Homework #3: Solutions

Problem 1 (2.9.1 Kuethe)

Streamlines are concentric circle about the origin.

and $|\vec{V}| = kr^n$

Show that: $\varepsilon_z = \frac{1}{2} k(n+1) r^{n-1}$

and: $\gamma_z = -\frac{2k}{r^2}$ if $\varepsilon_z = 0$

For circular streamlines about the origin:

$$u_r = 0$$

$$u_\theta = \pm k r^n$$

$$w_z = \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \quad \text{and} \quad \varepsilon_z = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\right)$$

$$\frac{\partial u_\theta}{\partial r} = \pm k n r^{n-1} \qquad \frac{u_\theta}{r} = \frac{\pm k r^n}{r} = \pm k r^{n-1}$$

$$\varepsilon_z = \pm \frac{1}{2}\left(k n r^{n-1} + k r^{n-1}\right)$$

assume (+) solution $\rightarrow$ $\boxed{\varepsilon_z = \frac{1}{2} k (n+1) r^{n-1}}$

$$\gamma_z = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} = k n r^{n-1} - k r^{n-1}$$

$$\gamma_z = k(n-1) r^{n-1}$$

If $\varepsilon_z = 0$ $\rightarrow$ $\frac{\partial u_\theta}{\partial r} = -\frac{u_\theta}{r}$

$$k n \cancel{r^{n-1}} = -k \cancel{r^{n-1}} \rightarrow n = -1$$

for $n = -1$ $\rightarrow$ $\gamma_z = k(-2) r^{-2}$ $\rightarrow$ $\boxed{\gamma_z = -\frac{2k}{r^2}}$

Problem 2

Compute the vorticity:

(a) $\vec{V} = x^2 yz\,\hat{\imath} + 3xyz^3\,\hat{\jmath} + (x^2 - z^2)\,\hat{k}$

$$\text{vorticity} \equiv \vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\imath} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$u = x^2 yz$ $\qquad \frac{\partial u}{\partial y} = x^2 z \qquad \frac{\partial u}{\partial z} = x^2 y$

$v = 3xyz^3$ $\qquad \frac{\partial v}{\partial x} = 3yz^3 \qquad \frac{\partial v}{\partial z} = 9xyz^2$

$w = (x^2 - z^2)$ $\qquad \frac{\partial w}{\partial x} = 2x \qquad \frac{\partial w}{\partial y} = 0$

$$\vec{\nabla}\times\vec{V} = -9xyz^2\,\hat{\imath} + (x^2 y - 2x)\,\hat{\jmath} + (3yz^3 - x^2 z)\,\hat{k}$$

(b) $\vec{V} = e^x\cos(y)\,\hat{\imath} + e^x\sin(y)\,\hat{\jmath} + z\,\hat{k}$

$u = e^x\cos(y) \qquad \frac{\partial u}{\partial y} = -e^x\sin(y) \qquad \frac{\partial u}{\partial z} = 0$

$v = e^x\sin(y) \qquad \frac{\partial v}{\partial x} = e^x\sin(y) \qquad \frac{\partial v}{\partial z} = 0$

$w = z \qquad \frac{\partial w}{\partial x} = 0 \qquad \frac{\partial w}{\partial y} = 0$

$$\vec{\nabla}\times\vec{V} = (0-0)\,\hat{\imath} + (0-0)\,\hat{\jmath} + \left(e^x\sin(y) - (-e^x\sin(y))\right)\hat{k}$$

$$\boxed{\vec{\nabla}\times\vec{V} = 2e^x\sin(y)\,\hat{k}}$$

Problem 3:

2-D laminar boundary layer:

$$u(y) = U \sin\left(\frac{\pi y}{2\delta}\right) \qquad U = 1.5\,m/s$$

$$\delta = 2\,mm$$

(a) Compute the vorticity:

$$\vec{\nabla}\times\vec{V} = \left(\cancel{\frac{\partial w}{\partial y}}^{0} - \cancel{\frac{\partial v}{\partial z}}^{0}\right)\hat{i} + \left(\cancel{\frac{\partial u}{\partial z}}^{0} - \cancel{\frac{\partial w}{\partial x}}^{0}\right)\hat{j} + \left(\cancel{\frac{\partial v}{\partial x}}^{0} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$w=0 \quad v=0 \quad u=u(y) \quad w=0 \quad v=0$

$$\vec{\nabla}\times\vec{V} = -\frac{\partial u}{\partial y}\hat{k} = \boxed{-\frac{U\pi}{2\delta}\cos\left(\frac{\pi y}{2\delta}\right)\hat{k}}$$

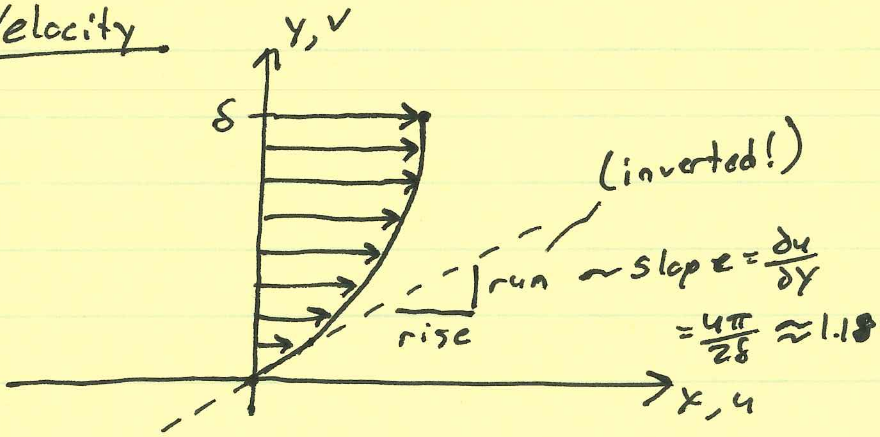
(b) Plot the vorticity and velocity from $y = 0$ to δ

Velocity: @ $y=0$, $u(0)=0$
$y=\delta$, $u(\delta)=U$

Vorticity: @ $y=0$, $\vec{\nabla}\times\vec{V} = -\frac{U\pi}{2\delta}$

$y=\delta$, $\vec{\nabla}\times\vec{V} = 0$

Velocity



Vorticity

(c) Compute angular rate of rotation @ $y = \delta/4$ and $y = \delta/2$

$$2\vec{\omega} = \vec{\nabla} \times \vec{V} \Rightarrow \vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V}$$

$$\vec{\omega} = -\frac{u\pi}{4\delta} \cos\left(\frac{\pi y}{2\delta}\right) \hat{k}$$

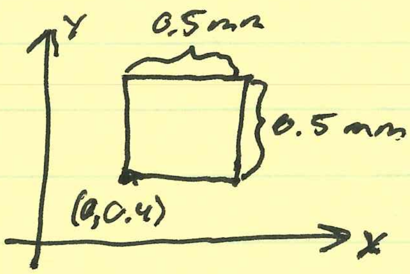
$$@ y = \delta/4 \rightarrow \vec{\omega}(\delta/4) = -\frac{u\pi}{4\delta} \cos\left(\frac{\pi}{8}\right)\hat{k} = -\frac{(1.5\,m/s)(\pi)}{4(2\,mm)}\left(\frac{1000\,mm}{1\,m}\right) \times \cos\left(\frac{\pi}{8}\right)$$

$$\boxed{\vec{\omega}(\delta/4) = -543.93 \text{ rad/s } \hat{k}}$$

$$@ y = \delta/2 \rightarrow \vec{\omega}(\delta/2) = -\frac{(1.5\,m/s)(\pi)}{4(2\,mm)}\left(\frac{1000\,mm}{1\,m}\right) \cos\left(\frac{\pi}{4}\right)$$

$$\boxed{\vec{\omega}(\delta/2) = -416.25 \text{ rad/s } \hat{k}}$$

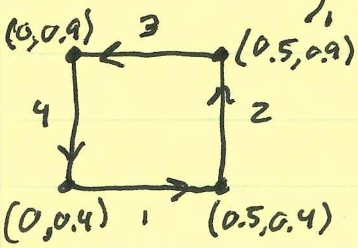
(d) Compute the circulation for the square contour below:



$$\Gamma = -\oint \vec{V}\cdot d\vec{s} = -\iint_A (\vec{\nabla}\times\vec{V})\cdot d\vec{A}$$

① ②

①: $\Gamma = -\int_1 \vec{V}\cdot d\vec{s} - \int_2 \vec{V}\cdot d\vec{s} - \int_3 \vec{V}\cdot d\vec{s} - \int_4 \vec{V}\cdot d\vec{s}$



1: $\vec{V}\cdot d\vec{s} = u\,dx$

2: $\vec{V}\cdot d\vec{s} = 0$

3: $\vec{V}\cdot d\vec{s} = -u\,dx$

4: $\vec{V}\cdot d\vec{s} = 0$

$$\Gamma = -\int_0^{0.5} u \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0.4} dx - \int_0^{0.5} -u \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0.9} dx$$

$$= -u\,x\Big|_0^{0.5} \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0.4} + u\,x\Big|_0^{0.5} \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0.9}$$

$$= -\frac{u}{2} \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0.4} + \frac{u}{2} \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{y=0.9}$$

$$\Gamma = \left(\frac{1.5}{2}\right)\left[\sin\left(\frac{\pi}{2}\,\frac{0.9\,mm}{2\,mm}\right) - \sin\left(\frac{\pi}{2}\,\frac{0.4\,mm}{2\,mm}\right)\right]\left(\frac{1}{1000}\,\frac{m}{mm}\right)$$

$\frac{mm \cdot m}{s}$ → multiply by $\left(\frac{1\,m}{1000\,mm}\right)$ to convert to m^2/s

$$\Gamma = 0.00026 \Rightarrow \boxed{\Gamma = 2.6 \times 10^{-4}\ m^2/s}$$

<u>2nd approach</u>

② $\Gamma = -\iint_A (\vec{\nabla} \times \vec{v}) \cdot d\vec{A}$ $\qquad d\vec{A} = dx\,dy\,\hat{k}$

$$\vec{\nabla} \times \vec{v} = \frac{-u\pi}{2\delta} \cos\left(\frac{\pi y}{2\delta}\right)\hat{k}$$

$$\Gamma = -\int_0^{0.5}\int_{0.4}^{0.9} \frac{-u\pi}{2\delta} \cos\left(\frac{\pi y}{2\delta}\right) dy\,dx$$

$$= +\int_0^{0.5} \frac{u\pi}{2\delta} \sin\left(\frac{\pi y}{2\delta}\right)\left(\frac{2\delta}{\pi}\right)\Big|_{0.4}^{0.9} dx$$

$$= u\int_0^{0.5} \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{0.4}^{0.9} dx = u\,x\Big|_0^{0.5} \sin\left(\frac{\pi y}{2\delta}\right)\Big|_{0.4}^{0.9}$$

$$= \frac{u}{2}\left[\sin\left(\frac{\pi}{2}\,\frac{0.9\,mm}{\delta}\right) - \sin\left(\frac{\pi}{2}\,\frac{0.4\,mm}{\delta}\right)\right] \frac{mm \cdot m}{s}$$

Same as before: $\therefore \boxed{\Gamma = 2.6 \times 10^{-4}\ m^2/s}$

Problem 4:

Rankine vortex:

$$V_\theta = \begin{cases} \omega_0 r & r \le a \\ \omega_0 \frac{a^2}{r} & r > a \end{cases}$$

$\omega_0 \equiv$ angular rate of the core.

(a) Compute the vorticity:

$$\vec{\nabla} \times \vec{v} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial z \\ v_r & r v_\theta & v_z \end{vmatrix} = \frac{1}{r}\left[\frac{\partial v_z}{\partial\theta} - \frac{\partial}{\partial z}(r v_\theta)\right]\hat{e}_r + \frac{1}{r}\left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{e}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{e}_z$$

$v_z = v_r = 0$ and $v_\theta = f(r)$, so:

$$\vec{\nabla} \times \vec{v} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta)\right]\hat{e}_z$$

for $r \le a \rightarrow \vec{\nabla}\times\vec{v} = \frac{1}{r}\left[\frac{\partial}{\partial r}(\omega_0 r^2)\right]$

$$= \frac{1}{r}\, 2\omega_0 r$$

$$\vec{\nabla}\times\vec{v} = 2\omega_0$$

for $r > a \rightarrow \vec{\nabla}\times\vec{v} = \frac{1}{r}\left[\frac{\partial}{\partial r}(\omega_0 a^2)\right]$

$$\vec{\nabla}\times\vec{v} = 0$$

$$\boxed{\vec{\nabla}\times\vec{v} = \begin{cases} 2\omega_0 & r \le a \\ 0 & r > a \end{cases}}$$

(b) Compute the circulation around a circular curve of radius $r = a$ centered at the origin.

$$\Gamma = -\oint \vec{v}\cdot d\vec{s}$$

$$\Gamma = -\int_0^{2\pi} (w_0 r)(r\,d\theta)\cancelto{1}{(\hat{e}_\theta \cdot \hat{e}_\theta)}$$

$$= -\int_0^{2\pi} w_0 r^2\, d\theta \quad \rightarrow \quad \Gamma = -w_0 r^2 (2\pi)$$

$$@\ r = a \ \rightarrow \ \boxed{\Gamma = -2\pi w_0 a^2}$$

(c) Argue that this is the largest circulation that can be calculated:

The circulation Γ can be related to the vorticity in a region bounded by curve C in the following way:

$$\Gamma = -\iint (\vec{\nabla} \times \vec{v}) \cdot d\vec{A}$$

This is an area integral of the vorticity. ~~Since the~~ The integral can be broken up into two integrals, one for $r \leq a$ and another for $r > a$ ($r = R$, $R > a$)

$$\Gamma = -\underbrace{\iint_0^a (\vec{\nabla} \times \vec{v}) \cdot d\vec{A}}_{= -2\pi w_0 a^2} - \underbrace{\iint_a^R (\vec{\nabla} \times \vec{v}) \cdot d\vec{A}}_{(R > a)}$$

$$\Gamma = -2\pi w_0 a^2 - \iint_a^R (\vec{\nabla} \times \vec{v}) \cdot d\vec{A}$$

$$\boxed{\text{Since } \vec{\nabla} \times \vec{v} = 0 \text{ for } r > a \text{ then the second integral} = 0 \text{ and } \Gamma = -2\pi w_0 a^2 \text{ is the largest circulation that can be calculated.}}$$

Problem 5:

Show that $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$

holds in 3D: (a) Cartesian coordinates

(b) Cylindrical coordinates

(a) $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

5

$$\vec{\nabla}\times(\vec{\nabla}\phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\cancelto{0}{\frac{\partial^2 \phi}{\partial y\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial y}}\right)\hat{i} - \left(\cancelto{0}{\frac{\partial^2 \phi}{\partial x\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial x}}\right)\hat{j}$$

$$+ \left(\cancelto{0}{\frac{\partial^2 \phi}{\partial x\,\partial y} - \frac{\partial^2 \phi}{\partial y\,\partial x}}\right)\hat{k}$$

$\therefore\ \vec{\nabla}\times(\vec{\nabla}\phi) = 0$ in 3D Cartesian coordinates.

(b) $\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \phi}{\partial z}\hat{e}_z$

5

$$\vec{\nabla}\times(\vec{\nabla}\phi) = \frac{1}{r}\begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial z \\ \frac{\partial \phi}{\partial r} & \frac{\partial \phi}{\partial \theta} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left(\cancelto{0}{\frac{\partial^2 \phi}{\partial \theta\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial \theta}}\right)\hat{e}_r - \left(\cancelto{0}{\frac{\partial^2 \phi}{\partial r\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial r}}\right)\hat{e}_\theta$$

$$+ \left(\cancelto{0}{\frac{\partial^2 \phi}{\partial r\,\partial z} - \frac{\partial^2 \phi}{\partial z\,\partial r}}\right)\hat{e}_z$$

$\therefore\ \vec{\nabla}\times(\vec{\nabla}\phi) = 0$ in 3D Cylindrical coordinates.

Problem 6:

$$\phi = Ax^2 + Bxy - Ay^2$$

(a) Verify that this is an incompressible flow.

Incompressibility condition: $\frac{D\rho}{Dt} \equiv 0$

Using the compressible continuity equation:

$$\cancel{\frac{D\rho}{Dt}}^{\,0} + \rho \vec{\nabla}\cdot\vec{V} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{V} = 0 \qquad 1$$

And, $\vec{V} = \vec{\nabla}\phi \Rightarrow \nabla^2\phi = 0$ (Laplace's equation)

1

Check if ϕ is incompressible:

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

2

$$\frac{\partial\phi}{\partial x} = 2Ax + By \qquad \frac{\partial^2\phi}{\partial x^2} = 2A$$

$$\frac{\partial\phi}{\partial y} = Bx - 2Ay \qquad \frac{\partial^2\phi}{\partial y^2} = -2A$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2A + (-2A) = 0 \checkmark$$

$\boxed{\text{It is incompressible}}$

(b) ~~ϕ~~ $u = \frac{\partial\phi}{\partial x}$ $\qquad v = \frac{\partial\phi}{\partial y}$

1 $\quad u = 2Ax + By \qquad v = Bx - 2Ay$

① $u = \frac{\partial\psi}{\partial y}$ $\qquad v = -\frac{\partial\psi}{\partial x}$ ② $\qquad$ 1

① $2Ax + By = \frac{\partial\psi}{\partial y}$

$$\psi = \int (2Ax + By)\,dy + C$$

1 $$\psi = 2Axy + \frac{By^2}{2} + f(x) + C \quad (3)$$

(2) $$-Bx + 2Ay = \frac{\partial \psi}{\partial x}$$

$$\psi = \int (-Bx + 2Ay)\,dx + C$$

1 $$\psi = \frac{-Bx^2}{2} + 2Axy + g(y) + C \quad (4)$$

Set arbitrary constant, $C = 0$ and compare (3) and (4)

$$\cancel{2Axy} + \frac{By^2}{2} + f(x) = -\frac{Bx^2}{2} + \cancel{2Axy} + g(y)$$

1 $$g(y) = \frac{By^2}{2} \quad , \quad f(x) = -\frac{Bx^2}{2}$$

Now either solution (3) or (4) is correct:

1 $$\boxed{\psi = \frac{By^2}{2} + 2Axy - \frac{Bx^2}{2}}$$