

Chapter 4

Vibration Under General Forcing Conditions

4.1 $F(t) = \frac{F_0}{\pi} + \frac{F_0}{2} \sin \omega t - \frac{2F_0}{\pi} \sum_{n=2,4,6,\dots} \frac{1}{(n^2-1)} \cos n\omega t \quad \text{where } \omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{\pi \kappa} + \frac{F_0}{2\kappa} \cdot \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi_1)$$

$$- \frac{2F_0}{\pi \kappa} \sum_{n=2,4,6,\dots} \frac{1}{(n^2-1)} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}} \cos(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$, $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$

4.2 From the solution of problem 1.109,

$$F(t) = \begin{cases} (2F_0t/\tau) & ; 0 \leq t \leq \tau/2 \\ -(2F_0t/\tau) + 2F_0 & ; \tau/2 \leq t \leq \tau \end{cases}$$

Fourier series representation of $F(t)$ is

$$F(t) = \frac{F_0}{2} - \frac{4F_0}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

$$\therefore x(t) = \frac{F_0}{2\kappa} - \frac{4F_0}{\pi^2 \kappa} \sum_{n=1,3,\dots}^{\infty} \frac{1}{n^2} \frac{\cos(n\omega t - \phi_n)}{\sqrt{(1-r^2n^2)^2 + (2\zeta nr)^2}}$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$.

4.3 $F(t) = \frac{8F_0}{\pi^2} \sum_{n=1,3,5,\dots} (-1)^{\frac{n-1}{2}} \sin \frac{n\omega t}{n^2} \quad \text{where } \omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{8F_0}{\pi^2 \kappa} \sum_{n=1,3,5,\dots} \frac{1}{n^2} (-1)^{\frac{n-1}{2}} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}} \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$

4.4 $F(t) = \frac{F_0}{2} + \frac{F_0}{\pi} \sum_{n=1,2,\dots}^{\infty} \frac{1}{n} \sin n\omega t \quad \text{where } \omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} + \frac{F_0}{\pi k} \sum_{n=1,2,\dots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \sin(n\omega t - \phi_n)$$

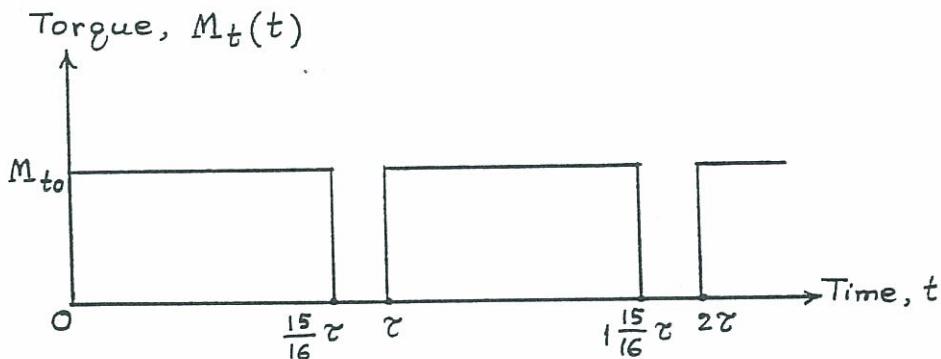
where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.5 From Example 1.19, $F(t) = \frac{F_0}{\pi} \left[\frac{\pi}{2} - \sum_{n=1,2,3,\dots}^{\infty} \frac{1}{n} \sin n\omega t \right]$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} - \frac{F_0}{\pi k} \sum_{n=1,2,3,\dots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.6



Torque transmitted to driven gear is shown in the figure. It can be expressed as:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n \omega t + b_n \sin n \omega t \right)$$

$$\text{where } \omega = 2 \pi \left(\frac{1000}{60} \right) = 104.72 \text{ rad/sec}$$

$$\tau = \frac{2 \pi}{\omega} = 0.06 \text{ sec} ; \quad \frac{15}{16} \tau = 0.05625 \text{ sec}$$

$$M_t(t) = \begin{cases} M_{t0} = 1000 \text{ N-m} ; 0 \leq t \leq \frac{15}{16} \tau \\ 0 ; \frac{15}{16} \tau \leq t \leq \tau \end{cases}$$

$$a_0 = \frac{2}{\tau} \int_0^\tau M_t(t) dt = \frac{2}{0.06} \int_0^{0.05625} (1000) dt = \frac{2000}{0.06} (0.05625) = 1875.0 \text{ N-m}$$

$$a_n = \frac{2}{\tau} \int_0^\tau M_t(t) \cos n \omega t dt = \frac{2}{\tau} M_{t0} \int_0^{0.05625} \cos n \omega t dt$$

$$= \frac{2 M_{t0}}{\tau} \left(\frac{\sin n \omega t}{n \omega} \right) \Big|_0^{0.05625} = \frac{318.3091}{n} \sin 5.8905 n \text{ N-m}$$

$$b_n = \frac{2}{\tau} \int_0^\tau M_t(t) \sin n \omega t dt = \frac{2 M_{t0}}{\tau} \int_0^{0.05625} \sin n \omega t dt \\ = \frac{2 M_{t0}}{\tau} \left(-\frac{\cos n \omega t}{n \omega} \right)_0^{0.05625} = \frac{318.3091}{n} (1 - \cos 5.8905 n) N-m$$

$$k_t = \frac{GJ}{\ell} = G \left(\frac{\pi d^4}{4} \right) \frac{1}{\ell} = (80 (10^9)) \left(\frac{\pi (0.05)^4}{4} \right) \frac{1}{1} = 392,700 \text{ N-m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{392700}{0.1}} = 1981.6660 \text{ rad/sec}$$

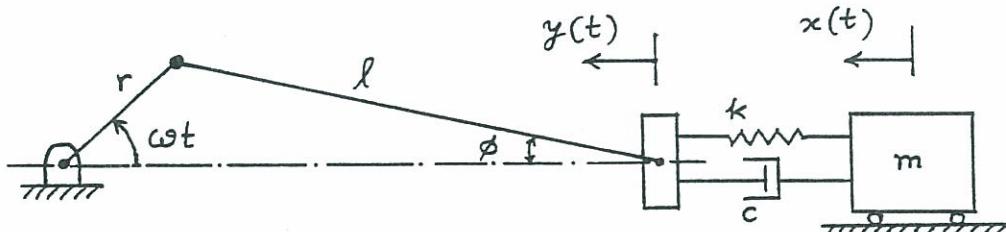
Equation of motion:

$$J_0 \ddot{\theta} + k_t \theta = M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n \omega t + b_n \sin n \omega t)$$

Response:

$$\theta(t) = \frac{a_0}{2 k_t} + \sum_{n=1}^{\infty} \left\{ \frac{a_n \cos n \omega t + b_n \sin n \omega t}{k_t - J_0 (n \omega)^2} \right\} \\ = 0.0023873 + \sum_{n=1}^{\infty} \left\{ \frac{318.3091 \sin 5.8905 n \cos \omega t + 318.3091 (1 - \cos 5.8905 n) \sin n \omega t}{n (392700.0 - 1096.6278 n^2)} \right\} \text{ rad}$$

4.7



Base motion is given by:

$$y(t) = r + l - r \cos \omega t - l \cos \phi = r + l - r \cos \omega t - l \sqrt{1 - \sin^2 \phi} \quad (1)$$

Using $l \sin \phi = r \sin \omega t$, Eq. (1) becomes

$$y(t) = r + l - r \cos \omega t - l \sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t} \quad (2)$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{l^2} \sin^2 \omega t} \approx 1 - \frac{r^2}{2 l^2} \sin^2 \omega t \quad (3)$$

Eq. (2) can be expressed as

$$y(t) = r + l - r \cos \omega t - l \left(1 - \frac{1}{2} \frac{r^2}{l^2} \sin^2 \omega t \right) \\ = r - r \cos \omega t + \frac{l}{4} \left(\frac{r}{l} \right)^2 - \frac{l}{4} \left(\frac{r}{l} \right)^2 \cos 2 \omega t \quad (4)$$

Equation of motion:

$$\begin{aligned}
 m \ddot{x} + c \dot{x} + k x &= k y + c \dot{y} \\
 = k r - k r \cos \omega t + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \dots \\
 + c r \omega \sin \omega t + \frac{c \ell}{4} \left(\frac{r}{\ell} \right)^2 (2 \omega) \sin 2 \omega t + \dots
 \end{aligned} \tag{5}$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$\begin{aligned}
 x(t) &= \frac{F_0}{k} \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \right] \\
 \text{where } \phi &= \tan^{-1} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \right)
 \end{aligned} \tag{6}$$

Solution due to sinusoidal term, $F_0 \sin \Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X \sin(\Omega t - \phi_0) \tag{7}$$

$$\text{where } X = \frac{F_0}{\left[(k - m \Omega^2)^2 + c^2 \Omega^2 \right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1} \left(\frac{c \Omega}{k - m \Omega^2} \right) \tag{8}$$

Solution due to cosine term, $F_0 \cos \Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X \cos(\Omega t - \phi_0) \tag{9}$$

$$\text{where } X = \frac{F_0}{\left[(k - m \Omega^2)^2 + c^2 \Omega^2 \right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1} \left(\frac{c \Omega}{k - m \Omega^2} \right) \tag{10}$$

For given data, $\zeta = \frac{c}{2 \sqrt{m k}} = \frac{10}{2 \sqrt{1 (100)}} = 0.5$, $\frac{r}{\ell} = 0.1$, $\omega = 100$, $2 \omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

4.8

Base motion can be represented by Fourier series as (from Example 1.19):

$$y(t) = \frac{Y}{\pi} \left[\frac{\pi}{2} - \left\{ \sin \omega t + \frac{1}{2} \sin 2 \omega t + \frac{1}{3} \sin 3 \omega t + \dots \right\} \right] \tag{1}$$

Equation of motion of mass:

$$m \ddot{x} + c (\dot{x} - \dot{y}) + k (x - y) = 0 \quad (2)$$

Since $y(t)$ is composed of several terms, the solution of Eq. (2) can be found by superposing the solutions corresponding to each of the terms appearing in Eq. (1). When $y(t) = Y/2$, constant, equation of motion becomes:

$$m \ddot{x} + c \dot{x} + k x = \frac{k Y}{2} = \text{constant} \quad (3)$$

The steady state solution of Eq. (3) is given by (see Example 4.9):

$$x(t) = \frac{Y}{2} \left[1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \right] \quad (4)$$

$$\text{where } \phi = \tan^{-1} \left(\frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \quad (5)$$

When $y(t) = A \sin \Omega t$, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin(\Omega t - \phi) \quad (6)$$

$$\text{where } A = \left[\frac{1 + (2 \zeta r)^2}{(1 - r^2)^2 + (2 \zeta r)^2} \right]^{\frac{1}{2}} \quad (7)$$

$$\phi = \tan^{-1} \left(\frac{2 \zeta r^3}{1 + r^2 (4 \zeta^2 - 1)} \right) \quad (8)$$

$$\text{and } r = \frac{\Omega}{\omega_n}$$

4.9

From solution of Problem 4.7, we can express the base motion as:

$$y(t) = r - r \cos \omega t + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \dots \quad (1)$$

Equation of motion:

$$m \ddot{x} + k (x - y) \pm \mu N = 0$$

or

$$\begin{aligned} & m \ddot{x} + k x \pm \mu N = k y \\ &= \left\{ k r + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \right\} - k r \cos \omega t - \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \dots \end{aligned} \quad (2)$$

For given numerical data, Eq. (2) becomes:

$$\ddot{x} + 100x \pm 0.981 = 100y$$

$$= \left\{ 100(0.1) + \frac{100(1)}{4} \left(\frac{0.1}{1} \right)^2 \right\} - (100)(0.1) \cos 100t - \frac{100(1)}{4} \left(\frac{0.1}{1} \right)^2 \cos 200t - \dots \\ = 10.25 - 10 \cos 100t - 0.25 \cos 200t - \dots \quad (3)$$

Using the definition of equivalent damping constant, c_{eq} , the solution of Eq. (3) can be found by superposition.

$$c_{eq} = \frac{4\mu N}{\pi \Omega X} = \frac{4\mu m g}{\pi \Omega X} \quad (4)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass.

Steady state solution due to constant term, F_0 , on the r.h.s. of Eq. (3) (from Example 4.9):

$$x(t) = \frac{F_0}{k} = \left\{ k r + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \right\} = r + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \quad (5)$$

Steady state solution due to harmonic term, $F_0 \cos \Omega t$, on the r.h.s. of Eq. (3) (from Eqs. (3.89), (3.93) and (3.96)):

$$x(t) = X \cos(\Omega t - \phi) \quad (6)$$

$$\text{where } X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4\mu N}{\pi F_0} \right)^2}{\left(1 - \left(\frac{\Omega^2}{\omega_n^2} \right)^2 \right)} \right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1} \left\{ \frac{\pm \frac{4\mu N}{\pi F_0}}{\left[1 - \left(\frac{4\mu N}{\pi F_0} \right)^2 \right]^{\frac{1}{2}}} \right\} \quad (7)$$

$N = m g = 1 (9.81) = 9.81$ Newtons, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10$ rad/sec, and + sign is to be used in Eq. (7) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (5) and (6) can be superposed suitably to find the complete steady state solution of Eq. (3).

4.10

Base motion can be represented by Fourier series as (see solution of Problem 4.8 or Example 1.19):

$$y(t) = \frac{Y}{\pi} \left\{ \frac{\pi}{2} - \left(\sin \omega t + \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t + \dots \right) \right\} \quad (1)$$

Equation of motion of mass:

$$m \ddot{x} + k((x - y) \pm \mu N) = 0$$

or

$$m \ddot{x} + kx \pm \mu N = k y \\ = \frac{k Y}{\pi} \left[\frac{\pi}{2} - \left\{ \sin \omega t + \frac{1}{2} \sin 2 \omega t + \frac{1}{3} \sin 3 \omega t + \dots \right\} \right] \quad (2)$$

Using the definition of equivalent viscous damping constant:

$$c_{eq} = \frac{4 \mu N}{\pi \Omega X} = \frac{4 \mu m g}{\pi \Omega X} \quad (3)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass, the solution of Eq. (2) can be determined using the superposition principle.

Steady state solution due to constant term, F_0 , on the r.h.s. of Eq. (2) (from Example 4.9):

$$x(t) = \frac{F_0}{k} = \frac{k Y}{2k} = \frac{Y}{2} \quad (4)$$

Steady state solution due to harmonic term, $F_0 \sin \Omega t$, on the r.h.s. of Eq. (2) (from Eqs. (3.89), (3.93) and (3.96)):

$$x(t) = X \sin (\Omega t - \phi)$$

where $X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2}{\left(1 - \left(\frac{\Omega^2}{\omega_n^2} \right)^2 \right)} \right]^{\frac{1}{2}}$ and $\phi = \tan^{-1} \left\{ \frac{\pm \frac{4 \mu N}{\pi F_0}}{\left[1 - \left(\frac{4 \mu N}{\pi F_0} \right)^2 \right]^{\frac{1}{2}}} \right\}$ (6)

and + sign is to be used in Eq. (6) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (4) and (5) can be superposed suitably to find the complete steady state solution of Eq. (2).

4.11

Base excitation: $y(t) = Y \cos \omega t$ (E.1)
with $Y = 0.05$ and $\omega = 5$

Equation of motion of the system:

$$m \ddot{x} + c \dot{x} + kx = ky + c \dot{y} = kY \cos \omega t - c \omega Y \sin \omega t \quad (E.2)$$

Equation (E.2) is similar to Eq. (4.8) with

$\omega_0 = 0$, $\omega_i = kY$, $b_1 = -c \omega Y$ and $a_i = b_i = 0$; $i = 2, 3, \dots$
steady state response of the system is given by

Eq. (E.9) of Example 4.9:

$$x_p(t) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \left\{ \frac{\omega_1}{\kappa} \cos(\omega t - \phi_1) + \frac{b_1}{\kappa} \sin(\omega t - \phi_1) \right\}$$

Here $\omega_n = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$

$$r = \frac{\omega}{\omega_n} = \frac{5}{20} = 0.25, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{\kappa m}} = \frac{20}{2\sqrt{4000(10)}} = 0.05$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = 19.975 \text{ rad/s}$$

$$\omega_1 = \kappa Y = 4000(0.05) = 200, \quad b_1 = -c\omega Y = -20(5)(0.05) = -5$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.05)(0.25)}{1-0.25^2}\right) = 0.02666 \text{ rad}$$

$$\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \sqrt{(1-0.25^2)^2 + (2*0.05*0.25)^2} \\ = 0.937833$$

Solution of the homogeneous equation is given by
Eq. (2.70):

$$x_h(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) \\ = X_0 e^{-t} \cos(19.975 t - \phi_0) \quad (E.3)$$

Where X_0 and ϕ_0 are unknown constants.

Total solution :

$$x(t) = x_h(t) + x_p(t) = X_0 e^{-t} \cos(19.975 t - \phi_0) \\ + \frac{1}{0.937833} \left\{ \frac{200}{4000} \cos(5t - 0.02666) - \frac{5}{4000} \sin(5t - 0.02666) \right\} \\ = X_0 e^{-t} \cos(19.975 t - \phi_0) + 0.053314 \cos(5t - 0.02666) \\ - 0.001333 \sin(5t - 0.02666) \quad (E.4)$$

where X_0 and ϕ_0 are to be found from the initial conditions. Differentiation of Eq. (E.4) gives the velocity as :

$$\dot{x}(t) = -X_0 e^{-t} \cos(19.975 t - \phi_0) \\ - X_0 (19.975) e^{-t} \sin(19.975 t - \phi_0)$$

$$\begin{aligned} & -0.26657 \sin(5t - 0.02666) \\ & -0.006665 \cos(5t - 0.02666) \end{aligned} \quad (E \cdot 5)$$

Using the initial conditions, we obtain

$$\begin{aligned} x_0 = x(t=0) = 0.1 = & X_0 \cos \phi_0 + 0.053314 \cos(0.02666) \\ & + 0.001333 \sin(0.02666) \end{aligned}$$

or

$$X_0 \cos \phi_0 = 0.04666947 \quad (E \cdot 6)$$

and

$$\begin{aligned} \dot{x}_0 = \dot{x}(t=0) = 1.0 = & -X_0 \cos \phi_0 + 19.975 X_0 \sin \phi_0 \\ & + 0.26657 \sin(0.02666) - 0.006665 \cos(0.02666) \end{aligned}$$

or

$$X_0 \sin \phi_0 = 0.05237678 \quad (E \cdot 7)$$

Solution of Eqs. (E·6) and (E·7) gives

$$X_0 = \left\{ (X_0 \cos \phi_0)^2 + (X_0 \sin \phi_0)^2 \right\}^{\frac{1}{2}} = 0.07015247 \quad (E \cdot 8)$$

$$\phi_0 = \tan^{-1} \left(\frac{X_0 \sin \phi_0}{X_0 \cos \phi_0} \right) = 0.84295916 \text{ rad} \quad (E \cdot 9)$$

Complete solution is given by Eq. (E·4):

$$\begin{aligned} x(t) = & 0.07015247 e^{-t} \cos(19.975 t - 0.84295916) \\ & + 0.053314 \cos(5t - 0.02666) \\ & - 0.001333 \sin(5t - 0.02666) \end{aligned} \quad (E \cdot 10)$$

4.13

From solution of problem 1.117,

$$F(t) = 9.9584 - 20.1587 \cos 10.472t + 23.5253 \sin 10.472t \\ + 3.3099 \cos 20.944t + 12.2646 \sin 20.944t \\ + 3.7719 \cos 31.416t - 0.4064 \sin 31.416t \quad (E1)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000}{1}} = 122.4745 \text{ rad/sec} ; \quad \zeta = 0.1 ; \quad r = 0.0855$$

The steady state solution is given by Eq. (E.9) of Example 4.9:

$$\begin{aligned}
x_p(t) = & \frac{9.9584}{\kappa} - \frac{20.1587}{\kappa} \frac{1}{\sqrt{(1-r^2)^2 + (25r)^2}} \cos(10.472t - \phi_1) \\
& + \frac{23.5253}{\kappa} \frac{1}{\sqrt{(1-r^2)^2 + (25r)^2}} \sin(10.472t - \phi_1) \\
& + \frac{3.3099}{\kappa} \frac{1}{\sqrt{(1-4r^2)^2 + (45r)^2}} \cos(20.944t - \phi_2) \\
& + \frac{12.2646}{\kappa} \frac{1}{\sqrt{(1-4r^2)^2 + (45r)^2}} \sin(20.944t - \phi_2) \\
& + \frac{3.7719}{\kappa} \frac{1}{\sqrt{(1-9r^2)^2 + (65r)^2}} \cos(31.416t - \phi_3) \\
& - \frac{0.4064}{\kappa} \frac{1}{\sqrt{(1-9r^2)^2 + (65r)^2}} \sin(31.416t - \phi_3) \quad (E_2)
\end{aligned}$$

where $\phi_1 = \tan^{-1}\left(\frac{25r}{1-r^2}\right) = \tan^{-1}(0.017226) = 0.0172 \text{ rad}$

$$\phi_2 = \tan^{-1}\left(\frac{45r}{1-4r^2}\right) = \tan^{-1}(0.035229) = 0.0352 \text{ rad}$$

$$\phi_3 = \tan^{-1}\left(\frac{65r}{1-9r^2}\right) = \tan^{-1}(0.054913) = 0.0549 \text{ rad}$$

Noting that $2\pi r = 2(0.1)(0.0855) = 0.0171$

and $(1-r^2) = 0.9927$, Eq. (E₂) can be rewritten as

$$\begin{aligned}
x_p(t) = & [6.6389 - 13.7821 \cos(10.472t - 0.0172) \\
& + 15.7965 \sin(10.472t - 0.0172) + 2.2715 \cos(20.944t - 0.0352) \\
& + 8.4168 \sin(20.944t - 0.0352) + 2.6876 \cos(31.416t - 0.0549) \\
& - 0.2896 \sin(31.416t - 0.0549)] \times 10^{-4} \text{ m.}
\end{aligned}$$

From problem 1.116,

$$\begin{aligned}
4.14 \quad F(t) = & 1137.5 - 414.9436 \cos 523.6t + 150.3139 \sin 523.6t \\
& + 28.6058 \cos 1047.2t - 146.1706 \sin 1047.2t \\
& + 35.7278 \cos 1570.8t + 55.1546 \sin 1570.8t \quad (E_1)
\end{aligned}$$

$$\omega_n = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{8000}{0.5}} = 126.4911 \text{ rad/sec} ; \quad \varsigma = 0.06$$

$$r = \frac{\omega}{\omega_n} = \frac{523.6}{126.4911} = 4.1394 ; \quad r^2 = 17.1348 ; \quad 1-r^2 = -16.1348$$

$$2\pi r = 2(0.06)(4.1394) = 0.4967$$

$$\begin{aligned}
x_p(t) = & \frac{1137.5}{\kappa} - \frac{414.9436}{\kappa} \frac{\cos(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (25r)^2}} \\
& + \frac{150.3139}{\kappa} \frac{\sin(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (25r)^2}} + \frac{28.6058}{\kappa} \frac{\cos(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (45r)^2}} \\
& - \frac{146.1706}{\kappa} \frac{\sin(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (45r)^2}} + \frac{35.7278}{\kappa} \frac{\cos(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (65r)^2}} \\
& + \frac{55.1546}{\kappa} \frac{\sin(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (65r)^2}}
\end{aligned} \tag{E2}$$

Since $\phi_1 = \tan^{-1}\left(\frac{25r}{1-r^2}\right) = \tan^{-1}(-0.0308) = -0.0308$

$\phi_2 = \tan^{-1}\left(\frac{45r}{1-4r^2}\right) = \tan^{-1}(-0.0147) = -0.0147$

$\phi_3 = \tan^{-1}\left(\frac{65r}{1-9r^2}\right) = \tan^{-1}(-0.00973) = -0.00973$

the steady-state solution, Eq. (E2), can be expressed as

$$\begin{aligned}
x_p(t) = & 0.1422 - 0.003211 \cos(523.6t + 0.0308) \\
& + 0.001163 \sin(523.6t + 0.0308) + 5.2921 \times 10^{-5} \cos(1047.2t + 0.0147) \\
& - 2.7042 \times 10^{-4} \sin(1047.2t + 0.0147) + 2.9145 \times 10^{-5} \cos(1570.8t + 0.00973) \\
& + 4.4992 \times 10^{-5} \sin(1570.8t + 0.00973) \quad m
\end{aligned}$$

4.15

$$\omega_n = \sqrt{\frac{5 \times 10^6}{10 \times 10^3}} = 22.3607 \frac{\text{rad}}{\text{sec}}$$

$$\tau = 0.15 \text{ sec}$$

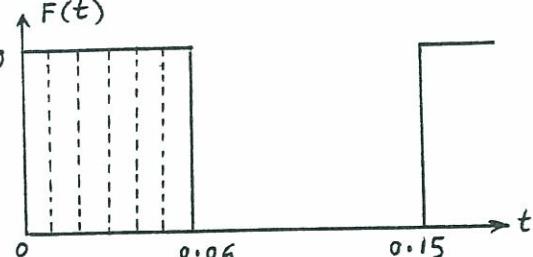
$$\omega = \frac{2\pi}{\tau} = 41.888 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = 1.8733, \quad r^2 = 3.5093$$

$$\zeta = 0$$

Following data is used in Program 1.F to find Fourier coefficients in the expansion of $F(t)$:

$t, \text{ sec}$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	...	0.15
$F(t), \text{ } \times N$	400	400	400	400	400	400	0	...	0



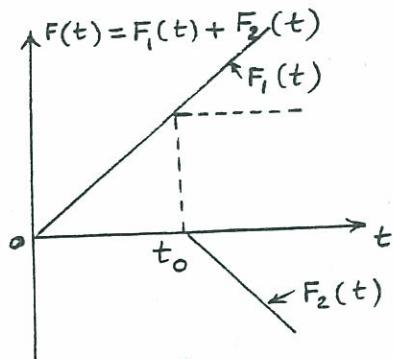
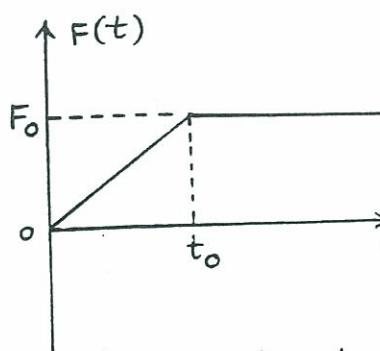
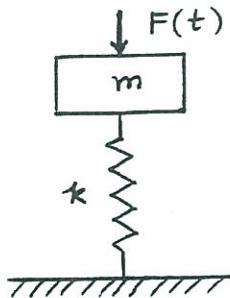
Result is:

$$\begin{aligned}
 F(t) = & 160.0 + 25.5002 \cos 41.888t + 242.6276 \sin 41.888t \\
 & - 75.3884 \cos 83.776t + 16.0237 \sin 83.776t \\
 & + 16.4806 \cos 125.664t + 50.7237 \sin 125.664t \\
 & - 62.3538 \cos 167.552t + 27.7604 \sin 167.552t \\
 & + \dots \quad KN
 \end{aligned}$$

Since $\zeta = 0$ and all $\phi_j = 0$, $j = 1, 2, \dots$, the steady-state response of the water tank, Eq. (4.13), becomes

$$\begin{aligned}
 x_p(t) = & 0.032 + 2.0325 \times 10^{-3} \cos 41.888t \\
 & + 19.3383 \times 10^{-3} \sin 41.888t - 1.1566 \times 10^{-3} \cos 83.776t \\
 & + 0.2458 \times 10^{-3} \sin 83.776t + 0.1078 \times 10^{-3} \cos 125.664t \\
 & + 0.3317 \times 10^{-3} \sin 125.664t - 0.2334 \times 10^{-3} \cos 167.552t \\
 & + 0.1007 \times 10^{-3} \sin 167.552t + \dots \quad m
 \end{aligned}$$

4.16



Forcing function can be considered as the sum of two ramp functions, $F_1(t) = \frac{F_0 t}{t_0}$ and $F_2(t) = -\frac{F_0(t-t_0)}{t_0}$.

Response of the casting (undamped spring-mass system) to F_1 is given by

$$x_1(t) = \frac{F_0}{k} \left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0} \right) \quad \text{for } t \geq 0 \quad (E_1)$$

Response due to F_2 can be obtained from Eq. (E1) by replacing t by $t-t_0$ and F_0 by $-F_0$:

$$x_2(t) = -\frac{F_0}{k} \left\{ \frac{t-t_0}{t_0} - \frac{\sin \omega_n(t-t_0)}{\omega_n t_0} \right\} \quad \text{for } t \geq t_0 \quad (E_2)$$

Total response of the casting is given by

$$x(t) = x_1(t) + x_2(t) = \frac{F_0}{k} \left\{ 1 + \frac{\sin \omega_n(t-t_0) - \sin \omega_n t}{\omega_n t_0} \right\}$$

for $t \geq t_0$ (E3)

4.17 From Eq. (4.31), $x(t) = \frac{1}{m\omega_d} \int_0^t F_0 e^{-\alpha\tau} e^{-\zeta\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \sin \omega_d(t-\tau) d\tau$$

$$\begin{aligned} x(t) &= \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \left\{ \sin \omega_d t \cdot \cos \omega_d \tau - \cos \omega_d t \cdot \sin \omega_d \tau \right\} d\tau \\ &= \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d} \sin \omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2 + \omega_d^2} \left\{ -(\alpha-\zeta\omega_n) \cos \omega_d \tau + \omega_d \sin \omega_d \tau \right\} \right]_0^t \\ &\quad - \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d} \cos \omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2 + \omega_d^2} \left\{ -(\alpha-\zeta\omega_n) \sin \omega_d \tau - \omega_d \cos \omega_d \tau \right\} \right]_0^t \end{aligned}$$

$$= \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d [(\alpha-\zeta\omega_n)^2 + \omega_d^2]} \left\{ \omega_d e^{-(\alpha-\zeta\omega_n)t} + (\alpha-\zeta\omega_n) \sin \omega_d t - \omega_d \cos \omega_d t \right\}$$

$$= \frac{F_0 e^{-\alpha t}}{m [(\alpha-\zeta\omega_n)^2 + \omega_d^2]} + \frac{F_0 e^{-\zeta\omega_n t}}{m\omega_d [(\alpha-\zeta\omega_n)^2 + \omega_d^2]} \sin(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\omega_d}{\alpha - \zeta\omega_n}\right)$.

4.18 Equation of motion:

$$m \ddot{x} + k x = A p(t)$$

$$\text{or } 10 \ddot{x} + 1000 x = \frac{\pi}{4} (0.1)^2 (50) (1 - e^{-3t}) = 0.3927 - 0.3927 e^{-3t}$$

Solution:

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{0.3927}{k} - \frac{0.3927}{k + m (3^2)} e^{-3t}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/sec. Thus $x(t)$ becomes

$$x(t) = C_1 \cos 10t + C_2 \sin 10t + 39.27 (10^{-5}) - 36.0275 (10^{-5}) e^{-3t} m$$

where C_1 and C_2 can be determined from the initial conditions. As $t \rightarrow \infty$, $e^{-3t} \rightarrow 0$ and the steady state response becomes

$$x(t) = C_1 \cos 10t + C_2 \sin 10t + 39.27 (10^{-5}) m$$

4.19

For $0 \leq t \leq \frac{\pi}{\omega}$: Equation of motion: $m\ddot{x} + kx = \frac{F_0}{2} - \frac{F_0}{2} \cos \omega t$

$$x(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{F_0}{2k} - \frac{F_0}{2k(1 - \frac{\omega^2}{\omega_n^2})} \cos \omega t$$

$$x(0) = 0 \quad \therefore A + \frac{F_0}{2k} - \frac{F_0}{2k(1 - \frac{\omega^2}{\omega_n^2})}, \quad A = \frac{F_0 (\frac{\omega}{\omega_n})^2}{2k \left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right\}}$$

$$\dot{x}(0) = 0 \quad \therefore B = 0$$

$$x(t) = \frac{F_0}{2k \left\{ 1 - \left(\frac{\omega}{\omega_n} \right)^2 \right\}} \left[1 - \cos \omega t - \left(\frac{\omega}{\omega_n} \right)^2 (1 - \cos \omega_n t) \right]$$

$$\text{At } t = \frac{\pi}{\omega}, \quad x(\frac{\pi}{\omega}) = \frac{F_0}{2k \left(1 - \frac{\omega^2}{\omega_n^2} \right)} \left[2 - \frac{\omega^2}{\omega_n^2} \left(1 - \cos \frac{\omega_n \pi}{\omega} \right) \right]$$

For $t > \frac{\pi}{\omega}$: Eq. (4.31) gives $x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau$

$$x(t) = \frac{1}{m\omega_n} \int_0^{\pi/\omega} F(\tau) \cdot \sin \omega_n(t-\tau) d\tau + \frac{1}{m\omega_n} \int_{\pi/\omega}^t F(\tau) \sin \omega_n(t-\tau) d\tau$$

$$= x\left(t = \frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n} \int_{\pi/\omega}^t \sin \omega_n(t-\tau) d\tau$$

$$= x\left(t = \frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n^2} \left[\cos \omega_n(t-\tau) \right]_{\tau=\pi/\omega}^t$$

$$= \frac{F_0}{2k \left(1 - \frac{\omega^2}{\omega_n^2} \right)} \left[2 - \frac{\omega^2}{\omega_n^2} \left(1 - \cos \frac{\omega_n \pi}{\omega} \right) \right] + \frac{F_0}{k} \left[1 - \cos \omega_n(t - \frac{\pi}{\omega}) \right]$$

4.20

For an undamped system Eq. (4.31) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau$$

$$F(\tau) = \begin{cases} F_0 & \text{for } 0 \leq \tau \leq t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

$$\text{For } 0 \leq t \leq t_0 : \quad x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n(t-\tau) d\tau = \frac{F_0}{m\omega_n^2} (1 - \cos \omega_n t)$$

$$= \frac{F_0}{\kappa} (1 - \cos \omega_n t)$$

$$\text{For } t > t_0 : \quad x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin \omega_n(t-\tau) d\tau = \frac{F_0}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{\tau=0}^{t_0}$$

$$= \frac{F_0}{\kappa} [\cos \omega_n(t-t_0) - \cos \omega_n t]$$

$$4.21 \quad F(\tau) = \begin{cases} F_0 \left(\frac{\tau}{t_0} \right) & \text{for } 0 \leq \tau \leq t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

$$\text{For } 0 \leq t \leq t_0 : \quad x(t) = \frac{F_0}{m\omega_n t_0} \int_0^t \tau \sin \omega_n(t-\tau) d\tau$$

$$\text{i.e., } x(t) = \frac{F_0}{m\omega_n t_0} \left[\int_0^t (t-\tau) \sin \omega_n(t-\tau) (-d\tau) - t \int_0^t \sin \omega_n(t-\tau) (-d\tau) \right]$$

$$= \frac{F_0}{m\omega_n t_0} \left[\frac{1}{\omega_n^2} \sin \omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n(t-\tau) \right]_{\tau=0}^t$$

$$+ \frac{F_0 t}{m\omega_n t_0} \left[\frac{\cos \omega_n(t-\tau)}{\omega_n} \right]_{\tau=0}^t$$

$$= \frac{F_0}{\kappa t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n(t-\tau) d\tau$$

$$= \frac{F_0}{m\omega_n t_0} \left[\frac{1}{\omega_n^2} \sin \omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n(t-\tau) \right]_{\tau=0}^t$$

$$+ \frac{F_0 t}{m\omega_n t_0} \left[\frac{\cos \omega_n(t-\tau)}{\omega_n} \right]_{\tau=0}^t$$

$$= \frac{F_0}{\kappa t_0} \left[\frac{1}{\omega_n} \sin \omega_n(t-t_0) + t_0 \cos \omega_n(t-t_0) - \frac{1}{\omega_n} \sin \omega_n t \right]$$

$$4.22 \quad F(t) = \begin{cases} F_0 \left(1 - \cos \frac{\pi t}{2t_0} \right) & ; \quad 0 \leq t \leq t_0 \\ 0 & ; \quad t > t_0 \end{cases} \quad (E_1)$$

For an undamped system, Eq. (4.31) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau$$

For $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \left(1 - \cos \frac{\pi\tau}{2t_0}\right) \sin \omega_n(t-\tau) d\tau \quad (E_2)$$

Noting that

$$\int_0^t \sin(\omega_n t - \omega_n \tau) d\tau = \left[\frac{1}{\omega_n} \cos(\omega_n t - \omega_n \tau) \right]_0^t = \frac{1}{\omega_n} (1 - \cos \omega_n t)$$

and

$$\begin{aligned} & \int_0^t \cos \frac{\pi\tau}{2t_0} (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau) d\tau \\ &= \sin \omega_n t \left[\frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right]_0^t \\ & - \cos \omega_n t \left[- \frac{\cos \left(\omega_n - \frac{\pi}{2t_0} \right) \tau}{2 \left(\omega_n - \frac{\pi}{2t_0} \right)} - \frac{\cos \left(\omega_n + \frac{\pi}{2t_0} \right) \tau}{2 \left(\omega_n + \frac{\pi}{2t_0} \right)} \right]_0^t \\ &= \frac{-1}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} \left[\cos \frac{\pi t}{2t_0} - \cos \omega_n t \right] + \frac{1}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \left[\cos \frac{\pi t}{2t_0} - \cos \omega_n t \right], \end{aligned}$$

Eq. (E₂) can be simplified as

$$x(t) = \frac{F_0}{m} (1 - \cos \omega_n t) + \frac{(F_0/m)}{\left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \left[\cos \frac{\pi t}{2t_0} - \cos \omega_n t \right] \quad (E_3)$$

For $t > t_0$:

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_n} \int_0^{t_0} \left(1 - \cos \frac{\pi\tau}{2t_0}\right) \sin \omega_n(t-\tau) d\tau \quad (E_4) \\ &= \frac{F_0}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-t_0) - \frac{1}{\omega_n} \cos \omega_n t \right] - \frac{F_0}{m\omega_n} \left\{ \sin \omega_n t * \right. \\ & \left[\frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) t_0}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) t_0}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right] + \cos \omega_n t * \\ & \left. \left[\frac{\cos \left(\frac{\pi}{2t_0} - \omega_n \right) t_0}{2 \left(\omega_n - \frac{\pi}{2t_0} \right)} + \frac{\cos \left(\frac{\pi}{2t_0} + \omega_n \right) t_0}{2 \left(\omega_n + \frac{\pi}{2t_0} \right)} - \frac{1}{2 \left(\omega_n - \frac{\pi}{2t_0} \right)} - \frac{1}{2 \left(\omega_n + \frac{\pi}{2t_0} \right)} \right] \right\} \end{aligned}$$

$$\text{i.e., } x(t) = \frac{F_0}{m} \left[\cos \omega_n(t-t_0) - \cos \omega_n t \right]$$

$$-\frac{F_0}{2m\omega_n\left(\frac{\pi}{2t_0} - \omega_n\right)} \sin \omega_n(t-t_0) - \frac{F_0}{2m\omega_n\left(\frac{\pi}{2t_0} + \omega_n\right)} \sin \omega_n(t-t_0) \\ + \frac{F_0}{2m\omega_n} \cos \omega_n t \cdot \left\{ -\frac{1}{\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{1}{\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\} \quad (E_5)$$

4.23 Base displacement = $y(s) = Y \sin \frac{\pi s}{s}$ (E₁)

i.e., $y(t) = Y \sin \frac{\pi t}{t_0}$ (E₂)

steady-state relative displacement can be found from Eq. (4.34) as

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau) e^{-\zeta \omega_n (t-\tau)} \sin \omega_d(t-\tau) d\tau \quad (E_3)$$

where $\ddot{y}(\tau) = -Y \left(\frac{\pi}{t_0}\right)^2 \sin \frac{\pi \tau}{t_0}$ (E₄)

$$z(t) = \frac{Y}{\omega_d} \left(\frac{\pi}{t_0}\right)^2 \int_0^t e^{-\zeta \omega_n t} e^{\zeta \omega_n \tau} \sin \frac{\pi \tau}{t_0} \sin \omega_d(t-\tau) d\tau \quad (E_5)$$

But $\sin \frac{\pi \tau}{t_0} \sin \omega_d(t-\tau) = \frac{1}{2} \cos\left(\frac{\pi \tau}{t_0} - \omega_d t + \omega_d \tau\right) - \frac{1}{2} \cos\left(\frac{\pi \tau}{t_0} + \omega_d t - \omega_d \tau\right)$
 $= \frac{1}{2} \left[\cos\left(\frac{\pi}{t_0} + \omega_d\right) \tau \cdot \cos \omega_d t + \sin\left(\frac{\pi}{t_0} + \omega_d\right) \tau \cdot \sin \omega_d t \right]$
 $- \frac{1}{2} \left[\cos\left(\frac{\pi}{t_0} - \omega_d\right) \tau \cdot \cos \omega_d t - \sin\left(\frac{\pi}{t_0} - \omega_d\right) \tau \cdot \sin \omega_d t \right] \quad (E_6)$

Eqs. (E₅) and (E₆) give

$$z(t) = \frac{1}{2} \frac{Y}{\omega_d} \left(\frac{\pi}{t_0}\right)^2 e^{-\zeta \omega_n t} \left[\cos \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cos\left(\frac{\pi}{t_0} + \omega_d\right) \tau \cdot d\tau \right. \\ + \sin \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cdot \sin\left(\frac{\pi}{t_0} + \omega_d\right) \tau \cdot d\tau \\ - \cos \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cdot \cos\left(\frac{\pi}{t_0} - \omega_d\right) \tau \cdot d\tau \\ \left. + \sin \omega_d t \cdot \int_0^t e^{\zeta \omega_n \tau} \cdot \sin\left(\frac{\pi}{t_0} - \omega_d\right) \tau \cdot d\tau \right] \quad (E_7)$$

Eq. (E₇) can be simplified

$$z(t) = \frac{Y}{2\omega_d} \left(\frac{\pi}{t_0}\right)^2 \frac{1}{(\zeta \omega_n)^2 + (\frac{\pi}{t_0} + \omega_d)^2} \left\{ \zeta \omega_n \cos \frac{\pi t}{t_0} \right. \\ + \left(\frac{\pi}{t_0} + \omega_d \right) \sin \frac{\pi t}{t_0} - \zeta \omega_n e^{-\zeta \omega_n t} \cos \omega_d t \\ + \left(\frac{\pi}{t_0} + \omega_d \right) e^{-\zeta \omega_n t} \sin \omega_d t \left. \right\} \\ + \frac{Y}{2\omega_d} \left(\frac{\pi}{t_0}\right)^2 \frac{1}{(\zeta \omega_n)^2 + (\frac{\pi}{t_0} - \omega_d)^2} \left\{ -\zeta \omega_n \cos \frac{\pi t}{t_0} \right.$$

$$-\left(\frac{\pi}{t_0} - \omega_d\right) \sin \frac{\pi t}{t_0} + \zeta \omega_n e^{-\zeta \omega_n t} \cos \omega_d t \\ + \left(\frac{\pi}{t_0} - \omega_d\right) e^{-\zeta \omega_n t} \sin \omega_d t \} \quad (\text{E}_8)$$

4.24 Base displacement:

$$y(t) = \begin{cases} \frac{Y v t}{\delta} ; & 0 \leq t \leq t_0 = \frac{\delta}{v} \\ 0 ; & t > t_0 = \frac{\delta}{v} \end{cases} \quad (1)$$

Equation of motion of vehicle:

$$m \ddot{x} + k(x - y) = 0 \quad (2)$$

Using Eq. (1), Eq. (2) can be expressed as

$$m \ddot{x} + k x = k y = \begin{cases} \frac{k v Y t}{\delta} ; & 0 \leq t \leq t_0 \\ 0 ; & t > t_0 \end{cases} \quad (3)$$

Steady state solution of Eq. (3) from Example 4.9:

$$x(t) = \begin{cases} \frac{v Y}{\delta \omega_n k} \left(\omega_n t - \sin \omega_n t \right) ; & 0 \leq t \leq t_0 \\ 0 ; & t > t_0 \end{cases} \quad (4)$$

Note that the homogeneous solution,

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \quad (5)$$

is to be added to Eq. (4) to obtain the complete solution. The constants C_1 and C_2 are to be evaluated from the initial conditions (at $t = 0$). In fact, the resulting complete solution is valid for all values of t , including values of $t > t_0$.

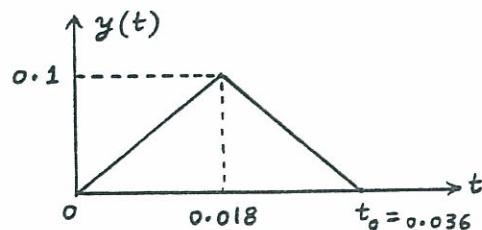
4.25 Speed of automobile = 50 km/hr

$$\text{Excitation frequency} = \left(\frac{50 \times 1000}{3600} \right) \frac{1}{0.5} = 27.7778 \text{ Hz}$$

$$\text{Natural frequency} = f_n = 1.0 \text{ Hz} \Rightarrow \omega_n = 2\pi \text{ rad/sec}$$

$$t_0 = \frac{0.5 \times 3600}{50 \times 1000} = 0.036 \text{ sec}$$

$$y(t) = \begin{cases} \frac{0.2 t}{t_0} ; & 0 \leq t \leq t_0/2 \\ -\frac{0.2 t}{t_0} + 0.2 ; & \frac{t_0}{2} \leq t \leq t_0 \\ 0 ; & t > t_0 \dots (\text{E}_1) \end{cases}$$



Equation of motion (for undamped case) :

$$m\ddot{x} + \kappa(x - y) = 0 \quad \text{or} \quad m\ddot{x} + \kappa x = \kappa y = F(t) \quad (E_2)$$

$$\text{where } F(t) = \kappa y(t) \quad (E_3)$$

Solution of Eq. (E₂) is : $x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \quad (E_4)$

For $0 \leq t \leq \frac{t_0}{2}$:

$$x(t) = \frac{\kappa}{m\omega_n} \int_0^t \left(\frac{0.2}{t_0} \right) \tau \sin \omega_n(t-\tau) d\tau \quad (E_5)$$

Since $\int_0^t \tau \sin \omega_n(t-\tau) d\tau = \left(\frac{t}{\omega_n} - \frac{1}{\omega_n^2} \sin \omega_n t \right)$,

Eg. (E₅) becomes

$$x(t) = 5.5556 \left(t - 0.1592 \sin 6.2832 t \right) \text{ m ; } 0 \leq t \leq 0.018 \text{ sec} \quad (E_6)$$

For $\frac{t_0}{2} \leq t \leq t_0$:

$$x(t) = \frac{\kappa}{m\omega_n} \left\{ \int_0^{t_0/2} \frac{0.2\tau}{t_0} \sin \omega_n(t-\tau) d\tau + \int_{t_0/2}^t \left(-\frac{0.2\tau}{t_0} + 0.2 \right) \sin \omega_n(t-\tau) d\tau \right\}$$

But

$$\frac{0.2\kappa}{m\omega_n t_0} \cdot \int_0^{t_0/2} \tau \sin \omega_n(t-\tau) d\tau = \frac{0.2\kappa}{m\omega_n t_0} \left\{ \sin \omega_n t \left[\frac{1}{\omega_n^2} \cos \omega_n \tau + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_0^{t_0/2} - \cos \omega_n t \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_0^{t_0/2} \right\}$$

$$= 5.5556 \left[0.1592 \sin 6.2832(t - 0.018) + 0.0180 \cos 6.2832(t - 0.018) - 0.1592 \sin 6.2832 t \right] \text{ m} \quad (E_7)$$

since $t_0 = 0.036$.

$$\begin{aligned} -\frac{0.2\kappa}{m\omega_n t_0} \int_{t_0/2}^t \tau \sin \omega_n(t-\tau) d\tau &= -\frac{0.2\kappa}{m\omega_n t_0} \left\{ \sin \omega_n t \left[\frac{1}{\omega_n^2} \cos \omega_n \tau + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_{t_0/2}^t - \cos \omega_n t \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_{t_0/2}^t \right\} \\ &= -5.5556 \left[t - 0.1592 \sin 6.2832(t - 0.018) - 0.0180 \cos 6.2832(t - 0.018) \right] \text{ m} \end{aligned} \quad (E_8)$$

$$\begin{aligned}
 \frac{0.2 k}{m \omega_n} \int_{t_0/2}^t \sin \omega_n(t-\tau) d\tau &= \frac{0.2 k}{m \omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{t_0/2}^t \\
 &= \frac{0.2 k}{m \omega_n^2} \left[1 - \cos \omega_n(t - \frac{t_0}{2}) \right] \\
 &= 0.2 \left[1 - \cos 6.2832(t - 0.018) \right] \text{ m} \tag{E9}
 \end{aligned}$$

Hence the solution can be expressed as

$$\begin{aligned}
 x(t) &= \left[1.7689 \sin 6.2832(t - 0.018) - 0.8845 \sin 6.2832 t \right. \\
 &\quad \left. - 5.5556 t + 0.2 \right] \text{ m} ; \quad 0.018 \leq t \leq 0.036 \text{ sec} \tag{E10}
 \end{aligned}$$

For $t > t_0$:

$$\begin{aligned}
 x(t) &= \frac{0.2 k}{m \omega_n t_0} \int_0^{t_0/2} \tau \sin \omega_n(t-\tau) d\tau - \frac{0.2 k}{m \omega_n t_0} \int_{t_0/2}^{t_0} \tau \sin \omega_n(t-\tau) d\tau \\
 &\quad + \frac{0.2 k}{m \omega_n} \int_{t_0/2}^{t_0} \sin \omega_n(t-\tau) d\tau \tag{E11}
 \end{aligned}$$

The first term on the right side of (E11) is given by (E7).

Second term on the right side of (E11) is

$$\begin{aligned}
 - \frac{0.2 k}{m \omega_n t_0} \int_{t_0/2}^{t_0} \tau \sin \omega_n(t-\tau) d\tau &= - \frac{0.2 k}{m \omega_n t_0} \left\{ \sin \omega_n t \cdot \left[\frac{1}{\omega_n^2} \cos \omega_n \tau \right. \right. \\
 &\quad \left. \left. + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_{t_0/2}^{t_0} - \cos \omega_n t \cdot \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_{t_0/2}^{t_0} \right\} \\
 &= -5.5556 \left[0.1592 \sin 6.2832(t - 0.036) + 0.0360 \cos 6.2832(t - 0.036) \right. \\
 &\quad \left. - 0.1592 \sin 6.2832(t - 0.018) - 0.0180 \cos 6.2832(t - 0.018) \right] \text{ m}
 \end{aligned}$$

The third term on the right side of (E11) is: (E12)

$$\begin{aligned}
 \frac{0.2 k}{m \omega_n} \int_{t_0/2}^{t_0} \sin \omega_n(t-\tau) d\tau &= \frac{0.2 k}{m \omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{t_0/2}^{t_0} \\
 &= \frac{0.2 k}{m \omega_n^2} \left[\cos \omega_n(t-t_0) - \cos \omega_n(t - \frac{t_0}{2}) \right] \\
 &= 0.2 \left[\cos 6.2832(t - 0.036) - \cos 6.2832(t - 0.018) \right] \text{ m} \tag{E13}
 \end{aligned}$$

$\therefore x(t)$ is given by the sum of Eqs. (E₇), (E₁₂) and (E₁₃), which can be simplified as

$$x(t) = 1.7689 \sin 6.2832(t - 0.018) - 0.8845 \sin 6.2832 t - 0.8845 \sin 6.2832(t - 0.036) \text{ m ; } t > 0.036 \text{ sec} \quad (\text{E14})$$

4.26

When the container strikes the floor, the velocity of the mass is given by

$$mgh = \frac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2gh} \quad (\text{E1})$$

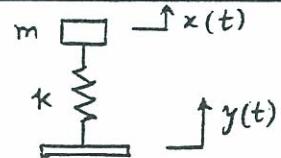
The displacement of the camcorder subjected to an initial velocity $\dot{x}_0 = v$ is given by Eq. (2.72), with $x_0 = 0$ and $\zeta \ll 1$,

$$x(t) = e^{-\zeta \omega_n t} \cdot \frac{\dot{x}_0}{\omega_n \sqrt{1-\zeta^2}} \cdot \sin \sqrt{1-\zeta^2} \omega_n t \quad (\text{E2})$$

4.27

System can be modeled as a spring-mass system subjected to base motion:

$$y(t) = \begin{cases} (Y t^2 / t_0^2) & ; 0 \leq t \leq t_0 \\ 0 & ; t > t_0 \end{cases} \quad \text{--- (E1)}$$



Relative displacement of mass, $\bar{z} = x - y$, is given by Eq. (4.34):

$$\bar{z}(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau) e^{-\zeta \omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau \quad (\text{E2})$$

$$\text{where } \begin{cases} y(\tau) = (Y \tau^2 / t_0^2) \\ \ddot{y}(\tau) = (2Y / t_0^2) \end{cases} ; \quad 0 \leq \tau \leq t_0 \quad (\text{E3})$$

$$\text{and } y(\tau) = 0 \quad ; \quad \tau > t_0$$

For $0 \leq t \leq t_0$:

Since the system is undamped, $\omega_d = \omega_n$ and $\zeta = 0$, and (E2) reduces to

$$\bar{z}(t) = -\frac{2Y}{\omega_n t_0^2} \int_0^t \sin \omega_n(t-\tau) d\tau \quad (\text{E4})$$

Here

$$\begin{aligned} \int_0^t \sin \omega_n(t-\tau) d\tau &= \int_0^t (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau) d\tau \\ &= \sin \omega_n t \int_0^t \cos \omega_n \tau d\tau - \cos \omega_n t \int_0^t \sin \omega_n \tau d\tau \end{aligned}$$

$$\begin{aligned}
 &= \sin \omega_n t \left(\frac{1}{\omega_n} \sin \omega_n \tau \right)_0^t - \cos \omega_n t \left(-\frac{1}{\omega_n} \cos \omega_n \tau \right)_0^t \\
 &= \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \tag{E5}
 \end{aligned}$$

Thus Eq. (E4) gives

$$x(t) = \frac{\gamma t^2}{t_0^2} - \frac{2\gamma}{t_0^2 \omega_n^2} (1 - \cos \omega_n t) ; 0 \leq t \leq t_0 \tag{E6}$$

For $t > t_0$:

Eg. (E2) gives

$$y(t) = -\frac{1}{\omega_n} \int_0^{t_0} \frac{2\gamma}{t_0^2} \sin \omega_n(t-\tau) d\tau \tag{E7}$$

But

$$\int_0^{t_0} \sin \omega_n(t-\tau) d\tau = \frac{1}{\omega_n} \{ \cos \omega_n(t-t_0) - \cos \omega_n t \}$$

$$\therefore z(t) = x(t) - y(t) = -\frac{2\gamma}{\omega_n^2 t_0^2} \{ \cos \omega_n(t-t_0) - \cos \omega_n t \} ; t > t_0 \tag{E8}$$

$$\begin{aligned}
 4.28 \quad I_1 &= \int_0^t (t-\tau) e^{-\zeta \omega_n(t-\tau)} \sin \omega_d(t-\tau) (-d\tau) \\
 &= \left[\frac{(t-\tau) e^{-\zeta \omega_n(t-\tau)}}{(\zeta \omega_n)^2 + \omega_d^2} \{ -\zeta \omega_n \sin \omega_d(t-\tau) - \omega_d \cos \omega_d(t-\tau) \} \right. \\
 &\quad \left. - \frac{e^{-\zeta \omega_n(t-\tau)}}{(\zeta \omega_n)^2 + \omega_d^2} \{ (\zeta^2 \omega_n^2 - \omega_d^2) \sin \omega_d(t-\tau) + 2\zeta \omega_n \omega_d \cos \omega_d(t-\tau) \} \right]_{\tau=0}^t \\
 &= -\frac{1}{\omega_n^4} (2\zeta \omega_n \omega_d) + \frac{t e^{-\zeta \omega_n t}}{\omega_n^2} (\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t) \\
 &\quad + \frac{e^{-\zeta \omega_n t}}{\omega_n^4} \{ \omega_n^2 (2\zeta^2 - 1) \sin \omega_d t + 2\zeta \omega_n \omega_d \cos \omega_d t \} \\
 I_2 &= \int_0^t e^{-\zeta \omega_n(t-\tau)} \sin \omega_d(t-\tau) (-d\tau) \\
 &= \left[\frac{e^{-\zeta \omega_n(t-\tau)}}{\zeta^2 \omega_n^2 + \omega_d^2} \{ -\zeta \omega_n \sin \omega_d(t-\tau) - \omega_d \cos \omega_d(t-\tau) \} \right]_{\tau=0}^t \\
 &= -\frac{\omega_d}{\omega_n^2} + \frac{e^{-\zeta \omega_n t}}{\omega_n^2} (\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t)
 \end{aligned}$$

$$x(t) = \frac{\delta F}{m\omega_d} \cdot I_1 - \frac{\delta F \cdot t}{m\omega_d} \cdot I_2 \\ = \frac{\delta F}{k} \left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t} \left\{ \frac{2\zeta}{\omega_n} \cos \omega_d t - \left(\frac{\omega_d^2 - \omega_n^2}{\omega_n^2 \omega_d} \right) \sin \omega_d t \right\} \right]$$

4.29 (1) $m\ddot{x} + c\dot{x} + kx = F(t)$

where $m(t) = M - m_0 t = (2000 - 10t) \text{ kg}$
and $F(t) = m_0 v = 10(2000) = 20000 \text{ N}$

(2) With $m = M - \frac{1}{2}m_0 t_0 = 2000 - \frac{1}{2}(10)(100) = 1500 \text{ kg}$,
equation of motion becomes

$$1500 \ddot{x} + 0.1 \times 10^6 \dot{x} + 7.5 \times 10^6 x = 20000 = \text{constant}$$

Maximum steady state displacement is

$$x_p(t) = \frac{F}{k} = \frac{20000}{7.5 \times 10^6} = 0.002667 \text{ m}$$

4.30 From Eq. (4.30), the response to unit step function can be obtained by setting $F(\tau) = 1$ as

$$h(t) = \int_0^t g(t-\tau) d\tau \quad \dots \dots (E_1)$$

By differentiating this equation with respect to t , we obtain

$$\frac{dh}{dt}(t) = g(t)$$

4.31 Equation (4.30) gives $x(t) = \int_0^t F(\tau) \cdot g(t-\tau) d\tau$

But $g(t-\tau) = \frac{dh}{d\tau}(t-\tau)$ from problem 4.30.

$$x(t) = \int_0^t F(\tau) \frac{dh}{d\tau}(t-\tau) d\tau$$

Integration by parts gives

$$x(t) = -F(t) \cdot h(t-\tau) \Big|_{\tau=0}^t + \int_0^t \frac{dF}{d\tau} \cdot h(t-\tau) d\tau \\ = -F(t) h(0) + F(0) h(t) + \int_0^t \frac{dF}{d\tau} h(t-\tau) d\tau$$

But $h(0) = 0$ from Eq. (E₁) of problem 4.30.

$$\therefore x(t) = F(0) h(t) + \int_0^t \frac{dF}{d\tau}(\tau) \cdot h(t-\tau) d\tau$$

Equation of motion for rotation about O:

4.32 $J_0 \ddot{\theta} + M \ddot{x}(\ell) + k_1 a^2 \theta + k_2 b^2 \theta = F_0 \ell e^{-t} \quad (1)$

where $\ddot{x} = \ell \ddot{\theta}$ and

$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{2} \right)^2 = \frac{1}{3} m \ell^2$$

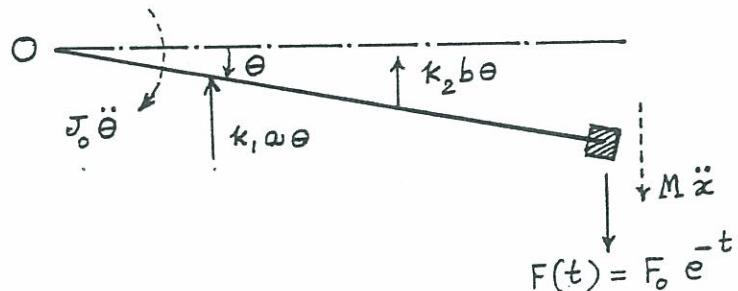
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3} m \ell^2 + M \ell^2 \right) \ddot{\theta} + (k_1 a^2 + k_2 b^2) \theta = F_0 \ell e^{-t} \quad (2)$$

For given data, Eq. (2) takes the form:

$$53.3333 \ddot{\theta} + 1562.5 \theta = 500 e^{-t} \quad (3)$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{1562.5}{53.3333}} = 5.4127 \text{ rad/sec}$ and



the forcing term as $500 e^{-t}$, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\begin{aligned} \theta(t) &= \frac{1}{(53.3333)(5.4127)} \int_0^t 500 e^{-\tau} \sin 5.4127 (t - \tau) d\tau \\ &= 1.7320 \int_0^t e^{-t} e^{(\tau-t)} \sin 5.4127 (\tau - t) d\tau \\ &= -1.7320 e^{-t} \int_0^t e^{(\tau-t)} \sin 5.4127 (\tau - t) (-d\tau) \end{aligned} \quad (4)$$

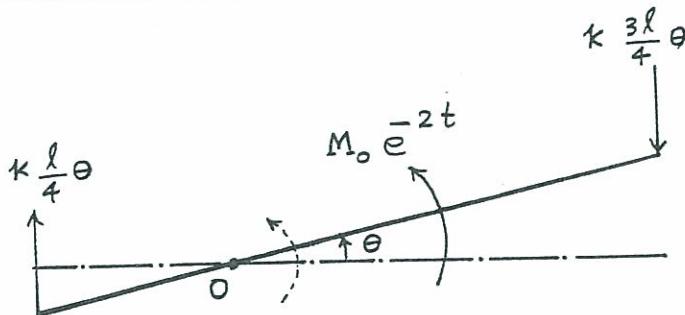
Using the formula:

$$\int e^{ax} \sin bx dx = \frac{1}{a^2 + b^2} e^{ax} [a \sin bx - b \cos bx] \quad (5)$$

Eq. (4) can be expressed as

$$\begin{aligned} \theta(t) &= -1.7320 e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 5.4127^2} \left\{ \sin 5.4127 (\tau - t) - 5.4127 \cos 5.4127 (\tau - t) \right\} \right]_{\tau=0}^{t=t} \\ &= 0.3094 e^{-t} + 0.05717 \sin 5.4127 t - 0.3094 \cos 5.4127 t \text{ radian} \end{aligned}$$

4.33



$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$$

Equation of motion for rotation about O:

$$\begin{aligned} J_0 \ddot{\theta} + k \frac{\ell^2}{16} \theta + k \frac{9}{16} \ell^2 \theta &= M_0 e^{-2t} \\ \text{or } J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta &= M_0 e^{-2t} \\ \text{or } 1.4583 \ddot{\theta} + 3125.0 \theta &= 100 e^{-2t} \end{aligned} \quad (1)$$

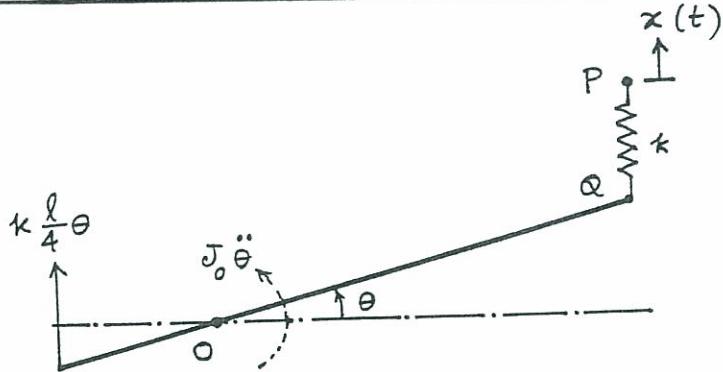
Noting that the system is undamped with $\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\begin{aligned} \theta(t) &= \frac{1}{(1.4583)(46.2915)} \int_0^t (100 e^{-2\tau}) \sin 46.2915(t-\tau) d\tau \\ &= -1.4813 e^{-2t} \int_0^t e^{2(t-\tau)} \sin 46.2915(t-\tau) (-d\tau) \end{aligned} \quad (2)$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (2) can be expressed as:

$$\begin{aligned} \theta(t) &= -1.4813 e^{-2t} \left[\frac{e^{2(t-\tau)}}{2^2 + 46.2915^2} \left\{ 2 \sin 46.2915(t-\tau) - 46.2915 \cos 46.2915(t-\tau) \right\} \right]_{\tau=0}^{t=t} \\ &= 0.03194 e^{-2t} + 13.7994 (10^{-4}) \sin 46.2915 t - 0.03194 \cos 46.2915 t \text{ radian} \end{aligned} \quad (4)$$

4.34



Net compression of spring PQ = $\frac{3\ell\theta}{4} - x(t)$. Equation of motion for rotation about O:

$$\begin{aligned} J_0 \ddot{\theta} &= -k \frac{\ell\theta}{4} \left(\frac{\ell}{4} \right) - k \left(\frac{3\ell\theta}{4} - x(t) \right) \left(\frac{3\ell}{4} \right) \\ \text{or } J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta &= \frac{3k\ell}{4} x(t) = \frac{3k\ell}{4} x_0 e^{-t} \end{aligned} \quad (1)$$

For given data, $J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4} \right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg-m}^2$, $\frac{5}{8} k \ell^2 = \frac{5}{8} (5000) (1^2) = 3125 \text{ N/m}$, and Eq. (1) becomes:

$$1.4583 \ddot{\theta} + 3125.0 \theta = 37.5 e^{-t} \quad (2)$$

Noting that the system is undamped with

$$\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$$

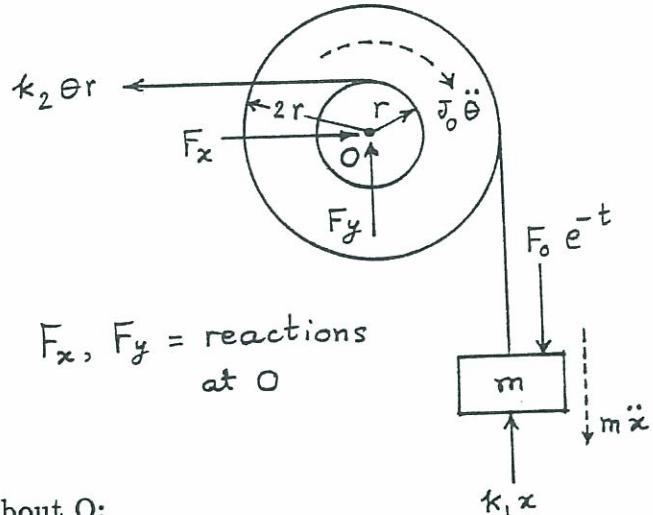
the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(1.4583)(46.2915)} \int_0^t (37.5 e^{-\tau}) \sin 46.2915(t - \tau) d\tau \quad (3)$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (3) can be expressed as

$$\begin{aligned} \theta(t) &= -0.5555 e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 46.2915^2} \left\{ \sin 46.2915(t-\tau) - 46.2915 \cos 46.2915(t-\tau) \right\} \right]_{\tau=0}^t \\ &= 0.01199 e^{-t} + 2.591 (10^{-4}) \sin 46.2915 t - 0.01199 \cos 46.2915 t \text{ radian} \end{aligned} \quad (4)$$

4.35



Equation of motion for rotation of pulley about O:

$$J_0 \ddot{\theta} + m \ddot{x} (2r) + k_1 x (2r) + k_2 (\theta r) r = 2r F_0 e^{-t} \quad (1)$$

where $\theta = \frac{x}{2r}$. Eq. (1) can be rewritten in terms of x only as:

$$\left(\frac{J_0}{2r} + 2m r \right) \ddot{x} + \left(2k_1 r + \frac{1}{2} k_2 r \right) x = 2r F_0 e^{-t} \quad (2)$$

For given data, Eq. (2) becomes

$$11.0 \ddot{x} + 112.5 x = 5 e^{-t} \quad (3)$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{112.5}{11.0}} = 3.1980 \text{ rad/sec}$, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$x(t) = -0.1421 e^{-t} \int_0^t e^{(t-\tau)} \sin 3.1980(t-\tau) (-d\tau) \quad (4)$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (4) can be expressed as

$$x(t) = -\frac{0.1421 e^{-t}}{1^2 + 3.1980^2} \left[e^{(t-\tau)} \left\{ \sin 3.1980(t-\tau) - 3.1980 \cos 3.1980(t-\tau) \right\} \right]_{\tau=0}^t \\ = 0.04048 e^{-t} + 0.01266 \sin 3.1980 t - 0.04048 \cos 3.1980 t \text{ m} \quad (5)$$

4.36 (a) Unit impulse response function for undamped case:

use $\zeta = 0$ and $\omega_d = \omega_n$ in Eq. (4.25):

$$x(t) = \frac{1}{m \omega_n} \sin \omega_n t \quad (\text{E.1})$$

(b) Unit impulse response function for underdamped case : Eq. (4.25):

$$x(t) = \frac{1}{m \omega_d} e^{-\zeta \omega_n t} \sin \omega_d t \quad (\text{E.2})$$

(c) Unit impulse response function for critically damped case :

Free vibration response of a critically damped system is given by Eq. (2.80) :

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)t\} e^{-\omega_n t} \quad (\text{E.3})$$

Using the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$,

Eg. (E.3) gives

$$x(t) = \frac{t}{m} e^{-\omega_n t} \quad (\text{E.4})$$

(d) Unit impulse response function for an over damped case :

Free vibration response of an overdamped system is given by Eq. (2.81) :

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1}) \omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1}) \omega_n t} \quad (\text{E.5})$$

where $C_1 = \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2 \omega_n \sqrt{\zeta^2 - 1}}$; $C_2 = \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2 \omega_n \sqrt{\zeta^2 - 1}}$

For the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$,

C_1 and C_2 become

$$C_1 = \frac{1}{2m\omega_n\sqrt{\zeta^2 - 1}} ; \quad C_2 = -\frac{1}{2m\omega_n\sqrt{\zeta^2 - 1}}$$

and hence Eq. (E·5) yields

$$x(t) = \frac{1}{2m\omega_n\sqrt{\zeta^2 - 1}} \left\{ e^{-(\zeta + \sqrt{\zeta^2 - 1})\omega_n t} - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \right\} \quad (\text{E} \cdot 6)$$

(4.37) $m = 2 \text{ kg}$, $c = 4 \text{ N-s/m}$, $k = 32 \text{ N/m}$, $F = 4 \delta(t)$,

$$x_0 = 0.01 \text{ m}, \quad \dot{x}_0 = 1 \text{ m/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{2}} = 4 \text{ rad/s}, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{4}{2\sqrt{32(2)}} = 0.25$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = 3.872983 \text{ rad/s}$$

Underdamped system.

Impulse response is given by Eq. (4.26):

$$x(t) = \frac{F}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$= 0.516398 e^{-t} \sin 3.872983 t \text{ m}$$

(4.38) The stiffness of the cantilever beam (wing) is given by

$$k = \frac{3EI}{l^3} = \frac{3(15 \times 10^9)}{10^3} = 45 \times 10^6 \text{ N/m}$$

System can be modeled as a single degree of freedom undamped system:

$$m\ddot{x} + kx = 0$$

where $m = 2500 \text{ kg}$, $k = 45 \times 10^6 \text{ N/m}$, and

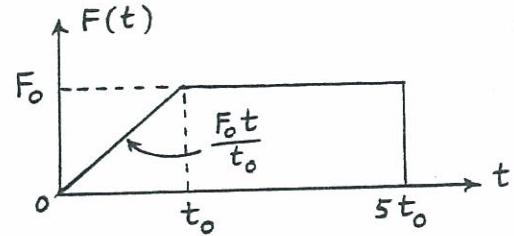
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45 \times 10^6}{2.5 \times 10^3}} = 134.1641 \text{ rad/s}$$

Response of mass due to impulse F is given by

Eq. (4.26) with $\zeta = 0$ and $\omega_d = \omega_n$:

$$\begin{aligned}x(t) &= \frac{\tilde{F}}{m \omega_n} \sin \omega_n t \\&= \frac{50}{2500 (134.1641)} \sin 134.1641 t \\&= 0.000149071 \sin 134.1641 t \text{ m}\end{aligned}$$

4.39 $F(t) = \begin{cases} (F_0 t / t_0) & ; 0 \leq t \leq t_0 \\ F_0 & ; t_0 \leq t \leq 5t_0 \\ 0 & ; t > 5t_0 \end{cases} \dots (E_1)$



Response of the anvil is given by [Eq. (4.31) for an undamped system]:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \quad (E_2)$$

For $0 \leq t \leq t_0$:

$$\begin{aligned} x(t) &= \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n(t-\tau) d\tau \\ &= \frac{F_0}{\kappa t_0} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right) \end{aligned} \quad (E_3)$$

For $t_0 \leq t \leq 5t_0$:

$$\begin{aligned} x(t) &= \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \\ &= \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n(t-\tau) d\tau + \frac{F_0}{m\omega_n} \int_{t_0}^t \sin \omega_n(t-\tau) d\tau \\ &= \frac{F_0}{m\omega_n t_0} \left[\sin \omega_n t \left\{ \frac{1}{\omega_n^2} \cos \omega_n t_0 + \frac{t_0}{\omega_n} \sin \omega_n t_0 - \frac{1}{\omega_n^2} \right\} \right. \\ &\quad \left. - \cos \omega_n t \left\{ \frac{1}{\omega_n^2} \sin \omega_n t_0 - \frac{t_0}{\omega_n} \cos \omega_n t_0 \right\} \right] \\ &\quad + \frac{F_0}{m\omega_n} \left\{ \frac{1}{\omega_n} \cos (\omega_n t - \omega_n \tau) \right\}_{t_0}^t \\ &= \frac{F_0}{\kappa \omega_n t_0} [\sin \omega_n(t-t_0) - \sin \omega_n t + \omega_n t_0] \end{aligned} \quad (E_4)$$

For $t > 5t_0$:

$$\begin{aligned}x(t) &= \frac{1}{m\omega_n} \left[\int_0^{t_0} \frac{F_0 \tau}{t_0} \sin \omega_n(t-\tau) d\tau + F_0 \int_{t_0}^{5t_0} \sin \omega_n(t-\tau) d\tau \right] \\&= \frac{F_0}{m\omega_n^2 t_0} \left[\frac{1}{\omega_n} \sin \omega_n(t-t_0) + t_0 \cos \omega_n(t-t_0) - \frac{1}{\omega_n} \sin \omega_n t \right] \\&\quad + \frac{F_0}{m\omega_n} \left[\frac{1}{\omega_n} \cos \omega_n(t-\tau) \right]_{t_0}^{5t_0} \\&= \frac{F_0}{m\omega_n t_0} \left[\sin \omega_n(t-t_0) - \sin \omega_n t + t_0 \omega_n \cos \omega_n(t-5t_0) \right] \quad (E5)\end{aligned}$$

4.47) $F(t) = \begin{cases} F_0 & ; \quad 0 \leq t \leq t_0 \\ 0 & ; \quad t > t_0 \end{cases}$ (E₁)

Eq. (4.31) gives, for an undamped system,

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n(t-\tau) d\tau \quad (E_2)$$

For $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n(t-\tau) d\tau = \frac{F_0}{\kappa} (1 - \cos \omega_n t) \quad (E_3)$$

using Eq. (E₅) in the solution of problem 4.27.

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin \omega_n(t-\tau) d\tau$$

Using the relation

$$\int_0^{t_0} \sin \omega_n(t-\tau) d\tau = \frac{1}{\omega_n} \left\{ \cos \omega_n(t-t_0) - \cos \omega_n t \right\}$$

the solution can be expressed as

$$x(t) = \frac{F_0}{\kappa} [\cos \omega_n(t-t_0) - \cos \omega_n t] \quad (E_4)$$

Response spectrum:

$$\text{For } 0 \leq t \leq t_0, \quad x(t) = \frac{F_0}{\kappa} (1 - \cos \omega_n t) \quad (E_5)$$

$$\frac{dx}{dt} = \frac{F_0 \omega_n}{\kappa} \sin \omega_n t = 0 \Rightarrow \omega_n t_{\max} = \pi$$

$$\therefore x_{\max} = x(t = t_{\max}) = \frac{F_0}{\kappa} (1 - \cos \omega_n t_{\max}) = \frac{2 F_0}{\kappa} \quad (E_6)$$

For $t > t_0$,

$$x(t) = \frac{F_0}{\kappa} \left\{ \cos \omega_n(t-t_0) - \cos \omega_n t \right\} \quad (E_7)$$

$$\frac{dx}{dt} = - \frac{F_0 \omega_n}{\kappa} \left\{ \sin \omega_n(t-t_0) - \sin \omega_n t \right\} = 0$$

$$\Rightarrow \sin \omega_n (t_{\max} - t_0) = \sin \omega_n t_{\max}$$

i.e., $\tan \omega_n t_{\max} = \left(\frac{\sin \omega_n t_0}{\cos \omega_n t_0 - 1} \right)$ (E₈)

i.e., $\sin \omega_n t_{\max} = \frac{\sin \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}}$ (E₉)

and $\cos \omega_n t_{\max} = \frac{\cos \omega_n t_0 - 1}{\sqrt{2(1 - \cos \omega_n t_0)}}$ (E₁₀)

$$\therefore x_{\max} = x(t = t_{\max}) = \frac{F_0}{k} \left\{ \cos \omega_n t_{\max} \cdot \cos \omega_n t_0 + \sin \omega_n t_{\max} \cdot \sin \omega_n t_0 - \cos \omega_n t_{\max} \right\}$$

$$\begin{aligned} &= \frac{F_0}{k} \left\{ \frac{(\cos \omega_n t_0 - 1)^2}{\sqrt{2(1 - \cos \omega_n t_0)}} + \frac{\sin^2 \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \right\} \\ &= \frac{2 F_0}{k} \sin \frac{\omega_n t_0}{2} \end{aligned} \quad (\text{E}_{11})$$

Plotting: $\omega_n = \frac{2\pi}{\tau_n}$

For $0 \leq t \leq t_0$, $\omega_n t_{\max} = \pi$ or $\frac{t_{\max}}{\tau_n} = \frac{1}{2}$

When $t \leq t_0$, $t_{\max} = \frac{\tau_n}{2} \leq t_0$

i.e., $\frac{t_0}{\tau_n} \geq \frac{1}{2}$ (E₁₂)

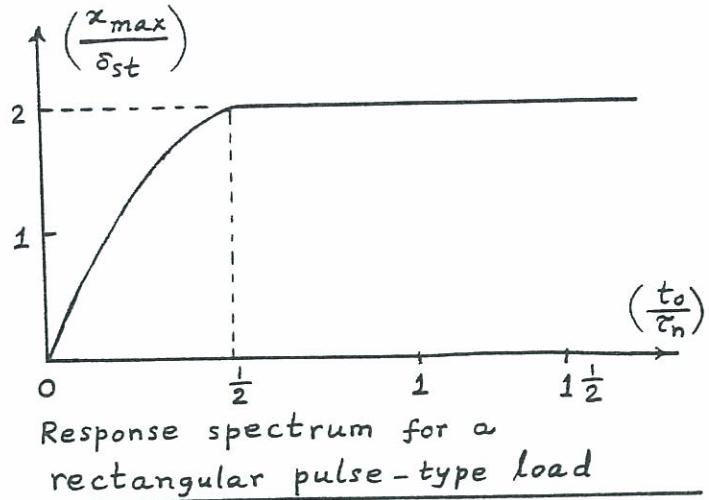
$$\therefore \frac{x_{\max}}{\delta_{st}} = 2 \quad \text{for} \quad \frac{t_0}{\tau_n} \geq \frac{1}{2}$$

For $t > t_0$, $\frac{t_0}{\tau_n} < \frac{1}{2}$

$$\frac{x_{\max}}{\delta_{st}} = 2 \sin \frac{\omega_n t_0}{2} = 2 \sin \frac{2\pi t_0}{2\tau_n}$$

$$\therefore \frac{x_{\max}}{\delta_{st}} = 2 \sin \frac{\pi t_0}{\tau_n} \quad \text{for} \quad \frac{t_0}{\tau_n} < \frac{1}{2} \quad (\text{E}_{13})$$

Eqs. (E₁₂) and (E₁₃) are plotted in the figure.



4.48

The response is found in problem 4.22.

For $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{\kappa} (1 - \cos \omega_n t) + \frac{F_0}{2m\omega_n} \cdot \frac{2\omega_n}{\left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \cdot \left\{ \cos \frac{\pi t}{2t_0} - \cos \omega_n t \right\} \quad (E_1)$$

For x_{max} , $\frac{dx}{dt} = 0$

$$\text{i.e., } \frac{F_0 \omega_n}{\kappa} \sin \omega_n t + \frac{F_0}{m \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \left(-\frac{\pi}{2t_0} \sin \frac{\pi t}{2t_0} + \omega_n \sin \omega_n t \right) = 0$$

which can be reduced to the form

$$\sin \omega_n t_{max} = \left[\frac{\pi}{2\omega_n t_0 \left\{ m \left(\frac{\pi}{2t_0} \right)^2 - m \omega_n^2 \right\} + \kappa} \right] \sin \frac{\pi t_{max}}{2t_0} \quad (E_2)$$

Once t_{max} is known from (E₂), Eq. (E₁) can be used to find x_{max} as:

$$x_{max} = x(t = t_{max}) = \frac{F_0}{\kappa} (1 - \cos \omega_n t_{max}) \\ + \frac{F_0}{m \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \left(\cos \frac{\pi t_{max}}{2t_0} - \cos \omega_n t_{max} \right) \quad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{k} \cos \omega_n(t - t_0) - \sin \omega_n(t - t_0) \left\{ \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\} + \cos \omega_n t \left\{ -\frac{F_0}{k} - \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m \omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}$$

For t_{max} , $\frac{dx}{dt} = 0$ (E4)

i.e.,

$$-\frac{F_0 \omega_n}{k} \sin \omega_n(t_{max} - t_0) - \frac{F_0 \pi \omega_n \cos \omega_n(t_{max} - t_0)}{2m \omega_n t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} + \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2 \omega_n}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n t_{max} = 0$$
(E5)

once t_{max} is found by solving Eq. (E5), x_{max} can be found from Eq. (E4) as

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k} \cos \omega_n(t_{max} - t_0) - \frac{\pi F_0}{2 \omega_n m t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n(t_{max} - t_0) - \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \cos \omega_n t_{max}$$
(E6)

Eqs. (E3) and (E6) can be used to plot x_{max} versus ω_n to get the displacement response spectrum.

Base acceleration = $\ddot{y}(t) = a_0 \left(1 - \sin \frac{\pi t}{2t_0} \right)$ (E1)

4.49 For an undamped system, the relative displacement is given by Eq. (4.34):

$$\begin{aligned} z(t) &= -\frac{1}{\omega_n} \int_0^t \ddot{y}(\tau) \sin \omega_n(t - \tau) d\tau \\ &= -\frac{1}{\omega_n} \left[\int_0^t a_0 \sin \omega_n(t - \tau) d\tau - \int_0^t a_0 \sin \frac{\pi t}{2t_0} \sin \omega_n(t - \tau) d\tau \right] \end{aligned}$$
(E2)

Here

$$\int_0^t \sin \omega_n(t-\tau) d\tau = \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \quad (E_3)$$

from Eq. (E₅) in the solution of problem 4-27.

and

$$\begin{aligned} & \int_0^t \sin \frac{\pi \tau}{2t_0} (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau) d\tau \\ &= \sin \omega_n t \left\{ - \frac{\cos \left(\frac{\pi}{2t_0} - \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} - \frac{\cos \left(\frac{\pi}{2t_0} + \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}_0^t \\ & - \cos \omega_n t \left\{ \frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} - \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) \tau}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}_0^t \end{aligned} \quad (E_4)$$

Thus the solution, Eq. (E₂), can be finally expressed as

$$\begin{aligned} z(t) = & - \frac{\omega_0}{\omega_n} \left(1 - \frac{\cos \omega_n t}{\omega_n} \right) - \frac{\omega_0}{\omega_n} \left\{ \sin \omega_n t \cdot \left[\frac{\cos \left(\frac{\pi}{2t_0} - \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} \right. \right. \\ & \left. \left. + \frac{\cos \left(\frac{\pi}{2t_0} + \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right] - 2 \right\} + \cos \omega_n t \cdot \left[\frac{\sin \left(\frac{\pi}{2t_0} - \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} - \omega_n \right)} - \frac{\sin \left(\frac{\pi}{2t_0} + \omega_n \right) t}{2 \left(\frac{\pi}{2t_0} + \omega_n \right)} \right] \end{aligned} \quad (E_5)$$

4.50

During $0 \leq t \leq t_0$:

$$x(t) = \frac{F_0}{k} \left(1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right) \quad (E_1)$$

$$\dot{x}(t) = 0 \text{ gives } \omega_n t_0 \sin \omega_n t_m = 1 - \cos \omega_n t_m \quad (E_2)$$

i.e. $\omega_n t_m = 2 \tan^{-1}(\omega_n t_0)$

$$(E_1) \text{ becomes } \frac{x_m}{(F_0/k)} = 1 - \frac{t_m}{t_0} - \cos \omega_n t_m + \frac{1}{\omega_n t_0} \sin \omega_n t_m \quad (E_3)$$

where t_m is given by (E₂).

During $t > t_0$:

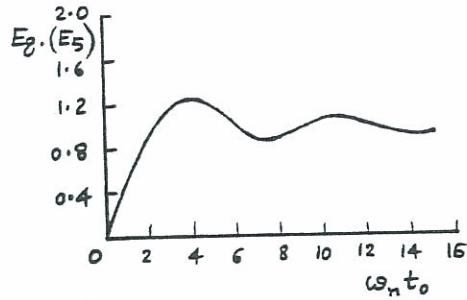
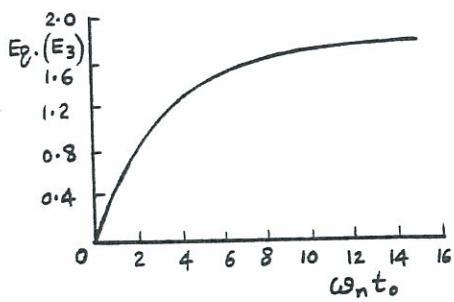
$$x(t) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right] \quad (E_4)$$

$$\text{i.e. } \frac{x(t) - \omega_n t_0}{F_0} = \ddot{x}(t) = A \sin \omega_n t + B \cos \omega_n t$$

$$\text{where } A = 1 - \cos \omega_n t_0 ; \quad B = -(\omega_n t_0 - \sin \omega_n t_0)$$

$$\text{Since } \ddot{x}|_{\max} = \sqrt{A^2 + B^2},$$

$$\frac{x_m}{(F_0/k)} = \frac{1}{\omega_n t_0} \left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2 \right]^{1/2} \quad (E_5)$$



4.51

From Example 4.13, the response of the building frame is given by

$$x(t) = \frac{F_0}{k} \left[1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right], \quad 0 \leq t \leq t_0 \quad (E_1)$$

and

$$x(t) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right], \quad t > t_0 \quad (E_2)$$

(i) For $0 \leq t \leq t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in (E_1) must be maximum. This implies that

$$\frac{d}{dt} \left[1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right] = 0$$

$$\text{i.e., } \omega_n t_0 \sin \omega_n t = 1 - \cos \omega_n t$$

$$\text{i.e., } \omega_n t_0 \cos \frac{\omega_n t}{2} = \sin \frac{\omega_n t}{2}$$

$$\text{i.e., } \tan \frac{\omega_n t}{2} = \omega_n t_0 \quad (E_3)$$

Thus, if $x(t)$ attains its maximum value at $t = t_{\max}$, $E_g(E_3)$ gives

$$t_{\max} = \frac{2}{\omega_n} \tan^{-1}(\omega_n t_0) \quad (E_4)$$

Once t_{\max} is known from (E_4) , (E_1) gives

$$\begin{aligned} x_{\max} = x(t = t_{\max}) &= \frac{F_0}{k} \left\{ 1 - \frac{t_{\max}}{t_0} - \cos \omega_n t_{\max} \right. \\ &\quad \left. + \frac{1}{\omega_n t_0} \sin \omega_n t_{\max} \right\} \end{aligned} \quad (E_5)$$

(ii) For $t > t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in $E_g(E_2)$ must be maximum. This implies that

$$\frac{d}{dt} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right] = 0$$

i.e.,

$$(1 - \cos \omega_n t_0) \omega_n \cos \omega_n t + (\omega_n t_0 - \sin \omega_n t_0) \omega_n \sin \omega_n t = 0$$

i.e.,

$$\tan \omega_n t = - \left(\frac{1 - \cos \omega_n t_0}{\omega_n t_0 - \sin \omega_n t_0} \right) \quad (E_6)$$

If $x(t)$ attains its maximum at $t = t_{\max}$, Eq. (E6) gives

$$t_{\max} = \frac{1}{\omega_n} \tan^{-1} \left(\frac{-1 + \cos \omega_n t_0}{\omega_n t_0 - \sin \omega_n t_0} \right) \quad (E_7)$$

Once t_{\max} is computed from (E7), (E2) gives

$$x_{\max} = x(t = t_{\max}) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2 \right]^{\frac{1}{2}} \quad (E_8)$$

Given data:

$$m = 5000 \text{ kg}, \quad F_0 = 4 \times 10^6 \text{ N}, \quad t_0 = 0.4 \text{ sec}, \quad x_{\max} \leq 0.01 \text{ m.}$$

$$\omega_n = \sqrt{k/m} = 0.01414 \sqrt{k}$$

Procedure:

1. Assume a series of values of k .

2. Find t_{\max} using Eqs. (E4) and (E7).

3. Find x_{\max} using Eqs. (E5) and (E8).

4. Select k such that $x_{\max} \leq 0.01 \text{ m}$ in Eq. (E5) or (E8).

Sample computer program and results are shown below.

```
XM=5000.0
F0=4.0E+6
T0=0.4
XK=0.0
DO 10 I=1,100
XK=XK+1.0E+7
OMN=0.01414*SQRT(XK)
TMAX1=(2.0/OMN)*ATAN(OMN*T0)
XMAX1=(F0/XK)/(1.0-(TMAX1/T0)-COS(OMN*TMAX1)+SIN(OMN*TMAX1)/
2*(OMN*T0))
XNR=-(1.0-COS(OMN*T0))
XDR=(OMN*T0-SIN(OMN*T0))/OMN
TMAX2=ATAN(XNR/XDR)/OMN
X1=(1.0-COS(OMN*T0))**2
X2=(OMN*T0-SIN(OMN*T0))**2
X3=(X1+X2)**0.5
XMAX2=X3*F0/(XK*OMN*T0)
PRINT 5,I,XK,TMAX1,XMAX1,TMAX2,XMAX2
FORMAT(I5.2X,E15.4,2X,2E12.4,2X,2E12.4)
CONTINUE
STOP
END
```

1	0. 1000E+08	0. 6776E-01	0. 7322E+00	-0. 5134E-03	0. 4185E+00
2	0. 2000E+08	0. 4843E-01	0. 3758E+00	-0. 8204E-05	0. 1987E+00
3	0. 3000E+08	0. 3773E-01	0. 2534E+00	-0. 3859E-04	0. 1387E+00
4	0. 4000E+08	0. 3450E-01	0. 1714E+00	-0. 4108E-03	0. 1027E+00
5	0. 5000E+08	0. 3092E-01	0. 1538E+00	-0. 4234E-03	0. 7857E-01

These results indicate that the required stiffness is

$$k = 8 \times 10^8 \text{ N/m.}$$

4.52 Let d = thickness of bracket. Then, from Example 4.18; self weight of beam = $w = 0.5$ d lb, total weight at free end of beam = $W = 0.5 d + 0.4$ lb, moment of inertia of beam cross section = $I = 0.04167 d^3$ in⁴, static deflection of beam under W :

$$\delta_{st} = \frac{W \ell^3}{3 E I} = \frac{(0.5 d + 0.4)}{d^3} 7.9994 (10^{-4}) \text{ in}$$

We need to use a trial and error procedure to find the correct value of d .

Let $d = 1$ in:

$$w = 0.5 \text{ lb}, W = 0.9 \text{ lb}, I = 0.04167 \text{ in}^4, \delta_{st} = 0.9 (7.9994) (10^{-4}) = 7.19946 (10^{-4}) \text{ in},$$

$$\tau_n = 2 \pi \sqrt{\frac{\delta_{st}}{g}} = 2 \pi \sqrt{\frac{7.19946 (10^{-4})}{386.4}} = 0.008577 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.008577} = 11.6591$$

From Fig. 4.65, shock amplification factor (A_a) corresponding to $t_0/\tau_n = 11.6591$ is $A_a \approx 2.0$.

Dynamic load at end of cantilever = $P_d = A_a M a_s = (2.0) (0.9/g) (100g) = 180.0 \text{ lb.}$

$$\sigma_{max} = \frac{M_b c}{I} = \frac{(180 (10)) (1.0/2)}{0.04167} = 21598.2721 \text{ lb/in}^2$$

Since this is smaller than the permissible value of 26000 psi, we choose a smaller value of d next.

Let $d = 0.9$ in:

$$w = 0.45 \text{ lb}, W = 0.85 \text{ lb}, I = 0.03038 \text{ in}^4, \delta_{st} = \frac{0.85 (7.9994 (10^{-4}))}{0.9^3} = 9.3271 (10^{-4}) \text{ in}$$

$$\tau_n = 2 \pi \sqrt{\frac{\delta_{st}}{g}} = 2 \pi \sqrt{\frac{9.3271 (10^{-4})}{386.4}} = 0.009762 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.009762} = 10.2438$$

$$A_a \approx 2.0, P_d = A_a M a_s = (2.0) (0.85/g) (100g) = 170.0 \text{ lb}$$

$$\sigma_{max} = \frac{M_b c}{I} = \frac{(170 (10)) (0.9/2)}{0.03038} = 25181.0402 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we take $d = 0.9$ in.

4.53

Let d = thickness of bracket. Then from Example 4.18, self weight of beam = $w = 0.5 d$ lb, total weight at free end of beam = $W = 0.5 d + 0.4$ lb, moment of inertia of beam cross section = $I = 0.04167 d^3 \text{ in}^4$, static deflection of beam under W :

$$\delta_{st} = \left(\frac{0.5 d + 0.4}{d^3} \right) 7.9994 (10^{-4}) \text{ in}$$

We need to use a trial and error procedure to find the correct value of d . However, since the shock amplification factor, for large values of t_0/τ_n , for the triangular pulse of Fig. 4.66 is similar to that of the pulse shown in Fig. 4.15, we start with $d = 0.6$ in. This gives:

$$w = 0.3 \text{ lb}, W = 0.7 \text{ lb}, I = (0.04167) (0.216) = 0.009001 \text{ in}^4,$$

$$\delta_{st} = \frac{0.7}{0.6^3} (7.9994 (10^{-4})) = 25.9240 (10^{-4}) \text{ in}, \tau_n = 2 \pi \sqrt{\frac{25.9240 (10^{-4})}{386.4}} =$$

0.01627 sec , $t_0/\tau_n = (0.1/0.01627) = 6.1445$. From Fig. 4.66, we find shock amplification factor as $A_a \approx 1.1$, dynamic load at end of beam = $P_d = A_a M a_s = (1.1) (0.7/g) (100g) = 77.0 \text{ lb}$, maximum bending stress at root of beam = $\sigma_{max} = \frac{M_b c}{I} = (77(10)) (0.6/2) / (0.009001) = 25663.8151 \text{ psi}$. Since this stress is very close to the maximum permissible value, we select $d = 0.6$ in as the design.

4.54

Let d = thickness of bracket (beam). Self weight of beam = $w = d (1/2) (16) (0.1) = 0.8 d$ lb, total load at middle of beam = $W = (1 + 0.8 d)$ lb, area moment of inertia of beam cross section = $I = \frac{1}{12} (\frac{1}{2}) d^3 \text{ in}^4$. Static deflection of beam at middle due to W :

$$\delta_{st} = \frac{W \ell^3}{192 E I} = \frac{(1 + 0.8 d) (16^3)}{192 (10^7) (0.04167 d^3)} = (1 + 0.8 d) (51.1959 (10^{-6})) \text{ in}$$

We need to use a trial and error procedure to determine the correct value of d .

Let $d = 0.4$ in:

$$w = 0.32 \text{ lb}, W = 1.32 \text{ lb}, I = 0.002667 \text{ in}^4, \delta_{st} = 67.5786 (10^{-6}) \text{ in},$$

$$\tau_n = 2 \pi \sqrt{\frac{67.5786 (10^{-6})}{386.4}} = 0.002628 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002628} = 38.0569$$

From Fig. 4.15(b), $A_a \approx 1.1$, and the dynamic load on beam is given by $P_d = A_a M a_s$ where M = total mass of beam and a_s = acceleration due to shock = 100 g. Thus $P_d = (1.1) (1.32/g) (100 \text{ g}) = 145.2 \text{ lb}$. Maximum bending moment in a fixed-fixed beam due to load (F) at the middle is given by $M_b = \frac{F \ell}{8}$ so that

$$\sigma_{max} = \frac{M_b c}{I} = \frac{\left(\frac{145.2 (16)}{8} \right) \left(\frac{0.4}{2} \right)}{0.002667} = 21777.2778 \text{ lb/in}^2$$

Since this stress is smaller than the maximum permissible value of 26000 psi, we next select a smaller value of d.

Let d = 0.35 in:

$$w = 0.28 \text{ lb}, W = 1.28 \text{ lb}, I = 0.001787 \text{ in}^4, \delta_{st} = 65.5307 (10^{-6}) \text{ in},$$

$$\tau_n = 2 \pi \sqrt{\frac{65.5307 (10^{-6})}{386.4}} = 0.002587 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002587} = 38.6469$$

From Fig. 4.15(b), $A_a \approx 1.1$, $P_d = (1.1) (1.28/g) (100g) = 140.8 \text{ lb}$

$$\sigma_{max} = \frac{M_b c}{I} = \frac{\left(\frac{140.8 (16)}{8} \right) \left(\frac{0.35}{2} \right)}{0.001787} = 27576.9446 \text{ lb/in}^2$$

Since this stress exceeds the permissible value, we increase the value of d.

Let d = 0.37 in:

$$w = 0.296 \text{ lb}, W = 1.296 \text{ lb}, I = 0.002111 \text{ in}^4, \delta_{st} = 66.3499 (10^{-6}) \text{ in}$$

$$\tau_n = 2 \pi \sqrt{\frac{66.3499 (10^{-6})}{386.4}} = 0.002604 \text{ sec}$$

$$\frac{t_0}{\tau_n} = 38.4024$$

From Fig. 4.15(b), $A_a \approx 1.1$, $P_d = (1.1) (1.296/g) (100 g) = 142.56 \text{ lb}$, and

$$\sigma_{max} = \frac{M_b c}{I} = \frac{\left(\frac{142.56 (16)}{8} \right) \left(\frac{0.37}{2} \right)}{0.002111} = 24986.8309 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we select the design as d = 0.37 in.

4.55

$$W = m g = 100000 \text{ lb}, \zeta = 0.05, \sigma_y = 30000 \text{ psi},$$

$$\sigma_{max} = \text{maximum permissible stress} = \frac{\sigma_y}{2} = \frac{30000}{2} = 15000 \text{ psi},$$

$$\tau_n = \frac{2 \pi}{\omega_n} = 2 \pi \sqrt{\frac{m}{k}} = 2 \pi \sqrt{\frac{100000}{386.4 k}} = \frac{101.0793}{\sqrt{k}} \text{ sec}$$

We need to use a trial and error procedure to find k.

Let k = 10000 lb/in:

$$k = 10^4 = \frac{3 E I}{\ell^3} = \frac{3 (30 (10^6)) I}{600^3}$$

$$I = 24000 \text{ in}^4 = \frac{\pi}{64} d_o^4 (0.5904) \text{ with } \frac{d_i}{d_o} = 0.8$$

$$d_o^4 = 82.8121 (10^4) \text{ in}^4$$

$$d_o = 30.1664 \text{ in} ; d_i = 24.1331 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{10^4}} \approx 1 \text{ sec}$$

From Fig. 4.18, for $\tau_n = 1 \text{ sec}$ and $\zeta = 0.05$, we find $S_v = 25 \text{ in/sec}$, $S_d = 4.2 \text{ in}$ and $S_a = 0.42 \text{ g}$.

Maximum shear force in column:

$$F_{\max} = \frac{W}{g} S_a = \frac{100000}{g} (0.42g) = 42000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{\max} h = (42000) (50 (12)) = 25.2 (10^6) \text{ lb-in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b c}{I} = \frac{(25.2 (10^6)) (30.1664/2)}{24 (10^3)} = 15837.36 \text{ lb/in}^2$$

Since this stress is slightly smaller than the maximum permissible value, we choose a larger value of k .

Let $k = 20000 \text{ lb/in}$:

$$k = 2 (10^4) = \frac{3 (30 (10^6)) I}{600^3}$$

$$I = 48000 \text{ in}^4 = \frac{\pi}{64} d_o^4 (0.5904)$$

$$d_o^4 = 165.6243 (10^4) \text{ in}^4$$

$$d_o = 35.8741 \text{ in} ; d_i = 28.6993 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{20000}} = 0.7147 \text{ sec}$$

From Fig. 4.18, we find

$$S_v = 26 \text{ in/sec}, S_d = 3 \text{ in}, S_a = 0.6 \text{ g}$$

Maximum shear force in column:

$$F_{\max} = \frac{W}{g} S_a = \frac{10^5}{g} (0.6g) = 60000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{\max} h = (60000) (600) = 36 (10^6) \text{ lb-in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b c}{I} = \frac{(36 (10^6)) (35.8741/2)}{48 (10^3)} = 13452.7875 \text{ lb/in}^2$$

Since this stress is less than the maximum permissible value, we choose the inner and outer diameters of the column as $d_i = 28.6993 \text{ in}$ and $d_o = 35.8741 \text{ in}$.

4.56

$m g = 5000 \text{ lb}$, $\zeta = 0.02$. From Fig. 4.19, in order to have $S_a \approx 1 \text{ g}$, we need to have τ_n = 0.2 sec. Thus

$$\tau_n = 0.2 = \frac{2\pi}{\omega_n} ; \quad \omega_n = 31.416 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$
$$k = (31.416)^2 m = (31.416^2) (5000/386.4) = 12771.2870 \text{ lb/in}$$

4.58

$$\text{Equation of motion is } \ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega t}$$

For zero initial conditions

$$(s^2 + \omega_n^2) \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{s - i\omega} ; \quad \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{(s^2 + \omega_n^2)} \cdot \frac{s + i\omega}{(s^2 + \omega^2)}$$

Inverse Laplace transformation gives

$$x(t) = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left\{ e^{i\omega t} - (\cos \omega_n t + \frac{i\omega}{\omega_n} \sin \omega_n t) \right\}$$

The terms containing $\cos \omega_n t$ and $\sin \omega_n t$ denote the transient

part of the response and hence can be neglected. Thus the steady state response can be expressed as

$$x(t) = \frac{F_0}{k} \left(\frac{1}{1 - r^2} \right) e^{i\omega t} \quad \text{where } r = \omega/\omega_n.$$

4.59

$$\bar{F}(s) = \frac{F_0}{s}$$

The Laplace transform of the response can be written as

$$\begin{aligned} \bar{x}(s) &= \frac{F_0}{ms(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) x_0 \\ &\quad + \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \dot{x}_0 \end{aligned}$$

Inverse transformation gives

$$\begin{aligned} x(t) &= \frac{F_0}{m} \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi_1) \right\} + \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) \\ &\quad + \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \end{aligned}$$

$$\text{where } \phi_1 = \cos^{-1}(\zeta).$$

The forcing function can be expressed as

4.60

$$F(t) = F_0 \{ u(t) - u(t - t_0) \}$$

where $u(t - \tau)$ is the unit step function applied at $t = \tau$. The Laplace transform of $F(t)$ is

$$\bar{F}(s) = F_0 \left(\frac{1}{s} - \frac{1}{s} e^{-s t_0} \right)$$

The equation of motion $m\ddot{x} + kx = F(t)$ gives

$$\bar{x}(s) = \frac{F_0}{s} \left(1 - e^{-st_0} \right) \frac{1}{ms^2 + k} = \frac{F_0}{m} \left\{ \frac{1}{s(s^2 + \omega_n^2)} - \frac{e^{-st_0}}{s(s^2 + \omega_n^2)} \right\}$$

Since $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 + \omega^2)} \right\} = \frac{1}{\omega^2} (1 - \cos \omega t)$,

$$x(t) = \frac{F_0}{m \omega_n^2} (1 - \cos \omega_n t) \cdot u(t) - \frac{F_0}{m \omega_n^2} \{ 1 - \cos \omega_n (t - t_0) \} u(t - t_0)$$

Hence

$$x(t) = \begin{cases} \frac{F_0}{m \omega_n^2} (1 - \cos \omega_n t) & \text{for } 0 \leq t \leq t_0 \\ \frac{F_0}{m \omega_n^2} \{ \cos \omega_n (t - t_0) - \cos \omega_n t \} & \text{for } t \geq t_0 \end{cases}$$

4.84

Method 1: $x_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left\{ 1 - \cos \omega_n (t_j - t_i) \right\}$

$$\dot{x}_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \omega_n \sin \omega_n (t_j - t_i)$$

Method 2:

$$x_j = \frac{F_j}{k} [1 - \cos \omega_n \Delta t_j] + x_{j-1} \cos \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j$$

$$\dot{x}_j = \frac{F_j \omega_n}{k} \sin \omega_n \Delta t_j + \omega_n \left\{ -x_{j-1} \sin \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \cos \omega_n \Delta t_j \right\}$$

Method 3:

$$x_j = \frac{\Delta F_j}{k \Delta t_j} \left\{ \Delta t_j - \frac{1}{\omega_n} \sin \omega_n \Delta t_j \right\} - \frac{F_{j-1}}{k} (1 - \cos \omega_n \Delta t_j) + x_{j-1} \cos \omega_n \Delta t_j$$

$$+ \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j$$

$$\dot{x}_j = \frac{\Delta F_j}{\kappa \Delta t_j} (1 - \cos \omega_n \Delta t_j) + \frac{F_{j-1}}{\kappa} \omega_n \sin \omega_n \Delta t_j + \dot{x}_{j-1} \cos \omega_n \Delta t_j \\ - \omega_n x_{j-1} \sin \omega_n \Delta t_j$$

VALUE OF	Method 1	Method 1	Method 2	Method 3
	Step variation with larger value	Step variation with smaller value	Step variation with mid-value	Linear variation
I	X(I)	X(I)	X(I)	X(I)
2	0.489135E+01	0.412870E+01	0.451034E+01	0.463829E+01
3	0.183327E+00	0.153639E+00	0.168413E+00	0.170863E+00
4	0.374871E+00	0.311494E+00	0.342969E+00	0.346380E+00
5	0.590262E+00	0.485757E+00	0.537537E+00	0.541621E+00
6	0.794774E+00	0.646981E+00	0.720014E+00	0.724433E+00
7	0.955998E+00	0.768556E+00	0.860896E+00	0.865287E+00
8	0.104732E+01	0.829582E+00	0.936446E+00	0.940461E+00
9	0.105081E+01	0.817133E+00	0.931268E+00	0.934605E+00
10	0.959172E+00	0.727695E+00	0.840009E+00	0.842438E+00
11	0.776638E+00	0.566437E+00	0.668027E+00	0.669410E+00
I	XD(T)	XD(I)	XD(I)	XD(I)
2	0.309017E+00	0.260676E+00	0.284772E+00	0.284646E+00
3	0.539444E+00	0.448685E+00	0.493775E+00	0.493286E+00
4	0.669917E+00	0.547974E+00	0.608327E+00	0.607283E+00
5	0.690013E+00	0.552279E+00	0.620126E+00	0.618406E+00
6	0.601222E+00	0.465650E+00	0.531993E+00	0.529561E+00
7	0.416706E+00	0.301948E+00	0.357498E+00	0.354412E+00
8	0.159909E+00	0.833536E-01	0.119506E+00	0.115921E+00
9	-0.137878E+00	-0.161957E+00	-0.152198E+00	-0.156045E+00
10	-0.440725E+00	-0.402734E+00	-0.423989E+00	-0.427799E+00
11	-0.711751E+00	-0.615408E+00	-0.661862E+00	-0.665302E+00

4.88 $m = 10 \text{ kg}$, $k = 4000 \text{ N/m}$, $c = 40 \text{ N-s/m}$, $F = 100 \text{ N-s}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{40}{2\sqrt{4000(10)}} = 0.1$$

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = \sqrt{1 - 0.1^2} (20) = 19.899749 \text{ rad/s}$$

Assuming that impact is given at $t=0$, we find the response of the system as (from Eq. (4.26)):

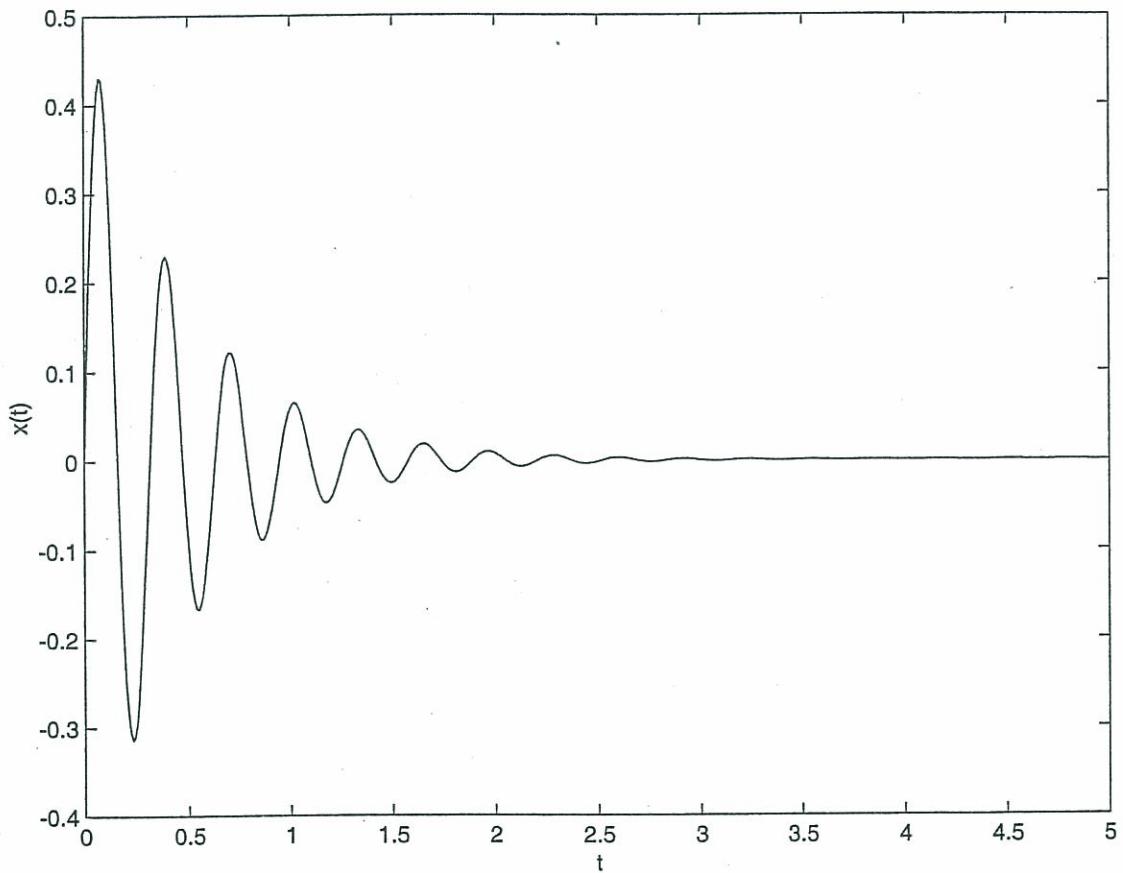
$$x_1(t) = \frac{F}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

$$= (100) \frac{e^{-(0.1)(20)t}}{10(19.899749)} \sin 19.899749 t$$

$$\therefore x_1(t) = 0.502519 e^{-2t} \sin 19.899749 t \quad (E \cdot 1)$$

Plotting of Eq. (E.1) using MATLAB:

```
% Ex4_88.m
for i = 1: 501
    t(i) = 5*(i-1)/500;
    x(i) = 0.502519 * exp(-2.0*t(i)) * sin( 19.899749*t(i) );
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



4.89 $\omega_n = 20 \text{ rad/s}$, $\zeta = 0.1$, $\omega_d = 19.899749 \text{ rad/s}$

Response due to $F_1 \delta(t)$ is given by Eq. (E.1) of Problem 4.88.

Response due to $F_2 \delta(t - 0.5)$ can be found from Eqs.

(4.27) and (4.26):

$$x_2(t) = F_2 \frac{e^{-\zeta \omega_n (t - \tau)}}{m \omega_d} \sin \omega_d (t - \tau) \quad (E \cdot 2)$$

For $\tau = 0.5$, Eq. (E.2) gives

$$x_2(t) = \frac{50 e^{-0.1(20)(t-0.5)}}{10 (19.899749)} \sin 19.899749(t-0.5)$$

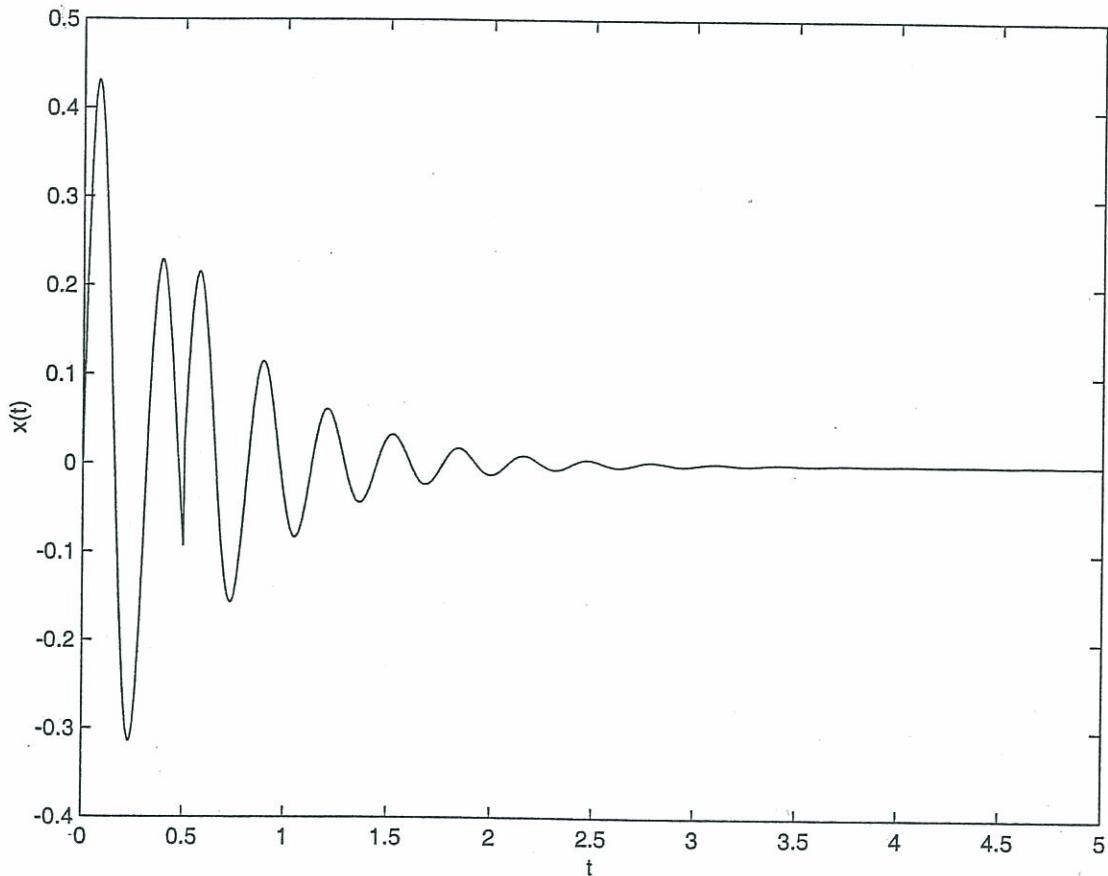
$$= 0.251259 e^{-2(t-0.5)} \sin 19.899749(t-0.5)$$

The response due to two impacts (in meters) is given by

$$x(t) = \begin{cases} 0.502519 e^{-2t} \sin 19.899749 t & ; 0 \leq t \leq 0.5 \\ 0.251259 e^{-2(t-0.5)} \sin 19.899749(t-0.5) & ; t > 0.5 \end{cases} \quad (E \cdot 3)$$

Plotting of Eq. (E·3) using MATLAB:

```
% Ex4_89.m
for i = 1: 1001
    t(i) = (i-1)*5/1000;
    if t(i) <= 0.5
        x(i) = 0.502519 * exp(-2*t(i)) * sin(19.899749*t(i));
    else
        x(i) = 0.251259 * exp(-2*(t(i)-0.5)) * sin(19.899749*(t(i)-0.5));
    end
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



4.90

Response due to step load (see Eq.(E-1) of Example 4.11):

$$x(t) = \frac{F_0 e^{-\zeta \omega_n t}}{\zeta \sqrt{1-\zeta^2}} \left\{ -\cos(\omega_d t - \phi) + e^{\zeta \omega_n t_0} \cos[\omega_d(t-t_0) - \phi] \right\}$$

with $\phi = \tan^{-1} \left\{ \frac{\zeta}{\sqrt{1-\zeta^2}} \right\}$

Data: $m = 100 \text{ kg}$, $k = 1200 \text{ N/m}$, $c = 50 \text{ N-s/m}$,

$F_0 = 100 \text{ N}$, $t_0 = 0.1 \text{ s}$ and 1.5 s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{12} = 3.464102 \text{ rad/s},$$

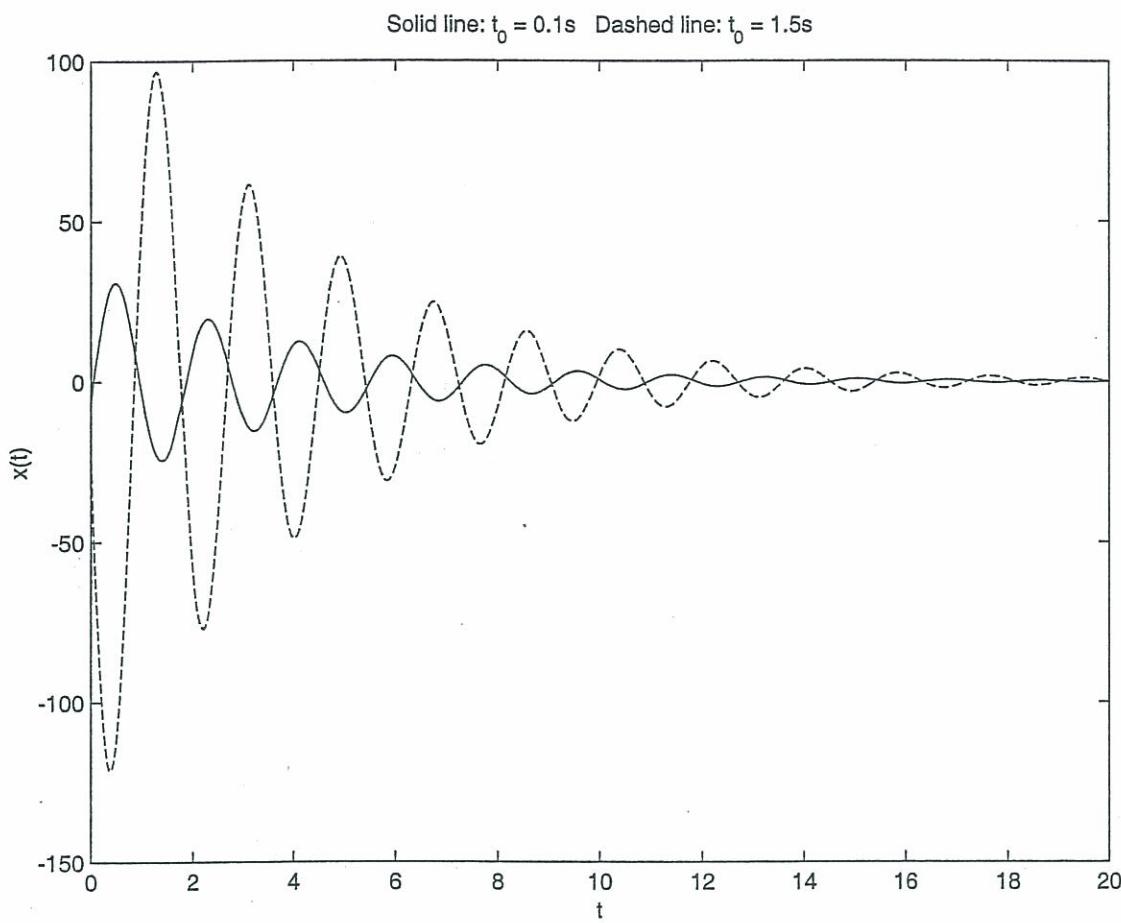
$$\zeta = \frac{c}{2\sqrt{km}} = \frac{50}{2\sqrt{(1200)(100)}} = \frac{25}{346.410161} = 0.072169$$

$$\omega_d = \sqrt{1-\zeta^2} \omega_n = 3.455069 \text{ rad/s}$$

$$\phi = \tan^{-1} \left(\frac{0.072169}{0.997392} \right) = 0.072232 \text{ rad}$$

Plotting:

```
% Ex4_90.m
F0 = 100;
m = 100;
k = 1200;
c = 50;
wn = 3.464102;
zeta = 0.072169;
wd = 3.455069;
phi = 0.072232;
t0 = 0.1;
for i = 1: 501
    t(i) = 20*(i-1)/500;
    x1(i) = F0 * exp(-zeta*wn*t(i)) * (-cos(wd*t(i)-phi) + ...
        exp(zeta*wn*t0) * cos(wd*(t(i)-t0)-phi));
end
t0 = 1.5;
for i = 1: 501
    t2(i) = 20*(i-1)/500;
    x2(i) = F0 * exp(-zeta*wn*t2(i)) * (-cos(wd*t2(i)-phi) + ...
        exp(zeta*wn*t0) * cos(wd*(t2(i)-t0)-phi));
end
plot(t,x1);
xlabel('t');
ylabel('x(t)');
title('Solid line: t_0 = 0.1s Dashed line: t_0 = 1.5s');
hold on;
plot(t2,x2,'--');
```



4.91

```
%=====
%
%Program4.m
%Main program which calls PERIOD
%
%=====
%
%Run "Program4" in MATLAB command window. Program4.m and period.m
%should be in the same folder, and set the path to this folder
%following seven lines contain problem-dependent data
xm=1.0;
xk=400.0;
xai=0.125;
n=20;
m=10;
time=1;
f=[62.5 125 187.5 250 312.5 375 437.5 500 zeros(1,12)]';
t=0.05:0.05:1.00;
%end of problem-dependent data
[xsin,xcos,psi,phi,fzero,fc,x,xpc,xps]=period(xm,xk,xai,n,m,time,f,t);
fprintf('Response of a single D.O.F. system under periodic force\n\n');
fprintf('xm    = %10.8e\n',xm);
fprintf('xk    = %10.8e\n',xk);
fprintf('xai   = %10.8e\n',xai);
fprintf('n     = %2.0f\n',n);
fprintf('m     = %2.0f\n',m);
fprintf('time = %10.8e\n\n',time);
```

```

fprintf('Applied force and response: \n\n');
fprintf('    i           t(i)           f(i)           x(i)')↵
;
fprintf('\n\n');
for i=1:n
    fprintf('  %2.0f    %10.8e    %10.8e    %10.8e\n',i,...)
    t(i),f(i),x(i));
end
subplot(121);
plot(t,f);
xlabel('t');
ylabel('F(t)');
%title('F(t)');
subplot(122);
plot(t,x);
%title('x(t)');
xlabel('t');
ylabel('x(t)');

%=====
%=====

%function period.m
%
%=====
%=====

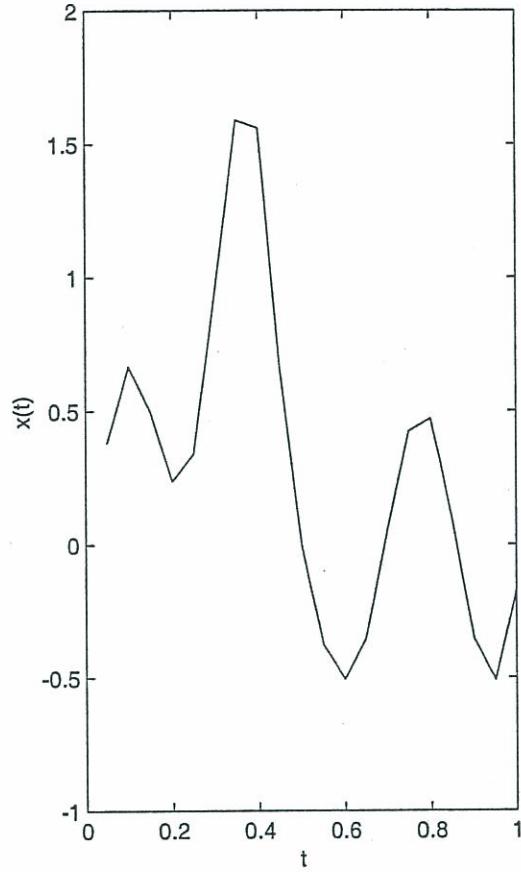
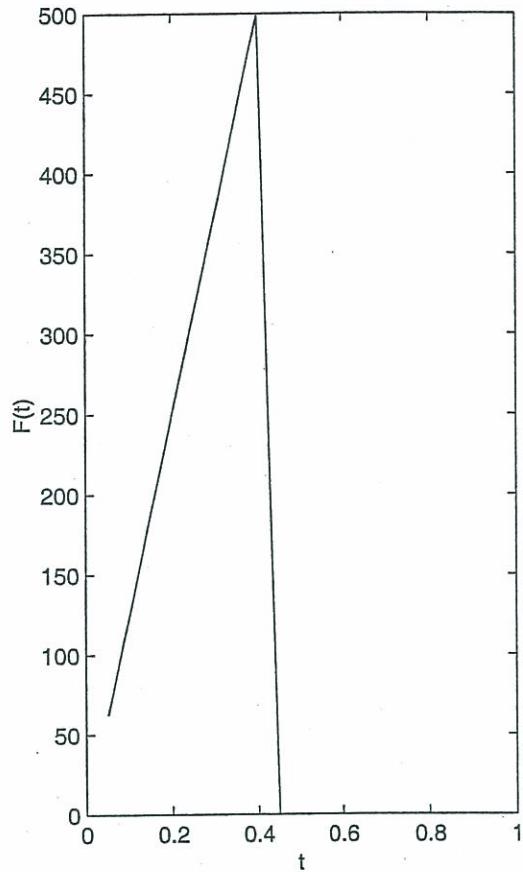
function [xsin,xcos,psi,phi,fzero,fc,x,xpc,xps]=period(xm,xk,xai,n,m,time,f,↵
,t)
omeg=2.0*3.1416/time;
omegn=sqrt(xk/xm);
sumz=0.0;
for i=1:n
    sumz=sumz+f(i);
end
fzero=2.0*sumz/n;
for j=1:m
    sums=0.0;
    sumc=0.0;
    for i=1:n
        theta=j*omeg*t(i);
        fsin=f(i)*sin(theta);
        fcos=f(i)*cos(theta);
        sums=sums+fsin;
        sumc=sumc+fcos;
    end
    aj=2.0*sumc/n;
    bj=2.0*sums/n;
    r=omeg/omegn;
    phi(j)=atan(2.0*xai*j*r/(1-(j*r)^2));
    con=sqrt((1.0-(j*r)^2)^2+(2.0*xai*j*r)^2);
    xpc(j)=(aj/xk)/con;
    xps(j)=(bj/xk)/con;
end
for i=1:n

```

```

x(i)=fzero/2/xk;
for j=1:m
    x(i)=x(i)+xpc(j)*cos(j*omeg*t(i)-phi(j))+xps(j)*sin(j*omeg*t(i)-phi(j))
);
end
end

```



Results of Ex4_91

```
*****

```

```

>> program4
xm = 1.0000000e+000
xk = 4.0000000e+002
xai = 1.2500000e-001
n = 20
m = 10
time = 1.0000000e+000

```

Applied force and response:

i	$t(i)$	$f(i)$	$x(i)$
1	5.0000000e-002	6.2500000e+001	3.78789292e-001
2	1.0000000e-001	1.2500000e+002	6.65286858e-001
3	1.5000000e-001	1.8750000e+002	5.00283762e-001
4	2.0000000e-001	2.5000000e+002	2.37031525e-001
5	2.5000000e-001	3.1250000e+002	3.39025654e-001
6	3.0000000e-001	3.7500000e+002	9.42110868e-001
7	3.5000000e-001	4.3750000e+002	1.58854150e+000
8	4.0000000e-001	5.0000000e+002	1.55862192e+000
9	4.5000000e-001	0.0000000e+000	6.56373440e-001
10	5.0000000e-001	0.0000000e+000	-5.23632679e-003

11	5.5000000e-001	0.00000000e+000	-3.77065247e-001
12	6.0000000e-001	0.00000000e+000	-5.06616825e-001
13	6.5000000e-001	0.00000000e+000	-3.57481271e-001
14	7.0000000e-001	0.00000000e+000	4.83338999e-002
15	7.5000000e-001	0.00000000e+000	4.21581779e-001
16	8.0000000e-001	0.00000000e+000	4.69545854e-001
17	8.5000000e-001	0.00000000e+000	1.02656116e-001
18	9.0000000e-001	0.00000000e+000	-3.54840551e-001
19	9.5000000e-001	0.00000000e+000	-5.10135040e-001
20	1.0000000e+000	0.00000000e+000	-1.71815377e-001

4.92

```
%=====
%
%Program5.m
%Response of a single D.O.F. system under arbitrary forcing function
%using the methods of section 4.11
%
%=====
% Run "Program5" in MATLAB command window. Program5.m should be in one
% folder, and set the path to this folder
% following 11 lines contain problem-dependent data
xai=0.1;
omn=31.622777;
xk=1e5;
delt=0.1;
np=21;
np1=20;
np2=19;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 (using step variation with larger value, Approach 1)
t(1)=0;
f(1) = 0;
ff(1) = 0;
for i=2:np
    t(i)=t(i-1)+delt;
    f(i) = 1000*(1-cos(pi*t(i)));
    ff(i) = 1000*( 1-cos( pi*( t(i)+t(i-1) )/2 ) );
end
for i=1:np1
    delf(i)=f(i+1)-f(i);
end
for j=2:np
    x(j)=0;
    xd(j)=0;
    jm1=j-1;
    for i=1:jm1
        x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*...
            (cos(pd*(t(j)-t(i)))+(xn/pd)*sin(pd*(t(j)-t(i))))));
        xd(j)=xd(j)+(delf(i)/xk)*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
    end
end
for i=2:np
    x1(i)=x(i);
```

```

    xd1(i)=xd(i);
end
%solution according to method 1 (using step variation with smaller value, Approach 2)
for k=2:np2
    delf(k)=delf(k+1);
end
delf(1)=f(3);
delf(np)=f(np);
for j=2:np
    x(j)=0;
    xd(j)=0;
    jm1=j-1;
    for i=1:jm1
        x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*...
            (cos(pd*(t(j)-t(i)))+(xn/pd)*sin(pd*(t(j)-t(i))))));
        xd(j)=xd(j)+(delf(i)/xk).*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
    end
end
for i=2:np
    x2(i)=x(i);
    xd2(i)=xd(i);
end
%solution according to method 2 (using step variation with mid-value, Approach 3)
x(1)=0;
xd(1)=0;
for j=2:np
    del=delt;
    x(j)=(ff(j)/xk)*(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del))...
        +exp(-xn*del)*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del)));
    xd(j)=(ff(j)*pd/xk)*exp(-xn*del)*(1+xn^2/(pd^2))*sin(pd*del) + pd*...
        exp(-xn*del)*(-x(j-1)*sin(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*cos(pd*del))...
        -xn*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del))/pd);
end
for i=2:np
    x3(i)=x(i);
    xd3(i)=xd(i);
end
%solution according to method 3 (using linear variation, Approach 4)
x(1)=0;
xd(1)=0;
for j=1:np1
    f(j)=f(j+1);
end
f(np)=0;
for j=2:np
    delf(j)=f(j)-f(j-1);
    x(j)=(delf(j)/(xk*del))*(del-(2*xai/omn)+exp(-xn*del)*((2*xai/omn)*...
        *cos(pd*del)-((pd^2-xn^2)/(omn*omn*pd))*sin(pd*del))+(f(j-1)/xk)*...
        *(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))+exp(-xn*del)*...
        (x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del)));
    % xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(((xn^2+pd^2)/(omn^2))*...
    %     cos(pd*del)+((xn^3+xn*pd*pd/(pd*(omn^2)))*sin(pd*del)))+(f(j-1)/xk).%
end

```

```

    % *exp(-xn*del)*(xn^2/pd)+pd)*sin(pd*del)+exp(-xn*del)*(xd(j-1)*...
    % cos(pd*del)-((xn*xd(j-1)+xn*xn*x(j-1)+pd*pd*x(j-1))/pd)*sin(pd*del)))*
;
    xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(cos(pd*del)+xn/pd*...
    sin(pd*del)))+f(j-1)/xk*exp(-xn*del)*omn^2/pd*sin(pd*del)+exp(-xn*del)...*
    *(xd(j-1)*cos(pd*del)-xn/pd*(xd(j-1)+omn/xai*x(j-1))*sin(pd*del));
end
for i=2:np
    x4(i)=x(i);
    xd4(i)=xd(i);
end
fprintf...
('Value      Approach #1      Approach #2      Approach #32      Approach #4\n' );
fprintf...
(' of      (Fig. 4.36)      (Fig. 4.36)      (Fig. 4.36)      (Fig. 4.36) ');
fprintf('\n\n');
fprintf...
(' I      x(i)      x(i)      x(i)      x(i)\n\n');
for i=2:np
    fprintf...
    ('%2.0f  %8.6e  %8.6e  %8.6e  %8.6e\n',i,x1(i),x2(i),x3(i),...
    x4(i));
end
fprintf...
(' \n I      xd(i)      xd(i)      xd(i)      xd(i)');
fprintf('\n\n');
for i=2:np
    fprintf...
    ('%2.0f  %8.6e  %8.6e  %8.6e  %8.6e\n',i,xd1(i),xd2(i),...
    xd3(i),xd4(i));
end
subplot(221);
plot(t,x1);
hold on;
plot(t,xd1);
xlabel('t');
ylabel('x(t)  xd(t)');
title('Method #1 (Fig. 4.36)');
subplot(222);
plot(t,x2);
hold on;
plot(t,xd2);
xlabel('t');
ylabel('x(t)  xd(t)');
title('Method #1 (Fig. 4.36)');
subplot(223);
plot(t,x3);
hold on;
plot(t,xd3);
xlabel('t');
ylabel('x(t)  xd(t)');
title('Method #2 (Fig. 4.36)');
subplot(224);

```

```

plot(t,x4);
hold on;
plot(t,xd4);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #3 (Fig. 4.36)');

```

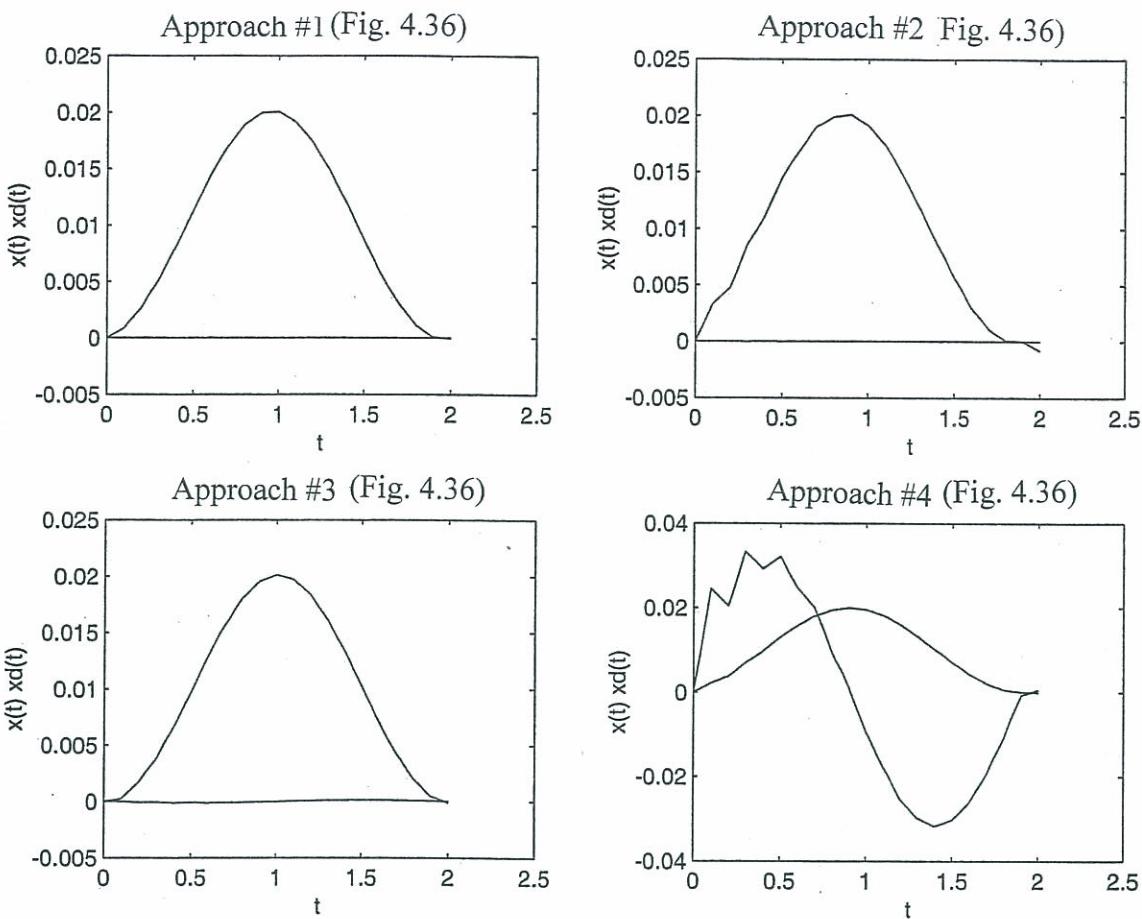
Results of Ex4_92

>> program5

Value of	Approach #1 (Fig. 4.36)	Approach #2 (Fig. 4.36)	Approach #3 (Fig. 4.36)	Approach #4 (Fig. 4.36)
-------------	----------------------------	----------------------------	----------------------------	----------------------------

I	x(i)	x(i)	x(i)	x(i)
2	8.463498e-004	3.302553e-003	2.128980e-004	2.112992e-003
3	2.685367e-003	4.719852e-003	1.729506e-003	3.731773e-003
4	5.169912e-003	8.506840e-003	3.803611e-003	6.890461e-003
5	8.178657e-003	1.108890e-002	6.668076e-003	9.675896e-003
6	1.132821e-002	1.454957e-002	9.724658e-003	1.298875e-002
7	1.437507e-002	1.684652e-002	1.290593e-002	1.564646e-002
8	1.697375e-002	1.899708e-002	1.573148e-002	1.801666e-002
9	1.890432e-002	1.988503e-002	1.804796e-002	1.940816e-002
10	1.995267e-002	2.008383e-002	1.953876e-002	2.002060e-002
11	2.003452e-002	1.909252e-002	2.012346e-002	1.954838e-002
12	1.912848e-002	1.735920e-002	1.969707e-002	1.821740e-002
13	1.733299e-002	1.479759e-002	1.833615e-002	1.602653e-002
14	1.481670e-002	1.184504e-002	1.614852e-002	1.328634e-002
15	1.183110e-002	8.654518e-003	1.336683e-002	1.019478e-002
16	8.664674e-003	5.637520e-003	1.024989e-002	7.105849e-003
17	5.630116e-003	3.017069e-003	7.112632e-003	4.284352e-003
18	3.022466e-003	1.102377e-003	4.254988e-003	2.033946e-003
19	1.098443e-003	4.244805e-005	1.961909e-003	5.548287e-004
20	4.531599e-005	-3.095846e-005	4.540486e-004	6.474018e-006
21	-3.304897e-005	-8.237715e-004	-1.182168e-004	-4.635301e-006

I	xd(i)	xd(i)	xd(i)	xd(i)
2	-1.724478e-006	-6.729107e-006	-1.378674e-005	2.450722e-002
3	-2.490738e-006	2.014596e-006	-8.816757e-005	2.042474e-002
4	-3.247781e-006	-9.183878e-006	-7.008044e-005	3.328193e-002
5	-3.764329e-006	1.429730e-006	-1.344394e-004	2.918735e-002
6	-3.674900e-006	-8.092879e-006	-9.999247e-005	3.213625e-002
7	-3.530832e-006	1.215821e-006	-1.331627e-004	2.478209e-002
8	-2.722577e-006	-5.267651e-006	-8.596171e-005	2.017583e-002
9	-1.950123e-006	2.028444e-006	-8.738285e-005	9.855085e-003
10	-7.153334e-007	-1.882855e-006	-3.287885e-005	1.272298e-003
11	3.543858e-007	3.391564e-006	-1.391042e-005	-9.388100e-003
12	1.587911e-006	1.060844e-006	3.774625e-005	-1.771827e-002
13	2.501541e-006	4.446538e-006	6.063053e-005	-2.533370e-002
14	3.304589e-006	2.776498e-006	9.749376e-005	-2.973266e-002
15	3.675776e-006	4.478047e-006	1.091071e-004	-3.175330e-002
16	3.773808e-006	2.884849e-006	1.223568e-004	-3.028136e-002
17	3.433694e-006	3.237571e-006	1.140193e-004	-2.612284e-002
18	2.811634e-006	1.542582e-006	1.019865e-004	-1.920680e-002
19	1.871907e-006	1.035826e-006	7.419301e-005	-1.055539e-002
20	7.820400e-007	-6.050694e-007	4.359287e-005	-7.662515e-004
21	-4.100735e-007	2.056210e-006	5.299422e-006	5.589628e-004

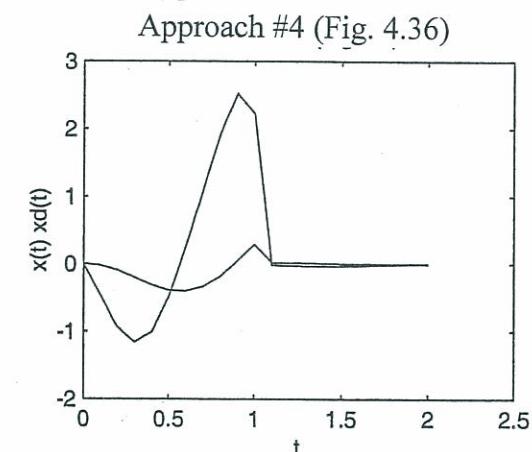
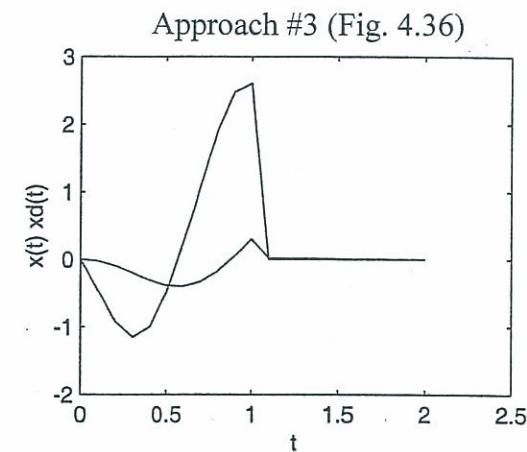
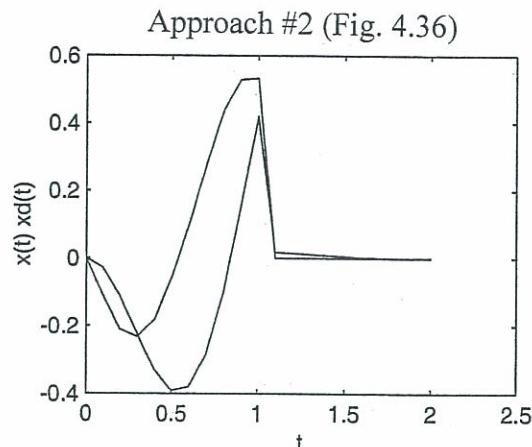
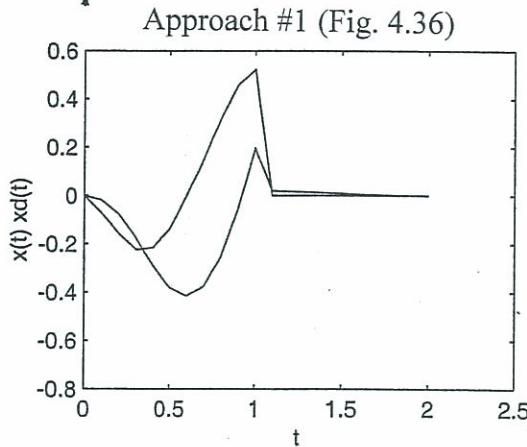


4.93

```
%=====
%
%Program5.m
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.8
%
%=====
%
% Run "Program5" in MATLAB command window. Program5.m should be in one
% folder, and set the path to this folder
% following 11 lines contain problem-dependent data
xai=0.1;
omn=5;
xk=50;
delt=0.1;
np=11;
np1=10;
np2=9;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 (using step variation with larger value, Approach 1)
t(1)=0;
for i=2:np
    t(i)=t(i-1)+delt;
end
```

```

f=[0.0 -8.0 -12.0 -15.0 -13.0 -11.0 -7.0 -4.0 3.0 10.0 15.0];
ff=[-8.0 -10.0 -13.5 -14.0 -12.0 -9.0 -5.5 -0.5 6.5 12.5 16.5];
for i=1:npl
    delf(i)=f(i+1)-f(i);
end
.
.
.
```



Results of Ex4_93

>> program5

Value of I	Approach #1 (Fig. 4.36)	Approach #2 (Fig. 4.36)	Approach #3 (Fig. 4.36)	Approach #4 (Fig. 4.36)
I	x(i)	x(i)	x(i)	x(i)
2	-1.895258e-002	-2.842886e-002	-2.369072e-002	-2.216415e-002
3	-7.844079e-002	-1.105540e-001	-9.449737e-002	-9.232748e-002
4	-1.770172e-001	-2.242651e-001	-2.006411e-001	-2.003034e-001
5	-2.901286e-001	-3.307430e-001	-3.104358e-001	-3.118343e-001
6	-3.807183e-001	-3.918096e-001	-3.862639e-001	-3.896705e-001
7	-4.157648e-001	-3.812882e-001	-3.985265e-001	-4.025740e-001
8	-3.761180e-001	-2.849867e-001	-3.305524e-001	-3.357629e-001
9	-2.546674e-001	-1.008869e-001	-1.777772e-001	-1.828250e-001
10	-5.487595e-002	1.476431e-001	4.638356e-002	4.342363e-002
11	1.971323e-001	4.198656e-001	3.061299e-001	2.920608e-001
I	xd(i)	xd(i)	xd(i)	xd(i)
2	-7.263201e-002	-1.089480e-001	-4.562370e-001	-4.597525e-001

3	-1.577455e-001	-2.093812e-001	-9.224406e-001	-9.313379e-001
4	-2.252429e-001	-2.333148e-001	-1.152170e+000	-1.162393e+000
5	-2.156793e-001	-1.824523e-001	-1.000343e+000	-1.007436e+000
6	-1.386163e-001	-5.760411e-002	-4.930222e-001	-4.919218e-001
7	-2.742288e-004	9.602153e-002	2.405741e-001	2.518514e-001
8	1.522037e-001	2.762085e-001	1.076426e+000	1.100724e+000
9	3.182622e-001	4.384471e-001	1.901304e+000	1.938544e+000
10	4.579186e-001	5.284869e-001	2.478437e+000	2.523778e+000
11	5.229886e-001	5.322201e-001	2.605688e+000	2.224476e+000

4.94

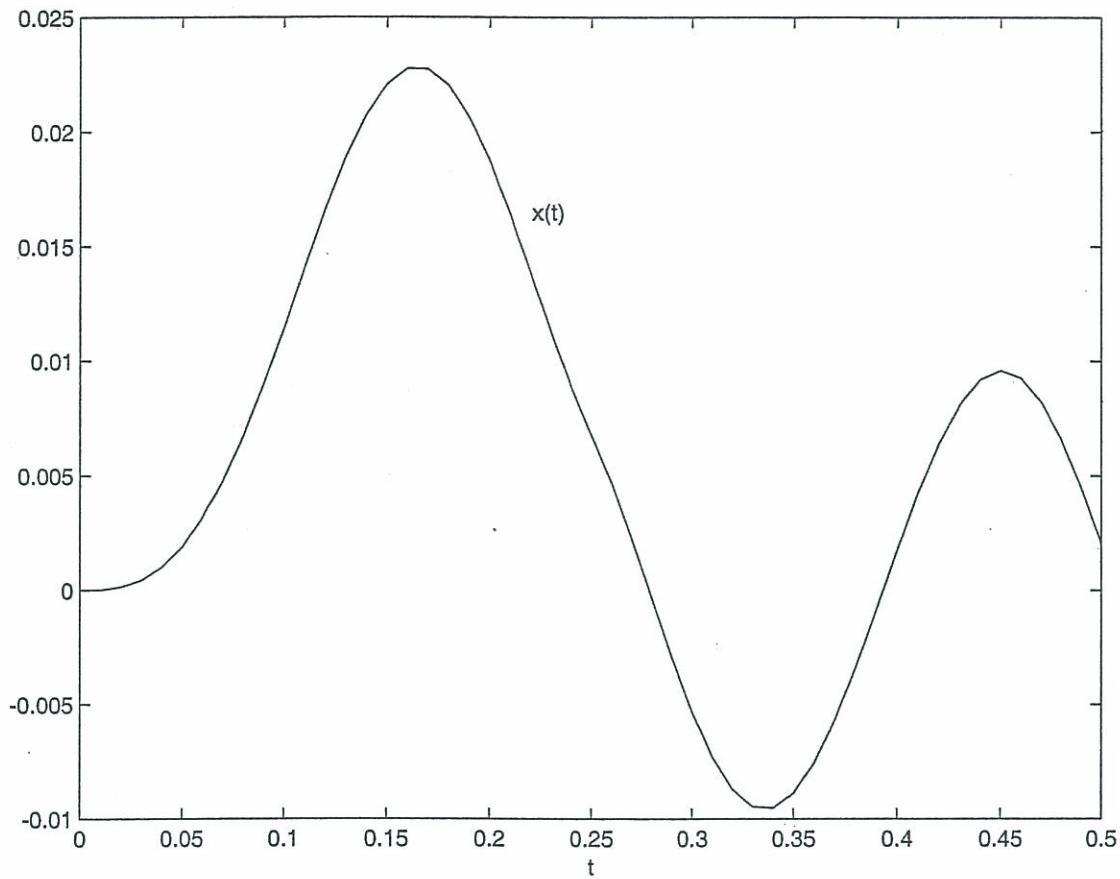
```
% Ex4_94.m
% This program will use the function dfunc4_94.m , they should
% be in the same folder
tspan = [0: 0.01: 0.5];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc4_94', tspan, x0);
disp('      t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');

% dfunc4_94.m
function f = dfunc4_94(t,x)
F = 200*t - 200*t*stepfun(t,0.1)...
    +20*stepfun(t,0.1)-20*stepfun(t,0.25);
f = zeros(2,1);
f(1) = x(2);
f(2) = F/2 - 1500 * x(1)/2;
```

Results of Ex4_94:

>> Ex4_94.m

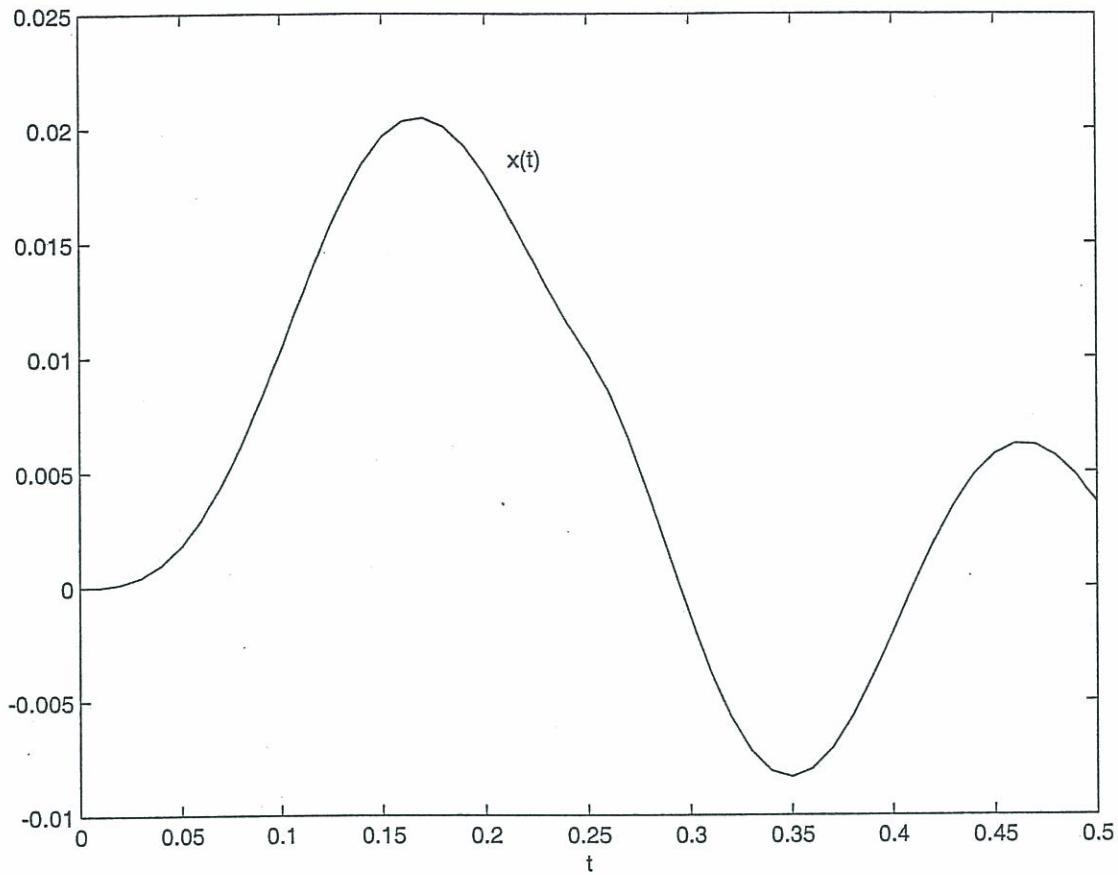
t	x(t)	xd(t)
0	0	0
0.0100	0.0000	0.0050
0.0200	0.0001	0.0195
0.0300	0.0004	0.0425
0.0400	0.0010	0.0723
0.0500	0.0019	0.1067
0.0600	0.0031	0.1430
.		
.		
.		
0.4100	0.0043	0.2352
0.4200	0.0064	0.1949
0.4300	0.0081	0.1400
0.4400	0.0092	0.0748
0.4500	0.0096	0.0039
0.4600	0.0093	-0.0671
0.4700	0.0083	-0.1332
0.4800	0.0066	-0.1893
0.4900	0.0045	-0.2313
0.5000	0.0021	-0.2560



4.95

```
% Ex4_95.m
% This program will use the function dfunc4_95.m , they should
% be in the same folder
tspan = [0: 0.01: 0.5];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc4_95 ', tspan, x0);
disp('      t          x(t)        xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');

% dfunc4_95.m
function f = dfunc4_95(t,x)
F = 200*t - 200*t*stepfun(t,0.1)...
    +20*stepfun(t,0.1)-20*stepfun(t,0.25);
f = zeros(2,1);
f(1) = x(2);
f(2) = F/2 - 10 * x(2)/2 - 1500 * x(1)/2;
```



Results of Ex4_95

>> Ex4_95

t	x(t)	xd(t)
0	0	0
0.0100	0.0000	0.0049
0.0200	0.0001	0.0189
0.0300	0.0004	0.0405
0.0400	0.0010	0.0678
0.0500	0.0018	0.0986
0.0600	0.0029	0.1303
0.0700	0.0044	0.1608
0.0800	0.0061	0.1877
0.0900	0.0081	0.2093
0.1000	0.0103	0.2244

0.4100	-0.0000	0.1962
0.4200	0.0019	0.1797
0.4300	0.0035	0.1510
0.4400	0.0049	0.1127
0.4500	0.0058	0.0680
0.4600	0.0062	0.0206
0.4700	0.0062	-0.0259
0.4800	0.0057	-0.0683
0.4900	0.0048	-0.1037
0.5000	0.0037	-0.1298

$$4.96 \quad x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\tau\omega_n(t-\tau)} \sin \omega_d(t-\tau) d\tau$$

But $\sin \omega_d(t-\tau) = \sin \omega_d t \cos \omega_d \tau - \cos \omega_d t \sin \omega_d \tau$

$$x(t) = \{A(t) \cdot \sin \omega_d t - B(t) \cdot \cos \omega_d t\} \frac{e^{-\tau\omega_n t}}{m\omega_d}$$

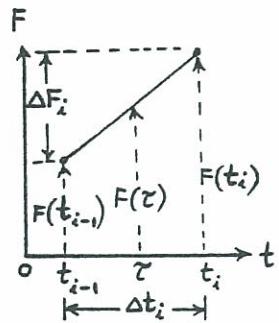
where

$$A(t) = \int_0^t F(\tau) e^{\tau\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t) = \int_0^t F(\tau) e^{\tau\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Let $F(\tau)$ be taken as a piecewise linear function during (t_{i-1}, t_i) as

$$F(\tau) = F_{i-1} + \left(\frac{\tau - t_{i-1}}{t_i - t_{i-1}} \right) (F_i - F_{i-1})$$



We can write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i \right) e^{\int \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) e^{\int \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$= A(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_1 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) P_2$$

where

$$P_1 = \int_{t_{i-1}}^{t_i} \tau e^{\int \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$P_2 = \int_{t_{i-1}}^{t_i} e^{\int \omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i} \Delta F_i \right) e^{\int \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) e^{\int \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$= B(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_3 + \left(F_{i-1} - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) P_4$$

where

$$P_3 = \int_{t_{i-1}}^{t_i} e^{\int \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$P_4 = \int_{t_{i-1}}^{t_i} \tau e^{\int \omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Assuming $A(t_i=0) = B(t_i=0) = 0$,

$$x(t_i) = \frac{e^{-\int \omega_n t_i}}{m \omega_d} [A(t_i) \sin \omega_d t_i - B(t_i) \cos \omega_d t_i]$$

The integrals in P_1 , P_2 , P_3 and P_4 can be evaluated in closed form.

The computer program and output are given.

```
C =====
C
C PROBLEM 4.96
C NUMERICAL INTEGRATION OF DUHAMEL INTEGRAL
C =====
C
C PROBLEM-DEPENDENT DATA
DIMENSION F(21), X(21), DELT(21), T(21), A(21), B(21)
NP=21
XAI=0.1
OMN=1.0
```

```

XM=1.0
DATA F/1.0,.8436,.6910,.5460,.4122,.2929,.1910,.1090,
2 .04894,.01231,.0,.0,.0,.0,.0,.0,.0,.0/
DO 10 I=1,21
10 DELT(I)=0.31416
C END OF PROBLEM-DEPENDENT DATA
A(1)=0.0
B(1)=0.0
OMD=OMN*SQRT(1.0-XAI**2)
T(1)=0.0
DO 20 I=2,NP
T(I)=T(I-1)+DELT(I)
TIME=T(I)
CALL PI1(TIME,XAI,OMN,OMD,PP1)
CALL PI2(TIME,XAI,OMN,OMD,PP2)
CALL PI3(TIME,XAI,OMN,OMD,PP3)
CALL PI4(TIME,XAI,OMN,OMD,PP4)
TIME=T(I-1)
CALL PI1(TIME,XAI,OMN,OMD,PM1)
CALL PI2(TIME,XAI,OMN,OMD,PM2)
CALL PI3(TIME,XAI,OMN,OMD,PM3)
CALL PI4(TIME,XAI,OMN,OMD,PM4)
P1=PP1-PM1
P2=PP2-PM2
P3=PP3-PM3
P4=PP4-PM4
DELF=F(I)-F(I-1)
A(I)=A(I-1)+(DELF/DELT(I))*P1+(F(I-1)-T(I-1)*DELF/DELT(I))*P2
B(I)=B(I-1)+(DELF/DELT(I))*P4+(F(I-1)-T(I-1)*DELF/DELT(I))*P3
X(I)=(EXP(-XAI*OMN*T(I))/(XM*OMD))*(A(I)*SIN(OMD*T(I))-
2 B(I)*COS(OMD*T(I)))
20 CONTINUE
PRINT 30
30 FORMAT (//,2X,41H NUMERICAL EVALUATION OF DUHAMEL INTEGRAL,
2 //,5X,2H I,6X,5H T(I),10X,5H F(I),10X,5H X(I),/)
DO 40 I=2,NP
40 PRINT 50, I,T(I),F(I),X(I)
50 FORMAT (2X,I5,3E15.8)
STOP
END
C =====
C
C SUBROUTINE PI1
C
C =====
SUBROUTINE PI1 (T,XAI,OMN,OMD,P)
DEN=(XAI*OMN)**2+OMD**2
P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*COS(OMD*T)
3 +2.0*XAI*OMN*OMD*SIN(OMD*T))
RETURN
END

```

```

C =====
C
C SUBROUTINE PI2
C
C =====
      SUBROUTINE PI2 (T, XAI, OMN, OMD, P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
      RETURN
      END
C =====
C
C SUBROUTINE PI3
C
C =====
      SUBROUTINE PI3 (T, XAI, OMN, OMD, P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      RETURN
      END
C =====
C
C SUBROUTINE PI4
C
C =====
      SUBROUTINE PI4 (T, XAI, OMN, OMD, P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*SIN(OMD*T))
      3 -2.0*XAI*OMN*OMD*COS(OMD*T))
      RETURN
      END

```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.31415999E+00	0.84359998E+00	0.45415938E-01
3	0.62831998E+00	0.69099998E+00	0.16377741E+00
4	0.94247997E+00	0.54600000E+00	0.32499743E+00
5	0.12566404E+01	0.41217997E+00	0.47747675E+00
6	0.15708008E+01	0.29290003E+00	0.65152758E+00
7	0.18849611E+01	0.19099998E+00	0.76237210E+00
8	0.21991215E+01	0.10900003E+00	0.81256396E+00
9	0.25132818E+01	0.48939997E-01	0.79324198E+00
10	0.28274422E+01	0.12309998E-01	0.70482814E+00
11	0.31416025E+01	0.00000000E+00	0.55646867E+00
12	0.34557629E+01	0.00000000E+00	0.36454141E+00
13	0.37699232E+01	0.00000000E+00	0.14971566E+00
14	0.40840836E+01	0.00000000E+00-0.66231847E-01	
15	0.43982439E+01	0.00000000E+00-0.26274461E+00	
16	0.47124043E+01	0.00000000E+00-0.42236131E+00	
17	0.50265644E+01	0.00000000E+00-0.53218621E+00	
18	0.53407249E+01	0.00000000E+00-0.58483124E+00	
19	0.56548853E+01	0.00000000E+00-0.57878375E+00	
20	0.59690456E+01	0.00000000E+00-0.51819199E+00	
21	0.62832060E+01	0.00000000E+00-0.41212624E+00	

4.97

The computer program of problem 4.96 can be used to find the relative displacement $z(t)$ of the water tank provided $-m\ddot{y}(t)$ is used in place of $F(t)$.

Here $\zeta = 0$, $\omega_n = 22.3607$ rad/sec and

$$F(t) = -10000 \times 9.81 \times \ddot{y}(t) \text{ if } \ddot{y} \text{ is in g's.}$$

The problem-dependent data for the program of problem 4.96 and the output are given below.

```
C =====
C
C PROBLEM 4.97
C
C =====
C PROBLEM-DEPENDENT DATA
DIMENSION F(15),X(15),DELT(15),T(15),A(15),B(15)
NP=15
XAI=0.0
OMN=22.3607
XM=10000.0
DATA F/.0,.45,-.8,-.9,-.6,-.75,-.7,.55,1.75,1.65,.25,
2 -1.1,-1.4,-1.05,.0/
DO 10 I=1,15
10 DELT(I)=0.025
DO 11 I=1,15
11 F(I)=-XM*9.81*F(I)
C END OF PROBLEM-DEPENDENT DATA
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.2499999E-01	0.44144996E+05	0.45271125E-03
3	0.4999997E-01	0.78480000E+05	0.17453337E-02
4	0.7499998E-01	0.88290000E+05	0.11150776E-02
5	0.9999994E-01	0.58860004E+05	0.86095147E-02
6	0.12499994E+00	0.73575000E+05	0.17519452E-01
7	0.14999992E+00	0.68670000E+05	0.25374368E-01
8	0.17499989E+00	0.53955000E+05	0.28478131E-01
9	0.19999987E+00	0.17167500E+06	0.19676924E-01
10	0.22499985E+00	0.16186494E+06	0.42601228E-02
11	0.24999982E+00	0.24525000E+05	0.35448216E-01
12	0.27499980E+00	0.10791006E+06	0.57387743E-01
13	0.29999977E+00	0.13734000E+06	0.56341648E-01
14	0.32499975E+00	0.10300506E+06	0.30433871E-01
15	0.34999973E+00	0.00000000E+00	0.10307007E-01

4.98

The problem-dependent data (to be used in the program of Problem 4.96) and output are given.

```
C PROBLEM-DEPENDENT DATA
DIMENSTON F(30),X(30),DELT(30),T(30),A(30),B(30)
NP=30
XA1=0.0
UMN=8.660254
XM=2.0
DATA F /60.0,60.0,60.0,60.0,60.0,100.0,100.0,100.0,100.0,
2 30.0,30.0,30.0,30.0,30.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
3 0.0,0.0,0.0,0.0,0.0,0.0/
DO 10 I=1,30
10 DELT(I)=0.01
C END OF PROBLE1-DEPENDANT DATA
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.99999998E-02	0.60000000E+02	0.14990796E-02
3	0.20000000E-01	0.60000000E+02	0.59849964E-02
4	0.29999999E-01	0.60000000E+02	0.13424230E-01
5	0.39999999E-01	0.60000000E+02	0.23760952E-01
6	0.50000001E-01	0.10000000E+03	0.37250735E-01
7	0.60000002E-01	0.10000000E+03	0.55125091E-01
8	0.70000000E-01	0.10000000E+03	0.77583194E-01
9	0.79999998E-01	0.10000000E+03	0.10445669E+00
10	0.89999996E-01	0.10000000E+03	0.13554408E+00
11	0.99999994E-01	0.30000000E+02	0.17002963E+00
12	0.10999999E+00	0.30000000E+02	0.20532255E+00
13	0.11999999E+00	0.30000000E+02	0.24057558E+00
14	0.13000000E+00	0.30000000E+02	0.27552453E+00
15	0.14000000E+00	0.30000000E+02	0.30990744E+00
16	0.15000001E+00	0.30000000E+02	0.34346649E+00
17	0.16000001E+00	0.00000000E+00	0.37570029E+00
18	0.17000002E+00	0.00000000E+00	0.40536803E+00
19	0.18000002E+00	0.00000000E+00	0.43199742E+00
20	0.19000003E+00	0.00000000E+00	0.45538884E+00
21	0.20000003E+00	0.00000000E+00	0.47536695E+00
22	0.21000004E+00	0.00000000E+00	0.49178207E+00
23	0.22000004E+00	0.00000000E+00	0.50451112E+00
24	0.23000005E+00	0.00000000E+00	0.51345861E+00
25	0.24000005E+00	0.00000000E+00	0.51855767E+00
26	0.25000006E+00	0.00000000E+00	0.51976985E+00
27	0.26000005E+00	0.00000000E+00	0.51708633E+00
28	0.27000004E+00	0.00000000E+00	0.51052707E+00
29	0.28000003E+00	0.00000000E+00	0.50014120E+00
30	0.29000002E+00	0.00000000E+00	0.48600671E+00

4.99

(i) Find natural frequency:

$$\bar{J}_0 = mb^2 = 9ma^2$$

$$\frac{1}{2}k_t \theta^2 = \frac{1}{2}k(\omega\theta)^2 = \frac{1}{2}ka^2\theta^2 \Rightarrow k_t = ka^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{ka^2}{9ma^2}} = \frac{1}{3}\sqrt{\frac{k}{m}} = 2\pi(10) = 20\pi \frac{\text{rad}}{\text{sec}} \quad (\text{E}_1)$$

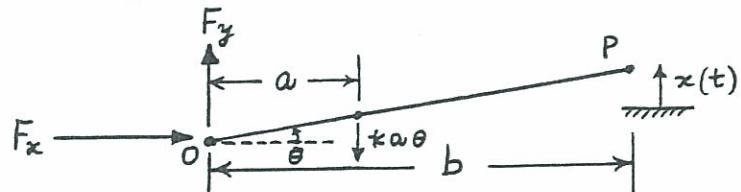
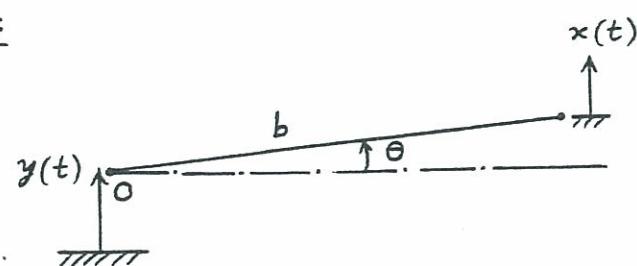
(ii) Find equation of motion:

When the base and hence the pivot O is displaced by $y(t)$, the displacement of mass m, $x(t)$, is given by

$$x(t) = y(t) + b\theta(t) \quad (\text{E}_2)$$

Relative displacement of mass is

$$z(t) = x(t) - y(t) = b\theta(t) \quad (\text{E}_3)$$



Equation of motion:

$$\sum \text{Forces along } y(\text{vertical}) \text{ direction} = 0 \Rightarrow F_y - ka\theta = m\ddot{z} \quad (\text{E}_4)$$

$$\sum \text{Moments about P} = 0 \Rightarrow F_y \cdot b - ka\theta(b-a) = 0 \quad (\text{E}_5)$$

Solve (E₄) for F_y and substitute the result in (E₅):

$$(ka\theta + m\ddot{z})b - ka\theta(b-a) = 0$$

$$\text{i.e., } mb\ddot{z} + \theta ka^2 = 0 \quad (\text{E}_6)$$

$$\text{But } x = y + z \text{ and } \theta(t) = \frac{1}{b}z(t)$$

Hence (E₆) becomes

$$mb\ddot{z} + \frac{ka^2}{b}z = -mb\ddot{y}$$

$$\text{or } m\ddot{z} + \frac{ka^2}{b^2}z = -m\ddot{y} \quad (\text{E}_7)$$

Eq. (E₇) can be compared with a standard forced vibration equation for an undamped system:

$$\tilde{m} \ddot{x} + \tilde{k} x = F(t) \quad (E_8)$$

Comparison of (E₇) and (E₈) shows that

$$x = z, \tilde{m} = m, \tilde{k} = \frac{k a^2}{b^2} \text{ and } \tilde{F} = -m \ddot{y} \quad (E_9)$$

(iii) Find solution:

Solution of Eq. (E₈) under a rectangular pulse is given in problem 4.20. Hence the solution of problem 4.20 can be used to find the relative displacement.

(iv) Design:

$$\text{Eq. (E}_1\text{)} \text{ gives } \sqrt{k} = 60\pi \sqrt{m} \quad \text{or} \quad k = 3600\pi^2 m \quad (E_{10})$$

Following procedure can be used to solve the problem:

- Assume a value of a .
- Assume a small value of m .
- Find k from Eq. (E₁₀).
- Evaluate the solution numerically as outlined in part (iii).
- If the maximum relative displacement is larger than or equal to 0.02 m, the design is complete.
- Otherwise, increase the value of m and go to step c.
- If necessary change the value of a in step a.

4.100

Let k and m denote the equivalent stiffness and mass of the cutting head.

Equation of motion is: $m \ddot{x} + k x = F(t) \quad (E_1)$

where $F(t)$ is given by

Figs. 4.75(a) and 4.75(b)..

Solution of Eq. (E₁) under the force given by Figs. 4.75(a) can be obtained as in problem 4.22. Solution of (E₁) under the force of Fig. (4.75(b)) can be determined using a procedure similar to that of problem 4.25.

The values of k and m can be determined as follows:

- Assume a small value for m .
- Assume a small value for k .
- Evaluate the response under $F(t)$ given by Fig. 4.75(a).

- d. Evaluate the response under $F(t)$ given by Fig. (4.75(b)).
- e. If the responses in steps c and d are approximately equal to 0.1 mm and 0.05 mm, the current values of m and κ are the desired values.
- f. Otherwise, increment the value of m and/or κ and go to step c.

4.101

Model the system as a single d.o.f. torsional system with:

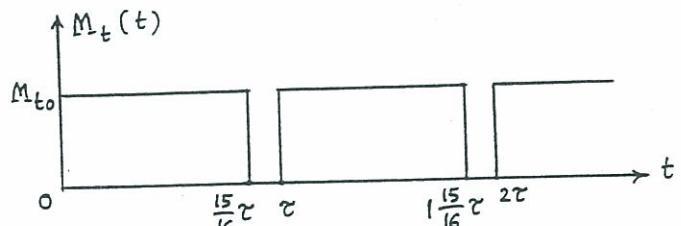
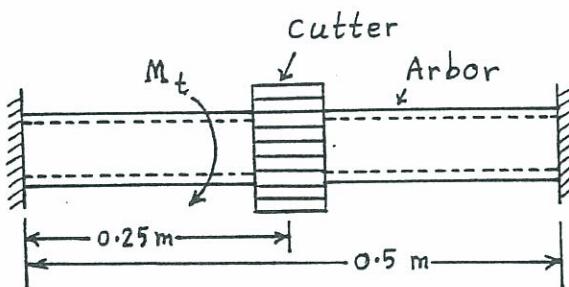
$$J_0 = 0.1 \text{ N-m}^2 ; c_t = 0 \text{ (no damping assumed)} ;$$

$$k_t = \frac{G J}{\ell} = \frac{1}{0.25} (80 (10^9)) \left(\frac{\pi}{32} (d_o^4 - d_i^4) \right)$$

where d_o = outer diameter of shaft, and d_i = inner diameter.

$$k_t = 62.832 (10^9) (d_o^4 - d_i^4) \quad (1)$$

Torque acting on the arbor due to breakage of one tooth can be modeled as shown in figure.



$M_{t0} = 500 \text{ N-m}$, $\tau = \frac{2\pi}{\omega} = \frac{2\pi}{N(2\pi)/60} = 60/N = 60/1000 = 0.06 \text{ sec}$ where $N =$ speed of cutter = 1000 rpm. Express $M_t(t)$ in Fourier series:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n \omega t + b_n \sin n \omega t \right) \quad (2)$$

(see solution of Problem 4.6 for procedure). where $\omega = \frac{2\pi N}{60} = 104.72 \text{ rad/sec.}$

Equation of motion:

$$J_0 \ddot{\theta} + k_t \theta = M_t(t) \quad (3)$$

where $M_t(t)$ is given by Eq. (2). Find solution of Eq. (3) and determine the maximum value of θ , θ_{\max} . This value must be less than 1° .

Note:

The solution requires an iterative procedure. Assume values for d_i and d_o . Compute k_t , find solution of $\theta(t)$, and the value of θ_{\max} . If θ_{\max} exceeds 1° , choose a different set of values for d_i and d_o and continue the process until θ_{\max} comes out to be less than 1° .