Engineering Vibrations & Systems

Module 5
Forced Response of
1st and 2nd Order Systems

ME 242
Professor M. Chew, PhD, PE

Module 5

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1. Introduction

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y \;=\; b_m \frac{d^m f}{dt^m} + b_{m-1} \frac{d^{m-1} f}{dt^{m-1}} + \ldots + b_1 \frac{df}{dt} + b_0 f \qquad [1]$$

where: $a_n \neq 0;\quad m \leq n$

Taking Laplace Transform of [1] (with n=3 and m=2)

initial conditions

$$\begin{aligned} &a_3\{s^3 Y(s) - s^2 y(0) - s\dot{y}(0) - \ddot{y}(0)\} + a_2\{s^2 Y(s) - s y(0) - \dot{y}(0)\} \\ &+ a_1\{sY(s) - y(0)\} + a_0 Y(s) \;=\; b_2\,\{s^2 F(s)\} + b_1\{sF(s)\} + b_0 F(s) \end{aligned} \qquad [2]$$

Therefore:

$$Y(s) = \underbrace{\frac{I(s)}{D(s)}}_{Y_{free}(s)} + \underbrace{\frac{\{b_2 s^2 + b_1 s + b_0\}F(s)}{D(s)}}_{Y_{forced}(s)} \qquad [3]$$

where:

$$D(s) = \; a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$I(s)$ contains the initial condition values

1. Introduction

$$Y(s) = \boxed{\underbrace{\frac{I(s)}{D(s)}}_{Y_{free}(s)} + \underbrace{\frac{\{b_2 s^2 + b_1 s + b_0\}F(s)}{D(s)}}_{Y_{forced}(s)}} \quad [3]$$

indep. of input

indep. of initial conditions

SYSTEM TRANSFER FUNCTION

$$T(s) = \frac{Y_{forced}}{F(s)} = \frac{b_2 s^2 + b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0} \quad [4]$$

1. Introduction

Example 1: Determine the transfer function ${}^{X(s)}/_{F(s)}$ for the following equation and compute the characteristic roots.

$$\ddot{x} + 14\dot{x} + 58x = 6\dot{f}(t) + 4f(t)$$

Taking Laplace Transform (assuming zero i.c.):

$$s^2X(s) + 14sX(s) + 58X(s) = 6sF(s) + 4F(s)$$

$$(s^2 + 14s + 58)X(s) = (6s + 4)F(s)$$

Therefore:

$$\frac{X(s)}{F(s)} = \frac{6s + 4}{s^2 + 14s + 58}$$

Characteristic roots are obtained from solving

$$s^2 + 14s + 58 = 0$$

so that:

$$s = -7 \pm 3j$$

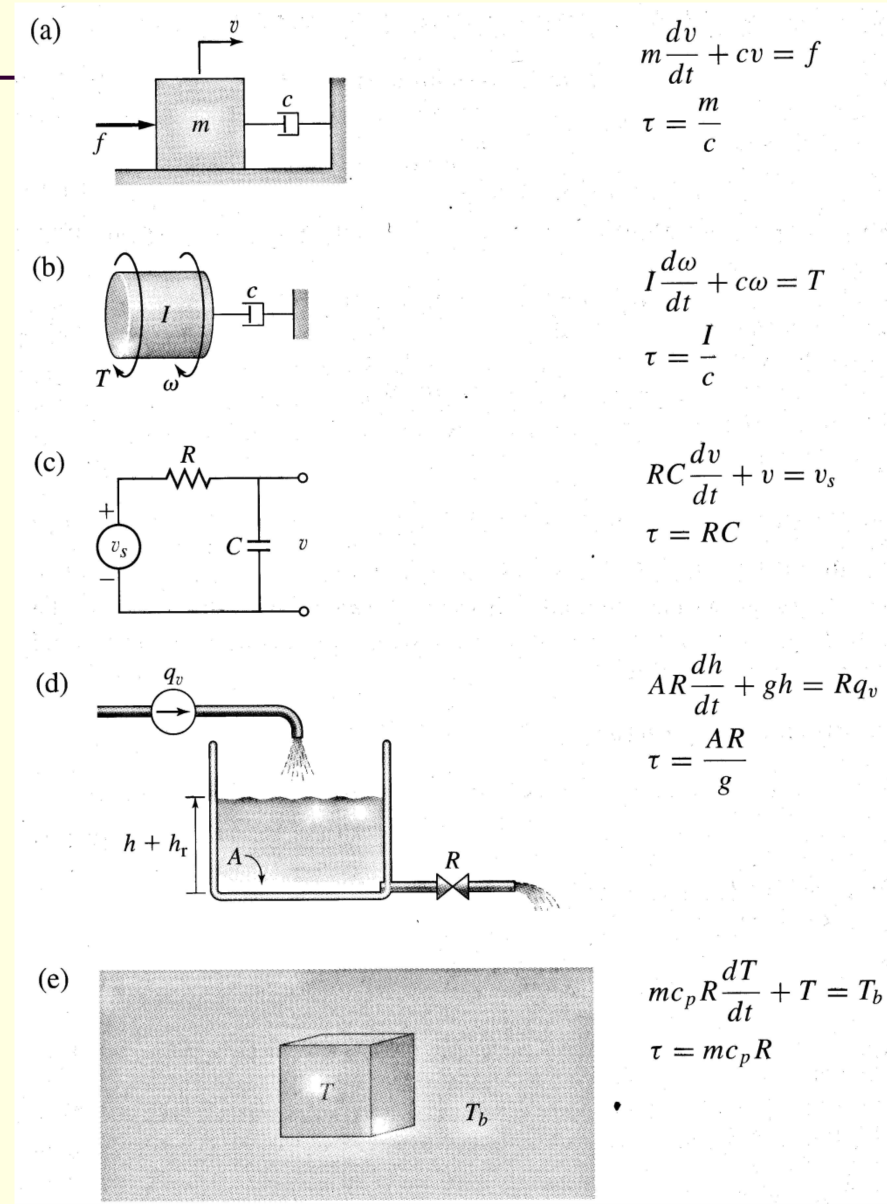
2. Response of First Order Systems

First order systems have the model form:

$$a\dot{x} + bx = f(t)$$

This can be rewritten in the form:

$$\frac{a}{b}\dot{x} + x = \frac{f(t)}{b}$$



2. Response of First Order Systems

2.1 Step Response & Time Constant (Sections 2.3, 2.5 & 8.1 of SD)

When an input is a step function, the response is called the STEP RESPONSE.
Consider the following first order system, where $f(t)$ is a step function:

$$\dot{x} + ax = f(t) \qquad [5]$$

Take Laplace Transform of D.E.

$$L(\dot{x} + ax) = L[f(t)]$$

Therefore:

$$sX(s) - x(0) + aX(s) = F(s)$$

$$X(s) = \frac{x(0)}{s+a} + \frac{1}{s+a}F(s)$$

Take Inv. Laplace Transform

$$x(t) = \underbrace{L^{-1}\left[\frac{x(0)}{s+a}\right]}_{\text{Free response}} + \underbrace{L^{-1}\left[\frac{1}{s+a}F(s)\right]}_{\text{Forced response}} \qquad [6]$$

2. Response of First Order Systems

$$x(t) = L^{-1}\left[\frac{x(0)}{s+a}\right] + L^{-1}\left[\frac{1}{s+a}F(s)\right]$$

Suppose:

$$f(t) = \begin{cases} b & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Then:

$$x(t) = L^{-1}\left[\frac{x(0)}{s+a}\right] + L^{-1}\left[\frac{1}{s+a}\frac{b}{s}\right]$$

$$\Rightarrow \quad x(t) = \frac{b}{a} + \left[x(0) - \frac{b}{a}\right]e^{-at}$$

Or:

$$\boxed{x(t) = \underbrace{x(0)e^{-at}}_{\substack{\text{Free response} \\ \text{(depends on i.c.)}}} + \underbrace{\overbrace{\frac{b}{a}}^{x_{ss}}(1 - e^{-at})}_{\substack{\text{Forced response} \\ \text{(depends on input)}}}} \qquad [7]$$

2. Response of First Order Systems

A. Look at the free response:

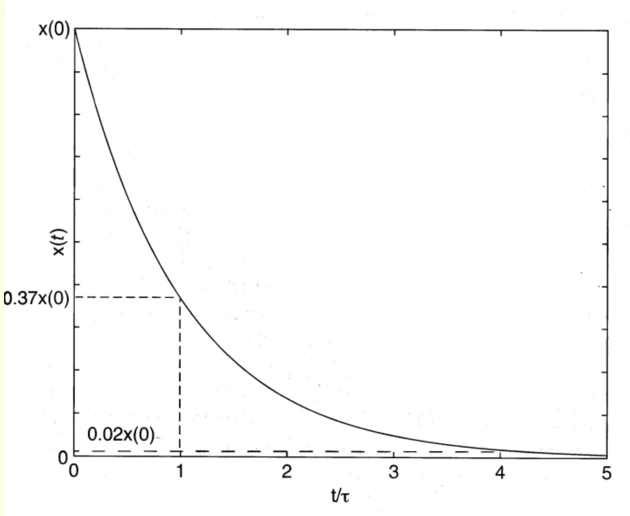
$$x(t) = x(0)e^{-at} \qquad [8]$$

Rewriting:

$$x(t) = x(0)e^{-\frac{t}{\tau}}$$

where τ is the time constant and is given by: $\tau = \frac{1}{a}$; $a > 0$

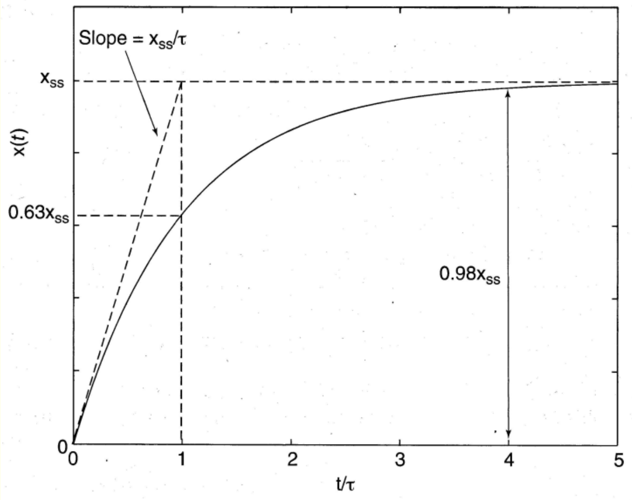
This means that after 1 *time constant*, the response $x(t)$ has decayed to just 37% of its initial value, or that $x(t)$ has decayed by 63%. At $t = 4\tau$ only 2% remains.



B. Look at the forced response:

$$x(t) = \frac{b}{a}\,(1 - e^{-at}) = \overbrace{b\tau}^{x_{ss}}(1 - e^{-\frac{t}{\tau}}) \qquad [9]$$

At $t = \tau$ the response $x(t)$ has risen 63% of x_{ss} the final steady-state value. The time constant is defined only for $a > 0$. At $t = 4\tau$, $x(t)$ has risen to 98% of steady-state value.



2. Response of First Order Systems

Example 2: Determine the free response and time constant for the following system.

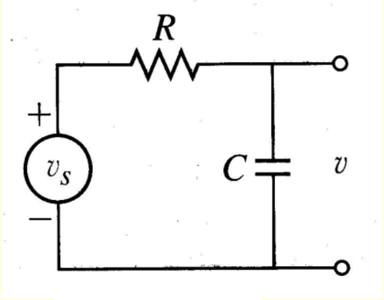
$$16\dot{x} + 14x = 15, \qquad x(0) = 6$$

Therefore: $\dot{x} + \frac{7}{8}x = \frac{15}{16}$

And the free response is: $x(t) = 6e^{-\frac{7}{8}t}$

Time constant: $\tau = \dfrac{8}{7}$

Example 3: RC circuit shown has $R = 3 \times 10^6 \Omega$ and $C = 1\mu F$. If initial capacitor voltage is 6V and the applied voltage is $v_s = 12u_s(t)$. Determine the capacitor voltage response $v(t)$.



$$RC\frac{dv}{dt} + v = v_s$$

$$\tau = RC$$

Time constant: $\tau = RC = (3 \times 10^6) \times (1 \times 10^{-6}) = 3\ s$

Capacitor voltage response $v(t)$ is the sum of the **free** and the **forced** responses (Eqs. 8 and 9):

$$v(t) = 6e^{-t/3} + 12(1 - e^{-t/3})$$

2. Response of First Order Systems

2.2a Impulse Response (Sections 2.7 & 8.1.6 of SD)

An impulse is a pulse function that is suddenly applied and removed after a *very short time*. It is a mathematical function that has an infinite magnitude for an infinitesimal time. When the area under that impulse function is 1, it is called a *unit impulse* (also called a *Dirac Delta function* $\delta(t)$. Impulse $\delta(t)$ starts at $t = 0^-$ and ends at $t = 0^+$.
Consider the following first order system, where $\delta(t)$ is an impulse function:

$$\dot{x} + 5x = \delta(t) \qquad [10]$$

Suppose i.c. $x(0^-) = 0$, what is the value of $x(0^+)$?

We know that $L[\delta(t)] = 1$, therefore:

$$X(s) = \frac{1}{s+5}$$

Response is: $x(t) = e^{-5t}$ for $t > 0$

$$\Rightarrow \quad x(0^+) = \lim_{t \to 0^+} x(t) = \lim_{t \to 0^+} e^{-5t} = 1$$

Therefore, impulse input changed x from 0 at $t \to 0^-$ to 1 at $t \to 0^+$. This is also obtained from the initial value theorem:

$$x(0^+) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s\frac{1}{s+5} = 1$$

2. Response of First Order Systems

Example 4: Compare the values of $x(0^+)$ and $x(0^-)$ for the impulse response.

$$7\dot{x} + 5x = 4\delta(t), \qquad x(0^-) = 3$$

$$7[sX(s) - 3] + 5X(s) = 4$$

$$X(s) = \frac{25}{7s+5} = \frac{25/7}{s+5/7}$$

$$x(t) = \frac{25}{7}e^{-5t/7}$$

Note that this gives $x(0+) = 25/7$. From the initial value theorem

$$x(0+) = \lim_{s\to\infty} s\frac{25/7}{s+5/7} = \frac{25}{7}$$

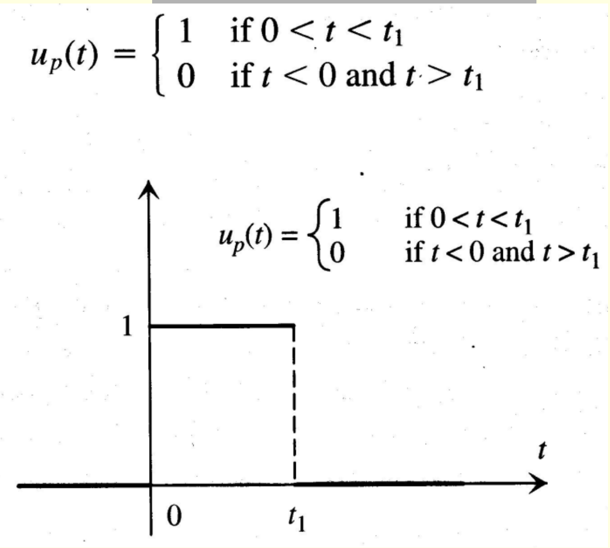
which is not the same as $x(0-)$.

2. Response of First Order Systems

2.2b Pulse Response (Example 2.8.6 of SD)

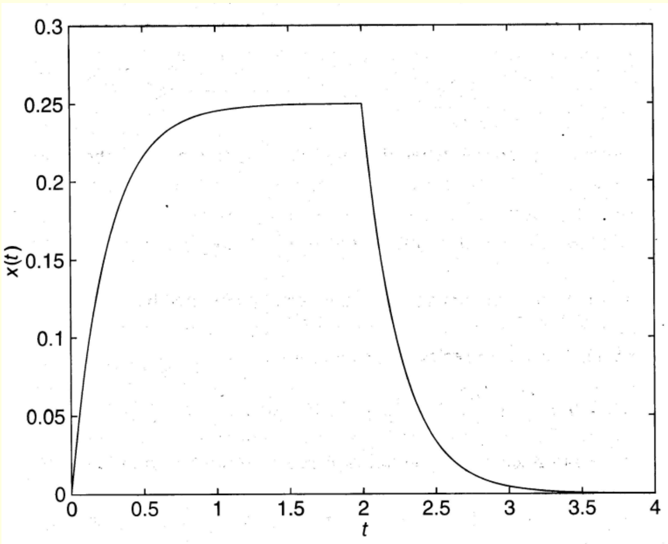
A pulse function one that is suddenly applied and removed after a *short time*. It is also called a rectangular function, which consists of a step function and then a delayed negative step function at a later time t_1 to bring the step back to zero after $t \geq t_1$. Consider the following first order system, where $u_p(t)$ is a pulse function of unit height and t_1 = 2:

$$\dot{x} + 4x = u_p(t) \text{ with } x(0^-) = 0 \qquad [11]$$



Response is:

$$x(t) = \tfrac{1}{4}(1 - e^{-4t}) - \tfrac{1}{4}\left[1 - e^{-4(t-3)}\right]u_s(t-2)$$



2. Response of First Order Systems

2.2 Ramp Response (Sections 2.7 & 8.1.7 of SD)

A ramp is an input function that is changing at a constant rate. Consider the following first order system, where $f(t) = mt$ is a ramp function:

$$\tau\dot{x} + x = f(t) \text{ with } x(0) = 0 \qquad [12]$$

Take Laplace Transform of D.E.

$$\tau sX(s) - x(0) + X(s) = F(s) = \frac{m}{s^2}$$

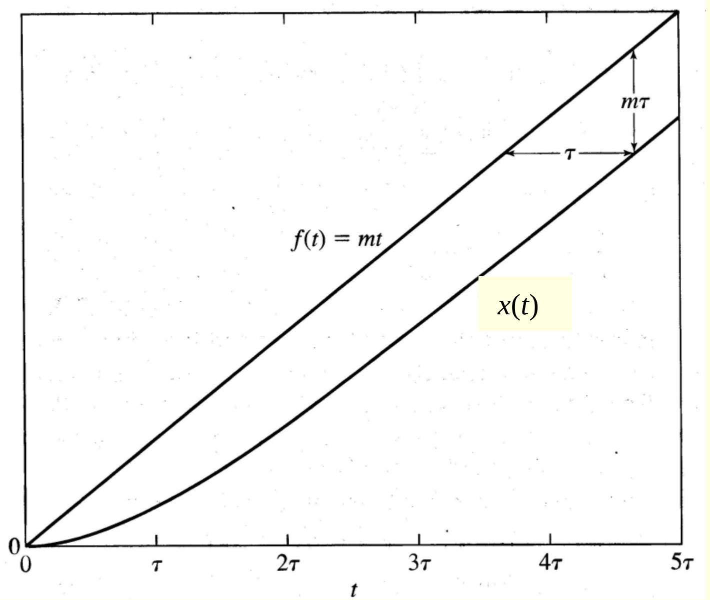
or

$$X(s) = \frac{m}{s^2(\tau s + 1)} = \frac{m}{s^2} - \frac{m\tau}{s} + \frac{m\tau}{s + \frac{1}{\tau}}$$

Inv. Laplace Transform:

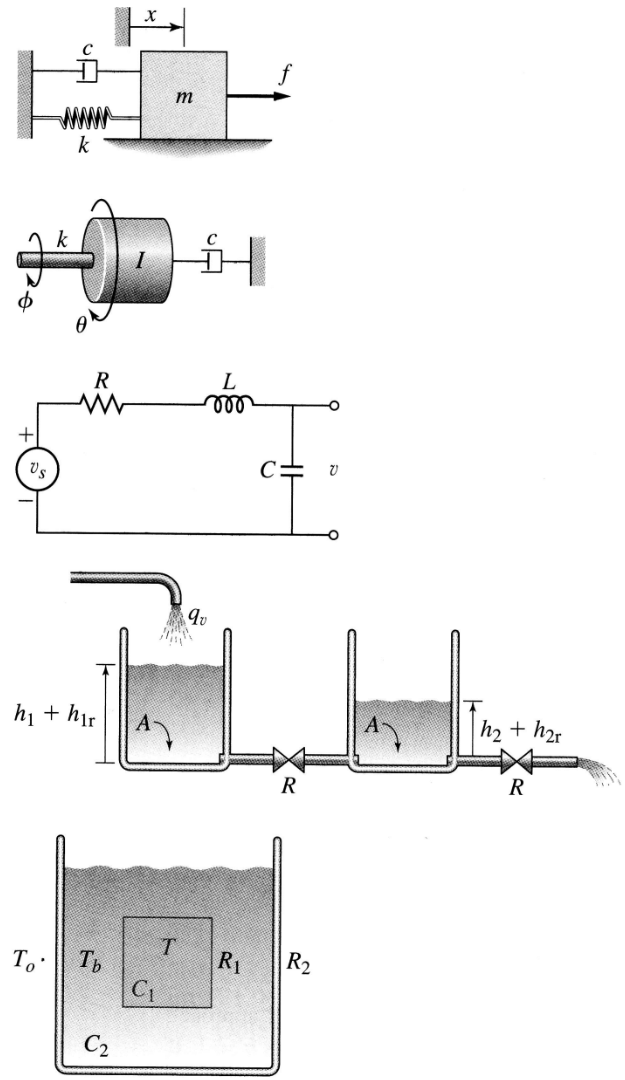
$$x(t) = m(t - \tau) + m\tau e^{-t/\tau}$$

Response is steady-state after $t \approx 4\tau$. At that state, $x(t) = m(t - \tau)$ so that the response is parallel to the input but lags behind by time τ .



3. Response of Second Order Systems

Figure 8.2.1 Some second-order systems.



(a) $$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$

(b) $$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = k\phi$$

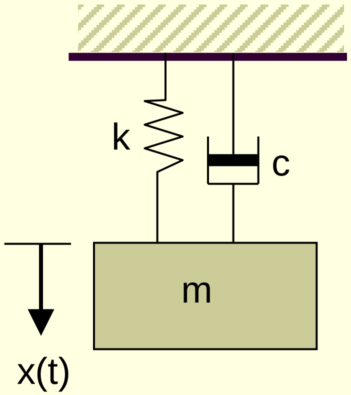
(c) $$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$

(d) $$RA\frac{dh_1}{dt} + g(h_1 - h_2) = Rq_v$$
$$RA\frac{dh_2}{dt} + g(h_2 - h_1) + gh_2 = 0$$

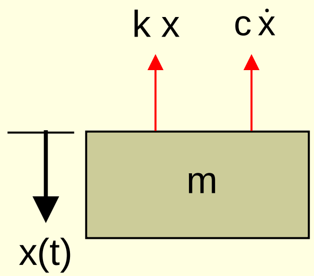
(e) $$R_1C_1\frac{dT}{dt} + T = T_b$$
$$R_1R_2C_2\frac{dT_b}{dt} + (R_1 + R_2)T_b = R_2T + R_1T_o$$

Recall: MODULE 2

Differential Equation; Characteristic Equation



Free Body Diagram



$$m\,\ddot{x}(t) = -\,c\,\dot{x}(t) - k\,x(t)$$

so that:

$$\boxed{m\,\ddot{x}(t) + c\,\dot{x}(t) + k\,x(t) = 0}$$

Solution:

Assume $$x(t) = Ae^{st} \;\rightarrow\; \dot{x} = A\,s\,e^{st}\,; \quad \ddot{x} = A\,s^2\,e^{st}$$

Into D.E.

$$m\,s^2 A e^{st} + csAe^{st} + kAe^{st} = 0$$

$$\rightarrow \qquad \boxed{m\,s^2 + c\,s + k = 0} \qquad \text{(Characteristic Eq.)}$$

Recall: MODULE 2

Solution of Differential Equation

Characteristic Equation: $\boxed{m s^2 + c s + k = 0}$

Roots: $$s_1 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}; \quad s_2 = \frac{-c - \sqrt{c^2 - 4mk}}{2m}$$

Giving 2 Solutions $x_1(t) = A_1 e^{s_1 t}$ and $x_2(t) = A_2 e^{s_2 t}$

General Solution:

$$x(t) = x_1(t) + x_2(t)$$
$$= A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$\boxed{x(t) = A_1 e^{\left(\frac{-c + \sqrt{c^2 - 4mk}}{2m}\right)t} + A_2 e^{\left(\frac{-c - \sqrt{c^2 - 4mk}}{2m}\right)t}}$$

Recall: MODULE 2

Solution of Differential Equation

Look closely at the roots of the characteristic equation:

	roots		*damping ratio*
1. Real and distinct	$-a, \ -b$	→	$\zeta > 1$
2. Real and equal	$-r, \ -r$	→	$\zeta = 1$
3. Complex conjugates	$-r + j\omega, \ -r - j\omega$	→	$0 < \zeta < 1$

rate of decay

frequency of oscillation

3. Response of Second Order Systems

Review of FREE VIBRATION

Three Cases:

1. Underdamped $0 < \zeta < 1$
2. Critical damped $\zeta = 1$
3. Overdamped $\zeta > 1$

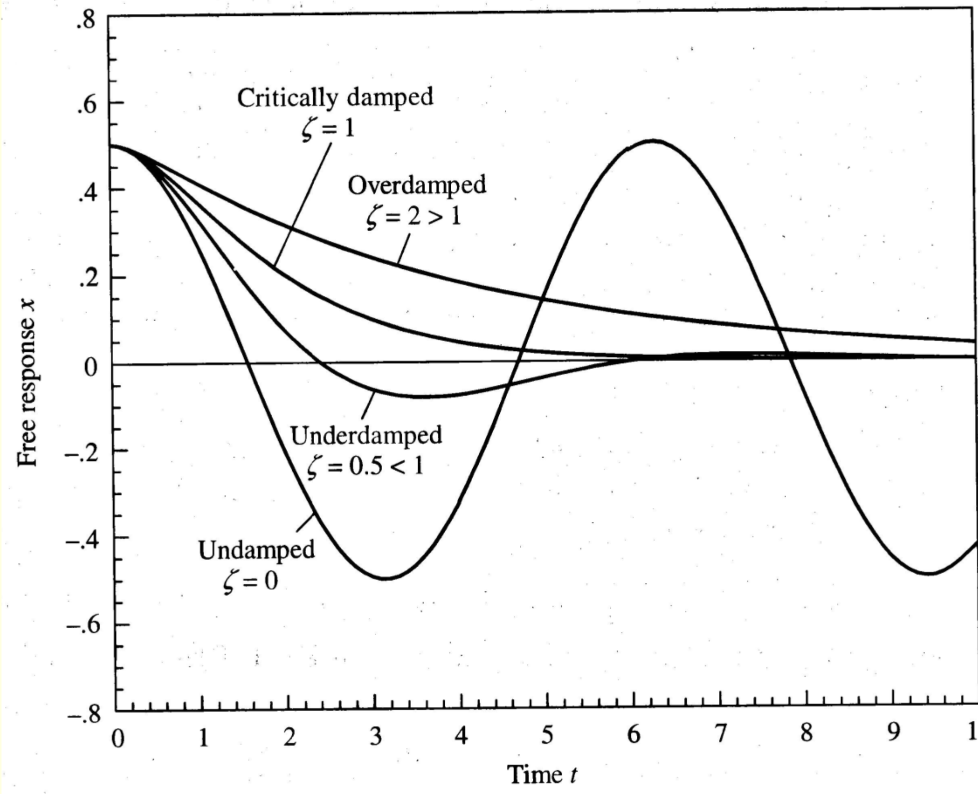
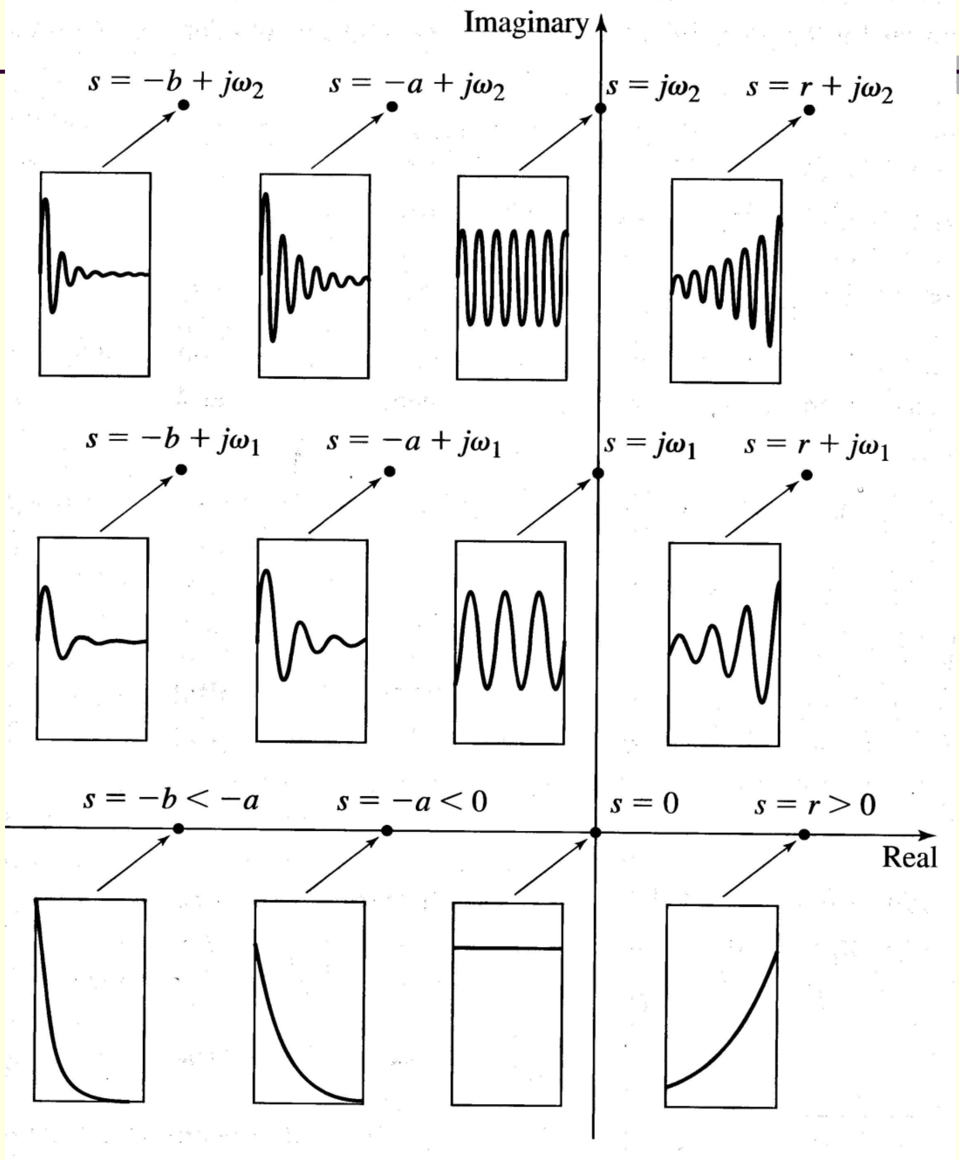


FIGURE Free response of overdamped, critically damped, underdamped, and undamped second-order systems; $x_0 \neq 0$, $v_0 = 0$.

3. Response of Second Order Systems

Effect of Root Location



3. Response of Second Order Systems

3.1 Step Response (Sections 2.3, 2.5 & 8.2 & 8.3 of SD)

When an input is a step function, the response is called the STEP RESPONSE.
Consider the following second order system, where $u_s(t)$ is a unit step function:

$$m\ddot{x} + c\dot{x} + kx = f(t) = u_s(t) \qquad [13]$$

i.c. $x(0^-) = 0; \quad \dot{x}(0^-) = 0$

$$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi\right) + 1\right] \qquad [14]$$

where

$$\phi = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$$

for $0 < \zeta < 1$

Response of a second order system to a unit step; zero i.c.

3. Response of Second Order Systems

Table 8.3.1 Unit step response of a stable second-order model.

Model: $m\ddot{x} + c\dot{x} + kx = u_s(t)$

Initial conditions: $x(0) = \dot{x}(0) = 0$

Characteristic roots: $s = \dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -r_1, -r_2$

1. $\boxed{\text{Overdamped case } (\zeta > 1)}$: distinct, real roots: $r_1 \neq r_2$

$$x(t) = A_1 e^{-r_1 t} + A_2 e^{-r_2 t} + \frac{1}{k} = \frac{1}{k}\left(\frac{r_2}{r_1 - r_2} e^{-r_1 t} - \frac{r_1}{r_1 - r_2} e^{-r_2 t} + 1\right) \quad [15a]$$

2. $\boxed{\text{Critically damped case } (\zeta = 1)}$: repeated, real roots: $r_1 = r_2$

$$x(t) = (A_1 + A_2 t)e^{-r_1 t} + \frac{1}{k} = \frac{1}{k}[(-r_1 t - 1)e^{-r_1 t} + 1] \quad [15b]$$

3. $\boxed{\text{Underdamped case } (0 \leq \zeta < 1)}$: complex roots: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}$

$$x(t) = Be^{-t/\tau} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \phi\right) + \frac{1}{k}$$

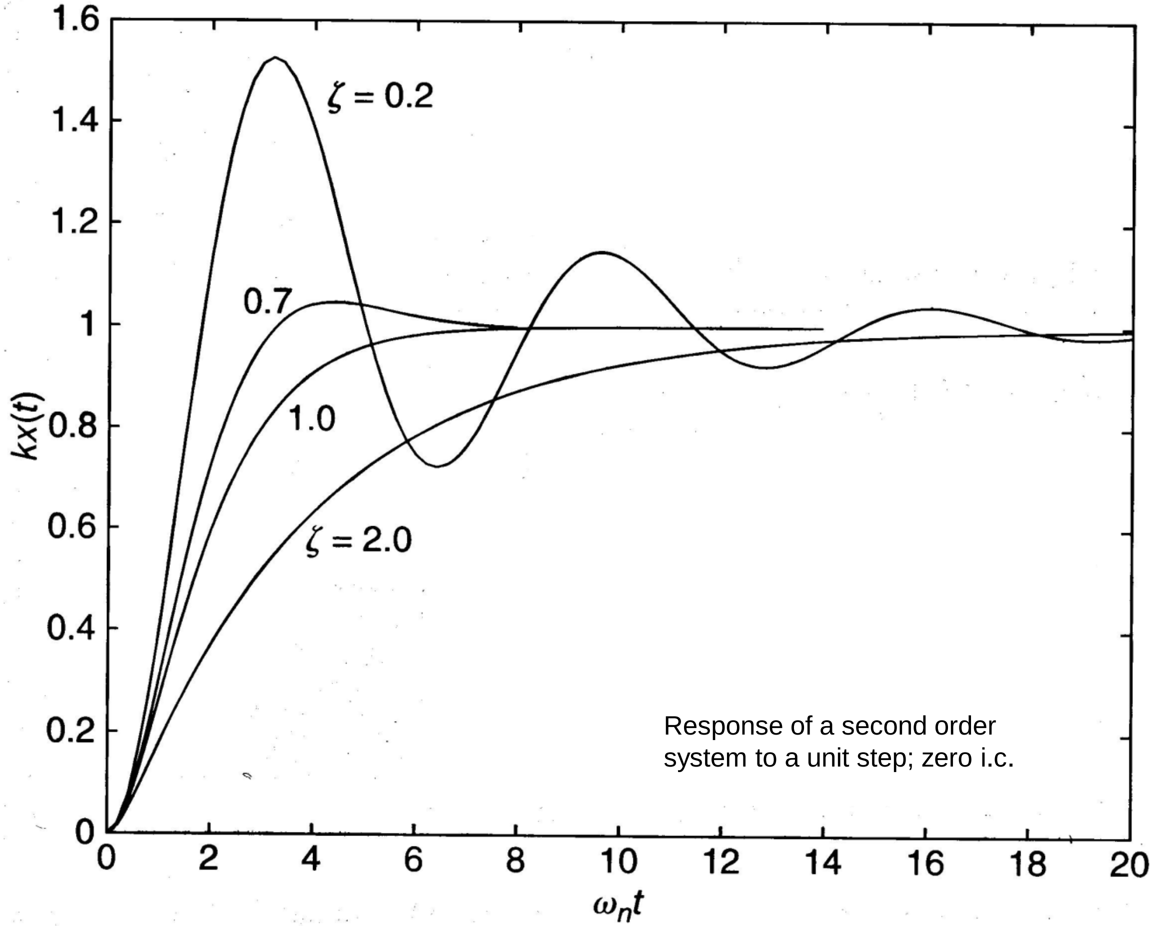
$$= \frac{1}{k}\left[\frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1 - \zeta^2}\, t + \phi\right) + 1\right] \quad [15c]$$

$$\phi = \tan^{-1}\left(\frac{\sqrt{1 - \zeta^2}}{\zeta}\right) + \pi \quad \text{(third quadrant)}$$

Time constant: $\tau = 1/\zeta\omega_n$

3. Response of Second Order Systems

Figure 8.3.2 Step response of second-order systems for various values of ζ.

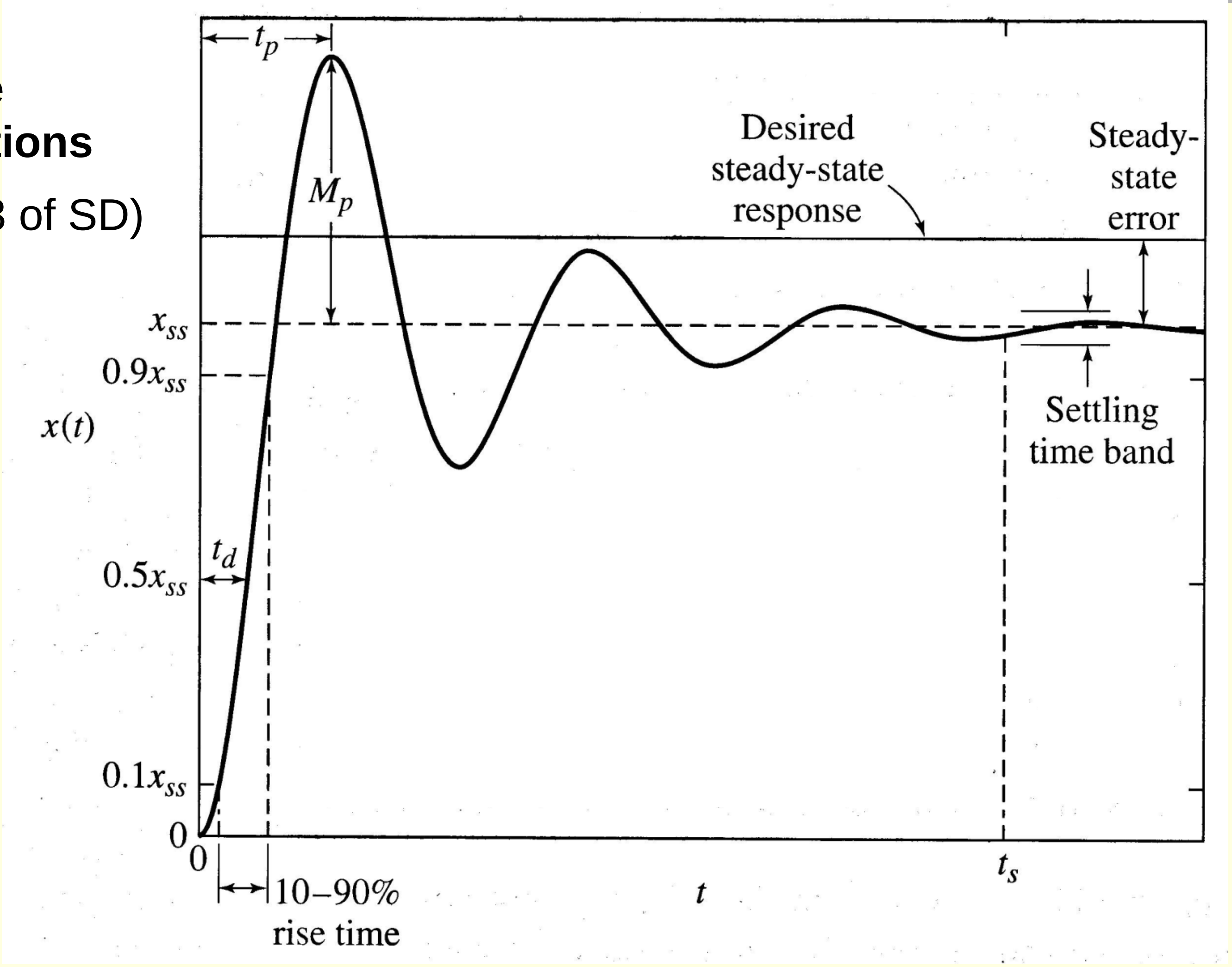


Response of a second order system to a unit step; zero i.c.

3. Response of Second Order Systems

3.2 Step Response Specifications

(Section 8.3 of SD)



3. Response of Second Order Systems

Table 8.3.2 Step response specifications for the underdamped model $m\ddot{x} + c\dot{x} + kx = f$.

Maximum percent overshoot	$M_{\%} = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$
	$\zeta = \dfrac{R}{\sqrt{\pi^2 + R^2}}, \quad R = \ln\dfrac{100}{M_{\%}}$
Peak time	$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Delay time	$t_d \approx \dfrac{1 + 0.7\zeta}{\omega_n}$
100% rise time	$t_r = \dfrac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$
	$\phi = \tan^{-1}\left(\dfrac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$

3. Response of Second Order Systems

Example 5: (Problem 8.33, p.520 of SD)

Compute the max percent overshoot, max overshoot, peak time, the 100% rise time, the delay time and the 2% settling time for the following model:

$$\ddot{x} + 4\dot{x} + 8x = 2u_s$$

8.33 $\zeta = 1/\sqrt{2} = 0.707$ which implies from Figure 8.3.5a that the maximum percent overshoot is $\approx 5\%$. Because $x_{ss} = 2/8 = 0.25$, the overshoot is $0.05(0.25) \approx 0.01$.

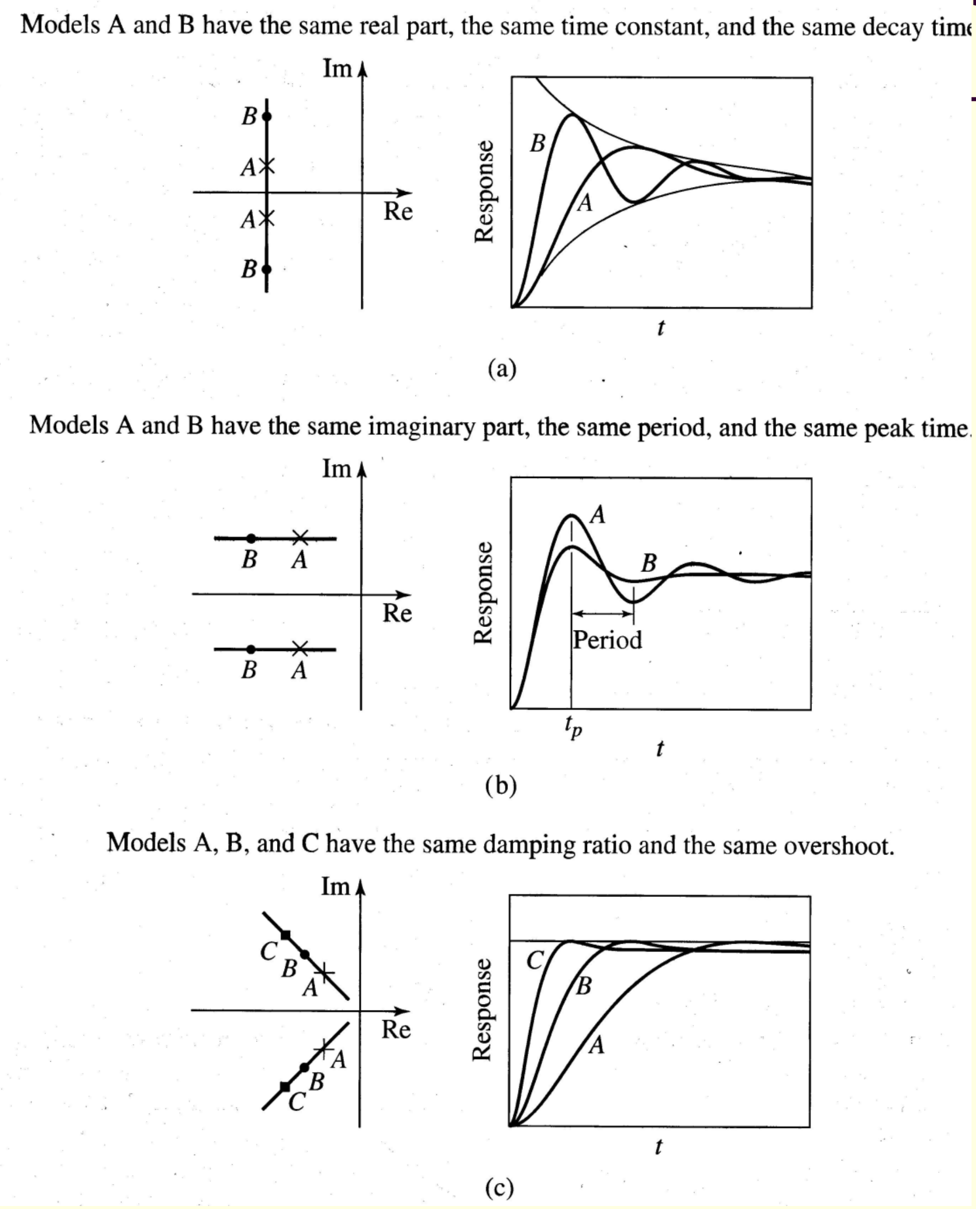
For $\zeta = 0.707$, Figure 8.3.5c shows that $\omega_n t_r \approx 3.2$. Because $\omega_n = \sqrt{8/1} = 2\sqrt{2}$, $t_r = 3.2/2\sqrt{2} = 1.1$.

For $\zeta = 0.707$, Figure 8.3.5a shows that $\omega_n t_p \approx 4.6$. Because $\omega_n = 2\sqrt{2}$, $t_p = 4.6/2\sqrt{2} = 1.6$.

For $\zeta = 0.707$ and $\omega_n = 2\sqrt{2}$, Table 8.3.2 gives $t_d = 0.53$.

The roots are $-2 \pm 2j$, so the time constant is $\tau = 0.5$. The 2% settling time is $4\tau = 2$.

3. Response of Second Order Systems



3. Response of Second Order Systems

3.3 Impulse Response (Section 2.7 & 2.10 of SD)

When an input is an impulse function, the response is called an IMPULSE RESPONSE.

Example 6:

Consider the following second order system, where $\delta(t)$ is an impulse function. Compare the values of $\dot{x}(0^+)$ and $\dot{x}(0^-)$.

$$3\ddot{x} + 30\dot{x} + 63x = 5\delta(t)\,; \qquad x(0^-) = \dot{x}(0^-) = 0$$

$$(3s^2 + 30s + 63)X(s) = 5$$

$$X(s) = \frac{5}{3s^2 + 30s + 63} = \frac{5/3}{s^2 + 10s + 21} = \frac{5}{12}\frac{1}{s+3} - \frac{5}{12}\frac{1}{s+7}$$

$$x(t) = \frac{5}{12}\left(e^{-3t} - e^{-7t}\right)$$

From the initial value theorem

$$x(0+) = \lim_{s\to\infty} s\frac{5/3}{s^2 + 10s + 21} = 0$$

which is the same as $x(0-)$. Also

$$\dot{x}(0+) = \lim_{s\to\infty} s^2\frac{5/3}{s^2 + 10s + 21} = \frac{5}{3}$$

which is not the same as $\dot{x}(0-)$.

3. Response of Second Order Systems

3.4 Stability (Section 2.5.8 of SD)

Unstable: when free response approaches ∞ as $t \rightarrow \infty$
Stable: when free response approaches 0
Neutral Stability: free response does not approach ∞ nor to 0, and is between unstable and stable.
Stability Properties: determined from linear system's characteristic roots

Learning from Examples of Second-Order Models: {with i.c. $x(0) = 1;\ \dot{x}(0) = 0$**}**

1. $\ddot{x} - 4x = f(t)$ ⇒ Characteristic roots are: $s = \pm 2$

Free response is: $$x(t) = \frac{1}{2}\left(e^{2t} + e^{-2t}\right)$$

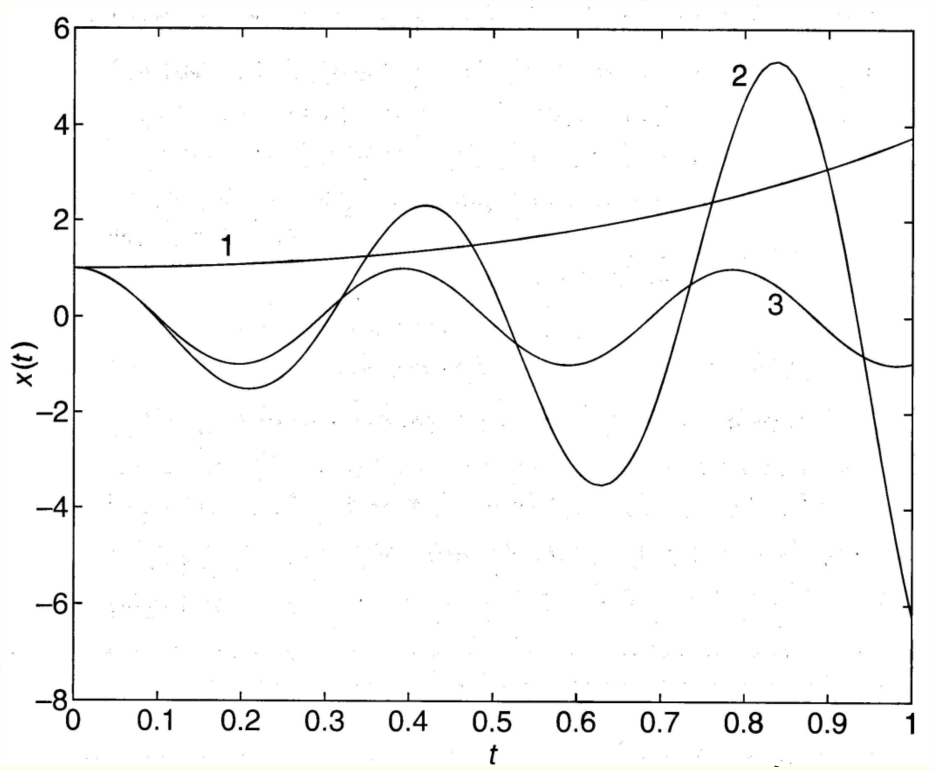
2. $\ddot{x} - 4\dot{x} + 229x = f(t)$ ⇒ Characteristic roots are: $s = 2 \pm 15j$

Free response is: $$x(t) = e^{2t}\left(\cos 15t - \frac{2}{15}\sin 15t\right)$$

3. Response of Second Order Systems

3. $\ddot{x} + 256x = f(t)$ ⇒ Characteristic roots are: $s = \pm 16j$

Free response is: $x(t) = \cos 16t$



Model 1: unstable
Model 2: unstable
Model 3: neutrally stable

3. Response of Second Order Systems

Observations:

Consider the following system model:

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Suppose the characteristic roots are: $s = \sigma \pm \omega j$

Free response is:

$$x(t) = e^{\sigma t}\left(C_1 \cos \omega t + C_2 \sin \omega t\right)$$

A. If real part is positive, ($\sigma > 0$), $e^{\sigma t}$ will grow very quickly with time.

➡ unstable

B. If real part is 0, ($\sigma = 0$), $e^{\sigma t} = 1$ and response oscillatory.

➡ neutrally stable

C. If real part is negative, ($\sigma < 0$), $e^{\sigma t} \rightarrow 0$ and response goes to zero.

➡ stable

3. Response of Second Order Systems

Conclusions on the Stability of Constant Coefficient Linear Systems:

1. A constant coefficient linear system model is **stable** iff *all* of its characteristic roots have negative real parts.
2. The model is **neutrally stable** if one or more roots have a zero real part with no roots on the imaginary axis of multiplicity ≥ 2 and the remaining roots have negative real parts.
3. The model is **unstable** if *any* root has a positive real part.
4. For a system such as: $m\ddot{x} + c\dot{x} + kx \ = f(t)$, the system is stable iff m, c and k have the same sign (Routh-Hurwitz Condition).

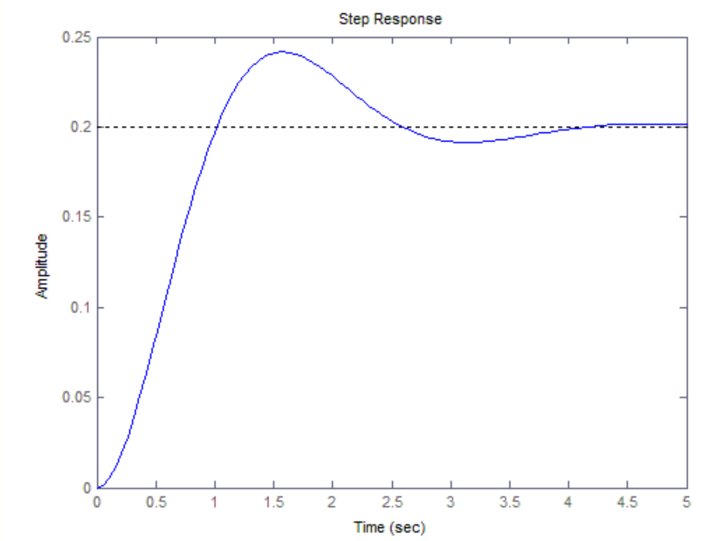
4. Matlab Applications

1. **step**

$$\ddot{x} + 2\dot{x} + 5x = f(t)$$

$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + 2s + 5}$$

```
>> sys=tf(1,[1,2,5]);
>> step(sys)
```



2. **Stepinfo**

```
>> sys=tf(1,[1,2,5]);
>> stepinfo(sys)
```

RiseTime: 0.6901
SettlingTime: 3.7352
SettlingMin: 0.1874
SettlingMax: 0.2416
Overshoot: 20.7875
Undershoot: 0
Peak: 0.2416
PeakTime: 1.5738