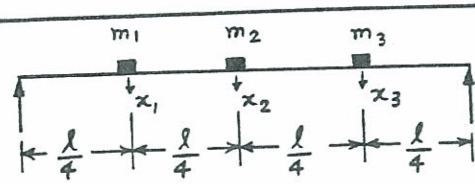
Chapter 7

Determination of Natural Frequencies and Mode Shapes

7.1 From Example 6.6,



$$a_{11} = a_{33} = \frac{3}{256} \frac{l^3}{EI}$$

$$a_{22} = \frac{1}{48} \frac{l^3}{EI}$$

(a) Eq. (7.6) gives

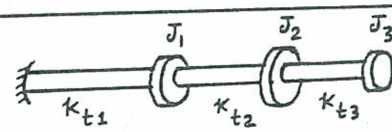
$$\frac{1}{\omega_1^2} \simeq \frac{m l^3}{EI} \left(\frac{3 \times 5}{256} + \frac{1}{48} \times 1 + \frac{3 \times 5}{256} \right) = \frac{106}{768} \frac{m l^3}{EI} = 0.13802 \frac{m l^3}{EI}$$

$$\omega_1 \simeq 2.6917 \sqrt{\frac{EI}{m l^3}}$$

(b)
$$\frac{1}{\omega_1^2} \simeq \frac{m l^3}{EI} \left(\frac{3}{256} + \frac{1 \times 5}{48} + \frac{3}{256} \right) = \frac{98}{768} \frac{m l^3}{EI} = 0.12760 \frac{m l^3}{EI}$$

$$\omega_1 \simeq 2.7994 \sqrt{\frac{EI}{m l^3}}$$

7.2 Flexibility influence coefficients:



a_{11} = rotation of J_1 when a unit torque is applied to $J_1 = 1/k_{t1}$

a_{22} = rotation of J_2 when a unit torque is applied to J_2

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}$$

a_{33} = rotation of J_3 when a unit torque is applied to J_3

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}$$

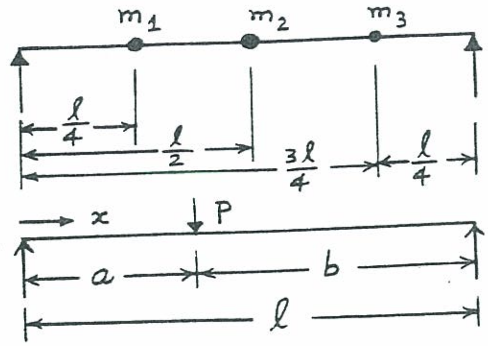
(a) Eq. (7.6) gives $\frac{1}{\omega_1^2} \simeq a_{11} J_1 + a_{22} J_2 + a_{33} J_3 = \frac{J_0}{k_t} (1+2+3)$

$$\omega_1 \simeq 0.4082 \sqrt{k_t / J_0}$$

(b)
$$\frac{1}{\omega_1^2} \simeq \frac{J_1}{k_{t1}} + J_2 \left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} \right) + J_3 \left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} \right)$$

$$\simeq \frac{J_0}{k_t} + 2 J_0 \left(\frac{1}{k_t} + \frac{1}{2 k_t} \right) + 3 J_0 \left(\frac{1}{k_t} + \frac{1}{2 k_t} + \frac{1}{3 k_t} \right) = \frac{9.5 J_0}{k_t}$$

$$\omega_1 \simeq 0.3244 \sqrt{k_t / J_0}$$



7.3 $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$;

$\ell_1 = \ell_2 = \ell_3 = \ell_4 = \ell/4$

$$w(x) = \frac{Pbx}{6EI\ell}(\ell^2 - b^2 - x^2); \quad 0 \le x \le a$$

$$= -\frac{Pa(\ell - x)}{6EI\ell}(a^2 + x^2 - 2\ell x); \quad a \le x \le \ell$$

Deflection due to weight of m_1: $(P = mg)$

At location of m_1 $\left(x = \frac{\ell}{4},\ b = \frac{3\ell}{4},\ \ell = \ell\right)$

$$w_1' = \frac{(mg)\left(\frac{3\ell}{4}\right)\left(\frac{\ell}{4}\right)}{6EI\ell}\left\{\ell^2 - \frac{9}{16}\ell^2 - \frac{1}{16}\ell^2\right\} = \frac{3mg\ell^3}{256\,EI}$$

At location of m_2 $\left(a = \frac{\ell}{4},\ x = \frac{\ell}{2},\ \ell = \ell\right)$

$$w_2' = -\frac{(mg)\left(\frac{\ell}{4}\right)\left(\ell - \frac{\ell}{2}\right)}{6EI\ell}\left(\frac{\ell^2}{16} + \frac{\ell^2}{4} - \ell^2\right) = \frac{11\,mg\ell^3}{768\,EI}$$

At location of m_3 $\left(a = \frac{\ell}{4},\ x = \frac{3\ell}{4},\ \ell = \ell\right)$

$$w_3' = -\frac{(mg)\left(\frac{\ell}{4}\right)\left(\ell - \frac{3\ell}{4}\right)}{6EI\ell}\left(\frac{\ell^2}{16} + \frac{9}{16}\ell^2 - \frac{6}{4}\ell^2\right) = \frac{7\,mg\ell^3}{768\,EI}$$

Deflection due to weight of m_2: $(P = 2mg)$

At location of m_1 $\left(x = \frac{\ell}{4},\ b = \frac{\ell}{2},\ \ell = \ell\right)$

$$w_1'' = \frac{(2mg)\left(\frac{\ell}{2}\right)\left(\frac{\ell}{4}\right)}{6EI\ell}\left(\ell^2 - \frac{\ell^2}{4} - \frac{1}{16}\ell^2\right) = \frac{11\,mg\ell^3}{384\,EI}$$

At location of m_2 $\left(x = \frac{\ell}{2},\ b = \frac{\ell}{2},\ \ell = \ell\right)$

$$w_2'' = \frac{(2mg)\left(\frac{\ell}{2}\right)\left(\frac{\ell}{2}\right)}{6EI\ell}\left(\ell^2 - \frac{\ell^2}{4} - \frac{\ell^2}{4}\right) = \frac{mg\ell^3}{24\,EI}$$

At location of m_3 $\left(x = \frac{3\ell}{4},\ a = \frac{\ell}{2},\ b = \frac{\ell}{2},\ \ell = \ell\right)$

$$w_3'' = -\frac{(2mg)\left(\frac{\ell}{2}\right)\left(\ell - \frac{3\ell}{4}\right)}{6EI\ell}\left(\frac{\ell^2}{4} + \frac{9}{16}\ell^2 - \frac{6}{4}\ell^2\right) = \frac{11\,mg\ell^3}{1024\,EI}$$

Deflection due to weight of m_3: $(P = 3mg)$

At location of m_1 $\left(x = \frac{\ell}{4},\ b = \frac{\ell}{4},\ \ell = \ell\right)$

$$w_1''' = \frac{(3mg)\left(\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)}{6EI\ell}\left(\ell^2 - \frac{\ell^2}{16} - \frac{\ell^2}{16}\right) = \frac{7mg\ell^3}{256\,EI}$$

At location of m_2 $\left(x = \frac{l}{2},\ b = \frac{l}{4},\ l = l\right)$

$$w_2''' = \frac{(3\,mg)\left(\frac{l}{4}\right)\left(\frac{l}{2}\right)}{6EIl}\left(l^2 - \frac{l^2}{16} - \frac{l^2}{4}\right) = \frac{11\,mg\,l^3}{256\,EI}$$

At location of m_3 $\left(x = \frac{3l}{4},\ b = \frac{l}{4},\ l = l\right)$

$$w_3''' = \frac{(3\,mg)\left(\frac{l}{4}\right)\left(\frac{3l}{4}\right)}{6EIl}\left(l^2 - \frac{1}{16}l^2 - \frac{9}{16}l^2\right) = \frac{9\,mg\,l^3}{256\,EI}$$

Total deflection of masses:

$$w_1 = w_1' + w_1'' + w_1''' = \frac{mg\,l^3}{EI}\left(\frac{3}{256} + \frac{11}{384} + \frac{7}{256}\right) = \frac{13}{192}\,\frac{mg\,l^3}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{mg\,l^3}{EI}\left(\frac{11}{768} + \frac{1}{24} + \frac{11}{256}\right) = \frac{19}{192}\,\frac{mg\,l^3}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{mg\,l^3}{EI}\left(\frac{7}{768} + \frac{11}{1024} + \frac{9}{256}\right) = \frac{169}{3072}\,\frac{mg\,l^3}{EI}$$

$$\omega = \left\{\frac{g\,\frac{m^2 g\,l^3}{EI}\left(\frac{13}{192} + 2\times\frac{19}{192} + 3\times\frac{169}{3072}\right)}{\frac{m^2 g^2 l^6\,m}{E^2 I^2}\left\{\left(\frac{13}{192}\right)^2 + 2\left(\frac{19}{192}\right)^2 + 3\left(\frac{169}{3072}\right)^2\right\}}\right\}^{1/2}$$

$$= 3.5987\sqrt{\frac{EI}{m\,l^3}}$$

7.4 ω_{11} = natural frequency of wing itself $= 20\ Hz = 125.664\ \frac{rad}{sec}$

ω_{22} = natural frequency of weapon attached at the tip of the wing (neglecting the effect of mass of wing)

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{12}\left(\frac{386.4}{2000}\right)} = 28.3725\ \frac{rad}{sec}$$

New frequency of vibration of the wing with weapon is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{1}{125.664^2} + \frac{1}{28.3725^2}$$

$$= 130.5563 \times 10^{-5}$$

$$\therefore\ \omega_1 = 27.6759\ \frac{rad}{sec} = 4.4047\ Hz$$

7.5 For a simply supported beam, natural frequency is given by (assuming its mass to be concentrated at the middle)

$$\omega_{11} = \sqrt{\frac{k}{m}} \qquad \text{where} \quad k = \frac{48EI}{l^3}$$

Neglecting the mass of beam, if trolley is placed at the middle of girder, its natural frequency is given by

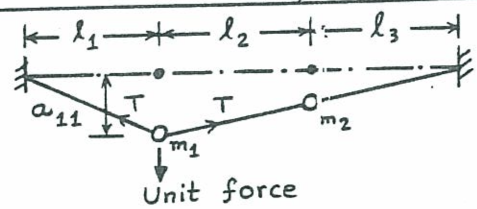
$$\omega_{22} = \sqrt{\frac{k}{10m}}$$

Fundamental natural frequency of the combined system is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{m}{k} + \frac{10m}{k} = \frac{11m}{k}$$

$$\therefore \omega_1 = 0.3015 \sqrt{\frac{k}{m}} = 30.15\% \text{ of the natural frequency of the girder (without the trolley)}$$

7.6 Flexibility coefficients:



Let T = tension in string

a_{11} = deflection of m_1 due to unit force applied to m_1

Unit force = sum of components of tension in vertical direction

i.e. $$1 = T\left(\frac{a_{11}}{l_1}\right) + T\left(\frac{a_{11}}{l_2 + l_3}\right) = T a_{11}\left(\frac{1}{l} + \frac{1}{2l}\right) = \frac{3Ta_{11}}{2l}$$

$$a_{11} = \frac{2l}{3T} = a_{22} \text{ (by symmetry)}$$

$$\frac{1}{\omega_1^2} \simeq a_{11} m_1 + a_{22} m_2 = \frac{2l}{3T}(m+m) = \frac{4ml}{3T}$$

$$\omega_1 \simeq 0.866 \sqrt{\frac{T}{ml}} \; ; \quad \text{Exact solution is } \omega_1 = \sqrt{\frac{T}{ml}} \text{ (Problem 5.22)}.$$

7.7 From Example 7.3, $\omega = 0.028222\sqrt{EI}$

Since $E = 2.07 \times 10^{11}$ N/m², $\omega = 12840.2346\sqrt{I}$ rad/sec

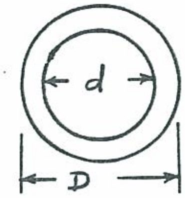
In order to have $\omega = 0.5$ Hz $= 3.1416$ rad/sec,

$12840.2346\sqrt{I} = 3.1416 \Rightarrow I = 59862.43022 \times 10^{-12}$ m⁴

For a tubular section,

$$I = \frac{\pi}{64}(D^4 - d^4) = 59862.43022 \times 10^{-12}$$

i.e. $D^4 - d^4 = 1.219504 \times 10^{-6}$ m⁴



To minimize weight, we need to minimize $(D^2 - d^2)$.

Problem is: Find D and d to minimize $(D^2 - d^2)$

subject to $D^4 - d^4 = 1.219504 \times 10^{-6}$

or Find d and $r = D/d$

to minimize $f = d^2(r^2 - 1)$ (E₁)

subject to $d^4(r^4 - 1) = 121.9504 \times 10^{-8}$ (E₂)

Eq. (E₂) gives
$$d^2 = \frac{11.0431 \times 10^{-4}}{\sqrt{r^4 - 1}}$$

Eq. (E₁) becomes
$$f = \frac{11.0431 \times 10^{-4}(r^2 - 1)}{\sqrt{r^4 - 1}}$$

For minimum of f,
$$\frac{df}{dr} = \frac{d}{dr}\left(\frac{r^2 - 1}{\sqrt{r^4 - 1}}\right) = 0$$

i.e.,
$$\frac{(r^2 - 1)\frac{1}{2}(r^4 - 1)^{-\frac{1}{2}}(4r^3) - \sqrt{r^4 - 1}\,(2r)}{r^4 - 1} = 0$$

i.e., $r^2(r^2 - 1) = (r^2 + 1)(r^2 - 1)$

i.e., $r^2 = 1$ or $r^2 = r^2 + 1$

i.e., $r = 1$ is the only feasible solution.

i.e., A solid circular section.

Since $I = \frac{\pi D^4}{64} = 0.059862 \times 10^{-6}$, $D = 0.03323$ m

7.10 From Example 6.3,

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assuming the mode shape as $\vec{X} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, Rayleigh's quotient becomes

$$R(\vec{X}) = \omega^2 = \frac{\vec{X}^T [k] \vec{X}}{\vec{X}^T [m] \vec{X}}$$

$$= \frac{(1 \quad 2 \quad 3)\, k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \quad 2 \quad 3)\, m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{6} \frac{k}{m}$$

$$\omega_1 = 0.4082 \sqrt{\frac{k}{m}}$$

Exact value is $\omega_1 = 0.3376 \sqrt{\frac{k}{m}}$ (Problem 6.51).

7.11 Stiffness matrix:

Give each disc a unit angular displacement holding other discs with zero rotation. Torque required will give stiffness coefficients.

$\theta_1 = 1,\ \theta_2 = \theta_3 = 0$: $M_{t1} = k_{t1} + k_{t2}$, $M_{t2} = -k_{t2}$, $M_{t3} = 0$

$\theta_2 = 1,\ \theta_1 = \theta_3 = 0$: $M_{t2} = k_{t2} + k_{t3}$, $M_{t1} = -k_{t2}$, $M_{t3} = -k_{t3}$

$\theta_3 = 1,\ \theta_1 = \theta_2 = 0$: $M_{t3} = k_{t3}$, $M_{t2} = -k_{t3}$, $M_{t1} = 0$

$$[k] = \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} & 0 \\ -k_{t2} & k_{t2} + k_{t3} & -k_{t3} \\ 0 & -k_{t3} & k_{t3} \end{bmatrix} = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

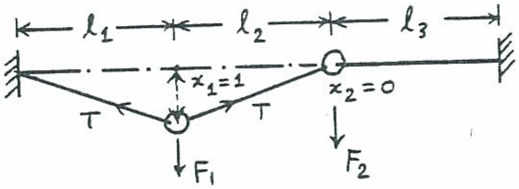
$$[m] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assume the mode shape as $\vec{\Theta} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$R(\vec{\Theta}) = \omega^2 = \frac{(1 \quad 2 \quad 3)\, k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \quad 2 \quad 3)\, J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{12} \frac{k_t}{J_0}$$

$$\omega_1 \simeq 0.2887 \sqrt{\frac{k_t}{J_0}}$$

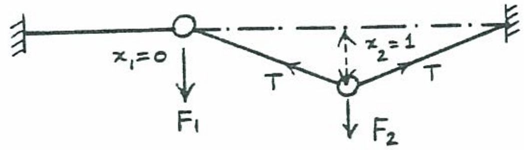
7.12 Stiffness matrix:



When $x_1 = 1$, $x_2 = 0$:

$$F_1 = k_{11} = \frac{T}{l_1} + \frac{T}{l_2}$$

$$F_2 = k_{21} = -\frac{T}{l_2}$$



When $x_1 = 0$, $x_2 = 1$:

$$F_2 = \frac{T}{l_2} + \frac{T}{l_3}$$

$$F_1 = k_{12} = -\frac{T}{l_2}$$

$$[k] = T\begin{bmatrix} \left(\frac{1}{l_1}+\frac{1}{l_2}\right) & -\frac{1}{l_2} \\ -\frac{1}{l_2} & \left(\frac{1}{l_2}+\frac{1}{l_3}\right) \end{bmatrix} = \frac{T}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume the mode shape as $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \quad 2)\, \frac{T}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \quad 2)\, m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{6}{5}\,\frac{T}{lm}$$

$$\omega_1 \simeq 1.0954\sqrt{\frac{T}{lm}}$$

Exact value is $\omega_1 = \sqrt{\frac{T}{lm}}$ (Problem 5.22).

7.13 From problem 7.12, for $l_1 = l_2 = l_3 = l$,

$$[k] = \frac{T}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and for $m_1 = m$, $m_2 = 5m$, $[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

Let $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \quad 2)\, \frac{T}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \quad 2)\, m\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{2}{7}\,\frac{T}{ml}$$

$$\omega_1 \simeq 0.5345\sqrt{\frac{T}{ml}}$$

7.14 Apply a unit load to masses m_1 and m_2 along x_1 and x_2, respectively:

$$a_{11} = \frac{1}{2k_1} \quad ; \quad a_{22} = \frac{1}{2k_1} + \frac{1}{2k_2} = \frac{k_2 + k_1}{2k_1 k_2}$$

Dunkerley's equation gives

$$\frac{1}{\omega_1^2} = a_{11} m_1 + a_{22} m_2 = \frac{m_1}{2k_1} + m_2 \left(\frac{k_2 + k_1}{2k_1 k_2} \right) \qquad (E_1)$$

Since $k_1 = k_2 = 3EI/h^3 \equiv k$, $m_1 = 2m$, $m_2 = m$ and hence (E_1) gives

$$\omega_1 = \sqrt{\frac{3EI}{2mh^3}}$$

7.15 Eq. (7.21) gives, for $r = n$,

$$R(\vec{x}) = \frac{c_n^2 \omega_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n}\right)^2 \omega_i^2}{c_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n}\right)^2} \qquad (E_1)$$

Let $\left|\frac{c_i}{c_n}\right| = \epsilon_i << 1$. Then Eq. (E_1) becomes

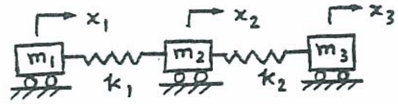
$$R(\vec{x}) = \frac{\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2}{1 + \sum_{i=1}^{n-1} \epsilon_i^2} \simeq \left(\omega_n^2 + \sum_{i=1} \epsilon_i^2 \omega_i^2\right)\left(1 - \sum_{i=1}^{n-1} \epsilon_i^2\right)$$

$$\simeq \omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 - \omega_n^2 \sum_{i=1}^{n-1} \epsilon_i^2$$

$$\simeq \omega_n^2 + \sum_{i=1}^{n-1} (\omega_i^2 - \omega_n^2)\, \epsilon_i^2 \qquad (E_2)$$

Since $\omega_i^2 < \omega_n^2$, in general, (E_2) shows that

$$R(\vec{x}) \le \omega_n^2$$



7.19 Equations of motion

$$m_1 \ddot{x}_1 + k_1 (x_1 - x_2) = 0 \quad \text{--- (E.1)}$$

$$m_2 \ddot{x}_2 + k_1 (x_2 - x_1) + k_2 (x_2 - x_3) = 0 \quad \text{--- (E.2)}$$

$$m_3 \ddot{x}_3 + k_2 (x_3 - x_2) = 0 \quad \text{--- (E.3)}$$

With $x_i(t) = X_i \cos \omega t$, Eqs. (E.1) and (E.2) give

$$m_1 \omega^2 X_1 = k_1 (X_1 - X_2) \; ; \qquad m_2 \omega^2 X_2 = -\omega^2 m_1 X_1 + k_2 (X_2 - X_3)$$

or $$X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1} \; ; \quad X_3 = X_2 - \frac{\omega^2}{k_2} (m_1 X_1 + m_2 X_2)$$

Since the system is free - free, the resultant force applied to mass 3,

$$F = \sum_{i=1}^{3} \omega^2 m_i X_i \; ,$$ must be zero.

The computer program, with the subroutine FUN and the output, are given below.

```
C ==========================================================================
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C ==========================================================================
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=2    <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.1 <----
      CALL FUN (OM,F)
      PRINT 60,NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
  100 DEL=10.0 <----
      NT=0
  200 CONTINUE
      F1=FB
   10 OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      F2=F1*F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
   20 NT=NT+1
      PRINT 30, OM,F,DEL,NFUN
   30 FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
     2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
   40 INR=INR+1
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      FB=F
      GO TO 100
   50 CONTINUE
   60 FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C ==========================================================================
C
C SUBROUTINE FUN
C
C ==========================================================================
```

```
      SUBROUTINE FUN (OM,F)
      XM1=100.0
      XM2=20.0
      XM3=200.0
      XK1=8000.0
      XK2=4000.0
      OMS=OM**2
      X1=1.0
      X2=(1.0-(OMS*XM1/XK1))*X1
      X3=X2-(OMS/XK2)*(XM1*X1+XM2*X2)
      F=OMS*(XM1*X1+XM2*X2+XM3*X3)
      RETURN
      END
```

$\omega_1 = 0$

```
 NFUN=    1   OM= 0.10000000E+00   F= 0.31991253E+01
 NFUN=    2   OM= 0.10000000E+02   F=-0.43000000E+05

CHANGE OF SIGN DETECTED AT OM= 0.10000000E+02
F=-0.43000000E+05
DEL= 0.10000000E+02
NFUN=    2

 NFUN=    3   OM= 0.10000000E+01   F= 0.31126251E+03
 NFUN=    4   OM= 0.20000000E+01   F= 0.11408000E+04
 NFUN=    5   OM= 0.30000000E+01   F= 0.21803625E+04
 NFUN=    6   OM= 0.40000000E+01   F= 0.29312000E+04
 NFUN=    7   OM= 0.50000000E+01   F= 0.27265625E+04
 NFUN=    8   OM= 0.60000000E+01   F= 0.76320044E+03
 NFUN=    9   OM= 0.70000000E+01   F=-0.38581377E+04

CHANGE OF SIGN DETECTED AT OM= 0.70000000E+01
F=-0.38581377E+04
DEL= 0.10000000E+01
NFUN=    9
    :
CHANGE OF SIGN DETECTED AT OM= 0.62220016E+01
F=-0.28649828E+00
DEL= 0.99999990E-01
NFUN=   27
```

$\omega_2 = 6.2220016$

```
 NFUN=   28   OM= 0.10000000E+02   F=-0.43000000E+05
 NFUN=   29   OM= 0.20000000E+02   F=-0.47200000E+06
 NFUN=   30   OM= 0.30000000E+02   F= 0.23130000E+07

CHANGE OF SIGN DETECTED AT OM= 0.30000000E+02
F= 0.23130000E+07
DEL= 0.10000000E+02
NFUN=   30

 NFUN=   31   OM= 0.21000000E+02   F=-0.48851225E+06
 NFUN=   32   OM= 0.22000000E+02   F=-0.47761113E+06
 NFUN=   33   OM= 0.23000000E+02   F=-0.42888006E+06
```

```
:
CHANGE OF SIGN DETECTED AT OM= 0.25715595E+02
F= 0.21467506E+02
DEL= 0.99999990E-04
NFUN=    58
```

$\leftarrow \quad \omega_3 = 25.715595$

7.20 Eigenvalue problem is

$$\begin{bmatrix} (\lambda-2) & 1 & 0 \\ 1 & (\lambda-2) & 1 \\ 0 & 1 & (2\lambda-3) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E_1)$$

where $\lambda = \frac{m\omega^2}{k}$.

If $x_1 = 1$, (E_1) gives $x_2 = -(\lambda-2)\,x_1$, $x_3 = -x_1 - (\lambda-2)\,x_2$,

$E = x_2 + (2\lambda-3)\,x_3$ (should be zero)

The computer program and results are given.

```
C ===========================================================================
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C ===========================================================================
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=3 <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.01 <-----
      CALL FUN (OM,F)
      WRITE(13,60) NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
100   DEL=0.25 <-----
      NT=0
200   CONTINUE
      F1=FB
10    OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      F2=F1*F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
20    NT=NT+1
      WRITE(13,30) OM,F,DEL,NFUN
30    FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
     2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
40    INR=INR+1
```

```
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      FB=F
      GO TO 100
50    CONTINUE
60    FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C ==========================================================================
C
C SUBROUTINE FUN
C ==========================================================================
C
C ==========================================================================
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=-(OM-2.0)*X1
      X3=-X1-(OM-2.0)*X2
      F=X2+(2.0*OM-3.0)*X3
      RETURN
      END
```

```
  NFUN=    1     OM= 0.99999998E-02     F=-0.68310986E+01
  NFUN=    2     OM= 0.25000000E+00     F=-0.34062500E+01
  NFUN=    3     OM= 0.50000000E+00     F=-0.10000000E+01
  NFUN=    4     OM= 0.75000000E+00     F= 0.40625000E+00

CHANGE OF SIGN DETECTED AT OM= 0.75000000E+00
F= 0.40625000E+00
DEL= 0.25000000E+00
NFUN=    4

  NFUN=    5     OM= 0.52499998E+00     F=-0.81746900E+00
  NFUN=    6     OM= 0.54999995E+00     F=-0.64475036E+00
  NFUN=    7     OM= 0.57499993E+00     F=-0.48165655E+00
  NFUN=    8     OM= 0.59999990E+00     F=-0.32800055E+00
  NFUN=    9     OM= 0.62499988E+00     F=-0.18359447E+00
  NFUN=   10     OM= 0.64999986E+00     F=-0.48250675E-01
  NFUN=   11     OM= 0.67499983E+00     F= 0.78217864E-01
   :
  NFUN=   28     OM= 0.65932715E+00     F=-0.38385391E-04
  NFUN=   29     OM= 0.65932965E+00     F=-0.25749207E-04
  NFUN=   30     OM= 0.65933216E+00     F=-0.12755394E-04
  NFUN=   31     OM= 0.65933466E+00     F=-0.11920929E-06
  NFUN=   32     OM= 0.65933716E+00     F= 0.12636185E-04

CHANGE OF SIGN DETECTED AT OM= 0.65933716E+00
F= 0.12636185E-04
DEL= 0.24999997E-05
NFUN=   32

  NFUN=   33     OM= 0.75000000E+00     F= 0.40625000E+00
```

← $\lambda_1 = 0.65933716$

```
NFUN=  34   OM= 0.10000000E+01    F= 0.10000000E+01
NFUN=  35   OM= 0.12500000E+01    F= 0.96875000E+00
NFUN=  36   OM= 0.15000000E+01    F= 0.50000000E+00
NFUN=  37   OM= 0.17500000E+01    F=-0.21875000E+00
 :
NFUN=  63   OM= 0.16789526E+01    F= 0.32067299E-04
NFUN=  64   OM= 0.16789551E+01    F= 0.24497509E-04
NFUN=  65   OM= 0.16789576E+01    F= 0.16927719E-04
NFUN=  66   OM= 0.16789601E+01    F= 0.93579292E-05
NFUN=  67   OM= 0.16789626E+01    F= 0.17881393E-05
NFUN=  68   OM= 0.16789651E+01    F=-0.57816505E-05

CHANGE OF SIGN DETECTED AT OM= 0.16789651E+01
F=-0.57816505E-05
DEL= 0.24999997E-05
NFUN=   68
```

← $\lambda_2 = 1.6789651$

```
NFUN=  69   OM= 0.17500000E+01    F=-0.21875000E+00
NFUN=  70   OM= 0.20000000E+01    F=-0.10000000E+01
NFUN=  71   OM= 0.22500000E+01    F=-0.16562500E+01
NFUN=  72   OM= 0.25000000E+01    F=-0.20000000E+01
NFUN=  73   OM= 0.27500000E+01    F=-0.18437500E+01
NFUN=  74   OM= 0.30000000E+01    F=-0.10000000E+01
NFUN=  75   OM= 0.32500000E+01    F= 0.71875000E+00
 :
NFUN=  95   OM= 0.31615264E+01    F=-0.13036728E-02
NFUN=  96   OM= 0.31615515E+01    F=-0.11179447E-02
NFUN=  97   OM= 0.31615765E+01    F=-0.93221664E-03
NFUN=  98   OM= 0.31616015E+01    F=-0.74636936E-03
NFUN=  99   OM= 0.31616266E+01    F=-0.56064129E-03
NFUN= 100   OM= 0.31616516E+01    F=-0.37479401E-03
NFUN= 101   OM= 0.31616766E+01    F=-0.18906593E-03
NFUN= 102   OM= 0.31617017E+01    F=-0.32186508E-05
NFUN= 103   OM= 0.31617267E+01    F= 0.18215179E-03

CHANGE OF SIGN DETECTED AT OM= 0.31617267E+01
F= 0.18215179E-03
DEL= 0.24999998E-04
NFUN=  103
```

← $\lambda_3 = 3.1617267$

7.21 The program listed in Problems 7.19 and 7.20 is used with NR = 1, initial value of OM = 0.01, DEL = 0.25 and

```
C ==================================================================
C
C SUBROUTINE FUN
C
C ==================================================================
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=(2.0-OM)*X1
      X3=-X1+(2.0-OM)*X2
      F=-X2+(1.0-OM)*X3
```

$\Theta_1 = 1$

$\Theta_2 = (-\lambda + 2)\,\Theta_1$; $\lambda = \left(\dfrac{J_0\,\omega^2}{k_t}\right)$

$\Theta_3 = -\Theta_1 + (-\lambda + 2)\,\Theta_2$

$E = -\Theta_2 + (-\lambda + 1)\,\Theta_3$

```
RETURN
END
```

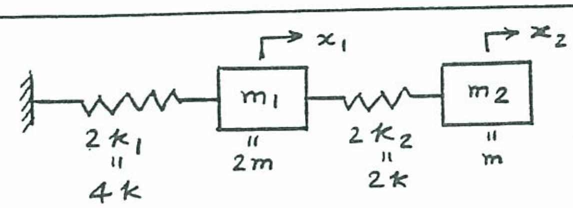
The output of the program is given below.

```
NFUN=    1    OM= 0.99999998E-02    F= 0.94049907E+00
NFUN=    2    OM= 0.25000000E+00    F=-0.20312500E+00

CHANGE OF SIGN DETECTED AT OM= 0.25000000E+00
F=-0.20312500E+00
DEL= 0.25000000E+00
NFUN=    2

NFUN=    3    OM= 0.25000000E-01    F= 0.85310948E+00
NFUN=    4    OM= 0.50000001E-01    F= 0.71237516E+00
NFUN=    5    OM= 0.75000003E-01    F= 0.57770300E+00
NFUN=    6    OM= 0.10000000E+00    F= 0.44899976E+00
NFUN=    7    OM= 0.12500000E+00    F= 0.32617188E+00
NFUN=    8    OM= 0.15000001E+00    F= 0.20912516E+00
NFUN=    9    OM= 0.17500001E+00    F= 0.97765565E-01
NFUN=   10    OM= 0.20000002E+00    F=-0.80001354E-02

CHANGE OF SIGN DETECTED AT OM= 0.20000002E+00
F=-0.80001354E-02
DEL= 0.25000000E-01
NFUN=   10

NFUN=   11    OM= 0.17750001E+00    F= 0.86938858E-01
NFUN=   12    OM= 0.18000001E+00    F= 0.76168060E-01
NFUN=   13    OM= 0.18250000E+00    F= 0.65452933E-01
NFUN=   14    OM= 0.18500000E+00    F= 0.54793477E-01
NFUN=   15    OM= 0.18750000E+00    F= 0.44189453E-01
NFUN=   16    OM= 0.19000000E+00    F= 0.33641100E-01
NFUN=   17    OM= 0.19250000E+00    F= 0.23147941E-01
NFUN=   18    OM= 0.19499999E+00    F= 0.12710214E-01
NFUN=   19    OM= 0.19749999E+00    F= 0.23275614E-02
NFUN=   20    OM= 0.19999999E+00    F=-0.79998970E-02

CHANGE OF SIGN DETECTED AT OM= 0.19999999E+00
F=-0.79998970E-02
DEL= 0.24999999E-02
NFUN=   20

NFUN=   21    OM= 0.19774999E+00    F= 0.12923479E-02
NFUN=   22    OM= 0.19799998E+00    F= 0.25773048E-03
NFUN=   23    OM= 0.19824998E+00    F=-0.77641010E-03

CHANGE OF SIGN DETECTED AT OM= 0.19824998E+00
F=-0.77641010E-03
DEL= 0.24999998E-03
NFUN=   23
```

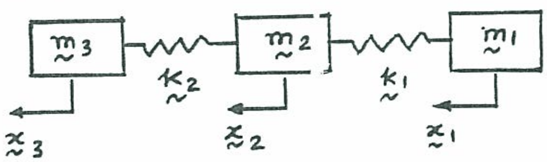
```
NFUN=  24   OM= 0.19802499E+00   F= 0.15425682E-03
NFUN=  25   OM= 0.19804999E+00   F= 0.50902367E-04
NFUN=  26   OM= 0.19807500E+00   F=-0.52571297E-04

CHANGE OF SIGN DETECTED AT OM= 0.19807500E+00
F=-0.52571297E-04
DEL= 0.24999998E-04
NFUN=   26

NFUN=  27   OM= 0.19805250E+00   F= 0.40531158E-04
NFUN=  28   OM= 0.19805500E+00   F= 0.30040741E-04
NFUN=  29   OM= 0.19805750E+00   F= 0.19669533E-04
NFUN=  30   OM= 0.19806001E+00   F= 0.92983246E-05
NFUN=  31   OM= 0.19806251E+00   F=-0.10728836E-05

CHANGE OF SIGN DETECTED AT OM= 0.19806251E+00
F=-0.10728836E-05
DEL= 0.24999997E-05
NFUN=   31
```

← $\lambda_1 = 0.19806251$

7.22 The system can be modeled as shown.



The system can be redrawn as follows:



Here $\underset{\sim}{m}_1 = m$, $\underset{\sim}{k}_1 = 2k$, $\underset{\sim}{m}_2 = 2m$, $\underset{\sim}{k}_2 = 4k$, $\underset{\sim}{x}_1 = x_2$, $\underset{\sim}{x}_2 = x_1$, $\underset{\sim}{x}_3 = 0$, $\underset{\sim}{m}_3$ = any value.

Let $\underset{\sim}{X}_1 = 1$

$$\underset{\sim}{X}_2 = \underset{\sim}{X}_1 - \frac{\omega^2 \underset{\sim}{m}_1 \underset{\sim}{X}_1}{\underset{\sim}{k}_1}$$

$$= \underset{\sim}{X}_1 \left(1 - \frac{m\omega^2}{2k}\right)$$

$$\underset{\sim}{X}_3 = \underset{\sim}{X}_2 - \frac{\omega^2}{\underset{\sim}{k}_2}\left(\underset{\sim}{m}_1 \underset{\sim}{X}_1 + \underset{\sim}{m}_2 \underset{\sim}{X}_2\right)$$

$$= \underset{\sim}{X}_2 \left(1 - \frac{m\omega^2}{2k}\right) - \underset{\sim}{X}_1 \left(\frac{m\omega^2}{4k}\right)$$

$$= \underset{\sim}{X}_1 \left(1 + \frac{m^2\omega^4}{4k^2} - \frac{5m\omega^2}{4k}\right)$$

Holzer's procedure involves assuming different values for ω^2 and finding, out of those values, the correct frequency ω as the one which gives $\underset{\sim}{X}_3 = 0$ (boundary condition to be satisfied).

From the expression for $\underset{\sim}{X}_3$, we can find the correct frequency (with out trial and error) by setting

$$\omega^4\left(\frac{m^2}{4k^2}\right) - \omega^2\left(\frac{5m}{4k}\right) + 1 = 0$$

or $$\omega^2 = \frac{\left(\frac{5m}{4k}\right) \pm \sqrt{\frac{25m^2}{16k^2} - \frac{m^2}{k^2}}}{(m^2/2k^2)} = \frac{\frac{5m}{4k} \pm \frac{3m}{4k}}{(m^2/2k^2)}$$

Thus the first natural frequency is given by

$$\omega_1^2 = \left(\frac{5m}{4k} - \frac{3m}{4k}\right)\Big/\left(\frac{m^2}{2k^2}\right) \quad \text{or} \quad \omega_1 = \sqrt{\frac{k}{m}}.$$

7.23 Eqs. (7.32) to (7.35) give

$$\Theta_2 = \Theta_1 - \frac{\omega^2 J_1}{k_{t1}}\Theta_1$$

$$\Theta_3 = \Theta_2 - \frac{\omega^2}{k_{t2}}(J_1\Theta_1 + J_2\Theta_2)$$

$$E = \sum_{i=1}^{3} \omega^2 J_i \Theta_i = \text{sum of inertia torques (should be zero)}$$

The computer program of Problems 7.19 and 7.20 is used with $NR = 3$, $OM = 0.01$ and $DEL = 10.0$.
The subroutine FUN and results are given.

```
C =================================================================
C
C SUBROUTINE FUN
C
C =================================================================
      SUBROUTINE FUN (OM,F)
      XJ1=10.0
      XJ2=5.0
      XJ3=1.0
      XKT1=1.0E+06
      XKT2=1.0E+06
      X1=1.0
      OMS=OM**2
      X2=X1-(OMS*XJ1/XKT1)*X1
      X3=X2-(OMS/XKT2)*(XJ1*X1+XJ2*X2)
      F=OMS*(XJ1*X1+XJ2*X2+XJ3*X3)
      RETURN
      END
```

$\omega_1 = 0$

```
NFUN=   1   OM= 0.99999998E-02   F= 0.16000000E-02
NFUN=   2   OM= 0.10000000E+02   F= 0.15992500E+04
NFUN=   3   OM= 0.20000000E+02   F= 0.63880034E+04
NFUN=   4   OM= 0.30000000E+02   F= 0.14339287E+05
NFUN=   5   OM= 0.40000000E+02   F= 0.25408205E+05
NFUN=   6   OM= 0.50000000E+02   F= 0.39532031E+05
NFUN=   7   OM= 0.60000000E+02   F= 0.56630336E+05
NFUN=   8   OM= 0.70000000E+02   F= 0.76605133E+05
NFUN=   9   OM= 0.80000000E+02   F= 0.99341109E+05
NFUN=  10   OM= 0.90000000E+02   F= 0.12470582E+06
  :
```

```
NFUN=  50   OM= 0.49000000E+03   F= 0.21006358E+06
NFUN=  51   OM= 0.50000000E+03   F= 0.93750000E+05
NFUN=  52   OM= 0.51000000E+03   F=-0.32486393E+05

CHANGE OF SIGN DETECTED AT OM= 0.51000000E+03
F=-0.32486393E+05
DEL= 0.10000000E+02
NFUN=   52
  :
NFUN=  69   OM= 0.50750012E+03   F= 0.93337460E+01
NFUN=  70   OM= 0.50750021E+03   F= 0.79214058E+01
NFUN=  71   OM= 0.50750031E+03   F= 0.66932836E+01
NFUN=  72   OM= 0.50750040E+03   F= 0.57721915E+01
NFUN=  73   OM= 0.50750049E+03   F= 0.45440674E+01
NFUN=  74   OM= 0.50750058E+03   F= 0.33159423E+01
NFUN=  75   OM= 0.50750067E+03   F= 0.20878162E+01
NFUN=  76   OM= 0.50750076E+03   F= 0.92109573E+00
NFUN=  77   OM= 0.50750085E+03   F=-0.30703202E+00

CHANGE OF SIGN DETECTED AT OM= 0.50750085E+03      ← ω2 = 507·50085
F=-0.30703202E+00
DEL= 0.99999990E-04
NFUN=   77

NFUN=  78   OM= 0.51000000E+03   F=-0.32486393E+05
NFUN=  79   OM= 0.52000000E+03   F=-0.16878158E+06
NFUN=  80   OM= 0.53000000E+03   F=-0.31524272E+06
  :
NFUN= 137   OM= 0.11000000E+04   F=-0.18694431E+07
NFUN= 138   OM= 0.11100000E+04   F=-0.62095219E+06
NFUN= 139   OM= 0.11200000E+04   F= 0.74758494E+06

CHANGE OF SIGN DETECTED AT OM= 0.11200000E+04
F= 0.74758494E+06
DEL= 0.10000000E+02
NFUN=  139
  :
NFUN= 168   OM= 0.11146489E+04   F=-0.37916328E+02
NFUN= 169   OM= 0.11146490E+04   F=-0.23697710E+02
NFUN= 170   OM= 0.11146492E+04   F=-0.94790859E+01
NFUN= 171   OM= 0.11146493E+04   F= 0.47395439E+01

CHANGE OF SIGN DETECTED AT OM= 0.11146493E+04      ← ω3 = 1114·6493
F= 0.47395439E+01
DEL= 0.99999990E-04
NFUN=  171
```

7.27 Eigenvector $\vec{X}^{(1)}$ corresponding to $\lambda_1 = 10.38068 \left(= \frac{1}{\omega_1^2}\right)$ is given by

$$\begin{bmatrix} (2.5-\lambda_1) & -1 & 0 \\ -1 & (5-\lambda_1) & -\sqrt{2} \\ 0 & -\sqrt{2} & (10-\lambda_1) \end{bmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

i.e. $x_2^{(1)} = -7.88068\, x_1^{(1)}$ and $x_3^{(1)} = -3.71497\, x_2^{(1)} = 29.27649\, x_1^{(1)}$

$$\vec{X}^{(1)} = \alpha \begin{Bmatrix} 1 \\ -7.88068 \\ 29.27649 \end{Bmatrix} \quad \text{where } \alpha = \text{value of } x_1^{(1)}$$

when $\vec{X}^{(1)}$ is normalized as $\vec{X}^{(1)T} [m] \vec{X}^{(1)} = 1$, $\alpha = 0.03296$

$$\therefore \vec{X}^{(1)} = \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix}$$

$$[D_2] = [D_1] - \lambda_1 \vec{X}^{(1)} \vec{X}^{(1)T} [m]$$

$$= \begin{bmatrix} 2.5 & -1.0 & 0.0 \\ -1.0 & 5.0 & -1.4142 \\ 0.0 & -1.4142 & 10.0 \end{bmatrix} - (10.38068) \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix} \{0.03296 \;\; -0.25975 \;\; 0.96495\}$$

$$= \begin{bmatrix} 2.4887228 & -0.9111273 & -0.3301549 \\ -0.9111273 & 4.2996149 & 1.1876738 \\ -0.3301549 & 1.1876738 & 0.3342530 \end{bmatrix}$$

If $\vec{X}_0 = \begin{Bmatrix} 1.0 \\ -7.88068 \\ 29.27649 \end{Bmatrix}$ is used, the iterative procedure $\vec{X}_{i+1} = [D_2] \vec{X}_i$ gives the following results (with $X_{1,i+1} = 1$):

Iter. No. (i)		$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1} = 1$)
ITER=	0	0.10000000E+01-0.78806801E+01 0.29276489E+02
ITER=	1	0.10000000E+01-0.10666667E+02 0.29333334E+02
ITER=	2	0.10000000E+01-0.47272644E+01-0.13066854E+01
ITER=	3	0.10000000E+01-0.31531227E+01-0.88291872E+00
ITER=	4	0.10000000E+01-0.27448514E+01-0.77301961E+00
ITER=	5	0.10000000E+01-0.25989382E+01-0.73374254E+00
ITER=	6	0.10000000E+01-0.25411220E+01-0.71817952E+00
ITER=	7	0.10000000E+01-0.25172873E+01-0.71176374E+00
ITER=	8	0.10000000E+01-0.25073018E+01-0.70907575E+00
ITER=	9	0.10000000E+01-0.25030899E+01-0.70794201E+00
ITER=	10	0.10000000E+01-0.25013082E+01-0.70746231E+00
ITER=	11	0.10000000E+01-0.25005538E+01-0.70725930E+00
ITER=	12	0.10000000E+01-0.25002341E+01-0.70717323E+00
ITER=	13	0.10000000E+01-0.25000987E+01-0.70713681E+00
ITER=	14	0.10000000E+01-0.25000415E+01-0.70712131E+00
ITER=	15	0.10000000E+01-0.25000172E+01-0.70711482E+00

converged value of $\lambda_2 = 5.00004216$ (or $\omega_2 = 0.44721171$)

By using a similar procedure, $[D_3]$ is found. With the same $\vec{X}_0$, the results obtained from $\vec{X}_{i+1} = [D_3] \vec{X}_i$ are given below.

Iter. no. (i)		$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1} = 1$)		
ITER=	0	0.10000000E+01	-0.78806801E+01	0.29276489E+02
ITER=	1	0.10000000E+01	0.19600000E+02	-0.72824997E+02
ITER=	2	0.10000000E+01	0.38067123E+00	0.68348452E-01
ITER=	3	0.10000000E+01	0.38067168E+00	0.68312615E-01
ITER=	4	0.10000000E+01	0.38067159E+00	0.68312593E-01

Converged value of $\lambda_3 = 2.11932243$ ($\omega_3 = 0.68691260$).

7.28 $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2.5 \end{bmatrix}$, $[D] = [k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$

Using $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained:

Iteration number (i)		$\vec{X}_{i+1} = [D]\,\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)		
ITER=	0	0.10000000E+01	0.10000000E+01	0.10000000E+01
ITER=	1	0.10000000E+01	0.17500000E+01	0.20000000E+01
ITER=	2	0.10000000E+01	0.18518518E+01	0.21481481E+01
ITER=	3	0.10000000E+01	0.18601036E+01	0.21606219E+01
ITER=	4	0.10000000E+01	0.18607503E+01	0.21616161E+01
ITER=	5	0.10000000E+01	0.18608015E+01	0.21616952E+01
ITER=	6	0.10000000E+01	0.18608055E+01	0.21617017E+01 $\leftarrow \vec{X}^{(1)}$

converged frequency $= 0.373088 = \omega_1\sqrt{\frac{m}{k}}$

Repetition of the procedure with $[D_2]$ and $[D_3]$ gives the following results:

$$\omega_2\sqrt{\frac{m}{k}} = 1.321324$$

$$\vec{X}^{(2)} = \{1.000000 \quad 0.254098 \quad -0.340706\}$$

$$\omega_3\sqrt{\frac{m}{k}} = 2.028523$$

$$\vec{X}^{(3)} = \{1.000000 \quad -2.114801 \quad 0.678847\}$$

7.29 $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$, $[k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$

$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)
0	0.10000000E+01 0.10000000E+01 0.10000000E+01
1	0.10000000E+01 0.13333334E+01 0.14443334E+01
2	0.10000000E+01 0.13676430E+01 0.14949605E+01
3	0.10000000E+01 0.13705537E+01 0.14994360E+01
4	0.10000000E+01 0.13708007E+01 0.14998223E+01
5	0.10000000E+01 0.13708220E+01 0.14998555E+01
6	0.10000000E+01 0.13708237E+01 0.14998584E+01 ← $\vec{X}^{(1)}$

converged frequency = 0.50828409 = ω_1

0	0.10000000E+01 0.10000000E+01 0.10000000E+01
1	0.10000000E+01-0.56849640E-01-0.61459929E+00
2	0.10000000E+01-0.22656711E-01-0.64602280E+00
3	0.10000000E+01-0.91092410E-02-0.65840477E+00
4	0.10000000E+01-0.38191630E-02-0.66323972E+00
5	0.10000000E+01-0.17635337E-02-0.66511846E+00
6	0.10000000E+01-0.96627331E-03-0.66584712E+00
7	0.10000000E+01-0.65728393E-03-0.66612953E+00
8	0.10000000E+01-0.53756207E-03-0.66623896E+00
9	0.10000000E+01-0.49118389E-03-0.66628140E+00
10	0.10000000E+01-0.47320157E-03-0.66629785E+00
11	0.10000000E+01-0.46624581E-03-0.66630417E+00
12	0.10000000E+01-0.46355074E-03-0.66630661E+00
13	0.10000000E+01-0.46252197E-03-0.66630769E+00
14	0.10000000E+01-0.46209697E-03-0.66630799E+00
15	0.10000000E+01-0.46194042E-03-0.66630810E+00
16	0.10000000E+01-0.46188448E-03-0.66630810E+00
17	0.10000000E+01-0.46187337E-03-0.66630816E+00
18	0.10000000E+01-0.46186216E-03-0.66630822E+00
19	0.10000000E+01-0.46185096E-03-0.66630816E+00
20	0.10000000E+01-0.46185098E-03-0.66630822E+00 ← $\vec{X}^{(2)}$

converged frequency = 1.7323176 = ω_2

0	0.10000000E+01 0.10000000E+01 0.10000000E+01
1	0.10000000E+01-0.23654659E+01 0.15024464E+01
2	0.10000000E+01-0.23733659E+01 0.15024524E+01
3	0.10000000E+01-0.23733664E+01 0.15024524E+01 ← $\vec{X}^{(3)}$

converged frequency = 2.783294 = ω_3

7.30 From problem 6.23 , $[k] = \frac{GJ}{l}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $[J_d] = J_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$[D] = \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix}$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.20000000E+01	0.30000000E+01	
1	0.10000000E+01	0.16000000E+01	0.14000000E+01	
2	0.10000000E+01	0.14736842E+01	0.11052632E+01	
3	0.10000000E+01	0.14328358E+01	0.10298507E+01	
4	0.10000000E+01	0.14199134E+01	0.10086581E+01	
5	0.10000000E+01	0.14159292E+01	0.10025285E+01	
6	0.10000000E+01	0.14147242E+01	0.10007399E+01	
7	0.10000000E+01	0.14143645E+01	0.10002166E+01	
8	0.10000000E+01	0.14142580E+01	0.10000634E+01	
9	0.10000000E+01	0.14142267E+01	0.10000186E+01	
10	0.10000000E+01	0.14142175E+01	0.10000055E+01	
11	0.10000000E+01	0.14142147E+01	0.10000015E+01	← $\vec{X}^{(1)}$

converged frequency = 0.76536608 = ω_1

0	0.10000000E+01	0.20000000E+01	0.30000000E+01	
1	0.10000000E+01	0.29291141E+00	-0.14141805E+01	
2	0.10000000E+01	0.15801974E+00	-0.12234681E+01	
3	0.10000000E+01	0.88471927E-01	-0.11251128E+01	
4	0.10000000E+01	0.50517395E-01	-0.10714371E+01	
5	0.10000000E+01	0.29161688E-01	-0.10412357E+01	
⋮				
27	0.10000000E+01	0.19669531E-05	-0.99999768E+00	
28	0.10000000E+01	0.18924472E-05	-0.99999750E+00	
29	0.10000000E+01	0.18179417E-05	-0.99999750E+00	
30	0.10000000E+01	0.18030403E-05	-0.99999738E+00	
31	0.10000000E+01	0.17732382E-05	-0.99999744E+00	
32	0.10000000E+01	0.17732380E-05	-0.99999726E+00	← $\vec{X}^{(2)}$

converged frequency = 1.4142134 = ω_2

0	0.10000000E+01	0.20000000E+01	0.30000000E+01	
1	0.10000000E+01	-0.14143940E+01	0.10000018E+01	
2	0.10000000E+01	-0.14142135E+01	0.10000004E+01	
3	0.10000000E+01	-0.14142135E+01	0.10000004E+01	← $\vec{X}^{(3)}$

converged frequency = 1.8477590 = ω_3

7.31 From Problem 7.12, $[k] = \frac{T}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[k]^{-1} = \frac{l}{3T}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$[D] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $x_{1,i} = 1$ (given as row vector)	
0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	0.12500000E+01
2	0.10000000E+01	0.10769231E+01
3	0.10000000E+01	0.10250001E+01
4	0.10000000E+01	0.10082645E+01
5	0.10000000E+01	0.10027474E+01
6	0.10000000E+01	0.10009151E+01
7	0.10000000E+01	0.10003049E+01

8	0.10000000E+01	0.10001017E+01
9	0.10000000E+01	0.10000340E+01
10	0.10000000E+01	0.10000113E+01
11	0.10000000E+01	0.10000038E+01 ← $\vec{X}^{(1)}$

Converged frequency = 0.99999809 = $\omega_1 \sqrt{ml/T}$

0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	-0.99991989E+00
2	0.10000000E+01	-0.99998856E+00
3	0.10000000E+01	-0.99998856E+00 ← $\vec{X}^{(2)}$

Converged frequency = 1.7320508 = $\omega_2 \sqrt{ml/T}$

7.32

$$[k]^{-1} = \frac{1}{k}\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 2.0 & 2.0 \\ 0.5 & 1.0 & 2.0 & 3.0 \end{bmatrix}, \quad [D] = \frac{m}{k}\begin{bmatrix} 1.5 & 1.0 & 0.5 & 0.5 \\ 1.5 & 2.0 & 1.0 & 1.0 \\ 1.5 & 2.0 & 2.0 & 2.0 \\ 1.5 & 2.0 & 2.0 & 3.0 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained for $\vec{X}^{(1)}$ and ω_1.

Iteration number (i)	$\vec{X}_{i+1} = [D]\,\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.15714285E+01	0.21428571E+01	0.24285715E+01
2	0.10000000E+01	0.17200000E+01	0.25733335E+01	0.30266669E+01
3	0.10000000E+01	0.17508308E+01	0.26810634E+01	0.31838319E+01
4	0.10000000E+01	0.17574103E+01	0.27059193E+01	0.32208292E+01
5	0.10000000E+01	0.17588729E+01	0.27116063E+01	0.32293591E+01
6	0.10000000E+01	0.17592046E+01	0.27129092E+01	0.32313194E+01
7	0.10000000E+01	0.17592806E+01	0.27132087E+01	0.32317696E+01
8	0.10000000E+01	0.17592978E+01	0.27132771E+01	0.32318730E+01
9	0.10000000E+01	0.17593019E+01	0.27132931E+01	0.32318966E+01

Converged frequency = 0.40058133 = $\omega_1 \sqrt{m/k}$

7.37 The problem-dependent data for Program 9.F and output are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/3.0,-2.0,0.0,-2.0,5.0,-3.0,0.0,-3.0,3.0/
C END OF PROBLEM-DEPENDENT DATA
---------------
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.300000E+01  -0.200000E+01   0.000000E+00
 -0.200000E+01   0.500000E+01  -0.300000E+01
  0.000000E+00  -0.300000E+01   0.300000E+01

EIGEN VALUES ARE
  0.774166E+01   0.300000E+01   0.258343E+00

EIGEN VECTORS
     FIRST           SECOND          THIRD
  0.335734E+00   0.832048E+00   0.441564E+00
 -0.796032E+00  -0.343903E-05   0.605254E+00
  0.503602E+00  -0.554704E+00   0.662336E+00
```

7.38 The problem-dependent data for Program 9.F and output are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/3.0,2.0,1.0,2.0,2.0,1.0,1.0,1.0,1.0/
C END OF PROBLEM-DEPENDENT DATA
-----------------
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
   0.300000E+01    0.200000E+01    0.100000E+01
   0.200000E+01    0.200000E+01    0.100000E+01
   0.100000E+01    0.100000E+01    0.100000E+01

 EIGEN VALUES ARE
   0.504892E+01    0.643104E+00    0.307979E+00

 EIGEN VECTORS
      FIRST            SECOND           THIRD
   0.736978E+00   -0.591004E+00    0.327991E+00
   0.591007E+00    0.327977E+00   -0.736982E+00
   0.327985E+00    0.736984E+00    0.590999E+00
```

7.39 The problem-dependent data to be used in Program 9.F and results are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(4,4),E(4,4)
      DATA N,ITMAX,EPS/4,200,1.0E-05/
      DATA D/4.,-2.,6.,4.,-2.,2.,-1.,3.,6.,-1.,22.,13.,4.,3.,13.,46./
C END OF PROBLEM-DEPENDENT DATA
-----------------------

     EIGENVALUE SOLUTION BY JACOBI METHOD

     GIVEN MATRIX
       0.400000E+01  -0.200000E+01   0.600000E+01
       0.400000E+01
      -0.200000E+01   0.200000E+01  -0.100000E+01
       0.300000E+01
       0.600000E+01  -0.100000E+01   0.220000E+02
       0.130000E+02
       0.400000E+01   0.300000E+01   0.130000E+02
       0.460000E+02

      EIGEN VALUES ARE
       0.525424E+02   0.178109E+02   0.346931E+01   0.177413E+00

      EIGEN VECTORS
         FIRST           SECOND          THIRD
        0.123182E+00  -0.274316E+00   0.718788E+00
        0.626834E+00   0.407039E-01   0.167114E+00
       -0.614583E+00   0.769873E+00   0.407591E+00
       -0.851753E+00  -0.316805E+00  -0.895647E-01
        0.903902E+00   0.413933E+00   0.725754E-01
       -0.797051E-01
```

7.40 The main program which calls DECOMP and results are given.

```
C ==================================================================
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS DECOMP
C
C ==================================================================
      DIMENSION A(4,4),U(4,4)
      DATA A/4.0,-2.0,6.0,4.0,-2.0,2.0,-1.0,3.0,6.0,-1.0,22.0,13.0,
     2 4.0,3.0,13.0,46.0/
      N=4
      CALL DECOMP (A,U,N)
      WRITE (17,10)
10    FORMAT (/,25H UPPER TRIANGULAR MATRIX:,/)
      DO 30 I=1,N
      WRITE (17,20) (U(I,J),J=1,N)
20    FORMAT (3E15.8)
30    CONTINUE
      STOP
      END
---------------------------------
UPPER TRIANGULAR MATRIX:

0.20000000E+01-0.10000000E+01 0.30000000E+01
0.20000000E+01
0.00000000E+00 0.10000000E+01 0.20000000E+01
0.00000000E+00
0.00000000E+00 0.00000000E+00 0.30000000E+01
0.23333333E+01
0.00000000E+00 0.00000000E+00 0.00000000E+00
0.60461192E+01
```

7.41 From Eq. (7.84), $u_{11} = \sqrt{5} = 2.236068$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{-1}{u_{11}} = -0.44721359$,

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{1}{u_{11}} = 0.44721359$$

$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = 2.408319$, $u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12} u_{13}\right) = -1.5778641$

$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = 0.55708611$

$$[U] = \begin{bmatrix} 2.236068 & -0.44721359 & 0.44721359 \\ 0 & 2.408319 & -1.5778641 \\ 0 & 0 & 0.55708611 \end{bmatrix}$$

$\alpha_{11} = \frac{1}{u_{11}} = 0.44721359$ | $\alpha_{12} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{22}\right) = 0.083045475$

$\alpha_{22} = \frac{1}{u_{22}} = 0.41522738$ | $\alpha_{23} = -\frac{1}{u_{22}}\left(u_{23}\alpha_{33}\right) = 1.1760702$

$\alpha_{33} = \frac{1}{u_{33}} = 1.7950547$ | $\alpha_{13} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{23} + u_{13}\alpha_{33}\right) = -0.12379687$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.44721359 & 0.083045475 & -0.12379687 \\ 0 & 0.41522738 & 1.1760702 \\ 0 & 0 & 1.7950547 \end{bmatrix}$$

7.42 From Eqs. (7.84) and (7.86),

$u_{11} = \sqrt{2} = 1.4142135$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{5}{u_{11}} = 3.5355339$, $u_{13} = \frac{a_{13}}{u_{11}} = 5.6568542$

$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = 1.8708287$, $u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12} u_{13}\right) = 4.2761798$

$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = 1.9272485$

$$[U] = \begin{bmatrix} 1.4142135 & 3.5355339 & 5.6568542 \\ 0 & 1.8708287 & 4.2761798 \\ 0 & 0 & 1.9272485 \end{bmatrix}$$

$\alpha_{11} = \frac{1}{u_{11}} = 0.70710677$ | $\alpha_{12} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{22}\right) = -1.3363062$

$\alpha_{22} = \frac{1}{u_{22}} = 0.53452247$ | $\alpha_{23} = -\frac{1}{u_{22}}\left(u_{23}\alpha_{33}\right) = -1.1859988$

$\alpha_{33} = \frac{1}{u_{33}} = 0.51887447$ | $\alpha_{13} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{23} + u_{13}\alpha_{33}\right) = 0.88949931$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.70710677 & -1.3363062 & 0.88949931 \\ 0 & 0.53452247 & -1.1859988 \\ 0 & 0 & 0.51887447 \end{bmatrix}$$

7.43 $[k] = k\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $[m] = m\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Eq. (7.84) gives, for $[A] = [k]$,

$$[U] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1.41421 & -0.707107 & 0 \\ 0 & 0 & 1.22474 & -0.816497 \\ 0 & 0 & 0 & 0.577350 \end{bmatrix}\sqrt{k}$$

Eq. (7.86) gives

$$[U]^{-1} = \begin{bmatrix} 0.5 & 0.35355338 & 0.20412414 & 0.28867510 \\ 0 & 0.70710677 & 0.40824828 & 0.57735020 \\ 0 & 0 & 0.81649655 & 1.1547004 \\ 0 & 0 & 0 & 1.7320507 \end{bmatrix}\frac{1}{\sqrt{k}}$$

standard eigenproblem is $[D]\vec{Y} = \lambda\vec{Y}$

where

$$[D] = \left([U]^{-1}\right)^T [m]\, [U]^{-1}$$

$$= \frac{m}{k}\begin{bmatrix} 0.75 & 0.53033006 & 0.30618620 & 0.43301266 \\ 0.53033006 & 1.3749999 & 0.79385662 & 1.1226827 \\ 0.30618620 & 0.79385662 & 1.125 & 1.5909899 \\ 0.43301266 & 1.1226827 & 1.5909899 & 5.2499990 \end{bmatrix}$$

$\lambda = 1/\omega^2$

and $\vec{Y} = [U]\vec{X}$

7.44 From Eq. (7.84),

$$u_{11} = \sqrt{a_{11}} = \sqrt{16} = 4, \quad u_{12} = \frac{a_{12}}{u_{11}} = -\frac{20}{4} = -5, \quad u_{13} = \frac{a_{13}}{u_{11}} = -\frac{24}{4} = -6$$

$$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = (89 - 25)^{1/2} = 8$$

$$u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}u_{13}\right) = \frac{1}{8}\left(-50 - (-5)(-6)\right) = -10$$

$$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = (280 - 36 - 100)^{1/2} = 12$$

$$[A] = [U]^T [U]$$

with

$$[U] = \begin{bmatrix} 4 & -5 & -6 \\ 0 & 8 & -10 \\ 0 & 0 & 12 \end{bmatrix}$$

7.45

```
% Ex7_45.m
>> A = [3 -2 0; -2 5 -3; 0 -1 1]

A =

     3    -2     0
    -2     5    -3
     0    -1     1

>> [V, D] = eig(A)

V =

   -0.4765   -0.8962    0.4154
    0.8656   -0.3444    0.5950
   -0.1537    0.2797    0.6880


D =

    6.6334         0         0
         0    2.2315         0
         0         0    0.1351
```

7.46

```
% Ex7_46.m
>> A = [-5 2 1; 1 -9 -1; 2 -1 7]

A =

    -5     2     1
     1    -9    -1
     2    -1     7

>> [V, D] = eig(A)

V =

   -0.0723   -0.9572    0.4172
    0.0570   -0.2514   -0.9027
   -0.9958    0.1431   -0.1048


D =

    7.2024         0         0
         0   -4.6241         0
         0         0   -9.5783
```

7.47

```
% Results of Ex7_47
>> program9
Eigenvalue solution by Jacobi Method

Given matrix
2.50000000e+000     -1.00000000e+000     0.00000000e+000
-1.00000000e+000     5.00000000e+000    -1.41421356e+000
0.00000000e+000     -1.41421356e+000     1.00000000e+001

Eigen values are
1.03806779e+001      5.00000000e+000     2.11932209e+000

Eigen vactors are
   First               Second              Third
3.29649826e-002      3.59210604e-001     9.32674140e-001
-2.59786425e-001     -8.98026512e-001     3.55048443e-001
9.65103271e-001     -2.54000247e-001     6.37146104e-002
```

7.48

```
% Results of Ex7_48
>> program10
Solution of eigenvalue problem by
matrix iteration method

Natural frequencies:

    4.450417e-001     1.246977e+000     1.801938e+000

Mode shapes (Columnwise):

    1.000000e+000     1.000000e+000     1.000000e+000
    8.019379e-001    -5.549503e-001    -2.246941e+000
    4.450421e-001    -1.246987e+000     1.801867e+000
```

7.49

```
% Results of Ex7_49
>> program11
Upper triangular matrix [U]:

 2.000000e+000   -1.000000e+000    0.000000e+000
 0.000000e+000    1.414214e+000   -7.071068e-001
 0.000000e+000    0.000000e+000    1.224745e+000
 0.000000e+000    0.000000e+000    0.000000e+000

Inverse of the upper triangular matrix:

 5.000000e-001    3.535534e-001    2.041241e-001
 0.000000e+000    7.071068e-001    4.082483e-001
 0.000000e+000    0.000000e+000    8.164966e-001
 0.000000e+000    0.000000e+000    0.000000e+000

Matrix [UMU]=[UTI][M][UI]:

 7.500000e-001    5.303301e-001    3.061862e-001
 5.303301e-001    1.375000e+000    7.938566e-001
 3.061862e-001    7.938566e-001    1.125000e+000
 4.330127e-001    1.122683e+000    1.590990e+000

Eigenvalues:

 6.231904e+000    1.431905e+000    5.000000e-001    3.361911e-001

Eigenvectors(Columnwise):

 4.804506e-001   -4.370337e-001    2.672613e-001   -8.209847e-002
 8.452583e-001   -4.162514e-001   -2.672612e-001    2.021007e-001
 1.303605e+000    2.067115e-001   -2.672612e-001   -4.318039e-001
 1.552770e+000    6.853100e-001    2.672613e-001    2.186930e-001
```

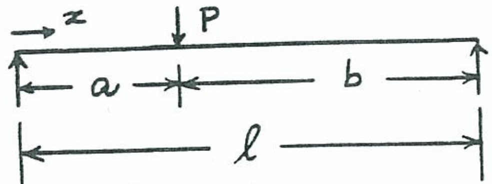
7.50

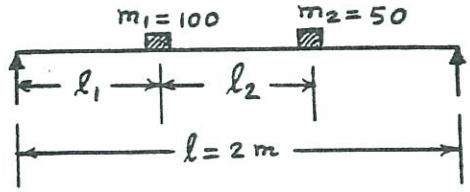
```
Results of Ex7_50
*********************

Please input N:
3

Please input matrix D row by row:
2 2 2
2 5 5
2 5 12


EIGENVALUE SOLUTION BY JACOBI METHOD

 GIVEN MATRIX
    2.000000    2.000000     2.000000
    2.000000    5.000000     5.000000
    2.000000    5.000000    12.000000

EIGEN VALUES ARE
     15.15868265      2.87890731       0.96241005

EIGEN VECTORS
           FIRST             SECOND           THIRD
      0.20164232       -0.49011982       0.84801117
      0.46419929       -0.71456420      -0.52337082
      0.86247284        0.49917989       0.08342689
```

7.51

```
Results of Ex7_51
*****************************************************************

Please input ND:
3
Please input BK matrix row by row:
10 -4 0
-4 6 -2
0 -2 2
Please input BM matrix row by row:
3 0 0
0 2 0
0 0 1

UPPER TRIANGULAR MATRIX [U]:

      3.16227766     -1.26491106       0.00000000
      0.00000000      2.09761770      -0.95346259
      0.00000000      0.00000000       1.04446594

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

      0.31622777      0.19069252       0.17407766
      0.00000000      0.47673129       0.43519414
      0.00000000      0.00000000       0.95742711

MATRIX [UMU] = [UTI][M][UI]:

      0.30000000      0.18090681       0.16514456
      0.18090681      0.56363636       0.51452725
      0.16514456      0.51452725       1.38636364


EIGENVALUES:

      1.67156962      0.38337151       0.19505887

EIGENVECTORS (COLUMNWISE):

      0.28923075     -0.23770397       0.16281604
      0.59330466     -0.12923308      -0.21898643
      0.84651373      0.42480537       0.14007703
```

7.52





Basic relationship:

$$w(x) = \begin{cases} \dfrac{P b x}{6EIl}(l^2 - b^2 - x^2); & 0 \le x \le a \qquad (E_1) \\[2ex] -\dfrac{Pa(l-x)}{6EIl}(a^2 + x^2 - 2lx); & a \le x \le l \qquad (E_2) \end{cases}$$

Deflection of mass m_1 due to load $m_1 g$:

Using $x = l_1$, $b = 2 - l_1$ and $l = 2$ in (E_1):

$$w_1' = \frac{(100 \times 9.81)(2 - l_1)\, l_1}{6EI(2)} \{4 - (2-l_1)^2 - l_1^2\} = \frac{981\, l_1^2 (2-l_1)^2}{6EI} \qquad (E_3)$$

Deflection of mass m_2 due to load $m_1 g$:

Using $x = l_1 + l_2$, $a = l_1$, $b = 2 - l_1$, $l = 2$ in (E_2):

$$w_2' = -\frac{(100 \times 9.81)\, l_1 (2 - l_1 - l_2)}{6EI(2)} \{l_1^2 + (l_1 + l_2)^2 - 2(2)(l_1 + l_2)\}$$

$$= -\frac{981\, l_1 (2 - l_1 - l_2)}{12EI}(2l_1^2 + l_2^2 + 2 l_1 l_2 - 4 l_1 - 4 l_2) \qquad (E_4)$$

Deflection of mass m_1 due to load $m_2 g$:

Using $x = l_1$, $l = 2$, $b = (2 - l_1 - l_2)$ in (E_1):

$$w_1'' = \frac{(50 \times 9.81)(2 - l_1 - l_2)\, l_1}{6EI(2)} \{4 - (2 - l_1 - l_2)^2 - l_1^2\}$$

$$= \frac{490.5\, l_1 (2 - l_1 - l_2)(-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1 l_2)}{12EI} \qquad (E_5)$$

Deflection of mass m_2 due to load $m_2 g$:

Using $x = l_1 + l_2$, $l = 2$ and $b = 2 - l_1 - l_2$ in (E_1):

$$w_2'' = \frac{(50 \times 9.81)(2-l_1-l_2)(l_1+l_2)}{6EI(2)} \left\{ 4 - (2-l_1-l_2)^2 - (l_1+l_2)^2 \right\}$$

$$= \frac{490.5\,(l_1+l_2)(2-l_1-l_2)(-2l_1^2-2l_2^2+4l_1+4l_2-4l_1l_2)}{12EI} \qquad (E_6)$$

Total deflection of masses m_1 and m_2 are:

$$w_1 = w_1' + w_1'' = \frac{981\, l_1^2 (2-l_1)^2}{6EI} + $$

$$+ \frac{490.5\, l_1 (2-l_1-l_2)(-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1l_2)}{12EI} \qquad (E_7)$$

$$w_2 = w_2' + w_2'' = \frac{-981\, l_1 (2-l_1-l_2)(2l_1^2 + l_2^2 + 2l_1l_2 - 4l_1 - 4l_2)}{12EI}$$

$$+ \frac{490.5\,(l_1+l_2)(2-l_1-l_2)(-2l_1^2-2l_2^2+4l_1+4l_2-4l_1l_2)}{12EI} \qquad (E_8)$$

Fundamental natural frequency is given by

$$\omega = \left\{ \frac{g(m_1 w_1 + m_2 w_2)}{(m_1 w_1^2 + m_2 w_2^2)} \right\}^{\frac{1}{2}} = 3.1321 \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right) \qquad (E_9)$$

To maximize ω, we can maximize ω^2.

Problem is: Find l_1 and l_2

to maximize $f = \left(\dfrac{2w_1 + w_2}{2w_1^2 + w_2^2} \right)$

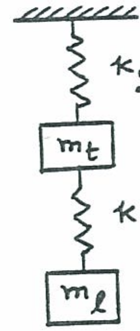
where w_1 and w_2 are given by (E_7) and (E_8).

Problem can be solved as follows:

Treat f as a function of l_1 and l_2.

Set $\dfrac{\partial f}{\partial l_1} = 0$ and $\dfrac{\partial f}{\partial l_2} = 0 \qquad (E_{10})$

Solve Eqs. (E_{10}) for l_1 and l_2.



7.53 Stiffness of one girder (simply supported beam)

$$k_{g1} = \frac{48\,E\,I}{\ell^3} = \frac{48\,(30\times10^6)\left(\frac{1}{12}a^4\right)}{(30\times12)^3} = 2.572\,a^4$$

where a = width and depth of cross-section (inch) of girder.

$$k_g = 2\,k_{g1} = 5.144\,a^4 \quad lb/in$$

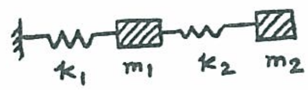
$$m_t = \text{mass of trolley} = \frac{40000}{386.4} = 103.5197 \ lb\text{-}sec^2/in$$

$$\text{stiffness of rope} = k_r = \frac{AE}{\ell} = \frac{\frac{\pi d^2}{4}(30\times10^6)}{(20\times12)} = 98175\,d^2$$

where d = diameter of the rope (inch).

$$m_\ell = \text{mass of lifted load} = \frac{10000}{386.4} = 25.8799 \ lb\text{-}sec^2/in$$

From section 5.3, the natural frequencies of a 2 d.o.f. system (shown in adjacent figure) are given by



$$\omega_{1,2}^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)\,m_2 + k_2 m_1}{m_1 m_2}\right\} \mp \left[\left\{\frac{(k_1+k_2)m_2 + k_2 m_1}{m_1 m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)k_2 - k_2^2}{m_1 m_2}\right\}\right]^{1/2} \qquad (E_1)$$

Here $k_1 = k_g$, $k_2 = k_r$, $m_1 = m_t$, $m_2 = m_\ell$ and hence ω_1^2 and ω_2^2 can be expressed as functions of a and d:

$$\omega_1^2 = \omega_1^2(a, d) \quad ; \quad \omega_2^2 = \omega_2^2(a, d) \qquad (E_2)$$

Requirement is

$$\omega_1^2(a, d) > (157.08)^2 \quad ; \quad \omega_2^2(a, d) > (157.08)^2 \qquad (E_3)$$

since 1500 rpm = 157.08 rad/sec.

Choose a and d such that the inequalities (E_3) are satisfied. A trial and error procedure can be used for this purpose.