9604671 \\ 0.2075095 \\ -2.0335920 \end{Bmatrix} = 2.7987180$$

$$\omega_3^2 = (0.7851722 \quad -0.5896857 \quad 0.1891445) \begin{Bmatrix} 7.0697761 \\ -9.0375462 \\ 1.1348671 \end{Bmatrix} = 11.094957$$

$$\therefore \omega_1 = 1.5429274, \quad \omega_2 = 1.6729369, \quad \omega_3 = 3.3309095.$$

6.67 From solution of Problem 6.1, we find

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad ; \quad [k] = k \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 \, [m] + [k] \right| = 0$$

$$\text{or} \quad \begin{vmatrix} (-\alpha+7) & -1 & -5 \\ -1 & (-\alpha+2) & -1 \\ -5 & -1 & (-\alpha+7) \end{vmatrix} = 0$$

where $\alpha = \dfrac{\omega^2 \, m}{k}$. Expansion of the frequency equation gives:

$$(-\alpha+7)\left\{(-\alpha+2)(-\alpha+7)-1\right\}+1\left\{-(-\alpha+7)-5\right\}-5\left\{1+5(-\alpha+2)\right\}=0$$

$$\text{or} \quad \alpha^3 - 16\,\alpha^2 + 50\,\alpha - 24 = 0$$

Roots of this equation give:

$$\alpha_1 = 0.58576 \;;\; \omega_1 = 0.7653\sqrt{\frac{k}{m}}$$

$$\alpha_2 = 4.41428 \;;\; \omega_2 = 1.8478\sqrt{\frac{k}{m}}$$

$$\alpha_3 = 12.0 \;;\; \omega_3 = 3.4641\sqrt{\frac{k}{m}}$$

6.68 From the solution of Problem 6.2, we obtain:

$$[m] = \begin{bmatrix} \frac{2\,m\,\ell^2}{3} & 0 & 0 \\ 0 & 2\,m & 0 \\ 0 & 0 & m \end{bmatrix} = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[k] = \begin{bmatrix} \frac{25\,k\,\ell^2}{8} & -\frac{k\,\ell}{2} & 0 \\ -\frac{k\,\ell}{2} & 3\,k & -k \\ 0 & -k & k \end{bmatrix} = \begin{bmatrix} 3125 & -500 & 0 \\ -500 & 3000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 \, [m] + [k] \right| = 0$$

$$\text{or} \quad \begin{vmatrix} (3125 - 0.6667\,\omega^2) & -500 & 0 \\ -500 & (-2\,\omega^2 + 3000) & -1000 \\ 0 & -1000 & (1000 - \omega^2) \end{vmatrix} = 0$$

or $$(3125 - 0.6667\,\omega^2)\left\{(3000 - 2\,\omega^2)(1000 - \omega^2) - 1000^2\right\} + 500\left\{-500\,(1000 - \omega^2) - 0\right\} = 0$$

or $$-1.3334\,\omega^6 + 9583.4\,\omega^4 - 16.7083\,(10^6)\,\omega^2 + 6.0\,(10^9) = 0$$

Defining $\alpha = \left(\dfrac{\omega^2}{1000}\right)$, the above equation can be rewritten as

$$\alpha^3 - 7.1872\,\alpha^2 + 12.5306\,\alpha - 4.4998 = 0$$

Roots of this equation are (using Program):

$$\alpha_1 = 0.484831 \;;\; \omega_1 = 22.0189 \text{ rad/sec}$$
$$\alpha_2 = 1.95501 \;;\; \omega_2 = 44.2155 \text{ rad/sec}$$
$$\alpha_3 = 4.74738 \;;\; \omega_3 = 68.9012 \text{ rad/sec}$$

6.75 $[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{bmatrix}$, $[k] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1+k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$

Equations of motion for harmonic motion:

$$\begin{bmatrix} (-m\omega^2+k) & -k & 0 \\ -k & (-2m\omega^2+2k) & -k \\ 0 & -k & (-3m\omega^2+k) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E.1)$$

Defining $\alpha = \frac{m\omega^2}{k}$, (E.1) can be rewritten as

$$\begin{bmatrix} (-\alpha+1) & -1 & 0 \\ -1 & (-2\alpha+2) & -1 \\ 0 & -1 & (-3\alpha+1) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E.2)$$

Frequency equation is

$$2\alpha\,(3\alpha^2 - 7\alpha + 3) = 0$$

Roots are

$\alpha_1 = 0 \quad ; \quad \omega_1 = 0$

$\alpha_2 = 0.565741 \quad ; \quad \omega_2 = 0.752158\sqrt{k/m}$

$\alpha_3 = 1.767592 \quad ; \quad \omega_3 = 1.329508\sqrt{k/m}$

Eqs. (E.2) give $\quad X_2^{(j)} = (-\alpha_j + 1)\, X_1^{(j)}, \quad X_3^{(j)} = \left(\frac{1}{-3\alpha_j+1}\right) X_2^{(j)}$

$$\therefore \quad \vec{X}^{(j)} = \begin{Bmatrix} 1.0 \\ (-\alpha_j + 1) \\ \left(\frac{-\alpha_j+1}{-3\alpha_j+1}\right) \end{Bmatrix} X_1^{(j)}$$

Hence

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.434259 \\ -0.622841 \end{Bmatrix} X_1^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -0.767592 \\ 0.178395 \end{Bmatrix} X_1^{(3)}$$

6.76 $[k_t] = k_t \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \quad [J] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

With $\alpha = \dfrac{\omega^2 J_0}{k_t}$, the equations of motion for harmonic motion become

$$\begin{bmatrix} (-\alpha+1) & -1 & 0 \\ -1 & (-\alpha+3) & -2 \\ 0 & -2 & (-\alpha+2) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E.1)$$

Frequency equation is $\quad \alpha(\alpha^2 - 6\alpha + 6) = 0$

Roots are: $\quad \alpha_1 = 0, \quad \alpha_2 = 1.267949, \quad \alpha_3 = 4.732051$

$$\omega_1 = 0, \quad \omega_2 = 1.126032\sqrt{k_t/J_0}, \quad \omega_3 = 2.175328\sqrt{k_t/J_0}$$

Eq. (E.1) gives

$$\Theta_2^{(j)} = (-\alpha_j + 1)\,\Theta_1^{(j)}, \quad \Theta_3^{(j)} = \frac{2}{(-\alpha_j+2)}\,\Theta_2^{(j)}$$

$$\vec{\Theta}^{(j)} = \begin{Bmatrix} 1 \\ (-\alpha_j+1) \\ \left(\dfrac{-2\alpha_j+2}{-\alpha_j+2}\right) \end{Bmatrix} \Theta_1^{(j)}$$

$$\therefore \quad \vec{\Theta}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}, \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} 1 \\ -0.267949 \\ -0.732050 \end{Bmatrix} \Theta_1^{(2)}, \quad \vec{\Theta}^{(3)} = \begin{Bmatrix} 1 \\ -3.732051 \\ 2.732051 \end{Bmatrix} \Theta_1^{(3)}$$

Normalize the eigenvectors with respect to the inertia matrix as

$$\vec{\Theta}^{(1)T}[J]\,\vec{\Theta}^{(1)} = \Theta_1^{(1)^2} J_0 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3\, J_0\, \Theta_1^{(1)^2} = 1$$

$$\Theta_1^{(1)} = 0.57735/\sqrt{J_0}$$

$$\vec{\Theta}^{(2)T}[J]\,\vec{\Theta}^{(2)} = \Theta_1^{(2)^2} J_0 \begin{pmatrix} 1 & -0.267949 & -0.732050 \end{pmatrix} \begin{Bmatrix} 1 \\ -0.267949 \\ -0.732050 \end{Bmatrix}$$

$$= \Theta_1^{(2)^2} J_0 (1.607694) = 1$$

$$\Theta_1^{(2)} = 0.788675/\sqrt{J_0}$$

$$\vec{\Theta}^{(3)T}[J]\,\vec{\Theta}^{(3)} = \Theta_1^{(3)^2} J_0 \begin{pmatrix} 1 & -3.732051 & 2.732051 \end{pmatrix} \begin{Bmatrix} 1 \\ -3.732051 \\ 2.732051 \end{Bmatrix}$$

$$= \Theta_1^{(3)^2} J_0 (22.392307)$$

$$\Theta_1^{(3)} = 0.211325/\sqrt{J_0}$$

Modal matrix is

$$[X] = \frac{1}{\sqrt{J_0}} \begin{bmatrix} 0.57735 & 0.788675 & 0.211325 \\ 0.57735 & -0.211325 & -0.788676 \\ 0.57735 & -0.577349 & 0.577351 \end{bmatrix}$$

6.77 From solution of Problem 6.57, the natural frequencies and mode shapes are given by:

$$\omega_1 = 0.482087 \sqrt{\frac{k}{m}} \;;\; \omega_2 = \sqrt{\frac{k}{m}} \;;\; \omega_3 = 1.197605 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.535184 \\ 1.0 \end{Bmatrix} \;;\; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \;;\; \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ 0.868516 \\ -1 \end{Bmatrix}$$

Initial conditions:

$$x_1(0) = x_{10},\; x_2(0) = 0,\; x_3(0) = 0,\; \dot{x}_i(0) = 0 \;;\; i = 1, 2, 3$$

Equations (6.98) and (6.99) yield:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = x_{10} \qquad (1)$$

$$1.5352\, A_1 \cos \phi_1 + 0.8685\, A_3 \cos \phi_3 = 0 \qquad (2)$$

$$A_1 \cos \phi_1 - A_2 \cos \phi_2 - A_3 \cos \phi_3 = 0 \qquad (3)$$

$$-0.4821 \sqrt{\frac{k}{m}} A_1 \sin \phi_1 - \sqrt{\frac{k}{m}} A_2 \sin \phi_2 - 1.1976 \sqrt{\frac{k}{m}} A_3 \sin \phi_3 = 0 \qquad (4)$$

$$-0.4821 \sqrt{\frac{k}{m}} (1.5352)\, A_1 \sin \phi_1 - 1.1976 \sqrt{\frac{k}{m}} (0.8685)\, A_3 \sin \phi_3 = 0 \qquad (5)$$

$$-0.4821 \sqrt{\frac{k}{m}} (1.0)\, A_1 \sin \phi_1 - \sqrt{\frac{k}{m}} (-1)\, A_2 \sin \phi_2 - 1.1976 \sqrt{\frac{k}{m}} (-1)\, A_3 \sin \phi_3 = 0 \qquad (6)$$

Solution of Eqs. (4) to (6):

$$\phi_i = 0 \;;\; i = 1, 2, 3$$

Solution of Eqs. (1) to (3) gives:

$$A_1 = 0.5\, x_{10} \;;\; A_2 = 0.3838\, x_{10} \;;\; A_3 = -\,0.8838\, x_{10}$$

Thus the free vibration solution of the system is given by:

$$x_1(t) = x_{10} \Big(0.5 \cos 0.4821 \sqrt{\frac{k}{m}}\, t + 0.3838 \cos \sqrt{\frac{k}{m}}\, t - 0.8838 \cos 1.1976 \sqrt{\frac{k}{m}}\, t\Big)$$

$$x_2(t) = x_{10} \left\{0.7676 \cos 0.4821 \sqrt{\frac{k}{m}}\, t - 0.7676 \cos 1.1976 \sqrt{\frac{k}{m}}\, t\right\}$$

$$x_3(t) = x_{10} \Big(0.5 \cos 0.4821 \sqrt{\frac{k}{m}}\, t - 0.3838 \cos \sqrt{\frac{k}{m}}\, t + 0.8838 \cos 1.1976 \sqrt{\frac{k}{m}}\, t\Big)$$

6.78 From solution of Problem 6.58, we find that:

$$\omega_1 = 0.6448\sqrt{\frac{g}{\ell}} \;;\; \omega_2 = 1.5147\sqrt{\frac{g}{\ell}} \;;\; \omega_3 = 2.5080\sqrt{\frac{g}{\ell}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix} \;;\; \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix} \;;\; \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

Equations (6.98) and (6.99) can be written, for the stated initial conditions, as:

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = 0 \tag{1}$$

$$1.2922\, A_1 \cos\phi_1 + 0.3527\, A_2 \cos\phi_2 - 1.6450\, A_3 \cos\phi_3 = 0 \tag{2}$$

$$1.6312\, A_1 \cos\phi_1 - 2.3978\, A_2 \cos\phi_2 + 0.7669\, A_3 \cos\phi_3 = \theta_{30} \tag{3}$$

$$-0.6448\sqrt{\frac{g}{\ell}}\, A_1 \sin\phi_1 - 1.5147\sqrt{\frac{g}{\ell}}\, A_2 \sin\phi_2 - 2.5080\sqrt{\frac{g}{\ell}}\, A_3 \sin\phi_3 = 0 \tag{4}$$

$$-0.6448\sqrt{\frac{g}{\ell}}\,(1.2922)\, A_1 \sin\phi_1 - 1.5147\sqrt{\frac{g}{\ell}}\,(0.3527)\, A_2 \sin\phi_2 - 2.5080\sqrt{\frac{g}{\ell}}\,(-1.6450)\, A_3 \sin\phi_3 = 0 \tag{5}$$

$$-0.6448\sqrt{\frac{g}{\ell}}\,(1.6312)\, A_1 \sin\phi_1 - 1.5147\sqrt{\frac{g}{\ell}}\,(-2.3978)\, A_2 \sin\phi_2 - 2.5080\sqrt{\frac{g}{\ell}}\,(0.7669)\, A_3 \sin\phi_3 = 0 \tag{6}$$

Equations (4) to (6) yield:

$$\phi_i = 0 \;;\; i = 1, 2, 3$$

Equations (1) to (3) give

$$A_1 = 0.1812\, x_{30} \;;\; A_2 = -0.2665\, x_{30} \;;\; A_3 = 0.08524\, x_{30}$$

Thus the free vibration solution can be expressed as (see Eq. (6.96)):

$$x_1(t) = x_{30}\left(0.1812 \cos 0.6448\sqrt{\frac{g}{\ell}}\, t - 0.2665 \cos 1.5147\sqrt{\frac{g}{\ell}}\, t + 0.08524 \cos 2.5080\sqrt{\frac{g}{\ell}}\, t\right) \tag{7}$$

$$x_2(t) = x_{30}\left(0.2341 \cos 0.6448\sqrt{\frac{g}{\ell}}\, t - 0.09399 \cos 1.5147\sqrt{\frac{g}{\ell}}\, t - 0.1402 \cos 2.5080\sqrt{\frac{g}{\ell}}\, t\right) \tag{8}$$

$$x_3(t) = x_{30}\left(0.2956 \cos 0.6448\sqrt{\frac{g}{\ell}}\, t + 0.6390 \cos 1.5147\sqrt{\frac{g}{\ell}}\, t + 0.06537 \cos 2.5080\sqrt{\frac{g}{\ell}}\, t\right) \tag{9}$$

6.79

From solution of Problem 6.61, we obtain

$$\omega_1 = 0.5626 \sqrt{\frac{P}{\ell m}} \; ; \; \omega_2 = 0.9158 \sqrt{\frac{P}{\ell m}} \; ; \; \omega_3 = 1.5848 \sqrt{\frac{P}{\ell m}}$$

$$\alpha_1 = \frac{\omega_1^2 \ell m}{4 P} = 0.079126 \; ; \; \alpha_2 = \frac{\omega_2^2 \ell m}{4 P} = 0.209671 \; ; \; \alpha_3 = \frac{\omega_3^2 \ell m}{4 P} = 0.627872$$

The mode shapes can be determined from the equations:

$$(6 \alpha - 1) X_1 + 2 \alpha X_2 + 3 \alpha X_3 = 0 \quad (1)$$
$$4 \alpha X_1 + (4 \alpha - 1) X_2 + 6 \alpha X_3 = 0 \quad (2)$$
$$2 \alpha X_1 + 2 \alpha X_2 + (9 \alpha - 1) X_3 = 0 \quad (3)$$

$$\text{Let } \; X_1 = 1 \quad (4)$$

Then Eqs. (1) and (2) yield

$$X_2 = 2 - 8 \alpha \quad (5)$$

$$X_3 = \frac{1 - 10 \alpha + 16 \alpha^2}{3 \alpha} \quad (6)$$

Using the values of α_1, α_2 and α_3, we obtain, from Eqs. (4) to (6):

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.3670 \\ 1.3014 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.3226 \\ -0.6253 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -3.0230 \\ 0.5462 \end{Bmatrix}$$

The stated initial conditions give, using Eqs. (6.98) and (6.99),

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \quad (7)$$
$$1.3670 A_1 \cos \phi_1 + 0.3226 A_2 \cos \phi_2 - 3.0230 A_3 \cos \phi_3 = x_{20} \quad (8)$$
$$1.3014 A_1 \cos \phi_1 - 0.6253 A_2 \cos \phi_2 + 0.5462 A_3 \cos \phi_3 = 0 \quad (9)$$

$$0.5626 \sqrt{\frac{P}{\ell m}} A_1 \sin \phi_1 + 0.9158 \sqrt{\frac{P}{\ell m}} A_2 \sin \phi_2 + 1.5848 \sqrt{\frac{P}{\ell m}} A_3 \sin \phi_3 = 0 \quad (10)$$

$$0.5626 \sqrt{\frac{P}{\ell m}} (1.3670) A_1 \sin \phi_1 + 0.9158 \sqrt{\frac{P}{\ell m}} (0.3226) A_2 \sin \phi_2 - 1.5848 \sqrt{\frac{P}{\ell m}} (3.0230) A_3 \sin \phi_3 = 0 \quad (11)$$

$$0.5626 \sqrt{\frac{P}{\ell m}} (1.3014) A_1 \sin \phi_1 - 0.9158 \sqrt{\frac{P}{\ell m}} (0.6253) A_2 \sin \phi_2 + 1.5848 \sqrt{\frac{P}{\ell m}} (0.5462) A_3 \sin \phi_3 = 0 \quad (12)$$

Equations (10) to (12) yield:

$$\phi_i = 0 \; ; \; i = 1, 2, 3$$

Equations (7) to (9) give

$$A_1 = 0.1527 \, x_{20} \; ; \; A_2 = 0.09847 \, x_{20} \; ; \; A_3 = -0.2512 \, x_{20}$$

$$x_1(t) = x_{20}\ (0.1527 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\ t + 0.09847 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\ t - 0.2512 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\ t) \qquad (13)$$

$$x_2(t) = x_{20}\ (0.2087 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\ t + 0.03177 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\ t + 0.7594 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\ t) \qquad (14)$$

$$x_3(t) = x_{20}\ (0.1987 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\ t - 0.06157 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\ t - 0.1372 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\ t) \qquad (15)$$

6.80 From solution of Problem 6.51, we obtain

$$\omega_1 = 0.3376 \sqrt{\frac{k}{m}} \ ; \ \omega_2 = 1.4142 \sqrt{\frac{k}{m}} \ ; \ \omega_3 = 2.0943 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.4430 \\ 1.6286 \end{Bmatrix} \ ; \ \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.5 \\ -0.5 \end{Bmatrix} \ ; \ \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -0.693 \\ 0.2047 \end{Bmatrix}$$

Initial conditions:

$$x_i(0) = 0 \ ; \ i = 1, 2, 3 \quad ; \quad \dot{x}_1(0) = \dot{x}_{10} \ , \ \dot{x}_2(0) = \dot{x}_3(0) = 0$$

Equations (6.98) and (6.99) can be expressed as:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \qquad (1)$$

$$1.4430\ A_1 \cos \phi_1 + 0.5\ A_2 \cos \phi_2 - 0.6930\ A_3 \cos \phi_3 = 0 \qquad (2)$$

$$1.6286\ A_1 \cos \phi_1 - 0.5\ A_2 \cos \phi_2 + 0.2047\ A_3 \cos \phi_3 = 0 \qquad (3)$$

$$-0.3376 \sqrt{\frac{k}{m}}\ A_1 \sin \phi_1 - 1.4142 \sqrt{\frac{k}{m}}\ A_2 \sin \phi_2 - 2.0943 \sqrt{\frac{k}{m}}\ A_3 \sin \phi_3 = \dot{x}_{10} \qquad (4)$$

$$-0.4872 \sqrt{\frac{k}{m}}\ A_1 \sin \phi_1 - 0.7071 \sqrt{\frac{k}{m}}\ A_2 \sin \phi_2 + 1.4513 \sqrt{\frac{k}{m}}\ A_3 \sin \phi_3 = 0 \qquad (5)$$

$$-0.5498 \sqrt{\frac{k}{m}}\ A_1 \sin \phi_1 + 0.7071 \sqrt{\frac{k}{m}}\ A_2 \sin \phi_2 - 0.4287 \sqrt{\frac{k}{m}}\ A_3 \sin \phi_3 = 0 \qquad (6)$$

Equations (1) to (3) give:

$$\phi_i = \frac{\pi}{2} \ ; \ i = 1, 2, 3$$

Treating $\sqrt{\frac{k}{m}}\ A_i \sin \phi_i$ (i = 1, 2, 3) as unknowns and noting that all $\phi_i = \frac{\pi}{2}$, Eqs. (4) to (6) can be solved to obtain

$$\sqrt{\frac{k}{m}}\ A_1 = -\,0.2257\ \dot{x}_{10}\ \ ;\ \ A_1 = -\,0.2257\sqrt{\frac{m}{k}}\ \dot{x}_{10}$$
$$\sqrt{\frac{k}{m}}\ A_2 = -\,0.3143\ \dot{x}_{10}\ \ ;\ \ A_2 = -\,0.3143\sqrt{\frac{m}{k}}\ \dot{x}_{10}$$
$$\sqrt{\frac{k}{m}}\ A_3 = -\,0.2289\ \dot{x}_{10}\ \ ;\ \ A_3 = -\,0.2289\sqrt{\frac{m}{k}}\ \dot{x}_{10}$$

The free vibration solution of the system can be expressed, using Eq. (6.96), as

$$x_1(t) = \dot{x}_{10}\sqrt{\frac{m}{k}}\ \Big(-\,0.2257\cos\Big(0.3376\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big) - 0.3143\cos\Big(1.4142\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big) - 0.2289\cos\Big(2.0943\sqrt{\frac{k}{m}} + \frac{\pi}{2}\Big)\Big) \quad (7)$$

$$x_2(t) = \dot{x}_{10}\sqrt{\frac{m}{k}}\ \Big(-\,0.3257\cos\Big(0.3376\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big) - 0.1571\cos\Big(1.4142\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big) + 0.1586\cos\Big(2.0943\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big)\Big) \quad (8)$$

$$x_3(t) = \dot{x}_{10}\sqrt{\frac{m}{k}}\ \Big(-\,0.3676\cos\Big(0.3376\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big) + 0.1571\cos\Big(1.4142\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big) - 0.0469\cos\Big(2.0943\sqrt{\frac{k}{m}}\ t + \frac{\pi}{2}\Big)\Big) \quad (9)$$

6.81 From solution of Problem 6.59, we find:

$$\omega_1 = 0.3863\sqrt{\frac{k}{m}}\ \ ;\ \ \omega_2 = 1.4142\sqrt{\frac{k}{m}}\ \ ;\ \ \omega_3 = 1.8305\sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.8508 \\ 2 \end{Bmatrix}\ ;\ \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -0.5 \end{Bmatrix}\ ;\ \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1.3508 \\ 2 \end{Bmatrix}$$

Initial conditions:

$$x_i(0) = 0\ ,\ i = 1,\ 2,\ 3\ \ ;\ \ \dot{x}_1(0) = \dot{x}_2(0) = 0,\ \ \dot{x}_3(0) = \dot{x}_{30}$$

Equations (6.98) and (6.99) can be expressed as:

$$A_1\cos\phi_1 + A_2\cos\phi_2 + A_3\cos\phi_3 = 0 \quad (1)$$
$$1.8508\ A_1\cos\phi_1 - 1.3508\ A_3\cos\phi_3 = 0 \quad (2)$$
$$2\ A_1\cos\phi_1 - 0.5\ A_2\cos\phi_2 + 2\ A_3\cos\phi_3 = 0 \quad (3)$$
$$-\,0.3863\sqrt{\frac{k}{m}}\ A_1\sin\phi_1 - 1.4142\sqrt{\frac{k}{m}}\ A_2\sin\phi_2 - 1.8305\sqrt{\frac{k}{m}}\ A_3\sin\phi_3 = 0 \quad (4)$$

$$-0.7150\sqrt{\frac{k}{m}}\,A_1\sin\phi_1+2.4726\sqrt{\frac{k}{m}}\,A_3\sin\phi_3=0 \qquad (5)$$

$$-0.7726\sqrt{\frac{k}{m}}\,A_1\sin\phi_1+0.7071\sqrt{\frac{k}{m}}\,A_2\sin\phi_2$$

$$-3.6610\sqrt{\frac{k}{m}}\,A_3\sin\phi_3=\dot{x}_{30} \qquad (6)$$

Equations (1) to (3) yield:

$$\phi_i=\frac{\pi}{2}\ ;\ i=1,\,2,\,3$$

and $\sqrt{\frac{k}{m}}\,A_i\sin\phi_i$ $(i=1,\,2,\,3)$ are the unknowns in Eqs. (4) to (6). The solution of Eqs. (4) to (6) gives, with $\phi_i=\frac{\pi}{2}$:

$$\sqrt{\frac{k}{m}}\,A_1=-0.4369\,\dot{x}_{30}\ ;\ A_1=-0.4369\,\dot{x}_{30}\sqrt{\frac{m}{k}}$$

$$\sqrt{\frac{k}{m}}\,A_2=0.2828\,\dot{x}_{30}\ ;\ A_2=0.2828\,\dot{x}_{30}\sqrt{\frac{m}{k}}$$

$$\sqrt{\frac{k}{m}}\,A_3=-0.1263\,\dot{x}_{30}\ ;\ A_3=-0.1263\,\dot{x}_{30}\sqrt{\frac{m}{k}}$$

Thus the free vibration response of the system can be expressed as (see Eq. (6.96)):

$$x_1(t)=\dot{x}_{30}\sqrt{\frac{m}{k}}\Big(-0.4369\cos\Big(0.3863\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)+0.2828\cos\Big(1.4142\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)$$

$$-0.1263\cos\Big(1.8305\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)\Big)$$

$$x_2(t)=\dot{x}_{30}\sqrt{\frac{m}{k}}\left\{-0.8086\cos\Big(0.3863\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)-0.1706\cos\Big(1.8305\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)\right\}$$

$$x_3(t)=\dot{x}_{30}\sqrt{\frac{m}{k}}\Big(-0.8738\cos\Big(0.3863\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)-0.1414\cos\Big(1.4142\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)$$

$$-0.2526\cos\Big(1.8305\sqrt{\frac{k}{m}}\,t+\frac{\pi}{2}\Big)\Big)$$

6.82 From Example 6.14, we obtain:

$$[m]=m\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\ ;\ [k]=k\begin{bmatrix}1&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$$

The natural frequencies and mode shapes are given by:

$$\omega_1=0\ ;\ \omega_2=\sqrt{\frac{k}{m}}\ ;\ \omega_3=\sqrt{\frac{3\,k}{m}}$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = b \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = c \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

Orthonormalization of mode shapes:

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = a^2 \, (1 \quad 1 \quad 1) \, m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3 \, a^2 \, m = 1 \quad \text{or} \quad a = \sqrt{\frac{1}{3\,m}}$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = b^2 \, m \, (1 \quad 0 \quad -1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 2 \, b^2 \, m = 1 \quad \text{or} \quad b = \sqrt{\frac{1}{2\,m}}$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = c^2 \, m \, (1 \quad -2 \quad 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 6 \, c^2 \, m = 1 \quad \text{or} \quad c = \sqrt{\frac{1}{6\,m}}$$

$$\text{Modal matrix:} \quad [X] = [\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)}] = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \tag{1}$$

Solution is given by Eq. (6.104):

$$\vec{x}(t) = [X] \, \vec{q}(t) \tag{2}$$

where $\vec{q}(t)$ is given by Eqs. (6.113):

$$\ddot{q}_i(t) + \omega_i^2 \, q_i(t) = Q_i(t) \; ; \; i = 1, 2, 3 \tag{3}$$

where

$$\vec{Q}(t) = [X]^T \, \vec{F}(t) = \vec{0} \quad \text{(no external forces)}$$

For the rigid body mode, $\omega_1^2 = 0$ and Eq. (3) reduces to:

$$\ddot{q}_1(t) = 0 \tag{4}$$

whose solution can be written as

$$q_1(t) = c_1 + c_2 \, t \tag{5}$$

where c_1 and c_2 are constants given by

$$c_1 = q_1(t = 0) \quad ; \quad c_2 = \dot{q}_1(t = 0) \tag{6}$$

Initial conditions of the problem:

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7}$$

$$\text{and} \quad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_0 \\ 0 \\ 0 \end{Bmatrix} \tag{8}$$

Using Eqs. (6.115) and (6.116), we find

$$\vec{q}(0) = [X]^T \; [m] \, \vec{x}(0) = \vec{0} \tag{9}$$

$$\dot{\vec{q}}(0) = [X]^T \; [m] \, \dot{\vec{x}}(0) = \sqrt{m} \; \dot{x}_0 \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{Bmatrix} \tag{10}$$

From Eqs. (6), (9) and (10), we obtain

$$c_1 = q_1(0) = 0 \quad ; \quad c_2 = \dot{q}_1(0) = \sqrt{\frac{m}{3}} \; \dot{x}_0 \tag{11}$$

and hence, from Eq. (5), we find

$$q_1(t) = \sqrt{\frac{m}{3}} \; \dot{x}_0 \; t \tag{12}$$

Solution of Eqs. (3) for $q_2(t)$ and $q_3(t)$ can be expressed as

$$q_2(t) = q_2(0) \cos \omega_2 \; t + \frac{\dot{q}_2(0)}{\omega_2} \sin \omega_2 \; t = \frac{m \, \dot{x}_0}{\sqrt{2 \, k}} \sin \sqrt{\frac{k}{m}} \; t \tag{13}$$

$$q_3(t) = q_3(0) \cos \omega_3 \; t + \frac{\dot{q}_3(0)}{\omega_3} \sin \omega_3 \; t = \frac{m \, \dot{x}_0}{3 \sqrt{2 \, k}} \sin \sqrt{\frac{3 \, k}{m}} \; t \tag{14}$$

The free vibration of the system is given by Eq. (2):

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{Bmatrix} \sqrt{\frac{m}{3}} \; \dot{x}_0 \; t \\ \frac{m \, \dot{x}_0}{\sqrt{2 \, k}} \sin \sqrt{\frac{k}{m}} \; t \\ \frac{m \, \dot{x}_0}{3 \sqrt{2 \, k}} \sin \sqrt{\frac{3 \, k}{m}} \; t \end{Bmatrix} \tag{15}$$

6.83 For given system, $m = 10$, $k = 100$ and hence $\sqrt{k/m} = \sqrt{10} = 3.1623$. For free vibration response, the initial conditions lead to Eqs. (6.98) and (6.99), which can be expressed as:

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = x_1(0) = 0.1 \qquad (1)$$

$$1.8019\, A_1 \cos\phi_1 + 0.4450\, A_2 \cos\phi_2 - 1.2468\, A_3 \cos\phi_3 = x_2(0) = 0.1 \qquad (2)$$

$$2.2470\, A_1 \cos\phi_1 - 0.8020\, A_2 \cos\phi_2 + 0.5544\, A_3 \cos\phi_3 = x_3(0) = 0.1 \qquad (3)$$

$$-0.44504\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 - 1.2471\sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 - 1.8025\sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = \dot{x}_1(0) = 0$$

or

$$-1.4073\, A_1 \sin\phi_1 - 3.9437\, A_2 \sin\phi_2 - 5.7000\, A_3 \sin\phi_3 = 0 \qquad (4)$$

$$-0.10192\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 - 0.55496\sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 + 2.2474\sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = \dot{x}_2(0) = 0$$

or

$$-0.3223\, A_1 \sin\phi_1 - 1.7549\, A_2 \sin\phi_2 + 7.1069\, A_3 \sin\phi_3 = 0 \qquad (5)$$

$$-\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 + \sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 - \sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = \dot{x}_3(0) = 0$$

or

$$-A_1 \sin\phi_1 + A_2 \sin\phi_2 - A_3 \sin\phi_3 = 0 \qquad (6)$$

Solution of Eqs. (1) to (3) is given by

$$A_1 \cos\phi_1 = 0.0543, \quad A_2 \cos\phi_2 = 0.0349, \quad A_3 \cos\phi_3 = 0.0108 \qquad (7)$$

Solution of Eqs. (4) to (6) is given by

$$A_1 \sin\phi_1 = 0, \quad A_2 \sin\phi_2 = 0, \quad A_3 \sin\phi_3 = 0 \qquad (8)$$

Equations (7) and (8) yield

$$\left.\begin{array}{l} A_1 = 0.0543, \quad A_2 = 0.0349, \quad A_3 = 0.0108 \\ \phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3 = 0 \end{array}\right\} \qquad (9)$$

Thus the free vibration response of the system is given by Eq. (6.96):

$$x_1(t) = 0.0543 \cos\omega_1 t + 0.0349 \cos\omega_2 t + 0.0108 \cos\omega_3 t$$

$$x_2(t) = 1.8019(0.0543) \cos\omega_1 t + 0.4450(0.0349) \cos\omega_2 t$$

$$- 1.2468\,(0.0108)\cos\omega_3 t$$

$$x_3(t) = 2.2470\,(0.0543)\cos\omega_1 t - 0.8020\,(0.0349)\cos\omega_2 t + 0.5544\,(0.0108)\cos\omega_3 t$$

where $\omega_1 = 0.44504\sqrt{10} = 1.4073$ rad/sec, $\omega_2 = 1.2471\sqrt{10} = 3.9437$ rad/sec, and $\omega_3 = 1.8025\sqrt{10} = 5.7000$ rad/sec.

6.84 From the solution of Problem 5.28, the natural frequencies and normal modes are given by

$$\omega_1 = 2, \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)} \tag{1}$$

$$\omega_2 = \sqrt{12}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)} \tag{2}$$

where $X_1^{(1)}$ and $X_1^{(2)}$ are arbitrary constants. By orthogonalizing the normal modes with respect to the mass matrix, we can find the values of $X_1^{(1)}$ and $X_1^{(2)}$:

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1 \Rightarrow \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 1$$

or $\quad X_1^{(1)} = \frac{1}{2}$

$$\vec{X}^{(2)T}[m]\vec{X}^{(2)} = 1 \Rightarrow \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 1$$

or $\quad X_1^{(2)} = \frac{1}{2}$

Thus the modal matrix becomes

$$[X] = \left[\vec{X}^{(1)} \quad \vec{X}^{(2)}\right] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3}$$

Using $\vec{x} = [X]\vec{q}$, the equations of motion can be written as

$$\ddot{\vec{q}} + [\omega^2_{\diagdown}]\vec{q} = \vec{Q} = \vec{0} \tag{4}$$

or

$$\ddot{q}_i + \omega_i^2 q_i = 0; \quad i = 1, 2 \tag{5}$$

Solution of Eqs. (5):

$$q_i(t) = q_{i0}\cos\omega_i t + \frac{\dot{q}_{i0}}{\omega_i}\sin\omega_i t \tag{6}$$

where q_{i0} and $\dot{q}_{i0}$ denote the initial values of q_i

and $\dot{q}_i$, respectively. From given initial conditions, we find:

$$\vec{q}(0) = \begin{Bmatrix} q_{10}(0) \\ q_{20}(0) \end{Bmatrix} = [X]^T [m] \vec{x}(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\dot{\vec{q}}(0) = \begin{Bmatrix} \dot{q}_{10}(0) \\ \dot{q}_{20}(0) \end{Bmatrix} = [X]^T [m] \dot{\vec{x}}(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Thus Eqs. (6) become

$$q_1(t) = \cos 2t + \frac{1}{2} \sin 2t$$

$$q_2(t) = \cos \sqrt{12}\, t - \frac{1}{\sqrt{12}} \sin \sqrt{12}\, t$$

The physical displacements can be found as

$$\vec{x}(t) = [X] \vec{q}(t) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

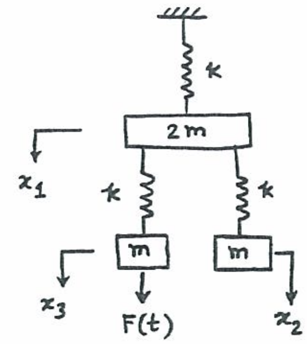
or

$$x_1(t) = \frac{1}{2}(q_1 + q_2), \qquad x_2(t) = \frac{1}{2}(q_1 - q_2)$$

$$\therefore x_1(t) = \frac{1}{2}\left\{\cos 2t + \frac{1}{2}\sin 2t + \cos\sqrt{12}\, t - \frac{1}{\sqrt{12}} \sin\sqrt{12}\, t\right\}$$

$$x_2(t) = \frac{1}{2}\left\{\cos 2t + \frac{1}{2}\sin 2t - \cos\sqrt{12}\, t + \frac{1}{\sqrt{12}} \sin\sqrt{12}\, t\right\}$$

6.89 Equations of motion are



$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F(t) \end{Bmatrix}.$$

where $m = 1$ kg, $k = 1000$ N/m, $F(t) = 5 \sin 10t$ N.

Eigenvalue analysis:

Frequency equation is

$$\left| -\omega^2 m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| = 0$$

or $$\begin{vmatrix} -2\lambda+3 & -1 & -1 \\ -1 & -\lambda+1 & 0 \\ -1 & 0 & -\lambda+1 \end{vmatrix} = 0 \qquad \text{where} \quad \lambda = \frac{\omega^2 m}{k}$$

or $2\lambda^3 - 7\lambda^2 + 6\lambda - 1 = 0$

Roots are

$\lambda_1 = 0.219220$, $\quad \lambda_2 = 1.0$, $\quad \lambda_3 = 2.28078$

$\omega_1 = 0.4682094\sqrt{\frac{k}{m}}$, $\quad \omega_2 = 1.0\sqrt{\frac{k}{m}}$, $\quad \omega_3 = 1.5102251\sqrt{\frac{k}{m}}$

Mode shapes are given by

$$\begin{bmatrix} -2\lambda_i+3 & -1 & -1 \\ -1 & -\lambda_i+1 & 0 \\ -1 & 0 & -\lambda_i+1 \end{bmatrix} \begin{Bmatrix} X_1^{(i)} \\ X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

or $\quad X_1^{(i)} = (-\lambda_i+1)\,X_2^{(i)}, \quad X_3^{(i)} = (-2\lambda_i+3)\,X_1^{(i)} - X_2^{(i)}$

$$\vec{X}^{(i)} = \begin{Bmatrix} -\lambda_i+1 \\ 1 \\ (-2\lambda_i+3)(-\lambda_i+1) - 1 \end{Bmatrix} X_2^{(i)}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 0.78078 \\ 1 \\ 1 \end{Bmatrix} X_2^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} X_2^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix} -1.28078 \\ 1 \\ 1 \end{Bmatrix} X_2^{(3)}$$

Normalization of mode shapes with respect to $[m]$:

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = (0.78078 \quad 1 \quad 1) \begin{Bmatrix} 1.56156 \\ 1 \\ 1 \end{Bmatrix} \left(X_2^{(1)}\right)^2 = 3.21923\left(X_2^{(1)}\right)^2 = 1$$

$$X_2^{(1)} = 0.55734 \; ; \qquad \vec{X}^{(1)} = \begin{Bmatrix} 0.43516 \\ 0.55734 \\ 0.55734 \end{Bmatrix}$$

$$\vec{X}^{(2)T}[m]\,\vec{X}^{(2)} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \left(X_2^{(2)}\right)^2 = 2\left(X_2^{(2)}\right)^2 = 1$$

$$X_2^{(2)} = 0.70711\ ; \qquad \vec{X}^{(2)} = \begin{Bmatrix} 0 \\ 0.70711 \\ -0.70711 \end{Bmatrix}$$

$$\vec{X}^{(3)T}[m]\,\vec{X}^{(3)} = \begin{pmatrix} -1.28078 & 1 & 1 \end{pmatrix} \begin{Bmatrix} -2.56156 \\ 1 \\ 1 \end{Bmatrix} \left(X_2^{(3)}\right)^2 = 5.28079\left(X_2^{(3)}\right)^2 = 1$$

$$X_2^{(3)} = 0.43516\ ; \qquad \vec{X}^{(3)} = \begin{Bmatrix} -0.55734 \\ 0.43516 \\ 0.43516 \end{Bmatrix}$$

$\vec{Q} = [X]^T\,\vec{F}(t) =$ vector of generalized forces

$$= \begin{bmatrix} 0.43516 & 0.55734 & 0.55734 \\ 0 & 0.70711 & -0.70711 \\ -0.55734 & 0.43516 & 0.43516 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ F_0 \sin\omega t \end{Bmatrix} = \begin{Bmatrix} 2.7867 \\ -3.5356 \\ 2.1758 \end{Bmatrix} \sin 10t$$

Uncoupled equations of motion are

$$\left.\begin{aligned} \ddot{q}_1 + 219.22\, q_1 &= 2.7867 \sin 10t \\ \ddot{q}_2 + 1000.00\, q_2 &= -3.5356 \sin 10t \\ \ddot{q}_3 + 2280.78\, q_3 &= 2.1758 \sin 10t \end{aligned}\right\} \qquad (E.1)$$

Particular solutions of (E.1) are

$$q_1(t) = \left(\frac{2.7867}{219.22 - 100}\right) \sin 10t = 0.0233744 \sin 10t$$

$$q_2(t) = \left(\frac{-3.5356}{1000 - 100}\right) \sin 10t = -0.0039284 \sin 10t$$

$$q_3(t) = \left(\frac{2.1758}{2280.78 - 100}\right) \sin 10t = 0.0009977 \sin 10t$$

Since $\vec{x} = [X]\,\vec{q} = \begin{bmatrix} 0.43516 & 0 & -0.55734 \\ 0.55734 & 0.70711 & 0.43516 \\ 0.55734 & -0.70711 & 0.43516 \end{bmatrix} \vec{q}$, we get

$$x_1(t) = 0.0096155 \sin 10t \quad m$$

$$x_2(t) = 0.0095333 \sin 10t \quad m$$

$$x_3(t) = 0.0162395 \sin 10t \quad m.$$

6.90 (a) From Example 6.8, for $k_{t1} = k_{t2} = k_{t3} = k_t$ and $J_1 = J_2 = J_3 = J_0$,

$$[m] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [k] = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Eigenvalue problem becomes

$$\begin{bmatrix} -\lambda + 2 & -1 & 0 \\ -1 & -\lambda + 2 & -1 \\ 0 & -1 & -\lambda + 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{where } \lambda = \frac{\omega^2 J_0}{k_t}. \qquad \text{--- (E.1)}$$

Frequency equation is

$$\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$$

Roots are

$$\lambda_1 = 0.198, \quad \lambda_2 = 1.555, \quad \lambda_3 = 3.247$$

$$\omega_1 = 0.44497\sqrt{\frac{k_t}{J_0}}, \quad \omega_2 = 1.24700\sqrt{\frac{k_t}{J_0}}, \quad \omega_3 = 1.80194\sqrt{\frac{k_t}{J_0}}$$

(E.1) gives $\quad \Theta_2^{(j)} = (-\lambda_j + 2)\,\Theta_1^{(j)}, \quad \Theta_3^{(j)} = \left(\frac{-\lambda_j + 2}{-\lambda_j + 1}\right)\Theta_1^{(j)}$

$$\vec{\Theta}^{(j)} = \begin{Bmatrix} 1 \\ -\lambda_j + 2 \\ \left(\dfrac{-\lambda_j + 2}{-\lambda_j + 1}\right) \end{Bmatrix} \Theta_1^{(j)}$$

$$\vec{\Theta}^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix}\Theta_1^{(1)}, \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix}\Theta_1^{(2)}, \quad \vec{\Theta}^{(3)} = \begin{Bmatrix} 1 \\ -1.247 \\ 0.555 \end{Bmatrix}\Theta_1^{(3)}$$

Normalization:

$$\vec{\Theta}^{(1)T}[m]\,\vec{\Theta}^{(1)} = (1 \quad 1.802 \quad 2.247)\begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix} J_0 \left(\Theta_1^{(1)}\right)^2 = 9.2962\left(\Theta_1^{(1)}\right)^2 J_0 = 1$$

$$\Theta_1^{(1)} = 0.328/\sqrt{J_0}$$

$$\vec{\Theta}^{(2)T}[m]\,\vec{\Theta}^{(2)} = (1 \quad 0.445 \quad -0.802)\begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix} J_0 \left(\Theta_1^{(2)}\right)^2 = 1.8412\left(\Theta_1^{(2)}\right)^2 J_0 = 1$$

$$\Theta_1^{(2)} = 0.737/\sqrt{J_0}$$

$$\vec{\Theta}^{(3)T}[m]\,\vec{\Theta}^{(3)} = (1 \quad -1.247 \quad 0.555)\begin{Bmatrix} 1 \\ -1.247 \\ 0.555 \end{Bmatrix} J_0 \left(\Theta_1^{(3)}\right)^2 = 2.863\left(\Theta_1^{(3)}\right)^2 J_0 = 1$$

$$\Theta_1^{(3)} = 0.591/\sqrt{J_0}$$

$[m]$ - orthonormal modal matrix is

$$[X] = \frac{1}{\sqrt{J_0}}\begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix}$$

For given data,

$$\omega_1 = 4.4497 \text{ rad/s}, \quad \omega_2 = 12.4700 \text{ rad/s}, \quad \omega_3 = 18.0194 \text{ rad/s}$$

$$[X] = \begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix}$$

(b) According to modal analysis, uncoupled equations of motion are

$$\ddot{q}_i(t) + \omega_i^2\, q_i(t) = Q_i(t) \; ; \quad i = 1, 2, 3 \tag{E.2}$$

where $\vec{Q}(t) = [X]^T \vec{M}_t(t) = [X]^T \begin{Bmatrix} 0 \\ 0 \\ M_{t0} \cos \omega t \end{Bmatrix}$

$$= \begin{Bmatrix} Q_{10} \\ Q_{20} \\ Q_{30} \end{Bmatrix} \cos 100t \equiv \begin{Bmatrix} 368.5 \\ -295.5 \\ 164.0 \end{Bmatrix} \cos 100t$$

Steady state solution of (E.2) is $q_i(t) = \left(\frac{Q_{i0}}{\omega_i^2 - \omega^2} \right) \cos \omega t$

$$\therefore \quad q_1(t) = -0.03692 \cos 100t$$
$$q_2(t) = 0.03002 \cos 100t$$
$$q_3(t) = -0.01695 \cos 100t$$

Angular deflections are

$$\vec{\theta}(t) = [X]\vec{q}(t) = \begin{Bmatrix} -0.0000025 \\ 0.0005190 \\ -0.0505115 \end{Bmatrix} \cos 100t \quad \text{radian}$$

(6.91) From problem 6.56, $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} X_1^{(2)}$, $\vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} X_1^{(3)}$

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Normalization gives

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = (1 \quad \sqrt{2} \quad 1) \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} \left(X_1^{(1)}\right)^2 m = 1 \Rightarrow X_1^{(1)} = \frac{1}{2\sqrt{m}}$$

$$\vec{X}^{(2)T}[m]\vec{X}^{(2)} = (1 \quad 0 \quad -1) \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \left(X_1^{(2)}\right)^2 m = 1 \Rightarrow X_1^{(2)} = \frac{1}{\sqrt{2m}}$$

$$\vec{X}^{(3)T}[m]\vec{X}^{(3)} = (1 \quad -\sqrt{2} \quad 1) \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} \left(X_1^{(3)}\right)^2 m = 1 \Rightarrow X_1^{(3)} = \frac{1}{2\sqrt{m}}$$

Modal matrix is

$$[X] = \frac{1}{\sqrt{m}} \begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$$

original coupled equations are $[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0}$

uncoupled equations with $\vec{x} = [X]\vec{q}$ are

$$\ddot{\vec{q}} + [\omega^2]\vec{q} = \vec{0}$$

or
$$\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

with $\omega_1^2 = 0.585786 \frac{k}{m}$, $\omega_2^2 = 2 \frac{k}{m}$, $\omega_3^2 = 3.414214 \frac{k}{m}$.

6.92 $[k]^{-1}\ \vec{F}(t) = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} F_0 \cos\omega t = \frac{1}{k}\begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix} F_0 \cos \omega t$

From Example 6.19,

$$q_{10} = \frac{Q_{10}}{\omega_1^2} \frac{1}{\left|1 - \left(\frac{\omega}{\omega_1}\right)^2\right|} = \frac{1.6561 \frac{F_0}{\sqrt{m}}}{\left(0.19806\ \frac{k}{m}\right)} * \frac{1}{\left|1 - \left(\frac{1.75}{0.44504}\right)^2\right|}$$

$$= 0.57816\ F_0 \sqrt{m}/k$$

and $q_1(t) = q_{10} \cos \omega t$; $\ddot{q}_1 = - q_{10}\ \omega^2 \cos \omega t$

$$-\frac{1}{\omega_1^2}\ \vec{X}^{(1)}\ \ddot{q}_1(t) = q_{10}\left(\frac{\omega^2}{\omega_1^2}\right)\frac{1}{\sqrt{m}}\begin{Bmatrix} 0.3280 \\ 0.5911 \\ 0.7370 \end{Bmatrix} \cos \omega t$$

$$= 8.93979\ \frac{F_0}{k}\begin{Bmatrix} 0.3280 \\ 0.5911 \\ 0.7370 \end{Bmatrix} \cos\omega t$$

Eq. (E.3) in problem statement gives

$$\vec{x}(t) = \begin{Bmatrix} 3 \\ 5 \\ 6 \end{Bmatrix}\frac{F_0}{k} \cos \omega t + \begin{Bmatrix} 2.93225 \\ 5.28431 \\ 6.58863 \end{Bmatrix}\frac{F_0}{k} \cos\omega t$$

$$= \begin{Bmatrix} 5.93225 \\ 10.28431 \\ 12.58863 \end{Bmatrix}\frac{F_0}{k} \cos \omega t$$

6.93 Equations of motion for free vibration

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\ddot{\vec{x}} + k\begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}\vec{x} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

From problem 6.51,

$\omega_1 = 0.337627$, $\omega_2 = 1.414221$, $\omega_3 = 2.094278$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.443004 \\ 1.628659 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ 0.49999 \\ -0.49998 \end{Bmatrix} X_1^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ -0.693 \\ 0.204666 \end{Bmatrix} X_1^{(3)}$$

Free vibratory motion is given by (see section 5.3)

$$\vec{x}(t) = \vec{x}^{(1)}(t) + \vec{x}^{(2)}(t) + \vec{x}^{(3)}(t)$$

$$= \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)}\cos(\omega_2 t + \phi_2) + X_1^{(3)} \cos(\omega_3 t + \phi_3) \\ 1.443\ X_1^{(1)} \cos(\omega_1 t + \phi_1) + 0.5\ X_1^{(2)} \cos(\omega_2 t + \phi_2) - 0.693\ X_1^{(3)} \cos(\omega_3 t + \phi_3) \\ 1.6287\ X_1^{(1)} \cos(\omega_1 t + \phi_1) - 0.5\ X_1^{(2)}\cos(\omega_2 t + \phi_2) + 0.2047\ X_1^{(3)} \cos(\omega_3 t + \phi_3) \end{Bmatrix} \quad \text{--- (E.1)}$$

Known initial conditions give

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.443 & 0.5 & -0.693 \\ 1.6287 & -0.5 & 0.2047 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \cos\phi_1 \\ X_1^{(2)} \cos\phi_2 \\ X_1^{(3)} \cos\phi_3 \end{Bmatrix} \quad \text{--- (E.2)}$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} = \begin{bmatrix} -\omega_1 & -\omega_2 & -\omega_3 \\ -1.443\,\omega_1 & -0.5\,\omega_2 & 0.693\,\omega_3 \\ -1.6287\,\omega_1 & 0.5\,\omega_2 & -0.2047\,\omega_3 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \sin\phi_1 \\ X_1^{(2)} \sin\phi_2 \\ X_1^{(3)} \sin\phi_3 \end{Bmatrix} \quad \text{--- (E.3)}$$

Solution of (E.2) and (E.3) gives

$X_1^{(1)} \cos\phi_1 = 0.8884$, $X_1^{(1)} \sin\phi_1 = 0.4514$

$X_1^{(2)} \cos\phi_2 = 0.6667$, $X_1^{(2)} \sin\phi_2 = -0.7857$

$X_1^{(3)} \cos\phi_3 = -0.5551$, $X_1^{(3)} \sin\phi_3 = 0.4578$

Hence $X_1^{(1)} = 0.9965$, $X_1^{(2)} = 1.0304$, $X_1^{(3)} = 0.7195$

$\phi_1 = 26.9353^\circ$, $\phi_2 = -49.6840^\circ$, $\phi_3 = 140.4870^\circ$

and the final solution is given by (E.1).

6.94

$$\vec{x}(t) = [X]\,\vec{q}(t) \quad ; \quad \dot{\vec{x}}(t) = [X]\,\dot{\vec{q}}(t) \tag{1}$$

If $\vec{x}(0)$ and $\dot{\vec{x}}(0)$ are the known initial conditions in terms of physical coordinates, to find the corresponding values of $\vec{q}(0)$ and $\dot{\vec{q}}(0)$, we premultiply both sides of Eq. (1) by $[X]^T\,[m]$ to obtain at t = 0:

$$[X]^T\,[m]\,\vec{x}(0) = [X]^T\,[m]\,[X]\,\vec{q}(0) \tag{2}$$

$$[X]^T\,[m]\,\dot{\vec{x}}(0) = [X]^T\,[m]\,[X]\,\dot{\vec{q}}(0) \tag{3}$$

Since the normal modes are normalized with respect to the mass matrix as

$$[X]^T\,[m]\,[X] = [I] \tag{4}$$

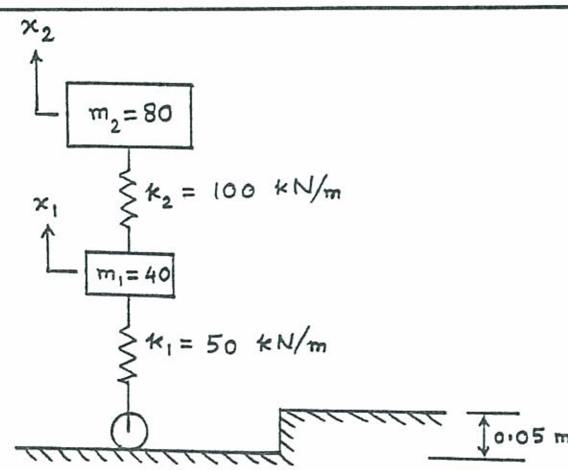
Eqs. (2) and (3) reduce to:

$$\vec{q}(0) = [X]^T\,[m]\,\vec{x}(0) \quad ; \quad \dot{\vec{q}}(0) = [X]^T\,[m]\,\dot{\vec{x}}(0) \tag{5}$$

6.95

$$[k] = \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix} (10^3)\ \text{N/m}$$

$$[m] = \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix}\ \text{kg}$$



Natural frequencies are given by (see Eq. (3) in the solution of Problem 5.5):

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2\, m_1} + \frac{k_2}{2\, m_2} \pm \left\{ \frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1\, k_2}{m_1\, m_2} \right\}^{\frac{1}{2}}$$

$$= \left[\frac{150}{80} + \frac{100}{160} \pm \left\{ \frac{1}{4} \left(\frac{150}{40} + \frac{100}{80} \right)^2 - \frac{(100)\,(50)}{(40)\,(80)} \right\}^{\frac{1}{2}} \right] (10^3)$$

$$= 334.9365,\ 4665.1$$

$$\omega_1 = 18.3013 \text{ rad/sec} \ ; \ \omega_2 = 68.3015 \text{ rad/sec}$$

Mode shapes are defined by Eqs. (4) and (5) in the solution of Problem 5.1:

$$\frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-\, m_2\, \omega_1^2 + k_2} = \frac{100\,(10^3)}{-\,(80)\,(334.9365) + (100)\,(10^3)} = 1.3660$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix}$$

where a is a constant.

$$\frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-\, m_2\, \omega_2^2 + k_2} = \frac{(100)\,(10^3)}{-\,(80)\,(4665.1) + (100)\,(10^3)} = -\,0.3660$$

$$\vec{X}^{(2)} = b \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}$$

where b is a constant.

Orthogonalization of modes:

$$\vec{X}^{(1)T}\, [m]\, \vec{X}^{(1)} = a^2\, (1.0 \quad 1.366) \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix} = 189.2765\ a^2 = 1$$

$$a = 0.07269$$

$$\vec{X}^{(2)T}\, [m]\, \vec{X}^{(2)} = b^2\, (1.0 \quad -0.366) \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix} = 50.7165\ b^2 = 1$$

$$b = 0.14042$$

Modal matrix:

$$[X] = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix}$$

Due to the elevation of 0.05 m, spring k_1 and hence m_1 will be subjected to additional compression of k_1 (0.05) = 2500 N.

$$\vec{F}(t) = \begin{Bmatrix} 2500 \\ 0 \end{Bmatrix} \text{ N}$$

Equation (6.111) gives:

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 181.725 \\ 351.05 \end{Bmatrix}$$

Solution is given by (without initial conditions) Eq. (6.114):

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t-\tau)\, d\tau \;\; ; \;\; i = 1, 2$$

$$\text{Since} \int_{\tau=0}^{t} \sin \Omega (t-\tau)\, d\tau = - \int_{\tau'=t-\tau=t}^{\tau'=t-\tau=0} \sin \Omega \tau' \, d\tau' = \frac{1}{\Omega} (1 - \cos \Omega t)$$

we find

$$q_1(t) = \frac{181.725}{(18.3013^2)} (1 - \cos 18.3013\, t) = 0.5426\, (1 - \cos 18.3013\, t)$$

$$q_2(t) = \frac{351.05}{(68.3015^2)} (1 - \cos 68.3015\, t) = 0.07525\, (1 - \cos 68.3015\, t)$$

Response of the masses can be found from Eq. (6.104):

$$\vec{x}(t) = [X]\, \vec{q}(t) = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix} \begin{Bmatrix} 0.5426\, (1 - \cos 18.3013\, t) \\ 0.07525\, (1 - \cos 68.3015\, t) \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.03944\, (1 - \cos 18.3013\, t) + 0.01057\, (1 - \cos 68.3015\, t) \\ 0.05387\, (1 - \cos 18.3013\, t - 0.00387\, (1 - \cos 68.3015\, t) \end{Bmatrix}$$

Note:

This problem can also be solved by specifying the initial conditions as

$$\vec{x}(0) = \begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix} \text{ m} \quad ; \quad \dot{\vec{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and solving the free vibration problem.

6.96 $\ell_i = 0.5$ m ; $m_i = 1$ kg ($i = 1, 2, 3$). Assume $m_1 = m_2 = m_3 = m = 1$; $\ell_1 = \ell_2 = \ell_3 = \ell = 0.5$.

From solution of Problem 6.42, we obtain:

$$[m] = m\, \ell^2 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0.25 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[k] = m\,g\,\ell \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4.905 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From solution of Problem 6.58, the natural frequencies and mode shapes can be found as

$$\omega_1 = 0.644798 \sqrt{\frac{g}{\ell}} \;;\; \omega_2 = 1.514698 \sqrt{\frac{g}{\ell}} \;;\; \omega_3 = 2.507977 \sqrt{\frac{g}{\ell}}$$

Since $\sqrt{\frac{g}{\ell}} = \sqrt{\frac{9.81}{0.5}} = 4.4294$, we find

$$\omega_1 = 2.8561 \text{ rad/sec} \;;\; \omega_2 = 6.7092 \text{ rad/sec} \;;\; \omega_3 = 11.1088 \text{ rad/sec}$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix} \;;\; \vec{X}^{(2)} = b \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix} \;;\; \vec{X}^{(3)} = c \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

Orthonormalization of mode shapes:

$$\vec{X}^{(1)^T} [m] \vec{X}^{(1)} = 5.4118\, a^2 = 1 \quad \text{or} \quad a = 0.4299$$
$$\vec{X}^{(2)^T} [m] \vec{X}^{(2)} = 0.9805\, b^2 = 1 \quad \text{or} \quad b = 1.0099$$
$$\vec{X}^{(3)^T} [m] \vec{X}^{(3)} = 0.3577\, c^2 = 1 \quad \text{or} \quad c = 1.6720$$

Modal matrix:

$$[X] = \begin{bmatrix} \vec{X}^{(1)} & \vec{X}^{(2)} & \vec{X}^{(3)} \end{bmatrix} = \begin{bmatrix} 0.4299 & 1.0099 & 1.6720 \\ 0.5555 & 0.3562 & -2.7504 \\ 0.7012 & -2.4215 & 1.2822 \end{bmatrix} \tag{1}$$

$$\vec{F}(t) = \begin{Bmatrix} 0 \\ 0 \\ M_{t3}(t) \end{Bmatrix} \tag{2}$$

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 0.7012\, M_{t3}(t) \\ -\,2.4215\, M_{t3}(t) \\ 1.2822\, M_{t3}(t) \end{Bmatrix} \tag{3}$$

Solution of $q_i(t)$ without initial conditions:

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i\,(t-\tau)\, d\tau \;;\; i = 1,\, 2,\, 3 \tag{4}$$

We denote $M_{t3}(t)$ as

$$M_{t3}(t) = M_0 \left(u(t) - u(t - t_0) \right) \tag{5}$$

where $M_0 = 0.1$ N-m, $t_0 = 0.1$ sec and $u(t)$ and $u(t - t_0)$ are the unit step functions:

$$u(t) = \begin{cases} 0 & ; \ t < 0 \\ 1 & ; \ t > 0 \end{cases}$$

$$u(t - t_0) = \begin{cases} 0 & ; \ t < t_0 \\ 1 & ; \ t > t_0 \end{cases}$$

Thus we obtain

$$q_1(t) = \frac{1}{\omega_1} \int_0^t Q_1(\tau) \sin \omega_1 \, (t - \tau) \, d\tau$$

$$= \frac{1}{2.8561} \int_0^t 0.7012 \, (0.1) \left\{ u(\tau) - u(\tau - 0.1) \right\} \sin 2.8561 \, (t - \tau) \, d\tau$$

$$= 0.02455 \left\{ \int_0^t u(\tau) \sin 2.8561 \, (t - \tau) \, d\tau - \int_0^t u(\tau - 0.1) \sin 2.8561 \, (t - \tau) \, d\tau \right\}$$

By noting that

$$\int_0^t u(\tau) \sin \Omega \, (t - \tau) \, d\tau = \frac{1}{\Omega} \, (1 - \cos \Omega \, t) \tag{6}$$

and $$\int_0^t u(\tau - t_0) \sin \Omega \, (t - \tau) \, d\tau = \frac{1}{\Omega} \left[1 - \cos \Omega \, (t - t_0) \right] \tag{7}$$

we can derive

$$\begin{aligned} q_1(t) &= 0.008596 \left[1 - \cos 2.8561 \, t \right] \quad ; \ 0 < t < 0.1 \text{ sec} \\ &= 0.008596 \left[\cos 2.8561 \, (t - 0.1) - \cos 2.8561 \, t \right] \quad ; \ t \geq 0.1 \text{ sec} \end{aligned} \tag{8}$$

Similarly, we can derive

$$\begin{aligned} q_2(t) &= -\,0.005379 \left[1 - \cos 6.7092 \, t \right] \quad ; \ 0 < t < 0.1 \text{ sec} \\ &= -\,0.005379 \left[\cos 6.7092 \, (t - 0.1) - \cos 6.7092 \, t \right] \quad ; \ t \geq 0.1 \text{ sec} \end{aligned} \tag{9}$$

$$\begin{aligned} q_3(t) &= 0.001039 \left[1 - \cos 11.1088 \, t \right] \quad ; \ 0 < t < 0.1 \text{ sec} \\ &= 0.001039 \left[\cos 11.1088 \, (t - 0.1) - \cos 11.1088 \, t \right] \quad : \ t \geq 0.1 \text{ sec} \end{aligned} \tag{10}$$

Thus the angular (physical) displacements of the pendulum can be expressed as:

$$\vec{x}(t) = [X] \, \vec{q}(t) \tag{11}$$

where $[X]$ is given by Eq. (1) and $\vec{q}(t)$ is given by Eqs. (8) to (10). Hence

$$\begin{aligned} x_1(t) &= 0.003675 \, (1 - \cos 2.8561 \, t) - 0.005432 \, (1 - \cos 6.7092 \, t) \\ &\quad + 0.001737 \, (1 - \cos 11.1088 \, t) \quad 0 < t < 0.1 \text{ sec} \\ &= 0.003695 \left[\cos 2.8561 \, (t - 0.1) - \cos 2.8561 \, t \right] \\ &\quad - 0.005432 \left[\cos 6.7092 \, (t - 0.1) - \cos 6.7092 \, t \right] \\ &\quad + 0.001737 \left[\cos 11.1088 \, (t - 0.1) - \cos 11.1088 \, t \right] \quad t \geq 0.1 \text{ sec} \end{aligned}$$

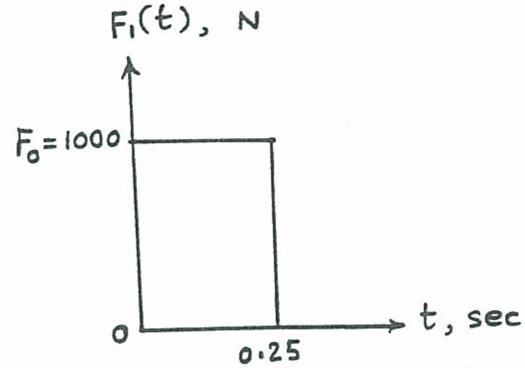
$$x_2(t) = 0.004775\,(1 - \cos 2.8561\, t) - 0.001916\,(1 - \cos 6.7092\, t) - 0.002858\,(1 - \cos 11.1088\, t) \quad 0 < t < 0.1 \text{ sec}$$

$$= 0.004775\left[\cos 2.8561\,(t - 0.1) - \cos 2.8561\, t\right] - 0.001916\left[\cos 6.7092\,(t - 0.1) - \cos 6.7092\, t\right] - 0.002858\left[\cos 11.1088\,(t - 0.1) - \cos 11.1088\, t\right] \quad t \geq 0.1 \text{ sec}$$

$$x_3(t) = 0.006027\,(1 - \cos 2.8561\, t) + 0.013025\,(1 - \cos 6.7092\, t) + 0.001332\,(1 - \cos 11.1088\, t) \quad 0 < t < 0.1 \text{ sec}$$

$$= 0.006027\left[\cos 2.8561\,(t - 0.1) - \cos 2.8561\, t\right] + 0.013025\left[\cos 6.7092\,(t - 0.1) - \cos 6.7092\, t\right] + 0.001332\left[\cos 11.1088\,(t - 0.1) - \cos 11.1088\, t\right] \quad t \geq 0.1 \text{ sec}$$

6.97 $m = 2$ kg, $k = 10{,}000$ N/m,



$$\sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{2}} = 70.7107.$$

$$[m] = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$[k] = 10^4\begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

From the solution of Problem 6.51, we find the natural frequencies and mode shapes as:

$$\omega_1 = 0.337627\sqrt{\frac{k}{m}} \;;\; \omega_2 = 1.414221\sqrt{\frac{k}{m}} \;;\; \omega_3 = 2.094278\sqrt{\frac{k}{m}}$$

or $\omega_1 = 23.8738$ rad/sec ; $\omega_2 = 100.0006$ rad/sec ; $\omega_3 = 148.0879$ rad/sec

$$\vec{X}^{(1)} = a\begin{Bmatrix} 1.0 \\ 1.4430 \\ 1.6286 \end{Bmatrix} \;;\; \vec{X}^{(2)} = b\begin{Bmatrix} 1.0 \\ 0.5 \\ -0.5 \end{Bmatrix} \;;\; \vec{X}^{(3)} = c\begin{Bmatrix} 1.0 \\ -0.6930 \\ 0.2047 \end{Bmatrix}$$

Orthonormalization of modes:

$$\vec{X}^{(1)T}\,[m]\,\vec{X}^{(1)} = 26.2430\, a^2 = 1 \quad \text{or} \quad a = 0.1952$$

$$\vec{X}^{(2)T}\,[m]\,\vec{X}^{(2)} = 4.5\, b^2 = 1 \quad \text{or} \quad b = 0.4714$$

$$\vec{X}^{(3)T}\,[m]\,\vec{X}^{(3)} = 4.1724\, c^2 = 1 \quad \text{or} \quad c = 0.4896$$

Modal matrix:

$$[X] = \left[\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)}\right] = \begin{bmatrix} 0.1952 & 0.4714 & 0.4896 \\ 0.2817 & 0.2357 & -0.3393 \\ 0.3179 & -0.2357 & 0.1002 \end{bmatrix} \quad (1)$$

$$\vec{F}(t) = \begin{Bmatrix} F_1(t) \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 0.1952\ F_1(t) \\ 0.4714\ F_1(t) \\ 0.4896\ F_1(t) \end{Bmatrix} \tag{3}$$

Solution of $q_i(t)$ without initial conditions:

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i\ (t - \tau)\ d\tau \ ;\ i = 1,\ 2,\ 3 \tag{4}$$

$$F_1(t) = F_0 \left\{ u(t) - u(t - t_0) \right\}$$

where $F_0 = 1000$ N, $t_0 = 0.25$ sec, and $u(t)$ and $u(t - t_0)$ are unit step functions:

$$u(t) = \begin{cases} 0 & ;\ t < 0 \\ 1 & ;\ t > t_0 \end{cases}$$

$$u(t - t_0) = \begin{cases} 0 & ;\ t < t_0 \\ 1 & ;\ t > t_0 \end{cases}$$

Equations (3) give:

$$\vec{Q}(t) = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{Bmatrix} = \begin{Bmatrix} 195.2 \left\{ u(t) - u(t - 0.25) \right\} \\ 471.4 \left\{ u(t) - u(t - 0.25) \right\} \\ 489.6 \left\{ u(t) - u(t - 0.25) \right\} \end{Bmatrix} \tag{5}$$

and Eqs. (4) yield:

$$\begin{aligned} q_1(t) &= 0.3425\ (1 - \cos 23.8738\ t) \ ;\ 0 < t < 0.25 \text{ sec} \\ &= 0.3425 \left[\cos 23.8738\ (t - 0.25) - \cos 23.8738\ t \right] \ ;\ t \geq 0.25 \text{ sec} \end{aligned}$$

$$\begin{aligned} q_2(t) &= 0.04714\ (1 - \cos 100.0006\ t) \ ;\ 0 < t < 0.25 \text{ sec} \\ &= 0.04714 \left[\cos 100.0006\ (t - 0.25) - \cos 100.0006\ t \right] \quad t \geq 0.25 \text{ sec} \end{aligned}$$

$$\begin{aligned} q_3(t) &= 0.02232\ (1 - \cos 148.0879\ t) \ ;\ 0 < t < 0.25 \text{ sec} \\ &= 0.02232 \left[\cos 148.0879\ (t - 0.25) - \cos 148.0879\ t \right] \ ;\ t \geq 0.25 \text{ sec} \end{aligned}$$

The physical displacements of the masses are given by:

$$\vec{x}(t) = [X]\,\vec{q}(t)$$

which can be explicitly expressed as:

$$\begin{aligned}
x_1(t) &= 0.06686\,(1 - \cos 23.8738\,t) + 0.02222\,(1 - \cos 100.0006\,t) \\
&\quad + 0.01093\,(1 - \cos 148.0879\,t)\ ;\ 0 < t < 0.25 \text{ sec} \\
&= 0.06686 \left[\cos 23.8738\,(t - 0.25) - \cos 23.8738\,t\right] \\
&\quad + 0.02222 \left[\cos 100.0006\,(t - 0.25) - \cos 100.0006\,t\right] \\
&\quad + 0.01093 \left[\cos 148.0879\,(t - 0.25) - \cos 148.0879\,t\right]\ ;\ t \geq 0.25 \text{ sec} \\
x_2(t) &= 0.09648\,(1 - \cos 23.8738\,t) + 0.01111\,(1 - \cos 100.0006\,t) \\
&\quad - 0.007573\,(1 - \cos 148.0879\,t)\ ;\ 0 < t < 0.25 \text{ sec} \\
&= 0.09648 \left[\cos 23.8738\,(t - 0.25) - \cos 23.8738\,t\right] \\
&\quad + 0.01111 \left[\cos 100.0006\,(t - 0.25) - \cos 100.0006\,t\right] \\
&\quad - 0.007573 \left[\cos 148.0879\,(t - 0.25) - \cos 148.0879\,t\right]\ ;\ t \geq 0.25 \text{ sec} \\
x_3(t) &= 0.1089\,(1 - \cos 23.8738\,t) - 0.01111\,(1 - \cos 100.0006\,t) \\
&\quad + 0.002236\,(1 - \cos 148.0879\,t)\ ;\ 0 < t < 0.25 \text{ sec} \\
&= 0.1089 \left[\cos 23.8738\,(t - 0.25) - \cos 23.8738\,t\right] \\
&\quad - 0.01111 \left[\cos 100.0006\,(t - 0.25) - \cos 100.0006\,t\right] \\
&\quad + 0.002236 \left[\cos 148.0879\,(t - 0.25) - \cos 148.0879\,t\right]\ ;\ t \geq 0.25 \text{ sec}
\end{aligned}$$

6.99 Equations of motion $\quad [m]\ \ddot{\vec{x}} + [c]\ \dot{\vec{x}} + [k]\ \vec{x} = \vec{F} \quad ---(E.1)$

where

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad [c] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix},$$

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}, \quad \vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \vec{F} = \begin{Bmatrix} F_1 = F_0 \cos \omega t \\ F_2 = 0 \\ F_3 = 0 \end{Bmatrix}$$

Since $F_j(t) = \mathrm{Re}\left[F_{j0}\ e^{i\omega t}\right]$ with $F_{10} = F_0$, and $F_{20} = F_{30} = 0$, we assume $x_j(t) = X_j\ e^{i\omega t}$; $j = 1,2,3$. Then (E.1) becomes

$$\left[Z_{rs}(i\omega)\right] \vec{X} = \vec{F_0} \quad ---(E.2)$$

where

$$Z_{11}(i\omega) = -m_1\omega^2 + (c_1+c_2)\, i\omega + (k_1+k_2) = -\omega^2 + 2i\omega + 200$$

$$Z_{12}(i\omega) = Z_{21}(i\omega) = -c_2\, i\omega - k_2 = -i\omega - 100$$

$$Z_{13}(i\omega) = Z_{31}(i\omega) = 0$$

$$Z_{22}(i\omega) = -m_2\omega^2 + (c_2+c_3)\, i\omega + (k_2+k_3) = -\omega^2 + 2i\omega + 200 \quad ---(E.3)$$

$$Z_{23}(i\omega) = Z_{32}(i\omega) = -c_3\, i\omega - k_3 = -i\omega - 100$$

$$Z_{33}(i\omega) = -m_3\omega^2 + (c_3+c_4)\, i\omega + (k_3+k_4) = -\omega^2 + 2i\omega + 200$$

(E.2) becomes

$$\begin{aligned}(2i+199)\,X_1 - (i+100)\,X_2 + (0)\,X_3 &= 10\\ -(i+100)\,X_1 + (2i+199)\,X_2 - (i+100)\,X_3 &= 0 \qquad \text{--- (E.4)}\\ (0)\,X_1 - (i+100)\,X_2 + (2i+199)\,X_3 &= 0\end{aligned}$$

Solution of (E.4) can be expressed as

$$X_j = \frac{\Delta_j}{\Delta} \; ; \quad j = 1, 2, 3 \qquad \text{--- (E.5)}$$

where

$$\Delta_1 = \begin{vmatrix} 10 & -(i+100) & 0 \\ 0 & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 295\,980 + 5960\, i$$

$$\Delta_2 = \begin{vmatrix} (2i+199) & 10 & 0 \\ -(i+100) & 0 & -(i+100) \\ 0 & 0 & (2i+199) \end{vmatrix} = 198\,980 + 3990\, i$$

$$\Delta_3 = \begin{vmatrix} (2i+199) & -(i+100) & 10 \\ -(i+100) & (2i+199) & 0 \\ 0 & -(i+100) & 0 \end{vmatrix} = 99\,990 + 2000\, i$$

$$\Delta = \begin{vmatrix} (2i+199) & -(i+100) & 0 \\ -(i+100) & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 118\,002\, i + 3\,899\,409$$

Using (E.5), we get

$$X_1 = \frac{87639.682}{1154850.392 + 11\,685.754\, i} \; ; \quad \begin{aligned}&\text{Amplitude} = 0.0758845 \text{ m}\\ &\text{Phase angle} = 0.5798^\circ\end{aligned}$$

$$X_2 = \frac{39\,608.961}{776375.228 + 7921.396\, i} \; ; \quad \begin{aligned}&\text{Amplitude} = 0.0510152 \text{ m}\\ &\text{Phase angle} = 0.5845^\circ\end{aligned}$$

$$X_3 = \frac{10\,002.000}{390\,137.914 + 4000.202\, i} \; ; \quad \begin{aligned}&\text{Amplitude} = 0.0256357 \text{ m}\\ &\text{Phase angle} = 0.5874^\circ\end{aligned}$$

Thus the steady state responses are:

$$x_1(t) = 0.0758845 \cos(\omega t + 0.5798^\circ) \text{ m}$$

$$x_2(t) = 0.0510152 \cos(\omega t + 0.5845^\circ) \text{ m}$$

$$x_3(t) = 0.0256357 \cos(\omega t + 0.5874^\circ) \text{ m}$$

6.100 Modal matrix can be expressed as $\underset{12\times 3}{[X]} = [\vec{X}^{(1)} \;\; \vec{X}^{(2)} \;\; \vec{X}^{(3)}]$

Using Eq. (6.122), $\underset{12\times 1}{\vec{x}(t)} = \underset{12\times 3}{[X]}\,\underset{3\times 1}{\vec{q}(t)}$ where $\vec{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$

equations of motion can be written as

$$[m][X]\ddot{\vec{q}} + [c][X]\dot{\vec{q}} + [k][X]\vec{q} = -[m]\vec{u}_1\ \ddot{x}_0(t)$$

Premultiplication by $[X]^T$ gives

$$[X]^T[m][X]\ddot{\vec{q}} + [X]^T[c][X]\dot{\vec{q}} + [X]^T[k][X]\vec{q} = -[X]^T[m]\vec{u}_1\ \ddot{x}_0(t) \quad \text{--- (E.1)}$$

Assuming that the mass matrix is diagonal and the damping matrix is proportional, (E.1) can be expressed in scalar form as

$$m_{ii}\ddot{q}_i + c_{ii}\dot{q}_i + k_{ii}q_i = -\ddot{x}_0(t)\sum_{j=1}^{12} m_j X_j^{(i)}\ ; \quad i = 1,2,3 \quad \text{--- (E.2)}$$

where m_{ii}, c_{ii} and k_{ii} are generalized mass, generalized damping, and generalized stiffness, m_j is the mass at the jth d.o.f. and $X_j^{(i)}$ is the jth component of the vector $\vec{X}^{(i)}$.

Here $$m_{ii} = \sum_{j=1}^{12} m_j\left(X_j^{(i)}\right)^2 = m\sum_{j=1}^{12}\left(X_j^{(i)}\right)^2$$

$$c_{ii} = 2\zeta_i\omega_i \quad \text{and} \quad \frac{k_{ii}}{m_{ii}} = \omega_i^2\ ; \quad i = 1,2,3$$

where m = mass at each d.o.f.

Eq.(E.2) can be written, noting that there is no damping in the system, as

$$\ddot{q}_i + \omega_i^2 q_i = -\frac{\ddot{x}_0(t)\left\{\sum_{j=1}^{12} m_j X_j^{(i)}\right\}}{\left\{\sum_{j=1}^{12} m_j\left(X_j^{(i)}\right)^2\right\}} = -\ddot{x}_0(t)\frac{\sum_{j=1}^{12} X_j^{(i)}}{\sum_{j=1}^{12}\left(X_j^{(i)}\right)^2}\ ; \quad i = 1,2,3 \quad \text{--- (E.3)}$$

By noting that

$$\sum_{j=1}^{12} X_j^{(i)} = 7.964 \text{ for } i=1,\ -2.67 \text{ for } i=2,\ 1.618 \text{ for } i=3,$$

$$\sum_{j=1}^{12}\left(X_j^{(i)}\right)^2 = 6.275658 \text{ for } i=1,\ 6.48612 \text{ for } i=2,\ 6.90962 \text{ for } i=3,$$

Eqs.(E.3) can be reduced to

$$\ddot{q}_1(t) + 50625.0\ q_1(t) = -1.26903\ \ddot{x}_0(t)$$

$$\ddot{q}_2(t) + 435600.0\ q_2(t) = 0.411648\ \ddot{x}_0(t) \quad \text{--- (E.4)}$$

$$q_3(t) + 1210000.0\ q_3(t) = -0.234166\ \ddot{x}_0(t)$$

6.101 Eigenvalue problem:

$$\lambda [m] \vec{X} = [k] \vec{X}$$

with $\lambda = \omega^2$, $[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[k] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Solution using MATLAB:

```
% Ex6_101.m
>> k = [1 -2 1;-2 4 -2; 1 -2 1]

k =

     1    -2     1
    -2     4    -2
     1    -2     1

>> m = [1 0 0; 0 2 0;0 0 1]

m =

     1     0     0
     0     2     0
     0     0     1

>> [V, D] = eig(k, m)

V =

   -0.6439   -0.5792    0.5000
   -0.4876    0.1105   -0.5000
   -0.3314    0.8001    0.5000


D =

   -0.0000         0         0
         0    0.0000         0
         0         0    4.0000
```

6.102

$$x_1(t) = x_{20}\left(0.1527 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\, t + 0.09847 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\, t - 0.2512 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\, t\right) \quad (1)$$

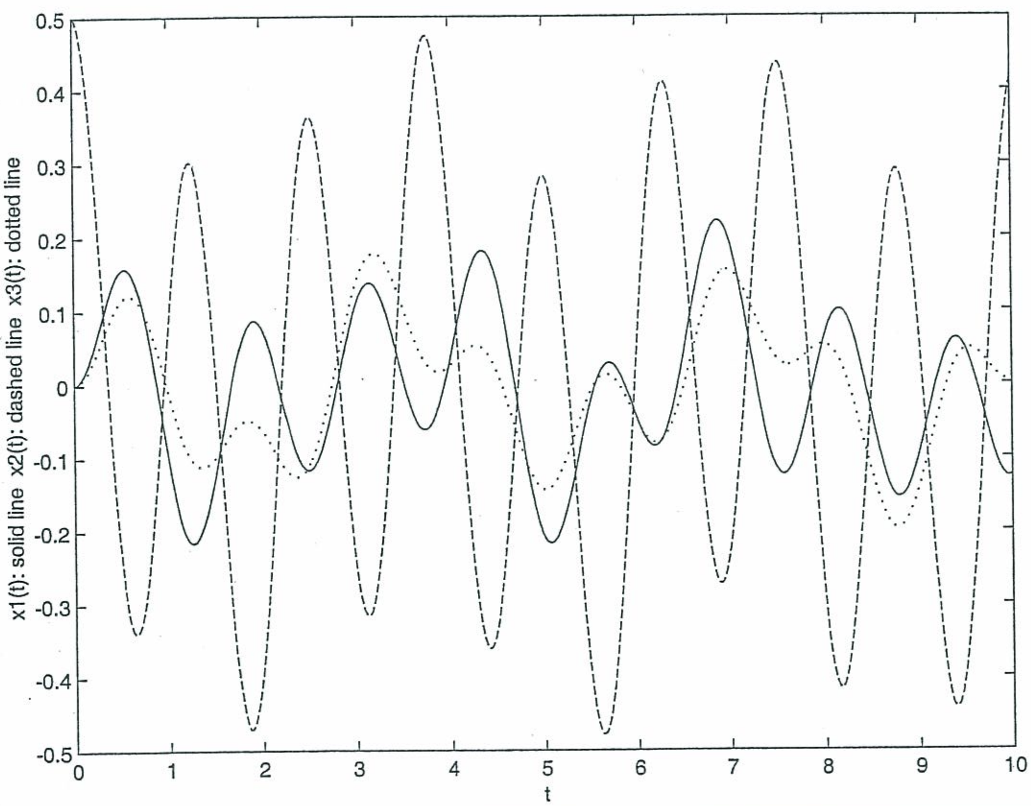
$$x_2(t) = x_{20}\left(0.2087 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\, t + 0.03177 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\, t + 0.7594 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\, t\right) \quad (2)$$

$$x_3(t) = x_{20}\left(0.1987 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\, t - 0.06157 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\, t - 0.1372 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\, t\right) \quad (3)$$

Data: $x_{20} = 0.5$, $P = 100$, $\ell = 5$, $m = 2$

MATLAB Solution of Eqs. (1) - (3):

```
% Ex6_102.m
x20 = 0.5;
p = 100;
l = 5;
m = 2;
c = sqrt(p/(l*m));
for i = 1: 501
    t(i) = 10*(i-1)/500;
    x1(i) = x20 * ( 0.1527*cos(0.5626*c*t(i)) + ...
        0.09847*cos(0.9158*c*t(i)) - 0.2512*cos(1.5848*c*t(i)) );
    x2(i) = x20 * ( 0.2087*cos(0.5626*c*t(i)) + ...
        0.03177*cos(0.9158*c*t(i)) + 0.7594*cos(1.5848*c*t(i)) );
    x3(i) = x20 * ( 0.1987*cos(0.5626*c*t(i)) - ...
        0.06157*cos(0.9158*c*t(i)) - 0.1372*cos(1.5848*c*t(i)) );
end
plot(t,x1);
hold on;
plot(t,x2,'--');
plot(t,x3,':');
xlabel('t');
ylabel('x1(t): solid line  x2(t): dashed line  x3(t): dotted line');
```



6.103 Equations of motion:

$$2m\ddot{x}_1 + 3kx_1 - kx_2 - kx_3 = 0 \qquad (1)$$

$$m\ddot{x}_2 - kx_1 + kx_2 = 0 \qquad (2)$$

$$m\ddot{x}_3 - kx_1 + kx_3 = F(t) = F_0 \sin\omega t \qquad (3)$$

Using the data: $m = 1$, $k = 1000$, $F_0 = 5$, $\omega = 10$,
Eqs. (1) – (3) can be expressed as

$$\ddot{x}_1 = -1500\, x_1 + 500\, x_2 + 500\, x_3 \qquad (4)$$

$$\ddot{x}_2 = 1000\, x_1 - 1000\, x_2 \qquad (5)$$

$$\ddot{x}_3 = 1000\, x_1 - 1000\, x_3 + 5 \sin 10t \qquad (6)$$

Let

$$\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{Bmatrix} \quad \text{and} \quad \vec{Y}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

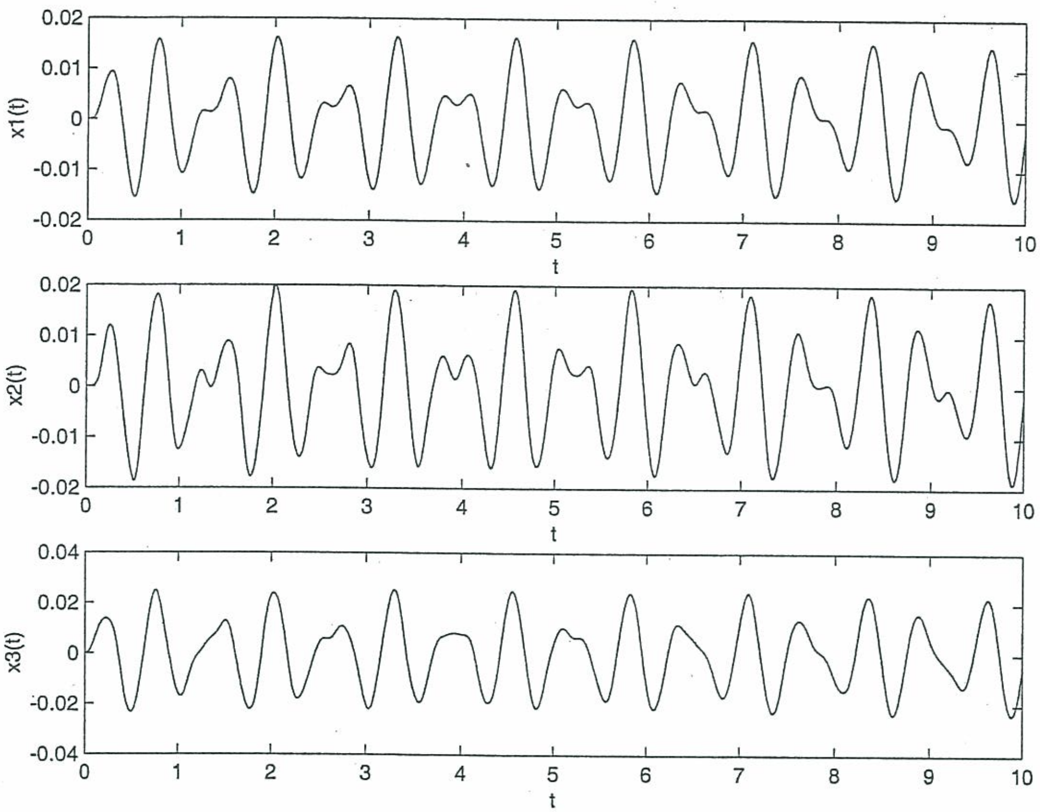
Eqs. (4) – (6) can be rewritten as

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -1500\, y_1 + 500\, y_3 + 500\, y_5 \\ y_4 \\ 1000\, y_1 - 1000\, y_3 \\ y_6 \\ 1000\, y_1 - 1000\, y_5 + 5 \sin 10t \end{Bmatrix} \qquad (7)$$

Solution of Eq. (7) using MATLAB:

```
% Ex6_103.m
% This program will use the function dfunc6_103.m ,they should
% be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.0; 0.0; 0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc6_103', tspan, x0);
subplot(311);
plot(t,x(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(312);
plot(t,x(:,3));
xlabel('t');
ylabel('x2(t)');
subplot(313);
plot(t,x(:,5));
xlabel('t');
ylabel('x3(t)');

% dfunc6_103.m
function f = dfunc6_103(t,x)
F0 = 5;
```

```
w = 10;
m = 1;
k = 1000;
f = zeros(6,1);
f(1) = x(2);
f(2) = -1500*x(1) + 500*x(3) + 500*x(5);
f(3) = x(4);
f(4) = 1000*x(1) - 1000*x(3);
f(5) = x(6);
f(6) = 1000*x(1) - 1000*x(5) + 5*sin(10*t);
```



6.104 Roots of $f(x) = x^{12} - 2 = 0$ using MATLAB:

```
% Ex6_104.m
>> x = roots([1 zeros(1,11) -2])

x =

  -1.0595
  -0.9175 + 0.5297i
  -0.9175 - 0.5297i
  -0.5297 + 0.9175i
  -0.5297 - 0.9175i
   0.0000 + 1.0595i
   0.0000 - 1.0595i
   0.5297 + 0.9175i
   0.5297 - 0.9175i
   1.0595
   0.9175 + 0.5297i
   0.9175 - 0.5297i
```

6.105 Equations of motion can be rewritten as

$$\ddot{x}_1 = -1.5\,\dot{x}_1 + 0.5\,\dot{x}_2 - 14\,x_1 + 6\,x_2 + 0.5\cos 2t \quad (1)$$

$$\ddot{x}_2 = 0.25\,\dot{x}_1 - \dot{x}_2 + 0.75\,\dot{x}_3 + 3x_1 - 5x_2 + 2x_3 \quad (2)$$

$$\ddot{x}_3 = 0.5\,\dot{x}_2 - 0.5\,\dot{x}_3 + 1.3333\,x_2 - 1.3333\,x_3 \quad (3)$$

Let $\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{Bmatrix}$ and $\vec{Y}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

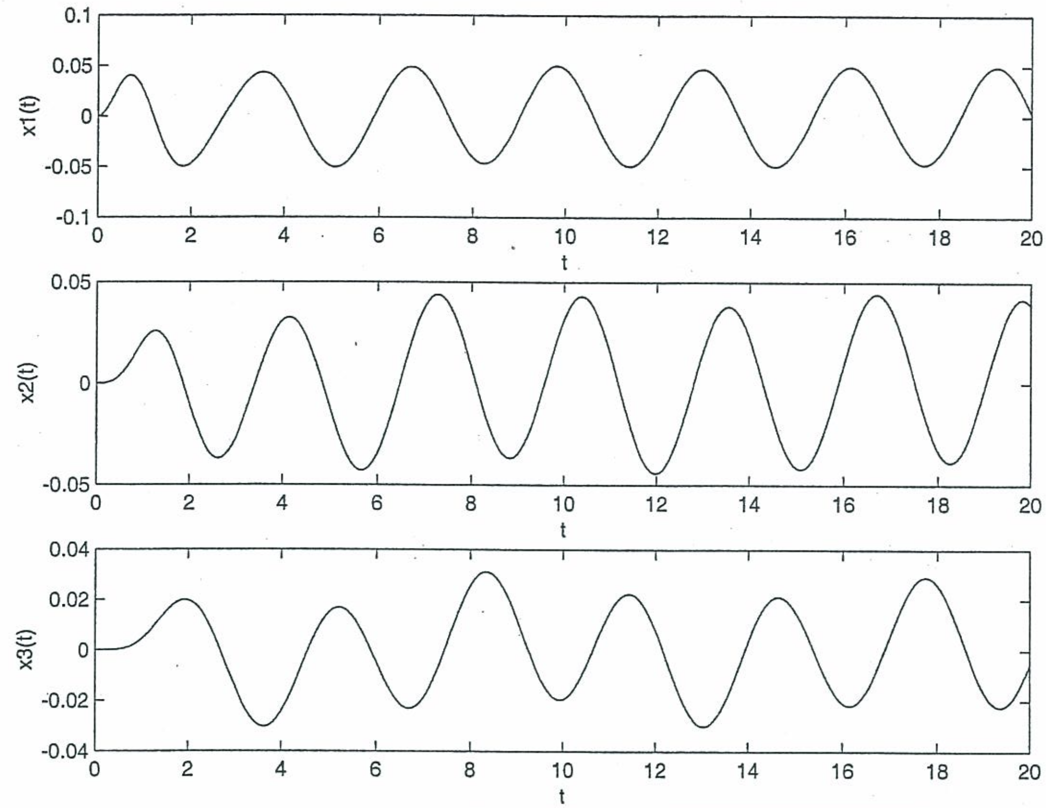
Eqs. (1) – (3) can be expressed as

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -1.5\,y_2 + 0.5\,y_4 - 14\,y_1 + 6\,y_3 + 0.5\cos 2t \\ y_4 \\ 0.25\,y_2 - y_4 + 0.75\,y_6 + 3y_1 - 5y_3 + 2y_5 \\ y_6 \\ 0.5\,y_4 - 0.5\,y_6 + 1.3333\,y_3 - 1.3333\,y_5 \end{Bmatrix} \quad (4)$$

Solution of Eq. (4) using MATLAB:

```
% Ex6_105.m
% This program will use the function dfunc6_105.m, they should
% be in the same folder
tspan = [0: 0.01: 20];
x0 = [0.0; 0.0; 0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc6_105', tspan, x0);
subplot(311);
plot(t,x(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(312);
plot(t,x(:,3));
xlabel('t');
ylabel('x2(t)');
subplot(313);
plot(t,x(:,5));
xlabel('t');
ylabel('x3(t)');
```

```
% dfunc6_105.m
function f = dfunc6_105(t,x)
f = zeros(6,1);
f(1) = x(2);
f(2) = -1.5*x(2) + 0.5*x(4) - 14*x(1) + 6*x(3) + 0.5*cos(2*t);
f(3) = x(4);
f(4) = 0.25*x(2) - x(4) + 0.75*x(6) + 3*x(1) - 5*x(3) + 2*x(5);
f(5) = x(6);
f(6) = 0.5*x(4) - 0.5*x(6) + 1.3333*x(3) - 1.3333*x(5);
```



6.106 Equations of motion:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\vec{x}} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \dot{\vec{x}} + \begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 200 \end{bmatrix} \vec{x} = \begin{Bmatrix} 10 \cos t \\ 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\ddot{x}_1 = -2\dot{x}_1 + \dot{x}_2 - 200 x_1 + 100 x_2 + 10 \cos t \qquad (1)$$

$$\ddot{x}_2 = \dot{x}_1 - 2\dot{x}_2 + \dot{x}_3 + 100 x_1 - 200 x_2 + 100 x_3 \qquad (2)$$

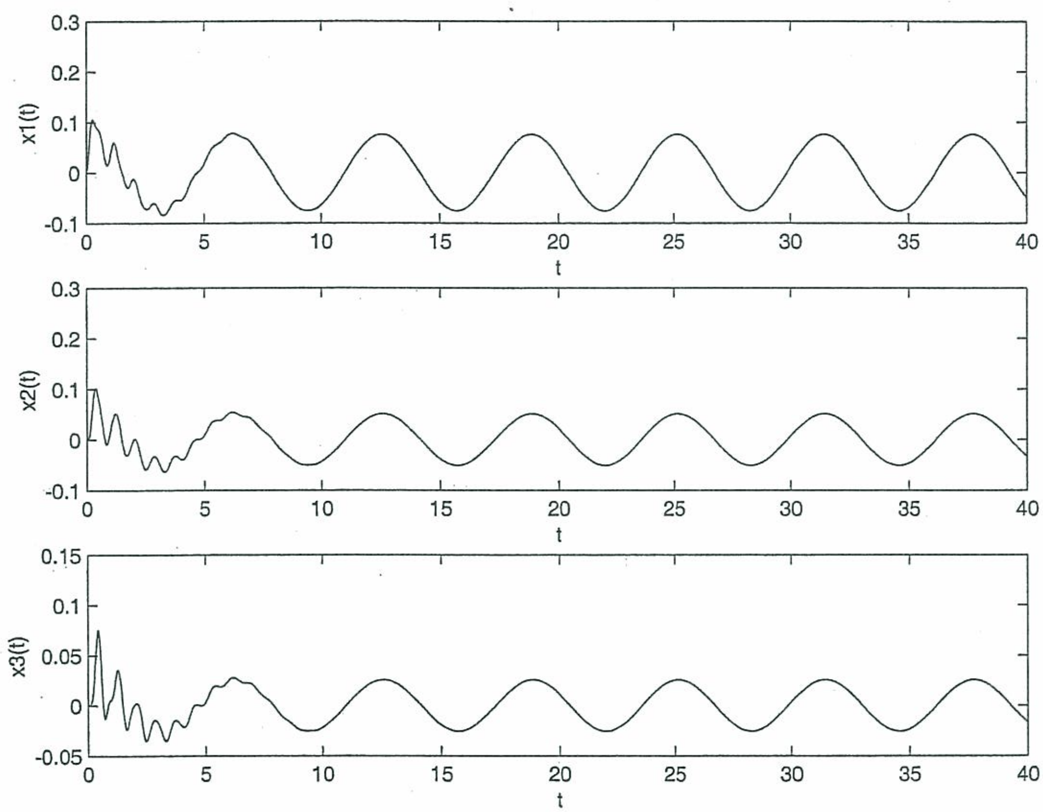
$$\ddot{x}_3 = \dot{x}_2 - 2\dot{x}_3 + 100 x_2 - 200 x_3 \qquad (3)$$

Defining $\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \vdots \\ \dot{x}_3 \end{Bmatrix}$ and using $\vec{Y}(0) = \vec{0}$

Eqs. (1) – (4) can be rewritten as

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -2y_2 + y_4 - 200y_1 + 100y_3 + 10\cos t \\ y_4 \\ y_2 - 2y_4 + y_6 + 100y_1 - 200y_3 + 100y_5 \\ y_6 \\ y_4 - 2y_6 + 100y_3 - 200y_5 \end{Bmatrix} \tag{4}$$

Solution of Eq. (4) using MATLAB:




```
% dfunc6_106.m
function f = dfunc6_106(t,x)
f = zeros(6,1);
f(1) = x(2);
f(2) = -2*x(2) + x(4) - 200*x(1) + 100*x(3) + 10*cos(t);
f(3) = x(4);
f(4) = x(2) - 2*x(4) + 100*x(1) - 200*x(3) + 100*x(5);
f(5) = x(6);
f(6) = x(4) - 2*x(6) + 100*x(3) - 200*x(5);
```

```
% Ex6_106.m
% This program will use the function dfunc6_106.m ,they should
% be in the same folder
tspan = [0: 0.01: 40];
x0 = [0.0; 0.0; 0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc6_106', tspan, x0);
subplot(311);
plot(t,x(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(312);
plot(t,x(:,3));
xlabel('t');
ylabel('x2(t)');
subplot(313);
plot(t,x(:,5));
xlabel('t');
ylabel('x3(t)');
```

6.107

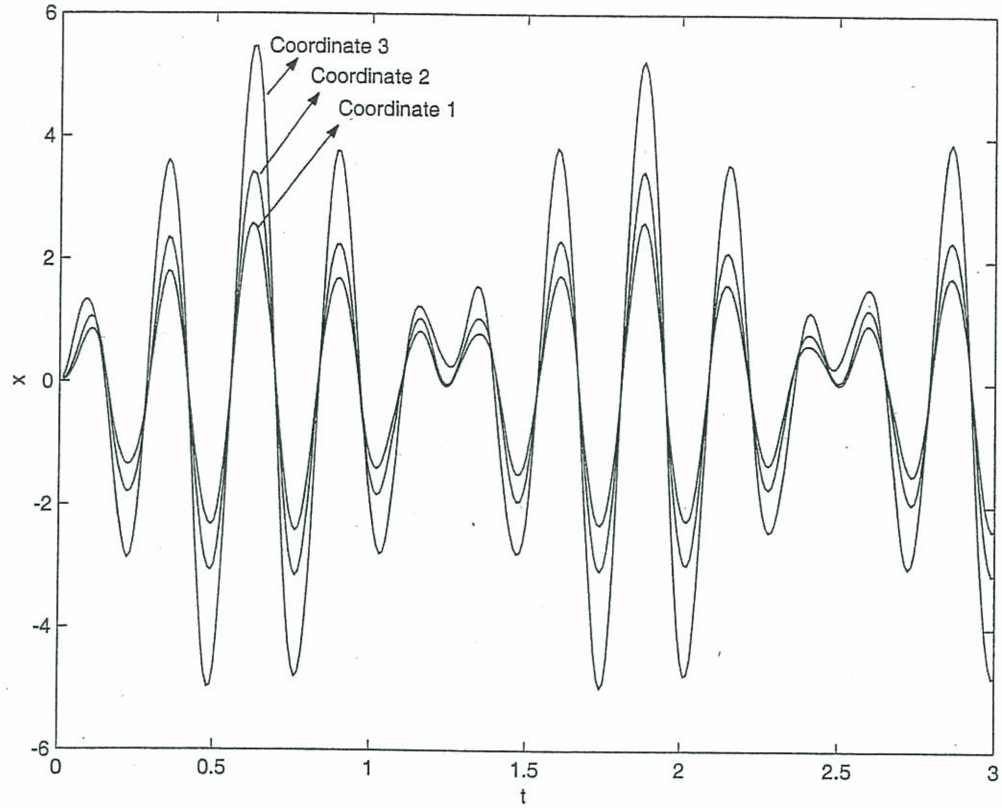
```
Results of Ex6_107
*****************
>> program7

 polynomial expansion of a determinantal equation


 data: determinant A:

    5.000000e+000      3.000000e+000      2.000000e+000
    3.000000e+000      6.000000e+000      4.000000e+000
    1.000000e+000      2.000000e+000      6.000000e+000

 result: polynomial coefficients in
 pcf(np)*(x^n)+pcf(n)*(x^(n-1))+...+pcf(2)+pcf(1)=0

 -9.800000e+001   7.700000e+001    -1.700000e+001  1.000000e+000
```

6.108

```
%======================================================================
%
% Program8.m
% Main program which calls the function MODAL
%
%======================================================================
% Run "Program8" in MATLAB command window, Progrm8.m and modal.m
% should be in the same folder,and set the path to this folder
% following line contain problem-dependent data
n=3;
nvec=3;
nstep=300;
delt=0.01;
xm=[41.4 0.0 0.0; 0.0 38.8 0.0; 0.0 0.0 25.88];
omf=[5 10 20];
om=[25.076 53.578 110.907];
z=[0.001 0.001 0.001];
x0=[0.0 0.0 0.0];
xd0=[0.0 0.0 0.0];
ev=[1.0 1.0 1.0;1.303 0.860 -1.0;1.947 -1.685 0.183];
%end of problem-dependent data
```

```
for i=1:nstep
   time=i*delt;
   f(1,i)=5000*cos(5*time);
   f(2,i)=10000*cos(10*time);
   f(3,i)=20000*cos(20*time);
end
for i=1:nvec
   for j=1:n
      evt(i,j)=ev(i,j);
   end
end
[x,t]=modal(xm,om,omf,z,x0,xd0,f,delt,ev,evt,nstep,n,nvec);
fprintf('\n Response of system using modal analysis \n\n');
for i=1:n
   fprintf('\n Coordinate %2.0f \n',i);
   fprintf(' %8.5e  %8.5e  %8.5e  %8.5e  %8.5e\n',x(i,1:nstep));
end
for i = 1: n
   plot(t,x(i,1:nstep));
   hold on;
end
xlabel('t');
ylabel('x');
gtext('Coordinate 3');
gtext('Coordinate 2');
gtext('Coordinate 1' );
%=====================================================================
%
% function modal.m
%
%=====================================================================
function [x,t]=modal(xm,om,omf,z,x0,xd0,f,delt,ev,evt,nstep,n,nvec);
t(1)=delt;
for i=2:nstep
   t(i)=t(i-1)+delt;
end
% normalization of modal matrix with respect to the mass matrix
% xmx=matmul(xm,ev,n,n,nvec);
% xtmx=matmul(evt,xmx,nvec,n,nvec);
xmx=xm*ev;
xtmx=evt*xmx;
for i=1:nvec
   for j=1:n
      ev(j,i)=ev(j,i)/sqrt(xtmx(i,i));
   end
end
% conversion of information to normal coordinates
for i=1:nvec
   y0(i)=0.0;
   yd0(i)=0.0;
end
for i=1:nvec
   for j=1:n
      yo(i)=y0(i)+ev(j,i)*x0(j);
      yd0(i)=yd0(i)+ev(j,i)*xd0(j);
   end
end
```

```
for i=1:nvec
   for j=1:n
      evt(i,j)=ev(j,i);
   end
end
% q=matmul(evt,f,nvec,n,nstep);
q=evt*f;
for i=1:nvec
   r=omf(i)/om(i);
   pp=y0(i);
   qq=yd0(i);
   zi=z(i);
   omeg=om(i);
   omd=omeg*sqrt(1-zi^2);
   for j=1:nstep
       if j~=1
          pp=u(i,j-1);
          qq=v(i,j-1);
       end
       c1=exp(-zi*omeg*delt);
       c2=cos(omd*delt);
       c3=sin(omd*delt);
       c4=(qq+omeg*zi*pp)/omd;
       c5=omeg*zi/omd;
       c6=q(i,j)/(omeg^2);
       u(i,j)=c1*(pp*c2+c3*c4)+c6*(1-c1*(c2+c3*c5));
       v(i,j)=omd*c1*(-pp*c3+c2*c4-c5*(pp*c2+c3*c4))+c6*omd*c1*c3*(1+c5^2);
   end
end
% finding the solution in the original coordinates
% x=matmul(ev,u,n,nvec,nstep);
x=ev*u;

%==================================================================
%
% function matmul.m
%
%==================================================================
function [a]=matmul(b,c,l,m,n)
% Matrix multipication subroutine: A=B*C
% b(l,m) and c(m,n) are input matrices, A(l,n) is output matrix
for i=1:l
   for j=1:n
      a(i,j)=0;
      for k=1:m
         a(i,j)=a(i,j)+b(i,k)*c(k,j);
      end
   end
end
```




```
Results of Ex6_108
******************
>> program8

 Response of system using modal analysis


 Coordinate  1
 1.16587e-002 4.77899e-002 1.10778e-001 2.01670e-001 3.17703e-001
 4.51127e-001 5.89314e-001 7.15610e-001 8.10748e-001 8.55251e-001
 8.33049e-001 7.35446e-001 5.63789e-001 3.29513e-001 5.16989e-002
 -2.46502e-001 -5.41192e-001 -8.10289e-001 -1.03560e+000 -1.20447e+000
 :
 .


 Coordinate  2
 1.94937e-002 7.72115e-002 1.71039e-001 2.97350e-001 4.49827e-001
 6.17690e-001 7.84560e-001 9.29302e-001 1.02905e+000 1.06321e+000
 1.01695e+000 8.83330e-001 6.64302e-001 3.71083e-001 2.35527e-002
 -3.51910e-001 -7.26226e-001 -1.07196e+000 -1.36651e+000 -1.59315e+000
 :
 .


 Coordinate  3
 5.07839e-002 1.94324e-001 4.06865e-001 6.55062e-001 9.01966e-001
 1.11328e+000 1.26237e+000 1.33292e+000 1.31883e+000 1.22192e+000
 1.04849e+000 8.06186e-001 5.01856e-001 1.41193e-001 -2.69837e-001
 -7.22160e-001 -1.20031e+000 -1.68000e+000 -2.12777e+000 -2.50340e+000
 :
 .
```

6.110

```
C ====================================================================
C
C SOLUTION OF PROBLEM 6.110
C
C ====================================================================
C
C N         = NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM
C NM        = N-1
C XK(N,N)   = STIFFNESS MATRIX
C XM(N,N)   = MASS MATRIX
C OM(N)     = VECTOR OF NATURAL FREQUENCIES
C X(N,N)    = MATRIX OF EIGENVECTORS. J TH EIGENVECTOR IS STORED IN THE
C             J TH COLUMN OF THE MATRIX X
C DIMENSIONS OF OTHER MATRICES: A(NM,NM),B(NM),LA(NM),LB(NM,2),S(NM)
C
      DIMENSION XK(3,3),XM(3,3),OM(3),X(3,3),A(2,2),B(2),LA(2),LB(2,2),
     2 S(2)
      DATA XK/2.,-1.,0.,-1.,2.,-1.,0.,-1.,2./
      DATA XM/2.,0.,0.,0.,3.,0.,0.,0.,2./
      DATA OM/.482087,1.,1.197605/
      N=3
      NM=N-1
      IND=1
      DO 10 I=1,N
      DO 20 J=1,NM
      DO 20 K=1,NM
      A(J,K)=XK(J,K+1)-(OM(I)**2)*XM(J,K+1)
20    B(J)=-XK(J,1)+(OM(I)**2)*XM(J,1)
      CALL SIMUL (A,B,NM,IND,LA,LB,S)
      X(1,I)=1.0
      DO 30 J=1,NM
30    X(J+1,I)=B(J)
      WRITE (44,110) OM(I),(X(J,I),J=1,N)
110   FORMAT (//,2X,19H NATURAL FREQUENCY=,E15.8,/,2X,13H EIGENVECTOR=,
     2 /,(2X,4E15.8))
10    CONTINUE
      STOP
      END
```

```
NATURAL FREQUENCY= 0.48208699E+00
EIGENVECTOR=
0.10000000E+01 0.15351844E+01 0.10000019E+01


NATURAL FREQUENCY= 0.10000000E+01
EIGENVECTOR=
0.10000000E+01 0.00000000E+00-0.10000000E+01


NATURAL FREQUENCY= 0.11976050E+01
EIGENVECTOR=
0.10000000E+01-0.86851549E+00 0.99999440E+00
```

6.111

```
C ====================================================================
C
C PROBLEM  6.111
C
C ====================================================================
C N       = NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM
C XM(N,N)= MASS MATRIX
C X(N,N) = MATRIX CONTAINING THE ORIGINAL I TH NORMAL MODE IN I TH
C          COLUMN
C XN(N,N)= MATRIX CONTAINING THE [M]-ORTHONORMAL I TH NORMAL MODE
C          IN I TH COLUMN
C DIMENSIONS OF OTHER VECTORS: A(N),B(N)
      DIMENSION XM(3,3),X(3,3),XN(3,3),A(3),B(3)
      N=3
      DATA XM /1.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,1.0/
      DATA X /1.0,-1.0,1.0,1.0,1.0,1.0,0.0,1.0,2.0/
      DO 10 I=1,N
      DO 20 J=1,N
 20   A(J)=X(J,I)
      CALL MULT (XM,A,B,N)
      SUM=0.0
      DO 30 K=1,N
 30   SUM=SUM+X(K,I)*B(K)
      SUM=SQRT(1.0/SUM)
      DO 40 K=1,N
 40   XN(K,I)=X(K,I)*SUM
      PRINT 50, I,(XN(J,I),J=1,N)
 50   FORMAT (/,2X,13H EIGENVECTOR:,I5,/,2X,(4E15.8))
 10   CONTINUE
      STOP
      END
C ====================================================================
C
C SUBROUTINE MULT
C
C ====================================================================
      SUBROUTINE MULT (XM,A,B,N)
      DIMENSION XM(N,N),A(N),B(N)
      DO 10 I=1,N
      B(I)=0.0
      DO 20 J=1,N
 20   B(I)=B(I)+XM(I,J)*A(J)
 10   CONTINUE
      RETURN
      END
```

```
 EIGENVECTOR:     1
 0.50000000E+00-0.50000000E+00 0.50000000E+00

 EIGENVECTOR:     2
 0.50000000E+00 0.50000000E+00 0.50000000E+00

 EIGENVECTOR:     3
 0.00000000E+00 0.40824831E+00 0.81649661E+00
```

6.112 The main program and results are shown below.

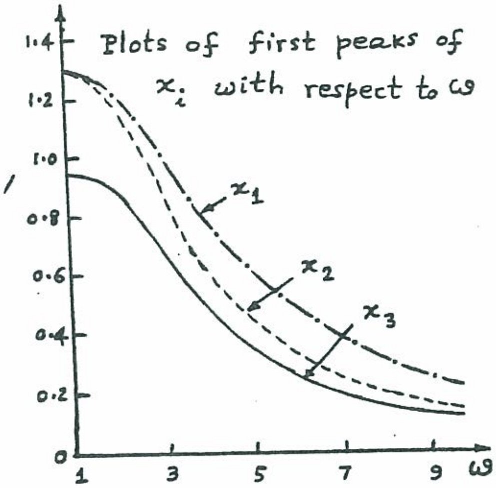
```
C ==================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MODAL
C
C ==================================================================
C
      DIMENSION XM(3,3),OM(3),Z(3),XO(3),XDO(3),YO(3),YDO(3),EV(3,3),
     2 EVT(3,3),XMX(3,3),XIMX(3,3),T(40),F(3,40),X(3,40),U(3,40),
     3 V(3,40),Q(3,40)
      DATA N,NVEC,NSTEP,DELT/3,3,40,0.1/
      DATA XM/2.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,2.0/
      OMF=5.0
      DATA OM/1.530734,2.828428,3.695518/
      DATA Z/0.0,0.0,0.0/
      DATA XO/0.0,0.0,0.0/
      DATA XDO/0.0,0.0,0.0/
      DATA (EV(I,1),I=1,3)/1.0,1.414214,1.0/
      DATA (EV(I,2),I=1,3)/1.0,0.0,-1.0/
      DATA (EV(I,3),I=1,3)/1.0,-1.414214,1.0/
      DO 5 I=1,NSTEP
      TIME=REAL(I)*DELT
5     F(1,I)=10.0*SIN(5.0*TIME)
      DO 10 I=1,NSTEP
      F(2,I)=0.0
10    F(3,I)=0.0
      DO 20 I=1,NVEC
      DO 20 J=1,N
20    EVT(I,J)=EV(J,I)
      CALL MODAL (XM,OM,OMF,T,Z,XO,XDO,YO,YDO,Q,F,DELT,EV,EVT,XMX,
     2 XTMX,X,U,V,NSTEP,N,NVEC)
      WRITE (29,30)
30    FORMAT (//,40H RESPONSE OF SYSTEM USING MODAL ANALYSIS,/)
      DO 40 I=1,N
40    WRITE (29,50) I,(X(I,J),J=1,NSTEP)
50    FORMAT (/,11H COORDINATE,I5,/,(1X,5E14.6))
      STOP
      END
```

```
RESPONSE OF SYSTEM USING MODAL ANALYSIS


COORDINATE    1
  0.119060E-01  0.556720E-01  0.140716E+00  0.262090E+00  0.400585E+00
  0.526842E+00  0.608511E+00  0.618689E+00  0.543493E+00  0.386804E+00
  0.170858E+00 -0.676210E-01 -0.284971E+00 -0.440109E+00 -0.503791E+00
 -0.465406E+00 -0.335646E+00 -0.144346E+00  0.660445E-01  0.249893E+00
  0.368518E+00  0.398672E+00  0.337414E+00  0.202231E+00  0.264541E-01
 -0.148937E+00 -0.285117E+00 -0.354495E+00 -0.346914E+00 -0.271541E+00
 -0.154034E+00 -0.297635E-01  0.652391E-01  0.103369E+00  0.723514E-01
 -0.216811E-01 -0.155688E+00 -0.295363E+00 -0.403527E+00 -0.449197E+00

COORDINATE    2
  0.397433E-04  0.655754E-03  0.356972E-02  0.119145E-01  0.297943E-01
  0.612223E-01  0.108686E+00  0.171738E+00  0.246047E+00  0.323279E+00
  0.391991E+00  0.439459E+00  0.454134E+00  0.428213E+00  0.359732E+00
  0.253658E+00  0.121640E+00 -0.196366E-01 -0.151233E+00 -0.255357E+00
 -0.318689E+00 -0.334940E+00 -0.306091E+00 -0.242021E+00 -0.158585E+00
 -0.745270E-01 -0.784475E-02  0.276338E-01  0.247827E-01 -0.156785E-01
 -0.856190E-01 -0.171361E+00 -0.256531E+00 -0.325272E+00 -0.365137E+00
 -0.369097E+00 -0.336327E+00 -0.271711E+00 -0.184302E+00 -0.851769E-01

COORDINATE    3
  0.526197E-07  0.338722E-05  0.398979E-04  0.232946E-03  0.912614E-03
  0.274891E-02  0.685317E-02  0.147839E-01  0.284018E-01  0.495493E-01
  0.795763E-01  0.118784E+00  0.165904E+00  0.217749E+00  0.269161E+00
```

0.313352E+00	0.342632E+00	0.349486E+00	0.327819E+00	0.274185E+00
0.188751E+00	0.758134E-01	-0.563065E-01	-0.195996E+00	-0.329661E+00
-0.443429E+00	-0.524977E+00	-0.565224E+00	-0.559583E+00	-0.508616E+00
-0.417951E+00	-0.297528E+00	-0.160267E+00	-0.204086E-01	0.108245E+00
0.213908E+00	0.288030E+00	0.326005E+00	0.327354E+00	0.295404E+00

6.113 The main program used to generate the numerical results and the graph are given.

```
C ==================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MODAL
C
C ==================================================================
C
      DIMENSION XM(3,3),OM(3),Z(3),X0(3),XD0(3),Y0(3),YD0(3),EV(3,3),
     2 EVT(3,3),XMX(3,3),XTMX(3,3),T(40),F(3,40),X(3,40),U(3,40),
     3 V(3,40),Q(3,40)
      DATA N,NVEC,NSTEP,DELT/3,3,40,0.1/
      DATA XM/2.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,2.0/
      DATA OM/1.530734,2.828428,3.695518/
      DATA Z/0.0,0.0,0.0/
      DATA X0/0.0,0.0,0.0/
      DATA XD0/0.0,0.0,0.0/
      DATA (EV(I,1),I=1,3)/1.0,1.414214,1.0/
      DATA (EV(I,2),I=1,3)/1.0,0.0,-1.0/
      DATA (EV(I,3),I=1,3)/1.0,-1.414214,1.0/
      OMF=0.0
      DO 55 IJK=1,10
      OMF=OMF+1.0
      DO 5 I=1,NSTEP
      TIME=REAL(I)*DELT
5     F(1,I)=10.0*SIN(OMF*TIME)
      DO 10 I=1,NSTEP
      F(2,I)=0.0
10    F(3,I)=0.0
      DO 20 I=1,NVEC
      DO 20 J=1,N
20    EVT(I,J)=EV(J,I)
      CALL MODAL (XM,OM,OMF,T,Z,X0,XD0,Y0,YD0,Q,F,DELT,EV,EVT,XMX,
     2 XTMX,X,U,V,NSTEP,N,NVEC)
      WRITE (29,30)
30    FORMAT (//,40H RESPONSE OF SYSTEM USING MODAL ANALYSIS,/)
      DO 40 I=1,N
40    WRITE (29,50) I,(X(I,J),J=1,NSTEP)
50    FORMAT (/,11H COORDINATE,I5,/,(1X,5E14.6))
55    CONTINUE
      STOP
      END
```



6.114 (a) Equation of motion:

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_h \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3(t) \end{Bmatrix}$$

Forcing function can be written as

$$F_3(t) = \mathrm{Re}\left(F_0\, e^{i\omega t}\right) = F_{30} \cos \omega t \equiv 1000 \cos 60t \ \ lb. \qquad (E_1)$$

steady-state solution can be assumed as

$$x_j(t) = X_j\, e^{i\omega t}, \quad j = 1,2,3 \qquad (E_2)$$

Equations of motion become

$$\begin{bmatrix} -m_f\omega^2 + (c_1+c_2)i\omega + k_1 + k_2 & -c_2 i\omega - k_2 & 0 \\ -c_2 i\omega - k_2 & -m_b\omega^2 + (c_2+c_3)i\omega + k_2 + k_3 & -c_3 i\omega - k_3 \\ 0 & -c_3 i\omega - k_3 & -m_h\omega^2 + c_3 i\omega + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{30} \end{Bmatrix} \qquad (E_3)$$

For known data, Eq. (E_3) becomes

$$[Z_{ij}]\vec{X} = \vec{F}_0 \qquad (E_4)$$

where

$$\begin{aligned} Z_{11} &= -50\,\omega^2 + i\,20\omega + 5500 = -174\,500 + i\,1200 \\ Z_{12} &= Z_{21} = -i\,10\omega - 500 = -500 - i\,600 \\ Z_{13} &= Z_{31} = 0 \\ Z_{22} &= -10\,\omega^2 + i\,20\,\omega + 2500 = -33\,500 + i\,1200 \\ Z_{23} &= Z_{32} = -i\,10\omega - 2000 = -2000 - i\,600 \\ Z_{33} &= -2\,\omega^2 + i\,10\omega + 2000 = -5200 + i\,600 \end{aligned} \qquad (E_5)$$

Solution of Eq. (E_4) can be expressed as

$$\vec{X} = [Z_{ij}]^{-1}\vec{F}_0 \qquad (E_6)$$

Using Cramer's rule, we get

$$X_1 = \begin{vmatrix} 0 & Z_{12} & Z_{13} \\ 0 & Z_{22} & Z_{23} \\ F_{30} & Z_{32} & Z_{33} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = \left(-0.1230\times 10^{-4} - i\,0.5284\times 10^{-4}\right)$$

$$X_2 = \begin{vmatrix} Z_{11} & 0 & Z_{13} \\ Z_{21} & 0 & Z_{23} \\ Z_{31} & F_{30} & Z_{33} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = \left(0.1087\times 10^{-1} + i\,0.5373\times 10^{-2}\right)$$

$$X_3 = \begin{vmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & Z_{32} & F_{30} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = (-0.1929 - i\, 0.02558)$$

$$\therefore x_1(t) = X_1 (\cos 60t + i \sin 60t)$$

$$= (-0.1230 \times 10^{-4} \cos 60t + 0.5284 \times 10^{-4} \sin 60t)$$

$$- i\,(0.5284 \times 10^{-4} \cos 60t + 0.1230 \times 10^{-4} \sin 60t)$$

Actual response of m_f = Real $[x_1(t)]$

$$= (-0.1230 \times 10^{-4} \cos 60t + 0.5284 \times 10^{-4} \sin 60t) \text{ in.}$$

Similarly, we find:

Actual response of m_b = Real $[x_2(t)]$

$$= (0.01087 \cos 60t - 0.005373 \sin 60t) \text{ in.}$$

Actual response of m_h = Real $[x_3(t)]$

$$= (-0.1929 \cos 60t + 0.02558 \sin 60t) \text{ in.}$$

(b) Change the stiffness k_2 from 100 lb/in in increments of 100 lb/in and find the response of the tool head (x_3) using the procedure outlined in part (a). Find the value of k_2 for which the maximum response of x_3 is 25% lower than the value found in part (a).

(c) Change the values of c_1, c_2, c_3, k_1 and k_3 individually and find whether any of these quantities can be used to achieve the goal of part (b).