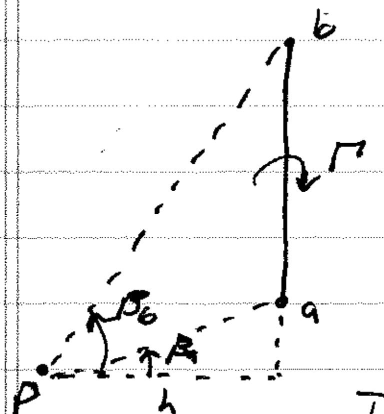
Problem #1

Show that $\rightarrow$ $w = \frac{\Gamma}{4\pi h}\left(\sin\beta_b - \sin\beta_a\right)$



Eq. (2.71) and (2.72): $u_\theta = \frac{\Gamma}{4\pi}\int_{-\infty}^{\infty}\frac{\cos\beta \, ds}{r^2}$

$r = h\sec\beta$

$s = h\tan\beta$

In our problem $w = u_\theta$ and the integration limits do not go from $-\infty$ to ∞. Instead lets transform our integral into β space and transform the limits with it. Then the limits will be from β_a to β_b.

$$w = \frac{\Gamma}{4\pi}\int_a^b \frac{\cos\beta \, ds}{r^2} = \frac{\Gamma}{4\pi}\int_a^b \frac{\cos\beta}{h^2\sec^2\beta}\, ds$$

$$s = h\tan\beta \rightarrow ds = h\sec^2\beta \, d\beta$$

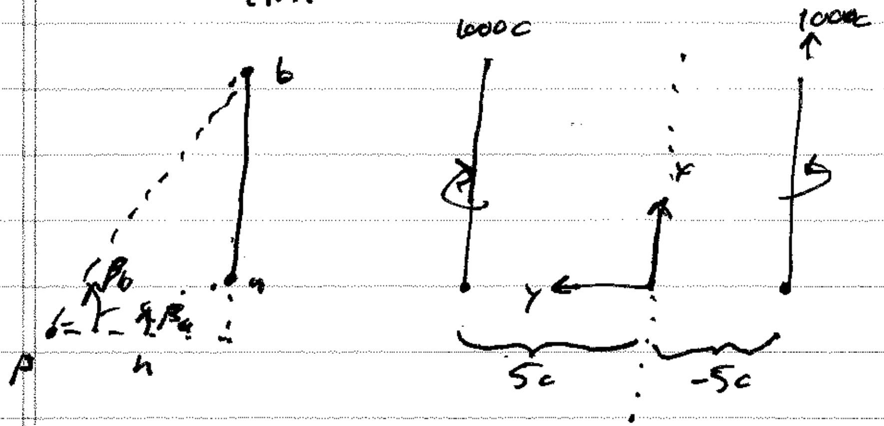
Substitute: $$w = \frac{\Gamma}{4\pi}\int_{\beta_a}^{\beta_b}\frac{\cos\beta}{h^{\cancel{2}}\cancel{\sec^2\beta}}\left(\cancel{h}\cancel{\sec^2\beta}\right)d\beta$$

$$w = \frac{\Gamma}{4\pi h}\int_{\beta_a}^{\beta_b}\cos\beta \, d\beta$$

So $\rightarrow$ $\boxed{w = \frac{\Gamma}{4\pi h}\left(\sin\beta_b - \sin\beta_a\right)}$

Problem #2

$$w = \frac{-\Gamma}{4\pi h}\left(\sin\beta_b - \sin\beta_a\right)$$

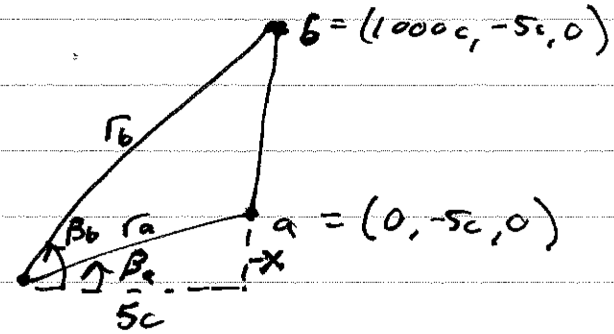


Let $h = 5c$ and we will double the induced velocity due to the superposition of both flow fields:

$$w = \frac{-\Gamma}{2\pi(5c)}\left(\sin\beta_b - \sin\beta_a\right)$$

Let $\Gamma = \pi U_\infty \alpha c \rightarrow \quad w = \frac{-\cancel{\pi} U_\infty \alpha \cancel{c}}{10\cancel{\pi}\cancel{c}}\left(\sin\beta_b - \sin\beta_a\right)$

$$\boxed{\frac{w}{U_\infty} = \frac{-\alpha}{10}\left(\sin\beta_b - \sin\beta_a\right)}$$



Let $\left(-5 \le x/c \le 15\right)$

$r_a = \sqrt{x^2 + 25c^2}$ $\qquad \sin\beta_a = \frac{-x}{\sqrt{x^2+25c^2}}$

$r_b = \sqrt{(1000c - x)^2 + 25c^2}$

$$\sin\beta_b = \frac{1000c - x}{\sqrt{(1000c - x)^2 + 25c^2}}$$

Let $x_0 = 1000c \qquad \sin\beta_b = \frac{x_0 - x}{\sqrt{(x_0 - x)^2 + 25c^2}}$

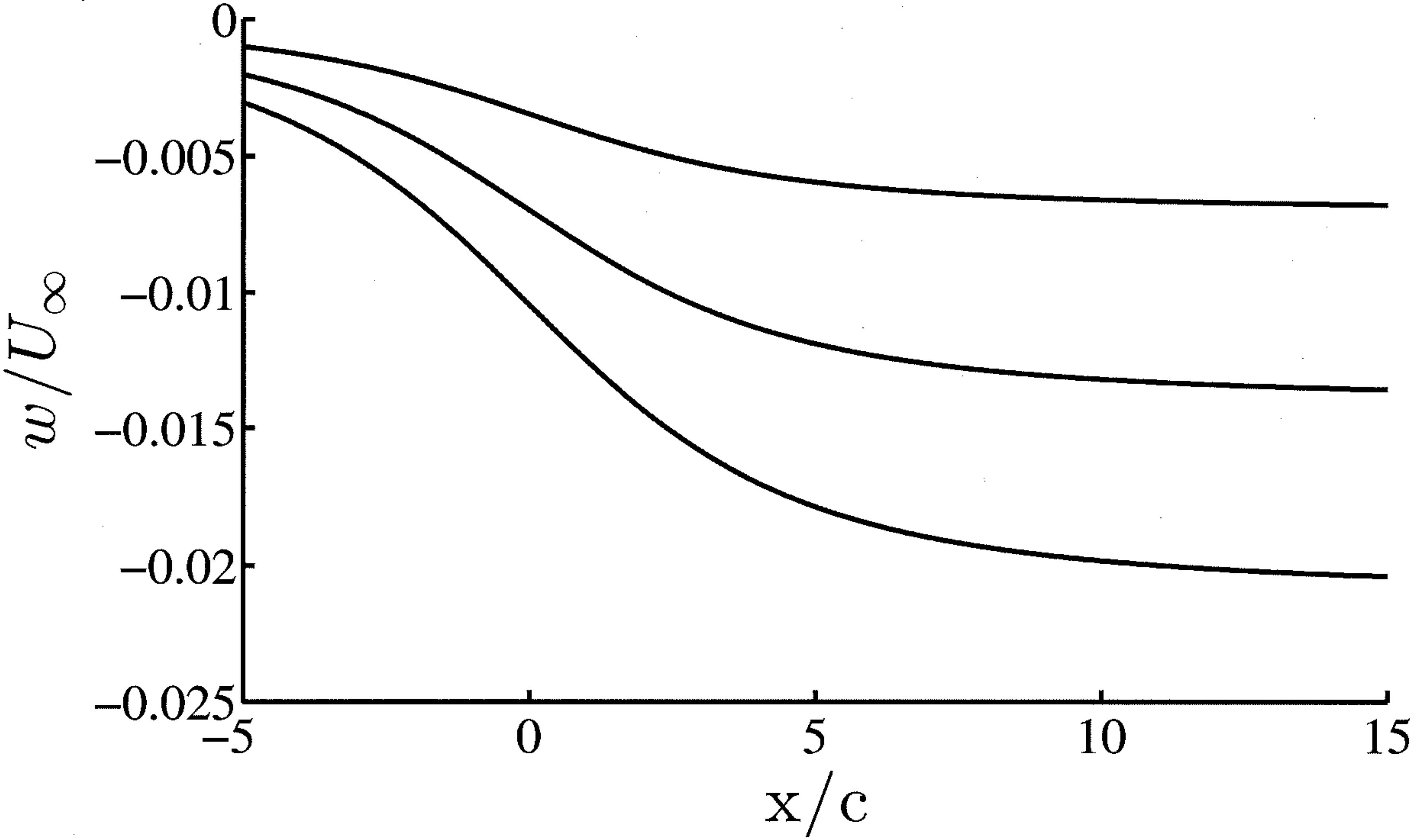
So $\rightarrow$ $$\boxed{\frac{w}{U_\infty} = \frac{-\alpha}{10}\left[\frac{x_0 - x}{\sqrt{(x_0-x)^2+25c^2}} + \frac{x}{\sqrt{x^2+25c^2}}\right]}$$

Non dimensionalize:

$$\boxed{\frac{w}{U_\infty} = \frac{-\alpha}{10}\left[\frac{(x_0/c) - (x/c)}{\sqrt{(x_0/c - x/c)^2 + 25}} + \frac{(x/c)}{\sqrt{(x/c)^2 + 25}}\right]}$$

$$\frac{x_0}{c} = 1000 \quad \text{and} \quad \left(-5 \le \frac{x}{c} \le 15\right)$$

```
clear
clc
close all

% This program is to solve P2 of homework #7 for Aerodynamics MECH 326 Fall
% 2014.

% Problem Parameters
alpha = [2 4 6]*(pi/180);        % Geometric angle of attack convert from degrees to
radians.
x0_c = 1000;                     % end of trailing vortices
N = 1000;                        % Number of points of interest
x_c = linspace(-5,15,N)';        % points of interest

% Calculate the induced velocity along the x-axis
influence = (x0_c - x_c)./sqrt((x0_c - x_c).^2 + 25) +  x_c./sqrt(x_c.^2 + 25);
w_Uinf = -(1/10)*repmat(alpha,N,1).*repmat(influence,1,3);

%% Plotting
FontSizeAx = 24;
FontSizeLb = 32;
afFigurePosition = [0 0 25 15];
axespos = [0.175 0.15 0.76 0.76];
ylabelpos = [-0.15 0.5];
xlabelpos = [0.5 -0.1];

% induced velocity
figure;
set(gcf, 'Units', 'centimeters','PaperPositionMode', 'auto','Position',
afFigurePosition);
set(gcf,'DefaultAxesFontSize',
FontSizeAx,'DefaultAxesFontName','TimesNewRoman','DefaultAxesGridLineStyle','-
.','DefaultAxesLineWidth',2,'DefaultAxesFontWeight','Normal')

hold on

    for i = 1:numel(alpha)
        plot(x_c,w_Uinf(:,i),'-k','linewidth',2)
    end

hold off

set(gca, 'FontName', 'TimesNewRoman', 'FontSize', FontSizeAx)
set(gca, 'Units', 'normalized', 'Position', axespos);
xlabel('x/c','FontName', 'TimesNewRoman','FontUnit', 'points','FontSize',
FontSizeLb,'FontWeight', 'normal','Rotation', 0,'Units', 'Normalize','Position',
xlabelpos,'Interpreter', 'LaTeX');
ylabel('$$w/U_\infty$$','FontName', 'TimesNewRoman','FontUnit', 'points','FontSize',
FontSizeLb,'FontWeight', 'normal','Rotation', 90,'Units', 'Normalize','Position',
ylabelpos,'Interpreter', 'LaTeX');

print('-depsc','-r300','downwash');
```



0
−0.005
−0.01
−0.015
−0.02
−0.025
w/U_∞
−5
0
5
10
15
x/c

Problem #3

$$L' = L'_s \left[1 - \frac{1}{2}\left(\frac{|y|}{b/2}\right)\right]$$

Show that $\rightarrow$ $\alpha_i(y_0) = \frac{L'_s}{4\pi\rho U_\infty^2 b} \ln \frac{y_0^2}{(b/2)^2 - y_0^2}$

Start with induced angle of attack equation:

$$\alpha_i(y_0) = -\frac{1}{4\pi U_\infty}\int_{-b/2}^{b/2} \frac{(d\Gamma/dy)_{wing}}{y_0 - y}\, dy$$

Use Kutta-Joukowski Theorem to determine circulation:

$$L' = \rho U_\infty \Gamma = L'_s \left[1 - \frac{1}{2}\left(\frac{|y|}{b/2}\right)\right]$$

$$\frac{d\Gamma}{dy} = \frac{d}{dy}\left[\frac{L'_s}{\rho U_\infty}\left(1 - \frac{1}{2}\left(\frac{|y|}{b/2}\right)\right)\right]$$

$$\frac{d\Gamma}{dy} = \begin{cases} -\frac{L'_s}{\rho U_\infty b} & y \geq 0 \\ \frac{L'_s}{\rho U_\infty b} & y \leq 0 \end{cases}$$

$$\alpha_i(y_0) = -\frac{1}{4\pi U_\infty}\left[\int_{-b/2}^{0} \frac{L'_s}{\rho U_\infty b}\left(\frac{1}{y_0 - y}\right) dy + \int_{0}^{b/2}\left(\frac{-L'_s}{\rho U_\infty b}\right)\left(\frac{1}{y_0 - y}\right) dy\right]$$

$$\alpha_i(y_0) = \frac{-L'_s}{4\pi\rho b U_\infty^2}\left[\int_{-b/2}^{0} \frac{1}{y_0 - y}\, dy + \int_{0}^{b/2} \frac{1}{-y_0 + y}\, dy\right]$$

Let $u_1 = y_0 - y \rightarrow du_1 = -dy$: limits $\Rightarrow y_0 + b/2 \rightarrow y_0$
$u_2 = y - y_0 \rightarrow du_2 = dy$: limits $\rightarrow -y_0 \rightarrow b/2 - y_0$

$$\text{So: } \alpha_i(y_0) = \frac{-L_s'}{4\pi\rho U_\infty^2 b}\left[\int_{y_0+b/2}^{y_0} \frac{-1}{u_1}\,du_1 + \int_{-y_0}^{b/2-y_0} \frac{1}{u_2}\,du_2\right]$$

$$\alpha_i(y_0) = \frac{+L_s'}{4\pi\rho U_\infty^2 b}\left[+\ln(y_0) \mp \ln(y_0+b/2) \mp \ln(b/2-y_0) + \ln(-y_0)\right]$$

$$\left[\cancel{\ln\left[(y_0+b/2)(b/2-y_0)\right]}\right]$$

$$\left[\ln\left[\frac{-y_0^2}{(y_0+b/2)(b/2-y_0)}\right]\right]$$

$$\ln\left[\frac{y_0^2}{(y_0+b/2)(y_0-b/2)}\right] = \ln\frac{y_0^2}{y_0^2-(b/2)^2}$$

$$\text{So} \rightarrow \boxed{\alpha_i(y_0) = \frac{L_s'}{4\pi\rho U_\infty^2 b}\ln\frac{y_0^2}{y_0^2-(b/2)^2}}$$

This is correct → the book has a minus sign error. This solution we can check. We know that $|y_0|$ varies from $0 \rightarrow b/2$. Unless $|y_0| = b/2$ the ln term is less than zero which leads to a negative α_i for positive lift (L_s'). This is correct.

Problem #4

Elliptical planform. Sealevel air. $U_\infty = 45$ m/s.
$W/S = 1000$ N/m². Untwisted wing. $m_0 = 5.7$

$b = 10$ m

$AR = 5$

(a) Find sectional-lift and induced drag coefficients.
(b) Find effective, induced and absolute angles of attack.
(c) What is the power that is required to overcome the induced drag of the wing?

$$AR = \frac{b^2}{S} \rightarrow S = \frac{b^2}{AR} = \frac{(10\text{ m})^2}{5}$$

$$S = 20\text{ m}^2$$

During steady level flight $\rightarrow L = W$

so $\rightarrow L/S = 1000$ N/m²

and $\rightarrow$ $$\frac{L}{\frac{1}{2}\rho U_\infty^2 S} = \underbrace{C_L = c_\ell}_{\text{elliptical wings}} = \frac{1000\text{ N/m}^2}{\frac{1}{2}\rho U_\infty^2}$$

$\rho = 1.23$ kg/m³ $\quad U_\infty = 45$ m/s $\rightarrow$ $$c_\ell = \frac{1000\text{ N/m}^2}{\frac{1}{2}(1.23\text{ kg/m}^3)(45\text{ m/s})^2}$$

$$\boxed{c_\ell = 0.803}$$

Also $\rightarrow$ $$c_{d_i} = C_{D_i} = \frac{C_L^2}{\pi AR} = \frac{(0.803)^2}{\pi(5)}$$

$$\boxed{c_{d_i} = 0.041}$$

(b) Induced AoA $\rightarrow$ $\alpha_i = -\frac{C_{di}}{C_\ell} \Rightarrow \boxed{\alpha_i = -0.051 \text{ rad}}$

$$\boxed{\alpha_i = -2.93^\circ}$$

Effective AoA $\rightarrow$ $\alpha_o = C_\ell / m_o = \frac{(0.803)}{5.7}$

$$\boxed{\alpha_o = 0.141 \text{ rad}}$$

$$\boxed{\alpha_o = 8.06^\circ}$$

Absolute AoA $\rightarrow$ $\alpha_a = \alpha_o - \alpha_i = 0.141 \text{ rad} + 0.051 \text{ rad}$

$$\boxed{\alpha_a = 0.192 \text{ rad}}$$

$$\boxed{\alpha_a = 11^\circ}$$

(c) Power required $\rightarrow$ $P = D_i U_\infty$

$$D_i = C_{Di}\left(\tfrac{1}{2}\rho U_\infty^2 S\right)$$

$$= (0.041)\left[\tfrac{1}{2}(1.23 \text{ kg/m}^3)(45 \text{ m/s})^2(20 \text{ m}^2)\right]$$

$$D_i = 1021.21 \text{ N}$$

$$P = (1021.21 \text{ N})(45 \text{ m/s})$$

$$P = 45{,}954.3 \ \frac{\text{N}\cdot\text{m}}{\text{s}}$$

$$\boxed{P = 45.95 \text{ kW}}$$