Chapter 10

Vibration Measurement and Applications

10.1 Voltage sensitivity = $v = 0.098$ volt-meter/Newton

thickness = $t = 2$ mm $= \frac{2}{1000}$ m

output voltage = 220 volts , pressure applied = $p_x = ?$

$E = v t p_x \quad \Rightarrow \quad 220 = (0.098)\left(\frac{2}{1000}\right) p_x$

$\therefore \; p_x = 1.1224 \times 10^6 \; N/m^2$

10.2 $m = 0.5$ kg, $k = 10000$ N/m, $c \approx 0$

amplitude = $Y = 4/1000$ m

total displacement of mass = $x = 12/1000$ m

relative displacement = $z = x - y = 8/1000$ m

$$Z = \frac{r^2 Y}{1-r^2} \quad \text{i.e.,} \quad \frac{8}{1000} = \left\{ \frac{(4/1000)\, r^2}{1-r^2} \right\} \Rightarrow r^2 = 2/3$$

$r = \frac{\omega}{\omega_n} = 0.8165$

$\omega = r\omega_n = 0.8165 \sqrt{\frac{10000}{0.5}} = 115.4705$ rad/sec $= 18.3777$ Hz

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10.3
$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad \text{where} \quad \zeta = \frac{c}{c_{cri}} = \frac{1}{\sqrt{2}}$$

$$= \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (\sqrt{2}\, r)^2}} = \frac{r^2 Y}{\sqrt{1+r^4}}$$

10.4 speed range:

500 rpm = 52.36 rad/sec − 1500 rpm = 157.08 rad/sec

$x = X \cos \omega t$;
$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad (E_1)$$

CASE (i): Let $\zeta = 0$

Eq. (E_1) gives, for 2% error,

$$\frac{Z}{Y} = \frac{r^2}{|r^2 - 1|} = 1.02 \quad \Rightarrow \quad r = \frac{\omega}{\omega_n} = 7.1414$$

$$\omega_n = \frac{\omega}{7.1414} = \frac{500}{7.1414} \text{ or } \frac{1500}{7.1414} = 70.0143 \text{ or } 210.0428 \text{ rpm}$$

$$= 1.1669 \text{ or } 3.5007 \text{ Hz} = 7.3319 \text{ or } 21.9957 \text{ rad/sec}$$

$\therefore \omega_n = 1.1669$ Hz

CASE (ii): Let $\zeta = 0.6$

Eq. (E_1) gives $\dfrac{Z}{Y} = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E = 1.02$

$\Rightarrow \quad 0.0404\, r^4 - 0.5826\, r^2 + 1.0404 = 0$

which gives $r = 2.2562,\ 3.0546$

Since the quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898 \quad \text{for } \zeta = 0.6$$

we use $r = 2.2562$.

$\therefore$ Maximum $\omega_n = \omega/r = 500/(2.2562 \times 60) = 3.6935$ Hz

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10.5 Error factor for vibrometer is $E = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Maximum of E occurs at $r^* = \dfrac{1}{\sqrt{1-2\zeta^2}}$

For $\zeta = 0$, $E = \dfrac{r^2}{|1-r^2|}$ and $r^* = 1$

Since the range is $4 \le r < \infty$, we use $r = 4$ for maximum error.

$$E\Big|_{r=4} = \frac{4^2}{|1-4^2|} = 1.0667$$

Percent error $= (E-1)\,100 = 6.67\%$

10.6 Error factor $= E = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r^* = 1/\sqrt{1-2\zeta^2}$

For $\zeta = 0.67$, $r^* = 3.1281$ and $E\Big|_{r=r^*,\ \zeta=0.67} = 1.0053$

Percent error $= 0.53\%$

10.7 Select the vibrometer on the basis of lowest frequency being measured.

(a) $\zeta = 0$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{|r^2-1|}$, $r^2 = \frac{1.03}{0.03} = 34.3333$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi(500)}{60\,\omega_n} = 5.8595$$

$$\omega_n = 8.9359 \text{ rad/sec} = 1.4222 \text{ Hz}$$

(b) $\zeta = 0.6$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = E$

$$(1.03)^2\{1 + r^4 - 2r^2 + 4r^2\zeta^2\} = r^4$$

i.e., $0.057404\, r^4 - 0.56\, r^2 + 1 = 0$

i.e., $r^2 = 2.3535, \quad 7.4018$

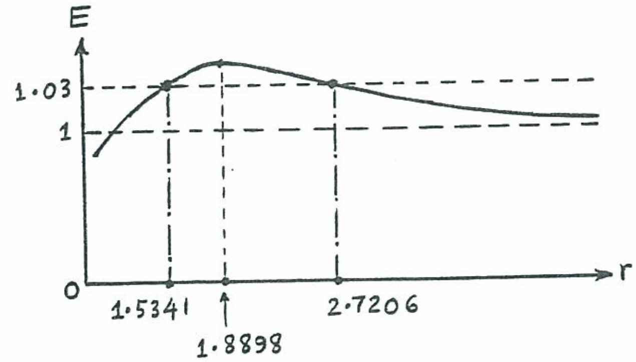
i.e., $r = 1.5341^{\dagger}, \quad 2.7206$

By selecting $r = 2.7206$,

$$\omega_n = \frac{1000\pi}{60(2.7206)}$$

$= 19.2458$ rad/sec

$= 3.0630$ Hz



† The quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898$$

for $\zeta = 0.6$. Hence we have to take $r = 2.7206$ to avoid peak of E.

10.8 $\delta_{st} = \frac{10}{1000}$ m , $\omega = 4000$ rpm $= 418.88$ rad/sec

$$\omega_n = \sqrt{g/\delta_{st}} = \sqrt{9.81\left(\frac{1000}{10}\right)} = 31.3209 \text{ rad/sec}$$

$$r = \omega/\omega_n = 418.88/31.3209 = 13.3738$$

Let $\zeta = 0$:

$$\text{Error factor} = \left.\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{13.3738^2}{|1-13.3738^2|}$$

$= 1.0056$

(i) Maximum displacement $= Y = Z/1.0056 = 1/1.0056 = 0.9944$ mm

(ii) Maximum velocity $= \omega Y = (418.88)(0.9944) = 416.5473$ mm/sec

(iii) Maximum acceleration $= \omega^2 Y = (418.88)^2 (0.9944) = 174483.35$ mm/sec².

10.9 $$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad ---- \quad (E_1)$$

Maximum of $\frac{Z}{Y}$ occurs when $r = r^* = \frac{1}{\sqrt{1-2\zeta^2}}$ (see Eq. (3.82)).

For $\zeta = 0.5$, $\quad r^* = \frac{1}{\sqrt{0.5}} = 1.4142$, $\quad (r^*)^2 = 2$

$$\left.\frac{Z}{Y}\right|_{r^*} = \frac{2}{\sqrt{(1-2)^2 + (2 \times 0.5 \times 1.4142)^2}} = \frac{2}{\sqrt{3}} = 1.1547$$

When error is one percent, $\frac{Z}{Y} = 1.01$ or $\frac{Y}{Z} = 0.9901$

Eq. (E_1) can be rewritten as:

$$\left|\frac{Y}{Z}\right|^2 = \frac{(1-r^2)^2 + 4\zeta^2 r^2}{r^4}$$

$$r^4 \left(1 - \left|\frac{Y}{Z}\right|^2\right) - r^2 (2 - 4\zeta^2) + 1 = 0 \quad ---- \quad (E_2)$$

For $\frac{Y}{Z} = 0.9901$ and $\zeta = 0.5$, (E_2) becomes

$$0.0197\, r^4 - r^2 + 1 = 0$$

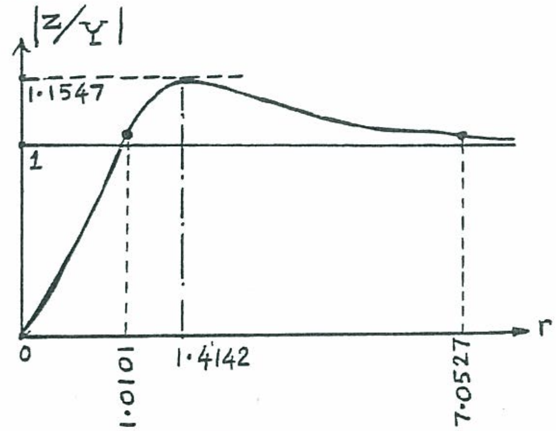
$$r^2 = 1.0203, \quad 49.7411$$

$$r = \frac{\omega}{\omega_n} = 1.0101, \quad 7.0527$$

Lowest frequency for one percent accuracy

$= 7.0527\,(5)$

$= 35.2635$ Hz



10.10 Frequency range > 100 Hz, maximum error $= 2\%$

$k = 4000$ N/m, $c = 0 \Rightarrow \zeta = 0$, $m = ?$

For vibrometer with $\zeta = 0$,
$$\frac{Z}{Y} = \left.\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right|_{\zeta = 0}$$
$$= \frac{r^2}{|1-r^2|} = 1.02$$

or $\quad r^2/(1-r^2) = -1.02$ since r must be greater than one for higher frequencies

or $\quad r^2 = 51 \quad$ i.e., $\quad r = 7.1414$

Minimum impressed frequency $= \omega = 10$ Hz

Since $r = \omega/\omega_n = 100/\omega_n = 7.1414$, $\omega_n = 14.0029$ Hz $= 87.9827 \frac{\text{rad}}{\text{sec}}$
$= \sqrt{k/m}$

$\therefore\ m = k/\omega_n^2 = 4000/(87.9827)^2 = 0.5167$ kg

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10.11 $\omega_n = 10$ Hz, $\omega_d = 8$ Hz $= \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-\zeta^2} \Rightarrow \zeta = 0.6$

Let the lowest frequency $= \omega_0 \Rightarrow r_0 = \omega_0/\omega_n$

Error $= 2\%$ in the range $r_0 \le r < \infty$

$$\text{Error} = E = 1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\times 0.6\ r)^2}}$$

$$\Rightarrow\quad 0.0404\ r^4 - 0.5826\ r^2 + 1.0404 = 0$$

$$\Rightarrow\quad r = 2.2562,\ 3.0546$$

Let $r_0 = 2.2562$:

$\omega_0 = r_0\,\omega_n = 2.2562\,(10) = 22.562$ Hz

Let $r_0 = 3.0546$:

$\omega_0 = r_0\,\omega_n = 3.0546\,(10) = 30.546$ Hz

$\therefore$ Lowest frequency $= 22.562$ Hz $= 141.7616$ rad/sec

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10.12 Error factor for accelerometer $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Maximum of E occurs at $r^* = \sqrt{1-2\zeta^2}$

When $\zeta = 0$, $r^* = 1$ and $E = \dfrac{1}{|1-r^2|}$

Since the range is $0 \le r \le 0.65$, we use $r = 0.65$:

$$E\Big|_{r=0.65} = \frac{1}{|1-0.65^2|} = 1.7316$$

Percent error $= (E-1)\,100 = 73.16\%$

10.13 Error factor $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Value of r at which E attains maximum is $r^* = \sqrt{1-2\zeta^2}$

When $\zeta = 0.75$, $E = \dfrac{1}{\sqrt{(1-r^2)^2 + 2.25\,r^2}}$ and $r^* = \sqrt{1-1.125}$
$=$ imaginary

Since the range is $0 \le r \le 0.6$, we use $r = 0.6$

$$E\Big|_{r=0.6} = \frac{1}{\sqrt{(1-0.36)^2 + 2.25\,(0.6)^2}} = 0.9055$$

Percent error $= (E-1)\,100 = -9.45\%$

10.14 $m = 0.05$ kg , max error = 3 % over frequency range of 0 to 100 Hz

Find k and c.

For accelerometer, error factor = $E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r = r^* = \sqrt{1 - 2\zeta^2}$

and $E_{max} = E\Big|_{r^*} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

(i) Consideration of maximum error:

let error $= e = E - 1 = 0.03 = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$

upon rearrangement, this leads to $\zeta^4 - \zeta^2 + 0.23565 = 0$

or $\zeta = 0.6164,\ 0.7874$

r^* at $\zeta = 0.6164 = \sqrt{1 - 2(0.6164)^2} = 0.49$

r^* at $\zeta = 0.7874 = \sqrt{1 - 2(0.7874)^2} =$ imaginary

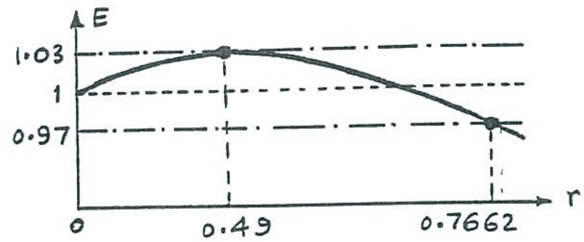
(ii) Consideration of minimum error:

let error $= e = E - 1 = -0.03 = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} - 1 \qquad (E_1)$

with $\zeta = 0.6164$, Eq. (E_1) can be simplified as

$r^4 - 0.4802\, r^2 - 0.0628 = 0$

$\Rightarrow r^2 = 0.5871,\ -0.1069 \qquad$ or $\quad r = 0.7662$



At the maximum frequency,

$\omega = 2\pi(100) = 628.32$ rad/sec

$\omega_n = \omega / r = 628.32 / 0.7662$

$= 820.0470$ rad/sec

$k = m\,\omega_n^2 = 0.05\,(820.047)^2$

$= 33\,623.8541$ N/m

$c_c = 2m\,\omega_n = \dfrac{c}{\zeta} \Rightarrow c = 2m\,\omega_n\,\zeta = 2(0.05)(820.047)(0.6164)$

$= 50.5477$ N-s/m

10.15 $m = 0.1$ kg , $k = 10\,000$ N/m , $c = 0 \Rightarrow \zeta = 0$

$\omega_n = \sqrt{k/m} = \sqrt{10000/0.1} = 316.2278$ rad/sec

Engine speed $= \omega = 1000$ rpm $= 104.72$ rad/sec

$r = \omega/\omega_n = 104.72/316.2278 = 0.3312$

peak-to-peak travel of mass = 10 mm

Find: Y, ωY, $\omega^2 Y$.

We have, from Eq. (10.19),

$$\left.\frac{Z}{Y}\right|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{0.3312^2}{|1-0.3312^2|} = 0.1232$$

Since peak-to-peak travel of mass = 10 mm, $Z = 5$ mm

$$Y = Z/0.1232 = 5/0.1232 = 40.5844 \text{ mm}$$

Max displacement of foundation = $Y = 40.5844$ mm

Max velocity of foundation = $\omega Y = 4249.9984$ mm/sec

Max acceleration of foundation = $\omega^2 Y = 445059.8291$ mm/sec^2

10.16 Maximum speed = 3000 rpm = 50 Hz ; $r = \frac{\omega}{\omega_n} = \frac{50}{100} = 0.5$

For accelerometer, $\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1 + \text{error} = 0.9$ (E_1)

Here $c = 20$ N-s/m ; $\zeta = \frac{c}{2m\omega_n} = \frac{20}{2m\,(100\times 2\pi)} = \frac{0.015915}{m}$

For $r = 0.5$, Eq. (E_1) gives $\zeta = 0.8198$

$$c_c = \frac{c}{\zeta} = 20/0.8198 = 24.3962 \text{ N-s/m}$$

$$m = \frac{c_c}{2\omega_n} = \frac{24.3962}{2\,(100\times 2\pi)} = 0.01941 \text{ kg} = 19.41 \text{ grams}$$

$$k = m\,\omega_n^2 = 0.01941\,(100\times 2\pi)^2 = 7622.7967 \text{ N/m}$$

10.17 $\omega_n = 2\pi\,(0.5) = \pi$ rad/sec ; $\omega_d = \omega_n \sqrt{1-\zeta^2}$

$$\sqrt{1-\zeta^2} = \omega_d/\omega_n = 0.48/0.5 \Rightarrow \zeta = 0.28$$

$r_1 = \omega_1/\omega_n = 4\pi/\pi = 4$; $r_2 = \omega_2/\omega_n = 8$; $r_3 = \omega_3/\omega_n = 12$

$$\phi_i = \tan^{-1}\left(\frac{2\zeta r_i}{1-r_i^2}\right)$$

$$\phi_1 = \tan^{-1}\left(\frac{2\,(0.28)\,4}{1-16}\right) = -8.4934^\circ$$

$$\phi_2 = \tan^{-1}\left(\frac{2\times 0.28\times 8}{1-64}\right) = -4.0675^\circ$$

$$\phi_3 = \tan^{-1}\left(\frac{2\times 0.28\times 12}{1-144}\right) = -2.6905^\circ$$

$$\frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} \times 20 = 21.0994$$

$$\frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} \times 10 = 10.1331$$

$$\frac{r_3^2}{\sqrt{(1-r_3^2)^2 + (2\zeta r_3)^2}} \times 5 = 5.0294$$

Record indicated by vibrometer is given by

$$z(t) = 21.0994 \sin(4\pi t + 8.4934°) + 10.1331 \sin(8\pi t + 4.0675°) + 5.0294 \sin(12\pi t + 2.6905°) \text{ mm}$$

10.18 $x(t) = 20 \sin 50t + 5 \sin 150t$ mm (E$_1$)

$$\ddot{x}(t) = -20(50)^2 \sin 50t - 5(150)^2 \sin 150t \text{ mm/sec}^2$$

$$= -50\,000 \sin 50t - 112\,500 \sin 150t \text{ mm/sec}^2 \quad (E_2)$$

$\omega_n = 100$ rad/sec, $\omega_d = \omega_n \sqrt{1-\zeta^2} = 80 \Rightarrow \zeta = 0.6$

$r_1 = \frac{\omega_1}{\omega_n} = \frac{50}{100} = 0.5$; $r_2 = \frac{\omega_2}{\omega_n} = \frac{150}{100} = 1.5$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 0.5}{1-0.25}\right) = 38.6598°$$

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 1.5}{1-2.25}\right) = -55.2222°$$

$$\frac{50\,000}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} = 52\,057.9206$$

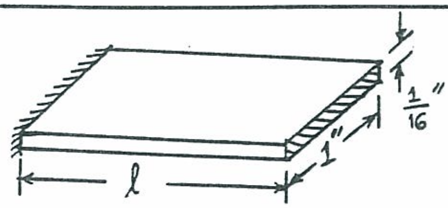
$$\frac{112\,500}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} = 51\,335.6229$$

output of the accelerometer is given by

$$\ddot{z}(t) = -52\,057.9206 \sin(50t - 38.6598°) - 51\,335.6229 \sin(150t + 55.2222°) \text{ mm/sec}^2 \quad (E_3)$$

It can be seen that Eq. (E$_3$) is substantially different from Eq. (E$_2$).

10.19 For given beam,



$A = 1\left(\frac{1}{16}\right) = 0.0625$ in^2

$I = \frac{1}{12}(1)\left(\frac{1}{16}\right)^3 = 20.35 \times 10^{-6}$ in^4

$l = 2''$ to $10''$

For a cantilever beam, Fig. 8.15 gives

$$\omega_n = (\beta_n l)^2 \left(\frac{E I}{\rho A l^4} \right)^{1/2}$$

where $(\beta_1 l)^2 = (1.875104)^2 = 3.516015$

$(\beta_2 l)^2 = (4.694091)^2 = 22.03449$

$(\beta_3 l)^2 = (7.854757)^2 = 61.69721$

$(\beta_4 l)^2 = (10.995541)^2 = 120.90192$

For spring steel, $E = 30 \times 10^6$ psi , $\rho g = 0.283$ lb/in^3

$$\left(\frac{E I}{\rho A l^4} \right)^{\frac{1}{2}} = \left\{ \frac{30 \times 10^6 \, (20.35 \times 10^{-6})}{\left(\frac{0.283}{386.4}\right) (0.0625) \, l^4} \right\}^{1/2} = \frac{3652.0}{l^2}$$

$$\omega_n = (\beta_n l)^2 \left(\frac{3652.0}{l^2} \right)$$

The first four frequencies are given below:

	l^2	ω_1	ω_2	ω_3	ω_4
$l = 2''$	4	3210.1020	20117.4894	56329.5527	110383.4530
$l = 10''$	100	128.4049	804.6996	2253.1821	4415.3381

Hence the range of frequencies that can be measured is given by $\omega > 128.4049$ rad/sec.

However, for first mode only (which is easiest to excite), the range of frequencies is $128.4049 \frac{\text{rad}}{\text{sec}} \le \omega \le 3210.102 \frac{\text{rad}}{\text{sec}}$.

10.20

$$\frac{k X_R}{F_0} = \frac{1 - r^2}{(1 - r^2)^2 + (2 \varsigma r)^2} = \frac{N}{D} \text{ (assume)}$$

$$\frac{d}{dr} \left(\frac{k X_R}{F_0} \right) = \frac{D \frac{dN}{dr} - N \frac{dD}{dr}}{D^2} = 0 \quad \text{i.e.,} \quad D \frac{dN}{dr} - N \frac{dD}{dr} = 0$$

$$\text{or} \quad \left\{ (1 - r^2)^2 + (2 \varsigma r)^2 \right\} (-2 r) - (1 - r^2) \left\{ 2 (1 - r^2)(-2 r) + 2 (2 \varsigma r)(2 \varsigma) \right\} = 0$$

This equation can be simplified as

$$r^4 - 2 r^2 + (1 - 4 \varsigma^2) = 0$$

and its solution is given by

$$r^2 = 1 \pm 2 \varsigma \quad \text{or} \quad r = \sqrt{1 + 2 \varsigma} \;;\; \sqrt{1 - 2 \varsigma}$$

Since

$$\frac{k X_R}{F_0} \Big|_{r = \sqrt{1 + 2 \varsigma}} = - \frac{1}{4 \varsigma (1 + \varsigma)} \qquad (1)$$

and

$$\frac{k\,X_R}{F_0}\Big|_{r=\sqrt{1-2\varsigma}} = \frac{1}{4\,\varsigma\,(1-\varsigma)} \tag{2}$$

we note that $r = R_1 = \sqrt{1-2\,\varsigma}$ corresponds to a maximum and $r = R_2 = \sqrt{1+2\,\varsigma}$ corresponds to a minimum of X_R.

(10.21)

$$\frac{k\,X_I}{F_0} = \frac{-2\,\varsigma\,r}{(1-r^2)^2 + 4\,\varsigma^2\,r^2}$$

$$\frac{d}{dr}\left(\frac{k\,X_I}{F_0}\right) = \frac{D\,\frac{dN}{dr} - N\,\frac{dD}{dr}}{D^2} = 0 \tag{1}$$

where $\frac{dN}{dr} = -2\,\varsigma$ and $\frac{dD}{dr} = -2\,(1-r^2)\,(2\,r) + 8\,\varsigma^2\,r$

By setting the numerator of Eq. (1) equal to zero, we obtain

$$\left\{1 + r^4 - 2\,r^2 + 4\,\varsigma^2\,r^2\right\}(-2\,\varsigma) - (-2\,\varsigma\,r)\left\{-4\,r + 4\,r^3 + 8\,\varsigma^2\,r\right\} = 0$$

which can be simplified to obtain

$$3\,r^4 + (4\,\varsigma^2 - 2)\,r^2 - 1 = 0 \tag{2}$$

The roots of Eq. (2) are given by

$$r^2 = \frac{1 - 2\,\varsigma^2 - 2\sqrt{\varsigma^4 - \varsigma^2 + 1}}{3}\ ;\ \frac{1 - 2\,\varsigma^2 + 2\sqrt{\varsigma^4 - \varsigma^2 + 1}}{3} \tag{3}$$

Since it is difficult to determine, from Eq. (3), the correct value of r that corresponds to the minimum of X_I, we use a numerical computation. For $\varsigma = 0.1$, for example, Eq. (3) gives

$$r^2 = -0.0030\ ;\ 0.6583$$

This shows that

$$r = \left\{\frac{1 - 2\,\varsigma^2 + 2\sqrt{\varsigma^4 - \varsigma^2 + 1}}{3}\right\}^{\frac{1}{2}} \tag{4}$$

corresponds to the minimum of X_I. For small values of ς, $\varsigma^2 \ll 1$ and Eq. (4) gives

$$r \approx 1 \tag{5}$$

Thus X_I attains its minimum value close to $r = 1$.

(10.22) Response of a single d.o.f. system with hysteretic damping is given by Eq. (3.106):

$$\frac{X}{F_0} = \frac{1}{k - m\,\omega^2 + i\,k\,\beta}$$

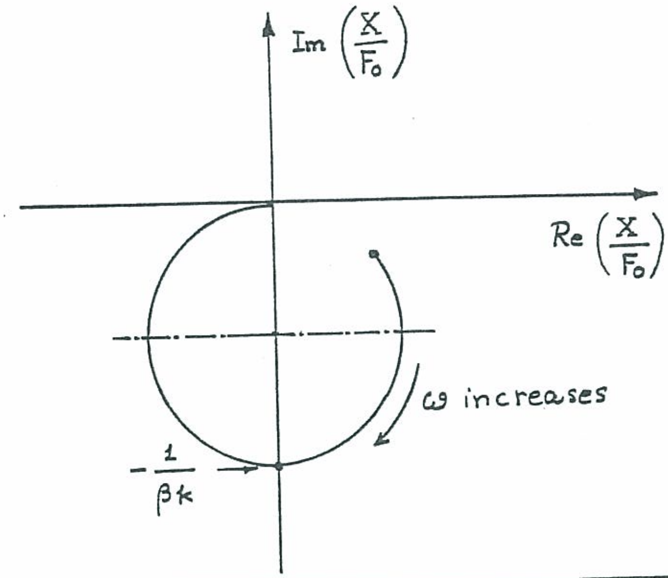
$$\mathrm{Re}\left(\frac{X}{F_0}\right) = \frac{k - m\,\omega^2}{(k - m\,\omega^2)^2 + (k\,\beta)^2} \quad , \quad \mathrm{Im}\left(\frac{X}{F_0}\right) = \frac{-k\,\beta}{(k - m\,\omega^2)^2 + (k\,\beta)^2}$$

$$\left[\mathrm{Re}\left(\frac{X}{F_0}\right)\right]^2 + \left[\mathrm{Im}\left(\frac{X}{F_0}\right)\right]^2 = \frac{1}{(k - m\,\omega^2)^2 + (k\,\beta)^2} \tag{1}$$

It can be verified that Eq. (1) can be rewritten as

$$\left[\mathrm{Re}\left(\frac{X}{F_0}\right)\right]^2 + \left[\mathrm{Im}\left(\frac{X}{F_0}\right) + \frac{1}{2\,\beta\,k}\right]^2 = \left(\frac{1}{2\,\beta\,k}\right)^2 \tag{2}$$

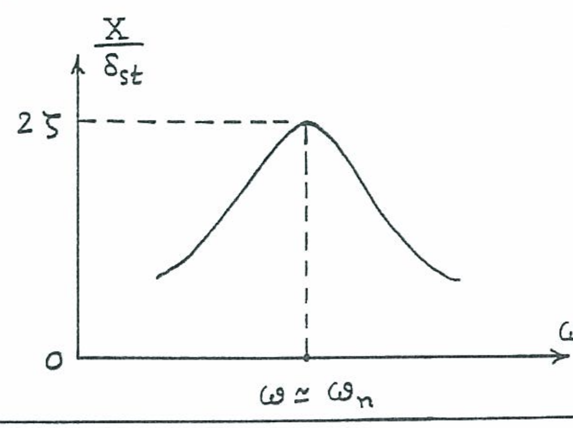
Eq. (2) shows that the locus of $\left(\frac{X}{F_0}\right)$ as ω increases from zero is part of a circle, with center $\left(0, -\frac{1}{2\,k\,\beta}\right)$ and radius $\frac{1}{2\,k\,\beta}$, as shown in the following figure.



10.23 The peak of Bode diagram is equal to $\approx \frac{1}{2\,\varsigma}$. In the present case, peak-to-peak value is plotted; hence $X \approx \frac{0.45}{2}$ mil $= 0.225$ mil.

$$\frac{X}{\delta_{st}} = \frac{0.225}{0.05} = 4.5 = \frac{1}{2\,\varsigma}$$

or $\varsigma = 0.1111$.

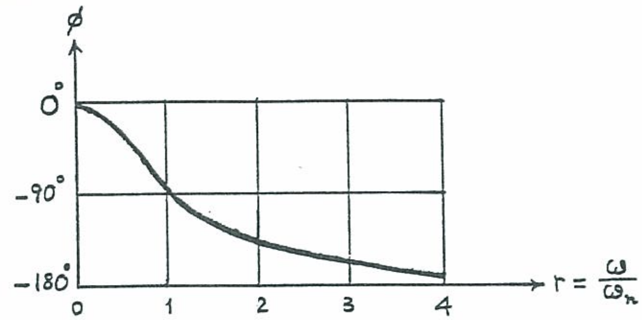


10.24 Reduction in amplitude from 6.8 in/sec to 0.8 in/sec in 7 cycles or 22 milliseconds. Eq. (2.92) gives:

$$\frac{1}{7} \ln \left(\frac{x_1}{x_8} \right) = \delta \quad \text{or} \quad \delta = \frac{1}{7} \ln \left(\frac{6.8}{0.8} \right) = 0.3057$$

Hence $\varsigma = \dfrac{\delta}{2\,\pi} = 0.04866.$

10.25 Typical Bode plot of phase angle is shown in the figure.



(a) $\phi = 90°$ at $r = \dfrac{\omega}{\omega_n} \approx 1$. Hence the value of ω_n can be determined from the value of r corresponding to $\phi = 90°$.

(b) Since

$$\phi = \tan^{-1} \left(- \frac{2\,\varsigma\, r}{1 - r^2} \right) = \tan^{-1} \left(- \frac{c\,\omega}{k - m\,\omega^2} \right)$$

we find

$$- \frac{c\,\omega_1}{k - m\,\omega_1^2} = -1 \quad \text{and} \quad - \frac{c\,\omega_2}{k - m\,\omega_2^2} = 1$$

where ω_1 and ω_2 correspond to the half power points. Hence, by finding the values of ω corresponding to $\phi = -45°$ and $\phi = -135°$, we obtain ω_1 and ω_2. From these values, the damping ratio can be found using the relation:

$$\varsigma = \frac{\omega_2 - \omega_1}{2\,\omega_n}$$

10.26 10.27 10.28

Characteristic	Problem 10.26	Problem 10.27	Problem 10.28
n	16	18	18
N	750 rpm	1000 rpm	1500 rpm
d	15 mm	2 cm	10 mm
D	100 mm	15 cm	80 mm
α	30°	20°	40°
$\frac{d}{D}\cos\alpha$	0.1299	0.1253	0.09576
Dominant frequency of vibration			
Inner race defect	6779.4 cycles/min (1078.97 Hz)	10127.7 cycles/min (1611.87 Hz)	14792.8 cycles/min (2354.33 Hz)
Outer race defect	5220.6 cycles/min (830.88 Hz)	7872.3 cycles/min (1252.91 Hz)	12207.2 cycles/min (1942.84 Hz)
Ball or roller defect	1214.6 cycles/min (193.31 Hz)	1761.7 cycles/min (280.39 Hz)	2188.1 cycles/min (348.25 Hz)
Cage defect	326.3 cycles/min (51.93 Hz)	437.3 cycles/min (69.61 Hz)	678.2 cycles/min (107.93 Hz)

10.29 $f(x) = \frac{1}{4}\ ;\ 1 \le x \le 5$

$$\bar{x} = \text{mean value of } x = \int_1^5 f(x)\, x\, dx = \frac{1}{4}\left(\frac{x^2}{2}\right)_1^5 = 3$$

$$\sigma^2 = (\text{standard deviation})^2 = \int_1^5 (x-\bar{x})^2 f(x)\, dx$$

$$= \frac{1}{4}\int_1^5 (x-3)^2\, dx = \frac{1}{4}\left(\frac{x^3}{3} - 3x^2 + 9x\right)_1^5 = \frac{4}{3}$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4}\int_{-\infty}^{\infty} (x-\bar{x})^4 f(x)\, dx = \frac{9}{16}\int_1^5 (x-3)^4 \left(\frac{1}{4}\right) dx$$

Let y = x - 3 so that dy = dx. This gives $k = \frac{9}{64}\int_{y=-2}^{2} y^4\, dy = \frac{9}{5}$

10.30

$$\bar{x} = \text{mean value of } x = \sum_i f(x_i)\, x_i$$

$$= \frac{1}{32}(1) + \frac{3}{32}(2) + \frac{3}{16}(3) + \frac{6}{16}(4) + \frac{3}{16}(5) + \frac{3}{32}(6) + \frac{1}{32}(7) = 4$$

$$\sigma^2 = (\text{standard deviation})^2 = \sum_i (x_i - \bar{x})^2 f(x_i)$$

$$= (1-4)^2 \frac{1}{32} + (2-4)^2 \frac{3}{32} + (3-4)^2 \frac{3}{16} + (4-4)^2 \frac{6}{16}$$

$$+ (5-4)^2 \frac{3}{16} + (6-4)^2 \frac{3}{32} + (7-4)^2 \frac{1}{32} = \frac{27}{16} = 1.6875$$

$$\sum_i (x_i - \bar{x})^4 f(x_i) = (1-4)^4 \frac{1}{32} + (2-4)^4 \frac{3}{32} + (3-4)^4 \frac{3}{16}$$

$$+ (4-4)^4 \frac{6}{16} + (5-4)^4 \frac{3}{16} + (6-4)^4 \frac{3}{32} + (7-4)^4 \frac{1}{32} = \frac{135}{16} = 8.4375$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4} \sum_i (x_i - \bar{x})^4 f(x_i) = \frac{8.4375}{1.6875^2} = 2.9630$$

10.31

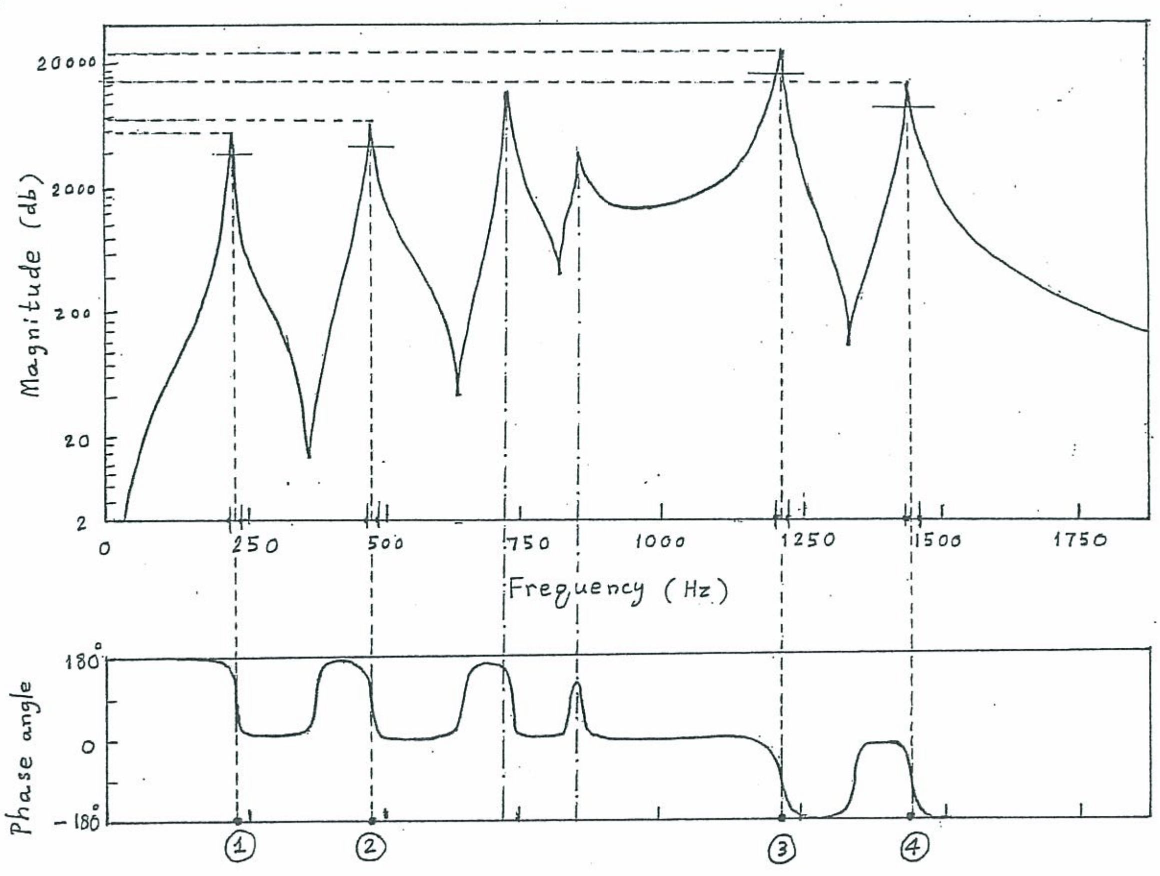


Fig. 10.46

Basic rules:

- At resonance, the magnitude will have a sharp peak.
- At resonance, phase will be 90° and the phase changes by 180° as frequency crosses the natural frequency.

Using these rules, we identify four resonant frequencies in Fig. 10.46.

Resonant frequency	Damping ratio
$\omega_1 = 228.4483$ Hz	$\zeta_1 = \dfrac{235.3448 - 219.8276}{2(228.4483)} = 0.033962$
$\omega_2 = 474.1379$ Hz	$\zeta_2 = \dfrac{482.7586 - 458.6207}{2(474.1379)} = 0.025454$
$\omega_3 = 1215.5172$ Hz	$\zeta_3 = \dfrac{1228.4482 - 1198.2758}{2(1215.5172)} = 0.012411$
$\omega_4 = 1448.27579$ Hz	$\zeta_4 = \dfrac{1453.4483 - 1431.0345}{2(1448.2759)} = 0.007738$

(10.32) Radius of circle

$$= 1.25 = \frac{1}{4\zeta}$$

$$\therefore \quad \zeta = \frac{1}{4(1.25)} = 0.2$$

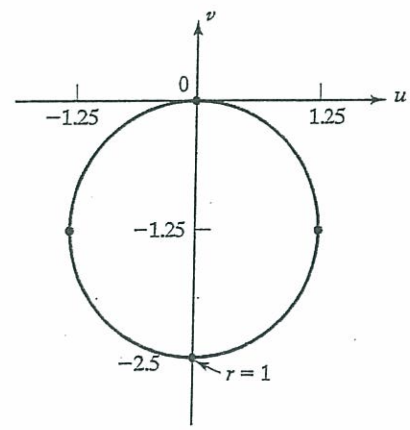


FIGURE 10.47

10.33 Range of $\omega = 62.832$ to 314.16 rad/sec = 600 to 3000 rpm

Max acceleration level = $10 g = 98.1$ m/sec^2

Max weight of specimen = 10 N

Max vibration amplitude = 0.0025 m

Frequency range:

Variable speed electric motor can be used to obtain the frequency range (for a mechanical shaker).

Vibration amplitude:

If $y(t) = A \sin \omega t$, (E$_1$)

acceleration = $A \omega^2$.

At $\omega = 314.16$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\text{amplitude } (A) = \text{acceleration}/\omega^2$$
$$= 98.1/(314.16)^2 = 0.9939 \times 10^{-3} \text{ m}$$

At $\omega = 62.832$ rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\text{amplitude } (A) = \text{acceleration}/\omega^2$$
$$= 98.1/(62.832)^2 = 0.02485 \text{ m}$$

(amplitude is too high; hence direct application of $y(t)$ is not permitted).

Mechanical shaker of the type shown in Fig. 10.18 can be used. Electrodynamic shaker of the type shown in Fig. 10.19 (a) can also be used.

Electromagnetic shaker:

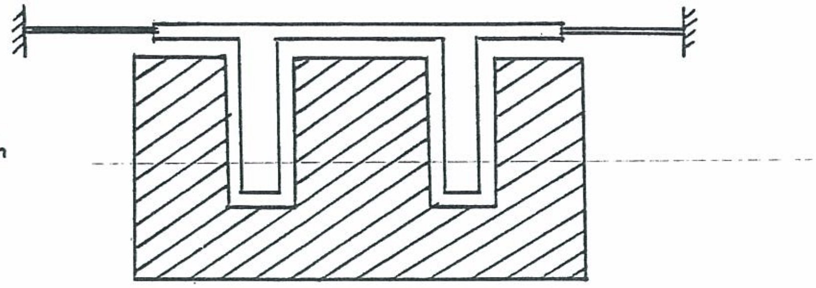
Max force (F_{max}) available depends on:

(a) magnetic field strength

(b) number of turns

(c) coil diameter

(d) current flowing



Limitations are: (a) material strength, (b) cooling provided.

Max acceleration = $\left(\dfrac{F_{max}}{m_s + m_t}\right)$

Where m_s = mass of specimen and m_t = mass of shaker table.

(10.34) Speed range: 300 - 600 rpm

Frequency range: 31.416 - 62.832 rad/sec

Number of reeds = 12

Uniform spacing of frequencies give the reed frequencies as:

$\Omega_1, \ldots, \Omega_{12} = 31.416, 34.272, 37.128, 39.984, 42.840,$
$45.696, 48.552, 51.408, 54.264, 57.120, 59.976, 62.832$ rad/sec

Let each reed be considered as a cantilever beam of cross section a x b inches. Let lengths of all reeds be same and the material be aluminum for light weight. The fundamental natural frequency of a reed is given by (Fig. 8.15):

$$\omega_1 = (\beta_1\, \ell)^2 \left\{\frac{EI}{\rho\, A\, \ell^4}\right\}^{\frac{1}{2}} = (1.8751^2) \left\{\frac{EI}{\rho\, A\, \ell^4}\right\}^{\frac{1}{2}}$$

$$= 3.516 \left\{\frac{(10^7)\, I}{(0.1/386.4)\, A\, \ell^4}\right\}^{\frac{1}{2}} = 69.1142\, (10^4) \left\{\frac{I}{A\, \ell^4}\right\}^{\frac{1}{2}} \qquad (1)$$

By equating ω_1 given by Eq. (1) to $\Omega_1, \ldots, \Omega_{12}$, in turn, the proper value of $\left\{\frac{I}{A\, \ell^4}\right\}^{\frac{1}{2}}$ needed for different reeds can be computed. By selecting a common value of ℓ for all reeds, the cross section of any reed can then be found to achieve the required value of $\left\{\frac{I}{A\, \ell^4}\right\}^{\frac{1}{2}}$.

10.35 Iterative process is to be used.

1. Select trial values of the design parameters (material of the beam and its dimensions).
2. Model the beam as a spring-mass system with:
   m = end mass = 50% of mass of beam:

$$m = \frac{1}{2}\,\rho\,A\,\ell \tag{1}$$

   k = stiffness of a cantilever beam:

$$k = \frac{3\,E\,I}{\ell^3} \tag{2}$$

3. Equations of motion:

$$m\,\ddot{x} + k\,(x - y) = 0$$

$$\text{or} \quad m\,\ddot{z} + k\,z = -\,m\,\ddot{y} \tag{3}$$

   where z = relative displacement of end mass.

4. Since $\ddot{y}_{max} = 0.2$ g, assume a constant force of - m (0.2 g) on the right hand side of Eq. (3) and solve the equation to find z(t).
5. From the known z_{max} value, compute the maximum stress (σ_{max}) induced in the beam. If σ_{max} is less than the yield stress of the material, the design is complete. Otherwise, go to step 1 and change one or more design parameters and repeat the procedure until a satisfactory design is found.