Engineering Vibrations & Systems

Module 9: Multi-Degree of Freedom Systems

ME 242
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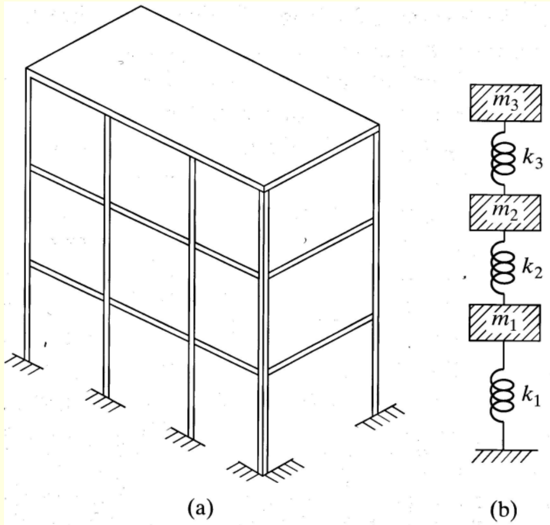
Module 9

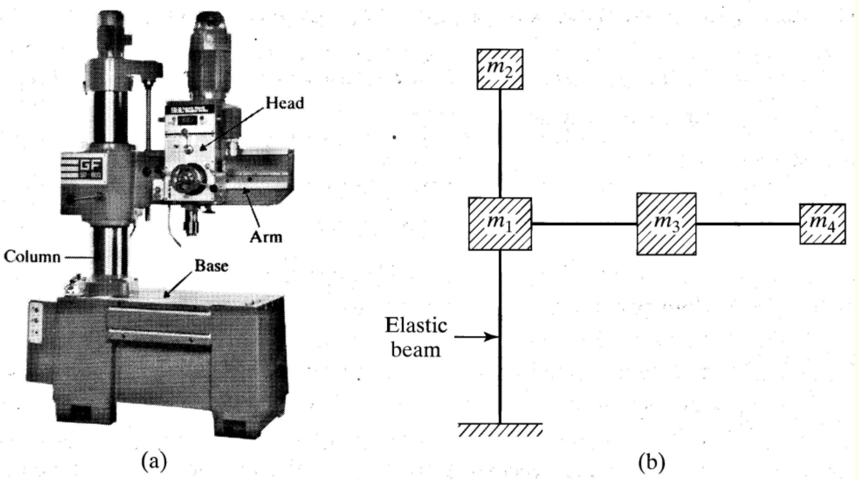
1. **Multi-Degree of Freedom Systems**
   - 1.1 Equations of Motion
   - 1.2 Flexibility Influence Coefficients
   - 1.3 Reciprocity Theorem
2. **Eigenvalue Problem**
   - 2.1 Equations of Motion
   - 2.2 Example
3. **Orthogonality of Normal Modes**
   - 3.1 Equations of Motion
   - 3.2 Example

1. Multi-Degree of Freedom Systems

Examples of Multi-Degree of Freedom Systems



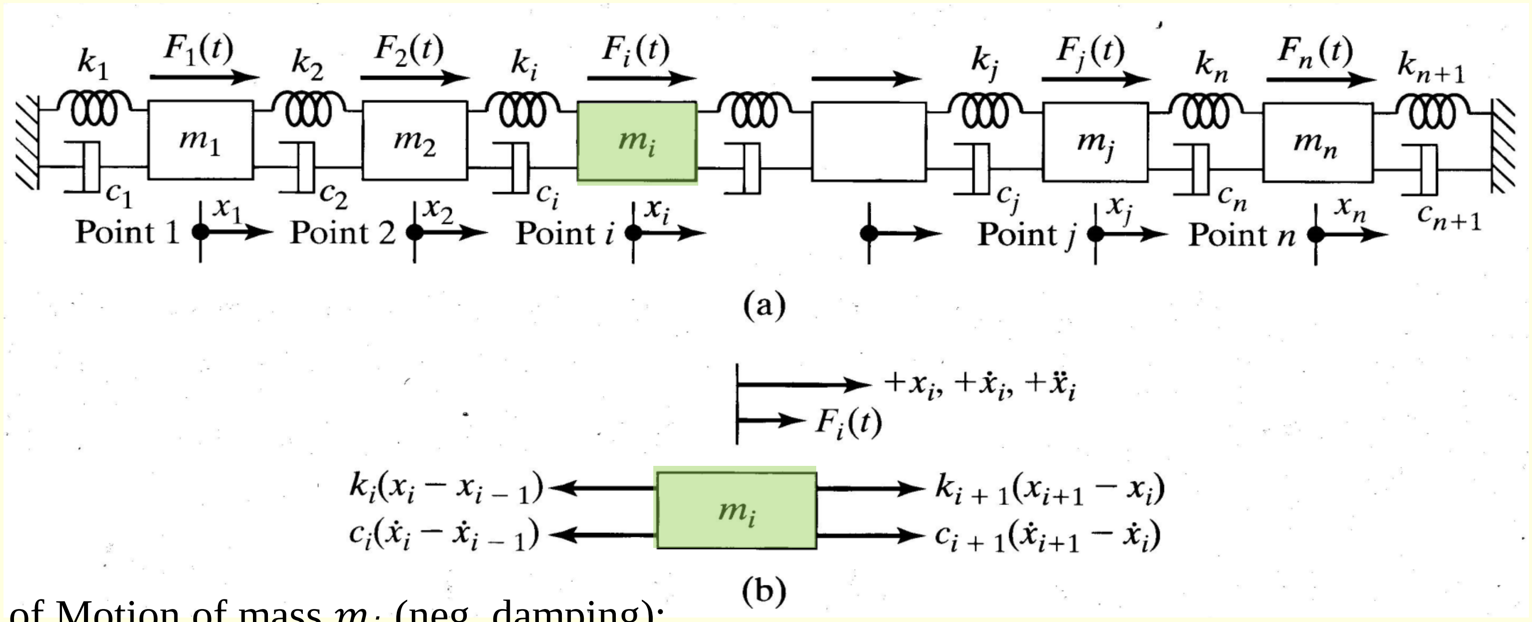
Three-story building.



Radial drilling machine. (Photo courtesy of South Bend Lathe Corp.)

1. Multi-Degree of Freedom Systems

1.1 Equations of Motion – Multi-Degree of Freedom Systems



Eqn. of Motion of mass m_i (neg. damping):

$$m_i\ddot{x}_i = -k_i\left(x_i - x_{i-1}\right) + k_{i+1}\left(x_{i+1} - x_i\right) + F_i \quad i = 2,\ldots,n-1$$

or

$$m_i\ddot{x}_i - k_i x_{i-1} + \left(k_i + k_{i+1}\right)x_i - k_{i+1}x_{i+1} = F_i \quad i = 2,\ldots,n-1 \qquad [1a]$$

1. Multi-Degree of Freedom Systems

1.1 Equations of Motion (Continue)

For i=1: $$m_1\ddot{x}_1+\left(k_1+k_2\right)x_1-k_2x_2=F_1 \qquad [1b]$$

Eqns. [1a] and [1b]:

$$[m]\{\ddot{x}\}+[k]\{x\}=\{F\} \qquad [2]$$

where:

$$[m]=\begin{bmatrix} m_1 & 0 & 0 & .... & 0 \\ 0 & m_2 & 0 & .... & 0 \\ 0 & 0 & m_3 & .... & 0 \\ \vdots & \vdots & \vdots & .... & 0 \\ 0 & 0 & 0 & .... & m_n \end{bmatrix};\{\ddot{x}\}=\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \vdots \\ \ddot{x}_n(t) \end{Bmatrix};\{F\}=\begin{Bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{Bmatrix} \qquad [3a]$$

$$[k]=\begin{bmatrix} (k_1+k_2) & -k_2 & 0 & ... & 0 & 0 \\ -k_2 & (k_2+k_3) & -k_3 & ... & 0 & 0 \\ 0 & -k_3 & (k_3+k_4) & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & -k_n & (k_n+k_{n+1}) \end{bmatrix} \qquad [3b]$$

In general form:

$$[m]=\begin{bmatrix} m_{11} & m_{12} & m_{13} & .... & m_{1n} \\ m_{12} & m_{22} & m_{23} & .... & m_{2n} \\ \vdots & & & & \\ m_{1n} & m_{2n} & m_{3n} & .... & m_{nn} \end{bmatrix} \qquad [k]=\begin{bmatrix} k_{11} & k_{12} & k_{13} & .... & k_{1n} \\ k_{12} & k_{22} & k_{23} & .... & k_{2n} \\ \vdots & & & & \\ k_{1n} & k_{2n} & k_{3n} & .... & k_{nn} \end{bmatrix} \qquad [4]$$

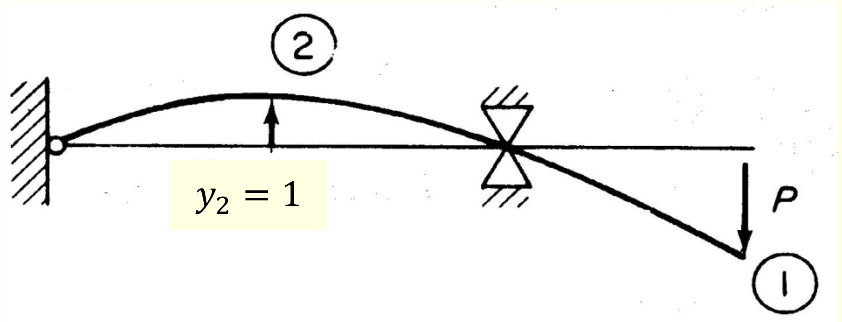
1. Multi-Degree of Freedom Systems

1.1 Equations of Motion (Continue)

Examine the stiffness matrix:

$$[k]=\begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{12} & k_{22} & k_{23} & \cdots & k_{2n} \\ \vdots & & & & \\ k_{1n} & k_{2n} & k_{3n} & \cdots & k_{nn} \end{bmatrix}$$

What does k_{12} mean? How do you even measure something like that in a structure?



If you want to determine that theoretically, it is not too difficult. However, to DIRECTLY MEASURE that, it is very difficult. So we will approach that a different way.

1. Multi-Degree of Freedom Systems

1.2 Flexibility Influence Coefficients (Section 7.6 SD)

Definition:

$a_{ij} =$ deflection at point *i* of the system, due to a unit load at point *j*

$x_{ij} =$ deflection (displacement) at point *i* due to force F_j applied at point *j*

Relationship: $x_{ij} = a_{ij} F_j$

In matrix form: $\{x\} = [a]\{F\}$ since $x_i = \sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} a_{ij} F_j \quad i = 1, 2, \ldots, n.$

$$\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}}_{\text{flexibility matrix}} \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \qquad [5]$$

Note: $[a] = [k]^{-1}$ flexibility matrix [6]

1. Multi-Degree of Freedom Systems

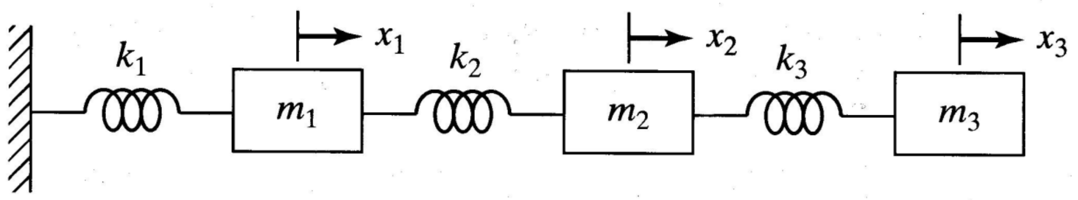
1.2 Flexibility Influence Coefficients

Steps:

1. Assume a unit load at point *j* (*j*=1 when starting). By definition the displacements of the various points *i* (*i*=1,2,…,*N*) resulting from this load give the flexibility influence coefficients, $a_{ij},\ i = 1,2,\dots,N$. Thus a_{ij} can be found using the principles of statics and solid mechanics.
2. After Step 1, for *j* = 1, the procedure is repeated for *j*=1,2,…,*N*.
3. Instead of applying Steps 1 and 2, the flexibility matrix, [a], can be determined by finding the inverse of the stiffness matrix, [k], if it is available.

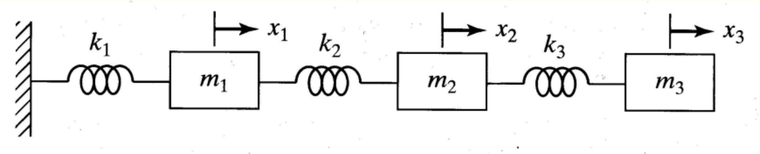
Example 1

Determine the flexibility influence coefficients for the spring-mass system shown below:



(a)

1. Multi-DoF Systems



(a)

Example 1 (Continue)

1. Apply unit force at mass m_1, no other force at other masses ($F_1 = 1,\ F_2 = F_3 = 0$). Deflections at (1), (2), (3), are by definition, a_{11}, a_{21} and a_{31} respectively. [Figs.(b) and (c)]

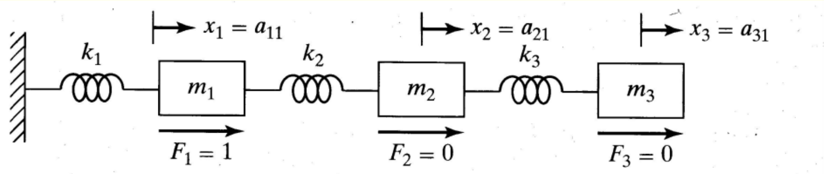
m_1: $k_1 a_{11} = k_2(a_{21} - a_{11}) + 1$ [7a]

m_2: $k_2(a_{21} - a_{11}) = k_3(a_{31} - a_{21})$ [7b]

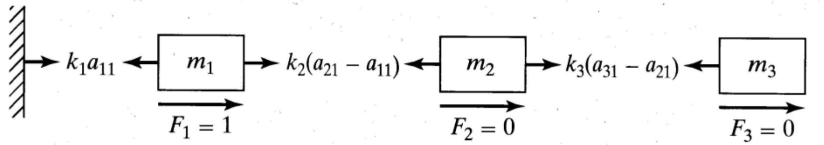
m_3: $k_3(a_{31} - a_{21}) = 0$ [7c]

Solution of [7a]—[7c]:

$$a_{11} = 1/k_1\ ; \quad a_{21} = 1/k_1\ ; \quad a_{31} = 1/k_1\ ;$$



(b)



(c)

2. Next apply unit force at mass m_2, no other force at other masses ($F_1 = 0,\ F_2 = 1,\ F_3 = 0$). Deflections at (1), (2), (3), are by definition, a_{12}, a_{22} and a_{32} respectively. [Fig.(d) and (e)]

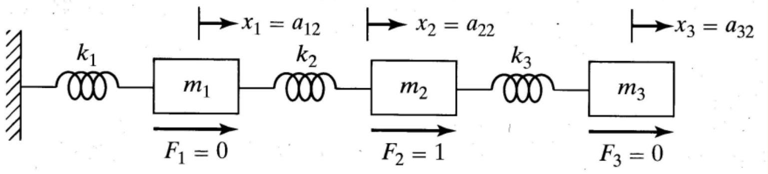
m_1: $k_1 a_{12} = k_2(a_{22} - a_{12})$ [7d]

m_2: $k_2(a_{22} - a_{12}) = k_3(a_{32} - a_{22}) + 1$ [7e]

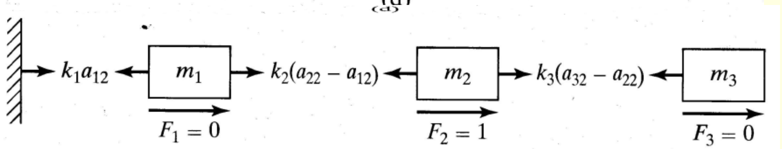
m_3: $k_3(a_{32} - a_{22}) = 0$ [7f]

Solution of [7d]—[7f]:

$$a_{12} = 1/k_1\ ; \quad a_{22} = \frac{1}{k_1} + \frac{1}{k_2}; \quad a_{32} = \frac{1}{k_1} + \frac{1}{k_2};$$



(d)



(e)

1. Multi-Degree of Freedom Systems

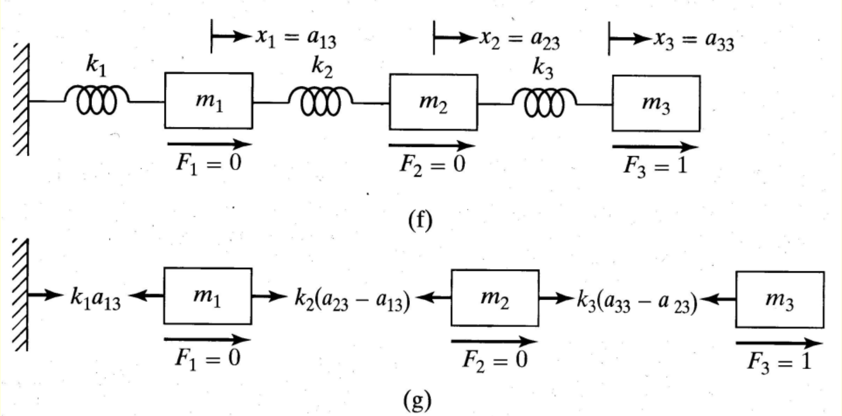
Example 1 (Continue)

3. Finally, apply unit force at mass m_3, no other force at other masses ($F_1 = F_2 = 0; F_3 = 1$). Deflections at (1), (2), (3), are by definition, a_{13}, a_{23} and a_{33} respectively. [Figs.(f) and (g)]

m_1: $k_1 a_{13} = k_2(a_{23} - a_{13})$ [7g]

m_2: $k_2(a_{23} - a_{13}) = k_3(a_{33} - a_{23})$ [7h]

m_3: $k_3(a_{33} - a_{23}) = 1$ [7i]



Solution of [7g]—[7i]:

$$a_{13} = 1/k_1\,; \quad a_{23} = \frac{1}{k_1} + \frac{1}{k_2}; \quad a_{33} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

Complete flexibility matrix is given by:

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{k_1} & \frac{1}{k_1} & \frac{1}{k_1} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} \\ \frac{1}{k_1} & \frac{1}{k_1} + \frac{1}{k_2} & \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix} \quad [7j]$$

Reciprocity Theorem:

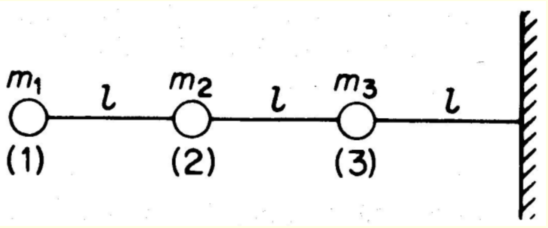
$a_{ij} = a_{ji}$

→ Flexibility matrix is SYMMETRIC

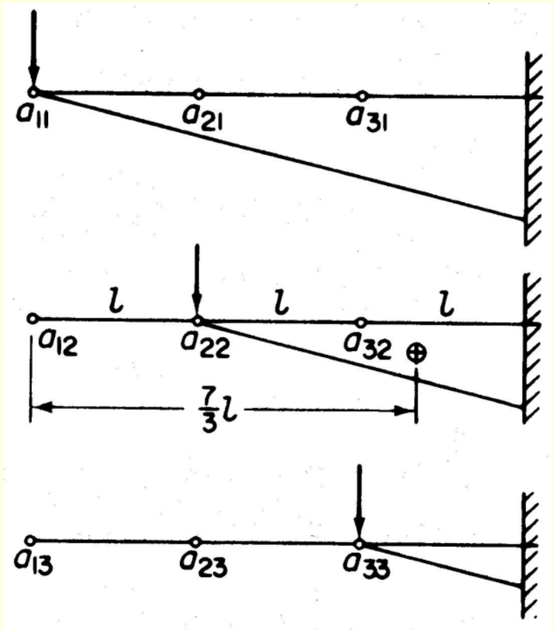
1. Multi-Degree of Freedom Systems

Example 2

Determine the flexibility influence coefficients at points (1), (2) and (3) of the uniform cantilever beam shown. You may either use solid mechanics equations if provided, or use the moment of area method where the deflection at a point is equal to the moment of the $\frac{M}{EI}$ area about the position in question.



For example, the value of a_{12} is found from the figure at the right and the calculation is as follows:



$$a_{12} = a_{21} = \frac{1}{EI}\left[\frac{1}{2}(2l)^2 \times \frac{7}{3}l\right] = \frac{14}{3}\frac{l^3}{EI}$$

Other values are:

$$a_{11} = \frac{27}{3}\frac{l^3}{EI} \quad a_{21} = a_{12} = \frac{14}{3}\frac{l^3}{EI}$$

$$a_{22} = \frac{8}{3}\frac{l^3}{EI} \quad a_{23} = a_{32} = \frac{2.5}{3}\frac{l^3}{EI}$$

$$a_{11} = \frac{1}{3}\frac{l^3}{EI} \quad a_{13} = a_{31} = \frac{4}{3}\frac{l^3}{EI}$$

1. Multi-Degree of Freedom Systems

Example 2 (Continue)

The flexibility matrix can now be written as:

$$[a] = \frac{l^3}{3EI}\begin{bmatrix} 27 & 14 & 4 \\ 14 & 8 & 2.5 \\ 4 & 2.5 & 1 \end{bmatrix}$$

From [7j], we saw that the deflection vector $\{x\}$ is equal to $[a]$ * the load vector, i.e.,

$$\{x\} = [a]\{f\} \qquad \text{[8a]}$$

or:

$$\begin{aligned} \{f\} &= [a]^{-1}\{x\} \\ &= [k]\,\{x\} \qquad \text{[8b]} \end{aligned}$$

Note that:

(i) the flexibility matrix is SYMMETRIC

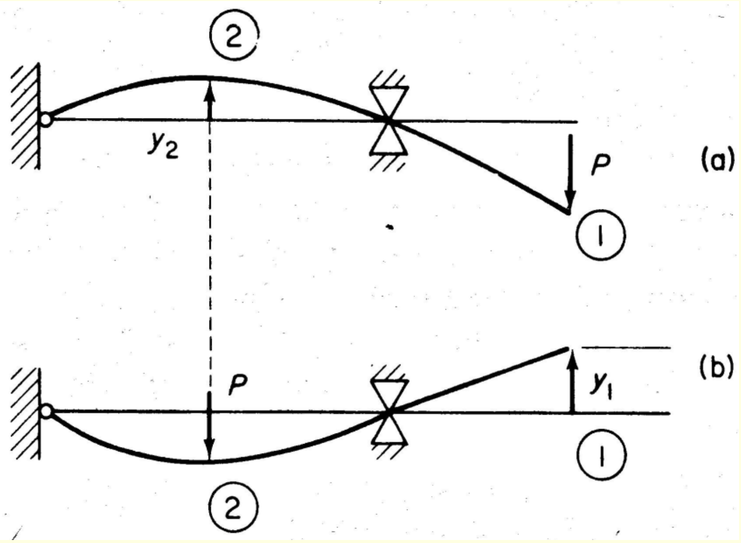
(ii) [a] = $[k]^{-1}$

1. Multi-Degree of Freedom Systems

1.3 Reciprocity Theorem

$$\boxed{a_{ij} = a_{ji}} \qquad [9]$$

Example 3



An overhanging beam with load P is first applied at (1) and then at (2). In the first figure (a), the deflection at (2) is:

$$y_2 = a_{21}P$$

In the second figure (b), the deflection at (1) is:

$$y_1 = a_{12}P$$

Since $a_{12} = a_{21}$, therefore $y_1 = y_2$. In a linear system, the deflection at (2) due to a load at (1) is equal to the deflection at (1) due to the same load at (2).

2. Eigenvalue Problem

2.1 Equations of Motion

For a conservative system with no non-conservative forces:

$$[m]\{\ddot{x}\} + [k]\{x\} = \{0\} \qquad \text{[10a]}$$

Solution of [10a] corresponds to the undamped free vibration of the system. If he system is given some energy in the form of initial displacements or initial velocities, or both, it will vibrate indefinitely since there is no dissipation of energy. We assume solution to be of the form:

$$\{x\} = \{X\}T(t) \qquad \text{[10b]}$$

where $T(t)$ is a function of time t, such as $\sin \omega t$ such as what we have been doing. Then, from [10b]:

$$\{\ddot{x}\} = -\{X\}\omega^2 T(t) \qquad \text{[10c]}$$

[10b] and [10c] into [10a]:

$$-[m]\{X\}\omega^2 T(t) + [k]\{X\}T(t) = \{0\} \qquad \text{[10d]}$$

Since $T \neq 0$

$$\therefore \underbrace{\left[[k] - \omega^2 [m]\right]}_{n\times n}\{X\} = \{0\} \qquad \text{[10e]}$$

where n is the number of DoF

[10e] is called the characteristic or eigenvalue problem. For non-trivial solution for $\{X\}$:

$$\Delta = \det\left|[k] - \omega^2 [m]\right| = 0 \qquad \text{[10f]}$$

2. Eigenvalue Problem

[10f] is called the characteristic equation, and ω^2 is the eigenvalue or characteristic value. ω is the natural frequency of the system. Solution of this polynomial gives $\omega_1{}^2, \omega_2{}^2, ..., \omega_n{}^2$ n roots in ascending order of magnitude $\omega_1 \le \omega_2 \le ... \le \omega_n$. Lowest value ω_1 is called the **fundamental frequency** or the first natural frequency.

2.2 Solution of Eigenvalue Problem

From [10e], rewriting, we have:

$$\left[\lambda[k]-[m]\right]\{X\}=\{0\} \qquad \text{[11a]}$$

where $\lambda = \dfrac{1}{\omega^2}$

Premultiplying by k^{-1}:

$$[\lambda[I]-\underbrace{[k]^{-1}[m]}_{[D]}]\{X\}=\{0\}$$

$$\left[\lambda[I]-[D]\right]\{X\}=\{0\} \qquad \text{[11b]}$$

For non-trivial $\{X\}$,

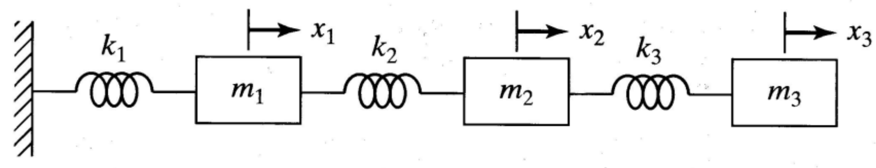
$$\det\left|\lambda[I]-[D]\right|=\{0\} \qquad \text{[11c]}$$

Solution of nth degree polynomial gives the frequencies of the system.

2. Eigenvalue Problem

Example 4

For the system used in Example 1, find the natural frequencies and mode shapes of the system for $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m_3 = m$.



(a)

The dynamical matrix is given by

$$[D] = [k]^{-1}[m] \equiv [a][m] \quad \text{(E.1)}$$

where the flexibility and mass matrices can be obtained from Example 1

$$[a] = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{(E.2)}$$

and

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(E.3)}$$

Thus

$$[D] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{(E.4)}$$

By setting the characteristic determinant equal to zero, we obtain the frequency equation

$$\Delta = |\lambda[I] - [D]| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \right| = 0 \quad \text{(E.5)}$$

2. Eigenvalue Problem

Example 4 (continue)

where

$$\lambda = \frac{1}{\omega^2} \tag{E.6}$$

By dividing throughout by λ, Eq. (E.5) gives

$$\begin{vmatrix} 1-\alpha & -\alpha & -\alpha \\ -\alpha & 1-2\alpha & -2\alpha \\ -\alpha & -2\alpha & 1-3\alpha \end{vmatrix} = \alpha^3 - 5\alpha^2 + 6\alpha - 1 = 0 \tag{E.7}$$

where

$$\alpha = \frac{m}{k\lambda} = \frac{m\omega^2}{k} \tag{E.8}$$

The roots of the cubic equation (E.7) are given by

$$\alpha_1 = \frac{m\omega_1^2}{k} = 0.19806, \qquad \omega_1 = 0.44504\sqrt{\frac{k}{m}} \tag{E.9}$$

$$\alpha_2 = \frac{m\omega_2^2}{k} = 1.5553, \qquad \omega_2 = 1.2471\sqrt{\frac{k}{m}} \tag{E.10}$$

$$\alpha_3 = \frac{m\omega_3^2}{k} = 3.2490, \qquad \omega_3 = 1.8025\sqrt{\frac{k}{m}} \tag{E.11}$$

2. Eigenvalue Problem

Example 4 (continue)

Once the natural frequencies are known, the mode shapes or eigenvectors can be calculated using Eq. 11b :

$$[\lambda_i[I] - [D]]\,\vec{X}^{(i)} = \vec{0}, \qquad i = 1, 2, 3 \tag{E.12}$$

where

$$\vec{X}^{(i)} = \begin{Bmatrix} X_1^{(i)} \\ X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix}$$

denotes the ith mode shape. The procedure is outlined below.

$\boxed{\textit{First Mode:}}$ By substituting the value of ω_1 $\left(\text{i.e., } \lambda_1 = 5.0489\frac{m}{k}\right)$ in Eq. (E.12), we obtain

$$\left[5.0489\frac{m}{k}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\right]\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \\ X_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

That is,

$$\begin{bmatrix} 4.0489 & -1.0 & -1.0 \\ -1.0 & 3.0489 & -2.0 \\ -1.0 & -2.0 & 2.0489 \end{bmatrix}\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \\ X_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.13}$$

2. Eigenvalue Problem

Example 4 (continue)

Equation (E.13) denotes a system of three homogeneous linear equations in the three unknowns $X_1^{(1)}$, $X_2^{(1)}$, and $X_3^{(1)}$. Any two of these unknowns can be expressed in terms of the remaining one. If we choose, arbitrarily, to express $X_2^{(1)}$ and $X_3^{(1)}$ in terms of $X_1^{(1)}$, we obtain from the first two rows of Eq. (E.13)

$$X_2^{(1)} + X_3^{(1)} = 4.0489\, X_1^{(1)}$$

$$3.0489 X_2^{(1)} - 2.0\, X_3^{(1)} = X_1^{(1)} \tag{E.14}$$

Once Eqs. (E.14) are satisfied, the third row of Eq. (E.13) is satisfied automatically. The solution of Eqs. (E.14) can be obtained:

$$X_2^{(1)} = 1.8019\, X_1^{(1)} \qquad \text{and} \qquad X_3^{(1)} = 2.2470\, X_1^{(1)} \tag{E.15}$$

Thus the first mode shape is given by

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.8019 \\ 2.2470 \end{Bmatrix} \tag{E.16}$$

where the value of $X_1^{(1)}$ can be chosen arbitrarily.

2. Eigenvalue Problem

Example 4 (continue)

$\boxed{\textit{Second Mode:}}$ The substitution of the value of $\omega_2\left(\text{i.e., } \lambda_2 = 0.6430\frac{m}{k}\right)$ in Eq. (E.12) leads to

$$\left[0.6430\frac{m}{k}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\right]\begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \\ X_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

that is,

$$\begin{bmatrix} -0.3570 & -1.0 & -1.0 \\ -1.0 & -1.3570 & -2.0 \\ -1.0 & -2.0 & -2.3570 \end{bmatrix}\begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \\ X_3^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.17}$$

As before, the first two rows of Eq. (E.17) can be used to obtain

$$\begin{aligned} -X_2^{(2)} - X_3^{(2)} &= 0.3570\, X_1^{(2)} \\ -1.3570 X_2^{(2)} - 2.0\, X_3^{(2)} &= X_1^{(2)} \end{aligned} \tag{E.18}$$

The solution of Eqs. (E.18) leads to

$$X_2^{(2)} = 0.4450\, X_1^{(2)} \quad \text{and} \quad X_3^{(2)} = -0.8020\, X_1^{(2)} \tag{E.19}$$

2. Eigenvalue Problem

Example 4 (continue)

Thus the second mode shape can be expressed as

$$\vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.4450 \\ -0.8020 \end{Bmatrix} \tag{E.20}$$

where the value of $X_1^{(2)}$ can be chosen arbitrarily.

Third Mode: To find the third mode, we substitute the value of $\omega_3 \left(\text{i.e., } \lambda_3 = 0.3078 \frac{m}{k}\right)$ in Eq. (E.12) and obtain

$$\left[0.3078 \frac{m}{k} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{m}{k} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \right] \begin{Bmatrix} X_1^{(3)} \\ X_2^{(3)} \\ X_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

that is,

$$\begin{bmatrix} -0.6922 & -1.0 & -1.0 \\ -1.0 & -1.6922 & -2.0 \\ -1.0 & -2.0 & -2.6922 \end{bmatrix} \begin{Bmatrix} X_1^{(3)} \\ X_2^{(3)} \\ X_3^{(3)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{E.21}$$

2. Eigenvalue Problem

Example 4 (continue)

The first two rows of Eq. (E.21) can be written as

$$-X_2^{(3)} - X_3^{(3)} = 0.6922\, X_1^{(3)}$$
$$-1.6922\, X_2^{(3)} - 2.0\, X_3^{(3)} = X_1^{(3)} \tag{E.22}$$

Equations (E.22) give

$$X_2^{(3)} = -1.2468\, X_1^{(3)} \quad \text{and} \quad X_3^{(3)} = 0.5544\, X_1^{(3)} \tag{E.23}$$

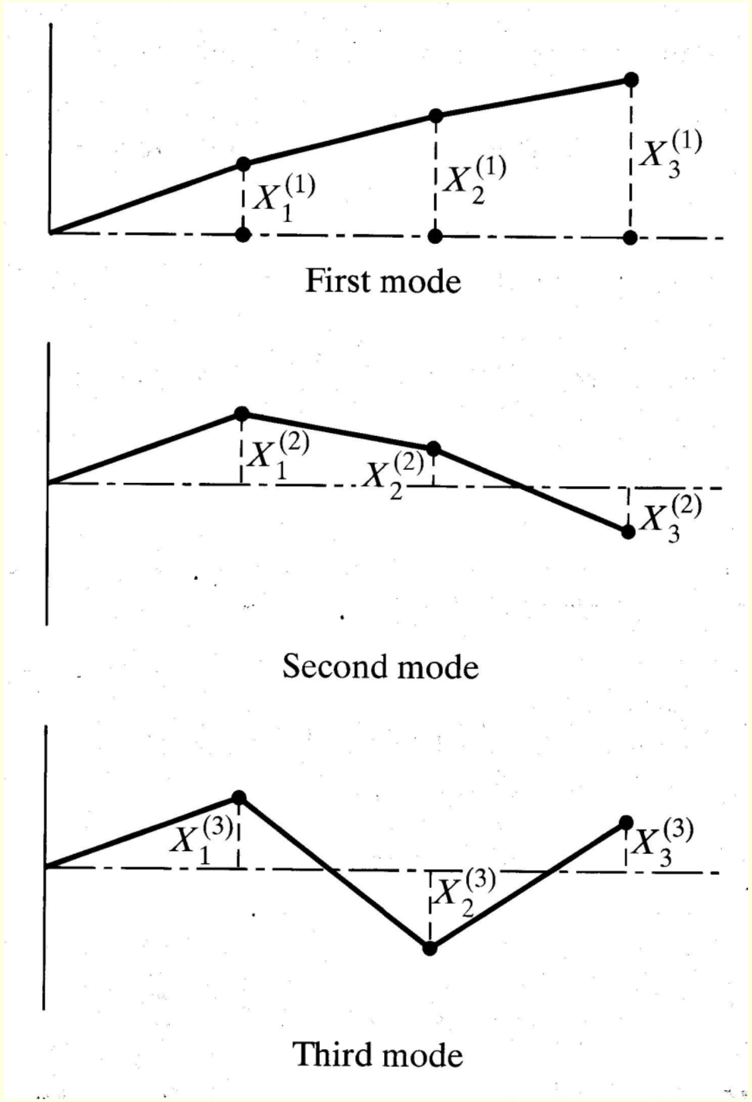
Hence the third mode shape can be written as

$$\vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.2468 \\ 0.5544 \end{Bmatrix} \tag{E.24}$$

where the value of $X_1^{(3)}$ is arbitrary. The values of $X_1^{(1)}$, $X_1^{(2)}$, and $X_1^{(3)}$ are usually taken as 1, and the mode shapes are shown

2. Eigenvalue Problem

Example 4 (continue)



3. Orthogonality of Normal Modes

From Eq. [10e]: $\therefore \underbrace{\left[[k]-\omega^2[m]\right]}_{n\times n}\{X\}=\{0\}$

Corresponding to frequency ω_j is the eigenvector $X^{(j)}$:

$$[k]\{X\}^{(j)}=\omega_j^{\,2}[m]\{X\}^{(j)} \qquad [12a]$$

Corresponding to another frequency ω_i is the eigenvector $X^{(i)}$:

$$[k]\{X\}^{(i)}=\omega_i^{\,2}[m]\{X\}^{(i)} \qquad [12b]$$

$$\{X\}^{(i)T}[12a]\rightarrow\{X\}^{(i)T}[k]\{X\}^{(j)}=\omega_j^{\,2}\{X\}^{(i)T}[m]\{X\}^{(j)} \qquad [13a]$$

$$\{X\}^{(j)T}[12b]\rightarrow\{X\}^{(j)T}[k]\{X\}^{(i)}=\omega_i^{\,2}\{X\}^{(j)T}[m]\{X\}^{(j)} \qquad [13b]$$

Subtracting: $\left(\omega_i^{\,2}-\omega_j^{\,2}\right)\{X\}^{(j)T}[m]\{X\}^{(i)}=0$

Since: $\omega_i\neq\omega_j$

Therefore: $\{X\}^{(j)T}[m]\{X\}^{(i)}=0 \qquad i\neq j$ [14a]

Similarly: $\{X\}^{(j)T}[k]\{X\}^{(i)}=0 \qquad i\neq j$ [14b]

i.e., $\{X\}^{(i)}$ and $\{X\}^{(j)}$ are **orthogonal** w.r.t. mass and stiffness matrices.

3. Orthogonality of Normal Modes

When $\boldsymbol{i} = \boldsymbol{j}$:

$$M_{ii} = \{X\}^{(i)T}[m]\{X\}^{(i)} \quad i = 1, 2, \ldots, n \qquad [15a]$$

and

$$K_{ii} = \{X\}^{(i)T}[k]\{X\}^{(i)} \quad i = 1, 2, \ldots, n \qquad [15b]$$

Writing the following matrix (**MODAL MATRIX**):

$$[X] = \left[\{X\}^{(1)}\{X\}^{(2)}\ldots\{X\}^{(n)}\right]_{n\times n} \qquad [16]$$

Then:

$$\left.\begin{aligned} [X]^T[m][X] &= \begin{bmatrix} M_{11} & & & 0 \\ & M_{22} & & \\ & & \ddots & \\ 0 & & & M_{nn} \end{bmatrix} \equiv [M] \\ [X]^T[k][X] &= \begin{bmatrix} K_{11} & & & 0 \\ & K_{22} & & \\ & & \ddots & \\ 0 & & & K_{nn} \end{bmatrix} \equiv [K] \end{aligned}\right\} \qquad [17]$$

3. Orthogonality of Normal Modes

3.1 Decoupling Force Vibration Equations

Given normal modes of a system, the modal matrix $[X]$ can be used to decouple the equations of motion:

$$[m]\{\ddot{x}\} + [k]\{x\} = \{F\}$$

Make a coordinate transformation $\{x\} = [X]\{y\}$

Then:

$$[m][X]\{\ddot{y}\} + [k][X]\{y\} = \{F\} \qquad [18]$$

Multiplying through by $[X]^T$:

$$\underbrace{[X]^T[m][X]}_{\textit{Diagonal Matrix}}\{\ddot{y}\} + \underbrace{[X]^T[k][X]}_{\textit{Diagonal Matrix}}\{y\} = [X]^T\{F\} \qquad [19]$$

New equation in terms of $\{y\}$ are *uncoupled* and can be solved as a system of 1 DOF systems. Original coordinate $\{x\}$ can then be found from $\{x\} = [X]\{y\}$.

3. Orthogonality of Normal Modes

3.2 Example 5

Equation of motion of system:

$$m\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+k\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ F_2\end{Bmatrix}$$

Let: $x_j = X_j \sin \omega t$

$$\therefore \begin{bmatrix}-m\omega^2 & 0\\ 0 & -2m\omega^2\end{bmatrix}\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}+\begin{bmatrix}2k & -k\\ -k & 2k\end{bmatrix}\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\begin{Bmatrix}0\\ F_2\end{Bmatrix}$$

Dividing through by k:

$$\begin{bmatrix}2-\dfrac{m\omega^2}{k} & -1\\ -1 & 2-\dfrac{2m\omega^2}{k}\end{bmatrix}\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\begin{Bmatrix}0\\ F_2/k\end{Bmatrix}$$

Let: $\lambda = m\omega^2 / k$

then the characteristic equation is $\begin{vmatrix}(2-\lambda) & -1\\ -1 & (2-2\lambda)\end{vmatrix}=0 \rightarrow \lambda^2-3\lambda+\dfrac{3}{2}=0$

$$\lambda=\frac{3}{2}\pm\sqrt{\frac{9}{4}-\frac{3}{2}}=1.5\pm 0.866=\begin{cases}0.634\\ 2.366\end{cases}$$

3. Orthogonality of Normal Modes

3.2 Example 5 (Continue)

$$\frac{X_1}{X_2} = \frac{2(1-\lambda)}{1} = \begin{cases} 0.732 \\ -2.732 \end{cases}$$

The 2 eigenvectors are: $\{X\}^{(1)} = \begin{Bmatrix} 0.732 \\ 1.000 \end{Bmatrix}$ and $\{X\}^{(2)} = \begin{Bmatrix} -2.732 \\ 1.000 \end{Bmatrix}$

So that: $[X] = \begin{bmatrix} 0.732 & -2.732 \\ 1.000 & 1.000 \end{bmatrix}$

From [18]:

$$[m][X]\{\ddot{y}\} + [k][X]\{y\} = \{F\}$$

$$\begin{bmatrix} 2.535 & 0 \\ 0 & 9.48 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 1.606 & 0 \\ 0 & 22.33 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_2 \end{Bmatrix}$$

Equations are uncoupled !!!

You can now solve the D.E.s individually for y_1 and y_2. To get back to x_1 and x_2 apply the modal matrix $[X]$:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = [X] \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix}$$