Engineering Vibrations & Systems

Module 6
Electrical & Electronic Systems

ME 242
Professor M. Chew, PhD, PE

Module 6

1. Introduction
2. Electrical Systems
   - 2.1 Passive Electrical Elements
   - 2.2 Kirchhoff's Laws
   - 2.3 Transfer Functions
   - 2.4 Impedances
   - 2.4 Examples
3. Electronic Systems
   - 3.1 Operational Amplifiers
   - 3.2 Applications of Op Amps
   - 3.3 Example

1. Introduction

Modern engineering systems such as robotics, aeronautical and automotive engineering have many electrical subsystems. These can be a power supply, sensor, motor or a controller. In this module, the fundamentals of modeling electrical, electronic and electromechanical systems are presented.

The concept of impedance is useful. Therefore electrical impedance and mechanical impedance will also be discussed.

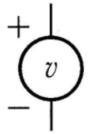
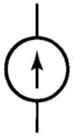
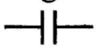
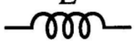
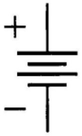
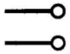
OHMS Law:

$$\boxed{V = I\,R} \qquad [1]$$

where V is the applied voltage
I is the current flowing through the circuit
R is the resistance in the circuit

1. Introduction

Electrical quantities.

Quantity	Units	Circuit symbol
Voltage	volt (V)	Voltage Source 
Charge	coulomb (C) = N · m/V	
Current	ampere (A) = C/s	Current Source i 
Resistance	ohm (Ω) = V/A	R 
Capacitance	farad (F) = C/V	C 
Inductance	henry (H) = V · s/A	L 
Battery	—	
Ground	—	
Terminals (input or output)	—	

2. Electrical Systems

2.1 Passive Electrical Elements

(Section 6.1 of SD)

Passive elements are resistors, capacitors and inductors, while sources that provide energy are call *active elements*. Elements that dissipate energy are called *loads*.

Relations of elements for currents are:

Resistor:

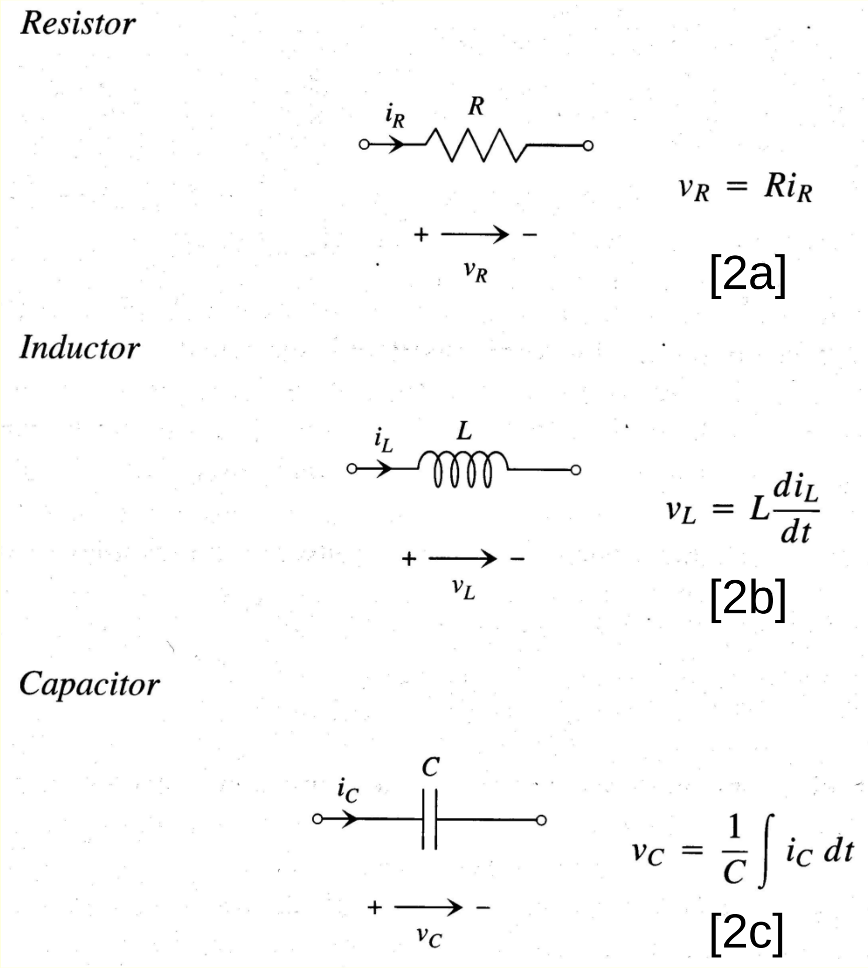
$$i_R = \frac{v_R}{R} \qquad [2a]$$

Inductor:

$$i_R = \frac{1}{L}\int v_L dt \qquad [2b]$$

Capacitor:

$$i_C = C\frac{dv_c}{dt} \qquad [2c]$$

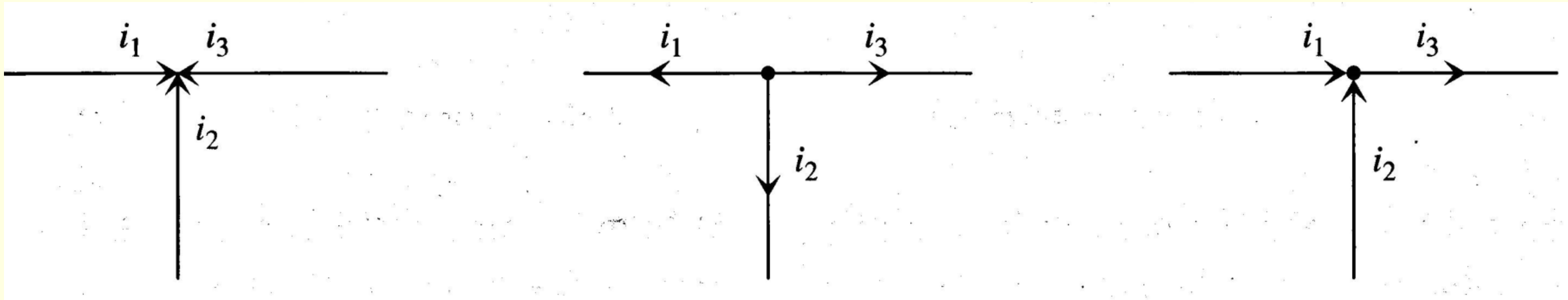


2. Electrical Systems

2.2 Kirchkoff's Laws

A. Kirchkoff's Current Law, KCL (Node Method):
The algebraic sum of all currents entering (*in*) a circuit **node** is zero.

$$\sum_k (i_k)_{in} = 0 \qquad [3]$$



(a) (b) (c)

$i_1 + i_2 + i_3 = 0$

$-i_1 - i_2 - i_3 = 0$

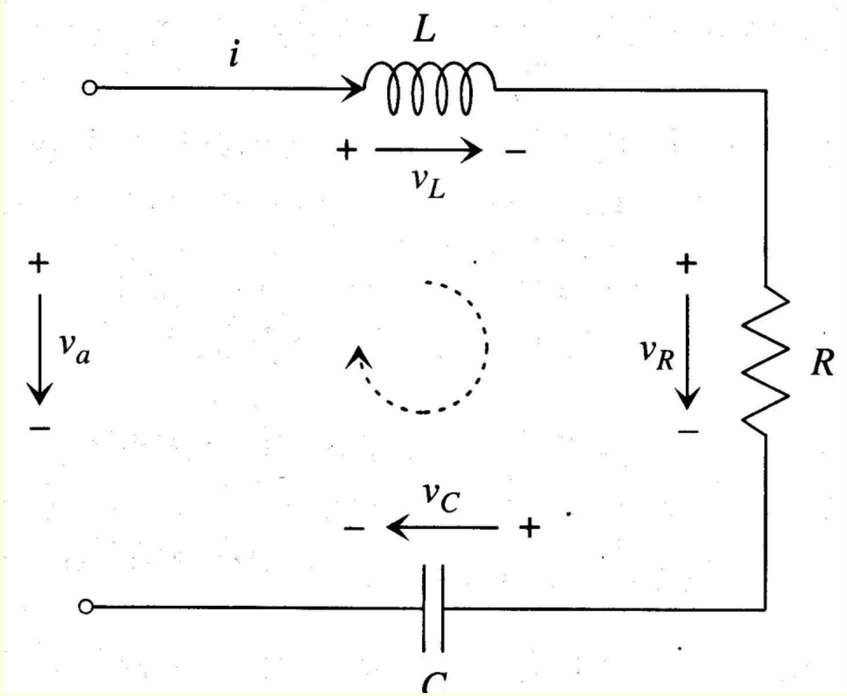
$i_1 + i_2 - i_3 = 0$

2. Electrical Systems

B. Kirchkoff's Voltage Law, KVL (Loop Method):

The algebraic sum of all voltage drops around a circuit loop is zero

$$\sum v_{drop} = 0 \qquad [4]$$



$$v_L + v_R + v_C - v_a = 0$$

2. Electrical Systems

2.3 Transfer Functions (Section 6.3 of SD)

The concept of *transfer functions* is very useful in the analysis of the response to inputs. Consider the system:

$$\tau \dot{x} + x = bf(t)$$

where τ is the time constant. Assuming zero i.c. and taking the Laplace Transform of the above equation:

$$\tau s X(s) + X(s) = bF(s)$$

so that the transfer function:

$$T(s) = \frac{X(s)}{F(s)} = \frac{b}{\tau s + 1} \qquad [5]$$

The transfer function is the transform of the forced response divided by the transform of the input.

2. Electrical Systems

2.3 Impedances (Section 6.3 of SD)

Impedance in electrical systems is a generalization of the concept of resistance and is denoted by the transfer function:

$$Z(s) = \frac{V(s)}{I(s)} \qquad [6a]$$

(a) *Impedances* of various electrical elements

(i) Resistances:

$$v_R = i_R\, R$$

$$V_R(s) = I_R(s)R$$

$$\boxed{Z_R(s) = \frac{V_R(s)}{I_R(s)} = R} \qquad [6b]$$

2. Electrical Systems

(ii) Capacitances:

$$v_C = \frac{1}{C}\int i_c \, dt$$

$$\dot{v}_c = \frac{1}{C} i_c$$

$$V_c(s)s = \frac{1}{C} I_c(s)$$

$$\boxed{Z_C(s) = \frac{V_c(s)}{I_c(s)} = \frac{1}{Cs}} \qquad [6c]$$

(iii) Inductances:

$$v_L = L\frac{di_L}{dt}$$

$$V_L(s) = L\, I_L(s)s$$

$$\boxed{Z_L(s) = \frac{V_L(s)}{I_L(s)} = Ls} \qquad [6d]$$

2. Electrical Systems

(b) Equivalence Impedances

(i) *n* impedances in SERIES:

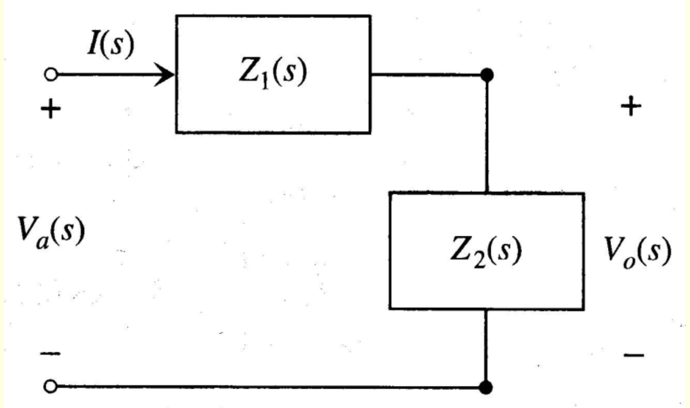
$$Z_{eq} = Z_1(s) + Z_2(s) + \cdots + Z_n(s) \qquad [6e]$$

(ii) *n* impedances in PARALLEL:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots + \frac{1}{Z_n(s)} \qquad [6f]$$

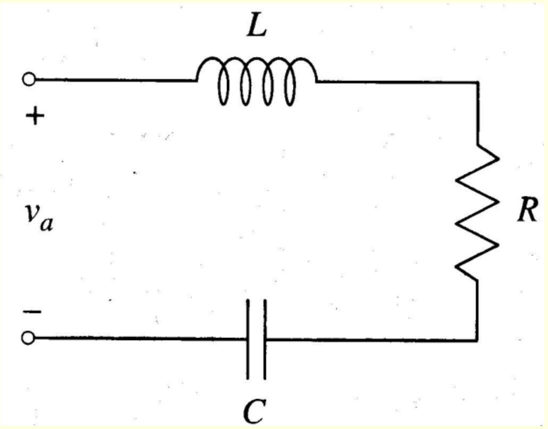
(iii) Voltage Divider Transfer Function:

$$\frac{V_o(s)}{V_a[s]} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} \qquad [6g]$$



2. Electrical Systems

2.4 Examples (Section 6.2, 6.3 of SD)



Example 1. Given a simple RLC circuit. Determine

(a) The differential equation

(b) The natural frequency and damping ratio

(c) The transfer function ${V_C(s)}/{V_a(s)}$

Using KVL (loop method):

$$\sum v_{drop} = 0$$

$$v_L + v_R + v_C - v_a = 0$$

From Eq. [2] and since all currents are the same:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = v_a$$

(a) Take time derivative and divide by L:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\frac{dv_a}{dt} \qquad [7a]$$

2. Electrical Systems

Example 1 (continue)

(b) Therefore: $\omega_n = \frac{1}{\sqrt{LC}}$ and $2\zeta\omega_n = \frac{R}{L}$ [7b]

$$\Rightarrow \quad \zeta = \frac{R}{2\omega_n L} = \frac{R}{2}\sqrt{\frac{C}{L}}$$

(c) Take Laplace Transform of Eq. [7]:

$$\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) I(s) = \frac{s}{L} V_a(s)$$

so that

$$\frac{I(s)}{V_a(s)} = \frac{s}{L\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]}$$

But for the capacitor:

$$v_c = \frac{1}{C}\int i_c dt \quad \Rightarrow \quad V_c(s) = \frac{I(s)}{Cs}$$

Therefore:

$$\frac{V_c(s)}{V_a(s)} = \frac{I(s)}{Cs\, V_a(s)} = \frac{1}{LC\left[s^2 + \frac{R}{L}s + \frac{1}{LC}\right]} \quad [7c]$$

2. Electrical Systems

Example 1 (continue)

Another way to work out part (c) is using the voltage divider method – Eq. [6g]:

$$\frac{V_c(s)}{V_a(s)} = \frac{Z_c(s)}{Z_L(s) + Z_R(s) + Z_C(s)} = \frac{{}^{1}/_{Cs}}{Ls + R + {}^{1}/_{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

which works out to be the same as [7c].

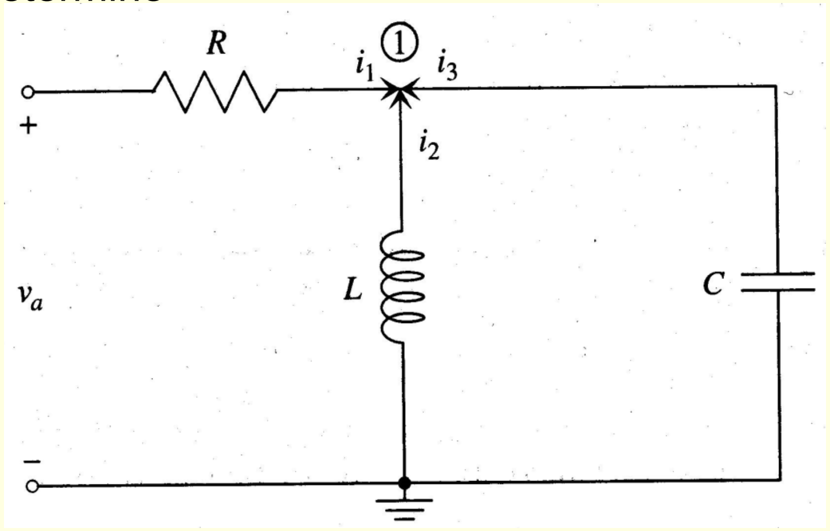
2. Electrical Systems

Example 2. Given the following RLC circuit. Determine

(a) The differential equation, natural frequency and damping ratio

(b) The transfer function ${V_L(s)}/{V_a(s)}$

(c) The transfer function ${V_R(s)}/{V_a(s)}$



Using KCL (node method). At node 1:

$$\sum_k (i_k)_{in} = 0$$

$$i_1 + i_2 + i_3 = 0$$

$$\frac{v_a - v_1}{R} + \frac{1}{L}\int (0 - v_1)dt + C\frac{d}{dt}(0 - v_1) = 0$$

(a) Differentiating w.r.t. time:

$$C\ddot{v_1} + \frac{1}{R}\dot{v_1} + \frac{1}{L}v_1 = \frac{1}{R}\dot{v_a} \qquad [8a]$$

2. Electrical Systems

Example 2 (continue)

so that: $\omega_n = \frac{1}{\sqrt{LC}}$ and $2\zeta\omega_n = \frac{1}{CR}$

$$\Rightarrow \quad \zeta = \frac{1}{2\omega_n CR} = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

(b) Transfer function:

$$\frac{V_L(s)}{V_a(s)} = \frac{V_1(s)}{V_a(s)} = \frac{s}{RC\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right]} \qquad [8b]$$

(c) Voltage across resistor R:

$$v_R = v_a - v_1$$

Therefore transfer function:

$$\frac{V_R(s)}{V_a(s)} = 1 - \frac{V_1(s)}{V_a(s)} = 1 - \frac{s}{RC\left[s^2 + \frac{1}{RC}s + \frac{1}{LC}\right]} \qquad [8c]$$

2. Electrical Systems

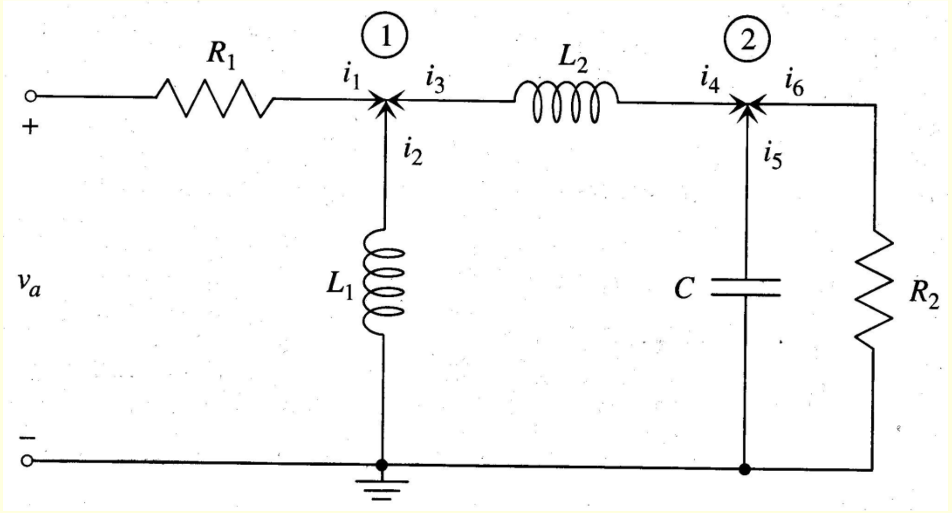
Example 3. Given the following RLC circuit. Determine

(a) The differential equations

(b) The transfer function $V_{L1}(s)/V_a(s)$

(c) The transfer function $V_C(s)/V_a(s)$

(d) The transfer function $I_1(s)/V_a(s)$



Use KCL (node method) → 2 D.E.s

At node 1:

$$i_1 + i_2 + i_3 = 0$$

$$\frac{v_a - v_1}{R_1} + \frac{1}{L_1}\int (0 - v_1)dt + \frac{1}{L_2}\int (v_2 - v_1)dt = 0 \qquad [9a]$$

Differentiating w.r.t. time:

$$\frac{\dot{v}_a}{R_1} - \frac{\dot{v}_1}{R_1} - \frac{v_1}{L_1} + \frac{v_2}{L_2} - \frac{v_1}{L_2} = 0$$

$$\Rightarrow \quad \frac{\dot{v}_1}{R_1} + \left(\frac{1}{L_1} + \frac{1}{L_2}\right)v_1 - \frac{v_2}{L_2} = \frac{\dot{v}_a}{R_1} \qquad [9b]$$

2. Electrical Systems

Example 3 (Continue):

At Node 2:

$$i_4 + i_5 + i_6 = 0$$

$$\frac{1}{L_2}\int (v_1 - v_2)dt + C\frac{d}{dt}(0 - v_2) + \frac{0 - v_2}{R_2} = 0$$

Differentiating w.r.t. time:

$$\frac{v_2 - v_1}{L_2} - C\ddot{v}_2 - \frac{\dot{v}_2}{R_2} = 0$$

$$\Rightarrow \quad C\ddot{v}_2 + \frac{\dot{v}_2}{R_2} - \frac{v_1}{L_2} + \frac{v_2}{L_2} = 0 \qquad [9c]$$

(a) Putting together Eqs. [9b] and [9c]:

$$\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}\begin{Bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{Bmatrix} + \begin{bmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{bmatrix}\begin{Bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{Bmatrix} + \begin{bmatrix} 1/L_1 + 1/L_2 & -1/L_2 \\ -1/L_2 & 1/L_2 \end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} \dot{v}_a/R_1 \\ 0 \end{Bmatrix} \qquad [9d]$$

Take Laplace Transform of Eq. [9d] with zero i.c. and simplifying:

$$\begin{bmatrix} s/R_1 + 1/L_1 + 1/L_2 & -1/L_2 \\ -1/L_2 & Cs^2 + s/R_2 + 1/L_2 \end{bmatrix}\begin{Bmatrix} V_1(s) \\ V_2(s) \end{Bmatrix} = \begin{Bmatrix} sV_a(s)/R_1 \\ 0 \end{Bmatrix} \qquad [9e]$$

2. Electrical Systems

Example 3 (Continue):

Eliminate denominator terms by multiplying top & bottom of Eq. [9e] by $R_1L_1L_2$ and R_2L_2:

$$\begin{bmatrix} L_1L_2s + R_1(L_1+L_2) & -R_1/L_1 \\ -R_2 & (R_2Cs+1)L_2s + R_2 \end{bmatrix} \begin{Bmatrix} V_1(s) \\ V_2(s) \end{Bmatrix} = \begin{Bmatrix} L_1L_2sV_a(s) \\ 0 \end{Bmatrix}$$

Applying Cramer's Rule:

$$V_1(s) = \frac{1}{\Delta(s)} \begin{vmatrix} L_1L_2sV_a(s) & -R_1/L_1 \\ 0 & (R_2Cs+1)L_2s + R_2 \end{vmatrix} = \frac{[(R_2Cs+1)L_2s + R_2]L_1L_2sV_a(s)}{\Delta(s)} \qquad \text{[9f]}$$

$$V_2(s) = \frac{1}{\Delta(s)} \begin{vmatrix} L_1L_2s + R_1(L_1+L_2) & L_1L_2sV_a(s) \\ -R_2 & 0 \end{vmatrix} = \frac{R_2L_1L_2sV_a(s)}{\Delta(s)} \qquad \text{[9g]}$$

where the characteristic polynomial is:

$$\Delta(s) = [L_1L_2s + R_1(L_1+L_2)][(R_2Cs+1)L_2s + R_2] - R_1R_2/L_1$$

Simplifying this polynomial into a cubic:

$$\begin{aligned}\Delta(s) = {} & L_1L_2R_2Cs^3 + [L_1L_2 + R_1(L_1+L_2)R_2C]s^2 \\ & + [R_1(L_1+L_2) + R_2L_1]s + R_1R_2\end{aligned} \qquad \text{[9h]}$$

2. Electrical Systems

Example 3 (Continue):

(b) and (c) Therefore the transfer functions are:

$$\frac{V_1(s)}{V_a(s)} = \frac{[(R_2Cs+1)L_2s+R_2]L_1L_2s}{\Delta(s)} \qquad [10a]$$

$$\frac{V_2(s)}{V_a(s)} = \frac{R_2L_1L_2s}{\Delta(s)} \qquad [10b]$$

(d) Transform function $I_1(s)$ can be obtained from: $i_1 = (v_a - v_1)/R_1$

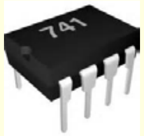
$$I_1(s) = \frac{V_a(s)-V_1(s)}{R_1} = \frac{V_a(s)}{R_1}\left[1-\frac{V_1(s)}{V_a(s)}\right]$$

Simplifying:

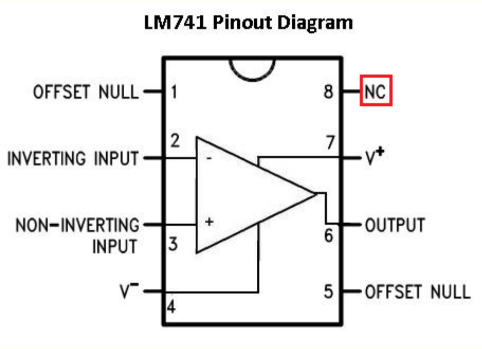
$$\frac{I_1(s)}{V_a(s)} = \frac{(L_1+L_2)R_2Cs^2+(L_1+L_2)s+R_2}{L_1L_2R_2Cs^3+[L_1L_2+R_1(L_1+L_2)R_2C]s^2+[R_1(L_1+L_2)+R_2L_1]s+R_1R_2} \qquad [10c]$$

Denominator of transfer function (characteristic polynomial) serves as a check that the system is *third* order. This result is expected because the first and second d.e. of the system are first and second orders, respectively. Also the system d.e. are coupled; they must be solved simultaneously, using Cramer's Rule in this example, being less prone to mistakes than say, the substitution method.

3. Electronic Systems

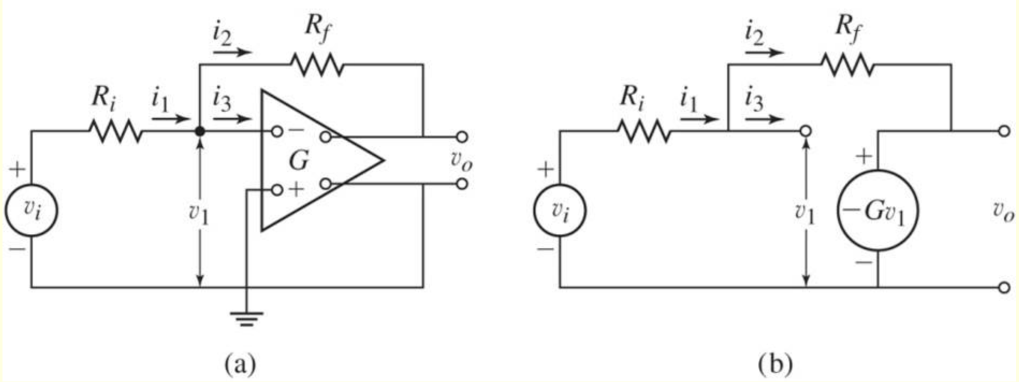


3.1 Operational Amplifiers (Section 6.4 of SD)



An Op-Amp is an electronic amplifier with a single output and very high voltage gain (10^5 to 10^6). They are available in hundreds of different types and have many uses in electronic circuits. Since they have very high impedances and so can only handle very small currents, their use is limited to low-current applications, such as in instruments and control systems.

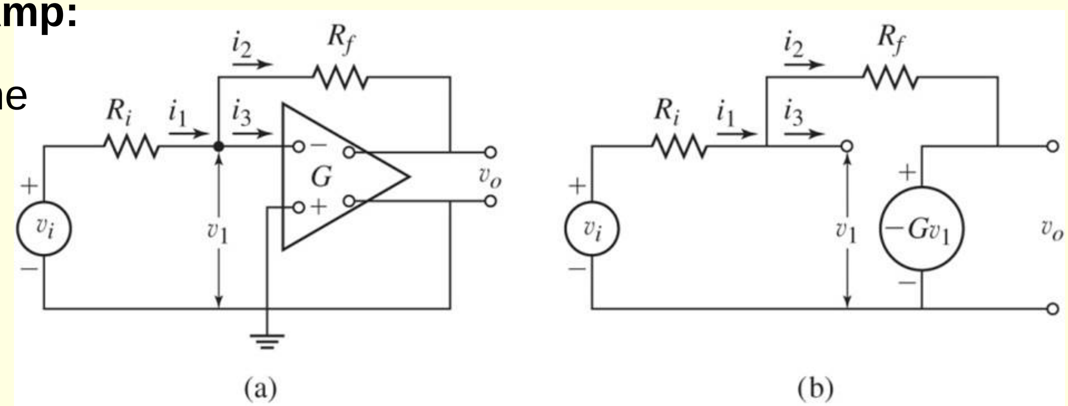
(A) Properties of Op-Amps:



1. The Op-Amp gain G, is very large
2. $v_o = -Gv_1$ [11a]
3. Input impedance is very large. So the current i_3 drawn by the op-amp is very small.
4. From [11a]: $v_- \approx v_+$ [11b]

3. Electronic Systems

(B) Input-Output Relation of Op-Amp:



Since i_3 is very small, the input to the neg. terminal can be considered to be an open circuit. Therefore:

$$i_1 = \frac{v_i - v_1}{R_i}$$

and
$$i_2 = \frac{v_1 - v_o}{R_f}$$

From KCL: $i_1 = i_2 + i_3$

Since i_3 is very small $i_3 \approx 0$, so that

$$i_1 \approx i_2$$

$$\frac{v_i - v_1}{R_i} = \frac{v_1 - v_o}{R_f}$$

But $v_1 = -v_o/G$ so that:

$$\frac{v_i}{R_i} + \frac{v_o}{R_i G} = -\frac{v_o}{R_f G} - \frac{v_o}{R_f}$$

G is very large so that those terms that are divided by G → 0. What is left is:

$$\frac{v_i}{R_i} = -\frac{v_o}{R_f}$$

Therefore

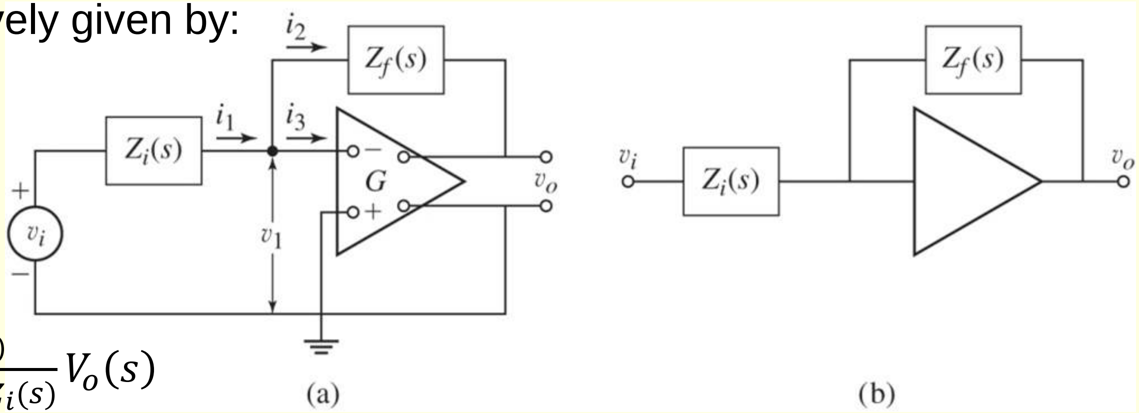
$$\boxed{v_o = -\frac{R_f}{R_i} v_i} \qquad [12]$$

This is a ***MULTIPLIER*** Op-Amp. For an ***INVERTER Op-Amp***, $R_f = R_i$

3. Electronic Systems

(C) General Input-Output Relation of Op-Amp:

Use the impedance concept to simplify the model of an op-amp. The general feedback and input elements $Z_f(s)$ and $Z_i(s)$ respectively, are now used, instead of resistors at the feedback and input. Therefore the impedances $Z_i(s)$ and $Z_f(s)$ of the input and feedback elements are respectively given by:



$$V_i(s) - V_1(s) = Z_i(s)I_1(s)$$

$$V_1(s) - V_o(s) = Z_f(s)I_2(s)$$

Eliminating $I_1(s)$ and $I_2(s)$:

$$V_1(s) = \frac{Z_f(s)}{Z_f(s)+Z_i(s)}V_i(s) + \frac{Z_i(s)}{Z_f(s)+Z_i(s)}V_o(s)$$

Also $V_o(s) = -GV_1(s)$ so that eliminating $V_1(s)$:

$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_f(s)+Z_i(s)}\frac{G}{1+GH}$$

where:

$$H(s) = \frac{Z_i(s)}{Z_f(s)+Z_i(s)}$$

Since G is very large $|GH(s)| \gg 1$:

$$\frac{V_o(s)}{V_i(s)} \approx -\frac{Z_f(s)}{Z_i(s)} \qquad [13]$$

Compare this result with Eq. [12] previously derived.

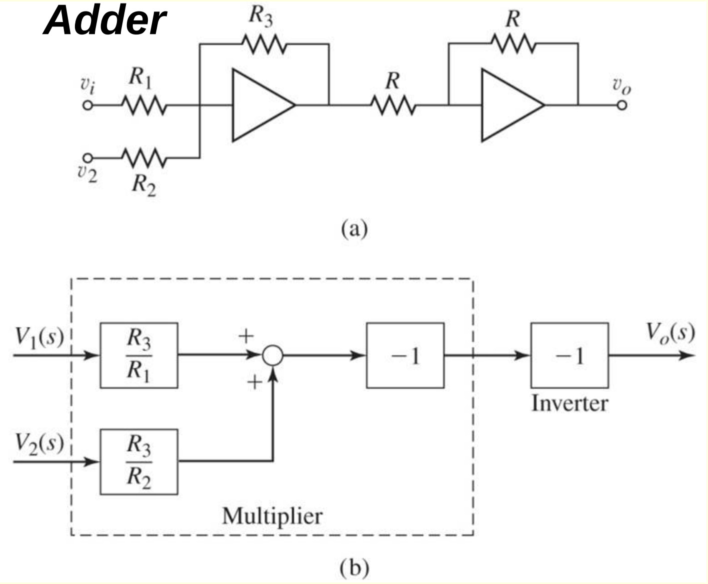
3. Electronic Systems

3.2 Applications of Op-Amps

(A) Adder, Subtractor & Comparator Op-Amp

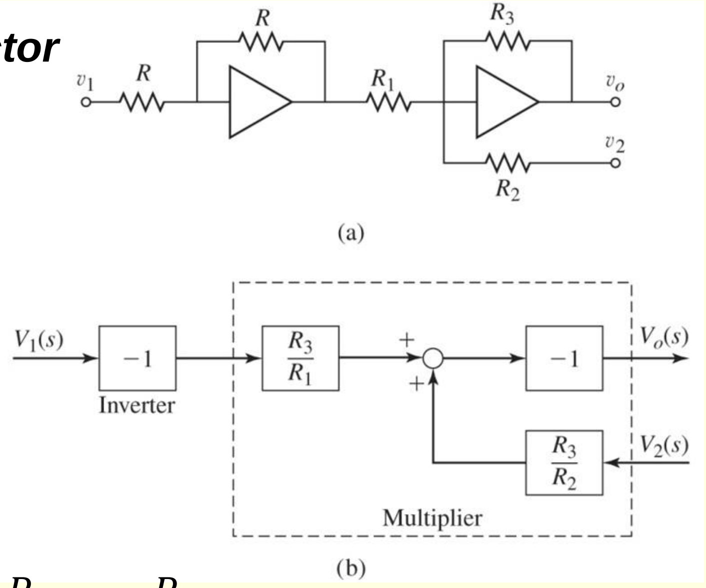
The *Multiplier* Op-Amp can be modified to act as an (i) *Adder* (ii) *Subtractor* (iii) *Comparator* if another inverter is included.

Adder



$$v_o = \frac{R_3}{R_1}v_1 + \frac{R_3}{R_2}v_2 \qquad [14a]$$

Subtractor



$$v_o = \frac{R_3}{R_1}v_1 - \frac{R_3}{R_2}v_2 \qquad [14b]$$

The ***Comparator*** Op-Amp is created by setting $R_1 = R_2 = R_3$ in [14b] giving:

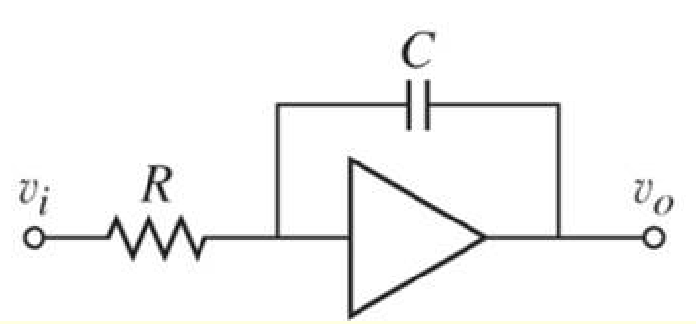
$$v_o = v_1 - v_2 \qquad [14c]$$

3. Electronic Systems

3.2 Applications of Op-Amps (Continue)

(B) Integration

The *Integration* Op-Amp is created by using a Capacitor for the feedback element. From Eq. [6c], The impedance of a resistor and capacitor is R and ${}^{1}/_{Cs}$ respectively, so that from Eq.[13], the transfer function:



$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} = -\frac{1}{RCs}$$

In the time domain:

$$v_o = -\frac{1}{RC}\int_0^t v_i \, dt \qquad [14d]$$

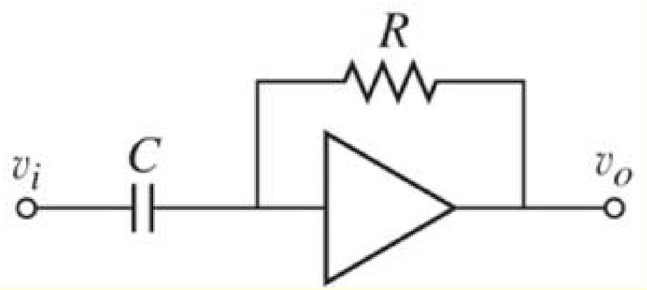
This circuit integrates the input voltage, reverses the sign and divides it by RC, so that it is called an op-amp ***integrator***.

3. Electronic Systems

3.2 Applications of Op-Amps (Continue)

(C) Differentiation

The *Differentiation* Op-Amp is created by using a Capacitor for the input element. From Eq. [6c], The impedance of a resistor and capacitor is R and ${}^{1}/_{Cs}$ respectively, so that from Eq.[13], the transfer function:



$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} = -RCs$$

In the time domain:

$$v_o = -RC\frac{dv_i(t)}{dt} \qquad [14e]$$

This circuit differentiates the input voltage, reverses the sign and multiplies it by RC, so that it is called an op-amp ***differentiator***. Since the input voltage v_i can be filled with high frequency noise, the output voltage may not be usable. Example 6.4.4 p. 351 of SD gives a design of an improved op-amp differentiator.

3. Electronic Systems

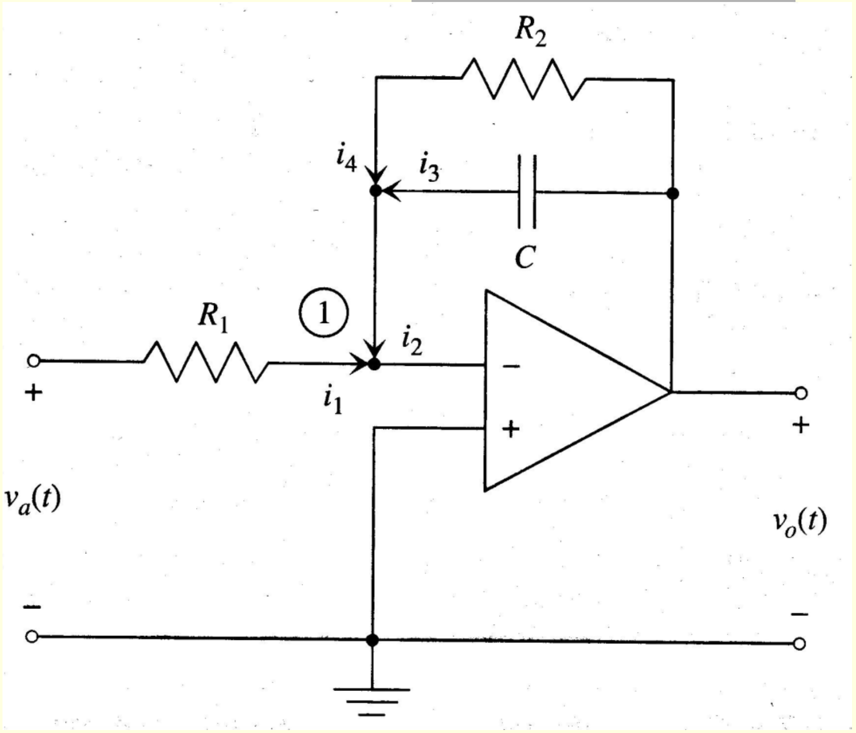
3.3 Example

Given the op-amp circuit, determine:

(a) The governing differential equation

(b) The time constant

(c) The transfer function $V_o(s)/V_a(s)$



(a) Apply KCL at node 1:

$$\sum_k i_{in} = 0 \qquad i_1 + i_2 = 0$$

but $i_2 = i_3 + i_4$

so that $\quad i_1 + i_3 + i_4 = 0$

$$\frac{v_a - v_1}{R_1} + C\frac{d}{dt}(v_o - v_1) + \frac{v_o - v_1}{R_2} = 0$$

From [11b]: $\quad v_- = v_1 = v_+ = v_{ground} = 0$

Therefore: $$C\frac{dv_o}{dt} + \frac{v_o}{R_2} = -\frac{v_a}{R_1} \qquad [15a]$$

3. Electronic Systems

3.3 Example (Continue)

(b) Multiply both sides of d.e. [15a] by R_2:

$$R_2 C \frac{dv_o}{dt} + v_o = -\frac{R_2 v_a}{R_1} \qquad [15b]$$

Therefore the time constant

$$\tau = R_2 C$$

(c) Taking the Laplace Transform of Eq.[15b]:

$$(R_2 Cs + 1)V_o(s) = -\frac{R_2}{R_1} V_s(s)$$

Therefore transfer function is

$$\frac{V_o(s)}{V_a(s)} = -\frac{R_2/R_1}{(R_2 Cs + 1)} \qquad [15c]$$

4. Electromechanical Systems