Exam 1

Mech 326: Aerodynamics
Friday, October 3, 2014: 1:10–2:00 PM

1. (Circulation in a boundary layer) (30%)
2. (Vortex near a wall) (40%)
3. (Stream function and velocity potential) (30%)

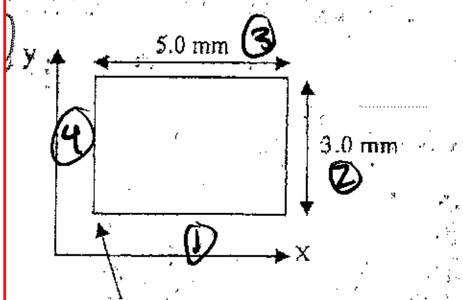
Bonus (+3)

1.) F-22 Raptor
2.) Wright Flyer
3.) SR-71 Blackbird

1. (Circulation in a boundary layer) (30%) A 2-D laminar boundary layer profile for a flow moving in the positive x-direction is given by the following expression:

$$u(y) = U\left(\frac{y}{\delta} - \frac{3y^3}{\delta^3} + \frac{y^4}{\delta^4}\right)$$

where $U = 0.5$ m/s is the free stream velocity of the **uniform** flow above the boundary layer and $\delta = 5$ mm is the boundary layer thickness. Compute the circulation for the following rectangular contour using **either** of the two approaches:

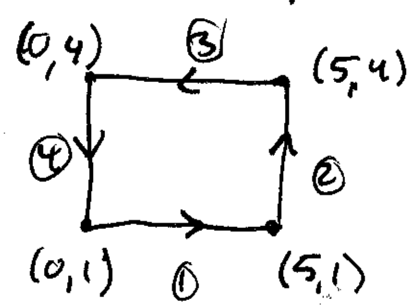


Origin of rectangle: x = 0 mm, y = 1.00 mm

First approach

(5)

$$\Gamma = -\oint \vec{V}\cdot d\vec{s} = -\oint_{①} \vec{V}\cdot d\vec{s} + -\oint_{②} \cdots -\oint_{③} + -\oint_{④}$$



$U = 0.5$ m/s $= 500$ mm/s

$\delta = 5$ mm

$\vec{V} = u\hat{\imath} + v\hat{\jmath}$ $v = 0$ $u(y) = U\left(\frac{y}{\delta} - \frac{3y^3}{\delta^3} + \frac{y^4}{\delta^4}\right)$

$\hookrightarrow \vec{V} = u\hat{\imath}$ (3)

$\oint_{②} = 0$ and $\oint_{4} = 0$

(2)

$$\Gamma = -\oint_{①}\left[U\left(\frac{1}{\delta} - \frac{3}{\delta^3} + \frac{1}{\delta^4}\right)\cancel{(\hat{\imath}\cdot\hat{\imath})}^{1}\right]dx - \int_{②}\left[U\left(\frac{4}{\delta} - \frac{192}{\delta^3} + \frac{256}{\delta^4}\right)\cancel{(\hat{\imath}\cdot -\hat{\imath})}^{-1}\right]dx$$

2

(5)

$$\Gamma = -U\underbrace{\left(\frac{1}{\delta} - \frac{3}{\delta^3} + \frac{1}{\delta^4}\right)}_{0.1776}(5\text{ mm}) + U\underbrace{\left(\frac{4}{\delta} - \frac{192}{\delta^3} + \frac{256}{\delta^4}\right)}_{-0.3264}(5\text{ mm})$$

$$\Gamma = U(5\,mm)\left[-0.1776 - 0.3264\right]$$

$$\Gamma = -1260\ \frac{mm^2}{s}\left(\frac{1\ m}{1000\ mm}\right)^2$$

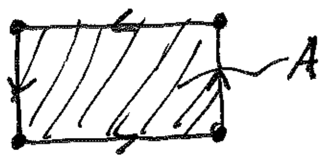
$$\boxed{\Gamma = -1.26 \times 10^{-3}\ m^2/s}$$

$$\boxed{\Gamma = -0.00126\ m^2/s}$$

$$\boxed{\Gamma = -1260\ \frac{mm^2}{s}}$$

3   2 units

Second Approach



$$\Gamma = -\oint \vec{V}\cdot d\vec{s} = -\iint_A (\vec{\nabla}\times\vec{V})\cdot d\vec{A}$$

10

$$d\vec{A} = dx\,dy\,\hat{k}$$

2

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & 0 & 0 \end{vmatrix} = -\frac{\partial u}{\partial y}\hat{k}$$

5

$$u(y) = U\left(\frac{y}{\delta} - \frac{3y^3}{\delta^3} + \frac{y^4}{\delta^4}\right)$$

$$\frac{\partial u}{\partial y} = U\left(\frac{1}{\delta} - \frac{9y^2}{\delta^3} + \frac{4y^3}{\delta^4}\right)$$

$$\Gamma = -\iint_A U\left(\frac{1}{\delta} - \frac{9y^2}{\delta^3} + \frac{4y^3}{\delta^4}\right)dx\,dy$$

5

$$= -\int_1^4\int_0^5 U\left(\frac{1}{\delta} - \frac{9y^2}{\delta^3} + \frac{4y^3}{\delta^4}\right)dx\,dy$$

$$= -U(5\,mm)\left[\frac{y}{\delta}\Big|_1^4 - \frac{3y^3}{\delta^3}\Big|_1^4 + \frac{y^4}{\delta^4}\Big|_1^4\right]$$

3

$$= -(500\,mm/s)(5\,mm)(-0.504)$$

$$\boxed{\Gamma = -1260\ mm^2/s} \qquad \boxed{\Gamma = -0.00126\ m^2/s}$$

3   2 units

$$\boxed{\Gamma = -1.26\times10^{-3}\ m^2/s}$$

2. (Vortex near a wall) (40%) Imagine that there is a two dimensional clockwise vortex that is a distance d from a wall ($y = 0$). The vortex is propagating to the left with a velocity $-U_\infty$. The method of images may be used to enforce the no-flux boundary condition along the wall. Place an image vortex beneath the wall a distance d. The image is a counterclockwise vortex that is also propagating to the left with velocity $-U_\infty$. Imagine that we are riding along with the vortex and its image. In the vortex frame of reference, the streamfunction for the clockwise vortex, the counterclockwise image and the uniform flow is,

$$\Psi(x, y) = U_\infty y + \frac{\Gamma}{2\pi} \ln\sqrt{x^2 + (y-d)^2} - \frac{\Gamma}{2\pi} \ln\sqrt{x^2 + (y+d)^2}$$

3

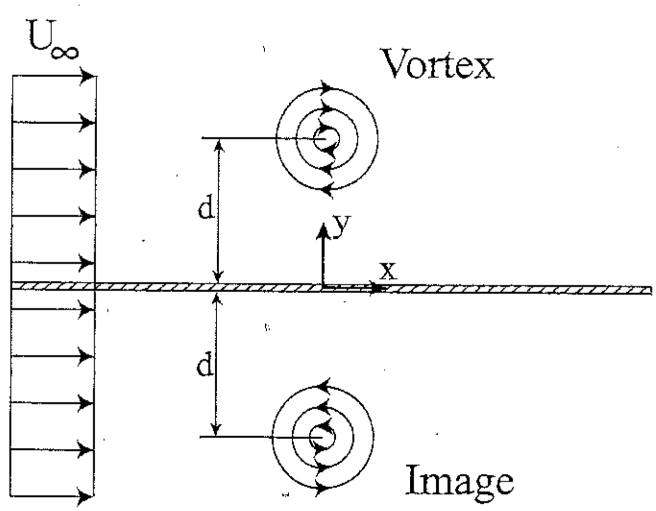
where $\Gamma > 0$ and $U_\infty = \Gamma/(4\pi d)$.

10 (a) Show with calculations that there is no flow through the wall.

10 (b) What is the velocity along the wall, $u(x, 0)$, in the vortex frame of reference?

10 (c) The stagnation points of the flow are along the wall. Where are they located?

10 (d) What is the pressure along the wall?



a.) $$\psi(x,y) = u_\infty y + \frac{\Gamma}{2\pi}\left[\ln\sqrt{x^2+(y-d)^2} - \ln\sqrt{x^2+(y+d)^2}\right]$$

$\Gamma > 0 \quad u_\infty = \dfrac{\Gamma}{4\pi d}$

$u = \dfrac{\partial\psi}{\partial y} \quad v = -\dfrac{\partial\psi}{\partial x}$ (if incorrect -5)

$$u(x,y) = u_\infty + \frac{\Gamma}{2\pi}\left[\frac{1}{\sqrt{x^2+(y-d)^2}}\left(\frac{1}{\cancel{2}}\frac{\cancel{2}(y-d)}{\sqrt{x^2+(y-d)^2}}\right) - \frac{1}{\sqrt{x^2+(y-d)^2}}\left(\frac{1}{\cancel{2}}\frac{\cancel{2}(y+d)}{\sqrt{x^2+(y-d)^2}}\right)\right]$$

If derivatives are incorrect -3.

$$u(x,y) = \cancel{u_\infty}^{\frac{\Gamma}{4\pi d}} + \frac{\Gamma}{2\pi}\left[\frac{y-d}{[x^2+(y-d)^2]} - \frac{y+d}{[x^2+(y-d)^2]}\right]$$

$$v(x,y) = -\frac{\Gamma}{2\pi}\left[\frac{1}{\sqrt{x^2+(y-d)^2}}\left(\frac{1}{2}\frac{2x}{\sqrt{x^2+(y-d)^2}}\right) - \frac{1}{\sqrt{x^2+(y+d)^2}}\left(\frac{1}{2}\frac{2x}{\sqrt{x^2+(y+d)^2}}\right)\right]$$

$$v(x,y) = -\frac{\Gamma}{2\pi}\left[\frac{x}{x^2+(y-d)^2} - \frac{x}{x^2+(y+d)^2}\right]$$

$$\boxed{v(x,0) = 0 \rightarrow \therefore \vec{V}\cdot\hat{n} = 0 \text{ on the wall}}$$

b.) $$u(x,0) = \frac{\Gamma}{4\pi d} + \frac{\Gamma}{2\pi}\left[\frac{-2d}{x^2+d^2}\right]$$

$$\boxed{u(x,0) = \frac{\Gamma}{4\pi d} - \frac{\Gamma}{\pi}\left(\frac{d}{x^2+d^2}\right)}$$

c.) Stagnation occurs when $u(x,0)=0$
$v(x,0)=0$

$v(x,0)=0 \rightarrow$ everywhere along wall.

$u(x,0)=0 \rightarrow \quad 0 = \frac{\Gamma}{4\pi d} - \frac{\Gamma}{\pi}\left(\frac{d}{x^2+d^2}\right)$

$$\frac{d}{x^2+d^2} = \frac{1}{4d}$$

$$4d^2 = x^2 + d^2$$

$$x^2 = 3d^2 \rightarrow x = \pm\sqrt{3}\,d$$

$$\boxed{\text{stag points: } (\sqrt{3}\,d, 0) \quad (-\sqrt{3}\,d, 0)}$$

d.) $P_\infty + \frac{1}{2}\rho U_\infty^2 = P(x,y) + \frac{1}{2}\rho V(x,y)^2$

$$P(x,0) = P_\infty + \frac{1}{2}\rho\left[U_\infty^2 - u(x,0)^2\right]$$

$$P(x,0) = P_\infty + \frac{1}{2}\rho\left[\cancel{\frac{\Gamma^2}{16\pi^2 d^2}} - \left(\cancel{\frac{\Gamma^2}{16\pi^2 d^2}} - \frac{2\Gamma^2}{4\pi^2 d}\left(\frac{d}{x^2+d^2}\right) + \frac{\Gamma^2}{\pi^2}\left(\frac{d}{x^2+d^2}\right)^2\right)\right]$$

$$P(x,0) = P_\infty + \frac{1}{2}\rho\left[\frac{\Gamma^2}{2\pi^2 d}\left(\frac{\cancel{d}}{x^2+d^2}\right) - \frac{\Gamma^2}{\pi^2}\left(\frac{d}{x^2+d^2}\right)^2\right]$$

$$\boxed{P(x,0) = P_\infty + \frac{1}{2}\rho\left(\frac{\Gamma}{\pi}\right)^2\left[\frac{1}{2}\left(\frac{1}{x^2+d^2}\right) - \left(\frac{d}{x^2+d^2}\right)^2\right]}$$

$$\boxed{P(x,0) = P_\infty + \frac{\rho}{4}\left(\frac{\Gamma}{\pi}\right)^2\left(\frac{x^2-d^2}{(x^2+d^2)^2}\right)}$$

3. (Stream function and velocity potential) (30%) The stream function for a certain flow is given by:

$$\Psi = x^2 - y^2$$

8 (a) Show that this flow is irrotational.

8 (b) Is this flow is an incompressible flow field? (support your answer with calculations)

8 (c) Determine the velocity potential for this flow.

(d) Determine the equation of the streamline that passes through the point $(x, y) = (5, 3)$.

$$\psi = x^2 - y^2 \qquad \vec{\nabla} \times \vec{U} = 0 \rightarrow \text{irrotational}$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$u = -2y \qquad v = -2x$$

$$\vec{\nabla} \times \vec{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = (-2) - (-2) = 0$$

$$\boxed{\text{Irrotational}}$$

b.) $\vec{\nabla} \cdot \vec{v} = 0 \rightarrow$ incompressible

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = (0) + (0) = 0$$

$$\boxed{\text{Incompressible}}$$

c.) $u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$

$$\int \partial \phi = \int u \, dx + f(y) \qquad \int \partial \phi = \int v \, dy + g(x)$$

$$\phi(x,y) = -2xy + f(y) \qquad \phi(x,y) = -2xy + g(x)$$

$$f(y) = g(x) = 0$$

$$\boxed{\phi(x,y) = -2xy + C} \qquad \boxed{C \text{ is arbitrary}}$$

d.) $(x, y) = (5, 3)$

$\psi = x^2 - y^2$

$\psi = 25 - 9 \rightarrow \psi = 16$

Equation of streamline: $\boxed{x^2 - y^2 = 16}$