Chapter 14

Random Vibration

14.1

$$p(x) = \begin{cases} k\left(1 - \frac{x}{30}\right) & ; \quad 20 \le x \le 30 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Normalization:

$$\int_{-\infty}^{\infty} p(x)\, dx = k \int_{20}^{30} \left(1 - \frac{x}{30}\right) dx = k\left(x - \frac{x^2}{60}\right)_{20}^{30} = 1$$

$$\Rightarrow \quad k = 0.6$$

$$P(x \ge 28) = \int_{28}^{\infty} p(x)\, dx = k \int_{28}^{30} \left(1 - \frac{x}{30}\right) dx = k\left(x - \frac{x^2}{60}\right)_{28}^{30}$$

$$= k\left(\frac{4}{60}\right) = 0.6\left(\frac{4}{60}\right) = 0.04$$

14.2

$$p(t) = \begin{cases} \lambda e^{-\lambda t} & ; \quad t \ge 0 \\ 0 & ; \quad t < 0 \end{cases}$$

(i) $$P(t) = \int_{-\infty}^{t} p(t')\, dt' = \lambda \int_0^t e^{-\lambda t'}\, dt' = 1 - e^{-\lambda t}$$

(ii) $$\overline{T} = \int_{-\infty}^{\infty} t\, p(t)\, dt = \lambda \int_0^{\infty} t\, e^{-\lambda t}\, dt = \frac{\lambda}{(-\lambda)^2}\left[e^{-\lambda t}(-\lambda t - 1)\right]_0^{\infty}$$

$$= \frac{1}{\lambda}$$

(iii) $$\sigma_T^2 = \int_0^{\infty} \left(t - \frac{1}{\lambda}\right)^2 p(t)\, dt$$

$$= \lambda \int_0^{\infty} t^2 e^{-\lambda t}\, dt + \frac{1}{\lambda} \int_0^{\infty} e^{-\lambda t}\, dt - 2\int_0^{\infty} t\, e^{-\lambda t}\, dt$$

$$= \lambda \left[-\frac{t^2}{\lambda} e^{-\lambda t} - \frac{2t}{\lambda^2} e^{-\lambda t} - \frac{2}{\lambda^3} e^{-\lambda t}\right]_0^{\infty}$$

$$-\frac{1}{\lambda^2}\left(e^{-\lambda t}\right)_0^{\infty} + \frac{2}{\lambda^2}\left(\lambda t\, e^{-\lambda t} + e^{-\lambda t}\right)_0^{\infty} = \frac{1}{\lambda^2}$$

$$\sigma_T = \frac{1}{\lambda}$$

14.3

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x \cdot p(x)\, dx = \int_0^2 0.5\, x \; dx = 1.0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 \; p(x)\, dx = \int_0^2 0.5\, x^2 \, dx = 1.3333$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 \; p(x)\, dx = \int_0^2 (x-1)^2 (0.5) \; dx = 0.5 \int_0^2 (x^2 - 2x + 1)\, dx$$

$$= 0.3333$$

$$\therefore \sigma_x = 0.5773$$

14.4

$$x(t) = x_0 \sin \frac{\pi t}{2}$$

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_0 \sin \frac{\pi t}{2} \; dt = \lim_{T\to\infty} \frac{1}{T} \left(\frac{-x_0 \cos \frac{\pi t}{2}}{\frac{\pi}{2}} \right) \Bigg|_{-T/2}^{T/2}$$

$$= \lim_{T\to\infty} -\frac{x_0}{T} \left(\frac{2}{\pi}\right) \left[\cos \frac{\pi T}{4} - \cos\left(-\frac{\pi T}{4}\right) \right] = 0$$

$$\langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x_0^2 \; \sin^2 \frac{\pi t}{2} \; dt$$

$$= \lim_{T\to\infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left\{ \frac{1}{2} - \frac{1}{2} \cos \pi t \right\} dt = \lim_{T\to\infty} \frac{x_0^2}{2T} \left[T - \frac{2}{\pi} \sin \frac{\pi T}{2} \right]$$

$$= \lim_{T\to\infty} \left(\frac{x_0^2}{2} - \frac{x_0^2}{2} \left\{ \frac{\sin \frac{\pi T}{2}}{\frac{\pi T}{2}} \right\} \right) = \frac{x_0^2}{2}$$

14.5 $p(x,y) = \begin{cases} \frac{xy}{9} & ; \quad 0 \le x \le 2, \quad 0 \le y \le 3 \\ 0 & ; \quad \text{elsewhere} \end{cases}$

Normalization:

$$\int_{x=0}^{2}\int_{y=0}^{3} p(x,y)\,dx\,dy = \int_{x=0}^{2} \frac{x}{9}\,dx \int_{y=0}^{3} y\,dy = \int_{x=0}^{2} \frac{x}{9}\,dx \left(\frac{y^2}{2}\right)_0^3$$

$$= \frac{1}{2}\int_{x=0}^{2} x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^2 = 1 \quad (\text{satisfied})$$

(a) Marginal density functions

$$p_X(x) = \int_{y=0}^{3} p(x,y)\,dy = \frac{x}{9}\int_{y=0}^{3} y\,dy = \frac{x}{2}$$

$$p_Y(y) = \int_{x=0}^{2} p(x,y)\,dx = \frac{y}{9}\int_{x=0}^{2} x\,dx = \frac{2}{9}y$$

(b) $$\mu_X = \bar{X} = \int_0^2 p(x)\cdot x\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_0^2 = \frac{4}{3}$$

$$\mu_Y = \bar{Y} = \int_0^3 p(y)\cdot y\,dy = \frac{2}{9}\int_0^3 y^2\,dy = \frac{2}{9}\left(\frac{y^3}{3}\right)_0^3 = 2$$

$$\sigma_X^2 = E\left[(x-\mu_X)^2\right] = \int_0^2 (x-\mu_X)^2\,p(x)\,dx = \int_0^2 \left(x-\frac{4}{3}\right)^2 \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4} + \frac{16}{9}\frac{x^2}{2} - \frac{8}{3}\frac{x^3}{3}\right)_0^2 = \frac{2}{9}$$

$\sigma_X = 0.4714$

$$\sigma_Y^2 = E\left[(y-\mu_Y)^2\right] = \int_0^3 (y-\mu_Y)^2\,p(y)\,dy = \int_0^3 (y-2)^2\,\frac{2y}{9}\,dy$$

$$= \frac{2}{9}\left(\frac{y^4}{4} + 2y^2 - \frac{4}{3}y^3\right)_0^3 = \frac{1}{2}$$

$\sigma_Y = 0.7071$

(c) $$\sigma_{X,Y} = E\left[(x-\mu_X)(y-\mu_Y)\right] = \int_{x=0}^{2}\int_{y=0}^{3}\left(x-\frac{4}{3}\right)(y-2)\,\frac{xy}{9}\,dx\,dy$$

$$= \int_{x=0}^{2}\left(x-\frac{4}{3}\right)\frac{x}{9}\,dx\left(\frac{y^3}{3} - 2\cdot\frac{y^2}{2}\right)_0^3 = 0$$

$\rho_{X,Y} = 0$

14.6 $z = x + y$, $E[z^2] = E[(x+y)^2] = E[x^2 + y^2 + 2xy]$

$$= E[x^2] + E[y^2] + 2\,E[xy]$$

Since x and y are independent, $E[xy] = E[x] \cdot E[y]$

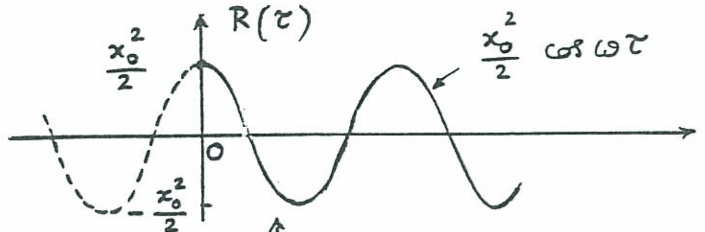
$$E[z^2] = E[x^2] + E[y^2] + 2\,E[x] \cdot E[y]$$

14.7 (a) $x(t) = x_0 \sin \omega t$, $x(t+\tau) = x_0 \sin \omega(t+\tau)$

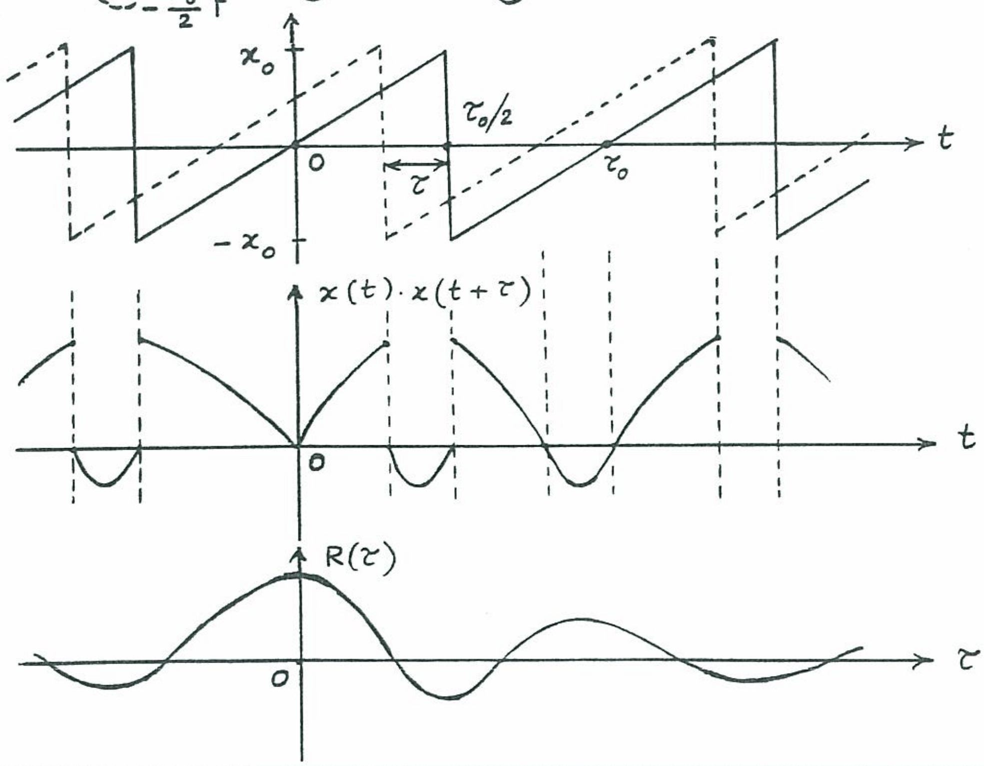
$$x(t)\,x(t+\tau) = x_0^2 \left(\sin^2 \omega t \cos \omega\tau + \sin \omega t \cdot \cos \omega t \cdot \sin \omega\tau \right)$$

$$R(\tau) = \lim_{T\to\infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left[\left(\frac{1 - \cos 2\omega t}{2} \right) \cos \omega\tau + \frac{\sin 2\omega t}{2} \cdot \sin \omega\tau \right] dt$$

$$= \lim_{T\to\infty} \frac{x_0^2}{T} \left[\frac{T}{2} \cos \omega\tau + (0) \sin \omega\tau \right] = \frac{x_0^2}{2} \cdot \cos \omega\tau$$

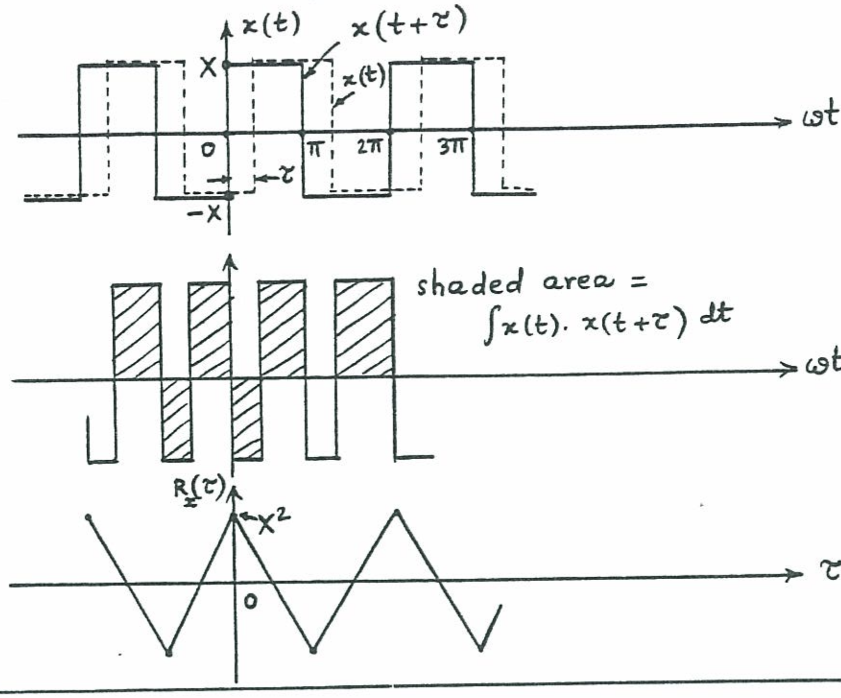


(b)



14.8 For $x(t) = X \sin \omega t$, $R_x(\tau) = \frac{X^2}{2} \cos \omega\tau$ [from Problem 14.7]

For square wave:



14.9 $R_x(\tau) = 20 + \dfrac{5}{1 + 3\tau^2}$

$E[x^2] = R(0) = 20 + 5 = 25$

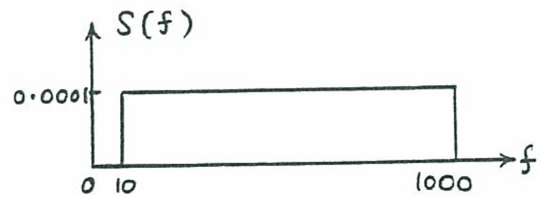
14.10 $S(f) = 0.0001$ m²/cycle per second for $10 \text{ Hz} \le f \le 1000 \text{ Hz}$

$\overline{x^2} = (0.0001)(1000 - 10) = 0.099 \text{ m}^2$

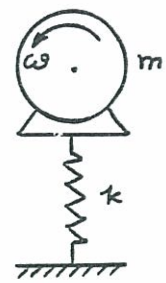
RMS value $= \sqrt{\overline{x^2}} = 0.3146$ m

$\sigma = \sqrt{\overline{x^2} - (\bar{x})^2} = \sqrt{0.099 - 0.0025}$

$= 0.3106$ m



14.11 $\omega = 2\pi(1800)/60 = 188.496$ rad/sec



$k = (\bar{k}, \sigma_k) = (2.25 \times 10^6,\ 0.225 \times 10^6)$ N/m

(normally distributed)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{100}} = \frac{\sqrt{k}}{10} \text{ rad/sec}$$

$$P(\omega_n \geq \omega) = P[\omega_n^2 \geq \omega^2]$$

$$= P[k \geq 100(188.496)^2] = P[k \geq 3553074.202] \quad (E_1)$$

Defining standard normal variate z as $z = \left(\frac{k - \bar{k}}{\sigma_k}\right)$,

Eq. (E_1) can be rewritten as

$$P[\omega_n > \omega] = P\left[\frac{k - \bar{k}}{\sigma_k} \geq \frac{3.5531 \times 10^6 - 2.25 \times 10^6}{0.225 \times 10^6}\right]$$

$$= P[z \geq 5.7916] = 0.3316 \times 10^{-8}$$

from standard normal distribution tables

[see, for example, Ref. 14.5]

14.12 $x(t) = \begin{cases} (2x_0 t/\tau) & ; \quad 0 \le t \le \frac{\tau}{2} \\ (2x_0 t/\tau) - 2x_0 & ; \quad \frac{\tau}{2} \le t \le \tau \end{cases}$

$$c_n = \frac{1}{\tau}\int_0^\tau x(t)\, e^{-in\omega t}\, dt = \frac{1}{\tau}\left[\int_0^{\tau/2} \frac{2x_0 t}{\tau} e^{-in\omega t}\, dt\right.$$

$$\left. + \int_{\tau/2}^{\tau}\left(\frac{2x_0 t}{\tau} - 2x_0\right) e^{-in\omega t}\, dt\right]$$

$$= \frac{2x_0}{\tau^2}\int_0^{\tau/2} t\, e^{-in\omega t}\, dt + \frac{2x_0}{\tau^2}\int_{\tau/2}^{\tau} t\, e^{-in\omega t}\, dt - \frac{2x_0}{\tau}\int_{\tau/2} e^{-in\omega t}\, dt$$

$$= \frac{2x_0}{\tau^2}\left(\frac{1}{n^2\omega^2}\right)\left[e^{-in\omega t}(in\omega t) + e^{-in\omega t}\right]_0^{\tau/2}$$

$$+ \frac{2x_0}{\tau^2}\left(\frac{1}{n^2\omega^2}\right)\left[e^{-in\omega t}(in\omega t) + e^{-in\omega t}\right]_{\tau/2}^{\tau}$$

$$- \frac{2x_0}{\tau}\left(\frac{1}{-in\omega}\right)\left(e^{-in\omega t}\right)_{\tau/2}^{\tau}$$

$$= \frac{2x_0}{n^2\pi^2}\left[in\pi\, e^{-in\pi} + e^{-in\pi} - 1\right] - \frac{2x_0 i}{n\pi}\left[e^{-in\pi} - e^{-in\frac{\pi}{2}}\right]$$

$$= \frac{2x_0}{n^2\pi^2}\cos n\pi + i\,\frac{2x_0}{n\pi}\cos\frac{n\pi}{2} + \frac{2x_0}{n\pi}\sin\frac{n\pi}{2} - \frac{2x_0}{n^2\pi^2}$$

$$= \frac{2x_0}{n^2\pi^2}\left[(-1)^n - 1\right] + i\,\underbrace{\frac{2x_0}{n\pi}(-1)^{\frac{n}{2}}}_{n = \text{even}} + \underbrace{\frac{2x_0}{n\pi}(-1)^{\frac{n-1}{2}}}_{n = \text{odd}}$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{2x_0}{n^2\pi^2}\left[(-1)^n - 1\right] e^{in\omega t}$$

$$+ i\,\frac{2x_0}{\pi}\sum_{\substack{n=-\infty \\ (n = \text{even})}}^{\infty} (-1)^{n/2}\,\frac{1}{n}\, e^{in\omega t}$$

$$+ \frac{2x_0}{\pi}\sum_{\substack{n=-\infty \\ (n = \text{odd})}}^{\infty} (-1)^{\frac{n-1}{2}}\,\frac{1}{n}\, e^{in\omega t}$$

14.13 $x(t) = \begin{cases} A & ; \quad -a \le t \le a \\ 0 & ; \quad \text{elsewhere} \end{cases}$

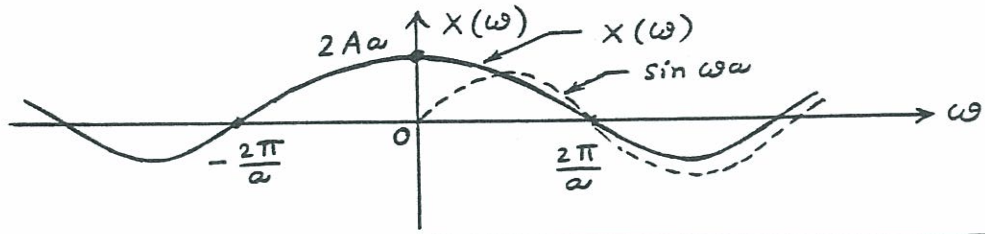
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt = A \int_{-a}^{a} e^{-i\omega t}\, dt = \frac{A}{-i\omega}\left(e^{-i\omega t}\right)_{-a}^{a}$$

$$= \frac{Ai}{\omega}\left(e^{-i\omega a} - e^{i\omega a}\right)$$

$$= \frac{Ai}{\omega}\left(\cos\omega a - i\sin\omega a - \cos\omega a - i\sin\omega a\right)$$

$$= \frac{2Aa}{\omega a}\sin\omega a \qquad (E_1)$$

Eq. (E_1) shows that $X(\omega) = 2Aa$ as $\omega \to 0$.



(14.14) $x(t) = \begin{cases} A\,e^{-at} & ;\ t \geq 0 \\ 0 & ;\ \text{elsewhere} \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-i\omega t}\,dt = A\int_0^{\infty} e^{-(a+i\omega)t}\,dt$$

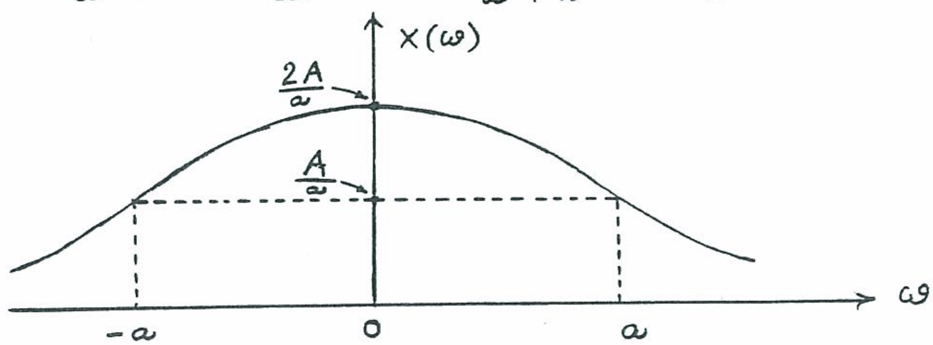
$$= \frac{-A}{(a+i\omega)}\left[e^{-(a+i\omega)t}\right]_0^{\infty} = \frac{A}{a+i\omega} = \frac{Aa}{a^2+\omega^2} - i\,\frac{A\omega}{a^2+\omega^2}$$

(14.15) $x(t) = A\,e^{-a|t|}$

$$X(\omega) = A\int_{-\infty}^{0} e^{at}\,e^{-i\omega t}\,dt + A\int_0^{\infty} e^{-at}\,e^{-i\omega t}\,dt$$

$$= \frac{A}{-(i\omega - a)}\left[e^{-(i\omega - a)t}\right]_{-\infty}^{0} - \frac{A}{(a+i\omega)}\left[e^{-(a+i\omega)}\right]_0^{\infty}$$

$$= \frac{A}{a-i\omega} + \frac{A}{a+i\omega} = \frac{A(a+i\omega)}{a^2+\omega^2} + \frac{A(a-i\omega)}{a^2+\omega^2} = \frac{2Aa}{a^2+\omega^2}$$



14.16 $x(t) = \delta(t-a)$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i\omega t}\, dt = e^{-i\omega a} = \cos \omega a - i \sin \omega a$$

since, by definition, the Dirac delta function is zero every where except at $t=a$. At $t=a$, $e^{-i\omega t} = e^{-i\omega a}$.

14.17

$$\int_{t=-\tau/2}^{\tau/2} \cos(n-m)\omega_0 t \; dt = \frac{1}{(n-m)\omega_0}\Big[\sin(n-m)\omega_0 t\Big]_{t=-\frac{\tau}{2}=-\frac{\pi}{\omega_0}}^{\frac{\tau}{2}=\frac{\pi}{\omega_0}}$$

$$= \frac{1}{(n-m)\omega_0}\Big[\sin(n-m)\pi + \sin(n-m)\pi\Big] = 0 \quad \text{for all } m \neq n$$

since $\tau = \dfrac{2\pi}{\omega_0}$.

$$\int_{-\tau/2}^{\tau/2} \sin(n-m)\omega_0 t \cdot dt = \frac{1}{-(n-m)\omega_0}\Big[\cos(n-m)\omega_0 t\Big]_{t=-\pi/\omega_0}^{\pi/\omega_0}$$

$$= \frac{-1}{(n-m)\omega_0}\Big[\cos(n-m)\pi - \cos(n-m)\pi\Big] = 0 \quad \text{for all } m \neq n$$

$$\int_{-\tau/2}^{\tau/2} \cos 0 \cdot dt = \Big(t\Big)_{-\tau/2}^{\tau/2} = \tau \quad \text{for } m=n$$

$$\int_{-\tau/2}^{\tau/2} \sin 0 \cdot dt = 0 \quad \text{for } m=n$$

$\therefore$ Eq. (14.45) becomes

$$\int_{-\tau/2}^{\tau/2} x(t)\, e^{-in\omega_0 t}\, dt = c_n \tau \Rightarrow c_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cdot e^{-in\omega_0 t} \cdot dt \qquad \text{----(14.46)}$$

14.18 $R_x(\tau) = A \cos \omega\tau$

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

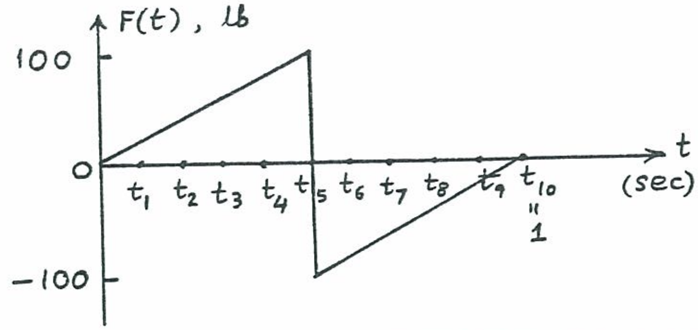
$$= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \cos \omega\tau \left(\cos \omega\tau - i \sin \omega\tau\right) d\tau$$

$$= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \left(\cos^2 \omega\tau - i\, \frac{1}{2} \sin 2\omega\tau\right) d\tau$$

$$= \frac{A}{2\pi} \left\{ \left[\frac{\tau}{2} + \frac{1}{4\omega} \sin 2\omega\tau \right]_{-\pi/2\omega}^{\pi/2\omega} - \frac{i}{2} \left(- \frac{\cos 2\omega\tau}{2\omega} \right)_{-\pi/2\omega}^{\pi/2\omega} \right\}$$

$$= \frac{A}{4\omega}$$

14.19



j	0	1	2	3	4	5	6	7	8	9	10
t_j (sec)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$F_j = F(t_j)$, lb	0	20	40	60	80	100	−80	−60	−40	−20	0

(i) Spectrum of $F(t)$:

From Eq. (14.46),

$$c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} F(t)\, e^{-i n \omega_0 t}\, dt \qquad (E_1)$$

where τ = time period = $\frac{2\pi}{\omega_0}$ and ω_0 = fundamental frequency, $\tau = 1.0$ sec, $\omega_0 = 2\pi$ rad/sec.

Since the integration in Eq. (E_1) is over one-time period, c_n can be expressed as

$$c_n = \frac{1}{\Delta t\, N} \sum_{j=1}^{N} F(t_j) \cdot e^{-in\left(\frac{2\pi}{\tau}\right) t_j} \cdot \Delta t \qquad \text{where } \frac{t_j}{\tau} = \frac{j}{N}$$

$$\therefore\ c_n = \frac{1}{N} \sum_{j=1}^{N} F_j\, e^{-i\left(\frac{2\pi n j}{N}\right)}$$

$$= \frac{1}{N} \sum_{j=1}^{N} F_j \left\{ \cos \frac{2\pi n j}{N} - i \sin \frac{2\pi n j}{N} \right\}$$

$$= \text{real}\,(c_n) + \text{imag}\,(c_n) \qquad (E_2)$$

Eq. (E_2) gives the following results for $N = 10$:

n	Real (c_n)	Imag (c_n)	Spectrum of $F(t) = \lvert c_n \rvert^2$ $= \text{Real}(c_n)^2 + \text{Imag}(c_n)^2$
0	10	0	100
1	−10	−30.7768	1047.21
2	10	13.7636	289.44
3	−10	−7.2652	152.79
4	10	3.2489	111.56
5	−10	0.0003	100.00
6	10	−3.2497	111.56
7	−10	7.2658	152.79
8	10	−13.7648	289.45
9	−10	30.7775	1047.21

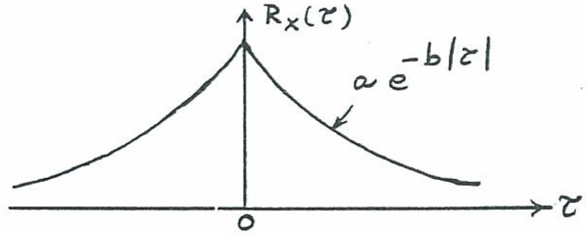
(ii) Mean square value of $F(t)$:

Using Eq. (14.49),

$$\overline{F^2(t)} = \sum_{n=0}^{N-1} |c_n|^2 = \sum_{n=0}^{9} |c_n|^2 = 3400.00$$

14.20

$$R_X(\tau) = a\, e^{-b|\tau|}$$



$$S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

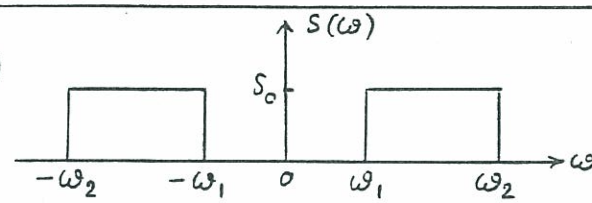
$$= \frac{1}{2\pi}\int_{-\infty}^{0} a\, e^{b\tau}\, e^{-i\omega\tau}\, d\tau + \frac{1}{2\pi}\int_{0}^{\infty} a\, e^{-b\tau}\, e^{-i\omega\tau}\, d\tau$$

$$= \left(\frac{a}{2\pi}\right)\frac{1}{-(i\omega-b)}\left[e^{-(i\omega-b)\tau}\right]_{-\infty}^{0} + \left(\frac{a}{2\pi}\right)\frac{1}{-(i\omega+b)}\left[e^{-(i\omega+b)}\right]_{0}^{\infty}$$

$$= \frac{-a}{2\pi(i\omega-b)} + \frac{a}{2\pi(i\omega+b)} = \frac{a\,b}{\pi(b^2+\omega^2)}$$

14.21

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega \quad \text{--- } (E_1)$$



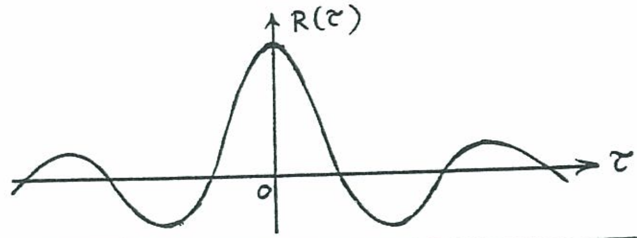
Since $S(\omega) = S_o$ is symmetric and real, we can neglect the imaginary component in (E_1) and write $R(\tau)$ as

$$R(\tau) = 2\, S_o \int_{\omega_1}^{\omega_2} \cos\omega\tau \cdot d\tau = \frac{2\, S_o}{\tau}\left(\sin\omega_2\tau - \sin\omega_1\tau\right) \quad \text{--- } (E_2)$$

$R(\tau)$ as $\tau \to 0$ is given by

$$\lim_{\tau \to 0} \left[2 S_0 \left(\frac{\omega_2 \sin \omega_2 \tau}{\omega_2 \tau} \right) - 2 S_0 \left(\frac{\omega_1 \sin \omega_1 \tau}{\omega_1 \tau} \right) \right] = 2 S_0 (\omega_2 - \omega_1) \quad \text{---}(E_3)$$

The variation of $R(\tau)$ is shown in the figure.



14.22 From Eq. (14.60),

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi} \sigma_x^2 \int_{-\infty}^{\infty} e^{-\alpha|v\tau|} \cos\beta v\tau\, e^{-i\omega\tau}\, d\tau$$

But $\cos \beta v \tau = \frac{1}{2} \left(e^{i\beta v\tau} + e^{-i\beta v\tau} \right)$

$$S_x(\omega) = \frac{\sigma_x^2}{4\pi} \int_{-\infty}^{\infty} \left\{ e^{-\alpha|v\tau|}\, e^{-i\omega\tau} \left(e^{i\beta v\tau} + e^{-i\beta v\tau} \right) d\tau \right\}$$

$$= \frac{\sigma_x^2}{4\pi} \left[\int_0^{\infty} e^{(-\alpha v - i\omega + i\beta v)\tau}\, d\tau + \int_0^{\infty} e^{(-\alpha v - i\omega - i\beta v)\tau}\, d\tau \right.$$

$$\left. + \int_{-\infty}^{0} e^{(\alpha v - i\omega + i\beta v)\tau}\, d\tau + \int_{-\infty}^{0} e^{(\alpha v - i\omega - i\beta v)\tau}\, d\tau \right]$$

$$= \frac{\sigma_x^2}{2\pi} \left[\frac{\alpha v}{\alpha^2 v^2 + (\omega - \beta v)^2} + \frac{\alpha v}{\alpha^2 v^2 + (\omega + \beta v)^2} \right]$$

$$= \frac{\alpha v\, \sigma_x^2}{\pi} \left[\frac{\omega^2 + \beta^2 v^2 + \alpha^2 v^2}{\{\alpha^2 v^2 + (\omega - \beta v)^2\}\{\alpha^2 v^2 + (\omega + \beta v)^2\}} \right]$$

For asphalt surface: $\sigma_x = 1.1$, $\alpha = 0.2$, $\beta = 0.4$

$$S_x(\omega) = \frac{(0.2)\, v\, (1.21)}{\pi} \left[\frac{\omega^2 + 0.16\, v^2 + 0.04\, v^2}{\{0.04\, v^2 + (\omega - 0.4 v)^2\}\{0.04\, v^2 + (\omega + 0.4 v)^2\}} \right]$$

$$= 0.07703\, v \left[\frac{\omega^2 + 0.2\, v^2}{\{0.04\, v^2 + (\omega - 0.4 v)^2\}\{0.04\, v^2 + (\omega + 0.4 v)^2\}} \right]$$

For paved surface: $\sigma_x = 1.6$, $\alpha = 0.3$, $\beta = 0.6$

$$S_x(\omega) = \frac{(0.3)\, v\, (2.56)}{\pi} \left[\frac{\omega^2 + 0.36\, v^2 + 0.09\, v^2}{\{0.09\, v^2 + (\omega - 0.6 v)^2\}\{0.09\, v^2 + (\omega + 0.6 v)^2\}} \right]$$

$$= 0.24446\, v \left[\frac{\omega^2 + 0.45\, v^2}{\{0.09\, v^2 + (\omega - 0.6 v)^2\}\{0.09\, v^2 + (\omega + 0.6 v)^2\}} \right]$$

For gravel surface: $\sigma_x = 1.8, \quad \alpha = 0.5, \quad \beta = 0.9$

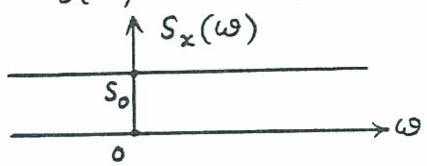
$$S_x(\omega) = \frac{(0.5)\, v\, (3.24)}{\pi} \left[\frac{\omega^2 + 0.81\, v^2 + 0.25\, v^2}{\{0.25\, v^2 + (\omega - 0.9 v)^2\}\{0.25\, v^2 + (\omega + 0.9 v)^2\}} \right]$$

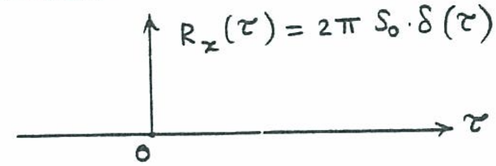
$$= 0.51566\, v \left[\frac{\omega^2 + 1.06\, v^2}{\{0.25\, v^2 + (\omega - 0.9 v)^2\}\{0.25\, v^2 + (\omega + 0.9 v)^2\}} \right]$$

14.23 From Eq. (14.61), $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega = S_0 \int_{-\infty}^{\infty} e^{i\omega\tau}\, d\omega$

$$= 2\pi\, S_0 \cdot \delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta function.





14.24 $S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau \quad$ or $\quad S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i 2\pi f \tau}\, d\tau \quad$ --- (E_1)

and

$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega \quad$ or $\quad R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{i 2\pi f \tau}\, df \quad$ --- (E_2)

where $S_x(f)$ is the two-sided power spectral density function for $-\infty < f < \infty$. Noting that

$S_x(f) = S_x(-f)$ and $R_x(\tau) = R_x(-\tau)$, (E_1) and (E_2) can be written as

$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \{\cos 2\pi f \tau - i \sin 2\pi f \tau\}\, d\tau \quad$ for $-\infty < f < \infty$ --- (E_3)

and

$R_x(\tau) = 2 \int_0^{\infty} S_x(f) \cdot \{\cos 2\pi f \tau + i \sin 2\pi f \tau\}\, d\tau \quad$ for $-\infty < f < \infty$ --- (E_4)

For a real random function $x(t)$, $S_x(f)$ and $R_x(\tau)$ are real-values functions and hence, by neglecting the imaginary parts in (E_3) and (E_4), we obtain

$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \cos 2\pi f \tau \cdot d\tau \quad$ for $-\infty < f < \infty$ ---(E_5)

and

$$R_x(\tau) = 2 \int_0^\infty S_x(f) \cdot \cos 2\pi f \tau \cdot df \quad \text{for } -\infty < f < \infty \quad --- (E_6)$$

If $S(f)$ denotes the one-sided power spectral density function, $S(f) = 2\ S_x(f)$ and $R(\tau) = R_x(\tau)$, Eqs. (E_5) and (E_6) become

$$S(f) = 4 \int_0^\infty R(\tau) \cos 2\pi f \tau \cdot d\tau \quad \text{for } 0 \le f < \infty \quad --- (E_7)$$

and

$$R(\tau) = \int_0^\infty S(f) \cos 2\pi f \tau \cdot df \quad \text{for } 0 \le f < \infty \quad --- (E_8)$$

14.25 Mean square value of a single d.o.f. system is given by Eq.(14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)\, d\omega \equiv \int_{-\infty}^{\infty} A(\omega)\, d\omega \qquad (E_1)$$

where

$$|H(\omega)| = \frac{1}{(k - m\omega^2)^2 + c^2 \omega^2} \qquad (E_2)$$

Let $-\omega^{(1)}$ and $+\omega^{(2)}$ denote finite lower and upper bounds to be used for integration in Eq. (E_1) [$\omega^{(1)}$ and $\omega^{(2)}$ should be sufficiently large].

We can use trapezoidal rule of integration for simplicity. In this method, we divide the interval $(\omega^{(2)} + \omega^{(1)})$ into $(n-1)$ equal divisions so that

$$\left.\begin{aligned}
\omega_1 &= -\omega^{(1)} \\
\omega_2 &= -\omega_1 + \Delta\omega \\
\omega_3 &= -\omega_1 + 2\,\Delta\omega \\
&\vdots \\
\omega_n &= -\omega_1 + (n-1)\Delta\omega = \omega^{(2)}
\end{aligned}\right\} \qquad (E_3)$$

Then

$$E[y^2] = \int_{-\omega^{(1)}}^{\omega^{(2)}} A(\omega)\, d\omega = \sum_{p=1}^{n-1} \left[A(\omega_p) + A(\omega_{p+1})\right] \frac{\Delta\omega}{2}$$

$$= \left[A(-\omega^{(1)}) + A(\omega^{(2)})\right] \frac{\Delta\omega}{2} + \sum_{p=2}^{n-1} A(\omega_p) \cdot \Delta\omega \ldots (E_4)$$

The computer program listing is given below. In this program, the following notation is used:

OM1 $= \omega^{(1)}$, OM2 $= \omega^{(2)}$, OMP $= \omega_p$, SX $= S_x(\omega)$,

HOM $= |H(\omega)|^2$, SUM $= E[y^2]$.

```
      REAL   M, K, C
      DATA   M, K, C/.../
      DATA   OM1, OM2/.../
      N = ...
      N1 = N - 1
      DOM = (OM2 + OM1)/REAL (N1)
      OM11 = -OM1
      CALL PSD (OM11, SX1, HOM1)
      CALL PSD (OM2,SX2, HOM2)
      SUM = [(SX1/HOM1) + (SX2/HOM2)] * DOM/2.0
      DO 11  I = 2, N1
      OMP = -OM1 + REAL (I-1) * DOM
      CALL PSD (OMP, SXP, HOMP)
   11 SUM = SUM + (SXP/HOMP) * DOM
      PRINT 12, SUM
   12 FORMAT (...)
      STOP
      END
C     SUBROUTINE TO EVALUATE POWER SPECTRAL DENSITY AND HOM
      SUBROUTINE PSD (OM, SX, HOM)
      SX = ...
      HOM = ...
      RETURN
      END
```

14.26 $m = 2000/386.4 = 5.1760$ lb-sec^2/in

$k = 4 \times 10^4$ lb/in, $\quad c = 1200$ lb-in/sec

Mean square response of the machine is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 \, S_x(\omega) \, dx = \sum_{n=0}^{N-1} \left| \frac{1}{-m\omega_n^2 + i c\omega_n + k} \right|^2 \cdot |c_n|^2 \qquad \text{----}(E_1)$$

where $\omega_n = n\,\omega_0 = n\left(\frac{2\pi}{\tau}\right) = 2\pi n$ rad/sec

and $|c_n|^2$ are given in the solution of problem 14.17.

Eq. (E_1) can be rewritten as

$$E[y^2] = \sum_{n=0}^{N-1} \frac{|c_n|^2}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \qquad \text{---}(E_2)$$

Computations:

n	$\omega_n = 2\pi n$ rad/sec	$(k-m\omega_n^2)^2 + c^2\omega_n^2$	$\lvert c_n\rvert^2$ lb^2	$\left\{\frac{\lvert c_n\rvert^2}{(k-m\omega_n^2)^2 + c^2\omega_n^2}\right\}$ in^2
0	0	16×10^8	100.00	6.25×10^{-8}
1	2π	16.4054×10^8	1047.21	63.8331×10^{-8}

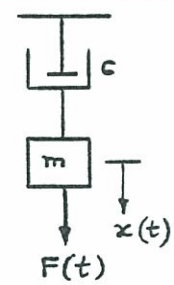
2	4π	17.6268×10^{8}	289.44	16.4205×10^{-8}
3	6π	19.6790×10^{8}	152.79	7.7641×10^{-8}
4	8π	22.5872×10^{8}	111.56	4.9391×10^{-8}
5	10π	26.3865×10^{8}	100.00	3.7898×10^{-8}
6	12π	31.1218×10^{8}	111.56	3.5846×10^{-8}
7	14π	36.8485×10^{8}	152.79	4.1464×10^{-8}
8	16π	43.6315×10^{8}	289.45	6.6340×10^{-8}
9	18π	51.5461×10^{8}	1047.21	20.3160×10^{-8}

Thus Eq. (E_2) gives the mean square value of the response as

$$E[y^2] = 137.6777 \times 10^{-8} \text{ in}^2$$

14.27 Equation of motion is

$$m\ddot{x} + c\dot{x} = F(t) \qquad (E_1)$$



Let $y = \dot{x}$ = velocity. Then (E_1) becomes

$$m\dot{y} + cy = F(t) \qquad (E_2)$$

Let the excitation be

$$F(t) = F_0 e^{i\omega t}$$

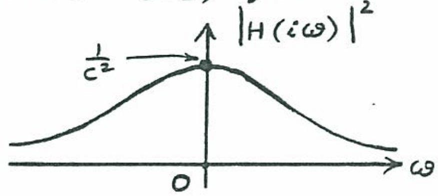
and the velocity response be

$$y(t) = Y e^{i\omega t}$$

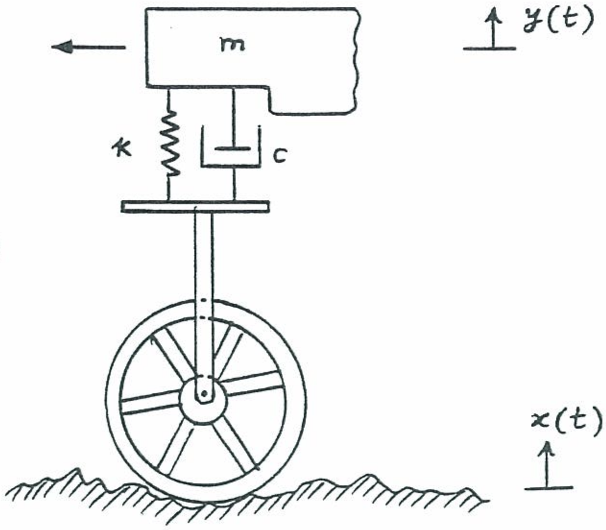
Hence (E_2) yields $\quad m i\omega Y e^{i\omega t} + cY e^{i\omega t} = F_0 e^{i\omega t}$

or $\quad \dfrac{Y}{Y_0} = \dfrac{1}{i\omega m + c} \equiv H(i\omega)$

$$|H(i\omega)|^2 = \frac{1}{m^2\omega^2 + c^2}$$



14.28 This system can be modeled as a single d.o.f. system with random base excitation. The equations of motion are given by:



$$m\ddot{z} + c\dot{z} + k\,z = -m\ddot{x} \qquad (E_1)$$

where $z = y - x$.

The frequency response function of the system can be derived as follows:

Let
$$\left.\begin{aligned} x(t) &= e^{i\omega t} \\ z(t) &= H(\omega)\, e^{i\omega t} \end{aligned}\right\} \qquad (E_2)$$

Substitution of (E_2) into (E_1) gives

$$\left(-\omega^2 m + i c\omega + k\right) H(\omega)\, e^{i\omega t} = m\omega^2\, e^{i\omega t}$$

i.e., $$H(\omega) = \frac{m\omega^2}{-m\omega^2 + k + i\omega c} = \frac{\omega^2}{(\omega_n^2 - \omega^2) + i\,2\zeta\omega\omega_n}$$

i.e., $$\left|H(\omega)\right|^2 = \frac{\omega^4}{\left(\omega_n^2 - \omega^2\right)^2 + 4\zeta^2\omega^2\omega_n^2} \qquad (E_3)$$

The power spectral density of the response $z(t)$ is given by

$$S_z(\omega) = \left| H(\omega) \right|^2 S_x(\omega) \qquad (E_4)$$

In the present case, $S_x(\omega) = S_0$ and Eq. (E_4) reduces to

$$S_z(\omega) = S_0 \left| H(\omega) \right|^2 \qquad (E_5)$$

The mean square value of the relative displacement of the mass can be found as

$$E[z^2] = \int_{-\infty}^{\infty} S_z(\omega) \cdot d\omega = S_0 \int_{-\infty}^{\infty} \left| \frac{\omega^2}{-\omega^2 + \omega_n^2 + \frac{i\omega c}{m}} \right|^2 \cdot d\omega$$

$$= S_0\, \omega^4\, \pi \left\{ \frac{\left(\frac{1}{\omega_n^2}\right)}{\left(\frac{c}{m}\right)} \right\} = \frac{\pi\, S_0\, \omega^4}{2\,\zeta\,\omega_n^3} \qquad (E_6)$$

14.29 $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} F \qquad --- (E_1)$

This gives $\quad S_x(\omega) = \frac{1}{m^2} \dfrac{S_F(\omega)}{\left\{ (\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2 \right\}} \qquad --- (E_2)$

$$E[x^2] = \frac{1}{m^2} \int_{-\infty}^{\infty} \frac{S_F(\omega) \cdot d\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2} \qquad --- (E_3)$$

For small damping, (E_3) becomes

$$E[x^2] \simeq \frac{1}{m^2} S_F(\omega_n) \cdot \frac{\pi}{2\zeta\omega_n^3} \qquad --- (E_4)$$

Here $\quad S_F(\omega) = A^2 \cdot \dfrac{1 + \left(\frac{L\omega}{v}\right)^2}{\left\{1 + \left(\frac{L\omega}{v}\right)^2\right\}^2 \left(1 + \frac{\pi\omega c}{v}\right)} \qquad --- (E_5)$

$$\therefore\ E[x^2] = \frac{\pi A^2}{2\zeta\omega_n^3 m^2} \left\{ \frac{1 + \left(\frac{L\omega_n}{v}\right)^2}{\left(1 + \frac{L^2\omega_n^2}{v^2}\right)^2 \left(1 + \frac{\pi\omega_n c}{v}\right)} \right\} \qquad --- (E_6)$$

14.30 $\omega_1 =$ undamped natural frequency $= \sqrt{k_{eq}/m_{eq}}$

or $\quad k_{eq} = \omega_1^2\, m_{eq} \qquad (E_1)$

$\omega_2 =$ damped natural frequency $= \omega_n \sqrt{1-\zeta^2}$

$$= \omega_1 \sqrt{1-\zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{1-\zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{\left(1 - \frac{c_{eq}^2}{4\, k_{eq}\, m_{eq}}\right)}$$

$$\omega_2^2 = \frac{k_{eq}}{m_{eq}} \left(1 - \frac{c_{eq}^2}{4\, k_{eq}\, m_{eq}} \right) = \omega_1^2 - \frac{c_{eq}^2}{4\, m_{eq}^2}$$

or $$\left(\frac{c_{eq}}{2\, m_{eq}} \right)^2 = \omega_1^2 - \omega_2^2$$

or $$c_{eq} = 2\, m_{eq} \sqrt{\omega_1^2 - \omega_2^2} \qquad (E_2)$$

Mean square value of the displacement of the wing (m_{eq}) $= \delta$.

$$E[y^2] = \frac{\pi S_0}{k_{eq}\, c_{eq}} = \delta \quad \text{or} \quad c_{eq} = \frac{\pi S_0}{\delta\, k_{eq}} \qquad (E_3)$$

Eqs. (E_1) and (E_3) give

$$c_{eq} = \frac{\pi S_0}{\delta\, \omega_1^2\, m_{eq}} \qquad (E_4)$$

Equating (E_2) and (E_4), $c_{eq} = 2\, m_{eq} \sqrt{\omega_1^2 - \omega_2^2} = \dfrac{\pi S_0}{\delta\, \omega_1^2\, m_{eq}}$

$$\therefore \quad m_{eq} = \left[\frac{\pi S_0}{2\, \delta\, \omega_1^2 \left(\omega_1^2 - \omega_2^2 \right)^{\frac{1}{2}}} \right]^{1/2} \qquad (E_5)$$

Eqs. (E_1) and (E_2), in view of (E_5), yield

$$k_{eq} = \left[\frac{\pi S_0\, \omega_1^2}{2\, \delta \left(\omega_1^2 - \omega_2^2 \right)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \qquad (E_6)$$

and

$$c_{eq} = 2 \left(\omega_1^2 - \omega_2^2 \right)^{\frac{1}{2}} \left[\frac{\pi S_0}{2\, \delta\, \omega_1^2 \left(\omega_1^2 - \omega_2^2 \right)^{1/2}} \right]^{\frac{1}{2}}$$

$$= \left[\frac{2\pi S_0 \left(\omega_1^2 - \omega_2^2 \right)^{1/2}}{\delta\, \omega_1^2} \right]^{\frac{1}{2}} \qquad (E_7)$$

14.31 In case of structural damping, the uncoupled equations of motion are given by

$$\ddot{q}_i(t) + (1 + i\, \beta)\, \omega_i^2\, q_i(t) = Q_i(t) \ ; \ i = 1, 2, \ldots, n \qquad (1)$$

where β denotes the structural damping coefficient. The mean square values of $x_i(t)$ are given by Eq. (14.113):

$$\overline{x_i^2(t)} = \sum_{r=1}^{n} \left(X_i^{(r)} \right)^2 \frac{N_r^2}{\omega_r^4} \int_{-\infty}^{\infty} | H_r(\omega) |^2\, S_r(\omega)\, d\omega \qquad (2)$$

where, from Eq. (3.106),

$$|H_r(\omega)| = \frac{1}{\left\{1 - \frac{\omega^2}{\omega_n^2}\right\}^2 + \beta^2} \tag{3}$$

For $\beta \ll 1$, Eq. (2) can be approximated as

$$\overline{x_i^2(t)} \approx \sum_{r=1}^{n} \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \int_{-\infty}^{\infty} |H_r(\omega)|^2 \, d\omega$$

$$\approx \sum_{r=1}^{n} \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \frac{\pi\,\omega_r}{\beta} \tag{4}$$

Using the computational details of Example 14.7, we can obtain (using g = 0.01 in place of $2\,\varsigma_r = 0.04$):

$$\overline{z_1^2(t)} = 0.0021253 \text{ m}^2 \tag{5}$$

$$\overline{z_2^2(t)} = 0.0055983 \text{ m}^2 \tag{6}$$

$$\overline{z_3^2(t)} = 0.0086582 \text{ m}^2 \tag{7}$$

14.32

$$S(\omega) = \frac{1}{4 + \omega^2} \frac{\text{m}^2/\text{s}^4}{\text{rad/sec}} \quad ; \quad \varsigma_i = 0.02$$

Since the natural frequencies (rad/sec) are given by $\omega_1 = 14.0734$, $\omega_2 = 39.4368$, and $\omega_3 = 57.0001$, we can approximate $S(\omega_r)$ for use in Eq. (14.115) as

$$S_r(\omega_1) = \frac{1}{4 + 14.0734^2} = 0.0049490 \tag{1}$$

$$S_r(\omega_2) = \frac{1}{4 + 39.4368^2} = 0.0006413 \tag{2}$$

$$S_r(\omega_3) = \frac{1}{4 + 57.0001^2} = 0.0003074 \tag{3}$$

and hence the mean square values of the relative displacements of the various floors can be computed as (using Eq. (E.18) of Example 14.7):

$$\overline{z_1^2(t)} = \frac{\pi}{2\,\varsigma_r} \left[\left(Z_1^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_1^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_1^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$

$$= \frac{\pi}{0.04} \left[0.0001058\,(0.004949) + 0.0000243\,(0.0006413) + 0.0000052\,(0.0003074) \right]$$

$$= 42.4744\,(10^{-6}) \text{ m}^2 \tag{4}$$

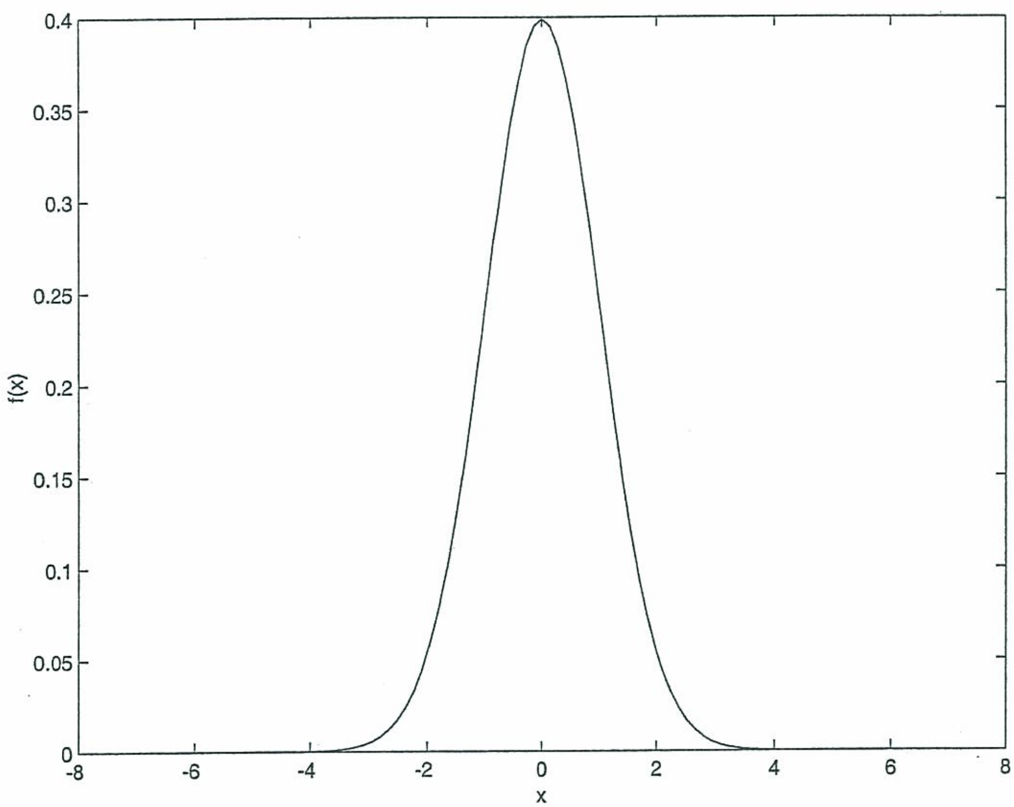
$$\overline{z_2^2(t)} = \frac{\pi}{2\,\varsigma_r} \left[\left(Z_2^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_2^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_2^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$

$$= \frac{\pi}{0.04}\ [0.0003436\ (0.004949) + 0.0000048\ (0.0006413) + 0.0000080\ (0.0003074)]$$
$$= 133.9971\ (10^{-6})\ \mathrm{m}^2 \qquad (5)$$

$$\overline{z_3^2(t)} = \frac{\pi}{2\ \zeta_r}\left[\left(Z_3^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3}\ S(\omega_1) + \left(Z_3^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3}\ S(\omega_2) + \left(Z_3^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3}\ S(\omega_3)\right]$$
$$= \frac{\pi}{0.04}\ [0.0005340\ (0.004949) + 0.0000156\ (0.0006413) + 0.0000016\ (0.0003074)]$$
$$= 208.3902\ (10^{-6})\ \mathrm{m}^2 \qquad (6)$$

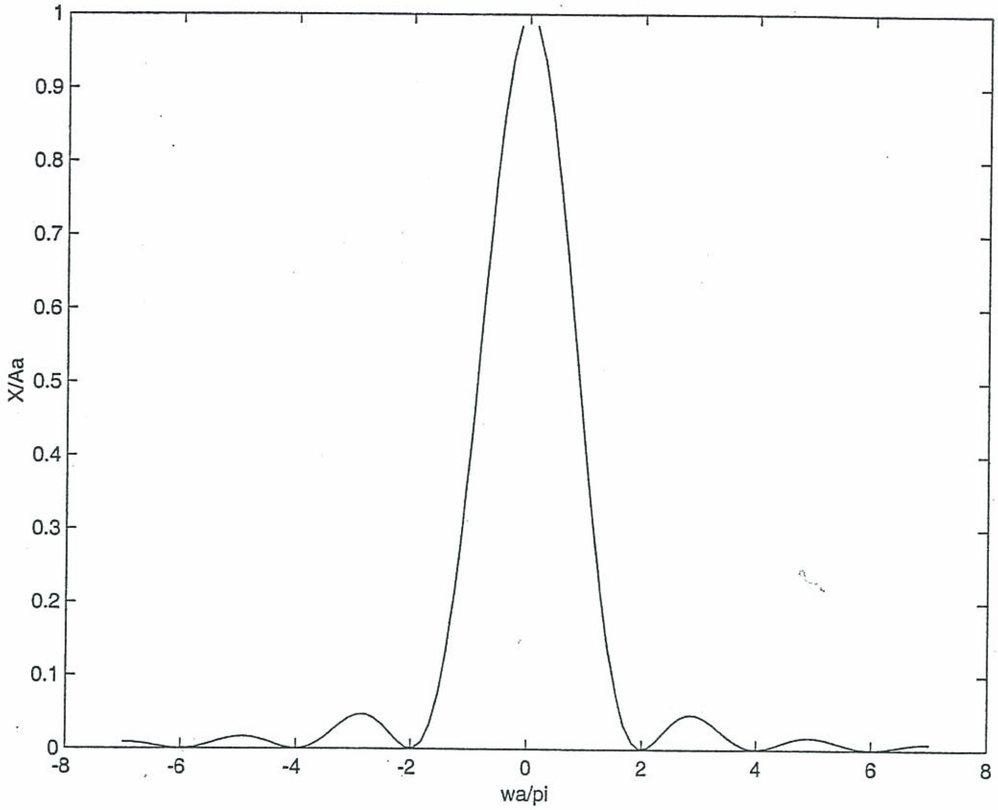
14.33

```
% Ex14_33.m
for i = 1: 101
    x(i) = -7 + 14 * (i-1)/100;
    f(i) = exp(-0.5* x(i)^2)/sqrt(2*pi);
end
plot(x,f);
xlabel('x');
ylabel('f(x)');
```



14.34

```
% Ex14_34.m
for i = 1: 101
    wa_pi(i) = -7 + 14 * (i-1)/100;
    x_Aa(i) = (4/(pi^2)) * (1/wa_pi(i))^2 * (sin(wa_pi(i)*pi/2.0))^2;
end
plot(wa_pi, x_Aa);
xlabel('wa/pi');
ylabel('X/Aa');
```



14.35

```
% Ex14_35.m
f = [0 20 40 60 80 100 -80 -60 -40 -20 0];
k = 4e4;
c = 1200;
m = 5.1760;
N = 10;
sumE = 0.0;
for i = 1: N
    n = i - 1;
    wn = 2*pi*n;
    sumC = 0.0;
    for j = 1: N
        sumC = sumC + f(j+1)*complex(cos(2*pi*n*j/N), -sin(2*pi*n*j/N))/N;
    end
    cn = sumC;
    sumE = sumE + ( abs(cn)^2 )/( (k - m*wn^2)^2 + (c*wn)^2 );
end
sumE

% Results: sumE =  1.3760e-006
```

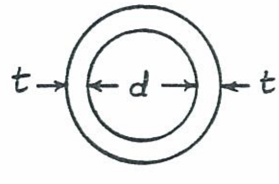
14.36 Let

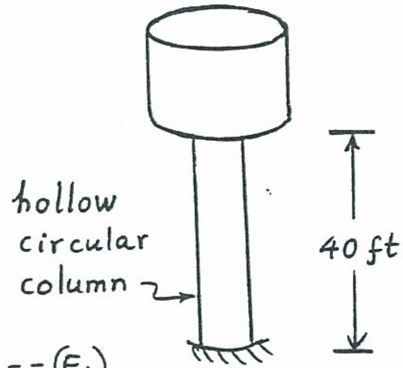
d = inner diameter of column

t = wall thickness

l = height = 480″

$E = 30 \times 10^6$ psi





k = stiffness of column (cantilever)

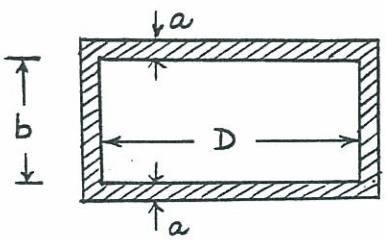
$$= \frac{3EI}{l^3} = \frac{3(30 \times 10^6)}{(480)^3} \frac{\pi}{64}\left[(d+t)^4 - d^4\right] \quad ----(E_1)$$

Let tank be a thin-walled cylindrical vessel with

D = mean diameter

a = thickness of shell (top & bottom)

b = axial length of shell.

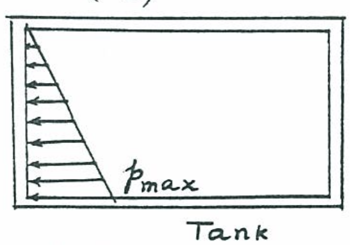


water volume = 10 000 gallons = 231×10^4 in^3

$$\text{Volume of tank} = \frac{\pi D^2}{4} b = 231 \times 10^4 \text{ in}^3 \quad ---- (E_2)$$

Max. pressure in tank $= p_{max} = \gamma h$

where γ = weight density of water = 62.4 lb/in^3 = 0.0361 lb/in^3



$$\text{Max. tangential stress in the tank} = \frac{p_{max} D}{2a} \quad ---- (E_3)$$

Let permissible stress in tank $= \sigma_p = \frac{\sigma_y}{2} = 15\,000$ psi $---- (E_4)$

(using a factor of safety of 2)

Equating (E_3) and (E_4),

$$15000 = \frac{0.0361\, h\, D}{2a} \quad ---- (E_5)$$

Weight of empty steel tank $= \gamma_s \, \pi D a b$

where γ_s = weight density of steel $= 0.283$ lb/in^3

m_t = mass of empty steel tank $= \frac{0.283}{386.4} \pi D a b$

$= 2.3009 \times 10^{-3} \, D a b$ ---- (E_6)

m_w = mass of water

$= (231 \times 10^4)\left(\frac{0.0361}{386.4}\right) = 215.8 \ \frac{lb - sec^2}{in}$ ---- (E_7)

Natural frequency of empty tank $= \sqrt{m_t / k} > 6.2832$ --- (E_8)

Natural frequency of full tank $= \sqrt{\frac{m_t + m_w}{k}} > 6.2832$ ---- (E_9)

where m_t, m_w and k are given by (E_6), (E_7) and (E_1).

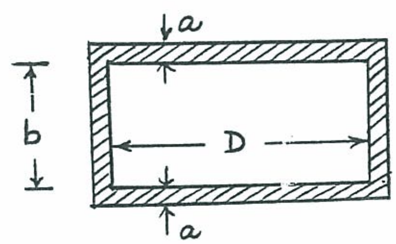
Due to ground acceleration with power spectral density $S(\omega)$, the mean square value of relative displacement of

Let tank be a thin-walled cylindrical vessel with

D = mean diameter

a = thickness of shell (top & bottom)

b = axial length of shell.



Water volume = 10 000 gallons $= 231 \times 10^4$ in^3

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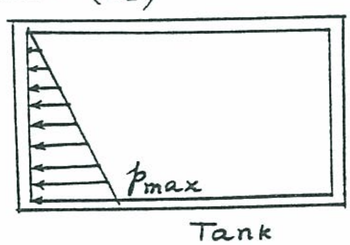
where γ = weight density of Water = 62.4 lb/in^3

$= 0.0361$ lb/in^3



Max. tangential stress in the tank $= \frac{p_{max} D}{2a}$ ---- (E_3)

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Weight of empty steel tank $= \gamma_s^t \, \pi D a b$

where $\gamma_s^t =$ weight density of steel $= 0.283 \ lb/in^3$

$m_t =$ mass of empty steel tank $= \dfrac{0.283}{386.4} \pi D a b$

$= 2.3009 \times 10^{-3} \, D a b \qquad \text{---- } (E_6)$

$m_w =$ mass of water

$= (231 \times 10^4)\left(\dfrac{0.0361}{386.4}\right) = 215.8 \ \dfrac{lb-sec^2}{in} \qquad \text{---- } (E_7)$

Natural frequency of empty tank $= \sqrt{m_t/k} > 6.2832 \quad \text{---- } (E_8)$

Natural frequency of full tank $= \sqrt{\dfrac{m_t + m_w}{k}} > 6.2832 \quad \text{---- } (E_9)$

where m_t, m_w and k are given by (E_6), (E_7) and (E_1).

Due to ground acceleration with power spectral density $S(\omega)$, the mean square value of relative displacement of tank is given by Eq. (E_{12}) of Example (14.6):

$$E[z^2] = \frac{S_0 \pi m^2}{kc} \qquad \text{---- } (E_{10})$$

where $c = 0.1 \, c_c = 0.2 \sqrt{k m}$ & $S_0 = 0.0002 / m^2/cycle/sec.$

When empty,

$$E[z^2] = \frac{\pi S_0 m_t^2}{k\left(0.2\sqrt{k m_t}\right)} \le 16 \ in^2 \qquad \text{---- } (E_{11})$$

and when full,

$$E[z^2] = \frac{\pi S_0 (m_t + m_w)^2}{k\left(0.2\sqrt{k(m_t + m_w)}\right)} \le 16 \ in^2 \qquad \text{---- } (E_{12})$$

$\therefore$ We need to find the values of d, t, D, a and b to satisfy the inequalities (E_8), (E_9), (E_{11}) and (E_{12}).