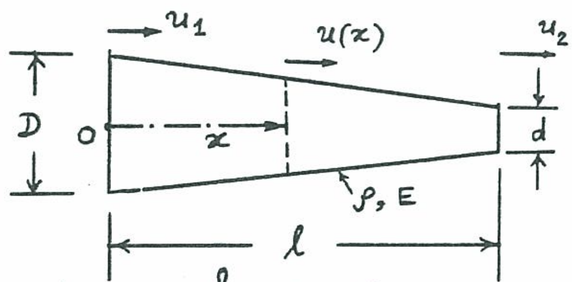
Chapter 12

Finite Element Method

12.1 As diameter $h(x) = D$ at $x=0$ and $= d$ at $x = \ell$, we have



$$h(x) = D + \left(\frac{d-D}{\ell}\right) x$$

<u>stiffness matrix:</u>

$V(t)$ = strain energy of element $= \frac{1}{2}\int_0^\ell EA\left(\frac{\partial u}{\partial x}\right)^2 dx$

with

$$u(x,t) = \left(1-\frac{x}{\ell}\right)\cdot u_1(t) + \left(\frac{x}{\ell}\right)\cdot u_2(t)$$

and

$$A(x) = \frac{\pi h^2}{4} = \frac{\pi}{4}\left[D^2 + \left(\frac{d-D}{\ell}\right)^2 x^2 + 2D\left(\frac{d-D}{\ell}\right)x\right]$$

Thus strain energy expression becomes

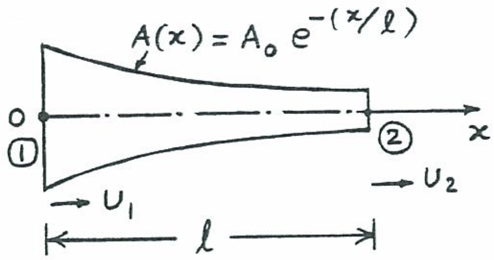
$$V = \frac{\pi E}{24\ell}\left(D^2 + d^2 + dD\right)\left(u_1^2 + u_2^2 - 2u_1u_2\right)$$

$$= \frac{1}{2}\vec{u}^T[k]\vec{u} \equiv \frac{1}{2}(u_1 \quad u_2)\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22}\end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

This gives the element stiffness matrix as

$$[k] = \frac{\pi E}{12\ell}\left(D^2 + d^2 + dD\right)\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}$$

12.2 Assume linear displacement model



$$u(x) = \alpha_1 + \alpha_2 \, x = U_1 + \left(\frac{U_2 - U_1}{l}\right) x$$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{U_2 - U_1}{l}$$

$$\sigma_x = E \, \epsilon_x = E \left(\frac{U_2 - U_1}{l} \right)$$

$$\text{strain energy} = \iiint\limits_{V^{(e)}} \frac{1}{2} \sigma_x \epsilon_x \, dV = \frac{1}{2} \int_{x=0}^{l} E \left(\frac{U_2 - U_1}{l} \right)^2 A(x) \, dx$$

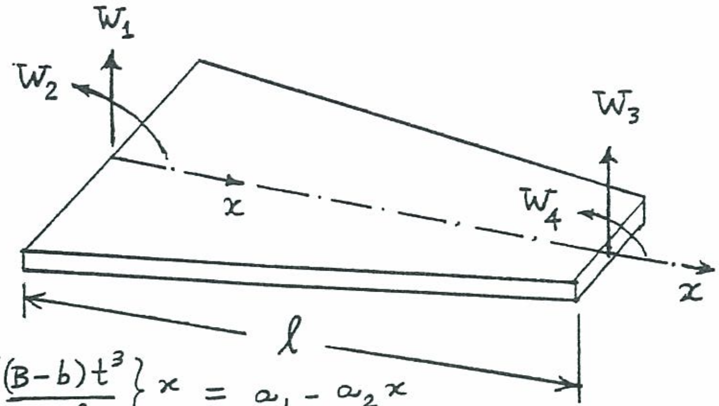
$$= \frac{E}{2 l^2} (U_2 - U_1)^2 \int_{x=0}^{l} A_0 \cdot e^{-\left(\frac{x}{l}\right)} \cdot dx = \frac{E}{2 l^2} (U_2 - U_1)^2 A_0 \left[- l \cdot e^{-\frac{x}{l}} \right]_0^l$$

$$= \frac{E}{2 l^2} (U_2 - U_1)^2 \, l \, A_0 \left(1 - \frac{1}{e} \right) = \frac{1}{2} \frac{E A_0}{l} (0.6321) (U_2 - U_1)^2$$

$$\equiv \frac{1}{2} \vec{U}^T [k] \vec{U} = \frac{1}{2} (U_1 \quad U_2) [k] \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

$$\therefore \; [k] = \frac{E A_0}{l} (0.6321) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

12.3 Derivation of element stiffness matrix:



At x,

thickness $= t$

width $= w(x) = B - \left(\frac{B-b}{\ell}\right) x$

$$I(x) = \frac{1}{12}\, w(x)\, t^3 = \frac{B t^3}{12} - \left\{\frac{(B-b)t^3}{12\ell}\right\} x = a_1 - a_2 x$$

with $a_1 = \frac{B t^3}{12}$ and $a_2 = \frac{(B-b)t^3}{12\ell}$

Deflection of beam: $$w(x) = \sum_{i=1}^{4} N_i(x) \cdot W_i \qquad (E_1)$$

where $N_i(x)$ are defined by Eqs. (12.33) – (12.36).

Strain energy of element is given by

$$V = \frac{1}{2} \int_0^{\ell} E\, I(x) \left\{\frac{d^2 w(x)}{dx^2}\right\}^2 dx \qquad (E_2)$$

with $\frac{d^2 N_1}{dx^2} = -\frac{6}{\ell^2} + \frac{12}{\ell^3} x$, $\frac{d^2 N_2}{dx^2} = -\frac{4}{\ell} + \frac{6}{\ell^2} x$,

$\frac{d^2 N_3}{dx^2} = \frac{6}{\ell^2} - \frac{12}{\ell^3} x$, $\frac{d^2 N_4}{dx^2} = -\frac{2}{\ell} + \frac{6}{\ell^2} x$

$$\left\{\frac{d^2 w(x)}{dx^2}\right\}^2 = c_1^2 + c_2^2 x^2 + 2 c_1 c_2 x = (c_1 + c_2 x)^2 \qquad (E_3)$$

where $c_1 = -\frac{6}{\ell^2} W_1 - \frac{4}{\ell} W_2 + \frac{6}{\ell^2} W_3 - \frac{2}{\ell} W_4$

and $c_2 = \frac{12}{\ell^3} W_1 + \frac{6}{\ell^2} W_2 - \frac{12}{\ell^3} W_3 + \frac{6}{\ell^2} W_4$

Integration in Eq. (E_2) gives

$$V = \frac{1}{2} E \Big\{ W_1^2 \Big[\frac{36}{\ell^4}(a_1 \ell - a_2 \ell^2/2) - \frac{72}{\ell^5}(a_1 \ell^2 - 2 a_2 \ell^3/3)$$
$$+ \frac{144}{\ell^6}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big] + W_2^2 \Big[\frac{16}{\ell^2}(a_1 \ell - a_2 \ell^2/2)$$
$$- \frac{24}{\ell^3}(a_1 \ell^2 - 2 a_2 \ell^3/3) + \frac{36}{\ell^4}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big]$$
$$+ W_3^2 \Big[\frac{36}{\ell^4}(a_1 \ell - a_2 \ell^2/2) - \frac{72}{\ell^5}(a_1 \ell^2 - 2 a_2 \ell^3/3)$$
$$+ \frac{144}{\ell^6}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big] + W_4^2 \Big[\frac{4}{\ell^2}(a_1 \ell - a_2 \ell^2/2)$$
$$- \frac{12}{\ell^3}(a_1 \ell^2 - 2 a_2 \ell^3/3) + \frac{36}{\ell^4}(a_1 \ell^3/3 - a_2 \ell^4/4)\Big]$$

$$+ 2W_1W_2\Big[\frac{24}{\ell^3}(a_1\ell - a_2\ell^2/2) - \frac{42}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3)$$
$$+ \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big] + 2W_1W_3\Big[-\frac{36}{\ell^4}(a_1\ell - a_2\ell^2/2)$$
$$+ \frac{72}{\ell^5}(a_1\ell^2 - 2a_2\ell^3/3) - \frac{144}{\ell^6}(a_1\ell^3/3 - a_2\ell^4/4)\Big]$$
$$+ 2W_1W_4\Big[\frac{12}{\ell^3}(a_1\ell - a_2\ell^2/2) - \frac{30}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3)$$
$$+ \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big] + 2W_2W_3\Big[-\frac{24}{\ell^3}(a_1\ell - a_2\ell^2/2)$$
$$+ \frac{42}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3) - \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big]$$
$$+ 2W_2W_4\Big[\frac{8}{\ell^2}(a_1\ell - a_2\ell^2/2) - \frac{18}{\ell^3}(a_1\ell^2 - 2a_2\ell^3/3)$$
$$+ \frac{36}{\ell^4}(a_1\ell^3/3 - a_2\ell^4/4)\Big] + 2W_3W_4\Big[-\frac{12}{\ell^3}(a_1\ell - a_2\ell^2/2)$$
$$+ \frac{30}{\ell^4}(a_1\ell^2 - 2a_2\ell^3/3) - \frac{72}{\ell^5}(a_1\ell^3/3 - a_2\ell^4/4)\Big]\Big\} \qquad (E_4)$$

By writing $V = \frac{1}{2}\vec{W}^T[k]\vec{W}$

with $\vec{W} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}$, the stiffness matrix can be identified.

Defining $a_1 = \frac{Bt^3}{12}$, $d_1 = \frac{(B-b)t^3}{12}$, $a_2 = \frac{d_1}{\ell}$, $a_2\ell = d_1$,

the elements of $[k]$ can be expressed as:

$k_{11} = E\left\{a_1\left(\frac{12}{\ell^3}\right) - d_1\left(\frac{6}{\ell^3}\right)\right\}$, $\quad k_{22} = E\left\{a_1\left(\frac{4}{\ell}\right) - d_1\left(\frac{1}{\ell}\right)\right\}$,

$k_{33} = E\left\{a_1\left(\frac{12}{\ell^3}\right) - d_1\left(\frac{6}{\ell^3}\right)\right\}$, $\quad k_{44} = E\left\{a_1\left(\frac{4}{\ell}\right) - d_1\left(\frac{3}{\ell}\right)\right\}$,

$k_{12} = E\left\{a_1\left(\frac{6}{\ell^2}\right) - d_1\left(\frac{2}{\ell^2}\right)\right\}$, $\quad k_{13} = E\left\{a_1\left(-\frac{12}{\ell^3}\right) + d_1\left(\frac{6}{\ell^3}\right)\right\}$,

$k_{14} = E\left\{a_1\left(\frac{6}{\ell^2}\right) - d_1\left(\frac{4}{\ell^2}\right)\right\}$, $\quad k_{23} = E\left\{a_1\left(-\frac{6}{\ell^2}\right) + d_1\left(\frac{2}{\ell^2}\right)\right\}$,

$k_{24} = E\left\{a_1\left(\frac{2}{\ell}\right) - d_1\left(\frac{1}{\ell}\right)\right\}$, $\quad k_{34} = E\left\{a_1\left(-\frac{6}{\ell^2}\right) + d_1\left(\frac{4}{\ell^2}\right)\right\}$

From given data,

$B = 0.25$ m, $b = 0.10$ m, $t = 0.025$ m, $\ell = 2$ m, $E = 2.07 \times 10^{11}$ N/m^2,

$P = 1000$ N, $a_1 = 32552.0833 \times 10^{-11}$, $d_1 = 19531.25 \times 10^{-11}$.

stiffness matrix can be computed as:

$$[k] = \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{bmatrix} 70750 & 80860 & -70750 & 60640 \\ 80860 & 114600 & -80860 & 47170 \\ -70750 & -80860 & 70750 & -60640 \\ 60640 & 47170 & -60640 & 74120 \end{bmatrix} \end{array} \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{array}$$

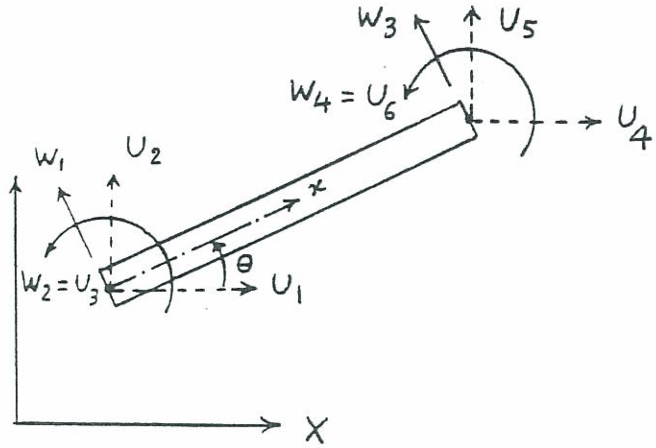
12.4

$$W_1 = U_1 \cos(90+\theta) + U_2 \cos\theta + U_3(0)$$

$$W_2 = U_1(0) + U_2(0) + U_3(1)$$

i.e.,

$$\begin{Bmatrix} W_1 \\ W_2 \end{Bmatrix} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix}$$



$$\vec{W} = [\lambda]\ \vec{U}$$

where $\vec{W} = \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{Bmatrix}$, $\vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix}$, and

$$\underset{4\times 6}{[\lambda]} = \begin{bmatrix} -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For a beam element with degrees of freedom $\vec{W}$,

$$[k^{(e)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

and

$$[m^{(e)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Matrices in global system (X, Y system) are given by

$$[\bar{k}^{(e)}] = [\lambda]^T [k^{(e)}] [\lambda]$$

and

$$[\bar{m}^{(e)}] = [\lambda]^T [m^{(e)}] [\lambda]$$

12.5 Assembled stiffness matrix is given in the solution of Problem 12.14. For the assembled mass matrix, we note

$$[m^{(e)}] = \frac{\rho^{(e)} A^{(e)} \ell^{(e)}}{420} \begin{bmatrix} 156 & 22\,\ell^{(e)} & 54 & -13\,\ell^{(e)} \\ 22\ell^{(e)} & 4\,\ell^{(e)2} & 13\ell^{(e)} & -3\,\ell^{(e)2} \\ 54 & 13\ell^{(e)} & 156 & -22\,\ell^{(e)} \\ -13\,\ell^{(e)} & -3\,\ell^{(e)2} & -22\,\ell^{(e)} & 4\,\ell^{(e)2} \end{bmatrix}$$

with

$\rho^{(e)} = \frac{0.283}{386.4} = 7.324 \times 10^{-4}$ lb-sec^2/in^4 ; $e = 1$ to 5

$\ell^{(e)} = \ell = 5''$; $e = 1$ to 5

$A^{(1)} = 1.5 \times 1.25 = 1.875$ in^2 $= 5 A_5$; $\quad A^{(2)} = 1.5 \times 1.0 = 1.5$ in^2 $= 4 A_5$

$A^{(3)} = 1.5 \times 0.75 = 1.125$ in^2 $= 3 A_5$; $\quad A^{(4)} = 1.5 \times 0.5 = 0.75$ in^2 $= 2 A_5$

$A^{(5)} = 1.5 \times 0.25 = 0.375$ in^2 $= A_5$

Assembled mass matrix, after applying boundary conditions, is:

$$[M] = \frac{\rho \ell A_5}{420} \begin{array}{c} \begin{array}{cccccccccc} W_3 & W_4 & W_5 & W_6 & W_7 & W_8 & W_9 & W_{10} & W_{11} & W_{12} \end{array} \\ \begin{bmatrix} 1404 & -22\ell & 216 & -52\ell & 0 & 0 & 0 & 0 & 0 & 0 \\ -22\ell & 36\ell^2 & 52\ell & -12\ell^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 216 & 52\ell & 1092 & -22\ell & 162 & -39\ell & 0 & 0 & 0 & 0 \\ -52\ell & -12\ell^2 & -22\ell & 28\ell^2 & 39\ell & -9\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 162 & 39\ell & 780 & -22\ell & 108 & -26\ell & 0 & 0 \\ 0 & 0 & -39\ell & -9\ell^2 & -22\ell & 20\ell^2 & 26\ell & -6\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 108 & 26\ell & 468 & -22\ell & 54 & -13\ell \\ 0 & 0 & 0 & 0 & -22\ell & -6\ell^2 & -22\ell & 12\ell^2 & 13\ell & -3\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 54 & 13\ell & 156 & -22\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \end{array}$$

with $\rho = 7.324 \times 10^{-4}$ lb-sec^2/in^4, $\ell = 5''$ and $A_5 = 0.375$ in^2.

12.9

$A^{(i)} = 2 \text{ in}^2 \quad ; \quad i = 1, 2, 3, 4$

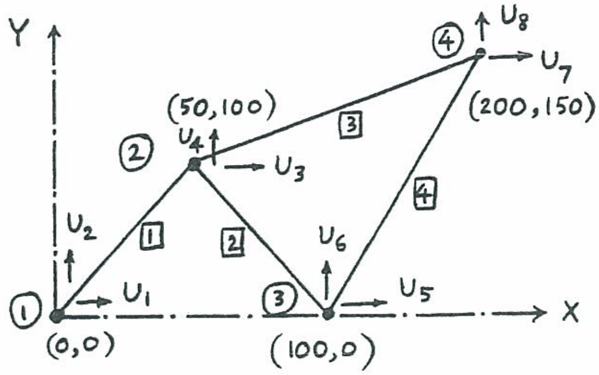
$E = 30 \times 10^6 \text{ psi}$

$\ell^{(1)} = \sqrt{50^2 + 100^2} = 111.8034 \text{ in}$

$\ell^{(2)} = \sqrt{50^2 + 100^2} = 111.8034 \text{ in}$

$\ell^{(3)} = \sqrt{150^2 + 50^2} = 158.1139 \text{ in}$

$\ell^{(4)} = \sqrt{100^2 + 150^2} = 180.2776 \text{ in}$



$$[k^{i}] = \frac{E^{(i)} A^{(i)}}{l^{(i)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

From Eq. (12.52), the global stiffness matrix of element i is

$$[\bar{k}^{(i)}] = [\lambda^{(i)}]^T [k^{(i)}] [\lambda^{(i)}] \quad --- (E_1)$$

where

$$[\lambda^{(i)}] = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & 0 \\ 0 & 0 & \cos\theta_i & \sin\theta_i \end{bmatrix} \quad --- (E_2)$$

and θ_i is the angle made by the element with respect to X-axis.

Thus (E_1) gives

$$[\bar{k}^{(i)}] = \frac{E^{(i)} A^{(i)}}{l^{(i)}} \left[\begin{array}{c|c|c|c} \cos^2\theta_i & \cos\theta_i \sin\theta_i & -\cos^2\theta_i & -\cos\theta_i \sin\theta_i \\ \cos\theta_i \sin\theta_i & \sin^2\theta_i & -\cos\theta_i \sin\theta_i & -\sin^2\theta_i \\ -\cos^2\theta_i & -\cos\theta_i \sin\theta_i & \cos^2\theta_i & \cos\theta_i \sin\theta_i \\ -\cos\theta_i \sin\theta_i & -\sin^2\theta_i & \cos\theta_i \sin\theta_i & \sin^2\theta_i \end{array}\right]$$

Here $\theta_1 = 63.4349°$ (line 1 2), $\theta_2 = 116.5651°$ (line 3 2),

$\theta_3 = 18.4350°$ (line 2 4), $\theta_4 = 56.3099°$ (line 3 4)

$$\frac{E^{(1)} A^{(1)}}{l^{(1)}} = \frac{(30 \times 10^6)(2)}{111.8034} = 0.53666 \times 10^6 \text{ lbf/in}, \quad \cos\theta_1 = 0.4472, \ \sin\theta_1 = 0.8944$$

$$[\bar{k}^{(1)}] = \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array}$$

$$[\bar{k}^{(1)}] = \begin{bmatrix} 107332 & 214664 & -107332 & -214664 \\ 214664 & 429328 & -214664 & -429328 \\ -107332 & -214664 & 107332 & 214664 \\ -214664 & -429328 & 214664 & 429328 \end{bmatrix} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}$$

$$\frac{E^{(2)} A^{(2)}}{l^{(2)}} = \frac{(30 \times 10^6)(2)}{111.8034} = 0.53666 \times 10^6 \text{ lbf/in}, \quad \cos\theta_2 = -0.4472, \ \sin\theta_2 = 0.8944$$

$$\begin{array}{cccc} U_5 & U_6 & U_3 & U_4 \end{array}$$

$$[\bar{k}^{(2)}] = \begin{bmatrix} 107332 & -214664 & -107332 & 214664 \\ -214664 & 429328 & 214664 & -429328 \\ -107332 & 214664 & 107332 & -214664 \\ 214664 & -429328 & -214664 & 429328 \end{bmatrix} \begin{array}{c} U_5 \\ U_6 \\ U_3 \\ U_4 \end{array}$$

$$\frac{E^{(3)} A^{(3)}}{l^{(3)}} = \frac{(30 \times 10^6)(2)}{158.1139} = 0.37947 \times 10^6 \text{ lbf/in}, \quad \cos\theta_3 = 0.9487, \ \sin\theta_3 = 0.3162$$

$$\left[\bar{K}^{(3)}\right] = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} U_3 & U_4 & U_7 & U_8 \\ \end{array}$$

$$\left[\bar{K}^{(3)}\right] = \begin{array}{cc} \begin{array}{cccc} U_3 & U_4 & U_7 & U_8 \end{array} & \\ \begin{bmatrix} 341526 & 113842 & -341526 & -113842 \\ 113842 & 37947 & -113842 & -37947 \\ -341526 & -113842 & 341526 & 113842 \\ -113842 & -37947 & 113842 & 37947 \end{bmatrix} & \begin{array}{c} U_3 \\ U_4 \\ U_7 \\ U_8 \end{array} \end{array}$$

$$\frac{E^{(4)}A^{(4)}}{\ell^{(4)}} = \frac{(30\times10^6)(2)}{180.2776} = 0.33282\times10^6 \text{ lbf/in}, \quad \cos\theta_4 = 0.5547, \quad \sin\theta_4 = 0.8321$$

$$\left[\bar{K}^{(4)}\right] = \begin{array}{cc} \begin{array}{cccc} U_5 & U_6 & U_7 & U_8 \end{array} & \\ \begin{bmatrix} 102405 & 153810 & -102405 & -153810 \\ 153810 & 230415 & -153810 & -230415 \\ -102405 & -153810 & 102405 & 153810 \\ -153810 & -230415 & 153810 & 230415 \end{bmatrix} & \begin{array}{c} U_5 \\ U_6 \\ U_7 \\ U_8 \end{array} \end{array}$$

12.11

$$\vec{U}^{(1)} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(1)}] \underset{\sim}{\vec{U}}$$

$$\vec{U}^{(2)} = \begin{Bmatrix} U_5 \\ U_6 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(2)}] \underset{\sim}{\vec{U}}$$

$$\vec{U}^{(3)} = \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(3)}] \underset{\sim}{\vec{U}}$$

$$\vec{U}^{(4)} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ \vdots \\ U_8 \end{Bmatrix} \equiv [A^{(4)}] \underset{\sim}{\vec{U}}$$

Assembled stiffness matrix of the truss is given by Eq. (12.63):

$$[\underset{\sim}{K}] = \sum_{e=1}^{4} [A^{(e)}]^T [\bar{k}^{(e)}] [A^{(e)}]$$

$$
= \begin{array}{c}
\begin{array}{cccccccc} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 & U_7 & U_8 \end{array} \\
\left[\begin{array}{c|c|c|c|c|c|c|c}
107332 & 214664 & -107332 & -214664 & 0 & 0 & 0 & 0 \\ \hline
214664 & 429328 & -214664 & -429328 & 0 & 0 & 0 & 0 \\ \hline
-107332 & -214664 & \begin{matrix}107332\\+107332\\+341526\end{matrix} & \begin{matrix}214664\\-214664\\+113842\end{matrix} & -107332 & 214664 & -341526 & -113842 \\ \hline
-214664 & -429328 & \begin{matrix}214664\\-214664\\+113842\end{matrix} & \begin{matrix}429328\\+429328\\+37947\end{matrix} & 214664 & -429328 & -113842 & -37947 \\ \hline
0 & 0 & -107332 & 214664 & \begin{matrix}107332\\+102405\end{matrix} & \begin{matrix}-214664\\+153810\end{matrix} & -102405 & -153810 \\ \hline
0 & 0 & 214664 & -429328 & \begin{matrix}-214664\\+153810\end{matrix} & \begin{matrix}429328\\+230415\end{matrix} & -153810 & -230415 \\ \hline
0 & 0 & -341526 & -113842 & -102405 & -153810 & \begin{matrix}341526\\+102405\end{matrix} & \begin{matrix}113842\\+153810\end{matrix} \\ \hline
0 & 0 & -113842 & -37947 & -153810 & -230415 & \begin{matrix}113842\\+153810\end{matrix} & \begin{matrix}37947\\+230415\end{matrix}
\end{array}\right]
\end{array}
\begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \\ U_8 \end{array}
$$

Since nodes 1 and 3 are fixed, $U_1 = U_2 = U_5 = U_6 = 0$ and the final equilibrium equations can be expressed as

$[K]\vec{U} = \vec{F}$ where

$$
[K] = \begin{bmatrix} 556190 & 113842 & -341526 & -113842 \\ 113842 & 896603 & -113842 & -37947 \\ -341526 & -113842 & 443931 & 267652 \\ -113842 & -37947 & 267652 & 268362 \end{bmatrix} \text{ lbf/in}, \quad \vec{U} = \begin{Bmatrix} U_3 \\ U_4 \\ U_7 \\ U_8 \end{Bmatrix} \text{ in},
$$

and $\vec{F} = \begin{Bmatrix} F_3 \\ F_4 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix}$ lbf.

12.13 stiffness matrix of the beam (element) is given in solution of Problem 12.3.

Stresses induced in the beam:

Equilibrium equations:

$$[K]\vec{W} = \vec{F}$$

i.e.

$$\begin{bmatrix} 70750 & -60640 \\ -60640 & 74120 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1000 \end{Bmatrix}$$

The solution of these equations is

$$W_3 = 0.03871 \text{ m}, \quad W_4 = 0.04516 \text{ rad}$$

stress at root:

$$\sigma_{max}\Big|_{x=0} = \frac{M\, y_{max}}{I}\Big|_{x=0} = \frac{E I(x) \dfrac{d^2 w(x)}{dx^2} y_{max}}{I(x)}\Bigg|_{x=0}$$

$$= E \frac{d^2 w(x)}{dx^2} \frac{t}{2}\Big|_{x=0}$$

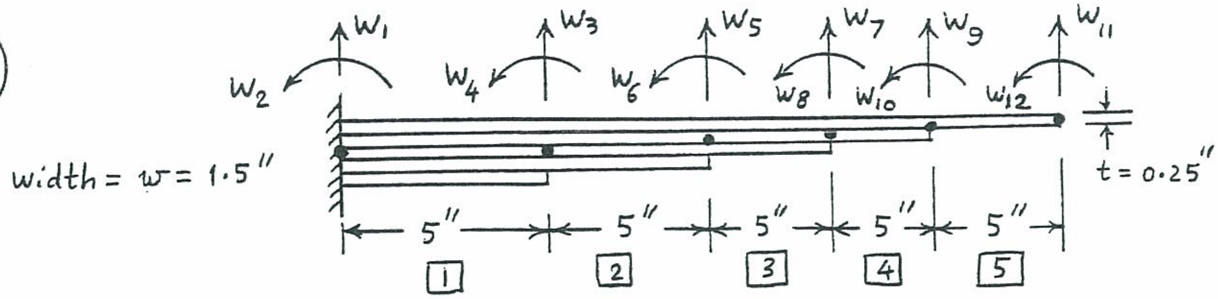
$$= \frac{Et}{2}\left[-\frac{6}{l^2} \overset{0}{W_1} - \frac{4}{l} \overset{0}{W_2} + \frac{6}{l^2} W_3 - \frac{2}{l} W_4\right]$$

$$= \frac{Et}{2}\left[\frac{6}{l^2} W_3 - \frac{2}{l} W_4\right]$$

$$= \frac{(2.07 \times 10^{11})(0.025)}{2}\left\{\frac{6}{4} \times 0.03871 - \frac{2}{2} \times 0.04516\right\}$$

$$= 3.3392 \times 10^7 \text{ N/m}^2$$

12.14



$$[K^{(e)}] = \frac{E^{(e)} I^{(e)}}{l^{(e)3}} \begin{bmatrix} 12 & 6l^{(e)} & -12 & 6l^{(e)} \\ 6l^{(e)} & 4l^{(e)2} & -6l^{(e)} & 2l^{(e)2} \\ -12 & -6l^{(e)} & 12 & -6l^{(e)} \\ 6l^{(e)} & 2l^{(e)2} & -6l^{(e)} & 4l^{(e)2} \end{bmatrix}$$

$E^{(e)} = E = 30 \times 10^6$ psi ; $e = 1$ to 5

$l^{(e)} = l = 5''$; $e = 1$ to 5

$I^{(1)} = \frac{1}{12}(1.5)(1.25)^3 = 0.24414 \text{ in}^4 = 125\, I_5$

$I^{(2)} = \frac{1}{12}(1.5)(1.00)^3 = 0.125 \text{ in}^4 = 64\, I_5$

$I^{(3)} = \frac{1}{12}(1.5)(0.75)^3 = 0.05273 \text{ in}^4 = 27\, I_5$

$I^{(4)} = \frac{1}{12}(1.5)(0.5)^3 = 0.01563 \text{ in}^4 = 8\, I_5$

$I^{(5)} = \frac{1}{12}(1.5)(0.25)^3 = 1.953125 \times 10^{-3} \text{ in}^4 = I_5$

Assembled stiffness matrix, after applying the boundary conditions, is given by Eq. (E_1).

Load vector :

$$\vec{F} = \begin{Bmatrix} F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ F_{11} \\ F_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2000 \\ 0 \end{Bmatrix} \text{ lb} \qquad (E_2)$$

$$[K] = \frac{E I_5}{\ell^3} \begin{array}{cc} \begin{array}{cccccccccc} w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 & w_{10} & w_{11} & w_{12} \end{array} & \\ \begin{bmatrix} 2268 & -366\ell & -768 & 384\ell & 0 & 0 & 0 & 0 & 0 & 0 \\ -366\ell & 756\ell^2 & -384\ell & 128\ell^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -768 & -384\ell & 1092 & -222\ell & -324 & 162\ell & 0 & 0 & 0 & 0 \\ 384\ell & 128\ell^2 & -222\ell & 364\ell^2 & -162\ell & 54\ell^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -324 & -162\ell & 420 & -114\ell & -96 & 48\ell & 0 & 0 \\ 0 & 0 & 162\ell & 54\ell^2 & -114\ell & 140\ell^2 & -48\ell & 16\ell^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -96 & -48\ell & 108 & -42\ell & -12 & 6\ell \\ 0 & 0 & 0 & 0 & 48\ell & 16\ell^2 & -42\ell & 36\ell^2 & -6\ell & 2\ell^2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -12 & -6\ell & 12 & -6\ell \\ 0 & 0 & 0 & 0 & 0 & 0 & 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} & \begin{array}{c} w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \\ w_{10} \\ w_{11} \\ w_{12} \end{array} \end{array} \quad (E_1)$$

with $E = 30 \times 10^6$ psi, $I_5 = 1.953125 \times 10^{-3}$ in^4 and $\ell = 5$ in.

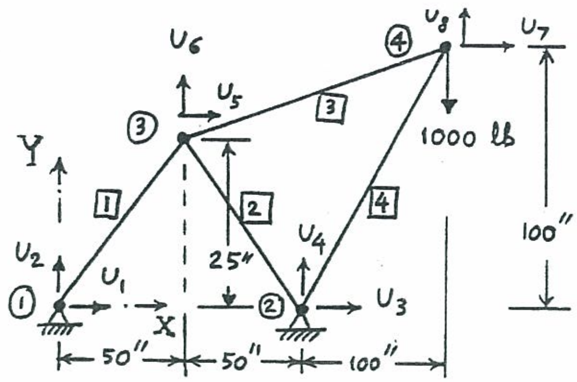
Solution of equilibrium equations, $[K]\vec{W} = \vec{F}$, gives

$$\vec{W}^T = \{ W_3, W_4, W_5, W_6, W_7, W_8, W_9, W_{10}, W_{11}, W_{12} \}^T$$

$$= \{ -0.07964, -0.03072, -0.3554, -0.07738, -0.9531, -0.1564, -2.179, -0.3164, -5.184, -0.7431 \}^T \text{ inch.}$$

12.15

element e	area of c/s, $A^{(e)}$	length $\ell^{(e)}$	Young's modulus, $E^{(e)}$
1	2 in^2	55.9017"	30×10^6 psi
2	2 in^2	55.9017"	30×10^6 psi
3	1 in^2	167.7051"	30×10^6 psi
4	1 in^2	141.4214"	30×10^6 psi



element e	global node corresponding to local node 1	global node corresponding to local node 2	X_i	Y_i	X_j	Y_j	l_{ij}	m_{ij}
			coordinates of local nodes (in)				direction cosines	
1	1	3	0	0	50	25	0.8944	0.4472
2	3	2	50	25	100	0	0.8944	−0.4472
3	3	4	50	25	200	100	0.8944	0.4472
4	2	4	100	0	200	100	0.7071	0.7071

$$[K^{(e)}] = \frac{A^{(e)}E^{(e)}}{l^{(e)}}\begin{bmatrix} l_{ij}^2 & l_{ij}m_{ij} & -l_{ij}^2 & -l_{ij}m_{ij} \\ l_{ij}m_{ij} & m_{ij}^2 & -l_{ij}m_{ij} & -m_{ij}^2 \\ -l_{ij}^2 & -l_{ij}m_{ij} & l_{ij}^2 & l_{ij}m_{ij} \\ -l_{ij}m_{ij} & -m_{ij}^2 & l_{ij}m_{ij} & m_{ij}^2 \end{bmatrix}$$

This gives

$$[K^{(1)}] = \frac{2(30\times10^6)}{55.9017}\begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.4 & -0.8 & -0.4 \\ 0.4 & 0.2 & -0.4 & -0.2 \\ -0.8 & -0.4 & 0.8 & 0.4 \\ -0.4 & -0.2 & 0.4 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}$$

$$[K^{(2)}] = \frac{2(30\times10^6)}{55.9017}\begin{array}{c} \begin{array}{cccc} U_5 & U_6 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 0.8 & -0.4 & -0.8 & 0.4 \\ -0.4 & 0.2 & 0.4 & -0.2 \\ -0.8 & 0.4 & 0.8 & -0.4 \\ 0.4 & -0.2 & -0.4 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} U_5 \\ U_6 \\ U_3 \\ U_4 \end{array}$$

$$[K^{(3)}] = \frac{1(30\times10^6)}{167.7051}\begin{array}{c} \begin{array}{cccc} U_5 & U_6 & U_7 & U_8 \end{array} \\ \begin{bmatrix} 0.8 & 0.4 & -0.8 & -0.4 \\ 0.4 & 0.2 & -0.4 & -0.2 \\ -0.8 & -0.4 & 0.8 & 0.4 \\ -0.4 & -0.2 & 0.4 & 0.2 \end{bmatrix} \end{array} \begin{array}{c} U_5 \\ U_6 \\ U_7 \\ U_8 \end{array}$$

$$[K^{(4)}] = \frac{1(30\times10^6)}{141.4214}\begin{array}{c} \begin{array}{cccc} U_3 & U_4 & U_7 & U_8 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array} \begin{array}{c} U_3 \\ U_4 \\ U_7 \\ U_8 \end{array}$$

Assembled stiffness matrix, after deleting the rows and columns corresponding to degrees of freedom U_1, U_2, U_3 and U_4, is:

$$[K] = 10^6 \begin{array}{c c} \begin{array}{cccc} U_5 & U_6 & U_7 & U_8 \end{array} & \\ \begin{bmatrix} 1.8603 & 0.0716 & -0.1431 & -0.0716 \\ 0.0716 & 0.4652 & -0.0716 & -0.0358 \\ -0.1431 & -0.0716 & 0.2492 & 0.1777 \\ -0.0716 & -0.0358 & 0.1777 & 0.1419 \end{bmatrix} & \begin{array}{c} U_5 \\ U_6 \\ U_7 \\ U_8 \end{array} \end{array}$$

Load vector: $\vec{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \end{Bmatrix}$ lb, $\quad \vec{U} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix}$

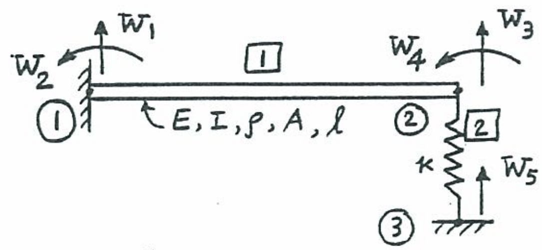
Equilibrium equations:

$$[K]\,\vec{U} = \vec{F}$$

Solution of these equations is

$$\vec{U} = \begin{Bmatrix} U_5 \\ U_6 \\ U_7 \\ U_8 \end{Bmatrix} = \begin{Bmatrix} 0.001168 \\ 0.002341 \\ 0.051650 \\ -0.070540 \end{Bmatrix} \text{ in.}$$

12.16 (a) ONE ELEMENT IDEALIZATION



$$[k^{(1)}] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \left[\begin{array}{cc|cc} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ \hline -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{array}\right] \end{array} \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array} \qquad (E_1)$$

where $I = \frac{1}{12} b h^3 = \frac{1}{12}\left(\frac{50}{100}\right)\left(\frac{25}{1000}\right)^3 = 6.5104 \times 10^{-8} \ m^4$.

$$[K^{(2)}] = k \begin{array}{c} \begin{array}{cc} W_5 & W_3 \end{array} \\ \left[\begin{array}{c|c} 1 & -1 \\ \hline -1 & 1 \end{array}\right] \end{array} \begin{array}{c} W_5 \\ W_3 \end{array} \qquad (E_2)$$

To incorporate the boundary conditions $W_1 = W_2 = W_5 = 0$, we delete the first two rows and columns in (E_1) and first row and column in (E_2). The assembled matrix becomes

$$[K] = \frac{EI}{\ell^3}\begin{array}{cc} w_3 & w_4 \\ \left[\begin{array}{cc} \left(12 + k\frac{\ell^3}{EI}\right) & -6\ell \\ -6\ell & 4\ell^2 \end{array}\right] & \begin{array}{c} w_3 \\ w_4 \end{array} \end{array}$$

$$= \frac{(2.07\times10^{11})(6.5104\times10^{-8})}{(0.25)^3}\begin{bmatrix} 12 + \frac{(10^5)(0.25)^3}{(2.07\times10^{11})(6.5104\times10^{-8})} & -6(0.25) \\ -6(0.25) & 4(0.25)^2 \end{bmatrix}$$

$$= 8.6250\times10^5\begin{bmatrix} 12.1159 & -1.5 \\ -1.5 & 0.25 \end{bmatrix}$$

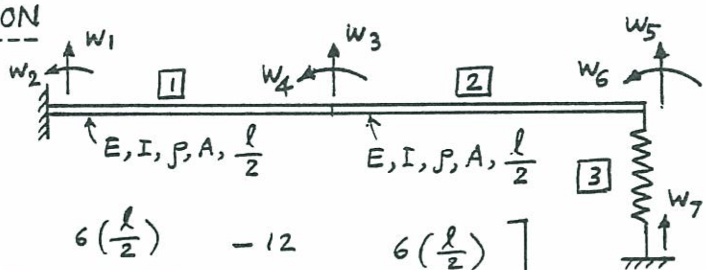
Load vector is $\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \end{Bmatrix} = \begin{Bmatrix} 500 \\ 0 \end{Bmatrix}$

Equilibrium equations become $[K]\vec{W} = \vec{P}$

i.e., $\begin{bmatrix} 12.1159 & -1.5 \\ -1.5 & 0.25 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 57.971\times10^{-5} \\ 0 \end{Bmatrix}$

The solution is given by $W_3 = 1.8605\times10^{-4}$ m; $W_4 = 11.1629\times10^{-4}$ rad.

(b) TWO ELEMENT IDEALIZATION



$$[K^{(1)}] = [K^{(2)}] = \frac{EI}{\left(\frac{\ell}{2}\right)^3}\begin{bmatrix} 12 & 6\left(\frac{\ell}{2}\right) & -12 & 6\left(\frac{\ell}{2}\right) \\ 6\left(\frac{\ell}{2}\right) & 4\left(\frac{\ell}{2}\right)^2 & -6\left(\frac{\ell}{2}\right) & 2\left(\frac{\ell}{2}\right)^2 \\ -12 & -6\left(\frac{\ell}{2}\right) & 12 & -6\left(\frac{\ell}{2}\right) \\ 6\left(\frac{\ell}{2}\right) & 2\left(\frac{\ell}{2}\right)^2 & -6\left(\frac{\ell}{2}\right) & 4\left(\frac{\ell}{2}\right)^2 \end{bmatrix} \qquad (E_1)$$

substituting $E = 2.07\times10^{11}$, $I = 6.5104\times10^{-8}$ and $\ell = 0.25$, Eq. (E_1) becomes

$$\begin{array}{ccccc} & W_3 & W_4 & W_5 & W_6 \text{ ---- for } i = 2 \\ & W_1 & W_2 & W_3 & W_4 \text{ --- for } i = 1 \end{array}$$

$$[K^{(i)}] = 69\times10^5\begin{array}{ccc} \begin{bmatrix} 12 & 0.75 & -12 & 0.75 \\ 0.75 & 0.0625 & -0.75 & 0.03125 \\ -12 & -0.75 & 12 & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} & \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array} & \begin{array}{c} W_3 \\ W_4 \\ W_5 \\ W_6 \end{array} \end{array}$$

$$[K^{(3)}] = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^5 \begin{array}{cc} \begin{array}{cc} W_7 & W_5 \end{array} & \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & \begin{array}{c} W_7 \\ W_5 \end{array} \end{array}$$

Assembled stiffness matrix is

$$[K] = 69 \times 10^5 \begin{array}{cc} \begin{array}{cccc} W_3 & W_4 & W_5 & W_6 \end{array} & \\ \begin{bmatrix} (12+12) & (-0.75+0.75) & -12 & 0.75 \\ (-0.75+0.75) & \begin{pmatrix} 0.0625+ \\ 0.0625 \end{pmatrix} & -0.75 & 0.03125 \\ -12 & -0.75 & \left(12+\frac{10^5}{69\times 10^5}\right) & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} & \begin{array}{c} W_3 \\ W_4 \\ W_5 \\ W_6 \end{array} \end{array}$$

Load vector is

$$\vec{P} = \begin{Bmatrix} P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

Equilibrium equations are $[K]\vec{W} = \vec{P}$

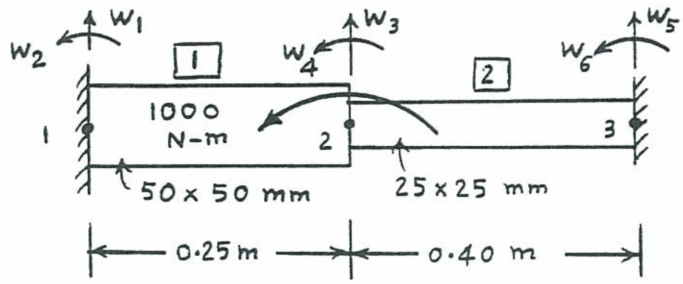
i.e.,

$$69 \times 10^5 \begin{bmatrix} 24 & 0 & -12 & 0.75 \\ 0 & 0.125 & -0.75 & 0.03125 \\ -12 & -0.75 & 12.0145 & -0.75 \\ 0.75 & 0.03125 & -0.75 & 0.0625 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 500 \\ 0 \end{Bmatrix}$$

The solution is given by :

$W_3 = 0.5814 \times 10^{-4}$ m, $W_4 = 8.372 \times 10^{-4}$ rad, $W_5 = 1.86 \times 10^{-4}$ m,
$W_6 = 11.16 \times 10^{-4}$ rad.

12.17 Element 1:



$$I = \frac{1}{12}\left(\frac{50}{1000}\right)\left(\frac{50}{1000}\right)^3$$

$$= 0.5208 \times 10^{-6} \ m^4$$

$$\frac{EI}{\ell^3} = \frac{(2.1\times 10^{11})(0.5208\times 10^{-6})}{(0.25)^3}$$

$$= 0.7 \times 10^7$$

$$[K^{(1)}] = 0.7 \times 10^7 \begin{array}{c} \begin{array}{cccc} w_1 & w_2 & w_3 & w_4 \end{array} \\ \begin{bmatrix} 12 & 1.5 & -12 & 1.5 \\ 1.5 & 0.25 & -1.5 & 0.125 \\ -12 & -1.5 & 12 & -1.5 \\ 1.5 & 0.125 & -1.5 & 0.25 \end{bmatrix} \end{array} \begin{array}{c} w_1 \\ w_2 \\ w_3 \\ w_4 \end{array}$$

$$= 0.7 \times 10^7 \begin{array}{c} \begin{array}{cc} w_3 & w_4 \end{array} \\ \begin{bmatrix} 12 & -1.5 \\ -1.5 & 0.25 \end{bmatrix} \end{array} \begin{array}{c} w_3 \\ w_4 \end{array}$$ after applying boundary conditions

Element 2:

$$I = \frac{1}{12}\left(\frac{25}{1000}\right)\left(\frac{25}{1000}\right)^3 = 0.3255 \times 10^{-7} \ m^4$$

$$\frac{EI}{l^3} = \frac{(2.1 \times 10^{11})(0.3255 \times 10^{-7})}{(0.4)^3} = 10.6805 \times 10^4$$

$$[K^{(2)}] = 10.6805 \times 10^4 \begin{array}{c} \begin{array}{cccc} w_3 & w_4 & w_5 & w_6 \end{array} \\ \begin{bmatrix} 12 & 2.4 & -12 & 2.4 \\ 2.4 & 0.64 & -2.4 & 0.32 \\ -12 & -2.4 & 12 & -2.4 \\ 2.4 & 0.32 & -2.4 & 0.64 \end{bmatrix} \end{array} \begin{array}{c} w_3 \\ w_4 \\ w_5 \\ w_6 \end{array}$$

$$= 10.6805 \times 10^4 \begin{array}{c} \begin{array}{cc} w_3 & w_4 \end{array} \\ \begin{bmatrix} 12 & 2.4 \\ 2.4 & 0.64 \end{bmatrix} \end{array} \begin{array}{c} w_3 \\ w_4 \end{array}$$ after applying boundary conditions

Assembled stiffness matrix:

$$[K] = \begin{bmatrix} (8.4 \times 10^7 + 0.1282 \times 10^7) & (-1.05 \times 10^7 + 0.0256 \times 10^7) \\ (-1.05 \times 10^7 + 0.0256 \times 10^7) & (0.175 \times 10^7 + 0.0068 \times 10^7) \end{bmatrix}$$

Equilibrium equations:

$$10^7 \begin{bmatrix} 8.5282 & -1.0244 \\ -1.0244 & 0.1818 \end{bmatrix} \begin{Bmatrix} w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10^3 \end{Bmatrix}$$

Solution is given by

$$w_3 = 2.0446 \times 10^{-4} \ m, \quad w_4 = 1.7021 \times 10^{-3} \ rad$$

Stresses in elements:

Bending moment $= M = EI \dfrac{d^2 w(x)}{dx^2}$

with $w(x) = \sum_{i=1}^{4} W_i \; N_i(x)$

$$\sigma_{max} = \frac{M \cdot c}{I} = \frac{M h}{2I} = \frac{E h}{2} \frac{d^2 w}{dx^2}$$

$$= \frac{E h}{2} \left[\frac{W_1}{l^3}(12x - 6l) + \frac{W_2}{l^2}(6x - 4l) + \frac{W_3}{l^3}(6l - 12x) + \frac{W_4}{l^2}(6x - 2l) \right] \quad \text{---}(E_1)$$

For element 1,

$W_1 = 0$, $W_2 = 0$, $W_3 = 2.0446 \times 10^{-4}$, $W_4 = 1.7021 \times 10^{-3}$,

$l = 0.25$, $h = 0.05$, $E = 2.1 \times 10^{11}$. Hence Eq. (E_1) gives

$$\sigma_{max}\Big|_{fixed\ end} = \sigma_{max}(x=0) = 3.1560 \times 10^{7} \ N/m^2$$

$$\sigma_{max}\Big|_{loaded\ end} = \sigma_{max}(x=0.25) = 3.9929 \times 10^{7} \ N/m^2$$

For element 2,

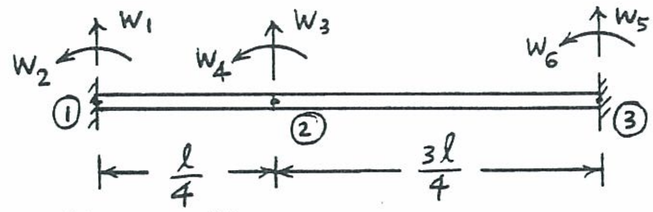
$W_1 = 2.0446 \times 10^{-4}$, $W_2 = 1.7021 \times 10^{-3}$, $W_3 = 0$, $W_4 = 0$,

$l = 0.4$, $h = 0.025$, $E = 2.1 \times 10^{11}$. Hence Eq. (E_1) gives

$$\sigma_{max}\Big|_{loaded\ end} = \sigma_{max}(x=0) = -6.4807 \times 10^{7} \ N/m^2$$

$$\sigma_{max}\Big|_{fixed\ end} = \sigma_{max}(x=0.4) = 4.2467 \times 10^{7} \ N/m^2$$

12.18



For element 1:

$$[k^{(1)}] = \frac{EI}{(l/4)^3} \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc|c} W_1 & W_2 & W_3 & W_4 & \end{array}$$

$$[k^{(1)}] = \frac{EI}{(l/4)^3} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \begin{bmatrix} 12 & \frac{3l}{2} & -12 & \frac{3l}{2} \\ \frac{3l}{2} & \frac{l^2}{4} & -\frac{3l}{2} & \frac{l^2}{8} \\ -12 & -\frac{3l}{2} & 12 & -\frac{3l}{2} \\ \frac{3l}{2} & \frac{l^2}{8} & -\frac{3l}{2} & \frac{l^2}{4} \end{bmatrix} \end{array} \begin{array}{c} \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{array}$$

For element 2:

$$[K^{(2)}] = \frac{EI}{(3\ell/4)^3} \begin{array}{c} \begin{array}{cccc} W_3 & W_4 & W_5 & W_6 \end{array} \\ \begin{bmatrix} 12 & \frac{9\ell}{2} & -12 & \frac{9\ell}{2} \\ \frac{9\ell}{2} & \frac{9\ell^2}{4} & -\frac{9\ell}{2} & \frac{9\ell^2}{8} \\ -12 & -\frac{9\ell}{2} & 12 & -\frac{9\ell}{2} \\ \frac{9\ell}{2} & \frac{9\ell^2}{8} & -\frac{9\ell}{2} & \frac{9\ell^2}{4} \end{bmatrix} \end{array} \begin{array}{c} \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{array}$$

Assembled stiffness matrix:

$$[K] = \frac{64\,EI}{\ell^3} \begin{bmatrix} \left(12 + \frac{12}{27}\right) & \left(-\frac{3\ell}{2} + \frac{9\ell}{54}\right) \\ \left(-\frac{3\ell}{2} + \frac{9\ell}{54}\right) & \left(\frac{\ell^2}{4} + \frac{9\ell^2}{108}\right) \end{bmatrix} = \frac{64EI}{3\ell^3} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} \frac{112}{3} & -4\ell \\ -4\ell & \ell^2 \end{bmatrix} \end{array} \begin{array}{c} \\ W_3 \\ W_4 \end{array}$$

Equilibrium equations:

$$[K]\vec{W} = \vec{P}$$

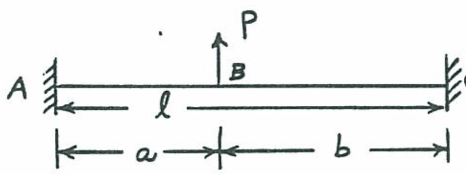
i.e.,

$$\frac{64\,EI}{3\,\ell^3} \begin{bmatrix} 112/3 & -4\ell \\ -4\ell & \ell^2 \end{bmatrix} \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

Solution is given by

$$W_3 = \frac{9\;P\ell^3}{4096\;EI} \quad \text{and} \quad W_4 = \frac{9\;P\ell^2}{1024\;EI}.$$

Simple beam deflection formula:



$$y_{AB} = -\frac{P b^2 x^2}{6EI\ell^3}\left[x(3a+b) - 3a\ell\right]$$

at $x = a$,

$$y_B = -\frac{P b^2 a^2}{6EI\ell^3}\left[3a^2 + ab - 3a\ell\right]$$

when $a = \ell/4$ and $b = 3\ell/4$,

$$y_B = \frac{9}{4096}\frac{P\ell^3}{EI}$$

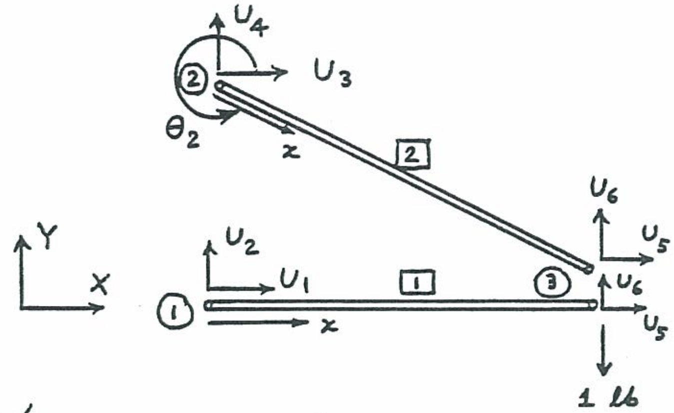
$$\text{slope} = \frac{dy_{AB}}{dx} = -\frac{Pb^2}{6EI\ell^2}\left[3x^2(3a+b) - 6xa\ell\right]$$

at $x = a$, $$\left.\frac{dy}{dx}\right|_B = -\frac{Pb^2}{6EI\ell^3}\left[9a^3 + 3a^2 b - 6a^2\ell\right]$$

When $a = \ell/4$ and $b = 3\ell/4$, $\left.\frac{dy}{dx}\right|_B = \frac{9P\ell^2}{1024\;EI}$

∴ Both results are same. Reason: shape functions = static deflection relations.

12.19



$E^{(1)} = E^{(2)} = 30 \times 10^6$ psi

$A^{(1)} = A^{(2)} = 1 \ in^2$

$\ell^{(1)} = 25''$

$\ell^{(2)} = \sqrt{25^2 + 10^2} = 26.9258''$

$\cos\theta_1 = \cos 0 = 1$

$\sin\theta_1 = 0$

$\cos\theta_2 = (X_3 - X_2)/\ell^{(2)} = 25/26.9258 = 0.9285$

$\sin\theta_2 = (Y_3 - Y_2)/\ell^{(2)} = (0-10)/26.9258 = -0.3714$

Element stiffness matrices:

$$[k^{(1)}] = \frac{A^{(1)}E^{(1)}}{\ell^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [\lambda^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$[K^{(1)}] = [\lambda^{(1)}]^T[k^{(1)}][\lambda^{(1)}] = 12 \times 10^5 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_5 & U_6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ U_1 \\ U_2 \\ U_5 \\ U_6 \end{array}$$

$$[k^{(2)}] = \frac{A^{(2)}E^{(2)}}{\ell^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 11.1417 \times 10^5 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[\lambda^{(2)}] = \begin{bmatrix} 0.9285 & -0.3714 & 0 & 0 \\ 0 & 0 & 0.9285 & -0.3714 \end{bmatrix}$$

$$[K^{(2)}] = [\lambda^{(2)}]^T[k^{(2)}][\lambda^{(2)}]$$

$$= 11.1417 \times 10^5 \begin{array}{c} \begin{array}{cccc} U_3 & U_4 & U_5 & U_6 \end{array} \\ \begin{bmatrix} 0.8621 & -0.3448 & -0.8621 & 0.3448 \\ -0.3448 & 0.1379 & 0.3448 & -0.1379 \\ -0.8621 & 0.3448 & 0.8621 & -0.3448 \\ 0.3448 & -0.1379 & -0.3448 & 0.1379 \end{bmatrix} \end{array} \begin{array}{c} \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{array}$$

Assembled stiffness matrix is:

$$[K] = \begin{bmatrix} (12\times 10^5 + 0.8621\times 11.1417\times 10^5) & (0 - 0.3448\times 11.1417\times 10^5) \\ (0 - 0.3448\times 11.1417\times 10^5) & (0 + 0.1379\times 11.1417\times 10^5) \end{bmatrix}$$

$$= \begin{bmatrix} 21.6053 & -3.8417 \\ -3.8417 & 1.5364 \end{bmatrix} \times 10^5$$

Load vector: $\vec{F} = \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$

Equilibrium equations:

$$[K]\,\vec{U} = \vec{F}$$

i.e.,

$$\left.\begin{aligned} 21.6053\ U_5 - 3.8417\ U_6 &= 0 \\ -3.8417\ U_5 + 1.5364\ U_6 &= -1\times 10^{-5} \end{aligned}\right\} \qquad (E_1)$$

Solution is:

$$U_5 = -0.20838\times 10^{-5}\ \text{in.},\quad U_6 = -1.1719\times 10^{-5}\ \text{in.}$$

Axial displacements of elements:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{(1)} = [\lambda^{(1)}]\,\vec{U}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} U_1 = 0 \\ U_2 = 0 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ U_5 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ -0.20838\times 10^{-5} \end{Bmatrix}\ \text{in.}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}^{(2)} = [\lambda^{(2)}]\,\vec{U}^{(2)} = \begin{bmatrix} 0.9285 & -0.3714 & 0 & 0 \\ 0 & 0 & 0.9285 & -0.3714 \end{bmatrix} \begin{Bmatrix} U_3 = 0 \\ U_4 = 0 \\ U_5 \\ U_6 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0.24176\times 10^{-5} \end{Bmatrix}\ \text{in.}$$

Stresses in elements:

$$\sigma^{(1)} = E^{(1)}\epsilon^{(1)} = E^{(1)}\left(u_2^{(1)} - u_1^{(1)}\right)/\ell^{(1)}$$

$$= (30\times 10^6)(-0.20838\times 10^{-5})/25 = -2.5056\ \text{psi}$$

$$\sigma^{(2)} = E^{(2)}\epsilon^{(2)} = E^{(2)}\left(u_2^{(2)} - u_1^{(2)}\right)/\ell^{(2)}$$

$$= (30\times 10^6)(0.24176\times 10^{-5})/26.9258 = +2.6936\ \text{psi}$$

12.20 Force vector:

$$\vec{F}^T = \{F_4, F_5, F_6, F_8, F_9, F_{10}, F_{10}\} = \{0, 0, 0, 0, 0, 5000, 500\}$$

Equilibrium equations: $[K]\vec{W} = \vec{F}$ ---(E_1)

where $[K]$ is given in the solution of Problem 12.27. The solution of Eqs. (E_1) is given by

$W_4 = 0.2066 \times 10^{-6}$ m, $W_5 = 0.3732 \times 10^{-4}$ m, $W_6 = 0.3731 \times 10^{-4}$ rad,
$W_8 = 0.5224 \times 10^{-4}$ m, $W_9 = 0.3731 \times 10^{-4}$ rad, $W_{10} = 0.1954 \times 10^{-3}$ m,
$W_{12} = 0.1046 \times 10^{-3}$ rad.

Bending stress in element "e":

$$\sigma_{max} = \left(E_e I_e \left.\frac{d^2 w}{dx^2}\right|_e\right) \cdot y_{max,e} \Big/ I_e = E_e\, y_{max,e} \left.\frac{d^2 w}{dx^2}\right|_e$$

$$= E_e\, y_{max,e} \sum_{i=1}^{4} W_i^{(e)} \frac{d^2 N_i^{(e)}}{dx^2}$$

$$= E_e\, y_{max,e} \Big[W_1^{(e)}\left(-\frac{6}{l^2} + \frac{12}{l^3}x\right) + W_2^{(e)}\left(-\frac{4}{l} + \frac{6}{l^2}x\right)$$

$$+ W_3^{(e)}\left(\frac{6}{l^2} - \frac{12}{l^3}x\right) + W_4^{(e)}\left(-\frac{2}{l} + \frac{6}{l^2}x\right)\Big] \text{---}(E_2)$$

Bending stress in element 1 ($e = 1$):

$W_1^{(1)} = W_2 = 0$, $W_2^{(1)} = W_3 = 0$, $W_3^{(1)} = W_5 = 0.3732 \times 10^{-4}$ m,
$W_4^{(1)} = W_6 = 0.3731 \times 10^{-4}$ rad

$l_1 = 2$ m, $y_{max,1} = 0.415$ m, $E_1 = 2.1 \times 10^{11}$ Pa

At $x = 0$ (at node 1):

$$\sigma_{max} = (2.1 \times 10^{11})(0.415)\left[0.3732 \times 10^{-4}\left(\frac{6}{4}\right) + 0.3731 \times 10^{-4}\left(-\frac{2}{2}\right)\right]$$

$$= 1.6262 \times 10^6 \text{ Pa}$$

At $x = l_1 = 2$ m (at node 2):

$$\sigma_{max} = (2.1 \times 10^{11})(0.415)\left[0.3732 \times 10^{-4}\left(\frac{6}{4} - \frac{12}{4}\right) + 0.3731 \times 10^{-4}\left(-\frac{2}{2} + \frac{6}{2}\right)\right]$$

$$= 1.6245 \times 10^6 \text{ Pa}$$

Axial stress in element 1 (at node 1):

$$\sigma = \left(\frac{W_4 - W_1}{l_1}\right)E_1 = \epsilon_1 E_1 = \left[\left(0.2066 \times 10^{-6} - 0\right)/2\right](2.1 \times 10^{11})$$

$$= 0.0217 \times 10^6 \text{ Pa}$$

Total stress in element 1 $= (1.6262 + 0.0217)10^6 = 1.6479 \times 10^6$ Pa

Bending stress in element 2 $(e=2)$:

$W_1^{(2)} = W_5 = 0.3732 \times 10^{-4}$ m, $\quad W_2^{(2)} = W_6 = 0.3731 \times 10^{-4}$ rad,

$W_3^{(2)} = W_8 = 0.5224 \times 10^{-4}$ m, $\quad W_4^{(2)} = W_9 = 0.3731 \times 10^{-4}$ rad,

$l_2 = 0.4$ m, $\quad y_{max,2} = 0.415$ m, $\quad E_2 = 2.1 \times 10^{11}$ Pa

At $x = 0$:

$$\sigma_{max} = (2.1 \times 10^{11})(0.415)\left[0.3732\left(-\frac{6}{0.16}\right) + 0.3731\left(-\frac{4}{0.4}\right) + 0.5224\left(\frac{6}{0.16}\right) - 0.3731\left(-\frac{2}{0.4}\right)\right]10^{-4} = 32.503 \times 10^6 \text{ Pa}$$

Bending stress in element 3 $(e=3)$:

$W_1^{(3)} = W_4 = 0.2066 \times 10^{-6}$ m, $\quad W_2^{(3)} = W_6 = 0.3731 \times 10^{-4}$ rad,

$W_3^{(3)} = W_{10} = 0.1954 \times 10^{-3}$ m, $\quad W_4^{(3)} = W_{12} = 0.1046 \times 10^{-3}$ rad

$l_3 = 2.4$ m, $\quad y_{max,3} = 0.275$ m, $\quad E_3 = 2.1 \times 10^{11}$ Pa

At $x = 0$ (node 2):

$$\sigma_{max} = (2.1 \times 10^{11})(0.275)\left[0.2066 \times 10^{-6}\left(-\frac{6}{5.76}\right) + 0.3731 \times 10^{-4}\left(-\frac{4}{2.4}\right) + 0.1954 \times 10^{-3}\left(\frac{6}{5.76}\right) + 0.1046 \times 10^{-3}\left(-\frac{2}{2.4}\right)\right]$$

$$= 3.1172 \times 10^6 \text{ Pa}$$

12.21

Fig. 1

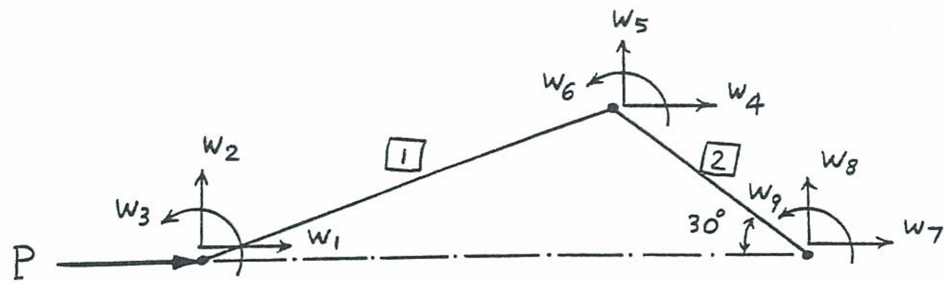
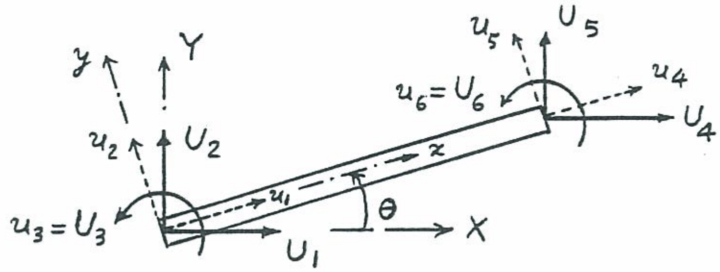


Fig. 2
General beam element in a plane



For a general beam element in XY plane, we consider two axial nodal displacements u_1 and u_4, and four bending nodal displacements u_2, u_3, u_5 and u_6. By superposing the stiffness matrices of a bar element and a beam element, the stiffness matrix of the element shown in Fig. 2 can be found as

$$[k] = \frac{EI}{l^3} \begin{array}{cc} \left[\begin{array}{cccccc} \frac{Al^2}{I} & & & & & \\ 0 & 12 & & & & \\ 0 & 6l & 4l^2 & & & \\ -\frac{Al^2}{I} & 0 & 0 & \frac{Al^2}{I} & & \\ 0 & -12 & -6l & 0 & 12 & \\ 0 & 6l & 2l^2 & 0 & -6l & 4l^2 \end{array} \right] & \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \\ \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} & \end{array} \qquad (E_1)$$

If U_i ; $i = 1, \cdots, 6$ denote the global nodal displacements, we find from Fig. 2,

$$\left.\begin{aligned} u_1 &= U_1 \cos\theta + U_2 \sin\theta \\ u_2 &= -U_1 \sin\theta + U_2 \cos\theta \\ u_3 &= U_3 \\ u_4 &= U_4 \cos\theta + U_5 \sin\theta \\ u_5 &= -U_4 \sin\theta + U_5 \cos\theta \\ u_6 &= U_6 \end{aligned}\right\} \qquad (E_2)$$

Defining $\lambda = \cos\theta$ and $\mu = \sin\theta$, Eq. (E_2) can be expressed as

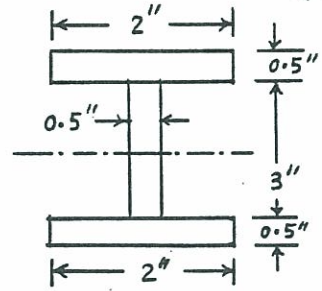
$$\vec{u} = [\underset{\sim}{\lambda}]\,\vec{U}$$

where

$$\vec{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}, \quad \vec{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} \quad \text{and} \quad [\underset{\sim}{\lambda}] = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (E_3)$$

Global stiffness matrix of the general beam element can be found as

$$[K^{(e)}] = [\underset{\sim}{\lambda}]^T [k] [\underset{\sim}{\lambda}] \qquad (E_4)$$



For element $\boxed{1}$:

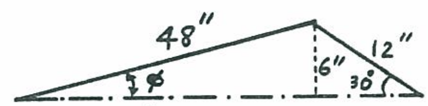
$E = 30 \times 10^6$ psi, $l = 48''$

$I = 2\left[\frac{1}{12}(2)(0.5)^3 + (2\times 0.5)(1.5+0.25)^2\right] + \frac{1}{12}(0.5)(3)^3 = 7.2917$ in^4

$A = 2(2\times 0.5) + 0.5\times 3 = 3.5$ in^2

$\lambda = \cos 7.1808^\circ = 0.9922$

$\mu = \sin 7.1808^\circ = 0.1250$



$\sin\phi = \frac{6}{48} = 0.125\,; \quad \phi = 7.1808^\circ$

For element $\boxed{2}$:

$E = 30 \times 10^6$ psi, $I = 7.2917$ in^4,

$A = 3.5$ in^2, $l = 12''$, $\lambda = \cos 330^\circ = 0.866$, $\mu = \sin 330^\circ = -0.5$

Boundary conditions:

$$W_2 = W_7 = W_8 = 0$$

Load on piston $= \frac{\pi}{4}(12)^2(200) = 22\,619.52$ lb

Load vector:

$$\vec{F} = \begin{Bmatrix} F_1 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_9 \end{Bmatrix} = \begin{Bmatrix} 22619.52 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Element matrices in global system (given by Eq. (E_4)):

$$[K^{(1)}] = \begin{array}{c} \begin{array}{cccccc} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{array} \\ \left[\begin{array}{cccccc} .2154E7 & & & & & \\ .2684E6 & .5755E5 & & \text{Symmetric} & & \\ -.7121E5 & .5652E6 & .1823E8 & & & \\ -.2154E7 & -.2684E6 & .7121E5 & .2154E7 & & \\ -.2684E6 & -.5755E5 & -.5652E6 & .2684E6 & .5755E5 & \\ -.7121E5 & .5652E6 & .9115E7 & .7121E5 & -.5652E6 & .1823E8 \end{array}\right] \end{array} \begin{array}{c} \\ w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{array}$$

$$[K^{(2)}] = \begin{array}{c} \begin{array}{cccccc} w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{array} \\ \left[\begin{array}{cccccc} .6942E7 & & & & & \\ -.3131E7 & .3327E7 & & \text{Symmetric} & & \\ .4557E7 & .7893E7 & .7292E8 & & & \\ -.6942E7 & .3131E7 & -.4557E7 & .6942E7 & & \\ .3131E7 & -.3327E7 & -.7893E7 & -.3131E7 & .3327E7 & \\ .4557E7 & .7893E7 & .3646E8 & -.4557E7 & -.7893E7 & .7292E8 \end{array}\right] \end{array} \begin{array}{c} \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \\ w_9 \end{array}$$

Assembled stiffness matrix (after incorporating boundary conditions):

$$[K] = 10^6 \begin{array}{c} \begin{array}{cccccc} w_1 & w_3 & w_4 & w_5 & w_6 & w_9 \end{array} \\ \left[\begin{array}{cccccc} 2.154 & & & & & \\ -.0712 & 18.23 & & \text{Symmetric} & & \\ -2.154 & .0712 & \begin{pmatrix} 2.154+ \\ 6.942 \end{pmatrix} & & & \\ -.2684 & -.5652 & \begin{pmatrix} .2684 \\ -3.131 \end{pmatrix} & \begin{pmatrix} .0575+ \\ 3.327 \end{pmatrix} & & \\ -.0712 & 9.115 & \begin{pmatrix} .0712+ \\ 4.557 \end{pmatrix} & \begin{pmatrix} -.5652+ \\ 7.893 \end{pmatrix} & \begin{pmatrix} 18.23+ \\ 72.92 \end{pmatrix} & \\ 0 & 0 & 4.557 & 7.893 & 36.46 & 72.92 \end{array}\right] \end{array} \begin{array}{c} \\ w_1 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_9 \end{array}$$

Solution of the equilibrium equations $[K]\vec{W} = \vec{F}$ gives the following:

$w_1 = 0.0866$ in

$w_3 = 0.007192$ rad

$w_4 = 0.06306$ in

$w_5 = 0.1047$ in

$w_6 = -0.007704$ rad

$w_9 = -0.01143$ rad

Element displacements in local coordinate system:

Element 1: $\lambda = 0.9922, \quad \mu = 0.1250$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}^{(1)} = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{(1)} \begin{Bmatrix} W_1 = 0.0866 \\ W_2 = 0 \\ W_3 = 0.00719 \\ W_4 = 0.06306 \\ W_5 = 0.1047 \\ W_6 = -0.0077 \end{Bmatrix} = \begin{Bmatrix} 0.08592 \text{ in} \\ -0.01083 \text{ in} \\ 0.00719 \text{ rad} \\ 0.07566 \text{ in} \\ 0.09599 \text{ in} \\ -0.0077 \text{ rad} \end{Bmatrix}$$

Element 2: $\lambda = 0.866, \quad \mu = -0.5$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}^{(2)} = \begin{bmatrix} \lambda & \mu & 0 & 0 & 0 & 0 \\ -\mu & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & \mu & 0 \\ 0 & 0 & 0 & -\mu & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{(2)} \begin{Bmatrix} W_4 = 0.06306 \\ W_5 = 0.1047 \\ W_6 = -0.0077 \\ W_7 = 0 \\ W_8 = 0 \\ W_9 = -0.01143 \end{Bmatrix} = \begin{Bmatrix} 0.00226 \text{ in} \\ 0.1222 \text{ in} \\ -.0077 \text{ rad} \\ 0 \\ 0 \\ -.01143 \text{ rad} \end{Bmatrix}$$

Axial stresses:

Element 1:

$$\sigma(x) = E\left(-\frac{1}{\ell} \quad \frac{1}{\ell}\right)\begin{Bmatrix} u_1 \\ u_4 \end{Bmatrix}^{(1)} = \frac{E}{\ell}(-u_1 + u_4)^{(1)} = \frac{30 \times 10^6}{48}(-.08592 + .07566)$$

$$= -6411 \text{ psi}$$

Element 2:

$$\sigma(x) = \frac{E}{\ell}(-u_1 + u_4)^{(2)} = \frac{30 \times 10^6}{12}(-.00226 + 0) = -5649 \text{ psi}$$

Bending stresses:

Element 1: $y_{max} = 2''$, $\ell = 48''$

$$\sigma(x) = E\,\frac{d^2 w(x)}{dx^2}\,y_{max} = E\,y_{max}\left\{\left(-\frac{6}{\ell^2} + \frac{12x}{\ell^3}\right)u_2 + \left(-\frac{4}{\ell^2} + \frac{6x}{\ell^3}\right)\ell u_3 + \left(\frac{6}{\ell^2} - \frac{12x}{\ell^3}\right)u_5 + \left(-\frac{2}{\ell^2} + \frac{6x}{\ell^3}\right)\ell u_6\right\}$$

at $x = 0$,

$$\sigma(0) = -12 \text{ psi}$$

at $x = 48$, $\sigma(48) = -37218$ psi

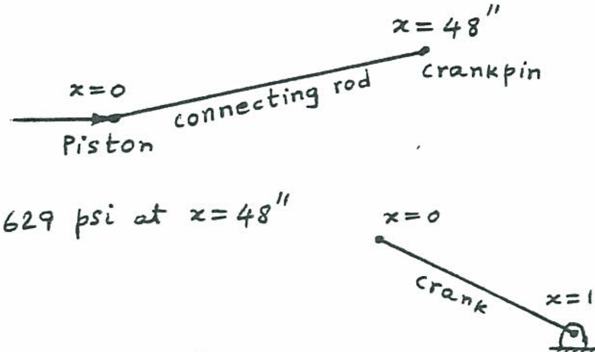
Element 2: $\quad y_{max} = 2''$, $l = 12''$

$$\sigma(x) = E\, y_{max} \left\{ \left(-\frac{6}{l^2} + \frac{12x}{l^3}\right) u_2 + \left(-\frac{4}{l^2} + \frac{6x}{l^3}\right) l\, u_3 + \left(\frac{6}{l^2} - \frac{12x}{l^3}\right) u_5 + \left(-\frac{2}{l^2} + \frac{6x}{l^3}\right) l\, u_6 \right\}$$

at $x = 0$, $\quad \sigma(0) = -37\,206$ psi

at $x = 12$, $\quad \sigma(12) = -114$ psi

Maximum stresses:



In element 1,

$\sigma_{max} = \max$ (axial $\pm$ bending)

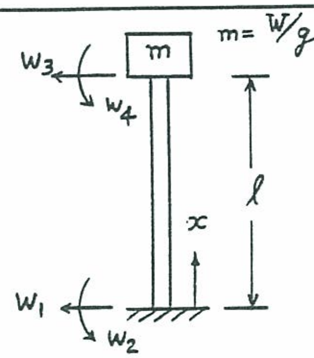
$= -6411 \pm 37\,218 = -43\,629$ psi at $x = 48''$

In element 2,

$\sigma_{max} = \max$ (axial $\pm$ bending)

$= -5649 \pm 37206 = -42\,855$ psi at $x = 0$.

12.22 $\quad l = 480''$, $E = 30 \times 10^6$ psi



$$I = \frac{\pi}{64}\left\{(d+2t)^4 - d^4\right\} = \frac{\pi}{64}\left\{(24+2)^4 - 24^4\right\}$$

$$= 6145 \cdot 755 \text{ in}^4$$

$\rho = 0.283/386.4 = 7.324 \times 10^{-4}$ lb-s²/in⁴

$$A = \frac{\pi}{4}\left\{(d+2t)^2 - d^2\right\} = \frac{\pi}{4}\left(26^2 - 24^2\right)$$

$$= 78.54 \text{ in}^2$$

$m = 10\,000/386.4 = 25.8799$ lb − s²/in , $\quad p_{max} = 100$ psi

pressure at $x = \left(\frac{x}{l}\, p_{max}\right)$ psi

load at $x = \frac{x}{l}\, p_{max}\, (d+2t)$ lb/in $= 5.4167\, x$ lb/in

Stiffness matrix:

$$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array}$$

where $\quad \dfrac{EI}{l^3} = \dfrac{(30 \times 10^6)(6145.755)}{(480)^3} = 1667.1427$ lb/in

stiffness matrix, after incorporating the conditions $W_1 = W_2 = 0$, is

$$[K] = \frac{EI}{\ell^3}\begin{bmatrix} 12 & -6\ell \\ -6\ell & 4\ell^2 \end{bmatrix} = 1667.1427 \begin{array}{cc} W_3 & W_4 \\ \left[\begin{array}{cc} 12 & -2880 \\ -2880 & 921600 \end{array}\right] & \begin{array}{c} W_3 \\ W_4 \end{array} \end{array}$$

Mass matrix:

$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A \ell}{420} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \\ \left[\begin{array}{cccc} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{array}\right] & \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array} \end{array}$$

where $\frac{\rho A \ell}{420} = \frac{(7.324 \times 10^{-4})(78.54)(480)}{420} = 657.4022 \times 10^{-4}$ lb-s²/in

Mass matrix, after incorporating $W_1 = W_2 = 0$,

$$[M] = \frac{\rho A \ell}{420}\begin{bmatrix} 156 & -22\ell \\ -22\ell & 4\ell^2 \end{bmatrix} = 657.4022 \times 10^{-4} \begin{array}{cc} W_3 & W_4 \\ \left[\begin{array}{cc} 156 & -10560 \\ -10560 & 921600 \end{array}\right] & \begin{array}{c} W_3 \\ W_4 \end{array} \end{array}$$

Consistent load vector:

$f_1 = \int_0^\ell f(x)\, N_1(x)\, dx$ where $f(x)$ = distributed transverse load

$$= \int_0^\ell (5.4167\, x)\left\{1 - 3\frac{x^2}{\ell^2} + 2\frac{x^3}{\ell^3}\right\} dx = 5.4167\left(\frac{3}{20}\ell^2\right)$$

$= 187\,201.152$ lb

$$f_3 = \int_0^\ell f(x)\, N_3(x)\, dx = \int_0^\ell (5.4167\, x)\left\{3\left(\frac{x}{\ell}\right)^2 - 2\left(\frac{x}{\ell}\right)^3\right\} dx$$

$$= 5.4167\left(\frac{7\ell^2}{20}\right) = 436\,802.688 \text{ lb}$$

As there is no distributed bending moment, $f_2 = f_4 = 0$.

STATIC ANALYSIS

Equilibrium equations are $[K]\,\vec{W} = \vec{f}$

i.e.,

$$1667.1427\begin{bmatrix} 12 & -2880 \\ -2880 & 921600 \end{bmatrix}\begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} 436802.688 \\ 0 \end{Bmatrix}$$

solution is

$W_3 = 87.3356$ in

$W_4 = 0.2729$ rad

Since $w(x) = N_1(x)\, W_1 + N_2(x)\, W_2 + N_3(x)\, W_3 + N_4(x)\, W_4$,

stress $= \sigma = \dfrac{M \cdot c}{I} = \left(EI\, \dfrac{d^2 w}{dx^2}\right) \dfrac{c}{I}$

with

$$\frac{d^2 w}{dx^2} = \sum_{i=1}^{4} \frac{d^2 N_i(x)}{dx^2}\, W_i .$$

Since $W_1 = W_2 = 0$, $N_3 = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$ and $N_4 = -\frac{x^2}{l} + \frac{x^3}{l^2}$,

and stress will be maximum at $x = 0$,

$$\sigma_{max}\Big|_{x=0} = Ec\, \frac{d^2 w}{dx^2}\Big|_{x=0} = (30 \times 10^6)\left(\frac{d+2t}{2}\right) \frac{d^2 w}{dx^2}\Big|_{x=0}$$

$$= (30 \times 10^6)(13)(1.1373 \times 10^{-3}) = 443547 \text{ psi}$$

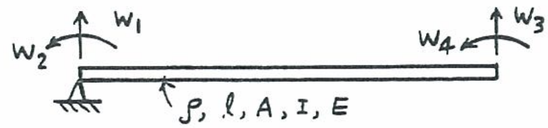
Direct compressive stress due to load W:

$$\sigma_c = \frac{W}{A} = 10000/78.54 = 127.3237 \text{ psi}$$

$\therefore$ Max. compressive stress $= \sigma_c + \sigma_{max} = 443674.3237$ psi

Max. tensile stress $= -\sigma_c + \sigma_{max} = 443419.6763$ psi

12.24

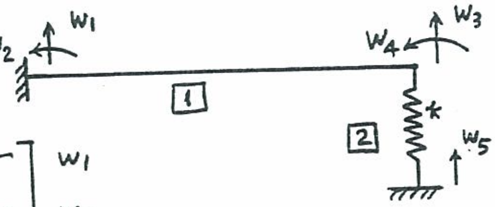


$$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \qquad (E_1)$$

$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \qquad (E_2)$$

Applying the boundary condition $W_1 = 0$, the eigenvalue problem can be expressed as
$$-\omega^2 [M]\vec{W} + [K]\vec{W} = \vec{0}$$
i.e.,
$$\left[-\omega^2 \frac{\rho A l}{420}\begin{bmatrix} 4l^2 & 13l & -3l^2 \\ 13l & 156 & -22l \\ -3l^2 & -22l & 4l^2 \end{bmatrix} + \frac{EI}{l^3}\begin{bmatrix} 4l^2 & -6l & 2l^2 \\ -6l & 12 & -6l \\ 2l^2 & -6l & 4l^2 \end{bmatrix}\right]\begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$
i.e.,
$$\begin{bmatrix} (4l^2 - \lambda l^2) & (-6l - 13l\lambda) & (2l^2 + 3l^2\lambda) \\ (-6l - 13l\lambda) & (12 - 156\lambda) & (-6l + 22l\lambda) \\ (2l^2 + 3l^2\lambda) & (-6l + 22l\lambda) & (4l^2 - 4l^2\lambda) \end{bmatrix}\begin{Bmatrix} W_2 \\ W_3 \\ W_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (E_3)$$
where $\lambda = \dfrac{\omega^2 \rho A l^4}{420\, EI}$.

Eq. (E_3) gives the frequency equation
$$\begin{vmatrix} 4l^2(1-\lambda) & -l(6+13\lambda) & l^2(2+3\lambda) \\ -l(6+13\lambda) & 12(1-13\lambda) & l(-6+22\lambda) \\ l^2(2+3\lambda) & l(-6+22\lambda) & 4l^2(1-\lambda) \end{vmatrix} = 0$$
which reduces to
$$-\lambda\,(196\lambda^2 - 2436\lambda + 1680) = 0$$
Roots are:
$$\lambda_1 = 0; \quad \lambda_{2,3} = \frac{2436 \pm \sqrt{(2436)^2 - 4(196)(1680)}}{2(196)}$$
$$= 0.7329;\ 11.6957$$
Thus
$$\omega_1 = 0$$
$$\omega_2 = 17.5447\sqrt{\frac{EI}{\rho A l^4}}$$
$$\omega_3 = 70.0870\sqrt{\frac{EI}{\rho A l^4}}$$
These values can be compared with the exact values (see Fig. 8.15): $\omega_1 = 0$, $\omega_2 = 15.4182\sqrt{\dfrac{EI}{\rho A l^4}}$, $\omega_3 = 49.9649\sqrt{\dfrac{EI}{\rho A l^4}}$.



12.25 Element matrices:

$$[K^{(1)}] = \frac{EI}{\ell^3} \begin{array}{c} \begin{matrix} w_1 & \quad w_2 & \quad w_3 & \quad w_4 \end{matrix} \\ \begin{bmatrix} \cancel{12} & \cancel{6\ell} & \cancel{-12} & \cancel{6\ell} \\ \cancel{6\ell} & \cancel{4\ell^2} & \cancel{-6\ell} & \cancel{2\ell^2} \\ \cancel{-12} & \cancel{-6\ell} & 12 & -6\ell \\ \cancel{6\ell} & \cancel{2\ell^2} & -6\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

$$[K^{(2)}] = k \begin{array}{c} \begin{matrix} w_3 & \quad w_5 \end{matrix} \\ \begin{bmatrix} 1 & \cancel{-1} \\ \cancel{-1} & \cancel{1} \end{bmatrix} \end{array} \begin{matrix} w_3 \\ w_5 \end{matrix}$$

$$[M^{(1)}] = \frac{\rho A \ell}{420} \begin{array}{c} \begin{matrix} w_1 & \quad w_2 & \quad w_3 & \quad w_4 \end{matrix} \\ \begin{bmatrix} \cancel{156} & \cancel{22\ell} & \cancel{54} & \cancel{-13\ell} \\ \cancel{22\ell} & \cancel{4\ell^2} & \cancel{13\ell} & \cancel{-3\ell^2} \\ \cancel{54} & \cancel{13\ell} & 156 & -22\ell \\ \cancel{-13\ell} & \cancel{-3\ell^2} & -22\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{matrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{matrix}$$

$$[M^{(2)}] = \begin{array}{c} \begin{matrix} w_3 & \quad w_5 \end{matrix} \\ \begin{bmatrix} 0 & \cancel{0} \\ \cancel{0} & \cancel{0} \end{bmatrix} \end{array} \begin{matrix} w_3 \\ w_5 \end{matrix}$$

(mass of spring is assumed to be zero)

Assembled matrices:

$$[K] = \frac{EI}{\ell^3} \begin{array}{c} \begin{matrix} w_3 & \qquad w_4 \end{matrix} \\ \begin{bmatrix} \left(12 + \frac{k\ell^3}{EI}\right) & -6\ell \\ -6\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

$$[M] = \frac{\rho A \ell}{420} \begin{array}{c} \begin{matrix} w_3 & \qquad w_4 \end{matrix} \\ \begin{bmatrix} 156 & -22\ell \\ -22\ell & 4\ell^2 \end{bmatrix} \end{array} \begin{matrix} w_3 \\ w_4 \end{matrix}$$

Eigenvalue equation:

$$\left| -\frac{\rho A \ell \omega^2}{420} \begin{bmatrix} 156 & -22\ell \\ -22\ell & 4\ell^2 \end{bmatrix} + \frac{EI}{\ell^3} \begin{bmatrix} \left(12 + \frac{k\ell^3}{EI}\right) & -6\ell \\ -6\ell & 4\ell^2 \end{bmatrix} \right| = 0$$

i.e.,
$$\begin{vmatrix} 12 + \frac{k\ell^3}{EI} - 156\lambda & -6\ell + 22\ell\lambda \\ -6\ell + 22\ell\lambda & 4\ell^2 - 4\ell^2\lambda \end{vmatrix} = 0$$

where $\lambda = \left(\dfrac{\rho A l^4 \omega^2}{420 EI}\right)$

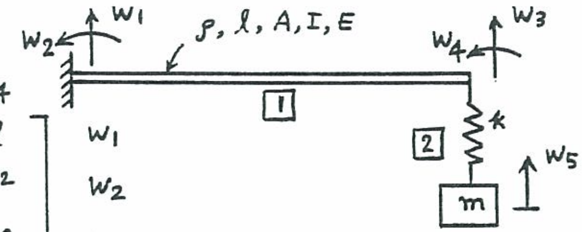
Upon expansion, the frequency equation reduces to

$$35\lambda^2 - \left(102 + \frac{kl^3}{EI}\right)\lambda + \left(3 + \frac{kl^3}{EI}\right) = 0$$

which implies that

$$\lambda_{1,2} = \left\{\frac{\left(102 + \frac{kl^3}{EI}\right) \pm \left(9984 + 64\frac{kl^3}{EI} + \frac{k^2 l^6}{E^2 I^2}\right)^{\frac{1}{2}}}{70}\right\}$$

12.26 Element matrices:



$$[K^{(1)}] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{array}$$

$$[M^{(1)}] = \frac{\rho A l}{420} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} \\ W_1 \\ W_2 \\ W_3 \\ W_4 \end{array}$$

$$[K^{(2)}] = k \begin{array}{c} \begin{array}{cc} W_3 & W_5 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ W_3 \\ W_5 \end{array} , \quad [M^{(2)}] = \begin{array}{c} \begin{array}{cc} W_3 & W_5 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 0 & m \end{bmatrix} \end{array} \begin{array}{c} \\ W_3 \\ W_5 \end{array}$$

Assembled matrices (after applying boundary conditions):

$$[K] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{ccc} W_3 & W_4 & W_5 \end{array} \\ \begin{bmatrix} 12 + \frac{kl^3}{EI} & -6l & -\frac{kl^3}{EI} \\ -6l & 4l^2 & 0 \\ -\frac{kl^3}{EI} & 0 & \frac{kl^3}{EI} \end{bmatrix} \end{array} \begin{array}{c} \\ W_3 \\ W_4 \\ W_5 \end{array}$$

$$= \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$[M] = \frac{\rho A l}{420} \begin{array}{cc} \begin{array}{ccc} w_3 & w_4 & w_5 \end{array} & \\ \begin{bmatrix} 156+0 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & m\left(\frac{420}{\rho A l}\right) \end{bmatrix} & \begin{array}{c} w_3 \\ w_4 \\ w_5 \end{array} \end{array}$$

$$= \frac{\rho A l}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 420 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 [M] + [K] \right| = 0$$

i.e.,

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 156 & -22l & 0 \\ -22l & 4l^2 & 0 \\ 0 & 0 & 420 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 13 & -6l & -1 \\ -6l & 4l^2 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 13-156\lambda & -6l+22l\lambda & -1 \\ -6l+22l\lambda & 4l^2-4l^2\lambda & 0 \\ -1 & 0 & 1-420\lambda \end{vmatrix} = 0 \qquad (E_1)$$

where $\lambda = \left(\frac{\rho A l^4 \omega^2}{420\, EI} \right)$

Eq. (E_1) can be simplified to obtain

$$-58800\,\lambda^3 + 173180\,\lambda^2 - 7128\,\lambda + 12 = 0 \qquad (E_2)$$

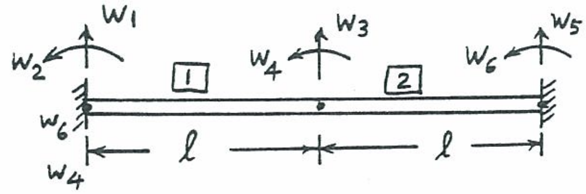
Roots of Eq. (E_2) given by:

$\lambda_1 = 0.00175555 \Rightarrow \omega_1 = 0.8587 \sqrt{EI/(\rho A l^4)}$

$\lambda_2 = 0.039951 \Rightarrow \omega_2 = 4.0965 \sqrt{EI/(\rho A l^4)}$

$\lambda_3 = 2.90351 \Rightarrow \omega_3 = 34.9210 \sqrt{EI/(\rho A l^4)}$

12.27 Element matrices:



$$\begin{array}{lcccc} \text{for } e=2 \text{ --- } & W_3 & W_4 & W_5 & W_6 \\ \text{for } e=1 \text{ --- } & W_1 & W_2 & W_3 & W_4 \end{array}$$

$$[K^{(e)}] = \frac{EI}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \begin{array}{cc} W_1 & W_3 \\ W_2 & W_4 \\ W_3 & W_5 \\ W_4 & W_6 \\ \text{for } e=1 & e=2 \end{array}$$

$$\begin{array}{ccccl} W_3 & W_4 & W_5 & W_6 & \text{----- for } e=2 \\ W_1 & W_2 & W_3 & W_4 & \text{-- for } e=1 \end{array}$$

$$[M^{(e)}] = \frac{\rho A \ell}{420} \begin{bmatrix} 156 & 22\ell & 54 & -13\ell \\ 22\ell & 4\ell^2 & 13\ell & -3\ell^2 \\ 54 & 13\ell & 156 & -22\ell \\ -13\ell & -3\ell^2 & -22\ell & 4\ell^2 \end{bmatrix} \begin{array}{cc} W_1 & W_3 \\ W_2 & W_4 \\ W_3 & W_5 \\ W_4 & W_6 \end{array}$$

Assembled matrices, after incorporating boundary conditions $W_1 = W_2 = W_5 = W_6 = 0$:

$$[K] = \frac{EI}{\ell^3} \begin{bmatrix} 12+12 & -6\ell+6\ell \\ -6\ell+6\ell & 4\ell^2+4\ell^2 \end{bmatrix} = \frac{EI}{\ell^3} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} 24 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \end{array}$$

$$[M] = \frac{\rho A \ell}{420} \begin{bmatrix} 156+156 & -22\ell+22\ell \\ -22\ell+22\ell & 4\ell^2+4\ell^2 \end{bmatrix} = \frac{\rho A \ell}{420} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} 312 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \end{array}$$

Eigenvalue problem:

$$\left[-\omega^2 [M] + [K] \right] \vec{W} = \vec{0}$$

For natural frequencies,

$$\left| -\omega^2 [M] + [K] \right| = 0$$

i.e.,

$$\left| -\frac{\omega^2 \rho A \ell}{420} \begin{bmatrix} 312 & 0 \\ 0 & 8\ell^2 \end{bmatrix} + \frac{EI}{\ell^3} \begin{bmatrix} 24 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} (24 - 312\lambda) & 0 \\ 0 & 8\ell^2(1-\lambda) \end{vmatrix} = 0 \quad \text{where } \lambda = \left(\frac{\rho A \ell^4 \omega^2}{420\, EI} \right).$$

i.e., $\quad 192\, \ell^2 (1-\lambda)(1-13\lambda) = 0$

$\therefore \quad \lambda_1 = \frac{1}{13}, \quad \omega_1 = 22.736 \sqrt{\frac{EI}{\rho A L^4}} \quad$ with $L = 2\ell$

$\lambda_2 = 1, \quad \omega_2 = 81.9756 \sqrt{\frac{EI}{\rho A L^4}}$

For mode shapes,

$$\left[-\omega_i^2 [M] + [K] \right] \vec{W}^{(i)} = \vec{0} \quad ; \quad i = 1, 2$$

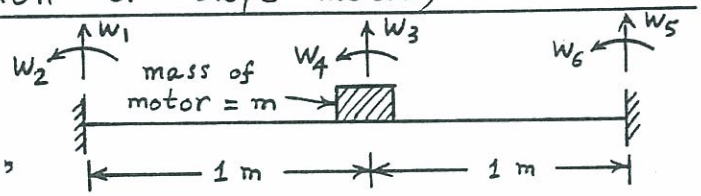
i.e.,

$$\begin{bmatrix} (24 - 312\,\lambda_i) & 0 \\ 0 & 8\ell^2 (1-\lambda_i) \end{bmatrix} \begin{Bmatrix} W_3^{(i)} \\ W_4^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $\lambda_1 = \frac{1}{13}$, $\quad W_3^{(1)}$ can have any value
(transverse displacement mode)

For $\lambda_2 = 1$, $\quad W_4^{(2)}$ can have any value
(rotation or slope mode)

12.28 $\quad m = 100$ kg, $\quad \ell_1 = \ell_2 = 1$ m



From solution of problem 12.27, we have

$$[K] = \frac{EI}{\ell^3} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} 24 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \end{array} \qquad (E_1)$$

$$[M] = \frac{\rho A \ell}{420} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} 312 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \end{array} \qquad (E_2)$$

When the mass of motor is added to d.o.f. W_3, the mass matrix becomes

$$[M'] = \frac{\rho A \ell}{420} \begin{bmatrix} 312 + m\left(\frac{420}{\rho A \ell}\right) & 0 \\ 0 & 8\ell^2 \end{bmatrix} \qquad (E_3)$$

Eqs. (E_1) and (E_3) yield the frequency equation

$$\left| -\omega^2 [M'] + [K] \right| = 0$$

i.e.,

$$\left| -\frac{\rho A \ell \omega^2}{420} \begin{bmatrix} \left(312+\frac{42000}{\rho A \ell}\right) & 0 \\ 0 & 8\ell^2 \end{bmatrix} + \frac{EI}{\ell^3} \begin{bmatrix} 24 & 0 \\ 0 & 8\ell^2 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 24-\left(312+\frac{42000}{\rho A \ell}\right)\lambda & 0 \\ 0 & 8\ell^2(1-\lambda) \end{vmatrix} = 0$$

i.e.,

$$192\,\ell^2 (1-\lambda)\left\{1-\left(13+\frac{1750}{\rho A \ell}\right)\lambda\right\} = 0$$

where $\lambda = \left(\frac{\rho A \ell^4 \omega^2}{420\, E I}\right)$.

$$\therefore\ \lambda_1 = 1 \quad \Rightarrow \quad \omega_1^2 = \frac{420\, EI\, \lambda_1}{\rho A \ell^4} = \frac{420\, EI}{\rho A \ell^4}$$

$$\lambda_2 = \frac{1}{\left(13+\frac{1750}{\rho A \ell}\right)} \quad \Rightarrow \quad \omega_2^2 = \frac{420\, EI}{\rho A \ell^4 \left(13+\frac{1750}{\rho A \ell}\right)}$$

For the steel beam,

$E = 2.1 \times 10^{11}$ Pa, $\ell = 1$ m, $\rho = 7.8 \times 10^3$ kg/m^3

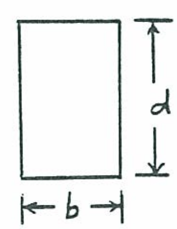
$$\omega_1^2 = \frac{420(2.1\times 10^{11})\, I}{(7.8\times 10^3)\, A\, (1)^4} \quad \Rightarrow \quad \omega_1 = 10.6338 \times 10^4 \sqrt{\frac{I}{A}} \ \text{rad/sec} \qquad (E_4)$$

$$\omega_2^2 = \frac{420\,(2.1\times 10^{11})\, I}{13(7.8\times 10^3)\, A\, (1)^4 + 1750\,(1)^3} = \frac{88.2\times 10^{12}\, I}{101.4\times 10^3\, A + 1750}$$

$$\Rightarrow \quad \omega_2 = \frac{9.3915\times 10^6 \sqrt{I}}{(101400\, A + 1750)^{1/2}} \ \text{rad/sec} \qquad (E_5)$$

Let depth = twice width for cross-section.

Then $A = 2b^2$ and $I = \frac{1}{12}(b)(2b)^3 = 0.6667\, b^4$



Operating speed of motor $= \Omega = \frac{1800\,(2\pi)}{60}$

$= 188.496$ rad/sec

Assume ω_1 = smaller than ω_2.

For $\omega_1 > 188.496$, we need to have

$$10^4 (10.6338) \sqrt{\frac{0.6667\ b^4}{2b^2}} > 188.496$$

i.e., $b > 3.07 \times 10^{-3}$ m

Let $b = 5$ mm and $d = 10$ mm so that

$A = 50 \times 10^{-6}$ m^2 and $I = 416.6667 \times 10^{-12}$ m^4

This gives

$$\omega_2 = \frac{9.3915 \times 10^6 \left(416.6667 \times 10^{-12}\right)^{1/2}}{\left(101400 \times 50 \times 10^{-6} + 1750\right)^{1/2}} = 4.5760 \ \frac{\text{rad}}{\text{sec}}$$

This violates the original assumption of $\omega_1 > \omega_2$.

Assume ω_2 = smaller than ω_1.

For $\omega_2 > 188.496$, we need to have

$$\frac{9.3915 \times 10^6 \left(0.6667\ b^4\right)^{1/2}}{\left\{101400\,(2b^2) + 1750\right\}^{1/2}} > 188.496$$

i.e.,

$$1654.8624\ b^4 - 0.2028\ b^2 - 0.00175 > 0 \qquad (E_6)$$

By setting the inequality in (E_6) to equality and solving for b^2, we find

$b^2 = 0.00109145$ or $b = 0.03304$ m

Let $b = 35$ mm and $d = 70$ mm, so that the left side of inequality (E_6) becomes

$(0.002483 - 0.000248 - 0.001750)$ which is positive.

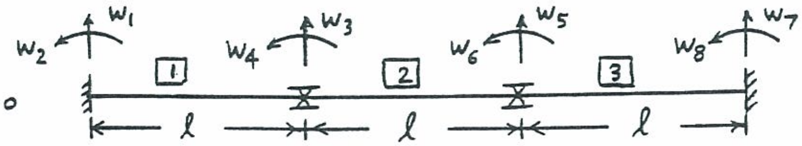
Hence the final design is given by

$b = 0.035$ m

$d = 0.070$ m

12.29 Boundary conditions:

$W_1 = W_2 = W_3 = W_5 = W_7 = W_8 = 0$



$$\begin{array}{cccccl} & W_5 & W_6 & W_7 & W_8 & \text{------ for } i=3 \\ & W_3 & W_4 & W_5 & W_6 & \text{---- for } i=2 \\ & W_1 & W_2 & W_3 & W_4 & \text{-- for } i=1 \end{array}$$

$$[K^{(i)}] = \frac{EI}{l^3} \left[\begin{array}{cccc} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{array} \right] \begin{array}{ccc} W_1 & W_3 & W_5 \\ W_2 & W_4 & W_6 \\ W_3 & W_5 & W_7 \\ W_4 & W_6 & W_8 \end{array}$$

$$\begin{array}{cccccl} & W_5 & W_6 & W_7 & W_8 & \text{--------- for } i=3 \\ & W_3 & W_4 & W_5 & W_6 & \text{---- for } i=2 \\ & W_1 & W_2 & W_3 & W_4 & \text{-- for } i=1 \end{array}$$

$$[M^{(i)}] = \frac{\rho A l}{420} \left[\begin{array}{cccc} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{array} \right] \begin{array}{ccc} W_1 & W_3 & W_5 \\ W_2 & W_4 & W_6 \\ W_3 & W_5 & W_7 \\ W_4 & W_6 & W_8 \end{array}$$

Assembled stiffness matrix, after applying boundary conditions:

$$\begin{array}{cc} W_4 & W_6 \end{array}$$

$$[K] = \frac{EI}{l^3} \left[\begin{array}{cc} (4l^2 + 4l^2) & 2l^2 \\ 2l^2 & (4l^2 + 4l^2) \end{array} \right] \begin{array}{c} W_4 \\ W_6 \end{array}$$

$$\begin{array}{cc} W_4 & W_6 \end{array}$$

$$[M] = \frac{\rho A l}{420} \left[\begin{array}{cc} (4l^2 + 4l^2) & -3l^2 \\ -3l^2 & (4l^2 + 4l^2) \end{array} \right] \begin{array}{c} W_4 \\ W_6 \end{array}$$

Frequency equation:

$$\left| -\frac{\rho A l \omega^2}{420} \begin{bmatrix} 8l^2 & -3l^2 \\ -3l^2 & 8l^2 \end{bmatrix} + \frac{EI}{l^3} \begin{bmatrix} 8l^2 & 2l^2 \\ 2l^2 & 8l^2 \end{bmatrix} \right| = 0 \qquad (E_1)$$

Defining $\lambda = \left(\dfrac{\rho A l^4 \omega^2}{420 EI} \right)$, Eq. ($E_1$) can be expressed as

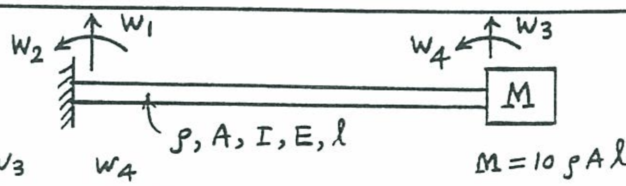
$$\begin{vmatrix} 8(1-\lambda) & (2+3\lambda) \\ (2+3\lambda) & 8(1-\lambda) \end{vmatrix} = 0$$

or $$11\lambda^2 - 28\lambda + 12 = 0$$

$$\therefore \quad \lambda_{1,2} = \frac{28 \pm \sqrt{784 - 4(11)(12)}}{2(11)} = \frac{6}{11},\ 2$$

i.e., $$\omega_1 = 15.1357\sqrt{\frac{EI}{\rho A l^4}} \quad \text{and} \quad \omega_2 = 28.9828\sqrt{\frac{EI}{\rho A l^4}}$$

12.30 Element matrices:



$$[K^{(1)}] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array}$$

$$[M^{(1)}] = \frac{\rho A l}{420} \begin{array}{c} \begin{array}{cccc} W_1 & W_2 & W_3 & W_4 \end{array} \\ \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} W_1 \\ W_2 \\ W_3 \\ W_4 \end{array}$$

Assembled matrices (after applying boundary conditions $W_1 = W_2 = 0$ and adding mass M at d.o.f. W_3):

$$[K] = \frac{EI}{l^3} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \end{array}$$

$$[M] = \frac{\rho A l}{420} \begin{array}{c} \begin{array}{cc} W_3 & W_4 \end{array} \\ \begin{bmatrix} 156 + 4200 & -22l \\ -22l & 4l^2 \end{bmatrix} \end{array} \begin{array}{c} W_3 \\ W_4 \end{array}$$

Defining $\lambda = \left(\dfrac{\rho A l^4 \omega^2}{420\, EI}\right)$, the frequency equation can be written as

$$\begin{vmatrix} (12 - 4356\,\lambda) & (-6\ell + 22\,\ell\,\lambda) \\ (-6\ell + 22\ell\,\lambda) & (4\ell^2 - 4\ell^2\,\lambda) \end{vmatrix} = 0$$

or

$$4235\,\lambda^2 - 4302\,\lambda + 3 = 0$$

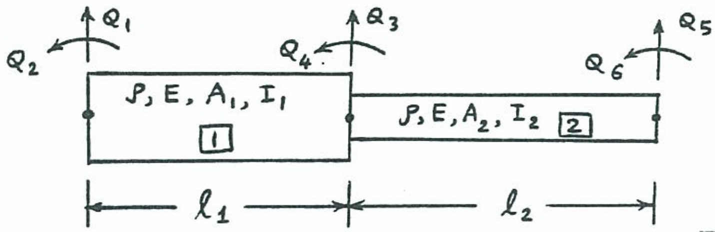
This gives

$$\lambda_{1,2} = \frac{4302 \pm \left(4302^2 - 4 \times 4235 \times 3\right)^{\frac{1}{2}}}{2(4235)}$$

$$= 0.0006978\,,\quad 1.01512272$$

$$\therefore\ \omega_1 = 0.5414\sqrt{\frac{EI}{\rho A \ell^4}}\,,\qquad \omega_2 = 20.6483\sqrt{\frac{EI}{\rho A \ell^4}}$$

12.31



$$[k^{(e)}] = [\bar{k}^{(e)}] = \frac{E^{(e)}\,I^{(e)}}{\ell^{(e)^3}} \begin{bmatrix} 12 & 6\ell^{(e)} & -12 & 6\ell^{(e)} \\ 6\ell^{(e)} & 4\ell^{(e)^2} & -6\ell^{(e)} & 2\ell^{(e)^2} \\ -12 & -6\ell^{(e)} & 12 & -6\ell^{(e)} \\ 6\ell^{(e)} & 2\ell^{(e)^2} & -6\ell^{(e)} & 4\ell^{(e)^2} \end{bmatrix};\ e=1,2$$

$$[m^{(e)}] = [\bar{m}^{(e)}] = \frac{\rho^{(e)}A^{(e)}\ell^{(e)}}{420} \begin{bmatrix} 156 & 22\ell^{(e)} & 54 & -13\ell^{(e)} \\ 22\ell^{(e)} & 4\ell^{(e)^2} & 13\ell^{(e)} & -3\ell^{(e)^2} \\ 54 & 13\ell^{(e)} & 156 & -22\ell^{(e)} \\ -13\ell^{(e)} & -3\ell^{(e)^2} & -22\ell^{(e)} & 4\ell^{(e)^2} \end{bmatrix};\ e=1,2$$

$$[A^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},\qquad [A^{(2)}] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{K}] = \sum_{e=1}^{2} [A^{(e)}]^T [\bar{k}^{(e)}][A^{(e)}]$$

$$
= E \begin{bmatrix}
12\, I_1/l_1^3 & 6 I_1/l_1^2 & -12\, I_1/l_1^3 & 6 I_1/l_1^2 & 0 & 0 \\
6 I_1/l_1^2 & 4 I_1/l_1 & -6 I_1/l_1^2 & 2 I_1/l_1 & 0 & 0 \\
-12 I_1/l_1^3 & -6 I_1/l_1^2 & \left(\frac{12 I_1}{l_1^3} + \frac{12 I_2}{l_2^3}\right) & \left(-\frac{6 I_1}{l_1^2} + \frac{6 I_2}{l_2^2}\right) & -12 I_2/l_2^3 & 6 I_2/l_2^2 \\
6 I_1/l_1^2 & 2 I_1/l_1 & \left(-\frac{6 I_1}{l_1^2} + \frac{6 I_2}{l_2^2}\right) & \left(\frac{4 I_1}{l_1} + \frac{4 I_2}{l_2}\right) & -6 I_2/l_2^2 & 2 I_2/l_2 \\
0 & 0 & -12 I_2/l_2^3 & -6 I_2/l_2^2 & 12 I_2/l_2^3 & -6 I_2/l_2^2 \\
0 & 0 & 6 I_2/l_2^2 & 2 I_2/l_2 & -6 I_2/l_2^2 & 4 I_2/l_2
\end{bmatrix}
$$

$$
[\underset{\sim}{M}] = \sum_{e=1}^{2} [A^{(e)}]^T [\bar{m}^{(e)}] [A^{(e)}]
$$

$$
= \frac{\rho}{420} \begin{bmatrix}
156 A_1 l_1 & 22 A_1 l_1^2 & 54 A_1 l_1 & -13 A_1 l_1^2 & 0 & 0 \\
22 A_1 l_1^2 & 4 A_1 l_1^3 & 13 A_1 l_1^2 & -3 A_1 l_1^3 & 0 & 0 \\
54 A_1 l_1 & 13 A_1 l_1^2 & 156(A_1 l_1 + A_2 l_2) & 22(-A_1 l_1^2 + A_2 l_2^2) & 54 A_2 l_2 & -13 A_2 l_2^2 \\
-13 A_1 l_1^2 & -3 A_1 l_1^3 & 22(-A_1 l_1^2 + A_2 l_2^2) & 4(A_1 l_1^3 + A_2 l_2^3) & 13 A_2 l_2^2 & -3 A_2 l_2^3 \\
0 & 0 & 54 A_2 l_2 & 13 A_2 l_2^2 & 156 A_2 l_2 & -22 A_2 l_2^2 \\
0 & 0 & -13 A_2 l_2^2 & -3 A_2 l_2^3 & -22 A_2 l_2^2 & 4 A_2 l_2^3
\end{bmatrix}
$$

where $I^{(e)} = I_e$ and $l^{(e)} = l_e$; $e = 1, 2$

Since $Q_1 = Q_2 = Q_5 = Q_6 = 0$, rows and columns 1, 2, 5 and 6 in $[\underset{\sim}{K}]$ and $[\underset{\sim}{M}]$ are deleted to obtain the frequency equation as

$$
\left| [K] - \omega^2 [M] \right| = 0
$$

i.e.

$$
\begin{vmatrix}
\left\{ 12 E \left(\frac{I_1}{l_1^3} + \frac{I_2}{l_2^3}\right) - \frac{156 \rho \omega^2}{420} (A_1 l_1 + A_2 l_2) \right\} & \left\{ 6E\left(-\frac{I_1}{l_1^2} + \frac{I_2}{l_2^2}\right) - \frac{22 \rho \omega^2}{420} \left(-A_1 l_1^2 + A_2 l_2^2\right) \right\} \\
\left\{ 6E\left(-\frac{I_1}{l_1^2} + \frac{I_2}{l_2^2}\right) - \frac{22 \rho \omega^2}{420} \left(-A_1 l_1^2 + A_2 l_2^2\right) \right\} & \left\{ 4E\left(\frac{I_1}{l_1} + \frac{I_2}{l_2}\right) - \frac{4 \rho \omega^2}{420} \left(A_1 l_1^3 + A_2 l_2^3\right) \right\}
\end{vmatrix} = 0
$$

Roots of this equation give ω_1 and ω_2.

<u>Load vector:</u>

$$
\vec{\bar{f}}^{(1)} = \vec{f}^{(1)} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} \int_0^{l_1} f_1(x,t)\, N_1(x)\, dx \\ \int_0^{l_1} f_2(x,t)\, N_2(x)\, dx \\ \int_0^{l_1} f_3(x,t)\, N_3(x)\, dx \\ \int_0^{l_1} f_4(x,t)\, N_4(x)\, dx \end{Bmatrix} = p \begin{Bmatrix} \int_0^{l_1} N_1 \cdot dx \\ \int_0^{l_1} N_2 \cdot dx \\ \int_0^{l_1} N_3 \cdot dx \\ \int_0^{l_1} N_4 \cdot dx \end{Bmatrix}
$$

$$= p \begin{Bmatrix} l_1/2 \\ l_1^2/12 \\ l_1/2 \\ l_1^2/12 \end{Bmatrix}$$

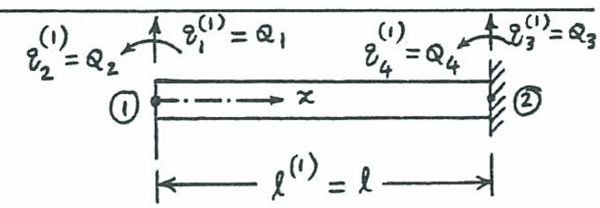
$$\vec{\underline{f}}^{(2)} = \vec{f}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\vec{\underset{\sim}{F}} = \sum_{e=1}^{2} [A^{(e)}]^T \, \vec{\underline{f}}^{(e)} = p \begin{Bmatrix} l_1/2 \\ l_1^2/12 \\ l_1/2 \\ l_1^2/12 \\ 0 \\ 0 \end{Bmatrix}$$

After applying the boundary conditions, load vector becomes

$$\vec{F} = \begin{Bmatrix} p\, l_1/2 \\ p\, l_1^2/12 \end{Bmatrix}.$$

(12.32) $$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Since node 1 is pin connected and node 2 is fixed, $Q_1 = Q_3 = Q_4 = 0$. By deleting the corresponding rows and columns in $[\underset{\sim}{K}]$ and $[\underset{\sim}{M}]$, the eigenvalue problem becomes

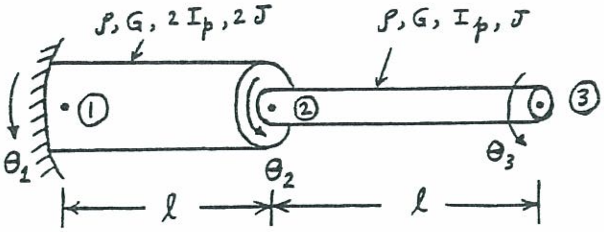
$$\left[[K] - \omega^2 [M] \right] \vec{Q} = \vec{0}$$

Frequency equation is

$$\left| \frac{EI}{l^3}(4l^2) - \omega^2 \frac{\rho A l}{420}(4l^2) \right| = 0$$

which gives $\omega_1^2 = \dfrac{420\, EI}{\rho A l^4}$ or $\omega_1 = 20.4939 \sqrt{\dfrac{EI}{\rho A l^4}}$.

12.33



$$[\bar{m}^{(1)}] = [m^{(1)}] = \frac{\rho_1 I_{p_1} l_1}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{\rho I_p l}{12}\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}$$

$$[\bar{m}^{(2)}] = [m^{(2)}] = \frac{\rho_2 I_{p_2} l_2}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = \frac{\rho I_p l}{12}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$[\bar{k}^{(1)}] = [k^{(1)}] = \frac{2G_1 J_1}{l_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{GJ}{l}\begin{bmatrix}4 & -4\\ -4 & 4\end{bmatrix}$$

$$[\bar{k}^{(2)}] = [k^{(2)}] = \frac{2G_2 J_2}{l_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{GJ}{l}\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}$$

$$[A^{(1)}] = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}, \quad [A^{(2)}] = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$[\underset{\sim}{M}] = \sum_{e=1}^{2}[A^{(e)}]^T[\bar{m}^{(e)}][A^{(e)}] = \frac{\rho I_p l}{12}\begin{bmatrix}4 & 2 & 0\\ 2 & 6 & 1\\ 0 & 1 & 2\end{bmatrix}$$

$$[\underset{\sim}{K}] = \sum_{e=1}^{2}[A^{(e)}]^T[\bar{k}^{(e)}][A^{(e)}] = \frac{2GJ}{l}\begin{bmatrix}2 & -2 & 0\\ -2 & 3 & -1\\ 0 & -1 & 1\end{bmatrix}$$

Since $\theta_1 = 0$, $[M] = \frac{\rho I_p l}{12}\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix}$ and $[K] = \frac{2GJ}{l}\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix}$

Frequency equation is

$$\left|\frac{2GJ}{l}\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix} - \frac{\rho I_p l \omega^2}{12}\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix}\right| = 0$$

i.e. $\left|\begin{bmatrix}3 & -1\\ -1 & 1\end{bmatrix} - \lambda\begin{bmatrix}6 & 1\\ 1 & 2\end{bmatrix}\right| = 0$ where $\lambda = \frac{\rho I_p l^2 \omega^2}{24\, GJ}$

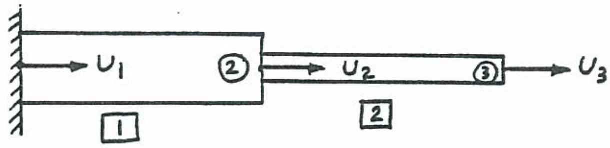
i.e. $11\lambda^2 - 14\lambda + 2 = 0$

i.e. $\lambda_1 = 0.1640$, $\lambda_2 = 1.1087$

$\omega_1^2 = 3.9360\, GJ/(\rho I_p l^2)$, $\omega_2^2 = 26.6088\, GJ/(\rho I_p l^2)$

$\therefore\ \omega_1 = 1.983935\sqrt{\frac{GJ}{\rho I_p l^2}}$ and $\omega_2 = 5.158372\sqrt{\frac{GJ}{\rho I_p l^2}}$

12.34 Element matrices:



$$[K^{(1)}] = \frac{A_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{4AE}{l}\begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array}$$

$$[K^{(2)}] = \frac{A_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE}{l}\begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

$$[M^{(1)}] = \frac{\rho_1 A_1 l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{2\rho A l}{3}\begin{array}{c} \begin{array}{cc} U_1 & U_2 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array}$$

$$[M^{(2)}] = \frac{\rho_2 A_2 l_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho A l}{6}\begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

Assembled matrices (with $U_1 = 0$):

$$[K] = \frac{AE}{l}\begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}, \quad [M] = \frac{\rho A l}{6}\begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

Eigenvalue problem:

$$\left[-\omega^2 [M] + [K]\right]\vec{U} = \vec{0} \qquad (E_1)$$

For natural frequencies,

$$\left| -\frac{\rho A l \omega^2}{6}\begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} + \frac{AE}{l}\begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 5 - 10\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0 \qquad (E_2)$$

Where $\lambda = \left(\dfrac{\rho l^2 \omega^2}{6E}\right) \qquad (E_3)$

i.e., $19\lambda^2 - 22\lambda + 4 = 0$

This gives

$$\lambda_1 = 0.2259 \quad , \quad \omega_1 = 1.1642 \left\{ E/(\rho l^2) \right\} \qquad (E_4)$$

$$\lambda_2 = 0.9320 \quad , \quad \omega_2 = 2.3648 \left\{ E/(\rho l^2) \right\} \qquad (E_5)$$

For eigenvectors,

Use of (E_2) leads to

$$(5 - 10\lambda_1) U_2 - (1 + \lambda_1) U_3 = 0$$

or $\quad U_3 = \left(\frac{5 - 10\lambda_1}{1+\lambda_1} \right) U_2 = 2.2359\ U_2 \quad \text{with } \lambda_1 = 0.2259$

$$\therefore \quad \underset{\sim}{\vec{U}}^{(1)} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.2359 \end{Bmatrix} \qquad (E_6)$$

Similarly,

$$U_3 = \left(\frac{5 - 10\lambda_2}{1+\lambda_2} \right) U_2 = -2.2360\ U_2 \quad \text{with } \lambda_2 = 0.9320$$

$$\therefore \quad \underset{\sim}{\vec{U}}^{(2)} = \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1.0 \\ -2.2360 \end{Bmatrix} \qquad (E_7)$$

Orthonormalization of normal modes with [M]-matrix:

Let $\vec{U}^{(1)} = \alpha_1 \underset{\sim}{\vec{U}}^{(1)}$ and $\vec{U}^{(2)} = \alpha_2 \underset{\sim}{\vec{U}}^{(2)}$ $\qquad (E_8)$

where α_1 and α_2 are constants to be determined.

$\vec{U}^{(1)T} [M] \vec{U}^{(1)} = 1$ gives $\quad \alpha_1^2 = \dfrac{1}{\underset{\sim}{\vec{U}}^{(1)T} [M] \underset{\sim}{\vec{U}}^{(1)}}$

Since $\underset{\sim}{\vec{U}}^{(1)T} [M] \underset{\sim}{\vec{U}}^{(1)} = (1.0 \quad 2.2359) \dfrac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 2.2359 \end{Bmatrix}$

$$= 4.0784\ \rho A l,$$

$\alpha_1 = 0.4952/\sqrt{\rho A l}$, and

$$\vec{U}^{(1)} = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.4952 \\ 1.1072 \end{Bmatrix} \qquad (E_9)$$

similarly, $\quad \alpha_2^2 = \dfrac{1}{\underset{\sim}{\vec{U}}^{(2)T} [M] \underset{\sim}{\vec{U}}^{(2)}}$

where $\underset{\sim}{\vec{U}}^{(2)T} [M] \underset{\sim}{\vec{U}}^{(2)} = (1.0 \quad -2.236) \dfrac{\rho A l}{6} \begin{bmatrix} 10 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -2.236 \end{Bmatrix}$

$$= 2.5879\ \rho A l$$

$\alpha_2 = 0.6216/\sqrt{\rho A l}$, and

$$\vec{U}^{(2)} = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.6216 \\ -1.3900 \end{Bmatrix} \qquad (E_{10})$$

Modal matrix:

$$[U] = \frac{1}{\sqrt{\rho A l}} \begin{bmatrix} 0.4952 & 0.6216 \\ 1.1072 & -1.3900 \end{bmatrix} \qquad (E_{11})$$

with

$$[U]^T[M][U] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [U]^T[K][U] = \frac{E}{\rho l^2}\begin{bmatrix} 1.3554 & 0 \\ 0 & 5.5921 \end{bmatrix}$$

Forced vibration equations:

Equations of motion are $[M]\ddot{\vec{U}} + [K]\vec{U} = \vec{P}$ $\quad (E_{12})$

Let $\vec{U}(t) = [U]\,\vec{\eta}(t)$ $\quad (E_{13})$

where $[U]$ = modal matrix and

$\vec{\eta}(t) = \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix}$ = vector of generalized coordinates

Substituting (E_{13}) into (E_{12}) and premultiplying by $[U]^T$ gives the uncoupled equations of motion:

$$\ddot{\eta}_i(t) + \omega_i^2\,\eta_i(t) = Q_i(t) \quad ; \quad i = 1,2 \qquad (E_{14})$$

Where the generalized loads $Q_i(t)$ are given by

$$\vec{Q} = [U]^T\,\vec{P} = \frac{1}{\sqrt{\rho A l}}\begin{bmatrix} 0.4952 & 0.6216 \\ 1.1072 & -1.3900 \end{bmatrix}\begin{Bmatrix} 0 \\ P(t) \end{Bmatrix}$$

$$= \frac{1}{\sqrt{\rho A l}}\begin{Bmatrix} 0.6216 \\ -1.3900 \end{Bmatrix} P(t) \qquad (E_{15})$$

Hence equations of motion become

$$\ddot{\eta}_1 + 1.3554\left(\frac{E}{\rho l^2}\right)\eta_1 = \frac{0.6216}{\sqrt{\rho A l}}\,P(t) \qquad (E_{16})$$

$$\ddot{\eta}_2 + 5.5921\left(\frac{E}{\rho l^2}\right)\eta_2 = -\frac{1.3900}{\sqrt{\rho A l}}\,P(t) \qquad (E_{17})$$

Assume all initial conditions to be zero:

$$\left.\begin{aligned} \vec{U}(t=0) &= [U]\,\vec{\eta}(0) = 0 \\ \dot{\vec{U}}(t=0) &= [U]\,\dot{\vec{\eta}}(0) = 0 \end{aligned}\right\} \Rightarrow \vec{\eta}(0) = \dot{\vec{\eta}}(0) = \vec{0} \qquad (E_{18})$$

Thus the solution of Eqs. (E_{16}) and (E_{17}) can be expressed as

$$\eta_1(t) = \frac{1}{\omega_1}\int_0^t Q_1(\tau)\,\sin\omega_1(t-\tau)\,d\tau$$

$$= \sqrt{\frac{l}{AE}}\,(0.5339)\int_0^t P(\tau)\,\sin\left\{1.1642\sqrt{\frac{E}{\rho l^2}}\,(t-\tau)\right\}.d\tau \qquad (E_{19})$$

$$\eta_2(t) = \frac{1}{\omega_2} \int_0^t Q_2(\tau) \sin \omega_2 (t-\tau)\, d\tau$$

$$= -\sqrt{\frac{l}{AE}}\,(0.5878) \int_0^t P(\tau) \cdot \sin\left\{\sqrt{\frac{E}{\rho l^2}}\; 2.3648\,(t-\tau)\right\} d\tau \qquad (E_{20})$$

Since

$$P(t) = \begin{cases} P_0 & \text{for } 0 \le t \le t_0 \\ 0 & \text{for } t > t_0 \end{cases} \qquad (E_{21})$$

we can express the solution as

$$\vec{U}(t) = \begin{Bmatrix} U_2(t) \\ U_3(t) \end{Bmatrix} = [U]\,\vec{\eta}(t) = \frac{1}{\sqrt{\rho A l}} \begin{Bmatrix} 0.4952\,\eta_1(t) + 0.6216\,\eta_2(t) \\ 1.1072\,\eta_1(t) - 1.3900\,\eta_2(t) \end{Bmatrix}$$

which becomes, in view of Eqs. $(E_{19}) - (E_{21})$:

For $t \le t_0$:

$$U_2(t) = \frac{P_0 l}{AE}\left\{0.3816 - 0.2271 \cos 1.1642\sqrt{\frac{E}{\rho l^2}}\,t - 0.1545 \cos 2.3648\sqrt{\frac{E}{\rho l^2}}\,t\right\}$$

$$U_3(t) = \frac{P_0 l}{AE}\left\{0.1622 - 0.5078 \cos 1.1642\sqrt{\frac{E}{\rho l^2}}\,t + 0.3456 \cos 2.3648\sqrt{\frac{E}{\rho l^2}}\,t\right\} \qquad (E_{22})$$

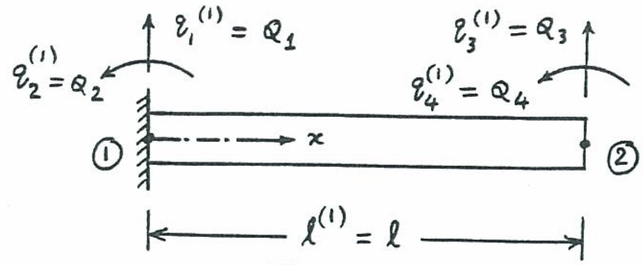
For $t > t_0$:

$$U_2(t) = \frac{P_0 l}{AE}\left\{0.2271 \cos 1.1642\sqrt{\frac{E}{\rho l^2}}\,(t-t_0) - 0.2271 \cos 1.1642\sqrt{\frac{E}{\rho l^2}}\,t + 0.1547 \cos 2.3648\sqrt{\frac{E}{\rho l^2}}\,(t-t_0) - 0.1547 \cos 2.3648\sqrt{\frac{E}{\rho l^2}}\,t\right\}$$

$$U_3(t) = \frac{P_0 l}{AE}\left\{0.5078 \cos 1.1642\sqrt{\frac{E}{\rho l^2}}\,(t-t_0) - 0.5078 \cos 1.1642\sqrt{\frac{E}{\rho l^2}}\,t - 0.3456 \cos 2.3648\sqrt{\frac{E}{\rho l^2}}\,(t-t_0) + 0.3456 \cos 2.3648\sqrt{\frac{E}{\rho l^2}}\,t\right\} \qquad (E_{23})$$

12.35

$$[\underset{\sim}{K}] = [K^{(1)}] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



$$[\underset{\sim}{M}] = [M^{(1)}] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

Since node 1 is fixed, $Q_1 = Q_2 = 0$. By deleting the first two rows and columns in $[\underset{\sim}{K}]$ and $[\underset{\sim}{M}]$, the eigenvalue problem becomes

$$\big[[K] - \omega^2 [M]\big]\begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\left| \frac{EI}{l^3}\begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix} - \frac{\rho A l \omega^2}{420}\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 \end{bmatrix} \right| = 0$$

Let $\lambda = \dfrac{\rho A l^4 \omega^2}{420\, EI}$. Then $\begin{vmatrix} 12 - 156\lambda & -6l + 22l\lambda \\ -6l + 22l\lambda & 4l^2 - 4l^2\lambda \end{vmatrix} = 0$

i.e. $\quad 140\, l^2 \lambda^2 - 408\, l^2 \lambda + 12\, l^2 = 0$

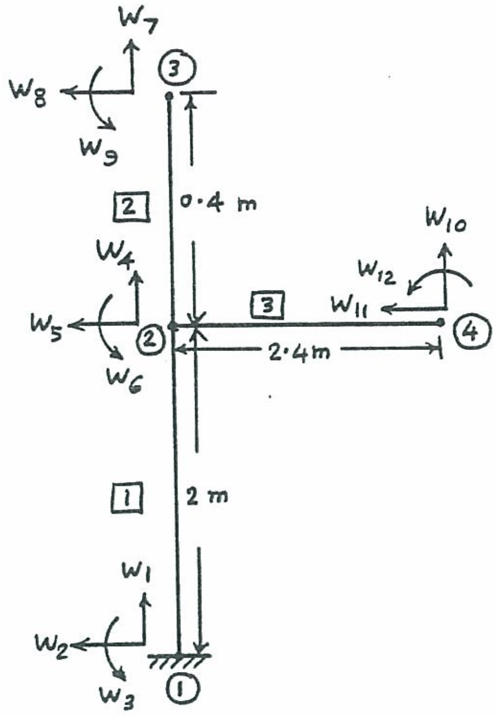
i.e. $\quad \lambda = 0.029715,\ 2.884571$

i.e. $\quad \omega_1^2 = 12.4803 \dfrac{EI}{\rho A l^4}, \quad \omega_2^2 = 1211.5198 \dfrac{EI}{\rho A l^4}$

$\therefore \quad \omega_1 = 3.5327 \sqrt{EI/\rho A l^4}, \quad \omega_2 = 34.8069 \sqrt{EI/\rho A l^4}$

12.36 $E_i = 2.1 \times 10^{11}$ Pa ; $i = 1,2,3$

$l_1 = 2.0$ m, $l_2 = 0.4$ m, $l_3 = 2.4$ m



$$I_1 = I_2 = \frac{\pi}{64}\left[\left(\frac{830}{1000}\right)^4 - \left(\frac{800}{1000}\right)^4\right]$$

$$= 3.189853 \times 10^{-3} \ m^4$$

$$I_3 = \frac{1}{12}\left[350(550)^3 - 320(520)^3\right] \times 10^{-12}$$

$$= 1.103058 \times 10^{-3} \ m^4$$

$\rho_i = 7.8 \times 10^3$ kg/m^3 ; $i = 1,2,3$

$$A_1 = A_2 = \frac{\pi}{4}\left[\left(\frac{830}{1000}\right)^2 - \left(\frac{800}{1000}\right)^2\right]$$

$$= 0.038406 \ m^2$$

$$A_3 = (350 \times 550 - 320 \times 520) \times 10^{-6}$$

$$= 0.0261 \ m^2$$

$$\frac{E_1 I_1}{l_1^3} = 8373.3641 \times 10^4 ; \quad \frac{E_2 I_2}{l_2^3} = 1046670.516 \times 10^4 ;$$

$$\frac{E_3 I_3}{l_3^3} = 1675.6523 \times 10^4$$

Element stiffness matrices:

$$[K^{(e)}] = \frac{E_e I_e}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccc} W_2 & W_3 & W_5 & W_6 \end{array} \\ [K^{(1)}] = & \begin{bmatrix} 10.0480 & 10.0480 & -10.0480 & 10.0480 \\ 10.0480 & 13.3974 & -10.0480 & 6.6987 \\ -10.0480 & -10.0480 & 10.0480 & -10.0480 \\ 10.0480 & 6.6987 & -10.0480 & 13.3974 \end{bmatrix} \times 10^8 \end{array} \begin{array}{c} W_2 \\ W_3 \\ W_5 \\ W_6 \end{array}$$

$$[K^{(2)}] = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} W_5 & W_6 & W_8 & W_9 \\ \end{array}$$

$$[K^{(2)}] = \begin{bmatrix} 1256.0052 & 251.2010 & -1256.0052 & 251.2010 \\ 251.2010 & 66.9869 & -251.2010 & 33.4935 \\ -1256.0052 & -251.2010 & 1256.0052 & -251.2010 \\ 251.2010 & 33.4935 & -251.2010 & 66.9869 \end{bmatrix} \times 10^8 \quad \begin{array}{c} W_5 \\ W_6 \\ W_8 \\ W_9 \end{array}$$

$$\begin{array}{cccc} W_4 & W_6 & W_{10} & W_{12} \end{array}$$

$$[K^{(3)}] = \begin{bmatrix} 2.0108 & 2.4129 & -2.0108 & 2.4129 \\ 2.4129 & 3.8607 & -2.4129 & 1.9303 \\ -2.0108 & -2.4129 & 2.0108 & -2.4129 \\ 2.4129 & 1.9303 & -2.4129 & 3.8607 \end{bmatrix} \times 10^8 \quad \begin{array}{c} W_4 \\ W_6 \\ W_{10} \\ W_{12} \end{array}$$

Axial stiffnesses are given by

$$\frac{A_1 E_1}{l_1} = 40.3263 \times 10^8, \quad \frac{A_2 E_2}{l_2} = 201.6315 \times 10^8, \quad \frac{A_3 E_3}{l_3} = 22.8375 \times 10^8$$

Considering the axial stiffnesses of elements 1 and 2 at degree of freedom W_4, the assembled stiffness matrix can be derived as

$$\begin{array}{ccccccc} W_4 & W_5 & W_6 & W_8 & W_9 & W_{10} & W_{12} \end{array}$$

$$[K] = 10^8 \begin{bmatrix} 243.9686 & 0 & 2.4129 & 0 & 0 & -2.0108 & 2.4129 \\ 0 & 1266.0532 & 241.1530 & -1256.0052 & 251.2010 & 0 & 0 \\ 2.4129 & 241.1530 & 84.245 & -251.2010 & 33.4935 & -2.4129 & 1.9303 \\ 0 & -1256.0052 & -251.2010 & 1256.0052 & -251.2010 & 0 & 0 \\ 0 & 251.2010 & 33.4935 & -251.2010 & 66.9869 & 0 & 0 \\ -2.0108 & 0 & -2.4129 & 0 & 0 & 2.0108 & -2.4129 \\ 2.4129 & 0 & 1.9303 & 0 & 0 & -2.4129 & 3.8607 \end{bmatrix} \begin{array}{c} W_4 \\ W_5 \\ W_6 \\ W_8 \\ W_9 \\ W_{10} \\ W_{12} \end{array}$$

$$\frac{\rho_1 A_1 l_1}{420} = 1.4265, \quad \frac{\rho_2 A_2 l_2}{420} = 0.2853, \quad \frac{\rho_3 A_3 l_3}{420} = 1.1633$$

Element mass matrices:

$$[M^{(e)}] = \frac{\rho_e A_e l_e}{420} \begin{bmatrix} 156 & 22 l_e & 54 & -13 l_e \\ 22 l_e & 4 l_e^2 & 13 l_e & -3 l_e^2 \\ 54 & 13 l_e & 156 & -22 l_e \\ -13 l_e & -3 l_e^2 & -22 l_e & 4 l_e^2 \end{bmatrix}$$

$$[M^{(1)}] = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc} w_2 & w_3 & w_5 & w_6 \end{array}$$

$$[M^{(1)}] = \left[\begin{array}{cccc} 222.534 & 62.766 & 77.031 & -37.089 \\ 62.766 & 22.824 & 37.089 & -17.118 \\ 77.031 & 37.089 & 222.534 & -62.766 \\ -37.089 & -17.118 & -62.766 & 22.824 \end{array}\right] \begin{array}{c} w_2 \\ w_3 \\ w_5 \\ w_6 \end{array}$$

$$[M^{(2)}] = \left[\begin{array}{cccc} 44.5068 & 2.5106 & 15.4062 & -1.4836 \\ 2.5106 & 0.1826 & 1.4836 & -0.1369 \\ 15.4062 & 1.4836 & 44.5068 & -2.5106 \\ -1.4836 & -0.1369 & -2.5106 & 0.1826 \end{array}\right] \begin{array}{c} w_5 \\ w_6 \\ w_8 \\ w_9 \end{array}$$

(columns: w_5, w_6, w_8, w_9)

$$[M^{(3)}] = \left[\begin{array}{cccc} 181.4748 & 61.4222 & 62.8182 & -36.2950 \\ 61.4222 & 26.8024 & 36.2950 & -20.1018 \\ 62.8182 & 36.2950 & 181.4748 & -61.4222 \\ -36.2950 & -20.1018 & -61.4222 & 26.8024 \end{array}\right] \begin{array}{c} w_4 \\ w_6 \\ w_{10} \\ w_{12} \end{array}$$

(columns: w_4, w_6, w_{10}, w_{12})

For axial motion,

$$\frac{\rho_1 A_1 l_1}{3} = 199.71, \quad \frac{\rho_2 A_2 l_2}{3} = 39.942, \quad \frac{\rho_3 A_3 l_3}{3} = 162.864.$$

Considering mass matrix terms (corresponding to axial motion of elements 1 and 2) at degree of freedom W_4, the assembled mass matrix can be obtained as:

$$[M] = \left[\begin{array}{ccccccc} 421.1268 & 0 & 61.4222 & 0 & 0 & 62.8182 & -36.2950 \\ 0 & 267.0408 & -60.2554 & 15.4062 & -1.4836 & 0 & 0 \\ 61.4222 & -60.2554 & 49.8090 & 1.4836 & -0.1369 & 36.2950 & -20.1018 \\ 0 & 15.4062 & 1.4836 & 44.5068 & -2.5106 & 0 & 0 \\ 0 & -1.4836 & -0.1369 & -2.5106 & 0.1826 & 0 & 0 \\ 62.8182 & 0 & 36.2950 & 0 & 0 & 181.4748 & -61.4222 \\ -36.2950 & 0 & -20.1018 & 0 & 0 & -61.4222 & 26.8024 \end{array}\right] \begin{array}{c} w_4 \\ w_5 \\ w_6 \\ w_8 \\ w_9 \\ w_{10} \\ w_{12} \end{array}$$

(columns: w_4, w_5, w_6, w_8, w_9, w_{10}, w_{12})

Once $[K]$ and $[M]$ are known, the natural frequencies can be found by solving the eigenvalue problem.

(12.37) NATURAL FREQUENCY ANALYSIS

The mass matrix of the column is given in the solution of Problem 12.22.

By adding the mass of tank, the mass matrix becomes

$$[M] = 657.4022 \times 10^{-4} \begin{bmatrix} 156 & -10560 \\ -10560 & 921600 \end{bmatrix} + \begin{bmatrix} 25.8799 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 36.1054 & -694.2167 \\ -694.2167 & 60586.1867 \end{bmatrix}$$

Natural frequencies are given by:

$$\left| -\omega^2 [M] + [K] \right| = 0$$

i.e.,

$$\begin{vmatrix} (2.0006 - 36.1054\,\lambda) & (-480.1371 + 694.2167\,\lambda) \\ (-480.1371 + 694.2167\,\lambda) & (153643.8712 - 60586.1867\,\lambda) \end{vmatrix} = 0$$

where $\lambda = \left(\omega^2 / 10^4\right)$.

This gives

$$170.5552\,\lambda^2 - 500.1944\,\lambda + 7.6848 = 0$$

Hence $$\lambda_{1,2} = \frac{500.1944 \pm (250194.4378 - 5242.7304)^{1/2}}{341.1104}$$

$\lambda_1 = 0.01544$, $\omega_1 = 12.4258$ rad/sec

$\lambda_2 = 2.9173$, $\omega_2 = 170.8011$ rad/sec

12.39 Consistent mass matrix:

$T(t)$ = kinetic energy of element $= \frac{1}{2} \int_0^l \rho A \left(\frac{\partial u}{\partial t}\right)^2 dx$

with

$$\dot{u}(x,t) \equiv \frac{\partial u}{\partial t}(x,t) = \left(1 - \frac{x}{l}\right) \dot{u}_1(t) + \left(\frac{x}{l}\right) \dot{u}_2(t)$$

Substituting for $A(x)$ and $\dot{u} = \frac{\partial u}{\partial t}$, kinetic energy expression can be derived as

$$T = \frac{1}{2} \int_0^l \rho \, \frac{\pi}{4} \left\{ D^2 + \left(\frac{d-D}{l}\right)^2 x^2 + 2D\left(\frac{d-D}{l}\right) x \right\} \left\{ \dot{u}_1^2 + x\left(-\frac{2}{l}\dot{u}_1^2 + \frac{2}{l}\dot{u}_1 \dot{u}_2\right) + x^2 \left(\frac{\dot{u}_1^2}{l^2} + \frac{\dot{u}_2^2}{l^2} - \frac{2\dot{u}_1 \dot{u}_2}{l^2} \right) \right\} dx$$

$$= \frac{\pi \rho l}{8} \left\{ \dot{u}_1^2 \left(\frac{D^2}{5} + \frac{d^2}{30} + \frac{D d}{10} \right) + \dot{u}_2^2 \left(\frac{D^2}{30} + \frac{d^2}{5} + \frac{D d}{10} \right) + 2\dot{u}_1 \dot{u}_2 \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{D d}{15} \right) \right\}$$

$$= \frac{1}{2} \vec{\dot{u}}^T [m_c] \vec{\dot{u}} \equiv \frac{1}{2} (\dot{u}_1 \;\; \dot{u}_2) \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix}$$

This gives the consistent mass matrix as

$$[m_c] = \frac{\pi \rho l}{4} \begin{bmatrix} \left(\frac{D^2}{5} + \frac{d^2}{30} + \frac{D d}{10}\right) & \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{D d}{15}\right) \\ \left(\frac{D^2}{20} + \frac{d^2}{20} + \frac{D d}{15}\right) & \left(\frac{D^2}{30} + \frac{d^2}{5} + \frac{D d}{10}\right) \end{bmatrix}$$

Lumped mass matrix:

Total mass of element $= \frac{\pi \rho}{4} \int_0^l h^2(x) \cdot dx$

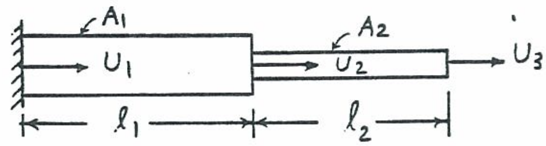
$$= \frac{\pi \rho}{4} \int_0^l \left\{ D^2 + \left(\frac{d-D}{l}\right)^2 x^2 + 2D\left(\frac{d-D}{l}\right) x \right\} \cdot dx$$

$$= \frac{\pi \rho l}{12} \left(D^2 + d^2 + D d \right)$$

Distributing the mass at the two nodes,

$$[m_l] = \frac{\pi \rho l}{24} \begin{bmatrix} (D^2 + d^2 + D d) & 0 \\ 0 & (D^2 + d^2 + D d) \end{bmatrix}$$

12.40 $A_1 = 2 \text{ in}^2, \quad A_2 = 1 \text{ in}^2$



$E = 30 \times 10^6$ psi

$\rho = 0.283 \text{ lb/in}^3, \quad l_1 = l_2 = 50$ in

Consistent mass matrices

$$[M^{(1)}]_c = \frac{\rho_1 A_1 l_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left(\frac{0.283}{386.4}\right)(2)(50)\frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.01221 \begin{array}{c c} U_1 & U_2 \\ \left[\begin{matrix} 2 & 1 \\ 1 & 2 \end{matrix}\right] & \end{array} \begin{array}{c} U_1 \\ U_2 \end{array}$$

$$[M^{(2)}]_c = \frac{\rho_2 A_2 l_2}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \left(\frac{0.283}{386.4}\right)(1)(50)\frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= 0.006105 \begin{array}{c} U_2 \quad U_3 \\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

Lumped mass matrices

$$[M^{(1)}]_l = \frac{\rho_1 A_1 l_1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.03662 \begin{array}{c} U_1 \quad U_2 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array}$$

$$[M^{(2)}]_l = \frac{\rho_2 A_2 l_2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.01831 \begin{array}{c} U_2 \quad U_3 \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

stiffness matrices

$$[K^{(1)}] = \frac{A_1 E_1}{l_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{2(30 \times 10^6)}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1.2 \times 10^6 \begin{array}{c} U_1 \quad U_2 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \end{array}$$

$$[K^{(2)}] = \frac{A_2 E_2}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1(30 \times 10^6)}{50}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.6 \times 10^6 \begin{array}{c} U_2 \quad U_3 \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

Assembled matrices (before applying boundary conditions)

$$[\underset{\sim}{K}] = 0.6 \times 10^6 \begin{array}{c} U_1 \quad\quad U_2 \quad\quad U_3 \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2+1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array}$$

$$[\underset{\sim}{M}]_c = 0.006105 \begin{array}{c} U_1 \quad\quad U_2 \quad\quad U_3 \\ \begin{bmatrix} 4 & 2 & 0 \\ 2 & 4+2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \end{array}$$

$$[\underset{\sim}{M}]_\ell = 0.01831 \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}$$

Assembled matrices after applying boundary conditions

$$[K] = 0.6 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[M]_c = 0.006105 \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[M]_\ell = 0.01831 \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Natural frequencies with consistent mass matrices:

$$\left| -\omega^2 (0.006105) \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} + 0.6 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 3 - 6\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0$$

where $\lambda = \omega^2 (0.006105)/0.6 \times 10^6 = 0.010175 \times 10^{-6}\, \omega^2$

i.e., $11\lambda^2 - 14\lambda + 2 = 0$

$$\therefore \quad \lambda_{1,2} = \frac{14 \pm \sqrt{196 - 88}}{22} = 0.163986,\quad 1.108741$$

or $\omega_1 = 4.0145 \times 10^3$ rad/sec, $\omega_2 = 10.4387 \times 10^3$ rad/sec

Natural frequencies with lumped mass matrices:

Here the frequency equation can be expressed as

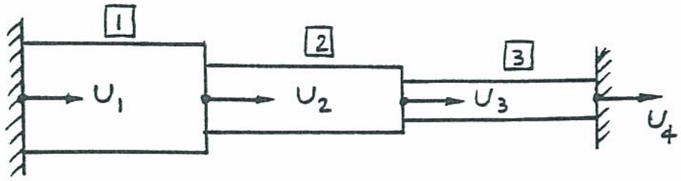
$$\begin{vmatrix} (3-3\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \quad \text{with } \lambda = \omega^2 (0.01831)/0.6 \times 10^6 = 0.0305167 \times 10^{-6}\, \omega^2$$

i.e., $3\lambda^2 - 6\lambda + 2 = 0$

$\therefore \quad \lambda_1 = 0.42265, \quad \lambda_2 = 1.57735$

or $\omega_1 = 6.4450 \times 10^3$ rad/sec, $\omega_2 = 12.4508 \times 10^3$ rad/sec

12.41 With consistent mass matrices:



$$\left[M^{(e)}\right]_c = \frac{\rho^{(e)} A^{(e)} \ell^{(e)}}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\left[M^{(1)}\right]_c = \frac{(7.8\times 10^3)(0.4\times 10^{-3})(0.2)}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = 0.104\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\left[M^{(2)}\right]_c = \frac{(7.8\times 10^3)(0.2\times 10^{-3})(0.2)}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = 0.052\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\left[M^{(3)}\right]_c = \frac{(7.8\times 10^3)(0.1\times 10^{-3})(0.2)}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix} = 0.026\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$\left[\underset{\sim}{M}\right]_c = 0.026\;\begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 8 & 4 & 0 & 0\\ 4 & 8+4 & 2 & 0\\ 0 & 2 & 4+2 & 1\\ 0 & 0 & 1 & 2\end{bmatrix}\end{array}\begin{array}{c} U_1\\ U_2\\ U_3\\ U_4\end{array}$$

$$\left[M\right]_c = 0.026\begin{bmatrix}12 & 2\\ 2 & 6\end{bmatrix} = \begin{array}{c}\begin{array}{cc} U_2 & U_3\end{array}\\ \begin{bmatrix}0.312 & 0.052\\ 0.052 & 0.156\end{bmatrix}\end{array}\begin{array}{c}U_2\\ U_3\end{array}$$

$$\left[K^{(e)}\right] = \frac{A^{(e)}E^{(e)}}{\ell^{(e)}}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\left[K^{(1)}\right] = \frac{(0.4\times 10^{-3})(2.1\times 10^{11})}{0.2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 4.2\times 10^8\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\left[K^{(2)}\right] = \frac{(0.2\times 10^{-3})(2.1\times 10^{11})}{0.2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 2.1\times 10^8\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\left[K^{(3)}\right] = \frac{(0.1\times 10^{-3})(2.1\times 10^{11})}{0.2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = 1.05\times 10^8\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$\left[\underset{\sim}{K}\right] = 1.05\times 10^8\;\begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 4 & -4 & 0 & 0\\ -4 & 4+2 & -2 & 0\\ 0 & -2 & 2+1 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}\end{array}\begin{array}{c} U_1\\ U_2\\ U_3\\ U_4\end{array}$$

$$[K] = 1.05 \times 10^8 \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} = 10^8 \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 6.3 & -2.1 \\ -2.1 & 3.15 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

Natural frequencies are given by

$$\left| -\omega^2 [M]_c + [K] \right| = 0$$

i.e.

$$\begin{vmatrix} 6.3 - 0.312\,\lambda & -2.1 - 0.052\,\lambda \\ -2.1 - 0.052\,\lambda & 3.15 - 0.156\,\lambda \end{vmatrix} = 0$$

where $\lambda = 10^{-8}\,\omega^2$

i.e.,

$$0.045968\,\lambda^2 - 2.184\,\lambda + 15.435 = 0$$

This gives $\lambda_1 = 8.6372$, $\lambda_2 = 38.8721$

or $\omega_1 = 2.9389 \times 10^4$ rad/sec, $\omega_2 = 6.2347 \times 10^4$ rad/sec.

With lumped mass matrices:

$$[M^{(e)}]_\ell = \frac{\rho A l}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(1)}]_\ell = \frac{(7.8 \times 10^3)(0.4 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.312 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(2)}]_\ell = \frac{(7.8 \times 10^3)(0.2 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.156 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[M^{(3)}]_\ell = \frac{(7.8 \times 10^3)(0.1 \times 10^{-3})(0.2)}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0.078 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{M}]_\ell = 0.078 \begin{array}{c} \begin{array}{cccc} U_1 & U_2 & U_3 & U_4 \end{array} \\ \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4+2 & 0 & 0 \\ 0 & 0 & 2+1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} U_1 \\ U_2 \\ U_3 \\ U_4 \end{array}$$

$$[M]_\ell = 0.078 \begin{bmatrix} 6 & 0 \\ 0 & 3 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} U_2 & U_3 \end{array} \\ \begin{bmatrix} 0.468 & 0 \\ 0 & 0.234 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \end{array}$$

Natural frequencies are given by

$$\left| -\omega^2 [M]_\ell + [K] \right| = 0$$

i.e.,

$$\left| -\omega^2 \begin{bmatrix} 0.468 & 0 \\ 0 & 0.234 \end{bmatrix} + 10^8 \begin{bmatrix} 6.3 & -2.1 \\ -2.1 & 3.15 \end{bmatrix} \right| = 0$$

i.e.,

$$\begin{vmatrix} 6.3 - 0.468\lambda & -2.1 \\ -2.1 & 3.15 - 0.234\lambda \end{vmatrix} = 0$$

where $\lambda = 10^{-8}\,\omega^2$

i.e.,

$$0.109512\,\lambda^2 - 2.9484\,\lambda + 15.435 = 0$$

This gives $\lambda_1 = 7.1157$, $\lambda_2 = 19.8074$

or $\omega_1 = 2.6675 \times 10^4$ rad/sec , $\omega_2 = 4.4506 \times 10^4$ rad/sec.

12.42

12.43

```
%------ Program Ex12_42_43.m
%------Initialization of values------------------------
A1 = 256e-4 ;
A2 = 16e-4 ;
A3 = 9e-4 ;

E1 = 20e10 ;
E2 = E1 ;
E3 = E1 ;

R1 = 7.8e3 ;
R2 = R1 ;
R3 = R1 ;

L1 = 3 ;
L2 = 2 ;
L3 = 1 ;

%------Definition of [K]-------------------------------

K11 = A1*E1/L1+A2*E2/L2 ;
K12 = -A2*E2/L2 ;
K13 = 0 ;

K21 = K12 ;
K22 = A2*E2/L2+A3*E3/L3 ;
K23 = -A3*E3/L3 ;

K31 = K13 ;
K32 = K23 ;
K33 = A3*E3/L3 ;

K = [ K11 K12 K13; K21 K22 K23; K31 K32 K33 ]
```

```
%-------- Calculation of matrix

P = [ 0 0 500]'

U = inv(K)*P

%------- Definition of [M]-------------------

M11 = (2*R1*A1*L1+2*R2*A2*L2)/6;
M12 = (R2*A2*L2)/6;
M13 = 0;
M21 = M12;
M22 = (2*R2*A2*L2+2*R3*A3*L3)/6;
M23 = R3*A3*L3;

M31 = M13;
M32 = M23;
M33 = 2*M23;

M= [M11 M12 M13; M21 M22 M23; M31 M32 M33 ]

MI = inv(M);

KM = MI*K;

%-------------Calculation of eigen vector and eigenvalue-------------------

[L,V] = eig(KM)
```

Results of Ex12_42_43.m
****

```
>> Ex12_42_43.m

K =

  1.0e+009 *

    1.8667   -0.1600         0
   -0.1600    0.3400   -0.1800
         0   -0.1800    0.1800


P =

     0
     0
   500


U =

  1.0e-005 *

    0.0293
    0.3418
    0.6196


M =

  208.0000    4.1600         0
    4.1600   10.6600    7.0200
         0    7.0200   14.0400
```

```
L =

   -0.0253    0.6914    0.0772
    0.8009   -0.1479    0.5712
   -0.5982   -0.7072    0.8172


V =

  1.0e+007 *

    9.0719         0         0
         0    0.9178         0
         0         0    0.2860
```

12.44

```
Results of Ex12_44
***********************
>> program17
 Natural frequencies of the stepped beams

 1.1508e+003 3.2589e+003 7.8111e+003 1.5986e+004

 Mode shapes

 1  3.1341e-003  5.0593e-005  1.5895e-003  -1.5864e-004

 2  -3.5755e-004  9.4254e-005  1.7410e-003  6.2691e-006

 3  -3.4813e-004  -1.2987e-004  5.9915e-004  3.8657e-005

 4  1.6669e-004  3.7347e-005  3.5876e-004  2.2350e-004
```

12.45

```
C=====================================================================
C
C      PROBLEM 12.45    PROGRAM FOR STRESS ANALYSIS OF PLANAR TRUSSES
C
C=====================================================================
C      DATA FOR PROBLEM 12.10 (TEST EXAMPLE)
       DIMENSION A(4),EL(4),GS(8,8),PP(8,1),P(8,1),GSS(4,4)
      2 ,STRS(4,1),X(4),Y(4),LOC(4,2),IFIX(4)
       DOUBLE PRECISION DIFF(2)
       DATA LOC/1,3,3,2,3,2,4,4/
       M=1
       DATA NN,NE,ND,NB,NFIX,E/4,4,8,4,4,30.0E+6/
       DATA IFIX/1,2,3,4/
       DATA X/0.,100.,50.,200./
       DATA Y/0.,0.,25.,100./
       DO 10 I=1,8
 10    P(I,1)=0.0
       P(8,1)=-1000.0
       DATA A/2.,2.,1.,1./
C      END OF PROBLEM-DEPENDENT DATA
       CALL TRUSS (NN,NE,ND,NB,M,LOC,X,Y,E,A,EL,NFIX,IFIX,P,GS,DIFF,
      2 GSS,ND2)
       PRINT 21
 21    FORMAT (2X,20H NODAL DISPLACEMENTS,/)
       DO 11 I=1,ND
       PP(I,1)=P(I,1)
 11    PRINT 12, I, P(I,1)
 12    FORMAT (4X,I5,2X,E15.6)
       DO 45 I=1,NFIX
       DO 45 J=1,M
       II=IFIX(I)
       PP(II,J)=0.0
45     CONTINUE
       PRINT 22
 22    FORMAT (//,2X, 21H STRESSES IN ELEMENTS,/)
       DO 80 K=1,NE
       DO 70 KK=1,M
       I=LOC(K,1)
       J=LOC(K,2)
       I1=2*I-2+1
       I2=I1+1
```

```
      I3=2*J-2+1
      I4=I3+1
      XL=(X(J)-X(I))/EL(K)
      XM=(Y(J)-Y(I))/EL(K)
      STRS(K,KK)=(E/EL(K))*(XL*(PP(I3,KK)-PP(I1,KK))+XM*(PP(I4,KK)-
     2 PP(I2,KK)))
 70   CONTINUE
 80   CONTINUE
      DO 120 I=1,NE
 120  PRINT 130, I,LOC(I,1),LOC(I,2),STRS(I,1)
 130   FORMAT (2X,3I4,2X,2E15.8)
      STOP
      END
C ====================================================================
C
C SUBROUTINE TRUSS
C
C ====================================================================
C     NN = NUMBER OF NODES,  NE = NUMBER OF ELEMENTS
C     ND = NUMBER OF DEGREES OF FREEDOM,  NB = SEMI-BANDWIDTH
C     M = NUMBER OF LOAD CONDITIONS, LOC = NODE CONNECTIVITY MATRIX
C     X,Y = X - AND Y- COORDINATES OF NODES
C     E = YOUNGS MODULUS, A = AREAS OF CROSS SECTION OF ELEMENTS
C     EL = LENGTHS OF ELEMENTS, NFIX = NUMBER OF D.O.F. WHICH ARE FIXED
C     IFIX = FIXED DEGREES OF FREEDOM NUMBERS, P = LOAD VECTOR
      SUBROUTINE TRUSS (NN,NE,ND,NB,M,LOC,X,Y,E,A,EL,NFIX,IFIX,P,GS,
     2 DIFF,GSS,ND2)
      DIMENSION LOC(NE,2),X(NN),Y(NN),A(NE),EL(NE),IFIX(NFIX),
     2 P(ND,M),GS(ND,NB),GSS(ND2,ND2)
      DIMENSION B(4,4),N(4)
      DOUBLE PRECISION DIFF(M)
      DO 5 I=1,ND
      DO 5 J=1,NB
5     GS(I,J)=0.0
      DO 6 I=1,ND2
      DO 6 J=1,ND2
 6    GSS(I,J)=0.0
      DO 200 K=1,NE
      I=LOC(K,1)
      J=LOC(K,2)
      EL(K)=SQRT((X(J)-X(I))**2+(Y(J)-Y(I))**2)
      CON=A(K)*E/EL(K)
      XL=(X(J)-X(I))/EL(K)
      XM=(Y(J)-Y(I))/EL(K)
      B(1,1)=XL**2
      B(1,2)=XL*XM
      B(1,3)=-B(1,1)
      B(1,4)=-B(1,2)
      B(2,2)=XM**2
      B(2,3)=-XL*XM
      B(2,4)=-XM**2
      B(3,3)=XL**2
      B(3,4)=XL*XM
      B(4,4)=XM**2
      DO 10 II=1,4
      DO 10 JJ=1,II
```

```
 10    B(II,JJ)=B(JJ,II)
       DO 20 II=1,4
       DO 20 JJ=1,4
 20    B(II,JJ)=B(II,JJ)*CON
       DO 90 II=1,2
       N(II)=2*I-2+II
 90    N(II+2)=2*J-2+II
       DO 100 II=1,4
       DO 100 JJ=1,4
       IK=N(II)
       JK=N(JJ)
       IF (IK .GT. ND2 .OR. JK .GT. ND2) GO TO 91
       GSS(IK,JK)=GSS(IK,JK)+B(II,JJ)
 91    CONTINUE
       IN=JK-IK+1
       IF (IN .LE. 0) GO TO 100
       GS(IK,IN)=GS(IK,IN)+B(II,JJ)
 100   CONTINUE
 200   CONTINUE
       DO 110 II=1,NFIX
       IX=IFIX(II)
 110   GS(IX,1)=GS(IX,1)*1.0E6
       CALL DECOMP (ND,NB,GS)
       CALL SOLVE (ND,NB,M,GS,P,DIFF)
       RETURN
       END
```

NODAL DISPLACEMENTS

1	-0.116462e-14
2	0.232924e-08
3	-0.128993e-14
4	0.170199e-07
5	0.116462e-02
6	0.232925e-02
7	0.514656e-01
8	-0.703219e-01

STRESSES IN ELEMENTS

1	1	3	0.11180378e+04
2	3	2	0.38109762e-02
3	3	4	0.22360752e+04
4	2	4	-0.28284324e+04

12.46

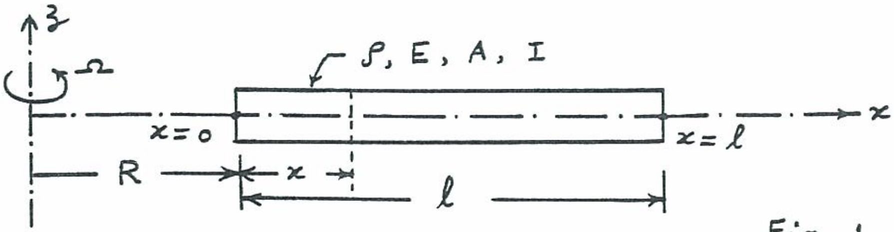


Fig. 1

(i) Rotating beam element

strain energy due to rotation:

The rotation of the beam induces an axial force P in the beam due to centrifugal action. If the beam bends in xz-plane as shown in Fig. 2, the change in the horizontal projection

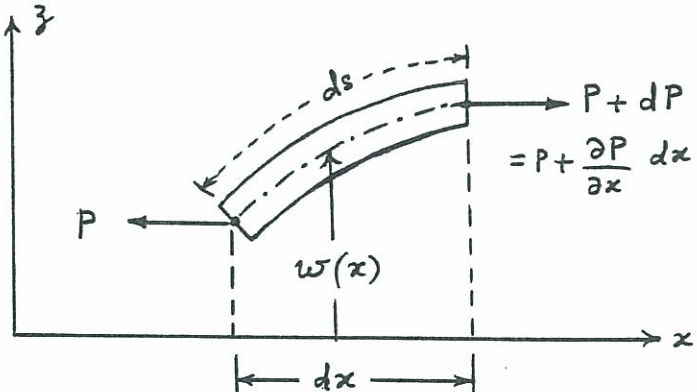


Fig. 2

of an element of length "ds" is given by

$$ds - dx = \left\{ (dx)^2 + \frac{\partial w}{\partial x} dx \right\}^{\frac{1}{2}} - dx \approx \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 dx \ldots (E_1)$$

Since the axial force P acts against the change in the horizontal projection, the work done by P is

$$\frac{1}{2} \int_0^l P(x) \left(\frac{\partial w}{\partial x} \right)^2 dx$$

where

$$P(x) = \int_{x_e + x}^{l + x_e} \frac{\rho A}{g} \Omega^2 \, \xi \, d\xi \qquad \text{where } \rho = \text{weight density}$$

$$\approx \frac{\rho A \Omega^2}{2g} \left[(l + R)^2 - (R + x)^2 \right] \qquad (E_2)$$

Work done by the transverse distributed force $p_w(x)$ can be expressed as

$$\int_0^l p_w(x) \, w(x) \, dx \qquad \text{where } p_w(x) = \frac{\rho A \Omega^2}{g} \cdot w(x) \qquad (E_3)$$

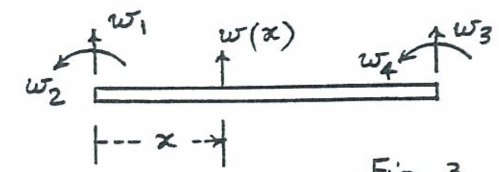


Fig. 3

If w_i, $i = 1, 2, 3, 4$ denote the nodal displacements of the beam element, the transverse displacement $w(x)$ can be expressed as

$$w(x,t) = \sum_{i=1}^{4} N_i(x)\, w_i(t) \qquad (E_4)$$

where $N_i(x)$ are the shape functions given by Eqs. (12.33) – (12.36). Introducing time variation of displacements, we get

kinetic energy of element is:

$$T(t) = \frac{1}{2}\int_0^l \rho A \left\{\frac{\partial w}{\partial t}(x,t)\right\}^2 dx \equiv \frac{1}{2}\,\dot{\vec{w}}^T [m]\, \dot{\vec{w}} \qquad (E_5)$$

Total strain energy of the element is

$$V(t) = \frac{1}{2}\int_0^l EI\left\{\frac{\partial^2 w(x,t)}{\partial x^2}\right\}^2 dx + \frac{1}{2}\int_0^l P(x)\left\{\frac{\partial w(x,t)}{\partial x}\right\}^2 dx - \int_0^l p_w(x,t)\cdot w(x,t)\, dx \equiv \frac{1}{2}\,\vec{w}^T [k]\, \vec{w} \qquad (E_6)$$

Total virtual work of element is

$$\delta W(t) = \int_0^l f(x,t)\, \delta w(x,t) \equiv \vec{f}(t)^T\, \delta\vec{w}(t) \qquad (E_7)$$

where $f(x,t)$ denotes the distributed force (which is zero in the present case).

Evaluation of integrals in Eqs. (E_5) – (E_7) enables us find the mass matrix, stiffness matrix and load vector.

(ii) For the helicopter blade, we can model it as one beam element for simplicity.

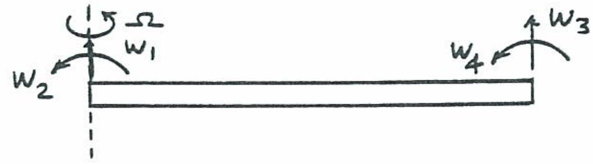
$\Omega = 300(2\pi)/60 = 31.416$ rad/sec

$A = 12\ in^2$, $I = \frac{1}{12}(12)(1)^3 = 1\ in^4$, $l = 48$ in,

$E = 10.3 \times 10^6$ psi, $\rho = 0.098\ lb/in^3$ (for aluminum).

Boundary conditions:

$$W_1 = W_2 = 0$$

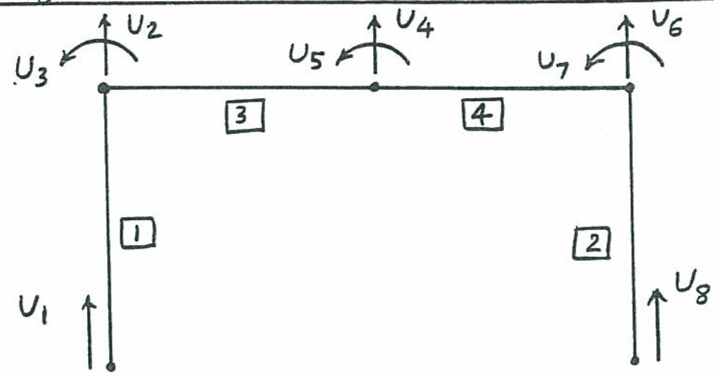


Solve the eigenvalue problem

$$\left[-\omega^2 [M] + [K]\right] \vec{w} = \vec{0} \qquad \text{where} \quad \vec{w} = \begin{Bmatrix} W_3 \\ W_4 \end{Bmatrix}$$

for the natural frequencies and mode shapes.

12.47



(a) Generate element matrices

$$\begin{array}{cc} & \begin{array}{cc} U_1 & U_2 \end{array} \\ [K^{(1)}] = \dfrac{A_1 E_1}{\ell_1} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} U_1 \\ U_2 \end{array} \end{array}, \qquad \begin{array}{cc} & \begin{array}{cc} U_8 & U_6 \end{array} \\ [K^{(2)}] = \dfrac{A_2 E_2}{\ell_2} & \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{array}{c} U_8 \\ U_6 \end{array} \end{array}$$

$$[K^{(3)}] = \frac{E_3 I_3}{\ell_3^3} \begin{array}{c} \begin{array}{cccc} U_2 & U_3 & U_4 & U_5 \end{array} \\ \begin{bmatrix} 12 & 6\ell_3 & -12 & 6\ell_3 \\ 6\ell_3 & 4\ell_3^2 & -6\ell_3 & 2\ell_3^2 \\ -12 & -6\ell_3 & 12 & -6\ell_3 \\ 6\ell_3 & 2\ell_3^2 & -6\ell_3 & 4\ell_3^2 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \\ U_4 \\ U_5 \end{array}$$

$$[K^{(4)}] = \frac{E_4 I_4}{\ell_4^3} \begin{array}{c} \begin{array}{cccc} U_4 & U_5 & U_6 & U_7 \end{array} \\ \begin{bmatrix} 12 & 6\ell_4 & -12 & 6\ell_4 \\ 6\ell_4 & 4\ell_4^2 & -6\ell_4 & 2\ell_4^2 \\ -12 & -6\ell_4 & 12 & -6\ell_4 \\ 6\ell_4 & 2\ell_4^2 & -6\ell_4 & 4\ell_4^2 \end{bmatrix} \end{array} \begin{array}{c} U_4 \\ U_5 \\ U_6 \\ U_7 \end{array}$$

$$\begin{array}{cc} & \begin{array}{cc} U_1 & U_2 \end{array} \\ [M^{(1)}] = \dfrac{\rho_1 A_1 \ell_1}{6} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{array}{c} U_1 \\ U_2 \end{array} \end{array}, \qquad \begin{array}{cc} & \begin{array}{cc} U_8 & U_6 \end{array} \\ [M^{(2)}] = \dfrac{\rho_2 A_2 \ell_2}{6} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{array}{c} U_8 \\ U_6 \end{array} \end{array}$$

$$[M^{(3)}] = \frac{\rho_3 A_3 l_3}{420} \begin{array}{c} \begin{array}{cccc} U_2 & U_3 & U_4 & U_5 \end{array} \\ \begin{bmatrix} 156 & 22 l_3 & 54 & -13 l_3 \\ 22 l_3 & 4 l_3^2 & 13 l_3 & -3 l_3^2 \\ 54 & 13 l_3 & 156 & -22 l_3 \\ -13 l_3 & -3 l_3^2 & -22 l_3 & 4 l_3^2 \end{bmatrix} \end{array} \begin{array}{c} U_2 \\ U_3 \\ U_4 \\ U_5 \end{array}$$

$$[M^{(4)}] = \frac{\rho_4 A_4 l_4}{420} \begin{array}{c} \begin{array}{cccc} U_4 & U_5 & U_6 & U_7 \end{array} \\ \begin{bmatrix} 156 & 22 l_4 & 54 & -13 l_4 \\ 22 l_4 & 4 l_4^2 & 13 l_4 & -3 l_4^2 \\ 54 & 13 l_4 & 156 & -22 l_4 \\ -13 l_4 & -3 l_4^2 & -22 l_4 & 4 l_4^2 \end{bmatrix} \end{array} \begin{array}{c} U_4 \\ U_5 \\ U_6 \\ U_7 \end{array}$$

where $\rho_1 = \rho_2 = 2.7 \times 10^{-3}$ lbm/in^3, $l_1 = l_2 = 108$ in,

$E_1 = E_2 = 4 \times 10^6$ psi, $A_1 = A_2 = \frac{\pi d^2}{4}$, $\rho_3 = \rho_4 = 8.8 \times 10^{-3}$ lbm/in^3,

$l_3 = l_4 = 108$ in, $E_3 = E_4 = 30 \times 10^6$ psi, $A_3 = A_4 = bh = 2b^2$,

$I_3 = I_4 = \frac{1}{12} bh^3 = \frac{2}{3} b^4$.

(b) Find the assembled stiffness and mass matrices after applying the boundary conditions $U_1 = U_8 = 0$.

(c) Select trial values of d and b and find the fundamental natural frequency (ω_1) by solving the eigenvalue problem

$$\left| -\omega^2 [M] + [K] \right| = 0$$

(d) change d and b until $\omega_1 > \frac{1500(2\pi)}{60} = 157.08$ rad/sec.