Exam 1

Mech 326: Aerodynamics
Friday, October 9, 2015: 8:00–9:00 AM

Name: Solution

1. (Desingularized vortex) (20%)
2. (Source near a wall) (20%)
3. (Lifting Cylinder) (20%)
4. (Concepts) (40%)

1. (Desingularized vortex) (20%) A desingularized clockwise vortex also known as a vortex blob (seriously) is centered at the origin and has the following two dimensional streamfunction,

$$\Psi = \frac{\Gamma_d}{2\pi} \ln\sqrt{r^2+\delta^2}$$

where $\delta =$ constant and $\Gamma_d > 0$.

(a) Calculate the velocity field of the desingularized vortex.

(b) Does a velocity potential exist for the vortex blob? If so, what is it? If not, why not?

(c) A circular contour of radius $r = \delta$ is centered around the vortex blob. Calculate the circulation around the contour using **either** of the two different approaches.

(a) $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$ $\quad$ or $\quad u_\theta = -\frac{\partial \psi}{\partial r}$ $\quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$

$\boxed{u_r = 0}$ $\quad u_\theta = -\frac{\partial}{\partial r}\left[\frac{\Gamma_d}{2\pi} \ln\sqrt{r^2+\delta^2}\right]$

$$= -\frac{\Gamma_d}{2\pi}\frac{\partial}{\partial r}\left[\ln f(r)\right] = -\frac{\Gamma_d}{2\pi}\frac{1}{f(r)}\frac{\partial f}{\partial r}$$

$$= -\frac{\Gamma_d}{2\pi}\frac{1}{\sqrt{r^2+\delta^2}}\left(\frac{1}{\cancel{2}}\right)\frac{\cancel{2}r}{\sqrt{r^2+\delta^2}}$$

$$\boxed{u_\theta = -\frac{\Gamma_d}{2\pi}\frac{r}{r^2+\delta^2}}$$

(b) $u_r = \frac{\partial \phi}{\partial r}$ $\quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \longrightarrow \phi = \int r u_\theta \, d\theta + g(r)$

$\downarrow$

$\phi = \int u_r \, dr + f(\theta) + C$ $\qquad \phi = \underbrace{-\frac{\Gamma_d}{2\pi}\frac{r^2}{r^2+\delta^2}\theta}_{\text{function of } r \text{ and } \theta} + g(r)$

$g(r) = 0$

$\phi = f(\theta) + C$ $\qquad f(\theta) \rightarrow$ does not exist

ϕ does not exist $\rightarrow$ there is no integration that can account for the term: $-\frac{\Gamma_d}{2\pi}\frac{r^2}{r^2+\delta^2}\theta$. Also, the flow is rotational, so ϕ does not exist.

(c) $r=\delta \quad \Gamma = -\oint \vec{v}\cdot d\vec{s} = -\iint_A (\vec{\nabla}\times\vec{v})\cdot d\vec{A}$

<ins>Approach 1</ins> : $\Gamma = -\oint \vec{v}\cdot d\vec{s}$

$$d\vec{s} = r\,d\theta\,\hat{e}_\theta = \delta\,d\theta\,\hat{e}_\theta \qquad \vec{v} = -\frac{\Gamma_d}{2\pi}\frac{r}{r^2+\delta^2}\,\hat{e}_\theta$$

$$\vec{v}\cdot d\vec{s} = -\frac{\Gamma_d}{2\pi}\frac{r\delta}{r^2+\delta^2}\,d\theta$$

$$@\, r=\delta \rightarrow \vec{v}\cdot d\vec{s} = -\frac{\Gamma_d}{2\pi}\frac{\cancel{\delta^2}}{2\cancel{\delta^2}}\,d\theta = \frac{-\Gamma_d}{4\pi}\,d\theta$$

$$\Gamma = -\oint_0^{2\pi} \frac{-\Gamma_d}{4\pi}\,d\theta \longrightarrow \boxed{\Gamma = +\frac{\Gamma_d}{2}}$$

<ins>Approach 2:</ins> $\Gamma = -\iint_A (\vec{\nabla}\times\vec{v})\cdot d\vec{A}$

$$\vec{\nabla}\times\vec{v} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r u_\theta\right)\right]\hat{e}_z$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{\Gamma_d}{2\pi}\frac{r^2}{r^2+\delta^2}\right)\right]\hat{e}_z$$

$$= -\frac{\Gamma_d}{2\pi r}\frac{\partial}{\partial r}\left[f(r)\,g(r)\right]\hat{e}_z \qquad f(r) = r^2, \quad g(r) = (r^2+\delta^2)^{-1}$$

$$= -\frac{\Gamma_d}{2\pi r}\left[g\frac{df}{dr} + f\frac{dg}{dr}\right]\hat{e}_z \qquad \frac{df}{dr} = 2r \qquad \frac{dg}{dr} = -(r^2+\delta^2)^{-2}(2r)$$

$$= -\frac{\Gamma_d}{\cancel{2}\pi\cancel{r}}\left[\frac{\cancel{2}\cancel{r}}{r^2+\delta^2} - \frac{\cancel{2}\cancel{r}\,r^2}{(r^2+\delta^2)^2}\right]\hat{e}_z = -\frac{\Gamma_d}{\pi}\left[\frac{1}{r^2+\delta^2} - \frac{r^2}{(r^2+\delta^2)^2}\right]\hat{e}_z$$

Now $d\vec{A} = r\,dr\,d\theta\,\hat{e}_z \qquad \Gamma = -\iint_A \frac{\Gamma_d}{\pi}\left[\frac{r^3}{(r^2+\delta^2)^2} - \frac{r}{r^2+\delta^2}\right]dr\,d\theta$

$$\Gamma = -\frac{\Gamma_d}{\pi}\int_0^\delta\int_0^{2\pi}\left[\frac{r^3}{(r^2+\delta^2)^2} - \frac{r}{r^2+\delta^2}\right]d\theta\,dr \xrightarrow{3} \Gamma = -2\Gamma_d\left[\frac{1}{2}\left(\frac{\delta^2}{\delta^2+r^2} + \ln(r^2+\delta^2)\right) - \frac{1}{2}\left(\ln(r^2+\delta^2)\right)\right]_0^\delta$$

$$\Gamma = -\Gamma_d \left[\frac{\delta^2}{r^2+\delta^2} \right]_0^{\delta} = -\Gamma_d \left[\frac{\cancel{\delta^2}}{2\cancel{\delta^2}} - \frac{\cancel{\delta^2}}{\cancel{\delta^2}} \right] = -\Gamma_d \left[\frac{1}{2} - 1 \right]$$

$$\boxed{\Gamma = \frac{\Gamma_d}{2}}$$

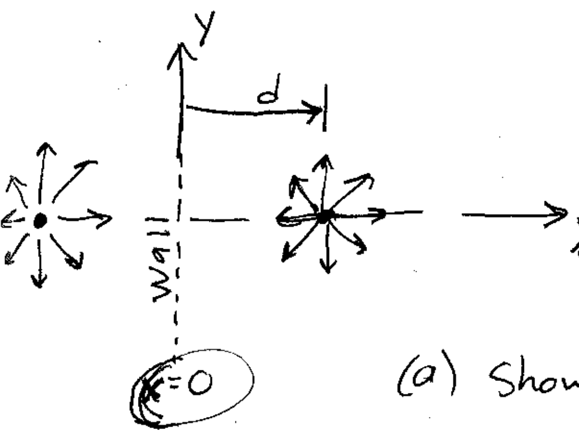
2. (Source near a wall) (20%) Consider a "source near a wall" flowfield. The flow consists of 2 sources of strength σ a distance d on either side of a wall location. The velocity potential for this flow is given by

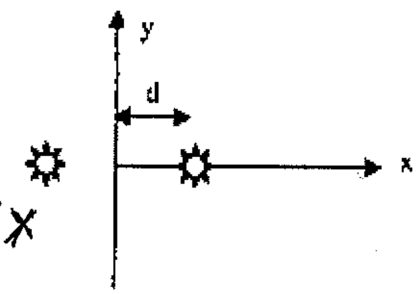
$$\phi = \frac{\sigma}{2\pi}\left\{ln\left[(x-d)^2+y^2\right] + ln\left[(x+d)^2+y^2\right]\right\}$$

(a) Show that there is no flow through the wall.

(b) Determine the velocity distribution along the wall.

(c) Determine the pressure distribution along the wall assuming $P = P_\infty$ far from the source.





(a) Show $\underline{u=0}$ @ $\underline{x=0}$

$$u = \frac{\partial \phi}{\partial x} = \frac{\sigma}{2\pi}\left\{\frac{1}{(x-d)^2+y^2}\,2(x-d) + \frac{2(x+d)}{(x+d)^2+y^2}\right\}$$

$$@x=0 \rightarrow u(0,y) = \frac{\sigma}{2\pi}\left\{\cancel{\frac{2(-d)}{d^2+y^2} + \frac{2d}{d^2+y^2}}\right\} \rightarrow 0$$

$$\boxed{u(0,y) = 0}$$

(b) Determine $v(0,y)$.

$$v(x,y) = \frac{\partial \phi}{\partial y} = \frac{\sigma}{2\pi}\left[\frac{2y}{(x-d)^2+y^2} + \frac{2y}{(x+d)^2+y^2}\right]$$

$$@x=0 \rightarrow v(0,y) = \frac{\sigma}{2\pi}\left(\frac{4y}{d^2+y^2}\right) \rightarrow \boxed{v(0,y) = \frac{2\sigma y}{\pi(d^2+y^2)}}$$

(c) Find $P(x,y)$ along the wall $\rightarrow P(0,y)$ when $P = P_\infty$ far from the wall

Bernoulli's: $P_\infty + \cancel{\frac{1}{2}\rho U_\infty^2}^{\rightarrow 0 \rightarrow U_\infty = 0} = P(0,y) + \frac{1}{2}\rho v^2(0,y) \rightarrow {}^{*}u(0,y) = 0$

$$P(0,y) = P_\infty - \frac{1}{2}\rho\left[\frac{4\sigma^2 y^2}{\pi^2(d^2+y^2)^2}\right]$$

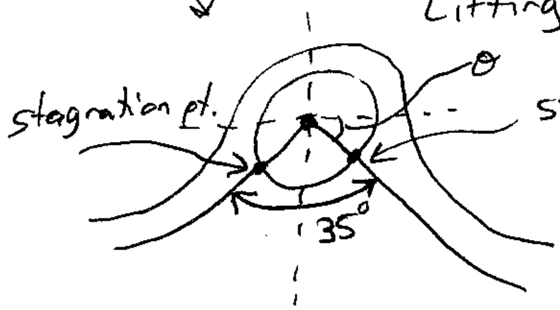
$$\boxed{P(0,y) = P_\infty - \frac{2\rho\sigma^2 y^2}{\pi^2(d^2+y^2)^2}}$$

3. (Lifting Cylinder) (20%) Consider the lifting flow over a circular cylinder with the following properties:

$$R = 0.2 \text{ m}$$
$$U_\infty = 2 \text{ m/s}$$
$$\rho = 1.23 \text{ kg/m}^3$$

The two stagnation points on the cylinder surface are 35 degrees apart. Determine the lift generated by this cylinder if it is 0.5 m long.

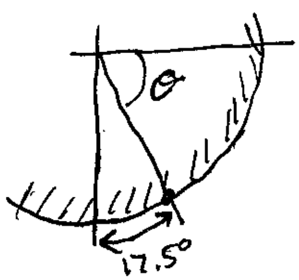
Lifting cylinder



Stagnation pts occur @ $\begin{pmatrix} \text{Uniform + Doublet} \\ \text{+ Vortex} \end{pmatrix}$ $\sin\theta = \dfrac{-\Gamma}{4\pi U_\infty R}$

Lift-per-unit-span: $L' = \rho U_\infty \Gamma$
(Kutta-Joukowski)

$\Gamma = -4\pi U_\infty R \sin\theta$



$\theta = -90° + 17.5° = -72.5° \rightarrow \theta = -1.265 \text{ rad}$

or $\theta = -90° - 17.5° = -107.5°$

$\sin\theta = -0.954$

$\Gamma = -4\pi(2)(0.2)(-0.954)$

$\Gamma = 4.793 \text{ m}^2/\text{s}$

$\dfrac{L}{b} = \rho U_\infty \Gamma \rightarrow L = \rho U_\infty \Gamma b = (1.23)(2)(4.793)(0.5)$

$\boxed{L = 5.89 N}$

4. (Concepts) (40%) How well do you conceptually understand aerodynamics:

(4.1) The application of what type of stress causes a fluid to continuously deform but not a solid?

A shear stress.

(4.2) What are the assumptions of potential flow?

(1) Inviscid (2) Irrotational (3) Incompressible

(4.3) In a potential flow, does the strength of a vortex filament vary along it's length or stay the same?

The strength stays the same. (Helmholtz's 1st theorem)

(4.4) In a potential flow, does the strength of a vortex filament vary in time or stay the same?

The strength stays the same (Kelvin's Circulation theorem)

(4.5) Mathematically, what does it mean to be incompressible?

$$\frac{D\rho}{Dt} \equiv 0 \rightarrow \vec{\nabla} \cdot \vec{V} = 0$$

(4.6) Mathematically, what does it mean to have a steady flow?

$$\frac{\partial}{\partial t}(\cdot) \equiv 0$$

(4.7) Mathematically, what does it mean to have an irrotational flow?

$$\vec{\nabla} \times \vec{V} = 0$$

(4.8) What are the four elementary potential flows?

(1) Uniform (2) Source/Sink (3) Doublet (4) Vortex

(4.9) In a potential flow, what is the drag on a cylinder?

Zero.

(4.10) What is the theorem that connects the circulation around a body to the lift produced by the body?

Kutta-Joukowski Theorem