Problem 1:

$$\psi = -\frac{\mu}{2\pi}\frac{\sin\theta}{r} \qquad V_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \Rightarrow -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2}$$

$$V_\theta = -\frac{\partial\psi}{\partial r} \Rightarrow -\frac{\mu}{2\pi}\frac{\sin\theta}{r^2}$$

$$① \; V_r = \frac{\partial\phi}{\partial r} \qquad V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \; ②$$

$$① \; -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2} = \frac{\partial\phi}{\partial r} \Longrightarrow \phi = \int -\frac{\mu}{2\pi}\frac{\cos\theta}{r^2}\,dr + C$$

$$\phi = -\frac{\mu}{2\pi}\left(\frac{1}{-1}\right)\frac{\cos\theta}{r} + f(\theta) + C$$

$$\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r} + f(\theta) + C \quad ③$$

$$② \; \cancel{-\frac{\mu}{2\pi}\frac{\sin\theta}{r^2} = \frac{\partial\varphi}{\partial r} \Rightarrow \psi =}$$

$$② \; -\frac{\mu}{2\pi}\frac{\sin\theta}{r^2} = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \Rightarrow \phi = \int -\frac{\mu}{2\pi}\frac{\sin\theta}{r}\,d\theta + C$$

$$\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r} + g(r) + C \quad ④$$

Compare ③ to ④:

$f(\theta) = g(r) = 0$

C is an arbitrary constant that we can set to zero.

$$\boxed{\phi = \frac{\mu}{2\pi}\frac{\cos\theta}{r}}$$

$\boxed{\text{Plot!}}$ Equipotential lines $\rightarrow$ $\boxed{r = \frac{\mu}{2\pi C}\cos\theta}$

Contents


```
clear
close all
clc

% Create r and theta vectors
r = linspace(1/2,10,50)';
theta = linspace(0,2*pi,51)';

% Create a grid of (r,theta) combinations
[R, Theta] = meshgrid(r,theta);

% Calculate V_r and V_theta
V_r = -cos(Theta)./R.^2;
V_theta = -sin(Theta)./R.^2;
V_r_neg = cos(Theta)./R.^2;
V_theta_neg = sin(Theta)./R.^2;

% Convert polar (r,theta) to Cartesian (x,y)
X = R.*cos(Theta);
Y = R.*sin(Theta);

% Find the matrix size of R and use for indices
[rows, cols] = size(R);

% Step through each (r,theta) combination
for i = 1:rows
    for j = 1:cols

        % Calculate the Cartesian velocity components using a
        % transformation matrix, Q:
        %
        % Q = [cos(theta) -sin(theta)
        %      sin(theta)  cos(theta)]
        %
        % then,
        %
        % [U; V] = Q*[V_r; V_theta]

        U(i,j) = V_r(i,j)*cos(Theta(i,j)) - V_theta(i,j)*sin(Theta(i,j));
        V(i,j) = V_r(i,j)*sin(Theta(i,j)) + V_theta(i,j)*cos(Theta(i,j));
        U_neg(i,j) = V_r_neg(i,j)*cos(Theta(i,j)) - V_theta_neg(i,j)*sin(Theta(i,j));
        V_neg(i,j) = V_r_neg(i,j)*sin(Theta(i,j)) + V_theta_neg(i,j)*cos(Theta(i,j));
    end
end

% Equipotential line calculations
theta = linspace(0,2*pi,1000)';
C_1 = 1/2;
C_2 = 1;
C_3 = 2;
```

```
C_4 = -1/2;
C_5 = -1;
C_6 = -2;


r_1 = 1/C_1*cos(theta);
r_2 = 1/C_2*cos(theta);
r_3 = 1/C_3*cos(theta);
r_4 = 1/C_4*cos(theta);
r_5 = 1/C_5*cos(theta);
r_6 = 1/C_6*cos(theta);


% Convert to Cartesian coordinates
[x_1,y_1] = pol2cart(theta,r_1);
[x_2,y_2] = pol2cart(theta,r_2);
[x_3,y_3] = pol2cart(theta,r_3);
[x_4,y_4] = pol2cart(theta,r_4);
[x_5,y_5] = pol2cart(theta,r_5);
[x_6,y_6] = pol2cart(theta,r_6);
```

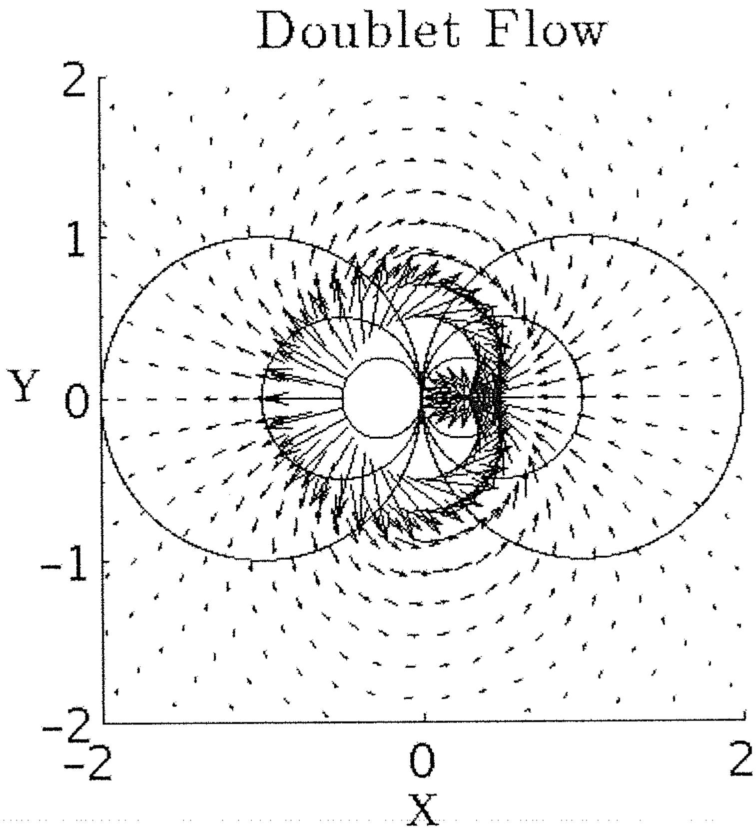
Plot the velocity field and equipotential lines: positive mu.

```
figure
set(gcf,'DefaultAxesfontsize',24,'DefaultAxesfontname','TimesNewRoman','DefaultAxesGridLineSty
le','-.')
num = 1;

hold on
    plot(x_1,y_1,'-k','LineWidth',num)
    plot(x_2,y_2,'-k','LineWidth',num)
    plot(x_3,y_3,'-k','LineWidth',num)
    plot(x_4,y_4,'-k','LineWidth',num)
    plot(x_5,y_5,'-k','LineWidth',num)
    plot(x_6,y_6,'-k','LineWidth',num)
    quiver(X,Y,U,V,'k')
hold off
axis equal
axis([-2 2 -2 2])

title('Doublet Flow','Interpreter','Latex','FontName','TimesNewRoman','FontSize',28)
xlabel('X','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24)
ylabel('Y','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24,'Rotation',0)

print('-depsc', '-r600','Doub_Flow_Pos.eps');
```



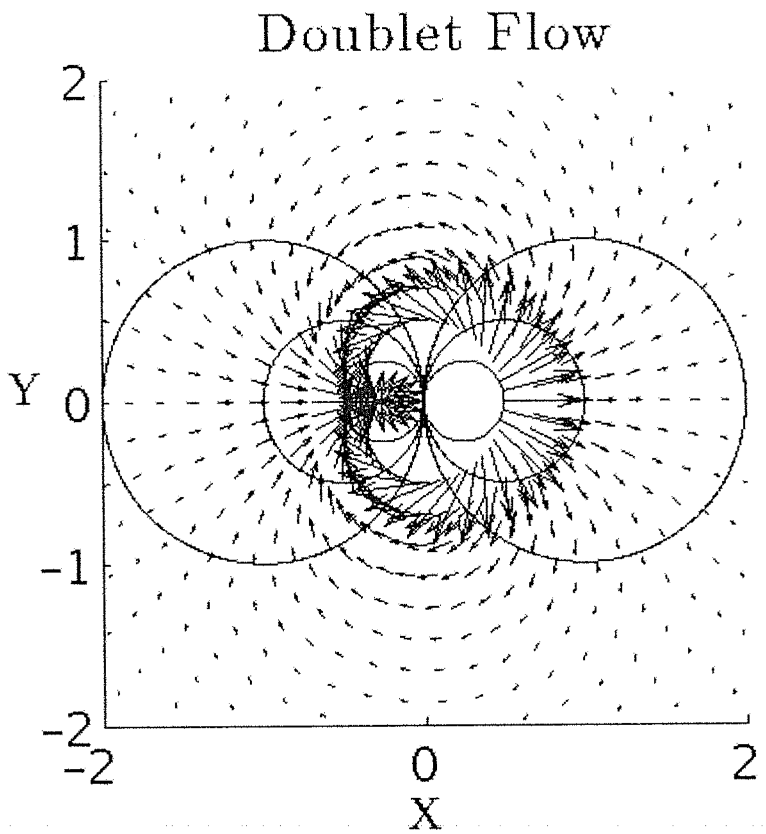
Plot the velocity field and equipotential lines: negative mu.

```
figure
set(gcf,'DefaultAxesfontsize',24,'DefaultAxesfontname','TimesNewRoman','DefaultAxesGridLineSty
le','-.')
num = 1;

hold on
    plot(x_1,y_1,'-k','LineWidth',num)
    plot(x_2,y_2,'-k','LineWidth',num)
    plot(x_3,y_3,'-k','LineWidth',num)
    plot(x_4,y_4,'-k','LineWidth',num)
    plot(x_5,y_5,'-k','LineWidth',num)
    plot(x_6,y_6,'-k','LineWidth',num)
    quiver(X,Y,U_neg,V_neg,'k')
hold off
axis equal
axis([-2 2 -2 2])

title('Doublet Flow','Interpreter','Latex','FontName','TimesNewRoman','FontSize',28)
xlabel('X','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24)
ylabel('Y','Interpreter','Latex','FontName','TimesNewRoman','FontSize',24,'Rotation',0)

print('-depsc', '-r600','Doub_Flow_Neg.eps');
```



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Homework #5 : Solutions

Problem 1

Follow Anderson text from page 270-273

or follow Koethe text from page 104-106

Problem #2

Uniform flow + vortex

Streamfunction: $\psi = u_\infty r \sin\theta + \frac{\Gamma}{2\pi} \ln r$

(a) $V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = u_\infty \cos\theta$

$V_\theta = -\frac{\partial \psi}{\partial r} = -u_\infty \sin\theta + \frac{-\Gamma}{2\pi r}$

To find the stagnation points set $V_r = 0$, and $V_\theta = 0$:

$V_r = 0 \Rightarrow u_\infty \cos\theta = 0 \longrightarrow \theta = \pm\frac{\pi}{2}$

$V_\theta = 0 \Rightarrow u_\infty \sin\theta + \frac{\Gamma}{2\pi r} = 0$

$2\pi r = -\frac{\Gamma}{u_\infty \sin\theta} \Rightarrow r = \frac{-\Gamma}{2\pi u_\infty \sin\theta}$

if $\theta = +\frac{\pi}{2} \rightarrow \sin\theta = 1 \rightarrow r < 0$

this is a non-physical solution!

if $\theta = -\frac{\pi}{2} \rightarrow \sin\theta = -1 \rightarrow r > 0$

$r = \frac{\Gamma}{2\pi u_\infty}$

There is one stagnation point @ $\boxed{(r,\theta) = \left(\frac{\Gamma}{2\pi u_\infty}, \frac{-\pi}{2}\right)}$

for a $\Gamma > 0$.

(b) The velocity of a vortex, V_θ, decays as r gets larger. At some point r, the velocity from the vortex will be larger enough to cancel the free-stream velocity at a point (Stag. pt.). If the flow speed is ~~higher~~ increased then the stagnation point should move closer to the vortex core. Equally if the vortex strength is increased we would expect ~~the~~ V_θ to be greater at some location r and the stagnation point should move away from the vortex core.

(c) Stagnation streamline equation:

$$\psi = U_\infty r \sin\theta + \frac{\Gamma}{2\pi}\ln r \quad @ \ (r,\theta) = \left(\frac{\Gamma}{2\pi U_\infty}, \frac{-\pi}{2}\right)$$

$$\psi = U_\infty\left(\frac{\Gamma}{2\pi U_\infty}\right)\cancel{\sin\left(\frac{-\pi}{2}\right)}^{-1} + \frac{\Gamma}{2\pi}\ln\left(\frac{\Gamma}{2\pi U_\infty}\right)$$

$$\psi = \frac{-\Gamma}{2\pi} + \frac{\Gamma}{2\pi}\ln\left(\frac{\Gamma}{2\pi U_\infty}\right)$$

$$\psi = \frac{\Gamma}{2\pi}\left[\ln\left(\frac{\Gamma}{2\pi U_\infty}\right) - 1\right] \leftarrow \text{Constant}$$

The stagnation streamline follows the relationship:

$$U_\infty r\sin\theta + \frac{\Gamma}{2\pi}\ln r - \frac{\Gamma}{2\pi}\left[\ln\left(\frac{\Gamma}{2\pi U_\infty}\right) - 1\right] = 0$$

or

$$\boxed{\theta = \sin^{-1}\left\{\frac{\Gamma}{2\pi U_\infty r}\left[\ln\left(\frac{\Gamma}{2\pi U_\infty}\right) - 1\right] - \frac{\Gamma}{2\pi U_\infty r}\ln r\right\}}$$

given $r \rightarrow$ find θ

(d) Using Bernoulli: $P_\infty + \frac{1}{2}\rho U_\infty^2 = P(r,\theta) + \frac{1}{2}\rho\left(V_r^2 + V_\theta^2\right)$

$$P(r,\theta) = P_\infty + \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho\left[U_\infty^2\cos^2\theta + U_\infty^2\sin^2\theta + \frac{\cancel{2}U_\infty\Gamma}{\cancel{2}\pi r}\sin\theta + \frac{\Gamma^2}{4\pi^2 r^2}\right]$$

$$P(r,\theta) = P_\infty + \cancel{\frac{1}{2}\rho U_\infty^2} - \cancel{\frac{1}{2}\rho U_\infty^2} - \frac{\rho U_\infty\Gamma}{2\pi r}\sin\theta - \frac{\rho\Gamma^2}{8\pi^2 r^2}$$

$$\boxed{P(r,\theta) = P_\infty - \frac{\rho U_\infty\Gamma}{2\pi}\frac{\sin\theta}{r} - \frac{\rho\Gamma^2}{8\pi^2 r^2}}$$