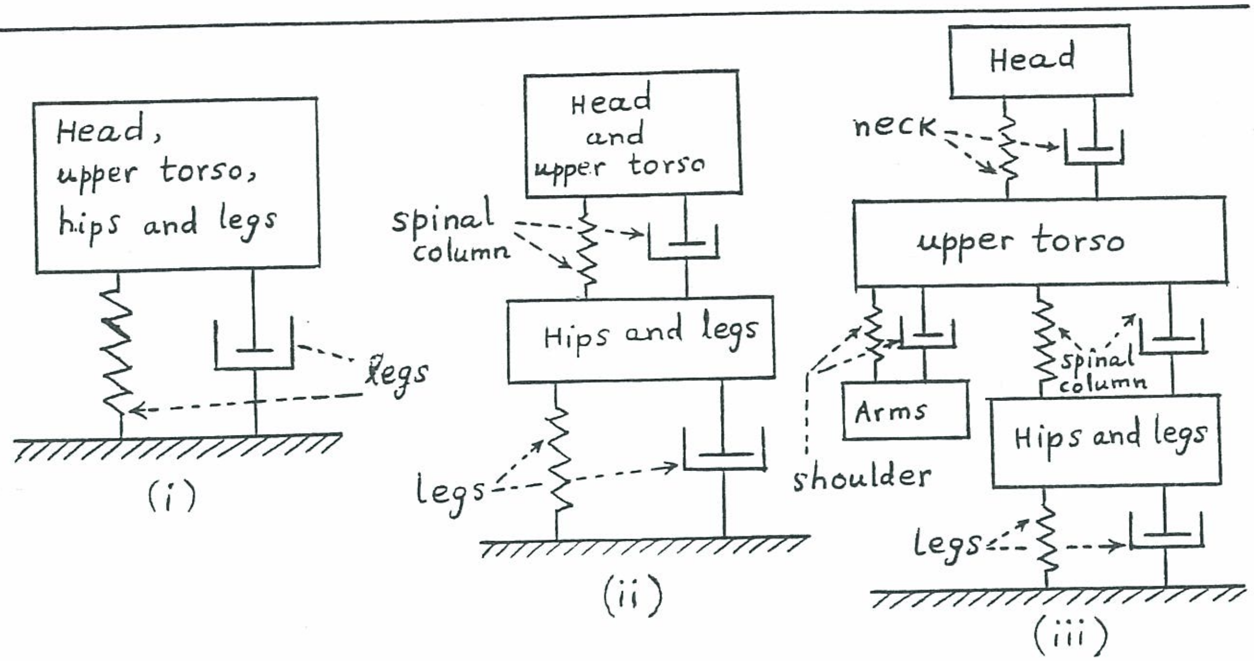
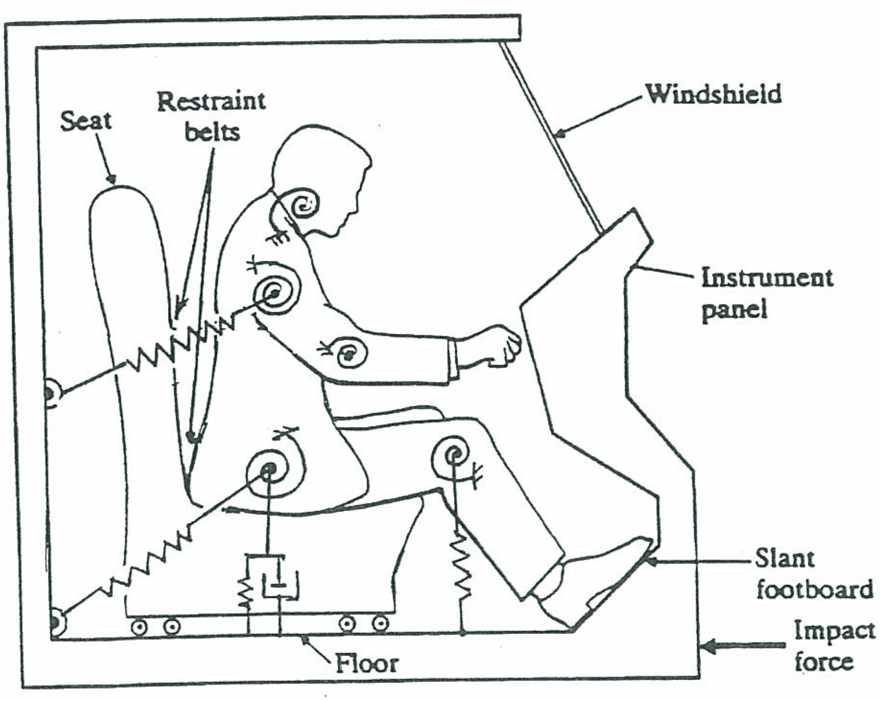
Chapter 1

Fundamentals of Vibration

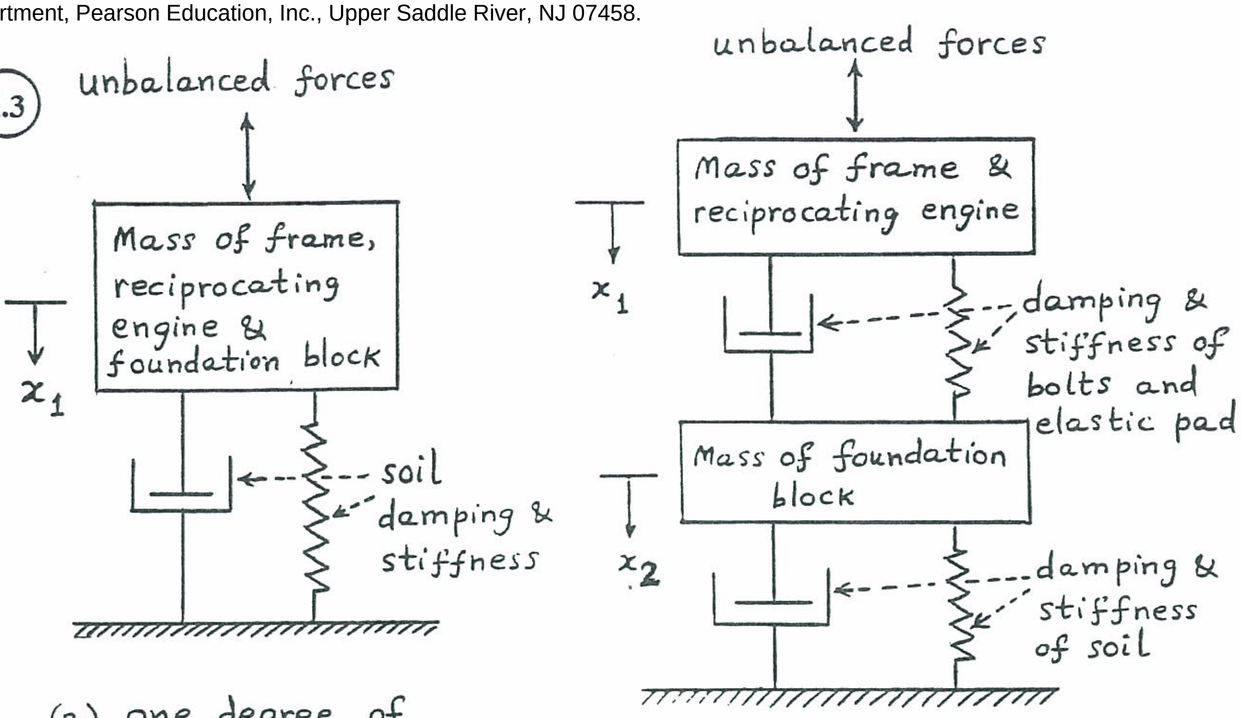
1.1



1.2



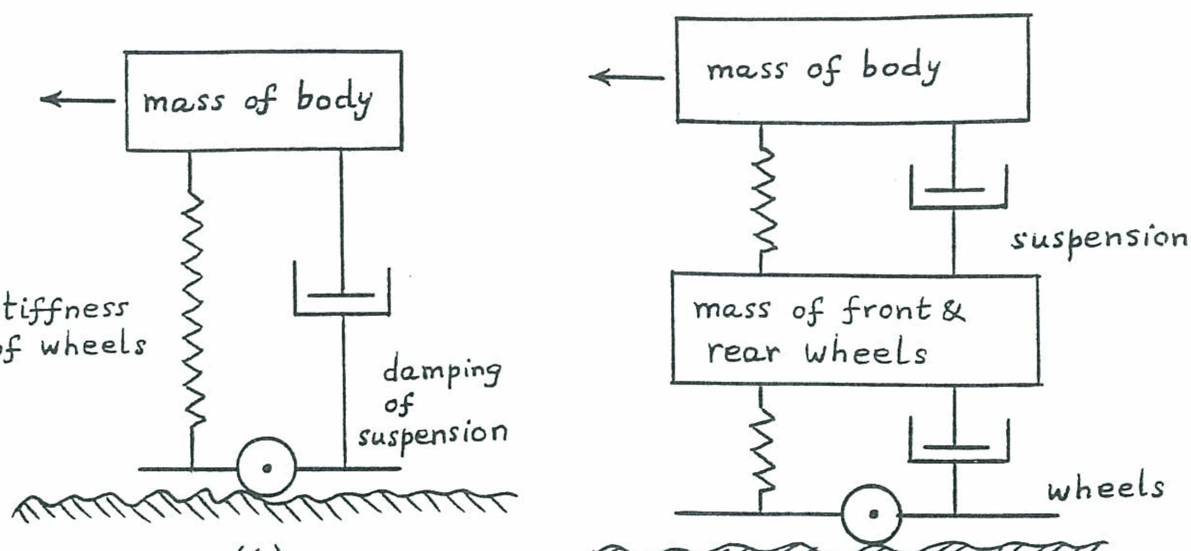
1.3



(a) one degree of freedom model

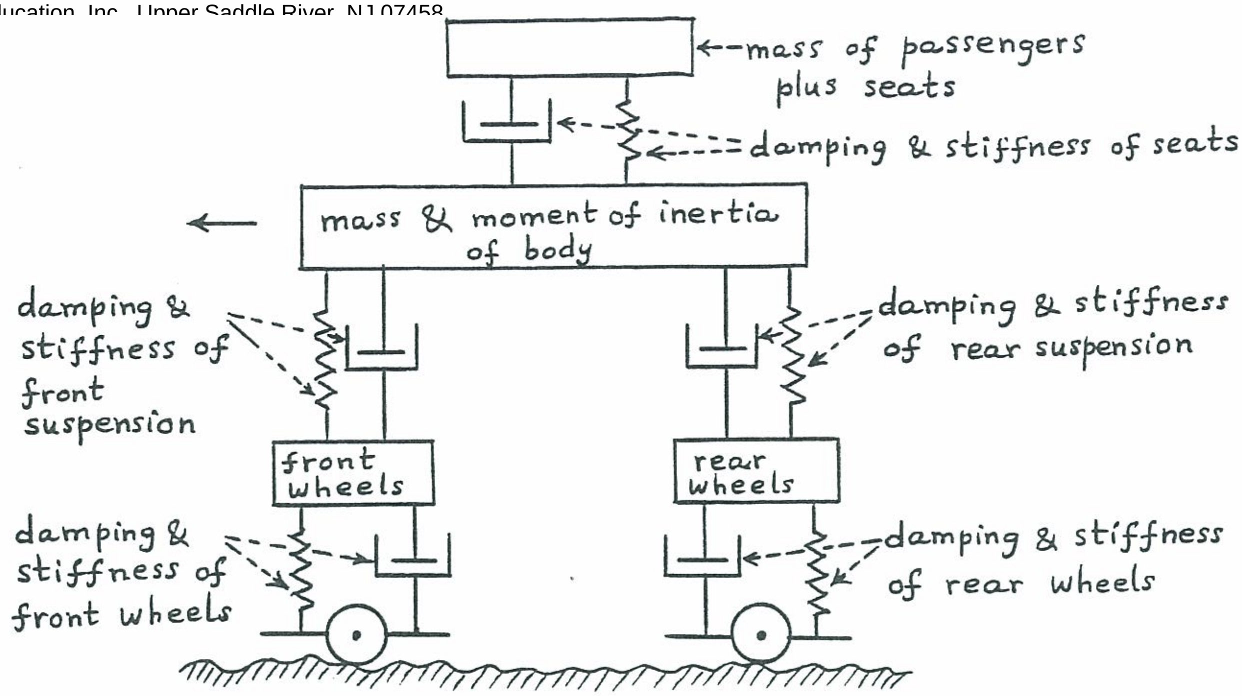
(b) Two degree of freedom model

1.4



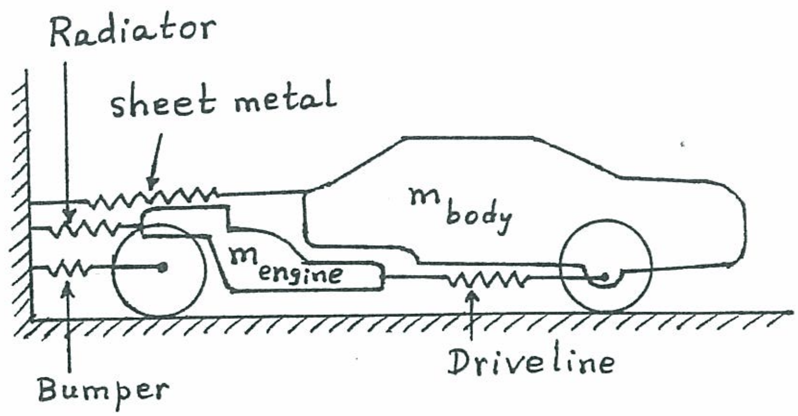
(i)

(ii)

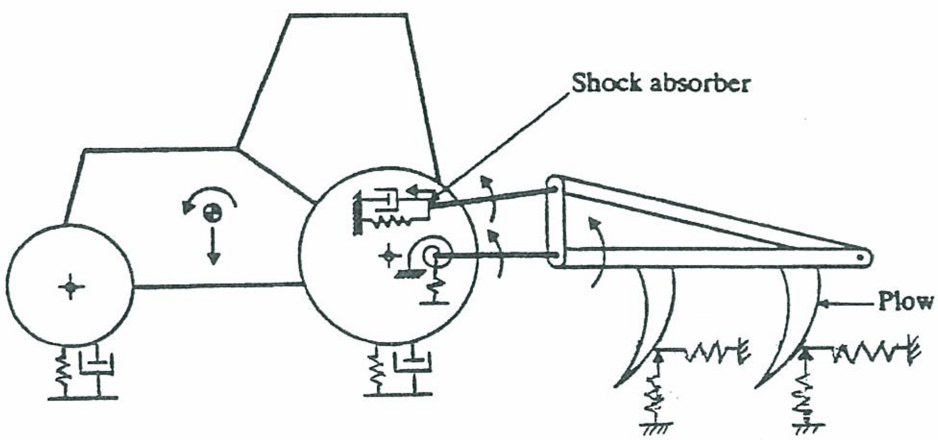

mass of passengers plus seats
damping & stiffness of seats
mass & moment of inertia of body
damping & stiffness of front suspension
damping & stiffness of rear suspension
front wheels
rear wheels
damping & stiffness of front wheels
damping & stiffness of rear wheels

(iii)

1.5

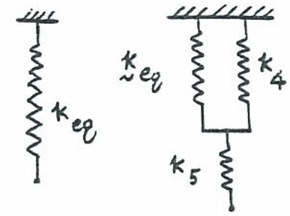

Radiator
sheet metal
m_{body}
m_{engine}
Bumper
Driveline

1.6


Shock absorber
Plow

1.7

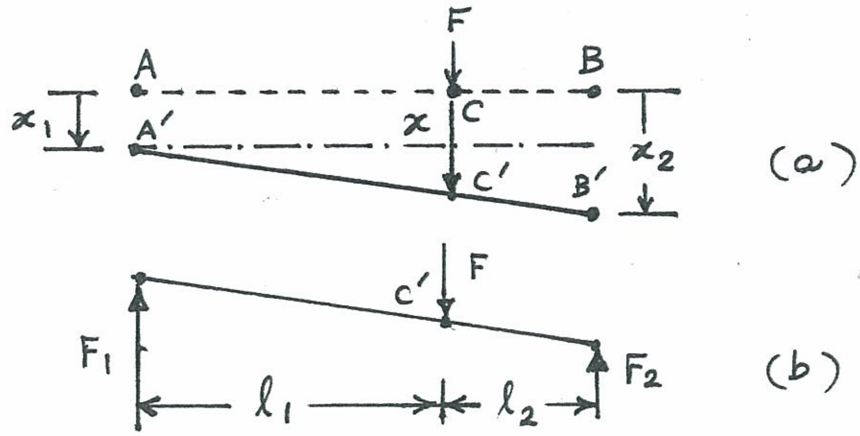
$$\frac{1}{\tilde{k}_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} \quad ; \quad \tilde{k}_{eq} = \left(\frac{2k_1k_2k_3}{k_2k_3 + 2k_1k_3 + k_1k_2}\right)$$



$$\frac{1}{k_{eq}} = \frac{1}{\tilde{k}_{eq} + k_4} + \frac{1}{k_5}$$

$$k_{eq} = \frac{k_5(\tilde{k}_{eq} + k_4)}{k_5 + k_4 + \tilde{k}_{eq}} = \frac{k_2k_3k_4k_5 + 2k_1k_3k_4k_5 + k_1k_2k_4k_5 + 2k_1k_2k_3k_5}{k_2k_3k_4 + k_2k_3k_5 + 2k_1k_3k_4 + 2k_1k_3k_5 + k_1k_2k_4 + k_1k_2k_5 + 2k_1k_2k_3}$$

1.8



From Fig. (a),
$$x = x_1 + \frac{l_1}{l_1+l_2}(x_2 - x_1)$$

$$= \frac{l_2}{l_1+l_2}\, x_1 + \frac{l_1}{l_1+l_2}\, x_2 \qquad (1)$$

Vertical force equilibrium from Fig. (b):
$$F = F_1 + F_2 \qquad (2)$$

Moment equilibrium about C′ (Fig. (b)):
$$F_2\, l_2 = F_1\, l_1 \qquad (3)$$

Solution of Eqs. (2) and (3):
$$F_1 = \frac{F\, l_2}{l_1 + l_2}, \quad F_2 = \frac{F\, l_1}{l_1 + l_2} \qquad (4)$$

Displacements of springs k_1 and k_2 are given by
$$x_1 = \frac{F_1}{k_1} = \frac{F\, l_2}{k_1\,(l_1+l_2)}, \quad x_2 = \frac{F_2}{k_2} = \frac{F\, l_1}{k_2\,(l_1+l_2)} \qquad (5)$$

Displacement of force F can be found using Eqs. (5) in Eq. (1):
$$x = \frac{l_2}{l_1+l_2}\cdot\frac{F\, l_2}{k_1\,(l_1+l_2)} + \frac{l_1}{l_1+l_2}\cdot\frac{F\, l_1}{k_2\,(l_1+l_2)}$$

$$= \frac{F}{(l_1+l_2)^2}\left(\frac{l_1^2\, k_1 + l_2^2\, k_2}{k_1\, k_2}\right) \qquad (6)$$

The equivalent spring constant of the system in the direction of x, k_e, is given by Eq. (6):

$$k_e = \frac{F}{x} = \frac{(l_1 + l_2)^2 \, k_1 k_2}{l_1^2 k_1 + l_2^2 k_2} \qquad (7)$$

1.9 Equivalence of potential energies gives

$$\frac{1}{2} k_{t1}\theta^2 + \frac{1}{2} k_{t2}\theta^2 + \frac{1}{2} k_1 (\theta l_1)^2 + \frac{1}{2} k_2 (\theta l_1)^2 + \frac{1}{2} k_3 (\theta l_2)^2 = \frac{1}{2} k_{eq}\theta^2$$

$$\therefore\ k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

1.10 k_{123} = for series springs k_1, k_2 and k_3:

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} \ ; \qquad k_{123} = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

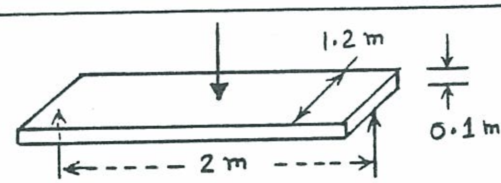
Using energy equivalence,

$$\frac{1}{2} k_{eq}\theta^2 = \frac{1}{2} k_4 \theta^2 + \frac{1}{2} k_{123}\theta^2 + \frac{1}{2} k_5 (\theta R)^2 + \frac{1}{2} k_6 (\theta R)^2$$

$$\therefore\ k_{eq} = k_4 + k_{123} + R^2 k_5 + R^2 k_6$$

$$= k_4 + \left(\frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}\right) + R^2 (k_5 + k_6)$$

1.11 For simply supported beam, for load at middle,



$$k_1 = \frac{48 EI}{l^3} = \frac{48 (2.06 \times 10^{11})(10^{-4})}{8}$$

$= 12.36 \times 10^7$ N/m where $I = \frac{1}{12}(1.2)(0.1)^3 = 10^{-4}\ m^4$.

$$\delta_1 = \text{original deflection} = \frac{mg}{k_1} = \frac{500 \times 9.81}{12.36 \times 10^7} = 396.8447 \times 10^{-7}\ m$$

When spring k is added, $k_{eq} = k + k_1$

(a) New deflection $= \dfrac{mg}{k_{eq}} = \dfrac{\delta_1}{4}$; $k_{eq} = \dfrac{4\,mg}{\delta_1} = 4 k_1$
$= k + k_1$

$\therefore\ k = 3 k_1 = 37.08 \times 10^7$ N/m

(b) New deflection $= \dfrac{mg}{k_{eq}} = \dfrac{\delta_1}{2}$; $k_{eq} = \dfrac{2\,mg}{\delta_1} = 2 k_1$
$= k + k_1$

$\therefore\ k = k_1 = 12.36 \times 10^7$ N/m

(c) New deflection $= \dfrac{mg}{k_{eq}} = \dfrac{3}{4}\delta_1$; $k_{eq} = \dfrac{4\,mg}{3\,\delta_1} = \dfrac{4}{3} k_1$
$= k + k_1$

$\therefore\ k = \frac{1}{3} k_1 = 4.12 \times 10^7$ N/m

1.12 For a bar with length L, Young's modulus E and cross-section A, the axial stiffness (k) is given by

$$k = \frac{AE}{L} \qquad (1)$$

when cross-section is solid circular with diameter d,

$$\text{area} = A_1 = \pi d^2/4 \qquad (2)$$

when cross-section is square with side d,

$$\text{area} = A_2 = d^2 \qquad (3)$$

When cross-section is hollow circular with mean dia. d and wall thickness $t = 0.1d$,

$$\text{area} = \pi d t = \pi d (0.1 d) = 0.1 \pi d^2 \qquad (4)$$

For specified value of $k = \bar{k}$, cross-section area required is:

$$A = \frac{\bar{k} L}{E} = c \text{ (constant)} \qquad (5)$$

Weight of bar :

with solid circular section:

$$W_1 = \frac{\pi d^2}{4} L = c L \quad \text{with} \quad d^2 = \frac{4C}{\pi} \qquad (6)$$

with hollow circular section:

$$W_3 = 0.1 \pi d^2 L = 0.1 \pi \left(\frac{4c}{\pi}\right) L = 0.4 c L \qquad (7)$$
$$= 0.4 W_1$$

with square section:

$$W_2 = d^2 L = \frac{4c}{\pi} L = \frac{4}{\pi} W_1 = 1.2732 W_1 \qquad (8)$$

$\therefore$ The shaft with the hollow circular cross-section corresponds to minimum weight.

1.13

Stiffness of a cantilever beam under a bending force at free end:

$$k = \frac{3EI}{l^3} \quad (1)$$

For a specified value of $k = \bar{k}$,

$$I = \frac{\bar{k}\, l^3}{3E} = c = \text{constant} \quad (2)$$

For a solid circular section with diameter d,

$$I_1 = \frac{\pi d^4}{64} = c \quad \Rightarrow \quad d^4 = \frac{64\, c}{\pi} \quad \text{or } d^2 = \sqrt{\frac{64\, c}{\pi}} \quad (3)$$

$$\text{weight of beam} = W_1 = \frac{\pi d^2 l}{4} = \frac{\pi l}{4} \sqrt{\frac{64\, c}{\pi}}$$

$$= 3.5449\, l \sqrt{c} \quad (4)$$

For a hollow circular section with mean diameter d and wall thickness $t = 0.1\, d$, weight of beam (W_2) is:

$$W_2 = \frac{\pi}{4} \left(d_o^4 - d_i^4 \right) l = \frac{\pi l}{4} \left\{ (d+t)^2 - (d-t)^2 \right\}$$

$$= \frac{\pi l}{4} (4\, d t) = \pi\, d t\, l = \pi l \left(0.1\, d^2 \right)$$

$$= 0.1\, \pi\, l \sqrt{\frac{64 c}{\pi}} = 1.4180\, l \sqrt{c} \quad (5)$$

For a square section with side d, weight of the beam (W_3) is:

$$W_3 = d^2 l = l \sqrt{\frac{64\, c}{\pi}} = 4.5135\, l \sqrt{c} \quad (6)$$

By comparing Eqs. (4), (5) and (6), the minimum weight beam corresponds to the hollow circular cross-section.

1.14 Spring force is given by $F = 800\,x + 40\,x^3$ (1)

static equilibrium of the rubber mounting (x^*) under the weight of the electronic instrument is given by

$$F = 200 = 800\,x^* + 40\,x^{*3}$$

or

$$40\,x^{*3} + 800\,x^* - 200 = 0 \quad (2)$$

The roots of the cubic equation (2) can be found from MATLAB as

$$x^* = 0.2492,\ -0.1246 \pm 4.4773\,i \quad (3)$$

Thus the static equilibrium position of the rubber mounting is given by the real root of Eq. (2):

$$x^* = 0.2492 \text{ in} \quad (4)$$

(a) Equivalent linear spring constant of rubber mounting at its static equilibrium position, using Eq. (1.7), is:

$$k_{eq} = \left.\frac{dF}{dx}\right|_{x^*} = 800 + 120\,x^{*2} = 800 + 1200\,(0.2492)^2$$

$$= 807.4521 \text{ lb/in} \quad (5)$$

(b) Deflection of rubber mounting corresponding to the equivalent linear spring constant is:

$$k = \frac{F}{k_{eq}} = \frac{200}{807.4521} = 0.2477 \text{ in} \quad (6)$$

1.15 $F(x) = 200x + 50x^2 + 10x^3$ (1)

When the spring undergoes a steady deflection of $x^* = 0.5$ in during the operation of the engine, the force exerted on the spring can be found as

$$F = 200(0.5) + 50(0.5)^2 + 10(0.5)^3 = 113.75 \text{ lb} \quad (2)$$

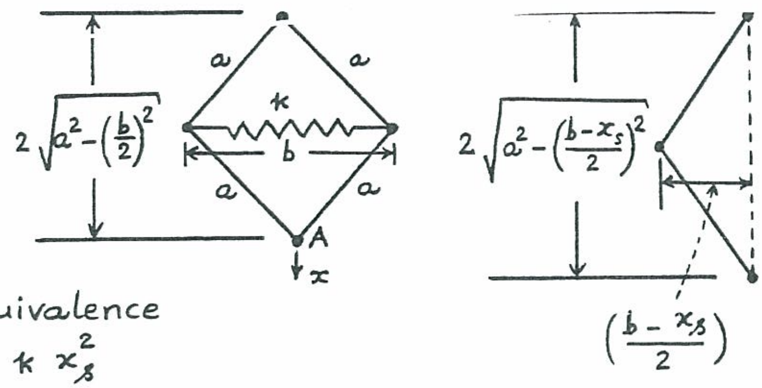
Equivalent linear spring constant at its steady deflection is given by Eq. (1.7):

$$k_{eq} = \left.\frac{dF}{dx}\right|_{x=x^*} = 200 + 100\,x^* + 30\,x^{*2}$$

$$= 200 + 100(0.5) + 30(0.5)^2$$

$$= 253.75 \text{ lb/in}$$

1.16 (a) x = downward deflection of point A,
x_s = resulting deformation of spring



Potential energy equivalence gives $\frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} k\, x_s^2$

$$k_{eq} = k\left(\frac{x_s}{x}\right)^2$$

But
$$x = 2\left[\sqrt{a^2-\left(\frac{b-x_s}{2}\right)^2} - \sqrt{a^2-\left(\frac{b}{2}\right)^2}\,\right]$$

$$= 2\sqrt{a^2-\left(\frac{b}{2}\right)^2}\left[\left\{\frac{a^2-\left\{\frac{b}{2}\left(1-\frac{x_s}{b}\right)\right\}^2}{a^2-\left(\frac{b}{2}\right)^2}\right\}^{1/2} - 1\right]$$

$$= 2\sqrt{a^2-\frac{b^2}{4}}\left[\left\{\frac{\left(a^2-\frac{b^2}{4}-\frac{x_s^2}{4}+\frac{b\,x_s}{2}\right)}{\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

$$= 2\sqrt{a^2-\frac{b^2}{4}}\left[\left\{1-\frac{\overbrace{x_s^2}^{\approx 0}}{\left(a^2-\frac{b^2}{4}\right)} + \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

Using the relation $(1+\theta)^{1/2} \simeq 1+\frac{\theta}{2}$, we obtain

$$x = 2\left(a^2-\frac{b^2}{4}\right)^{1/2}\left[1+\frac{b\,x_s}{4\left(a^2-\frac{b^2}{4}\right)} - 1\right] = \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)^{1/2}}$$

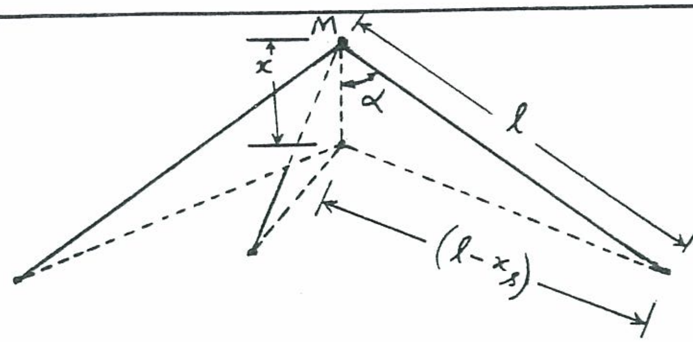
$$\therefore\ k_{eq} = k\left(\frac{x_s}{x}\right)^2 = 4k\left(\frac{a^2-\frac{b^2}{4}}{b^2}\right) = k\left(\frac{4a^2-b^2}{b^2}\right)$$

(b) Here $x = x_s$ (spring deflection)

$$\therefore\ k_{eq} = k$$

1.17 Let x = vertical displacement of mass M,

x_s = resulting deformation of each inclined spring.



From equivalence of potential energy,

$$\frac{1}{2} k_{eq}\, x^2 = 3 \left(\frac{1}{2} k\, x_s^2 \right) \quad ; \quad k_{eq} = 3 k \left(\frac{x_s}{x} \right)^2$$

From geometry, $$(l - x_s)^2 = l^2 + x^2 - 2 l x \cos\alpha$$

$$x^2 - 2x\, l \cos\alpha + 2 l\, x_s - x_s^2 = 0 \qquad (E_1)$$

Solving (E_1), $$x = l \cos\alpha \left[1 \pm \left\{ 1 - \frac{(2 l x_s - x_s^2)}{l^2 \cos^2\alpha} \right\}^{1/2} \right] \qquad (E_2)$$

Using the relation $\sqrt{1-\theta} \approx 1 - \frac{\theta}{2}$, (E_2) can be rewritten as

$$x = l \cos\alpha \left[1 \pm \left\{ 1 - \left(\frac{2 l x_s - x_s^2}{2 l^2 \cos^2\alpha} \right) \right\} \right] \qquad (E_3)$$

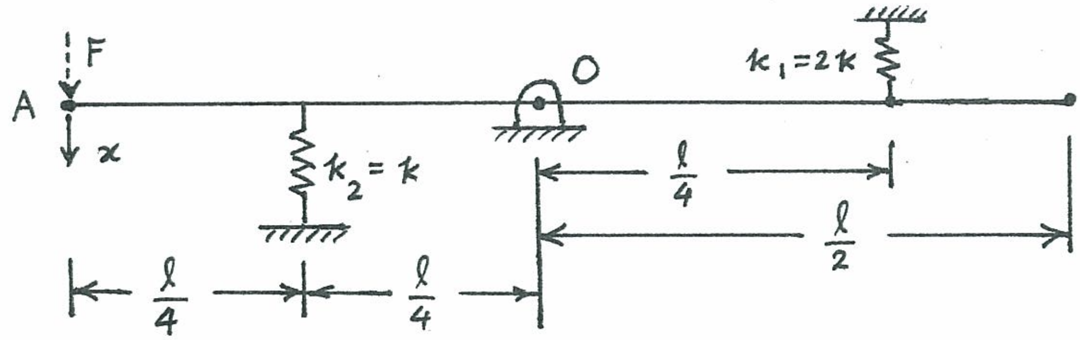
Assuming x to be small, we use minus sign and neglect x_s^2 compared to $2 l\, x_s$ in (E_3). This gives

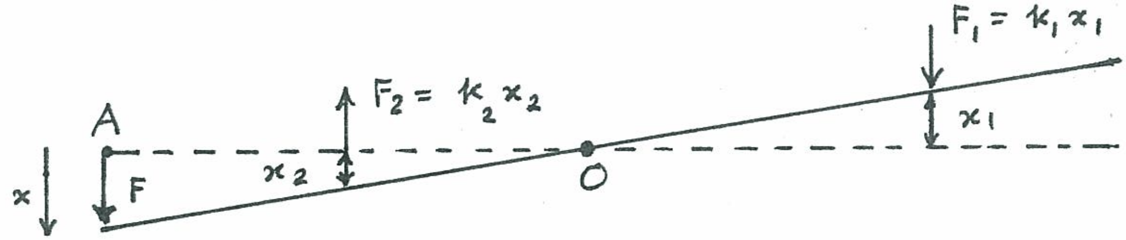
$$x = \frac{x_s}{\cos\alpha}$$

$$\therefore \quad k_{eq} = 3 k \cos^2\alpha$$

In a similar manner, $c_{eq} = 3\, c \cos^2\alpha$

1.18





$$x_2 = \frac{x}{2}, \quad x_1 = \frac{x}{2}$$

$$F_2 = k_2 x_2 = \frac{kx}{2}, \quad F_1 = k_1 x_1 = 2k\left(\frac{x}{2}\right) = kx$$

Equivalent spring constant of the system (k_{eq}) at point A can be determined by considering the moment equilibrium of forces about the pivot point O:

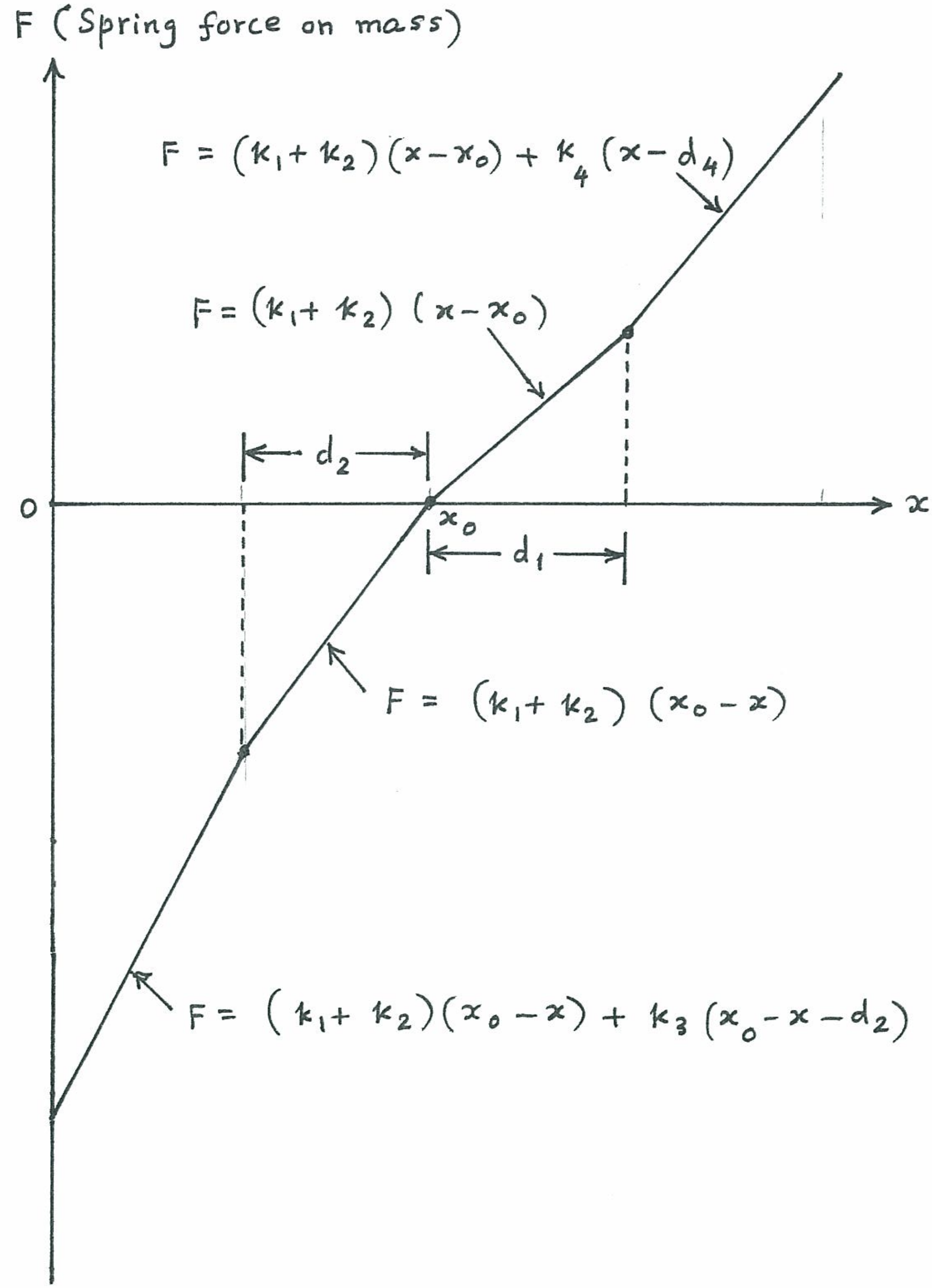
$$F\left(\frac{l}{2}\right) - F_2\left(\frac{l}{4}\right) - F_1\left(\frac{l}{4}\right) = 0$$

$$F = \frac{F_2}{2} + \frac{F_1}{2} = \frac{kx}{4} + \frac{kx}{2} = \frac{3}{4} kx$$

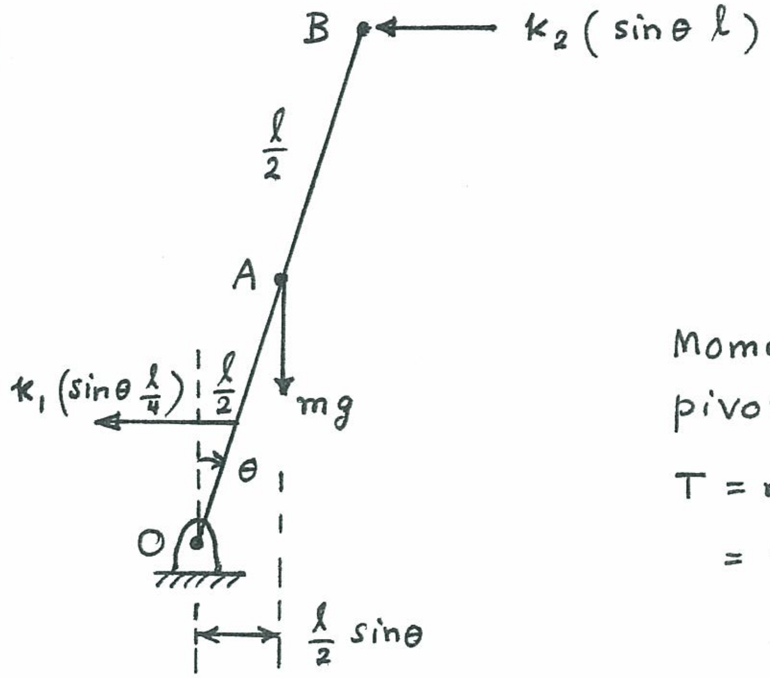
$$= k_{eq} x$$

$$\therefore \quad k_{eq} = \frac{3}{4} k$$

1.19


F (Spring force on mass)
$F = (k_1 + k_2)(x - x_0) + k_4 (x - d_4)$
$F = (k_1 + k_2)(x - x_0)$
d_2
0
x
x_0
d_1
$F = (k_1 + k_2)(x_0 - x)$
$F = (k_1 + k_2)(x_0 - x) + k_3 (x_0 - x - d_2)$

1.20



Moment about the pivot point O:

T = moment

$$= mg\frac{l}{2}\sin\theta - \left(k_1 \frac{l}{4}\sin\theta\right)\frac{l}{4} - (k_2\, l\sin\theta)\, l$$

$$\simeq \left(\frac{mgl}{2} - k_1\frac{l^2}{16} - k_2 l^2\right)\theta \quad (1)$$

Denoting the equivalent torsional spring constant of the system as k_t, the moment T can be expressed as

$$T = k_t\,\theta \quad (2)$$

By equating Eqs. (1) and (2), we obtain

$$k_t = \frac{mgl}{2} - \frac{k_1 l^2}{16} - k_2 l^2 \quad (3)$$

1.21 When mercury is displaced by an amount x in one leg of the manometer (Fig. 1.77), the mercury column will undergo a total displacement of $2x$. The magnitude of the force, due to the weight of the displaced mercury, acts on the rest of the fluid. The restoring force is given by

$$F = 2\gamma A x \tag{1}$$

where γ is the specific weight of mercury and A is the cross-sectional area of the manometer tube. If k_{eq} denotes the spring constant associated with the restoring force, the restoring force can be expressed as

$$F = k_{eq}\, x \tag{2}$$

Equations (1) and (2) yield the equivalent spring constant as

$$k_{eq} = 2\gamma A \tag{3}$$

1.22 When the drum is displaced by an amount x from its static equilibrium position, the weight of the fluid (sea water) displaced is given by

$$W = \rho_w g \left(\frac{\pi d^2}{4}\right) x \qquad (1)$$

where ρ_w is the density of sea water and g is the acceleration due to gravity. The weight, W, given by Eq. (1) also denotes the restoring force F. By expressing the restoring force as

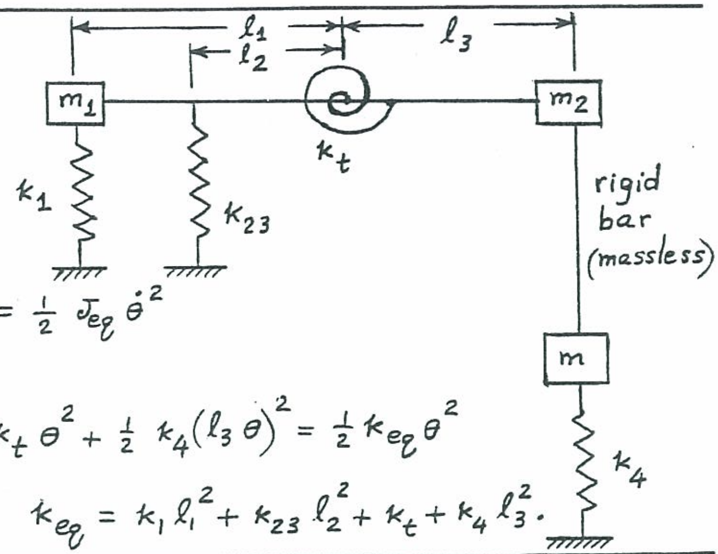
$$F = k_{eq}\, x \qquad (2)$$

where k_{eq} denotes the equivalent spring constant associated with the restoring force. Equating (1) and (2), we obtain

$$k_{eq} = \rho_w g \frac{\pi d^2}{4} \qquad (3)$$

1.23

$$k_{23} = \frac{k_2 k_3}{k_2 + k_3}$$



$$k_4 = A \rho g = \frac{\pi d^2}{4} \rho g$$

From kinetic energy,

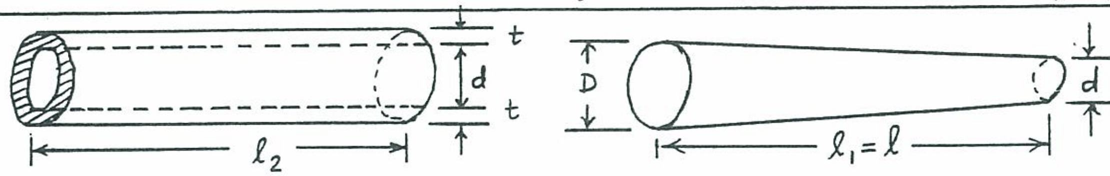
$$\frac{1}{2} m_1 (l_1 \dot{\theta})^2 + \frac{1}{2}(m_2 + m)(l_3 \dot{\theta})^2 = \frac{1}{2} J_{eq} \dot{\theta}^2$$

From potential energy,

$$\frac{1}{2} k_1 (l_1 \theta)^2 + \frac{1}{2} k_{23} (l_2 \theta)^2 + \frac{1}{2} k_t \theta^2 + \frac{1}{2} k_4 (l_3 \theta)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore\ J_{eq} = m_1 l_1^2 + (m_2 + m) l_3^2 \ ; \quad k_{eq} = k_1 l_1^2 + k_{23} l_2^2 + k_t + k_4 l_3^2 .$$

1.24



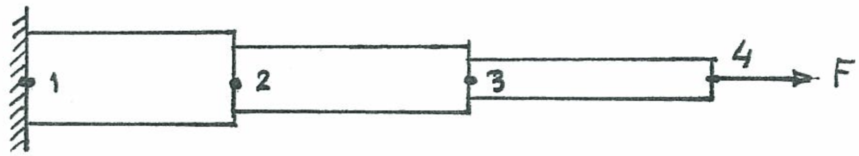
$$k_2 = \frac{E A}{l_2} = \frac{\pi E t (d + t)}{l_2} \qquad k_1 = \frac{\pi E D d}{4 l}$$

$$k_2 = k_1 \text{ gives } \quad l_2 = \frac{4 t (d+t)}{D d}$$

1.25 (a) Spring constant (stiffness) of step i in the axial direction:

$$k_i = \frac{A_i E_i}{l_i} = \frac{A_i E}{l_i}, \quad i = 1, 2, 3 \qquad (1)$$

(b)



The reaction at any point along the stepped shaft due to an axial force (F) applied at point 4 will be same as F. Hence the springs (stiffnesses) corresponding to the three steps 12, 23 and 34 are to be considered as series springs. In view of Eq.(1), the equivalent spring constant given by Eq. (1.17) becomes

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right)$$

$$= \frac{1}{E}\,\frac{(l_1 A_2 A_3 + l_2 A_1 A_3 + l_3 A_1 A_2)}{A_1 A_2 A_3}$$

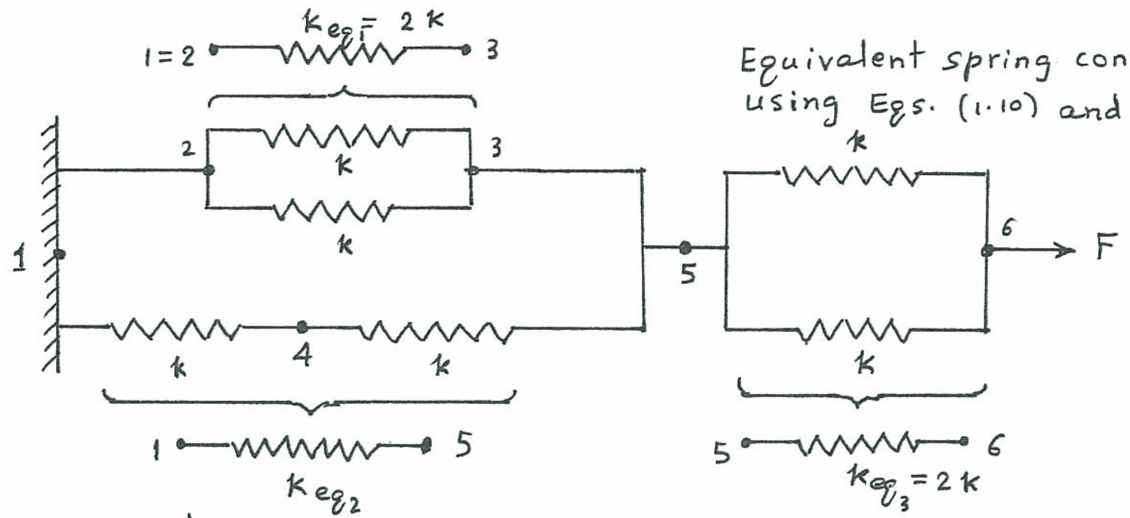
or

$$k_{eq} = \frac{E A_1 A_2 A_3}{l_1 A_2 A_3 + l_2 A_1 A_3 + l_3 A_1 A_2} \qquad (2)$$

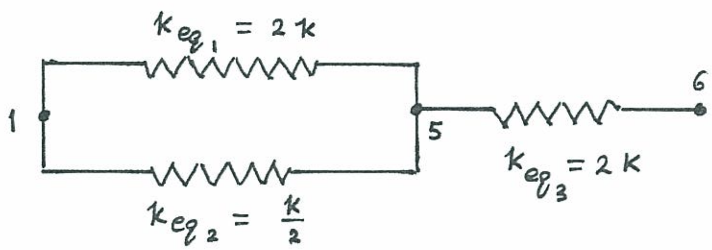
(c) steps behave as series springs.

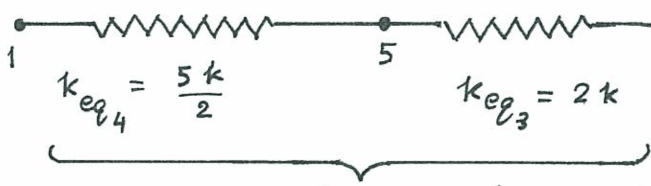
1.26

Equivalent spring constants using Eqs. (1.10) and (1.16).



$$\frac{1}{k_{eq_2}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{eq_2} = \frac{k}{2}$$





$$k_{eq5} \Rightarrow \frac{1}{k_{eq_5}} = \frac{1}{k_{eq_4}} + \frac{1}{k_{eq_3}} = \frac{2}{5k} + \frac{1}{2k}$$

$$k_{eq_5} = \frac{10}{9}k$$

$$k_{eq} = k_{eq_5} = \frac{10}{9}k$$

PRELIMINARY VERSION 12/14/2010

1.27 (a) Torsional spring constant or stiffness of step i is

$$k_{t_i} = \frac{G_i J_i}{l_i} = \frac{G_i \pi D_i^4}{32\, l_i}, \quad i = 1, 2, 3 \qquad (1)$$

(b) The reactive torque at any point along the stepped shaft due to an applied torque T at the free end will be T. Hence the torsional stiffnesses (springs) corresponding to the three steps 12, 23 and 34 are to be considered as series springs. In view of Eq. (1), the equivalent torsional spring constant given by Eq. (1.17) becomes (Eq. (1.17) is to be interpreted for torsional springs):

$$\frac{1}{k_{eq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} = \frac{32}{\pi G}\left(\frac{l_1}{D_1^4} + \frac{l_2}{D_2^4} + \frac{l_3}{D_3^4}\right)$$

$$= \frac{32}{\pi G}\left(\frac{l_1 D_2^4 D_3^4 + l_2 D_1^4 D_3^4 + l_3 D_1^4 D_2^4}{D_1^4 D_2^4 D_3^4}\right)$$

or

$$k_{eq} = \frac{\pi G D_1^4 D_2^4 D_3^4}{32\left(l_1 D_2^4 D_3^4 + l_2 D_1^4 D_3^4 + l_3 D_1^4 D_2^4\right)} \qquad (2)$$

(c) steps behave as series springs.

1.28 (a) $F \simeq F\big|_{x_0} + \dfrac{dF}{dx}\Big|_{x_0} \cdot (x - x_0) = \left(500\,x + 2x^3\right)_{x=10} + \left(500 + 6x^2\right)_{x=10} \cdot (x-10)$

$\simeq 1100\,x - 4000$

(b) at $x = 9$ mm :

Exact $F_9 = 500 \times 9 + 2\,(9)^3 = 5958$ N

Approximate $F_9 = 1100 \times 9 - 4000 = 5900$ N

Error $= -0.9735\,\%$

(c) at $x = 11$ mm :

Exact $F_{11} = 500 \times 11 + 2\,(11)^3 = 8162$ N

Approximate $F_{11} = 1100 \times 11 - 4000 = 8100$ N

Error $= +0.7596\,\%$

1.29 $p v^{\gamma} = \text{constant}$ --- (E_1) ; Differentiation of (E_1) gives

$$dp\; v^{\gamma} + p\,\gamma\, v^{\gamma-1}\, dv = 0$$

$$dp = -\frac{p\gamma}{v}\, dv \quad --- (E_2)$$

change in volume when mass moves by dx, $\quad dv = -A \cdot dx \quad --- (E_3)$

Eqs. (E_2) and (E_3) give $\quad dp = \dfrac{p\gamma A}{v}\, dx$

Force due to pressure change $= dF = dp \cdot A = \dfrac{p\gamma A^2}{v} \cdot dx$

spring constant of air spring $= k = \dfrac{dF}{dx} = \left(\dfrac{p\gamma A^2}{v}\right)$.

1.30 Equivalent spring constants in diffent directions are

$$k_{e1} = \left(\frac{k_5\, k_6\, k_7}{k_5 k_6 + k_5 k_7 + k_6 k_7}\right), \qquad k_{e2} = \left(\frac{k_8\, k_9}{k_8 + k_9}\right).$$

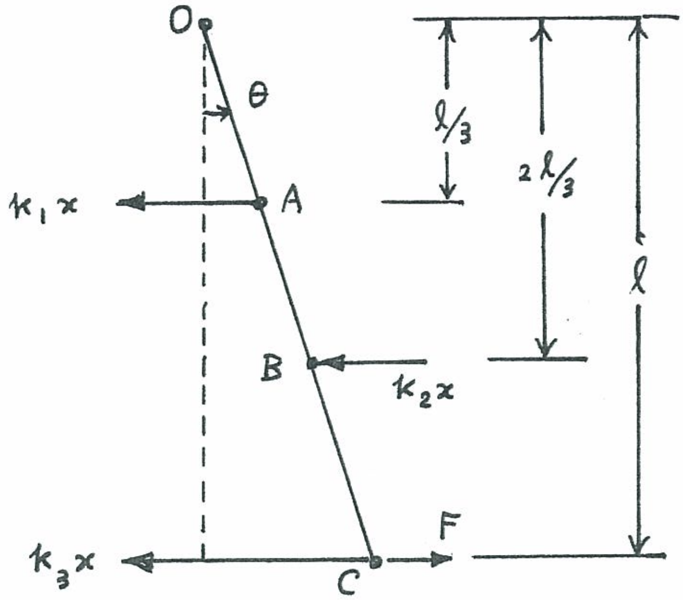
$$k_{e3} = \left(\frac{k_1\, k_2}{k_1 + k_2}\right), \qquad k_{e4} = \left(\frac{k_3\, k_4}{k_3 + k_4}\right)$$

If the force P moves by x, spring located at θ_i undergoes a displacement of $x_i = x \cos\theta_i$ (derivation as in problem 1.17).

Equivalence of potential energy gives $\quad \frac{1}{2} k_{eq}\, x^2 = \frac{1}{2} \sum_{i=1}^{4} k_{ei}\, x_i^2$

$$k_{eq} = \sum_{i=1}^{4} \left(k_{ei}\, \cos^2\theta_i\right)$$

1.31



Let the link OABC undergo a small angular displacement θ as shown in above figure. The spring reaction forces are also indicated in the figure. Equilibrium of moments about the pivot point O gives:

$$-k_1 x\left(\frac{l}{3}\right) - k_3 x(l) - k_2 x\left(\frac{2l}{3}\right) + F(l) = 0$$

or

$$F = \left(\frac{k_1}{3} + \frac{2}{3}k_2 + k_3\right) x \qquad (1)$$

If k_{eq} denotes the equivalent spring constant of the link along the direction of F at point C, we have

$$F = k_{eq}\, x \qquad (2)$$

Equations (1) and (2) give

$$k_{eq} = \frac{k_1}{3} + \frac{2k_2}{3} + k_3 = \frac{k}{3} + \frac{2}{3}(2k) + (3k)$$

$$\therefore\ k_{eq} = \frac{14}{3} k \qquad (3)$$

1.32 Spring constant of a helical spring is

$$k = \frac{G d^4}{8 N D^3} \tag{1}$$

Assuming the shear modulus of steel as $G = 79.3$ GPa, Eq. (1) gives, for $D = 0.2$ m, $d = 0.005$ m and $N = 10$,

$$k = \frac{(79.3 \times 10^9)(0.005)^4}{8\,(10)\,(0.2)^3} = 77.4414 \text{ N/m}$$

1.33 (a) D and d: same for both helical springs

Weight of a helical spring is:

$$W = \pi D \left(\frac{\pi d^2}{4}\right) N \gamma \qquad (1)$$

where γ = specific weight of material of spring.

For a steel spring with $\gamma_s = 76.5$ kN/m³, the weight is (for $N_s = 10$):

$$W_s = \pi D \left(\frac{\pi d^2}{4}\right) N_s \gamma_s = \frac{\pi^2 D d^2}{4}(10)(76.5 \times 10^3)$$

$$= 19.125 \times 10^4 \pi^2 D d^2 \qquad (2)$$

For an aluminum spring with $\gamma_a = 26.6$ kN/m³, the weight is (for number of turns N_a),

$$W_a = \pi D \left(\frac{\pi d^2}{4}\right) N_a \gamma_a = \frac{\pi^2 D d^2 N_a}{4}(26.6 \times 10^3)$$

$$= 6.65 \times 10^3 \pi^2 D d^2 N_a \qquad (3)$$

Equating (2) and (3),

$$19.125 \times 10^4 \pi^2 D d^2 = 6.65 \times 10^3 \pi^2 D d^2 N_a$$

or $$N_a = \frac{19.125 \times 10^4}{6.65 \times 10^3} = 28.7594 \qquad (4)$$

(b) Spring constant of a helical spring is:

$$k = G d^4 / (8 N D^3)$$

For a steel spring with G = 793.3 GPa,

$$k_s = (79.3 \times 10^9) d^4 / \{8(10) D^3\}$$

$$= 0.99125 \times 10^9 d^4 / D^3 \qquad (5)$$

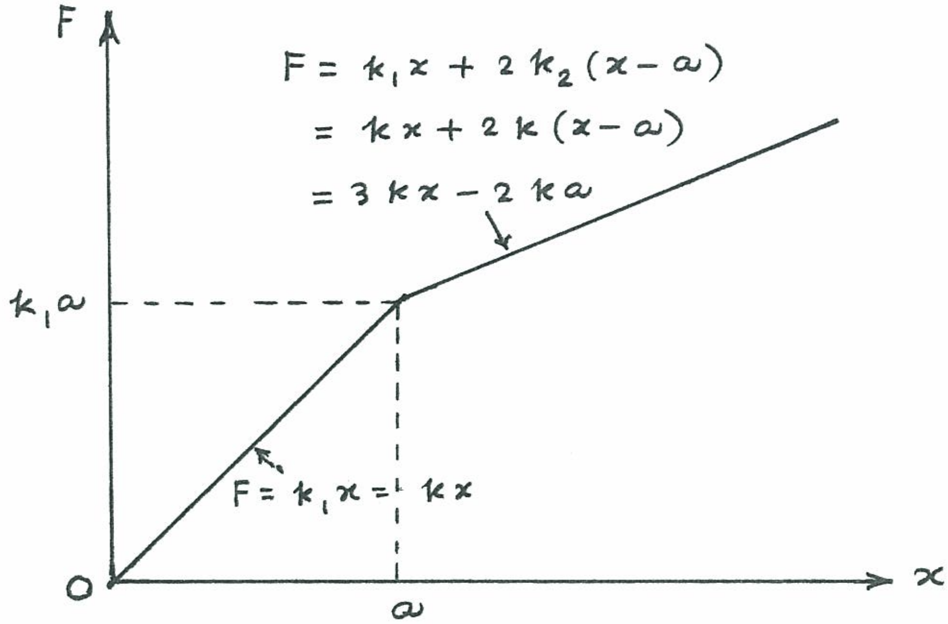
For an aluminum spring with G = 26.2 GPa,

$$k_a = (26.2 \times 10^9) d^4 / \{8(28.7594) D^3\}$$

$$= 0.1139 \times 10^9 d^4 / D^3 \qquad (6)$$

Eqs. (5) and (6) indicate that the spring constant of steel spring is $0.99125/0.1139 = 8.7046$ times larger than that of aluminum spring.

1.34



1.35 From Problem 1.29, $k = \dfrac{p \gamma A^2}{v}$ with $\gamma = 1.4$ for air

Let $p = 200$ psi

$$k = 75 \text{ lb/in} = \frac{(200)(1.4)A^2}{v} \Rightarrow \frac{A^2}{v} = 0.2679$$

Let diameter of piston $= d = 2$ inch ; $A = \frac{\pi}{4}(2)^2 = 3.1416 \text{ in}^2$

$$v = A^2/0.2679 = 36.8408 \text{ in}^3$$

Let $h = 2$ inch ; $\frac{\pi}{4} D^2 (2) = v \Rightarrow D = 4.8429$ inch

---

1.36 $F = a\,x + b\,x^3 = 2\,(10^4)\,x + 4\,(10^7)\,x^3$

Around x^*: $F(x) \approx F(x^*) + \frac{dF}{dx}\big|_{x^*}\,(x - x^*)$

When $x^* = 10^{-2}$ m, $F(x^*) = 2\,(10^4)\,(10^{-2}) + 4\,(10^7)\,(10^{-6}) = 240$ N

$\frac{dF}{dx}\big|_{x^*} = a + 3\,b\,x^2 = 2\,(10^4) + 3\,(4)\,(10^7)\,(10^{-4}) = 32000$

Hence $F(x) = 240 + 32000\,(x - 0.01) = (32000\,x - 80)$ N

Since the linearized spring constant is given by $F(x) = k_{eq}\,x$, we have $k_{eq} = 32{,}000$ N/m.

1.37 $F_i = a_i x_i + b_i x_i^3 \quad ; \; i = 1, 2$

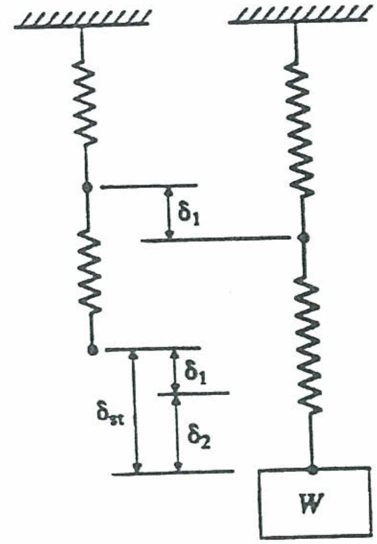
Springs in series:

$$W = a_1 \delta_1 + b_1 \delta_1^3 \qquad (1)$$

$$W = a_2 \delta_2 + b_2 \delta_2^3 \qquad (2)$$

$$W = k_{eq} \delta_{st} \qquad (3)$$

$$\delta_{st} = \delta_1 + \delta_2 \qquad (4)$$



Solve Eqs. (1) and (2) for δ_1 and δ_2, respectively. Substitute the result in Eq. (4) and then in Eq. (3) to find k_{eq}.

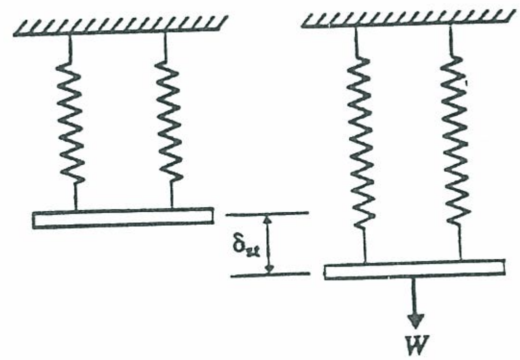
Springs in parallel:

$$W = F_1 + F_2$$

$$= a_1 \delta_{st} + b_1 \delta_{st}^3 + a_2 \delta_{st} + b_2 \delta_{st}^3$$

$$= k_{eq} \delta_{st}$$

$$k_{eq} = a_1 + b_1 \delta_{st}^2 + a_2 + b_2 \delta_{st}^2$$



1.38 $k = \dfrac{G d^4}{8 D^3 N} \geq 8 \times 10^6$ N/m ; $\dfrac{D}{d} \geq 6$; $N \geq 10$

$W = \pi D N \rho \left(\dfrac{\pi d^2}{4}\right)$ where ρ = weight per unit volume

$$f_1 = \frac{1}{2}\sqrt{\frac{k g}{W}} = \frac{1}{2}\sqrt{\frac{G d^2 g}{2 \pi^2 D^4 N^2 \rho}} \geq 0.4 \text{ Hz}$$

Using $G = 73.1 \times 10^9$ N/m^2 , $\rho = 76000$ N/m^3 , $g = 9.81$ m/sec^2 , $\dfrac{D}{d} = 6, 8, 10$; $N = 10, 15, 20$; $d = 0.4, 0.6, \ldots,$ values of k and f_1 are computed.

Combination of $\dfrac{D}{d} = 6$, $N = 10$ and $d = 2.0$ m, corresponding to $k = 8.4606 \times 10^6$ N/m and $f_1 = 0.4801$ Hz, can be taken as an acceptable design.

1.39 Total elongation (strain) is same in each material:

$$\epsilon_s = \epsilon_a = \frac{x}{\ell} \tag{1}$$

where x is the total elongation. Equation (1) can be expressed as

$$\frac{\sigma_s}{E_s} = \frac{\sigma_a}{E_a} = \frac{x}{\ell} \tag{2}$$

$$\text{or} \quad \sigma_s = \frac{E_s\, x}{\ell} \tag{3}$$

$$\sigma_a = \frac{E_a\, x}{\ell} \tag{4}$$

Total axial force is:

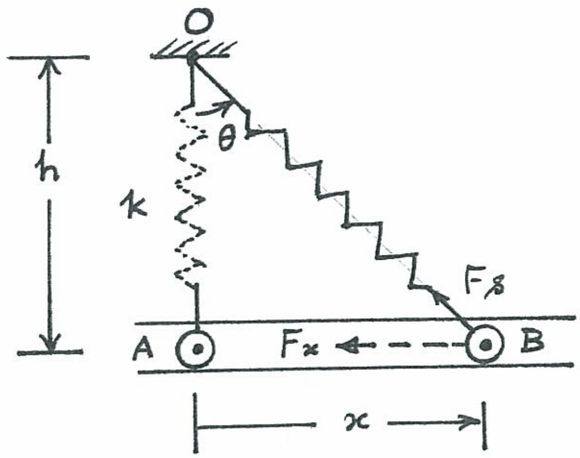
$$F = F_s + F_a = \sigma_s A_s + \sigma_a A_a \tag{5}$$

where F_s and F_a denote the axial forces acting on steel and aluminum, respectively, and A_s and A_a represent the cross-sectional areas of the two materials. Equating F to k_{eq} x where k_{eq} denotes the equivalent spring constant of the bimetallic bar, we obtain from Eqs. (3) to (5):

$$F = k_{eq}\, x = \left(\frac{E_s\, x}{\ell}\right) A_s + \left(\frac{E_a\, x}{\ell}\right) A_a$$

$$\text{or} \quad k_{eq} = \frac{E_s A_s}{\ell} + \frac{E_a A_a}{\ell} \tag{6}$$

1.40 Let the length of the spring be h. Spring is undeformed at $\theta = 0$. When the end A of the spring is displaced by an amount x as shown in the figure,



the spring is stretched by the amount $\left(\sqrt{h^2+x^2} - h\right)$ so that the force in the spring (F_s) is given by

$$F_s = k\left(\sqrt{h^2+x^2} - h\right) \tag{1}$$

The component of the spring force F_s along the direction of x is given by

$$F_x = F_s \sin\theta = F_s \frac{x}{\sqrt{h^2+x^2}} = \frac{k\left(\sqrt{h^2+x^2} - h\right)x}{\sqrt{h^2+x^2}}$$

$$= k\left(1 - \frac{h}{\sqrt{h^2+x^2}}\right)x \tag{2}$$

Equation (2) shows that the force - displacement relation (in the x- direction) is nonlinear. If the relation is linear, we could write

$$F_x = \underset{\sim}{k}\, x \tag{3}$$

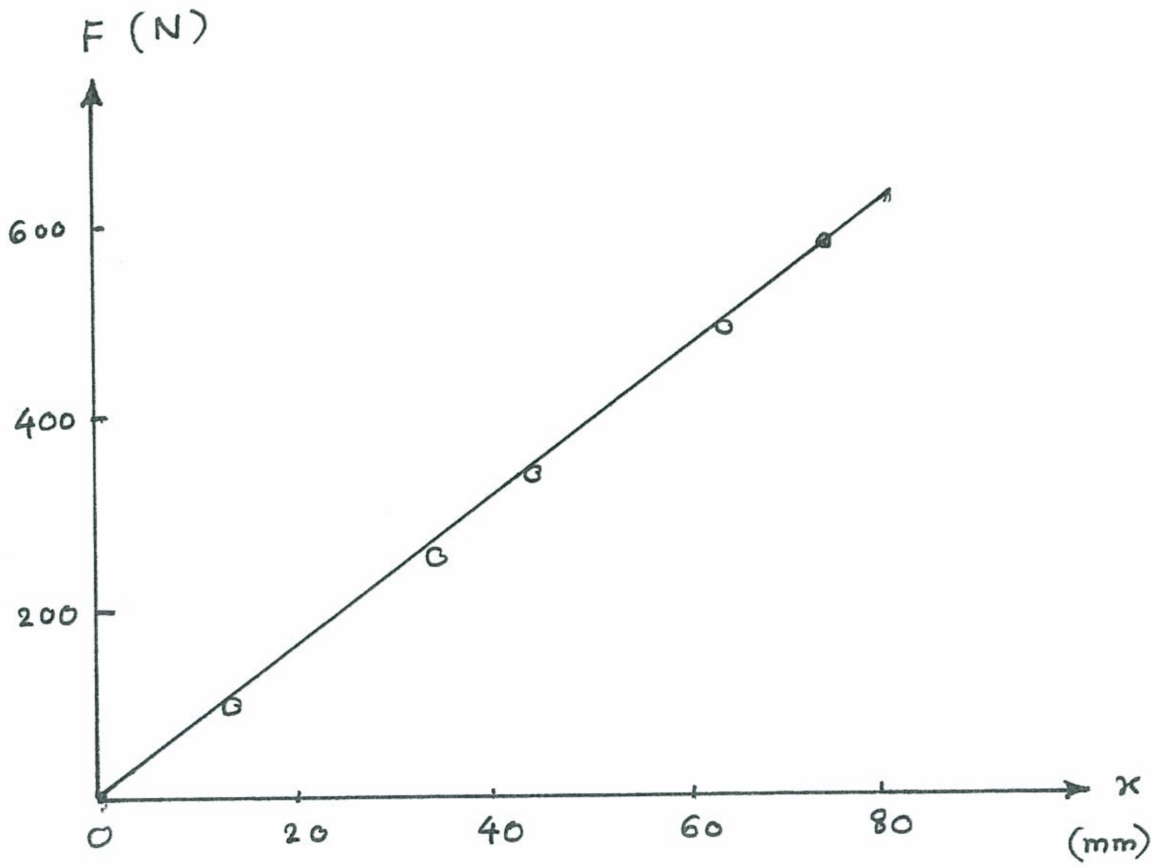
A comparison of Eqs. (2) and (3) shows that the spring constant $\underset{\sim}{x}$ is not a constant, but depends on the displacement x.

1.41 From the given data, the force - deformation relation of the spring can be obtained as :

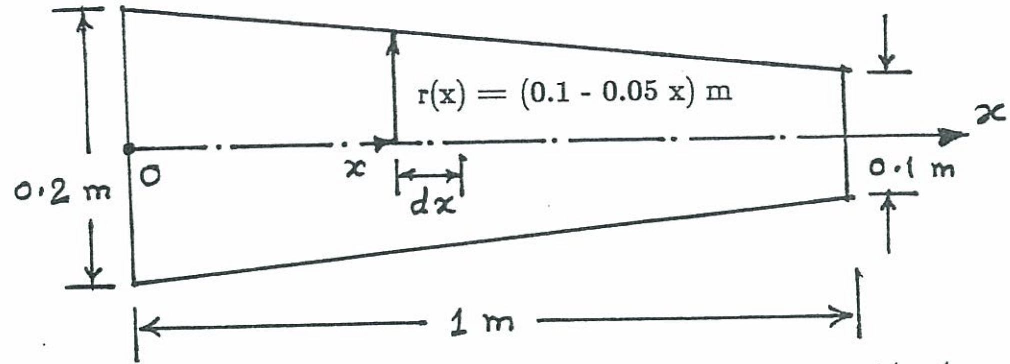
Tensile force (F), N	0	100	250	330	480	570
Deformation of spring (x), mm (change in length)	0	13	33	44	64	76

The force - deformation relation is plotten in the figure shown below. The relation can be seen to be nearly linear with the spring constant given by

$$k = \frac{F}{x} \simeq \frac{570}{76} = 7.5 \text{ N/mm} = 7500 \text{ N/m}.$$



1.42



$J = \frac{\pi}{2} r^4$ = area polar moment of inertia at section x = $1.5708\ (0.1 - 0.05\ x)^4\ m^4$

Knowing that the angle of twist, θ, between the ends of a uniform shaft of length ℓ under a torque T is given by $\theta = \frac{T\,\ell}{GJ}$, the angle of twist for an element of length dx can be expressed as

$$d\theta = \frac{T\ dx}{GJ} = \frac{T\ dx}{(80\ (10^9))\ 1.5708\ (0.1 - 0.05\ x)^4} \tag{1}$$

The total angle of twist can be determined by integrating Eq. (1) from x=0 to 1 as:

$$\theta = \int_0^1 \frac{T\ dx}{(12.5664\ (10^{10}))\ (0.1 - 0.05\ x)^4} = \left(\frac{T}{12.5664\ (10^{10})} \right) \int_0^1 \frac{dx}{(0.1 - 0.05\ x)^4} \tag{2}$$

$$\text{But} \int_0^1 \frac{dx}{(0.1 - 0.05\ x)^4} = -\frac{1}{0.05} \int_0^1 \frac{(-0.05\ dx)}{(0.1 - 0.05\ x)^4} = -20 \int_0^{-0.05} \frac{dy}{(0.1 + y)^4}$$

$= 4.6667\ (10^4)$ where $y = -0.05\ x$

$$\text{Hence}\ \ \theta = \frac{T\ (4.6667)\ (10^4)}{12.5664\ (10^{10})} = T\ (0.3714\ (10^{-6}))\ \text{rad}$$

This gives $k_t = \frac{T}{\theta} = 2.6925\ (10^6)$ N-m/rad

1.43 The steel and aluminum hollow shafts can be treated as two torsional springs in parallel. For a hollow shaft,

$$k_t = \frac{\pi\ G}{32\ \ell}\ (D^4 - d^4)$$

For the steel shaft, $G = 80\ (10^9)$ Pa, $\ell = 5$ m, $D = 0.25$ m, $d = 0.15$ m, and hence

$$k_{t_1} = \frac{\pi\ (8\ (10^{10}))}{32\ (5)}\ (0.25^4 - 0.15^4) = 5.34072\ (10^6)\ \text{N–m/rad}$$

(a) For the aluminum shaft, $G = 26\ (10^9)$ Pa, $\ell = 5$ m, $D = 0.15$ m, $d = 0.1$ m, and hence

$$k_{t_2} = \frac{\pi\ (26\ (10^9))}{32\ (5)}\ (0.15^4 - 0.10^4) = 0.207395\ (10^6)\ \text{N–m/rad}$$

$$k_{eq} = k_{t_1} + k_{t_2} = 5.34072\ (10^6) + 0.20739\ (10^6) = 5.54811\ (10^6)\ \text{N–m/rad}$$

(b) With $G = 26\ (10^9)$ Pa, $\ell = 5$ m, $D = 0.15$ m and $d = 0.05$ m,

$$k_{t_2} = \frac{\pi\ (26 \times 10^9)}{32\ (5)} \left(0.15^4 - 0.05^4\right) = 0.255255 \times 10^6 \text{ N-m/rad}$$

$$k_{eq} = k_{t_1} + k_{t_2} = 5.34072 \times 10^6 + 0.255255 \times 10^6 = 5.595975 \times 10^6 \text{ N-m/rad}$$

1.44 For helical spring: $k = \dfrac{G\ d^4}{64\ n\ R^3}$

Spring 1: $k_1 = \dfrac{(12 \times 10^6)(2^4)}{64\ (10)(6^3)} = 1{,}388.89$ lb/in

Spring 2: $k_2 = \dfrac{(4 \times 10^6)(1^4)}{64\ (10)(5^3)} = 50.00$ lb/in

(a) Spring 2 inside spring 1 (parallel): $k_{eq} = k_1 + k_2 = 1{,}438.89$ lb/in
(b) Spring 2 on top of spring 1 (series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_2 + k_1}{k_1\ k_2}$$

which gives $k_{eq} = 48.2625$ lb/in.

1.45 For a helical spring, $k = \dfrac{G\ d^4}{64\ n\ R^3}$

$$k_1 = \frac{(12 \times 10^6)(1)^4}{64\ (10)\ (6^3)} = 86.806 \text{ lb/in}$$

$$k_2 = \frac{(4 \times 10^6)(0.5)^4}{64\ (10)\ (5^3)} = 3.125 \text{ lb/in}$$

(a) Spring 2 inside spring 1: $k_{eq} = k_1 + k_2 = 89.931$ lb/in

(b) Spring 2 on top of spring 1: $\dfrac{1}{k_{eq}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

or $k_{eq} = \dfrac{k_1\ k_2}{k_1 + k_2} = \dfrac{86.806\ (3.125)}{86.806 + 3.125} = 3.0164$ lb/in

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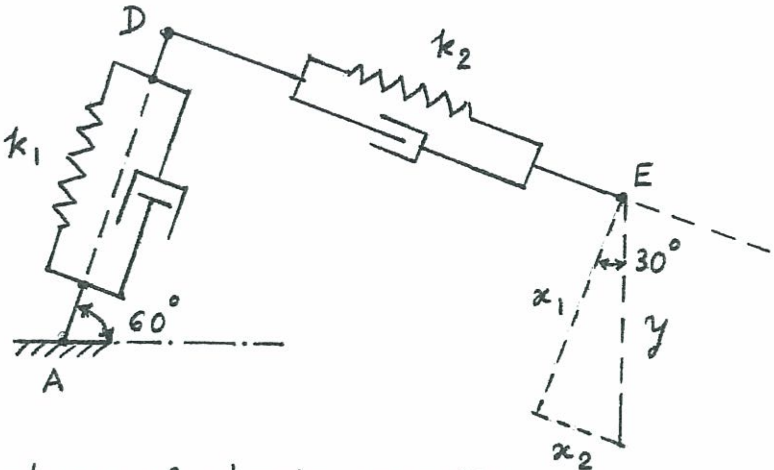
1.46



$x_2 = y \sin 30°$

$x_1 = y \cos 30°$

Equivalence of strain energies:

$$\frac{1}{2} k_{eq} y^2 = \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_1 x_1^2 = \frac{1}{2} k_1 y^2 \cos^2 30° + \frac{1}{2} k_2 y^2 \sin^2 30°$$

i.e., $k_{eq} = \frac{3}{4} k_1 + \frac{1}{4} k_2$

with $k_1 = \frac{A_1 E_1}{l_1} = \frac{\pi}{4} \frac{(10^2 - 9.5^2)(30 \times 10^6)}{100} = 2.297295 \times 10^6$ lb/in

and $k_2 = \frac{A_2 E_2}{l_2} = \frac{\pi}{4} \frac{(7^2 - 6.5^2)(30 \times 10^6)}{75} = 2.12058 \times 10^6$ lb/in

$$\therefore k_{eq} = \frac{3}{4}(2.297295 \times 10^6) + \frac{1}{4}(2.12058 \times 10^6)$$
$$= 2.25311625 \times 10^6 \text{ lb/in}$$

Similarly, the equivalent damping constant can be found as (using equivalence of kinetic energies):

$$c_{eq} = \frac{3}{4} c_1 + \frac{1}{4} c_2 = \frac{3}{4}(0.4) + \frac{1}{4}(0.3) = 0.375 \text{ lb-sec/in.}$$

1.47 Stainless steel: $E = 30 \times 10^6$ lb/in^2, $G = 11.5 \times 10^6$ lb/in^2

For each tube:

$D = 0.30''$, $d = 0.29''$, $l = 50''$

Axial stiffness $= \frac{AE}{l} = \frac{\pi}{4}(D^2 - d^2)\frac{E}{l}$

$$= \frac{\pi}{4}(0.30^2 - 0.29^2)\left(\frac{30 \times 10^6}{50}\right) = 2780.316 \text{ lb/in} = k_a$$

$$\text{Torsional stiffness} = \frac{\pi G}{32\,l}\left(D^4 - d^4\right)$$

$$= \frac{\pi\,(11.5 \times 10^6)}{32\,(50)}\left(0.30^4 - 0.29^4\right) = 23.1942 \text{ lb-in/rad} = k_t$$

For heat exchanger with 6 tubes:

Axial stiffness $= 6\,k_a = 16{,}681.896$ lb/in

Torsional stiffness $= 6\,k_t = 139.1652$ lb-in/rad

1.48 Assume small angles θ_1 and θ_2; $\quad \theta_2 = \left(\frac{p_1}{p_2}\right)\theta_1$

x_1 = horizontal displacement of C.G. of mass $m_1 = \theta_1 r_1$

x_2 = vertical displacement of C.G. of mass $m_2 = \theta_2 r_2 = p_1 \theta_1 r_2 / p_2$

y_1 = horizontal displacement of springs k_1 and $k_2 = \theta_1 (r_1 + l_1)$

y_2 = vertical displacement of springs k_3 and $k_4 = \theta_2 l_2 = p_1 l_2 \theta_1 / p_2$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq} (\dot{\theta}_1)^2 = \frac{1}{2} J_1 (\dot{\theta}_1)^2 + \frac{1}{2} J_2 (\dot{\theta}_2)^2 + \frac{1}{2} m_1 (\dot{x}_1)^2 + \frac{1}{2} m_2 (\dot{x}_2)^2$$

$$\therefore\ J_{eq} = J_1 + J_2 \left(p_1/p_2\right)^2 + m_1 r_1^2 + m_2 r_2^2 \left(p_1/p_2\right)^2$$

Equivalence of potential energies gives

$$\frac{1}{2} k_{eq}\, \theta_1^2 = \frac{1}{2} k_{12}\, y_1^2 + \frac{1}{2} k_{34}\, y_2^2 + \frac{1}{2} k_{t1}\, \theta_1^2 + \frac{1}{2} k_{t2}\, \theta_2^2$$

with $k_{12} = k_1 + k_2$, $\quad k_{34} = k_3 k_4 / (k_3 + k_4)$

$y_1 = \theta_1 (r_1 + l_1)$, $\ y_2 = p_1 l_2 \theta_1 / p_2$ and $\theta_2 = p_1 \theta_1 / p_2$.

$$\therefore\quad k_{eq} = (k_1 + k_2)(r_1 + l_1)^2 + \left(\frac{k_3 k_4}{k_3 + k_4}\right)\frac{p_1^2 l_2^2}{p_2^2} + k_{t1} + k_{t2}\,\frac{p_1^2}{p_2^2}.$$

1.49 $\theta = \frac{x}{b}$, $\quad x_1 = \frac{x\,a}{b}$.

From equivalence of kinetic energies,

$$\frac{1}{2} m_{eq}\, \dot{x}^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} J_0\, \dot{\theta}^2$$

$$m_{eq} = m_1 \left(\frac{a}{b}\right)^2 + m_2 + J_0 \left(\frac{1}{b}\right)^2$$

1.50 Let $\dot{\theta}_i$ = angular velocity of the motor (input)

Angular velocities of different gear sets are:

J_{motor}, J_1	J_2, J_3	J_4, J_5	...	J_{2N}, J_{load}
$\dot{\theta}_i$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\right)$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\right)$		$\dot{\theta}_i\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_i^2 = \frac{1}{2} J_{motor}\,\dot{\theta}_i^2 + \frac{1}{2}\sum_{k=1}^{2N} J_k\,\dot{\theta}_k^2 + \frac{1}{2} J_{load}\,\dot{\theta}_{load}^2$$

$$\therefore J_{eq} = \left(J_{motor} + J_1\right) + \left(J_2 + J_3\right)\left(\frac{n_1}{n_2}\right)^2 + \left(J_4 + J_5\right)\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\right)^2$$
$$+ \cdots + \left(J_{2N} + J_{load}\right)\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)^2$$

1.51 Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_1^2 = \frac{1}{2} J_1\,\dot{\theta}_1^2 + \frac{1}{2} J_2\,\dot{\theta}_2^2 \quad \text{where} \quad \dot{\theta}_2 = \dot{\theta}_1\left(\frac{n_1}{n_2}\right)$$

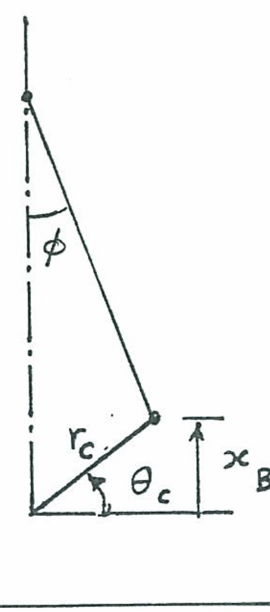
$$J_{eq} = J_1 + J_2\left(\frac{n_1}{n_2}\right)^2$$

1.52 When point A moves by distance $x = x_h$, the walking beam rotates by the angle $\theta_b = \frac{x_h}{\ell_3}$.

This corresponds to a linear motion of point B: $x_B = \theta_b\,\ell_2 = \frac{x_h\,\ell_2}{\ell_3}$

and the angular rotation of crank can be found from the relation:

$$x_B = r_c \sin\theta_c + \ell_4 \cos\phi = r_c \sin\theta_c + \ell_4\sqrt{1 - \frac{r_c^2}{\ell_4^2}\sin^2\theta_c}$$



For large values of ℓ_4 compared to r_c and for small values of x and θ_c, we have

$$x_B \approx r_c \sin\theta_c = r_c\,\theta_c \text{ or } \theta_c = \frac{x_B}{r_c} = \frac{x_h\,\ell_2}{\ell_3\,r_c}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m_h\,\dot{x}_h^2 + \frac{1}{2} J_b\,\dot{\theta}_b^2 + \frac{1}{2} J_c\,\dot{\theta}_c^2$$

Equating this to $T = \frac{1}{2} m_{eq}\,\dot{x}^2 = \frac{1}{2} m_{eq}\,\dot{x}_h^2$, we obtain

$$m_{eq} = m_h + \frac{J_b}{\ell_3^2} + J_c\left(\frac{\ell_2}{\ell_3\,r_c}\right)^2.$$

1.53 When mass m is displaced by x, the bell crank lever rotates by the angle $\theta_b = \frac{x}{\ell_1}$. This makes the center of the sphere displace by $x_s = \theta_b\, \ell_2$. Since the sphere rotates with out slip, it rotates by an angle

$$\theta_s = \frac{x_s}{r_s} = \frac{\theta_b\, \ell_2}{r_s} = \frac{x\, \ell_2}{\ell_1\, r_s}$$

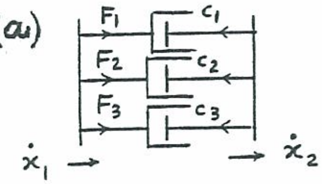
The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} J_s \dot{\theta}_s^2 + \frac{1}{2} m_s \dot{x}_s^2$$
$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{\ell_1}\right)^2 + \frac{1}{2}\left(\frac{2}{5} m_s r_s^2\right) \dot{x}^2 \left(\frac{\ell_2}{\ell_1 r_s}\right)^2 + \frac{1}{2} m_s \left(\frac{\dot{x}\, \ell_2}{\ell_1}\right)^2$$

since for a sphere, $J_s = \frac{2}{5}\, m_s\, r_s^2$. Equating this to $T = \frac{1}{2}\, m_{eq}\, \dot{x}^2$, we obtain

$$m_{eq} = m + J_0\, \frac{1}{\ell_1^2} + \frac{7}{5}\, m_s\, \frac{\ell_2^2}{\ell_1^2}$$

1.55 (a)

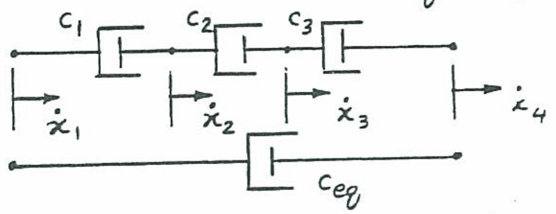


F_i = damping force of $c_i = c_i(\dot{x}_2 - \dot{x}_1)$; $i = 1,2,3$

F_{eq} = damping force of $c_{eq} = c_{eq}(\dot{x}_2 - \dot{x}_1)$
$\equiv F_1 + F_2 + F_3$

$\therefore \; c_{eq} = c_1 + c_2 + c_3$

(b)



$F_1 = c_1(\dot{x}_2 - \dot{x}_1)$

$F_2 = c_2(\dot{x}_3 - \dot{x}_2)$

$F_3 = c_3(\dot{x}_4 - \dot{x}_3)$

$$\dot{x}_4 - \dot{x}_1 = \dot{x}_4 - \dot{x}_3 + \dot{x}_3 - \dot{x}_2 + \dot{x}_2 - \dot{x}_1$$

$$\frac{F_{eq}}{c_{eq}} = \frac{F_3}{c_3} + \frac{F_2}{c_2} + \frac{F_1}{c_1}$$

since $F_{eq} = F_1 = F_2 = F_3$, $\quad \dfrac{1}{c_{eq}} = \dfrac{1}{c_1} + \dfrac{1}{c_2} + \dfrac{1}{c_3}$

(c) Equating the energies dissipated in a cycle,

$$\pi\, c_{eq}\, \omega\, X_1^2 = \pi c_1 \omega X_1^2 + \pi c_2 \omega X_2^2 + \pi c_3 \omega X_3^2$$

where $X_1 = \theta l_1$, $X_2 = \theta l_2$ and $X_3 = \theta l_3$

$$\therefore \; c_{eq} = c_1 + c_2\left(\frac{l_2}{l_1}\right)^2 + c_3\left(\frac{l_3}{l_1}\right)^2$$

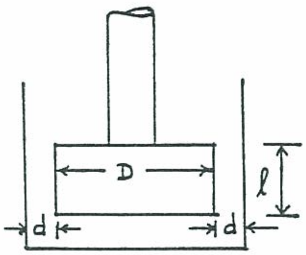
(d) Equating the energies dissipated in a cycle,

$$\pi\, c_{t_{eq}}\, \omega\, \theta_1^2 = \pi\, c_{t1}\, \omega\, \theta_1^2 + \pi\, c_{t2}\, \omega\, \theta_2^2 + \pi\, c_{t3}\, \omega\, \theta_3^2$$

where $\theta_2 = \theta_1\left(\frac{n_1}{n_2}\right)$ and $\theta_3 = \theta_1\left(\frac{n_1}{n_3}\right)$.

$$\therefore \; c_{t_{eq}} = c_{t1} + c_{t2}\left(\frac{n_1}{n_2}\right)^2 + c_{t3}\left(\frac{n_1}{n_3}\right)^2.$$

1.57 Damping constant desired $= c = 1$ lb-sec/in, viscosity of the fluid $= \mu = 4\ \mu$ reyn $= 4\ (10^{-6})$ lb—sec/in^2.



$$c = \mu \left\{ \frac{3\ \pi\ D^3\ \ell\ (1 + \frac{2\ d}{D})}{4\ d^3} \right\} \tag{1}$$

Assuming $x = D/d$ as the unknown with $\ell = 2$ in, Eq. (1) can be written as

$$c = \mu \left(\frac{3\ \pi\ \ell\ x^3}{4} \right) (1 + \frac{2}{x}) \quad \text{or} \quad 1 = (4\ (10^{-6}))\ (\frac{3\ \pi\ (2)}{4})\ x^3\ (1 + \frac{2}{x}) \tag{2}$$

This gives $x^3 + 2\ x^2 - 53{,}051.52 = 0$

Using a trial and error procedure, the solution of this cubic equation can be found as $x \approx 36.92$. Using $D = 3$ in, we get $d = 3/36.92 = 0.08126$ in.

1.58 $c = \mu \left\{ \dfrac{3 \pi D^3 \ell}{4 d^3} \left(1 + 2 \dfrac{d}{D}\right) \right\};$

D = diameter of piston
ℓ = axial length of piston
d = radial clearance

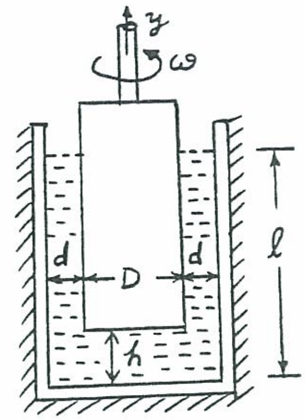
$\mu = 45\ \mu$ reynolds
(from Shigley's Mechanical Engineering Design)

Let $d = 0.001''$, $D = 2.4''$ and above equation gives

$$10^5 = (45 \times 10^{-6}) \left\{ \frac{3\pi (2.4)^3 \ell}{4 (0.001)^3} \left(1 + \frac{2 \times 0.001}{2.4}\right) \right\}$$

$\therefore\ \ell = 0.6817''$

1.59 Tangential velocity of inner cylinder $= \frac{D}{2}\omega$



For small d, rate of change of velocity of fluid is $\frac{dv}{dr} = \frac{\frac{D}{2}\omega}{d}$

shear stress between cylinders is

$$\tau = \mu \frac{dv}{dr} = \mu \frac{D\omega}{2d}$$

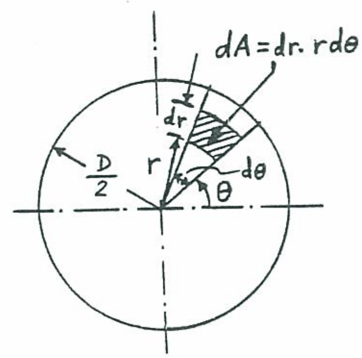
and shear force is

$$F = \tau \cdot \text{Area} = \tau\, \pi D(l-h) = \frac{\pi\mu D^2\omega(l-h)}{2d}$$

Torque developed $= M_{t1} = F \cdot \frac{D}{2}$

For small h, rate of change of velocity of fluid in vertical direction is

$$\frac{dv}{dy} = \frac{r\omega}{h}$$



Shear stress is $\tau = \mu \frac{dv}{dy} = \frac{\mu r \omega}{h}$

Force on area $dA = dF = \tau\, dA$

Torque between bottom surfaces of cylinders is

$$M_{t2} = \iint_{\text{area}} dM_{t2} \cdot dA \quad \text{where } dM_{t2} = dF \cdot r = \frac{\mu r^3 \omega}{h}\, dr\, d\theta$$

i.e., $M_{t2} = \frac{\mu\omega}{h} \int_{r=0}^{D/2} \int_{\theta=0}^{2\pi} r^3 . dr\, d\theta = \frac{\mu\omega\pi D^4}{64h}$

Total torque $= M_t = M_{t1} + M_{t2} = \frac{\pi\mu D^3\omega(l-h)}{4d} + \frac{\pi\mu\omega D^4}{64h}$

Expressing M_t as $c_t\, v = c_t\, \omega D/2$, we get damping constant:

$$c_t = \frac{\pi\mu D^2(l-h)}{2d} + \frac{\pi\mu D^3}{32h}$$

1.69

$F = a\,\dot{x} + b\,\dot{x}^2 = 5\,\dot{x} + 0.2\,\dot{x}^2$

$F(\dot{x}) \approx F(\dot{x}_0) + \frac{dF}{d\dot{x}}\Big|_{\dot{x}_0} (\dot{x} - \dot{x}_0)$

At $\dot{x}_0 = 5$ m/s, $F(\dot{x}_0) = 5\,(5) + 0.2\,(25) = 30$ N , $\frac{dF}{d\dot{x}}\Big|_{\dot{x}_0} = (5 + 0.4\,\dot{x})\Big|_5 = 7$ and hence

$F(\dot{x}) = 30 + 7\,(\dot{x} - 5) = 7\,\dot{x} - 5.$

Thus the linearized damping constant is given by $F(\dot{x}) \approx 7\,\dot{x} = c_{eq}\,\dot{x}$ or $c_{eq} = 7$ N–s/m.

1.70

Damping constant due to skin friction drag is:

$$c = 100\,\mu\,\ell^2\,d \tag{1}$$

Damping constant of a plate-type damper is:

$$c_p = \frac{\mu\,A}{h} \tag{2}$$

where A = area of plates and h = distance between the plates. If the area of plates (A) in Fig. 1.42 is taken to be same as the area of the plate shown in Fig, 1.107, we have $A = \ell\,d$. Equating (1) and (2) gives

$$100\,\mu\,\ell^2\,d = \frac{\mu\,\ell\,d}{h} \tag{3}$$

from which the clearance between the plates can be determined as $h = \frac{1}{100\,\ell}$.

1.71

$$c = \frac{6\,\pi\,\mu\,\ell}{h^3}\left\{\left(a - \frac{h}{2}\right)^2 - r^2\right\}\left(\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right)$$

When $\mu = 0.3445$ Pa-s, $\ell = 0.1$ m, $h = 0.001$ m, $a = 0..02$ m, and $r = 0.005$ m:

$$c = \frac{6\,\pi\,(0.3445)\,(0.1)}{(10^{-3})^3}\left\{(0.02 - 0.0005)^2 - 0.005^2\right\}\left(\frac{0.02^2 - 0.005^2}{0.02 - 0.0005} - 0.001\right)$$

$$= 4{,}205.6394 \text{ N–s/m}$$

1.72

$$c = \frac{6\pi\mu l}{h^3}\left[\left(a - \frac{h}{2}\right)^2 - r^2\right]\left[\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right]$$

Basic data: $l = 10$ cm, $h = 0.1$ cm, $a = 2$ cm, $r = 0.5$ cm, $\mu = 0.3445$

Damping constant with basic data:

$c = 4{,}205.6230$ N-s/m

(a) r changed to 1 cm; new $c = 2{,}617.7920$ N-s/m

(b) h changed to 0.05 cm; new $c = 35{,}060.8910$ N-s/m

(c) a changed to 4 cm; new $c = 38{,}754.5860$ N-s/m

1.75 $\vec{x} = 5 + 2i = A e^{i\theta} = A\cos\theta + i A \sin\theta$

$A\cos\theta = 5$

$A\sin\theta = 2$

$$A = \sqrt{(A\cos\theta)^2 + (A\sin\theta)^2} = \sqrt{5^2 + 2^2} = 5.3852$$

$$\theta = \tan^{-1}\left(\frac{A\sin\theta}{A\cos\theta}\right) = \tan^{-1}\left(\frac{2}{5}\right) = 21.8014^\circ$$

1.76 $\vec{x}_1 = 1 + 2i = a_1 + a_2 i \quad , \quad \vec{x}_2 = 3 - 4i = b_1 + b_2 i$

$$\vec{x} = \vec{x}_1 + \vec{x}_2 = (a_1 + b_1) + i(a_2 + b_2) = 4 - 2i$$

$$= A e^{i\theta} = A\cos\theta + i A\sin\theta$$

$$A = \sqrt{4^2 + (-2)^2} = 4.4721$$

$$\theta = \tan^{-1}\left(\frac{-2}{4}\right) = -26.5651^\circ$$

1.77 $z_1 = (3 - 4i),\ z_2 = (1 + 2i)$

$z = z_1 - z_2 = (3 - 4i) - (1 + 2i) = 2 - 6i = A e^{i\theta}$

where $A = \sqrt{2^2 + (-6)^2} = 6.3246$ and $\theta = \tan^{-1}\left(\frac{-6}{2}\right) = \tan^{-1}(-3) = -1.2490$ rad

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1.78 $z_1 = 1 + 2i,\quad z_2 = 3 - 4i$

$z = z_1 z_2 = (1 + 2i)(3 - 4i) = 11 + 2i = A e^{i\theta}$

where $A = \sqrt{11^2 + 2^2} = 11.1803$ and $\theta = \tan^{-1}(2/11) = 0.1798$ rad

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1.79 $z = \dfrac{z_1}{z_2} = \dfrac{1 + 2i}{3 - 4i} = \dfrac{(1 + 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \dfrac{-5 + 10i}{25} = -0.2 + 0.4i = A e^{i\theta}$

where $A = \sqrt{(-0.2)^2 + (0.4)^2} = 0.4472$

and $\theta = \tan^{-1}\left(\frac{-0.4}{0.2}\right) = \tan^{-1}(-2) = -1.1071$ rad

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1.80 $x(t) = X \cos \omega t\,,\quad y(t) = Y \cos(\omega t + \phi)$

(a) $\dfrac{x^2}{X^2} = \cos^2 \omega t,\ \dfrac{y^2}{Y^2} = \cos^2(\omega t + \phi),$

$$2\,\frac{x\,y}{X\,Y}\cos\phi = 2\cos\omega t\cos(\omega t+\phi)\cos\phi$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos\phi$$
$$= \cos^2\omega t + \cos^2(\omega t+\phi) - 2\cos\omega t\cos\phi\cos(\omega t+\phi) \qquad (1)$$

Noting that $\cos^2\alpha = \dfrac{1}{2}(1+\cos 2\alpha)$, Eq. (1) can be rewritten as

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos\phi$$
$$= \frac{1}{2} + \frac{1}{2}\cos 2\omega t + \frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\phi) - 2\cos\omega t\cos\phi\cos(\omega t+\phi)$$
$$= 1 + \frac{1}{2}\left\{2\cos\frac{2\omega t + 2\omega t + 2\phi}{2}\cos\frac{2\omega t - 2\omega t - 2\phi}{2}\right\}$$
$$- 2\cos\omega t\cos\phi\cos(\omega t+\phi)$$
$$= 1 + \cos(2\omega t+\phi)\cos\phi - 2\cos\omega t\cos\phi\cos(\omega t+\phi)$$
$$= 1 + \cos(2\omega t+\phi)\cos\phi - 2\cos\phi\left\{\frac{1}{2}\left[\cos(\omega t+\phi-\omega t) + \cos(\omega t+\phi+\omega t)\right]\right\}$$
$$= 1 + \cos\phi\cos(2\omega t+\phi) - \cos\phi\left\{\cos\phi + \cos(2\omega t+\phi)\right\}$$
$$= 1 - \cos^2\phi = \sin^2\phi \qquad (2)$$

(b) When $\phi = 0$, Eq. (2) reduces to

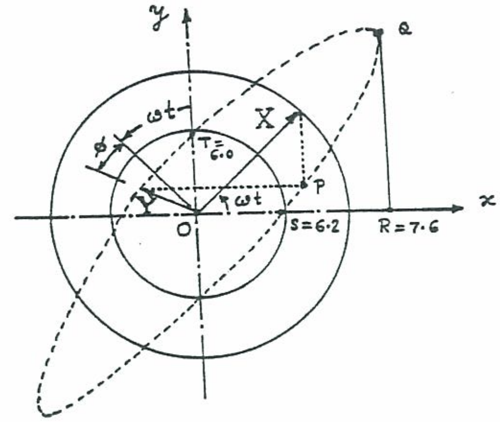
$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y} = \left(\frac{x}{X} - \frac{y}{Y}\right)^2 = 0$$

which gives $X = \pm\dfrac{X}{Y}\,y$. This indicates that the locus of the resultant motion is a straight line. When $\phi = \dfrac{\pi}{2}$, Eq. (2) reduces to

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$

which denotes an ellipse with its major and minor axes along x and y directions, respectively. When $\phi = \pi$, Eq. (2) reduces to that of a straight line as in the case of $\phi = 0$.

1.81



Equation for resultant motion:

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos^2\phi = \sin^2\phi \qquad (1)$$

When $y = 0$, Eq. (1) reduces to $\frac{x^2}{X^2} = \sin^2\phi$ and hence:

$$x = \pm X \sin\phi = \pm\, 6.2 = \text{OS in figure} \qquad (2)$$

When $x = 0$, Eq. (1) reduces to $\frac{y^2}{Y^2} = \sin^2\phi$ and hence:

$$y = \pm Y \sin\phi = \pm\, 6.0 = \text{OT in figure} \qquad (3)$$

It can be seen that

$$\text{OR} = X\cos\phi = 7.6 \text{ in figure} \qquad (4)$$

$$\frac{\text{OS}}{\text{OR}} = \frac{X\sin\phi}{X\cos\phi} = \tan\phi = \frac{6.2}{7.6} = 0.8158 \ \text{ or } \ \phi = 39.2072° \qquad (5)$$

From Eqs. (2) and (4), we find

$$X = \sqrt{(X\sin\phi)^2 + (X\cos\phi)^2} = \sqrt{(6.2)^2 + (7.6)^2} = 9.8082 \text{ mm}$$

Equations (3) and (5) give

$$Y = \frac{6.0}{\sin\phi} = \frac{6.0}{\sin 39.2072°} = 9.4918 \text{ mm}$$

1.82 (a) $x(t) = \frac{A}{1000}\cos(50t+\alpha)$ m where A is in mm ---- (E_1)

$x(0) = \frac{A}{1000}\cos\alpha = 0.003$, $A\cos\alpha = 3$ ---- (E_2)

$\dot{x}(0) = -\frac{50A}{1000}\sin\alpha = 1$, $A\sin\alpha = -20$ ---- (E_3)

$A = \{(A\cos\alpha)^2 + (A\sin\alpha)^2\}^{1/2} = 20.2237$ mm

$\alpha = \tan^{-1}\left(\frac{A\sin\alpha}{A\cos\alpha}\right) = \tan^{-1}(-6.6667) = -81.4692^\circ = -1.4219$ rad

$x(t) = 20.2237\cos(50t - 1.4219)$ mm

(b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Eq. (E_1) can be expressed as $x(t) = A\cos 50t\cdot\cos\alpha - A\sin 50t\cdot\sin\alpha$

$= A_1\cos\omega t + A_2\sin\omega t$

where $\omega = 50$, $A_1 = A\cos\alpha$, $A_2 = -A\sin\alpha$

$\therefore x(t) = (3\cos 50t + 20\sin 50t)$ mm

1.83 $x(t) = A_1\cos\omega t + A_2\sin\omega t$

$\frac{dx}{dt}(t) = -A_1\omega\sin\omega t + A_2\omega\cos\omega t$, $\frac{d^2x}{dt^2} = -A_1\omega^2\cos\omega t - A_2\omega^2\sin\omega t$

$\frac{d^2x}{dt^2} = -\omega^2 x(t)$ where ω^2 is a constant

Hence $x(t)$ is a simple harmonic motion.

1.84 (a) Using trigonometric relations:

$x_1(t) = 5(\cos 3t\cos 1 - \sin 3t\sin 1)$

$x_2(t) = 10(\cos 3t\cos 2 - \sin 3t\sin 2)$

$x(t) = x_1(t) + x_2(t) = \cos 3t(5\cos 1 + 10\cos 2) - \sin 3t(5\sin 1 + 10\sin 2)$

If $x(t) = A\cos(\omega t+\alpha) = A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha$,

$\omega = 3$, $A\cos\alpha = 5\cos 1 + 10\cos 2 = -1.4599$,

$A\sin\alpha = 5\sin 1 + 10\sin 2 = 13.3003$

$A = \sqrt{(A\cos\alpha)^2 + (A\sin\alpha)^2} = 13.3802$

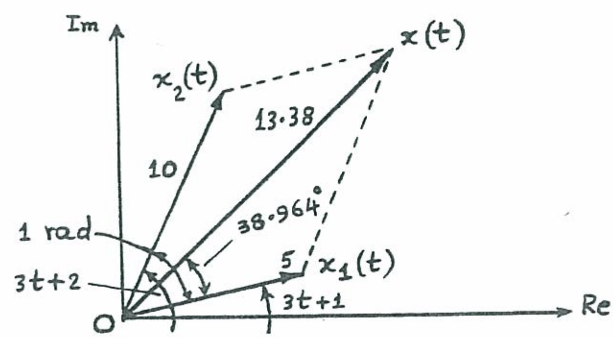
$\alpha = \tan^{-1}\left(\frac{A\sin\alpha}{A\cos\alpha}\right) = \tan^{-1}(-9.1104) = 96.2640^\circ = 1.68$ rad

Angle between $x_1(t)$ and $x(t)$ is $96.2640^\circ - 57.3^\circ = 38.964^\circ$

(b) Using vector addition:

For an arbitrary value of $(\omega t+1)$, harmonic motions $x_1(t)$ and $x_2(t)$ can be shown as in the figure. From vector addition, we find

$x(t) \simeq 13.38\cos(\omega t + 1.68)$



(c) Using complex numbers:

$$x_1(t) = \text{Re}\left\{A_1\ e^{i(\omega t+1)}\right\} = \text{Re}\left\{5\ e^{i(\omega t+1)}\right\}$$

$$x_2(t) = \text{Re}\left\{A_2\ e^{i(\omega t+2)}\right\} = \text{Re}\left\{10\ e^{i(\omega t+2)}\right\}$$

If $x(t) = \text{Re}\left\{A\ e^{i(\omega t+\alpha)}\right\}$,

$$A\cos(3t+\alpha) = A_1\cos(3t+1) + A_2\cos(3t+2)$$

i.e. $A(\cos 3t\cos\alpha - \sin 3t\sin\alpha) = 5(\cos 3t\cdot\cos 1 - \sin 3t\cdot\sin 1) + 10(\cos 3t\cdot\cos 2 - \sin 3t\cdot\sin 2)$

i.e. $A\cos\alpha = 5\cos 1 + 10\cos 2$, $\quad A\sin\alpha = 5\sin 1 + 10\sin 2$

$A = 13.3802$, $\alpha = 1.68$ rad

$$x(t) = \text{Re}\left\{13.3802\ e^{i(3t+1.68)}\right\}$$

1.85 $x(t) = 10\sin(\omega t + 60°) = x_1(t) + x_2(t)$

where $x_1(t) = 5\sin(\omega t + 30°)$ and $x_2(t) = A\sin(\omega t + \alpha°)$

$$10(\sin\omega t\cos 60° + \cos\omega t\sin 60°) = 5(\sin\omega t\cdot\cos 30° + \cos\omega t\cdot\sin 30°) + A(\sin\omega t\cdot\cos\alpha° + \cos\omega t\cdot\sin\alpha°)$$

$10\cos 60° = 5\cos 30° + A\cos\alpha°$; $\quad A\cos\alpha° = 0.6699$

$10\sin 60° = 5\sin 30° + A\sin\alpha°$; $\quad A\sin\alpha° = 6.1603$

$$A = \sqrt{0.6699^2 + 6.1603^2} = 6.1966$$

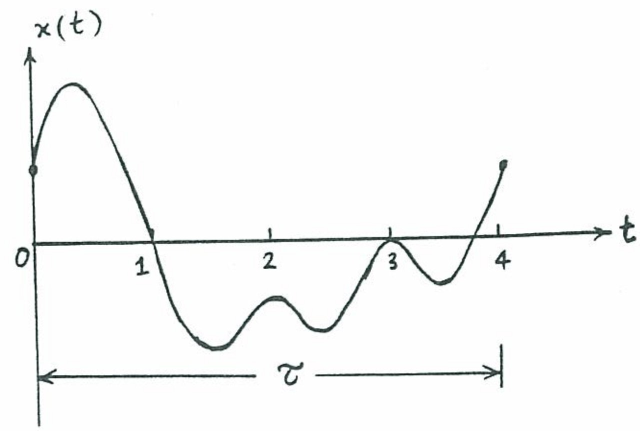
$$\alpha = \tan^{-1}(6.1603/0.6699) = 83.7938°$$

$$x_2(t) = 6.1966\sin(\omega t + 83.7938°)$$

1.86 $x(t) = \frac{1}{2}\cos\frac{\pi}{2}t + \sin\pi t$

$$= \frac{1}{2}\cos\frac{\pi}{2}t\left(1 + 4\sin\frac{\pi}{2}t\right)$$

From the nature of the graph of $x(t)$, it can be seen that $x(t)$ is periodic with a time period of $\tau = 4$.



1.87 If $x(t)$ is harmonic, $\ddot{x}(t) = -\omega^2\ x(t)$

Here $x(t) = 2\cos 2t + \cos 3t$

$\ddot{x}(t) = -8\cos 2t - 9\cos 3t \neq -$ constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

1.88

$x(t) = \frac{1}{2} \cos \frac{\pi}{2} t - \cos \pi t$

$\ddot{x}(t) = -\frac{\pi^2}{8} \cos \frac{\pi}{2} t + \pi^2 \cos \pi t \neq -$ constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

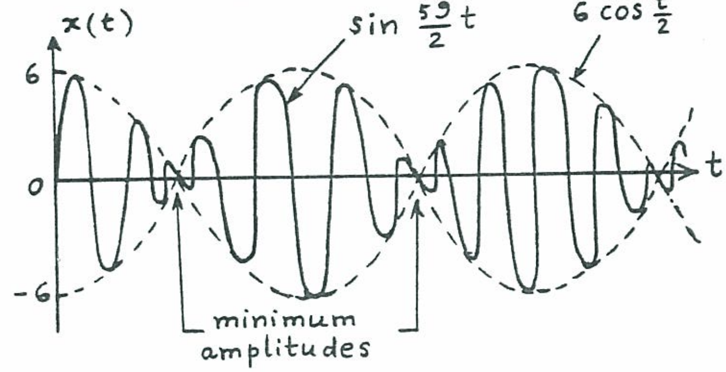
1.89

$x(t) = x_1(t) + x_2(t) = 3 \sin 30t + 3 \sin 29t$

since $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$,

$x(t) = \left(6 \cos \frac{t}{2}\right) \sin \frac{59}{2} t$

This equation shows that the amplitude $(6 \cos \frac{t}{2})$ varies with time between a maximum value of 6 and a minimum value of 0. The frequency of this oscillation (beat frequency) is $\omega_b = 1$.



Note: Beat frequency is twice the frequency of the term $6 \cos \frac{t}{2}$ since two peaks pass in each cycle of $(6 \cos \frac{t}{2})$.

1.90

The resultant motion of two harmonic motions having identical amplitudes (X) but slightly different frequencies (ω and $\omega + \delta\omega$) is given by Eq. (1.67):

$$x(t) = 2\,X \cos \left(\omega\, t + \frac{\delta\omega\, t}{2}\right) \cos \left(\frac{\delta\, \omega\, t}{2}\right)$$

Thus the maximum amplitude of the resultant motion is equal to 2X and the beat frequency is equal to $\delta\omega$. From Fig. 1.113, we find that $2X \approx 5$ mm or $X = 2.5$ mm and

$$\frac{\delta\omega}{2} = \frac{2\,\pi}{\tau_{\text{beat}}} = \frac{2\,\pi}{\tau_{\text{larger}}} = \frac{2\,\pi}{2\,(12.6 - 4.2)} = 0.374 \text{ rad/sec}$$

or $\delta\omega = 0.748$ rad/sec and $\omega + \frac{\delta\,\omega}{2} = \frac{2\,\pi}{\tau_{\text{smaller}}} = \frac{2\,\pi}{1} = 6.2832$ rad/sec

Hence $\omega = 6.2832 - 0.3740 = 5.9092$ rad/sec. Thus the amplitudes of the two motions $= X = 2.5$ mm and their frequencies are $\omega = 5.9092$ rad/sec and $\omega + \delta\omega = 5.9092 + 0.7480 = 6.6572$ rad/sec.

1.91

$A = 0{\cdot}05$ m, $\omega = 10$ Hz $= 62{\cdot}832$ rad/sec

period $= \tau = \frac{2\pi}{\omega} = \frac{2\pi}{62{\cdot}832} = 0{\cdot}1$ sec

maximum velocity $= A\omega = 0{\cdot}05 \times 62{\cdot}832 = 3{\cdot}1416$ m/s

maximum acceleration $= A\omega^2 = 0{\cdot}05\,(62{\cdot}832)^2 = 197{\cdot}393$ m/s^2

1.92

$\omega = 15$ cps $= 94.248$ rad/sec

$\ddot{x}_{max} = 0.5g = 0.5(9.81) = 4.905$ m/s² $= A\omega^2$

$A =$ amplitude $= 4.905/(94.248)^2 = 0.0005522$ m

$\dot{x}_{max} =$ max. velocity $= A\omega = 0.05204$ m/s

1.93

$x = A \cos \omega t$, $x_{max} = A = 0.25$ mm , $\ddot{x} = -\omega^2 A \cos \omega t$

$\ddot{x}_{max} = A\omega^2 = 0.4g = 3924$ mm/s² ; $\omega^2 = 2924/A = 15696$ (rad/s)²

operating speed of pump $= \omega = 125.2837$ rad/s $= 19.9395$ rpm

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1.104 $x(t) = X \sin \frac{2\pi t}{\tau}$; $x_{rms} = \left[\frac{1}{\tau} \int_0^\tau X^2 \text{ sun}^2 \frac{2\pi t}{\tau} dt \right]^{\frac{1}{2}}$

Using $\sin^2 \frac{2\pi t}{\tau} = \frac{1 - \cos \frac{4\pi t}{\tau}}{2}$, we obtain

$$x_{rms} = \left[\frac{X^2}{\tau} \int_0^\tau \left(\frac{1}{2} - \frac{1}{2} \cos \frac{4\pi t}{\tau} \right) dt \right]^{\frac{1}{2}} = \left[\frac{X^2}{\tau} \left\{ \frac{t}{2} - \frac{1}{2} \frac{\tau}{4\pi} \sin \frac{4\pi t}{\tau} \right\} \Big|_0^\tau \right]^{\frac{1}{2}}$$

$$= \left[\frac{X^2}{\tau} \left\{ \frac{\tau}{2} - \frac{\tau}{8\pi} \sin 4\pi - 0 + 0 \right\} \right]^{\frac{1}{2}} = \frac{X}{\sqrt{2}}$$

1.105 $x(t) = \frac{A\,t}{\tau}$; $0 \le t \le \tau$

$$x_{rms} = \left\{ \frac{1}{\tau} \int_0^\tau \frac{A^2}{\tau^2} t^2 \, dt \right\}^{\frac{1}{2}} = \left\{ \frac{1}{\tau} \frac{A^2}{\tau^2} \left(\frac{t^3}{3} \right)_0^\tau \right\}^{\frac{1}{2}} = \left\{ \frac{A^2}{\tau^3} \frac{\tau^3}{3} \right\}^{\frac{1}{2}} = \left(\frac{A^2}{3} \right)^{\frac{1}{2}} = \frac{A}{\sqrt{3}}$$

1.106 For even functions, $x(-t) = x(t)$.

From Eq. (1.73), $b_n = \frac{2}{\tau} \int_0^\tau x(t) \sin n\omega t \cdot dt = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \sin n\omega t \, dt$

$$= \frac{2}{\tau} \left[\int_{-\frac{\tau}{2}}^{0} x(t) \sin n\omega t \cdot dt + \int_0^{\tau/2} x(t) \sin n\omega t \cdot dt \right] \quad \text{---- } (E_1)$$

Since $\sin(-n\omega t) = -\sin(n\omega t)$ = odd function of t, the product of $x(t)$ and $\sin n\omega t$ is an odd function.

Further, for an odd function $f(t)$, $f(-t) = -f(t)$, and

$$\int_{-a}^{a} f(t)\, dt = \int_{-a}^{0} f(t)\, dt + \int_{0}^{a} f(t)\, dt = \int_{0}^{a} f(-t)\, dt + \int_{0}^{a} f(t)\, dt$$

$$= -\int_{0}^{a} f(t)\, dt + \int_{0}^{a} f(t)\, dt = 0 \qquad \text{----}(E_2)$$

Equations (E_1) and (E_2) lead to $b_n = 0$.

Also, since $\cos n\omega t$ is an even function, we get

$$a_n = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cos n\omega t\, dt = \frac{4}{\tau} \int_{0}^{\tau/2} x(t) \cos n\omega t\, dt$$

For odd functions, $x(-t) = -x(t)$.

From Eq. (1.72), $a_n = \frac{2}{\tau} \int_{0}^{\tau} x(t) \cos n\omega t\, dt = \frac{2}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cos n\omega t\, dt$

Since $\cos n\omega t$ is an even function, $\cos(-n\omega t) = \cos(n\omega t)$, the product of $x(t)$ and $\cos n\omega t$ is an odd function.

Hence $a_n = 0$.

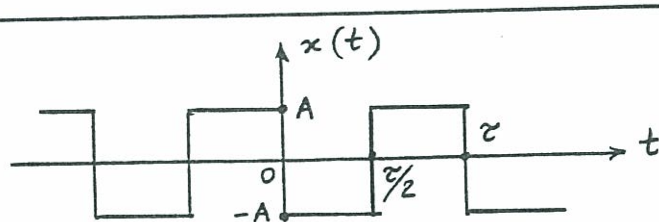
Further, since $\sin n\omega t$ is an odd function, $x(t) \sin n\omega t$ is an even function and hence

$$b_n = \frac{4}{\tau} \int_{0}^{\tau/2} x(t) \sin n\omega t\, dt$$

1.107

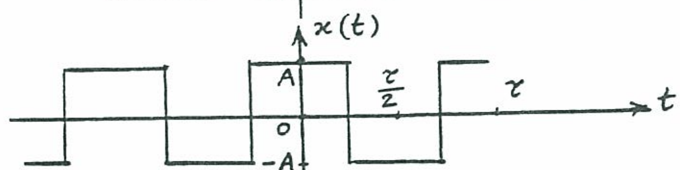
$$x(t) = \begin{cases} -A, & 0 \le t \le \frac{\tau}{2} \\ A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

(a)



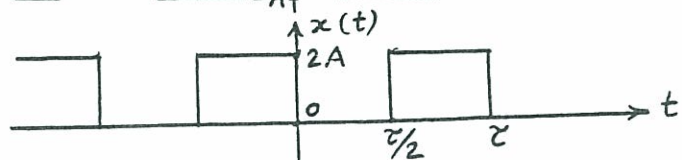
$$x(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ -A, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ A, & \frac{3\tau}{4} \le t \le \tau \end{cases}$$

(b)



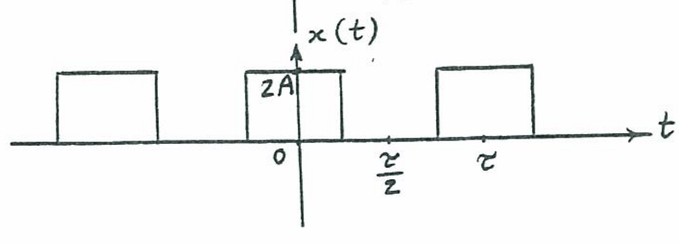
$$x(t) = \begin{cases} 0, & 0 \le t \le \tau/2 \\ 2A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

(c)



$$x(t) = \begin{cases} 2A, & 0 \le t \le \frac{\tau}{4} \\ 0, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ 2A, & \frac{3\tau}{4} \le t \le \tau \end{cases}$$

(d)



(a) $x(-t) = -x(t)$, odd function, hence $a_0 = a_n = 0$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t \cdot dt = \frac{2}{\tau}\left[-A\int_0^{\tau/2}\sin n\omega t \cdot dt + A\int_{\tau/2}^{\tau}\sin n\omega t \cdot dt\right]$$

$$= -\frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_0^{\pi/\omega} + \frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_{\pi/\omega}^{2\pi/\omega}$$

$$= \frac{2A}{\tau n\omega}\left(2\cos n\pi - \cos 0 - \cos 2n\pi\right)$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t$$

(b) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[A\cdot(t)_0^{\tau/4} - A(t)_{\tau/4}^{3\tau/4} + A(t)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t \cdot dt$$

$$= \frac{2A}{\tau n\omega}\left[\sin n\omega t\Big|_0^{\tau/4} - \sin n\omega t\Big|_{\tau/4}^{\frac{3\tau}{4}} + \sin n\omega t\Big|_{3\tau/4}^{\tau}\right]$$

$$= \frac{A}{n\pi}\left[2\sin\frac{n\pi}{2} - 2\sin\frac{3n\pi}{2} + \sin 2\pi n\right] = \begin{cases} 4A/n\pi & \text{for } n = 1,5,9,\ldots \\ -4A/n\pi & \text{for } n = 3,7,11,\ldots \end{cases}$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(n-1)t}{\tau}$$

(c) $a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[0 + 2A\ (t)\Big|_{\tau/2}^{\tau}\right] = 2A$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{4A}{n\omega\tau}\left(\sin n\omega t\right)_{\tau/2}^{\tau} = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = -\frac{4A}{n\omega\tau}\left(\cos n\omega t\right)_{\tau/2}^{\tau}$$

$$= -\frac{4A}{n\omega\tau}\left(\cos 2\pi n - \cos n\pi\right)$$

$$\therefore\ x(t) = -\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t \qquad \text{with } \omega = 2\pi/\tau.$$

(d) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[2A\left(\frac{\tau}{4} - 0\right) + 2A\left(\tau - \frac{3\tau}{4}\right)\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{4A}{n\omega\tau}\left[\left(\sin n\omega t\right)_0^{\tau/4} + \left(\sin n\omega t\right)_{\frac{3\tau}{4}}^{\tau}\right]$$

$$= \frac{4A}{n\omega\tau}\left(\sin\frac{n\pi}{2} + \sin 2n\pi - \sin\frac{3n\pi}{2}\right)$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(2n-1)t}{\tau} \quad \text{with } \omega = 2\pi/\tau.$$

1.108

$$x(t) = \begin{cases} A\sin\frac{2\pi t}{\tau}, & 0 \le t \le \frac{\tau}{2} \\ 0, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\,dt = \frac{2A}{\tau}\left(-\frac{\tau}{2\pi}\cdot\cos\frac{2\pi t}{\tau}\right)_0^{\tau/2}$$

$$= \frac{2A}{\pi}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cdot\cos n\omega t\,.\,dt \qquad \text{----} (E_1)$$

Using the relation $\sin m\omega t\cdot\cos n\omega t = \frac{\sin(m+n)\omega t + \sin(m-n)\omega t}{2}$,
Eq. (E_1) can be rewritten as

$$a_n = \frac{A}{\tau}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

when $n = 1$, $a_1 = \frac{A}{\tau}\int_0^{\pi/\omega}\sin 2\omega t\,.\,dt = 0$

when $n = 2, 3, 4, \ldots$, $a_n = \frac{A}{\tau}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$

$$= \frac{A}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n} + \frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{2A}{(n-1)(n+1)\pi} & \text{if } n \text{ is even} \end{cases}$$

Similarly

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cos n\omega t\,dt$$

$$= \frac{A}{\tau}\int_0^{\tau/2}\left[\cos(1-n)\omega t - \cos(1+n)\omega t\right]dt$$

when $n = 1$, $b_1 = \frac{A}{\tau}\int_0^{\pi/\omega}(dt - \cos 2\omega t)\,dt = \frac{A}{2}$

when $n = 2, 3, 4, \ldots$, $b_n = \frac{A}{\tau}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^{\pi/\omega} = 0$

$$\therefore\ x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega t - \frac{2A}{\pi}\sum_{n=2,4,6,\ldots}\frac{\cos n\omega t}{(n^2-1)}$$

1.109

$$x(t) = \begin{cases} \dfrac{2At}{\tau}, & 0 \le t \le \dfrac{\tau}{2} \\ -\dfrac{2At}{\tau} + 2A, & \dfrac{\tau}{2} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2} \frac{2At}{\tau}\cdot dt + \int_{\tau/2}^{\tau}\left(-\frac{2At}{\tau} + 2A\right)\cdot dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_0^{\tau/2} - \frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_{\tau/2}^{\tau} + 2A\cdot t\Big|_{\tau/2}^{\tau}\right]$$

$$= \frac{2}{\tau}\left[\frac{A\tau}{4} - \frac{3A\tau}{4} + A\tau\right] = A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt$$

$$= \frac{2}{\tau}\left[\int_0^{\frac{\tau}{2}} \frac{2A}{\tau}\,t\cos n\omega t\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}t + 2A\right)\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\sin n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

As $\tau = \frac{2\pi}{\omega}$,

$$a_n = \frac{\omega}{\pi}\left[\frac{A\omega}{\pi n^2\omega^2}\cos n\pi - \frac{A\omega}{\pi n^2\omega^2} - \frac{A\omega}{\pi n^2\omega^2}\cos 2\pi n + \frac{A\omega}{\pi n^2\omega^2}\cos n\pi\right]$$

$$= \frac{2A}{n^2\pi^2}\left(\cos n\pi - 1\right) = \begin{cases} -\dfrac{4A}{n^2\pi^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2}\frac{2A}{\tau}\,t\sin n\omega t\,dt\right.$$

$$\left. + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}t + 2A\right)\sin n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[-\frac{A}{n\omega}\cos n\pi + \frac{2A}{n\omega}\cos 2n\pi - \frac{A}{n\omega}\cos n\pi - \frac{2A}{n\omega}\cos 2n\pi + \frac{2A}{n\omega}\cos n\pi\right] = 0$$

$$\therefore \; x(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

1.110

$$x(t) = \begin{cases} \dfrac{4At}{\tau} & , \; 0 \le t \le \dfrac{\tau}{4} \\ -\dfrac{4At}{\tau} + 2A & , \; \dfrac{\tau}{4} \le t \le \dfrac{3\tau}{4} \\ \dfrac{4At}{\tau} - 4A & , \; \dfrac{3\tau}{4} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t)\, dt = \frac{2}{\tau} \left[\frac{4A}{\tau} \frac{t^2}{2} \Big|_0^{\tau/4} + \left(-\frac{4A}{\tau} \frac{t^2}{2} + 2At \right)_{\tau/4}^{3\tau/4} + \left(\frac{4A}{\tau} \frac{t^2}{2} - 4At \right)_{3\tau/4}^{\tau} \right] = 0$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t \, dt = \frac{2}{\tau} \left[\frac{4A}{\tau} \int_0^{\tau/4} t \cos n\omega t \cdot dt - \frac{4A}{\tau} \int_{\tau/4}^{3\tau/4} t \cos n\omega t \, dt + 2A \int_{\tau/4}^{3\tau/4} \cos n\omega t \, dt + \frac{4A}{\tau} \int_{3\tau/4}^{\tau} t \cos n\omega t \, dt - 4A \int_{3\tau/4}^{\tau} \cos n\omega t \, dt \right]$$

$$= \frac{2}{\tau} \left[\frac{4A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2 \omega^2} \right\}_0^{\tau/4} - \frac{4A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2} \right\}_{\tau/4}^{3\tau/4} + 2A \left(\frac{\sin n\omega t}{n\omega} \right)_{\tau/4}^{3\tau/4} + \frac{4A}{\tau} \left\{ t \frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2} \right\}_{3\tau/4}^{\tau} - 4A \left(\frac{\sin n\omega t}{n\omega} \right)_{3\tau/4}^{\tau} \right]$$

$$= \frac{\omega}{\pi} \left[\sin \frac{n\pi}{2} \left(\frac{A}{n\omega} + \frac{A}{n\omega} - \frac{2A}{n\omega} \right) + \cos \frac{n\pi}{2} \left(\frac{2A}{\pi n^2 \omega} + \frac{2A}{\pi n^2 \omega} \right) + \sin \frac{3n\pi}{2} \left(-\frac{3A}{n\omega} + \frac{2A}{n\omega} - \frac{3A}{n\omega} + \frac{4A}{n\omega} \right) + \cos \frac{3n\pi}{2} \left(-\frac{2A}{\pi n^2 \omega} - \frac{2A}{\pi n^2 \omega} \right) + \cos 2\pi n \left(\frac{2A}{\pi n^2 \omega} \right) - \cos 0 \left(\frac{2A}{\pi n^2 \omega} \right) \right] = 0$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t \, dt = \frac{2}{\tau} \left[\frac{4A}{\tau} \int_0^{\tau/4} t \sin n\omega t \cdot dt - \frac{4A}{\tau} \int_{\tau/4}^{3\tau/4} t \sin n\omega t \cdot dt + 2A \int_{\tau/4}^{3\tau/4} \sin n\omega t \cdot dt + \frac{4A}{\tau} \int_{3\tau/4}^{\tau} t \sin n\omega t \cdot dt - 4A \int_{3\tau/4}^{\tau} \sin n\omega t \cdot dt \right]$$

$$= \frac{2}{\tau} \left[\frac{4A}{\tau} \left\{ \frac{1}{n^2\omega^2} \sin n\omega t - \frac{t}{n\omega} \cos n\omega t \right\}_0^{\tau/4 = \frac{\pi}{2\omega}} - \frac{4A}{\tau} \left\{ \frac{1}{n^2\omega^2} \sin n\omega t - \frac{t}{n\omega} \cos n\omega t \right\}_{\frac{\tau}{4} = \frac{\pi}{2\omega}}^{\frac{3\tau}{4} = \frac{3\pi}{2\omega}} + 2A \left(-\frac{\cos n\omega t}{n\omega} \right)_{\frac{\tau}{4} = \frac{\pi}{2\omega}}^{\frac{3\tau}{4} = \frac{3\pi}{2\omega}} + \frac{4A}{\tau} \left\{ \frac{1}{n^2\omega^2} \sin n\omega t \right. \right.$$

$$- \frac{t}{n\omega} \cos n\omega t \Big|_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} - 4A \left(- \frac{\cos n\omega t}{n\omega} \right)\Big|_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} \Big]$$

$$= \frac{4A}{\pi^2 n^2} \left(\sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8A}{\pi^2 n^2} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore\ x(t) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\sin n\omega t}{n^2}$$

1.111

$$x(t) = A\left(1 - \frac{t}{\tau}\right), \quad 0 \le t \le \tau$$

$$a_0 = \frac{2}{\tau} \int_0^\tau x(t)\, dt = \frac{2A}{\tau} \int_0^\tau \left(1 - \frac{t}{\tau}\right) dt = \frac{2A}{\tau}\left(t - \frac{t^2}{2\tau}\right)\Big|_0^\tau = A$$

$$a_n = \frac{2}{\tau} \int_0^\tau x(t) \cos n\omega t\, dt = \frac{2A}{\tau}\left(\frac{\sin n\omega t}{n\omega} - \frac{t}{\tau} \frac{\sin n\omega t}{n\omega} - \frac{\cos n\omega t}{\tau n^2 \omega^2} \right)\Big|_0^{2\pi/\omega}$$

$$= 0$$

$$b_n = \frac{2}{\tau} \int_0^\tau x(t) \sin n\omega t\, dt = \frac{2A}{\tau}\left(- \frac{\cos n\omega t}{n\omega} + \frac{t}{\tau} \frac{\cos n\omega t}{n\omega} - \frac{\sin n\omega t}{\tau n^2 \omega^2} \right)\Big|_0^{2\pi/\omega}$$

$$= \frac{A}{\pi n}$$

$$\therefore\ x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega t}{n}$$

1.112 The truncated series of k terms can be denoted as

$$\bar{x}(t) = \frac{\bar{a}_0}{2} + \sum_{n=1}^{k} \bar{a}_n \cos n\,\omega\,t + \sum_{n=1}^{k} \bar{b}_n \sin n\,\omega\,t \tag{1}$$

with $\bar{x}(t)$ denoting an approximation to the exact $x(t)$ given by Eq. (1.70). The error to be minimized is given by

$$E = \int_{-\pi/\omega}^{\pi/\omega} e^2(t)\, dt \tag{2}$$

$$\text{where} \quad e(t) = x(t) - \bar{x}(t) \tag{3}$$

and $x(t)$ is the exact value (with infinite series on the right hand side of Eq. (1)). Treating E as a function of the unknowns $\bar{a}_n$ and $\bar{b}_n$, it can be minimized by setting:

$$\frac{\partial E}{\partial \bar{a}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{x(t) - \bar{x}(t)\right\} \left(- \cos n\,\omega\,t\right) dt = 0 \tag{4}$$

$$\frac{\partial E}{\partial \bar{b}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{x(t) - \bar{x}(t)\right\} \left(- \sin n\,\omega\,t\right) dt = 0 \tag{5}$$

Rearranging Eq. (4) gives

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\,t\;dt = \int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n\,\omega\,t\;dt \tag{6}$$

Using orthogonalty property, the right hand side of Eq. (6) can be expressed as

$$\int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n\,\omega\,t\;dt = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{\bar{a}_n\,\pi}{\omega} & \text{for } m = n \end{cases} \tag{7}$$

This leads to

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\,t\;dt = \frac{\bar{a}_n\,\pi}{\omega} \tag{8}$$

$$\text{or } \bar{a}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\,t\;dt \quad ; \quad n = 0, 1, 2, \ldots, k \tag{9}$$

In a similar manner, we can derive:

$$\bar{b}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \sin n\,\omega\,t\;dt \quad ; \quad n = 1, 2, \ldots, k \tag{10}$$

It can be observed that Eqs. (9) and (10) are simlar to those of Eqs. (E.3) and (E.4).

1.113

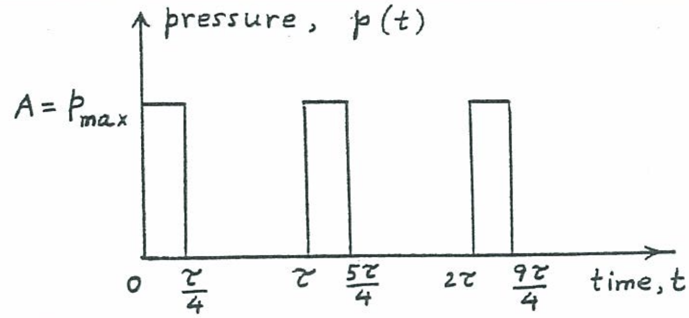
i	t_i	x_i	n = 1: $x_i \cos \frac{2\pi t_i}{0.32}$	n = 1: $x_i \sin \frac{2\pi t_i}{0.32}$	n = 2: $x_i \cos \frac{4\pi t_i}{0.32}$	n = 2: $x_i \sin \frac{4\pi t_i}{0.32}$	n = 3: $x_i \cos \frac{6\pi t_i}{0.32}$	n = 3: $x_i \sin \frac{6\pi t_i}{0.32}$
1	0.02	9	8.3149	3.4442	6.3639	6.3640	3.4441	8.3149
2	0.04	13	9.1924	9.1924	0.0000	13.0000	−9.1924	9.1923
3	0.06	17	6.5056	15.7060	−12.0209	12.0208	−15.7059	−6.5057
4	0.08	29	0.0000	29.0000	−29.0000	0.0000	0.0000	−29.0000
5	0.10	43	−16.4556	39.7267	−30.4053	−30.4059	39.7271	−16.4548
6	0.12	59	−41.7195	41.7191	0.0000	−59.0000	41.7187	41.7199
7	0.14	63	−58.2045	24.1087	44.5482	−44.5472	−24.1101	58.2040
8	0.16	57	−57.0000	0.0000	57.0000	0.0000	−57.0000	0.0000
9	0.18	49	−45.2700	−18.7518	34.6477	34.6487	−18.7505	−45.2705
10	0.20	35	−24.7485	−24.7489	0.0000	35.0000	24.7493	−24.7482
11	0.22	35	−13.3936	−32.3359	−24.7493	24.7482	32.3354	13.3950
12	0.24	41	0.0000	−41.0000	−41.0000	0.0000	0.0000	41.0000
13	0.26	47	17.9866	−43.4221	−33.2333	−33.2347	−43.4229	17.9847
14	0.28	41	28.9917	−28.9911	0.0000	−41.0000	−28.9905	−28.9923
15	0.30	13	12.0105	−4.9747	9.1927	−9.1921	4.9755	−12.0102
16	0.32	7	7.0000	0.0000	7.0000	0.0000	7.0000	0.0000

$\sum_{i=1}^{16}(\ )$	558	−166.7897	−31.3278	−11.6552	−91.5984	−43.2234	26.8281
$\frac{1}{8}\sum_{i=1}^{16}(\ )$	69.75	−20.8487	−3.9160	−1.4569	−11.4498	−5.4029	3.3535

1.114

Speed = 100 rpm

In a minute, a point will be subjected to the maximum pressure, $A = p_{max} = 100$ psi, $100 \times 4 = 400$ times. Hence



period $= \tau = \frac{60}{400} = 0.15$ sec.

$$p(t) = \begin{cases} A & , \ 0 \le t \le \tau/4 \\ 0 & , \ \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} p(t)\, dt = \frac{2}{\tau} A \left(t\right)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau}\int_0^{\tau} p(t) \cos m\omega t \, dt = \frac{2A}{\tau}\left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m} \sin \frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau}\int_0^{\tau} p(t) \sin m\omega t \, dt = -\frac{2A}{\tau}\left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos \frac{m\pi}{2} - 1\right)$$

Evaluation of a_m and b_m:

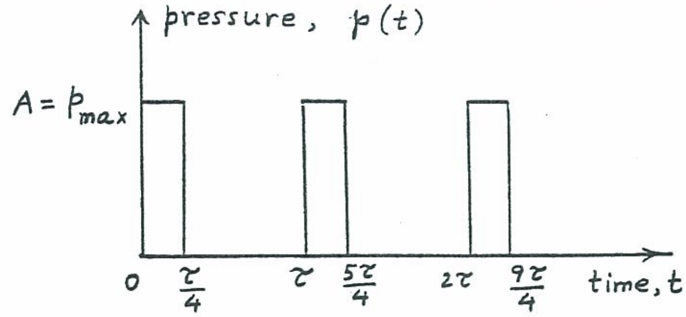
$m = 1$	$m = 2$	$m = 3$
$a_1 = \frac{A}{\pi} \sin \frac{\pi}{2} = \frac{A}{\pi}$ $= 31.8309$ psi	$a_2 = \frac{A}{2\pi} \sin \pi = 0$	$a_3 = \frac{A}{3\pi} \sin \frac{3\pi}{2}$ $= -10.6103$ psi
$b_1 = -\frac{A}{\pi}\left(\cos \frac{\pi}{2} - 1\right)$ $= 31.8309$ psi	$b_2 = -\frac{A}{2\pi}\left(\cos \pi - 1\right)$ $= 31.8309$ psi	$b_3 = -\frac{A}{3\pi}\left(\cos \frac{3\pi}{2} - 1\right)$ $= 10.6103$ psi

$$\therefore\ p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos m\omega t + b_m \sin m\omega t\right) \text{ psi}$$

1.115 Speed = 200 rpm

In a minute, a point will be subjected to the maximum pressure, $A = p_{max} = 100$ psi, $200 \times 6 = 1200$ times. Hence

period $= \tau = \frac{60}{1200} = 0.05$ sec.



$$p(t) = \begin{cases} A & , \ 0 \le t \le \tau/4 \\ 0 & , \ \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau} \int_0^\tau p(t)\, dt = \frac{2}{\tau} A \left(t\right)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau} \int_0^\tau p(t) \cos m\omega t\, dt = \frac{2A}{\tau} \left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m} \sin \frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau} \int_0^\tau p(t) \sin m\omega t\, dt = -\frac{2A}{\tau} \left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos \frac{m\pi}{2} - 1\right)$$

Evaluation of a_m and b_m:

$m = 1$	$m = 2$	$m = 3$
$a_1 = \frac{A}{\pi} \sin \frac{\pi}{2} = \frac{A}{\pi} = 31.8309$ psi	$a_2 = \frac{A}{2\pi} \sin \pi = 0$	$a_3 = \frac{A}{3\pi} \sin \frac{3\pi}{2} = -10.6103$ psi
$b_1 = -\frac{A}{\pi}\left(\cos \frac{\pi}{2} - 1\right) = 31.8309$ psi	$b_2 = -\frac{A}{2\pi}\left(\cos \pi - 1\right) = 31.8309$ psi	$b_3 = -\frac{A}{3\pi}\left(\cos \frac{3\pi}{2} - 1\right) = 10.6103$ psi

$$\therefore\ p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos m\omega t + b_m \sin m\omega t\right) \quad \text{psi}$$

1.116

i	t_i	M_{t_i}	$n=1$: $M_{t_i} \cos \frac{2\pi t_i}{0.012}$	$n=1$: $M_{t_i} \sin \frac{2\pi t_i}{0.012}$	$n=2$: $M_{t_i} \cos \frac{4\pi t_i}{0.012}$	$n=2$: $M_{t_i} \sin \frac{4\pi t_i}{0.012}$	$n=3$: $M_{t_i} \cos \frac{6\pi t_i}{0.012}$	$n=3$: $M_{t_i} \sin \frac{6\pi t_i}{0.012}$
1	0.0005	770	743.7627	199.2912	666.8391	385.0010	544.4712	544.4731
2	0.0010	810	701.4802	405.0007	404.9988	701.4812	0.0000	810.0000
3	0.0015	850	601.0398	601.0417	0.0000	850.0000	−601.0442	601.0373
4	0.0020	910	454.9978	788.0845	−455.0041	788.0808	−910.0000	0.0000

5	0.0025	1010	261.4043	975.5859	−874.689	504.995	−714.171	−714.184
6	0.0030	1170	0.0000	1170.0000	−1170.000	0.000	0.000	−1170.000
7	0.0035	1370	−354.5874	1323.3169	−1186.449	−685.010	968.748	−968.725
8	0.0040	1610	−805.0073	1394.2966	−804.987	−1394.309	1610.000	0.000
9	0.0045	1890	−1336.4407	1336.4229	0.000	−1890.000	1336.410	1336.454
10	0.0050	1750	−1515.5491	874.7922	875.019	−1515.534	0.000	1750.000
11	0.0055	1630	−1574.4619	421.8647	1411.634	−814.979	−1152.608	1152.560
12	0.0060	1510	−1510.0000	0.0000	1510.000	0.000	−1510.000	0.000
13	0.0065	1390	−1342.6345	−359.7671	1203.767	695.014	−982.858	−982.898
14	0.0070	1290	−1117.1677	−645.0088	644.982	1117.183	0.000	−1290.000
15	0.0075	1190	−841.4492	−841.4648	0.000	1190.000	841.479	−841.435
16	0.0080	1110	−554.9897	−961.2942	−555.021	961.276	1110.000	0.000
17	0.0085	1050	−271.7498	−1014.2249	−909.337	524.982	742.440	742.485
18	0.0090	990	0.0000	−990.0000	−990.000	0.000	0.000	990.000
19	0.0095	930	240.7123	−898.3081	−805.393	−465.018	−657.633	657.586
20	0.0100	890	445.0095	−770.7571	−444.981	−770.773	−890.000	0.000
21	0.0105	850	601.0478	−601.0337	0.000	−850.000	−601.022	−601.060
22	0.0110	810	701.4868	−404.9895	405.022	−701.468	0.000	−810.000
23	0.0115	770	743.7659	−199.2798	666.851	−384.980	544.500	−544.444
24	0.0120	750	750.0000	0.0000	750.000	0.000	750.000	0.000
$\sum_{i=1}^{24}(\ )$		27,300	−4,979.3242	1,803.7673	343.270	−1,754.047	428.734	661.855
$\frac{1}{12}\sum_{i=1}^{24}$		2,275	−414.9436	150.3139	28.606	−146.171	35.728	55.155

1.117

i	t_i	x_i	$n=1$: $x_i \cos \frac{2\pi t_i}{0.6}$	$n=1$: $x_i \sin \frac{2\pi t_i}{0.6}$	$n=2$: $x_i \cos \frac{4\pi t_i}{0.6}$	$n=2$: $x_i \sin \frac{4\pi t_i}{0.6}$	$n=3$: $x_i \cos \frac{6\pi t_i}{0.6}$	$n=3$: $x_i \sin \frac{6\pi t_i}{0.6}$
1	0.025	9.00	8.69	2.33	7.79	4.50	6.36	6.36
2	0.050	17.00	14.72	8.50	8.50	14.72	0.00	17.00
3	0.075	23.00	16.26	16.26	0.00	23.00	−16.26	16.26
4	0.100	25.00	12.50	21.65	−12.50	21.65	−25.00	0.00
5	0.125	26.00	6.73	25.11	−22.52	13.00	−18.38	−18.38
6	0.150	28.00	0.00	28.00	−28.00	0.00	0.00	−28.00
7	0.175	33.00	−8.54	31.88	−28.58	−16.50	23.33	−23.33
8	0.200	35.00	−17.50	30.31	−17.50	−30.31	35.00	0.00
9	0.225	34.00	−24.04	24.04	0.00	−34.00	24.04	24.04
10	0.250	29.00	−25.11	14.50	14.50	−25.11	0.00	29.00
11	0. 75	24.00	−23.18	6.21	20.78	−12.00	−16.97	16.97
12	0.300	26.00	−26.00	0.00	26.00	0.00	−26.00	0.00
13	0.325	32.00	−30.91	−8.28	27.71	16.00	−22.63	−22.63
14	0.350	40.00	−34.64	−20.00	20.00	34.64	0.00	−40.00
15	0.375	18.00	−12.73	−12.73	0.00	18.00	12.73	−12.73
16	0.400	8.00	−4.00	−6.93	−4.00	6.93	8.00	0.00
17	0.425	−5.00	1.29	4.83	4.33	−2.50	−3.54	−3.54
18	0.450	−14.00	0.00	14.00	14.00	0.00	0.00	−14.00
19	0.475	−28.00	−7.25	27.05	24.25	14.00	19.80	−19.80
20	0.500	−37.00	−18.50	32.04	18.50	32.04	37.00	0.00
21	0.525	−33.00	−23.33	23.33	0.00	33.00	23.33	23.34
22	0.550	−29.00	−25.11	14.50	−14.50	25.11	0.00	29.00

23	0.575	−22.00	−21.25	5.69	−19.05	11.00	−15.56	15.56
24	0.600	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{i=1}^{24}(\ )$		239.00	−241.90	282.30	39.72	147.18	45.26	−4.88
$\frac{1}{12}\sum_{i=1}^{24}(\ )$		19.92	−20.16	23.53	3.31	12.26	3.77	−0.41

1.118

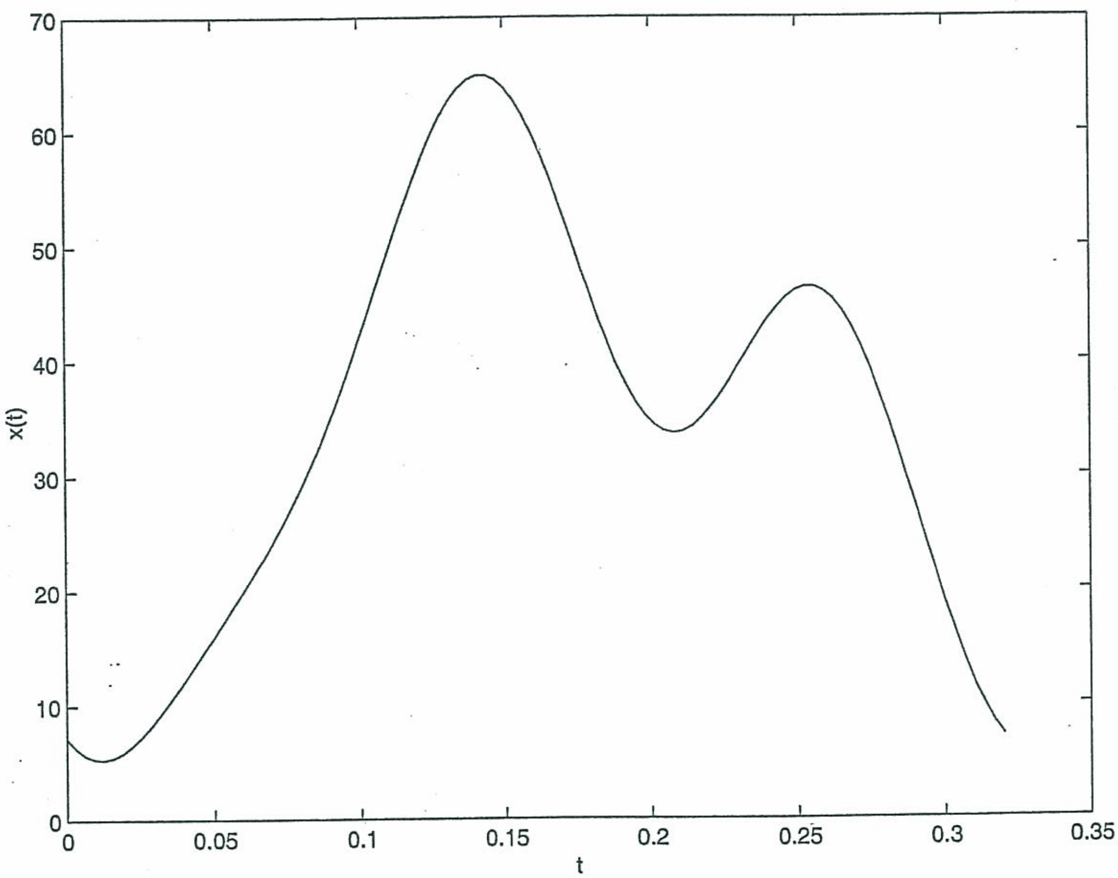
```
%==========================================================================
%
%Program1.m
%Program for calling the subroutine FORIER
%
%==========================================================================
%Run "Program1.m" in MATLAB Command Window. Program1.m and forier.m should be
%in the same file folder, and set the path to this folder
%Following 6 lines contain problem-dependent data
n=16;
m=3;
time=0.32;
x=[9 13 17 29 43 59 63 57 49 35 35 41 47 41 13 7];
t=0.02:0.02:0.32;
%end of problem-dependent data
%Following line calls subroutine forier.m
[azero,a,b,xsin,xcos]=forier(n,m,time,x,t);
%following outputs data
fprintf('Fourier series expansion of the function x(t)\n\n');
fprintf('Data:\n\n');
fprintf('Number of data points in one cycle = %3.0f \n',n);
fprintf(' \n');
fprintf('Number of Fourier Coefficients required = %3.0f \n',m);
fprintf(' \n');
fprintf('Time period = %8.6e \n\n',time);
fprintf('Station i        ')
fprintf('Time at station i: t(i)          ')
fprintf('x(i) at t(i)')
for i=1:n
   fprintf('\n %8d%25.6e%27.6e ',i,t(i),x(i));
end
fprintf(' \n\n');
fprintf('Results of Fourier analysis:\n\n');
fprintf('azero=%8.6e \n\n',azero);
fprintf('values of i        a(i)                 b(i)\n');
for i=1:m
   fprintf('%10.0g   %8.6e%20.6e \n',i,a(i),b(i));
end
```

```
%=================================================================
%
%Subroutine forier.m
%
%=================================================================
function [azero,a,b,xsin,xcos]=forier(n,m,time,x,t)
pi=3.1416;
sumz=0.0;
for i=1:n
   sumz=sumz+x(i);
end
azero=2.0*sumz/n;
for ii=1:m
   sums=0.0;
   sumc=0.0;
   for i=1:n
      theta=2.0*pi*t(i)*ii/time;
      xcos(i)=x(i)*cos(theta);
      xsin(i)=x(i)*sin(theta);
      sums=sums+xsin(i);
      sumc=sumc+xcos(i);
   end
   a(ii)=2.0*sumc/n;
   b(ii)=2.0*sums/n;
end
```

```
>> program1
Fourier series expansion of the function x(t)

Data:

Number of data points in one cycle =  16

Number of Fourier Coefficients required =   3

Time period = 3.200000e-001

Station i       Time at station i: t(i)        x(i) at t(i)
       1             2.000000e-002             9.000000e+000
       2             4.000000e-002             1.300000e+001
       3             6.000000e-002             1.700000e+001
       4             8.000000e-002             2.900000e+001
       5             1.000000e-001             4.300000e+001
       6             1.200000e-001             5.900000e+001
       7             1.400000e-001             6.300000e+001
       8             1.600000e-001             5.700000e+001
       9             1.800000e-001             4.900000e+001
      10             2.000000e-001             3.500000e+001
      11             2.200000e-001             3.500000e+001
      12             2.400000e-001             4.100000e+001
      13             2.600000e-001             4.700000e+001
      14             2.800000e-001             4.100000e+001
      15             3.000000e-001             1.300000e+001
      16             3.200000e-001             7.000000e+000

Results of Fourier analysis:

azero=6.975000e+001

values of i        a(i)              b(i)
        1    -2.084870e+001      -3.915985e+000
        2    -1.456887e+000      -1.144979e+001
        3    -5.402900e+000       3.353473e+000
```

1.119

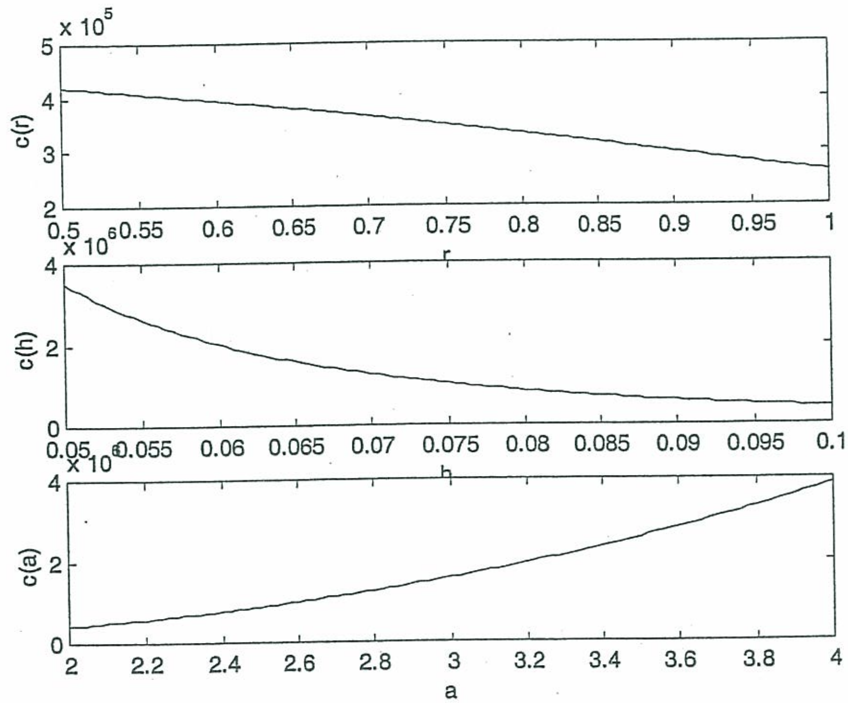
```
% Ex1_119.m
for i = 1: 101
    t(i) = 0.32*(i-1)/100;
    x(i) = 34.875 - 20.8487*cos(19.635*t(i)) - 3.9160*sin(19.635*t(i))...
         - 1.4569*cos(39.27*t(i)) - 11.4498*sin(39.27*t(i))...
         - 5.4029*cos(58.905*t(i)) + 3.3535*sin(58.905*t(i));
end
plot(t,x)
xlabel('t');
ylabel('x(t)');
```



1.120

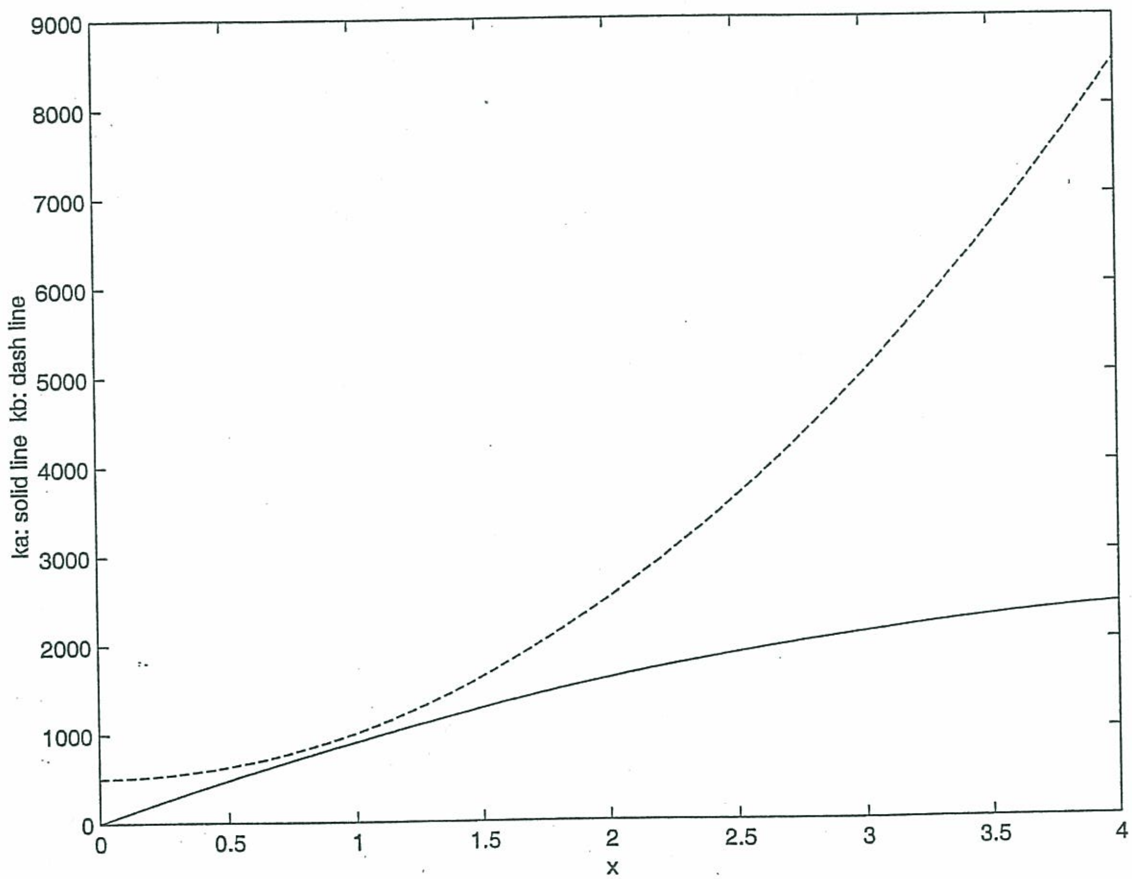
```
% Ex1_1.120.m
u = 0.3445;
l = 10;
h0 = 0.1;
a0 = 2;
r0 = 0.5;
% First case, r changes
for i = 1:101
    r(i) = 0.5 + (i-1)*0.5/100;
    c1(i) = ( 6*pi*u*l/(h0^3) ) * ( (a0 - h0/2)^2 - r(i)^2 )...
        * ( (a0^2-r(i)^2)/(a0-h0/2) - h0 );
end
```

```
% Second case, h changes
for i = 1:101
    h(i) = 0.05 + (i-1)*0.05/100;
    c2(i) = ( 6*pi*u*l/(h(i)^3) ) * ( (a0 - h(i)/2)^2 - r0^2 )...
        * ( (a0^2-r0^2)/(a0-h(i)/2) - h(i) );
end
% Third case, a changes
for i = 1:101
    a(i) = 2 + (i-1)*2/100;
    c3(i) = ( 6*pi*u*l/(h0^3) ) * ( (a(i) - h0/2)^2 - r0^2 )...
        * ( (a(i)^2-r0^2)/(a(i)-h0/2) - h0 );
end
subplot(311);
plot(r,c1);
xlabel('r');
ylabel('c(r)');
subplot(312);
plot(h,c2);
xlabel('h');
ylabel('c(h)');
subplot(313);
plot(a,c3);
xlabel('a');
ylabel('c(a)');
```



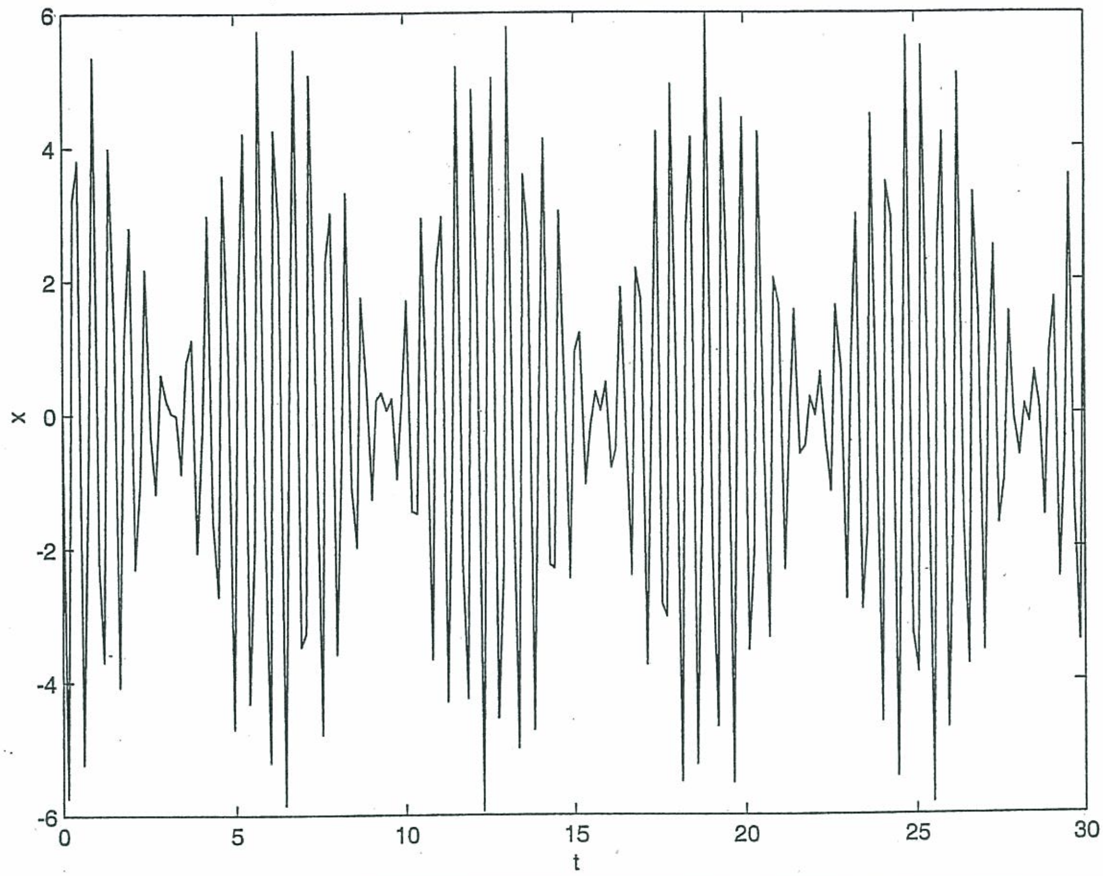
1.121

```
% Ex1_121.m
for i = 1:101
    x(i) = (i-1)*4/100;
    ka(i) = 1000*x(i) - 100*x(i)^2;
    kb(i) = 500 + 500 *x(i)^2;
end
plot(x,ka);
hold on
plot(x,kb, '--');
xlabel('x');
ylabel('ka: solid line  kb: dash line');
```



1.122

```
% Ex1_122.m
for i = 1:201
    t(i) = (i-1)*30/200;
    x1(i) = 3*sin(30*t(i));
    x2(i) = 3*sin(29*t(i));
    x(i) = x1(i) + x2(i);
end
plot(t,x);
xlabel('t');
ylabel('x');
```


6
4
2
x 0
-2
-4
-6
0
5
10
15
20
25
30
t

1.123 $x_p = r + l - r\cos\theta - l\cos\phi = r + l - r\cos\omega t - l\sqrt{1-\sin^2\phi}$ $\quad (E_1)$

But $\quad l\sin\phi = r\sin\theta, \quad \cos\phi = \left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{\frac{1}{2}}$ $\quad (E_2)$

Using (E_2) in (E_1), $\quad x_p = r + l - r\cos\omega t - l\left(1 - \frac{r^2}{l^2}\sin^2\omega t\right)^{\frac{1}{2}}$ $\quad (E_3)$

Let $\frac{r}{l}$ = small $(< \frac{1}{4})$. Using $\sqrt{1-\epsilon} \approx 1 - \frac{1}{2}\epsilon$, (E_3) becomes

$$x_p \approx r\left(1 + \frac{r}{2l}\right) - r\left(\cos\omega t + \frac{r}{4l}\cos 2\omega t\right) \qquad (E_4)$$

(a) Eq. (E_4) gives $\quad y_p = x_p - r\left(1 + \frac{r}{2l}\right) \simeq -r\left(\cos\omega t + \frac{1}{4}\frac{r}{l}\cos 2\omega t\right)$ --- (E_5)

If $\frac{r}{l}$ is very small, $y_p \simeq -r\cos\omega t \Rightarrow$ harmonic motion.

(b) To have amplitude of second harmonic smaller than that of first harmonic in Eq. (E_5), we need to have

$$\frac{1}{4}\frac{r}{l} \leq \frac{1}{25}, \quad \text{i.e.,} \quad \frac{r}{l} \leq \frac{4}{25}, \quad \text{i.e.,} \quad \frac{l}{r} \geq 6.25$$

Once the amplitude of second harmonic is smaller by a factor of 25, the amplitudes of higher harmonics arising from the expansion of square-root-term in (E_3) are expected to be still smaller.

1.124 Unbalanced force developed $= P = 2\,m\,\omega^2\,r\cos\omega t$, range of force = 0 - 100 N, range of frequency = 25 - 50 Hz = 157.08 - 314.16 rad/sec.
Parameters to be determined: m, r, ω.
Let r = 0.1 m. To generate 100 N force at 25 Hz, set:

$$P_{max} = 100 = 2\,m\,(157.08)^2\,(0.1)$$

which gives

$$m = \frac{100}{2\,(157.08)^2\,(0.1)} = 0.0202641 \text{ kg} = 20.2641 \text{ g}$$

To generate 100 N force at 50 Hz, set:

$$P_{max} = 100 = 2\,m\,(314.16)^2\,(0.1)$$

which yields

$$m = \frac{100}{2\,(314.16)^2\,(0.1)} = 0.0050660 \text{ kg} = 5.0660 \text{ g}$$

1.125 Goal: Weight to be maintained at 10 ± 0.1 lb/min
Parameters to be determined: Angular velocity of crank (ω), lengths of crank and connecting rod, dimensions of the wedge, dimensions of the orifice in the hopper, dimensions of the actuating rod, and dimensions of the lever arrangement.
Given: Density of the material in the hopper.
Procedure:
Select ω based on available motor. Determine the dimensions of the orifice in the hopper which delivers approximately 10 lb/min (assuming continuous flow of material). For trial dimensions of the wedge, determine the increase/decrease in the size (diameter) of the orifice. Choose the final dimensions of the wedge such that the material flow rate delivered by the orifice lies within the specified range.

1.126 Force to be applied = 200 lb, frequency = 50 Hz = 314.16 rad/sec.

Procedure:

1. Select a motor that provides, either directly or through a gear system, the desired frequency. Assume that it is connected to the cam.
2. D ~~Setermine~~ the sizes and dimensions of the plate cam and the roller.
3. Choose the dimensions of the follower.
4. Select the weight as 200 lb. From the geometry, determine the range of displacement (vertical motion) of the weight.
5. Determine the force exerted due to the falling weight.

1.127 Considerations to be taken in the design of vibratory bowl feeders:

1. Suitable design of the electromagnet and its coil.
2. Radius of the bowl and the pitch of the spiral (helical) delivery track.
3. Tooling to be fixed along the spiral track to reject the defective or out-of-tolerance or incorrectly oriented parts.
4. Design of elastic supports.
5. Size and location of the outlet.

1.128 Axial spring constant of each tube $= k = \frac{A\,E}{\ell}$.

Let diameter of each tube be 0.01 m (1 cm) with thickness 0.001 m (1 mm). Then

$$A = \frac{\pi}{4}\left(D^2 - d^2\right) = \frac{\pi}{4}\left(0.01^2 - 0.008^2\right) = 28.27\left(10^{-6}\right)\ \mathrm{m}^2$$

This gives

$$k = \frac{\left(28.27\left(10^{-6}\right)\right)\left(2.07\left(10^{11}\right)\right)}{2} = 29.26\left(10^5\right)\ \mathrm{N/m}$$

Since 76 tubes are in parallel, we have the total axial stiffness as:

$$k_{eq} = 76\ k = (76)\left(29.26\left(10^5\right)\right) = 222.38\left(10^6\right)\ \mathrm{N/m}$$

The polar area moment of inertia of each tube is

$$J = \frac{\pi}{32}\left(D^4 - d^4\right) = \frac{\pi}{32}\left(0.01^4 - 0.008^4\right) = 580\left(10^{-8}\right)\ \mathrm{m}^4$$

Torsional stiffness of each tube is given by

$$\frac{G\,J}{\ell} = \frac{\left(79.6154\left(10^9\right)\right)\left(580\left(10^{-8}\right)\right)}{2} = 231\left(10^3\right)\ \mathrm{N{-}m/rad}$$

For 76 tubes in parallel, equivalent torsional stiffness will be:

$$k_{t_{eq}} = (76)\left(231\left(10^3\right)\right) = 17.56\left(10^6\right)\ \mathrm{N{-}m/rad}$$
