PHYS 569 HW6

1) a) R is defined as the radius of the circle formed at $w=0$, since $x|_{w=0} = R\cos\varphi$, $y|_{w=0} = R\sin\varphi$, $z|_{w=0} = 0$. Consider $\forall w \{-\frac{W}{2} < w < \frac{W}{2}\}$ to form a Möbius strip of radius R and width W. To show twisting consider the value of (x, y, z) at $(w, \varphi) = \lim_{b \to \frac{W}{2}} (b, 0)$, which sits on the perimeter of the Möbius strip. Here,

$$\boxed{x = R + \frac{W}{2},\ y = 0,\ z = 0}$$

Meanwhile, as $a \to 2\pi$, (x, y, z) at $(w, \varphi) = \lim_{a\to 2\pi} \lim_{b \to \frac{W}{2}} (b, a)$ gives the value

$$\boxed{x = R - \frac{W}{2},\ y = 0,\ z = 0}$$

This demonstrates a "twist", as a particle traveling along the perimeter of the Möbius strip travels from $x = R + \frac{W}{2}$ to $x = R - \frac{W}{2}$ after a full revolution and requires two revolutions to return to its starting point, which is expected behavior for a Möbius strip.

b) Given that $g_{AB} = \hat{e}^i_{(A)}\hat{e}^j_{(B)} g_{ij}$ and $\hat{e}^i_{(w)} = \frac{\partial x^i}{\partial w}$, $\hat{e}^i_{(\varphi)} = \frac{\partial x^i}{\partial \varphi}$, $g_{AB} = \hat{e}^i_{(w)}\hat{e}^j_{(\varphi)} g_{ij}$. $g_{ij} = \delta^i_j$, so

$$g_{ww} = \frac{\partial x}{\partial w}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}\frac{\partial y}{\partial w} + \frac{\partial z}{\partial w}\frac{\partial z}{\partial w} = (\cos\tfrac{\varphi}{2}\cos\varphi)^2 + (\cos\tfrac{\varphi}{2}\sin\varphi)^2 + (\sin\tfrac{\varphi}{2})^2 = 1$$

$$g_{w\varphi} = \frac{\partial x^i}{\partial w}\frac{\partial x^i}{\partial \varphi} = \frac{\partial x}{\partial w}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial w}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial w}\frac{\partial z}{\partial \varphi} = (\cos\tfrac{\varphi}{2}\cos\varphi)(-R\sin\varphi - \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\cos\varphi - w\cos\tfrac{\varphi}{2}\sin\varphi)$$
$$+ (\cos\tfrac{\varphi}{2}\sin\varphi)(R\cos\varphi - \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\sin\varphi + w\cos\tfrac{\varphi}{2}\cos\varphi) + (\sin\tfrac{\varphi}{2})(\tfrac{1}{2}w\cos\tfrac{\varphi}{2})$$

$$= \cancel{-R\sin\varphi\cos\varphi\cos\tfrac{\varphi}{2}} - \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\cos^2\varphi \cancel{- w\cos^2\tfrac{\varphi}{2}\sin\varphi\cos\varphi} + \cancel{R\sin\varphi\cos\varphi\cos\tfrac{\varphi}{2}} - \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\sin^2\varphi$$
$$+ \cancel{w\cos^2\tfrac{\varphi}{2}\sin\varphi\cos\varphi} + \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}$$
$$= \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}(1 - \cos^2\varphi - \sin^2\varphi) = 0$$

(It makes sense to define orthogonal coordinates, so I'm sure this is right, but "proof by prediction" isn't very rigorous)

$$g_{\varphi\varphi} = (-R\sin\varphi - \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\cos\varphi - w\cos\tfrac{\varphi}{2}\sin\varphi)^2 + (R\cos\varphi - \tfrac{1}{2}w\sin\tfrac{\varphi}{2}\sin\varphi + w\cos\tfrac{\varphi}{2}\cos\varphi)^2 + (\tfrac{1}{2}w\cos\tfrac{\varphi}{2})^2$$
$$= R^2\sin^2\varphi + \tfrac{1}{4}w^2\sin^2\tfrac{\varphi}{2}\cos^2\varphi + w^2\cos^2\tfrac{\varphi}{2}\sin^2\varphi + \cancel{Rw\sin\tfrac{\varphi}{2}\sin\varphi\cos\varphi} + 2Rw\cos\tfrac{\varphi}{2}\sin^2\varphi + \cancel{w^2\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\sin\varphi\cos\varphi}$$
$$+ R^2\cos^2\varphi + \tfrac{1}{4}w^2\sin^2\tfrac{\varphi}{2}\sin^2\varphi + w^2\cos^2\tfrac{\varphi}{2}\cos^2\varphi \cancel{- Rw\sin\tfrac{\varphi}{2}\sin\varphi\cos\varphi} + 2Rw\cos\tfrac{\varphi}{2}\cos^2\varphi \cancel{- w^2\sin\tfrac{\varphi}{2}\cos\tfrac{\varphi}{2}\sin\varphi\cos\varphi}$$
$$+ \tfrac{1}{4}w^2\cos^2\tfrac{\varphi}{2}$$
$$= R^2(\sin^2\varphi + \cos^2\varphi) + \frac{w^2}{4}(\cos^2\tfrac{\varphi}{2} + \sin^2\tfrac{\varphi}{2}(\cos^2\varphi + \sin^2\varphi)) + w^2\cos^2\tfrac{\varphi}{2}(\sin^2\varphi + \cos^2\varphi) + 2Rw\cos\tfrac{\varphi}{2}(\sin^2\varphi + \cos^2\varphi)$$

$$= R^2 + 2Rw\cos\tfrac{\varphi}{2} + w^2\cos^2\tfrac{\varphi}{2} + \frac{w^2}{4} = (R + w\cos\tfrac{\varphi}{2})^2 + \frac{w^2}{4}$$

Therefore, the metric is $\boxed{g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & (R + w\cos\frac{\varphi}{2})^2 + \frac{w^2}{4} \end{pmatrix}}$ and the line element is $\boxed{ds^2 = dw^2 + ((R + w\cos\tfrac{\varphi}{2})^2 + \tfrac{w^2}{4})\,d\varphi^2}$

c) For there to be a continuous scalar field on the strip, the function must also satisfy the periodicity of the Möbius strip incorporating its characteristic "twist". We should be careful here; $\varphi = 2\pi$ is not defined, so like part a, we must speak of $\lim_{a \to 2\pi} a$. To satisfy the geometry of the Möbius strip, a function with a continuous scalar field is bound by the equation $\boxed{f(w, 0) = \lim_{a\to 2\pi} f(-w, a)\ \{-\tfrac{W}{2} < w < \tfrac{W}{2}\}}$

d) $\partial_w = \left(\frac{\partial x}{\partial w}\ \ \frac{\partial y}{\partial w}\ \ \frac{\partial z}{\partial w}\right) = \left(\cos\frac{\varphi}{2}\cos\varphi\ \ \cos\frac{\varphi}{2}\sin\varphi\ \ \sin\frac{\varphi}{2}\right)$ so $\partial_w|_{\varphi=0} = (1, 0, 0)$ and $\lim_{a\to 2\pi} \partial_w|_{\varphi = a} = (-1\ 0\ 0)$.

Therefore, $\boxed{\partial_w|_{(w,0)} = -\lim_{a\to 2\pi} \partial_w|_{(-w,a)}}$, which means a continuous scalar function (as specified in part c cannot have a continuous basis

e) g_{ww} and $g_{w\varphi}$ are constant

$$g_{\varphi\varphi}|_{(w,0)} = \left(R + w\cos\tfrac{0}{2}\right)^2 + \frac{w^2}{4} = (R+w)^2 + \frac{w^2}{4}$$

$$\lim_{a\to 2\pi} g_{\varphi\varphi}|_{(-w,a)} = \left(R + (-w)\cos\tfrac{2\pi}{2}\right)^2 + \frac{(-w)^2}{4} = (R+w)^2 + \frac{w^2}{4} = g_{\varphi\varphi}|_{(w,0)}$$

Thus, the induced metric is continuous on the Möbius strip

2) a) For fixed $\vec{r}$, the line element reduces to $ds^2 = -(1-\frac{2M}{r})dt^2$

The observer's proper time is given by $d\tau = \sqrt{-ds^2} = \sqrt{1-\frac{2M}{r}}\,dt$, so, setting $\tau = 0$ when $t=0$,

$\boxed{\tau = \sqrt{1-\frac{2M}{r}}\,t}$. A clock at the top of a tower of height h will have a proper time of $\sqrt{1-\frac{2M}{r+h}}$

Because $r+h > r$ and $1-\frac{2M}{r+h} > 1-\frac{2M}{r}$, more proper time will pass on the clock on top of the building, so $\boxed{\text{the clock on top of the building runs faster}}$ To measure the difference between the proper times of the clock on the ground ($\Delta\tau_1$) and on top of the building ($\Delta\tau_2$), use the equation

$\Delta\tau_2 - \Delta\tau_1 = \left(\sqrt{1-\frac{2M}{r+h}} - \sqrt{1-\frac{2M}{r}}\right)\Delta t$. This cannot be simplified as is, but we can approximate $r >> h$,

so $\sqrt{1-\frac{2M}{r+h}} = \sqrt{1-\frac{2M}{r}(1+\frac{h}{r})^{-1}} \approx \sqrt{1-\frac{2M}{r}(1-\frac{h}{r})} = \sqrt{1-\frac{2M}{r}+\frac{2Mh}{r^2}} = \sqrt{1-\frac{2M}{r}}\sqrt{1+\frac{2Mh/r^2}{1-2M/r}}$. Here, $d\tau = \sqrt{1-\frac{2M}{r}}\,dt > 0$,

so the additional $\frac{h}{r}$ term in the numerator means $\sqrt{1-\frac{2M}{r}}\sqrt{1+\frac{2Mh/r^2}{1-2M/r}} \approx \sqrt{1-\frac{2M}{r}}\left(1+\frac{Mh}{r^2-2Mr}\right)$

Therefore, $\Delta\tau_2 - \Delta\tau_1 \approx \left(\sqrt{1-\frac{2M}{r}}\left(1+\frac{Mh}{r^2-2Mr}\right) - \sqrt{1-\frac{2M}{r}}\right)\Delta t = \sqrt{1-\frac{2M}{r}}\frac{Mh}{r^2-2Mr}\Delta t = \frac{Mh\Delta t}{r\sqrt{r^2-2Mr}}$

$G=c=1$, so $M = \frac{GM}{c^2}$ and using $M = 6\cdot10^{24}$ kg, $r = 6\cdot10^6$ m, $h = 10^2$ m, and $\Delta\tau_2 - \Delta\tau_1 = 1$ s, we get

$\Delta t = \frac{r\sqrt{r^2-2GMr/c^2}}{GMh/c^2}(\Delta\tau_2-\Delta\tau_1) \approx 8\cdot10^{13}\text{ s} = \boxed{2.5\text{ Myr}}$

b) For constant rotation $\Omega = \frac{d\varphi}{dt}$, the line element becomes $ds^2 = -(1-\frac{2M}{r})dt^2 + r^2\sin^2\theta\, d\varphi^2$.

$r = r_0, \theta = \frac{\pi}{2}, d\varphi^2 = \Omega^2 dt^2$, so $ds^2 = -(1-\frac{2M}{r_0} - r_0^2\Omega^2)dt^2$ and similar to part a, $\boxed{\tau = -\int ds = \sqrt{1-\frac{2M}{r_0}-r_0^2\Omega^2}\,t}$

The proper time of a moving clock includes an additional $r_0^2\Omega^2$ term subtracted from 1, so it will have a lower value. Thus, $\boxed{\text{the moving clock runs slower}}$ than the stationary clock

c) Repeating the calculation from part a (without the approximations, since $h > r$. Therefore,

$\Delta t = (\Delta\tau_2-\Delta\tau_1)\left(\sqrt{1-\frac{2M}{r_0+h}} - \sqrt{1-\frac{2M}{r_0}}\right)^{-1} = (\Delta\tau_2-\Delta\tau_1)\left(\sqrt{1-\frac{2GM/c^2}{r_0+h}} - \sqrt{1-\frac{2GM/c^2}{r_0}}\right)^{-1} = \boxed{175\text{ s} \approx 3\text{ min}}$

meaning it takes 3 minutes for the clocks to desynchronize. Note that $\Delta\tau_2 - \Delta\tau_1 = 100$ ns and $h = 2\cdot10^7$ m

For motion-induced time shifts, we have

$\Delta t = (\Delta\tau_2-\Delta\tau_1)\left(\sqrt{1-\frac{2M}{r_0+h}} - \sqrt{1-\frac{2M}{r_0+h}-(r_0+h)^2\Omega^2}\right)^{-1} = (\Delta\tau_2-\Delta\tau_1)\left(\sqrt{1-\frac{2GM/c^2}{r_0+h}-\frac{(r_0+h)^2\Omega}{c^2}} - \sqrt{1-\frac{2GM/c^2}{r_0+h}}\right)^{-1}$

$= \boxed{2\cdot10^3\text{ s} \approx 33\text{ min}}$ where $\Omega = \frac{2\pi}{12\text{ hr}} \approx 1.5\cdot10^{-4}\text{ s}^{-1}$

The gravitational time shift desynchronizes the clocks at a rate of 600 ps/s and exceeds the 100 ns threshold in roughly 3 minutes

The motion induced time shift desynchronizes the clocks at a rate of 50 ps/s and exceeds the 100 ns threshold in roughly 33 minutes

Both relativistic time shifts need to be considered by the GPS, but the gravitational time shift is more important

d) Using proper time as our λ parameter, the geodesic equation becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

For the r coordinate, considering the only nonzero components of $d\tau^2$ are dt^2 and $d\varphi^2$, we get

$\frac{d^2r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{dt}{d\tau}\right)^2 + \Gamma^r_{\varphi\varphi}\left(\frac{d\varphi}{dt}\right)^2$. The metric near the Earth's surface is symmetric, so

$\Gamma^\mu_{\rho\rho} = -\frac{1}{2}(g_{\mu\mu})^{-1}\partial_\mu g_{\rho\rho}$. Therefore,

$\Gamma^r_{tt} = -\frac{1}{2}(g_{rr})^{-1}\frac{\partial g_{tt}}{\partial r} = -\frac{1}{2}\left(1+\frac{2M}{r}\right)^{-1}\frac{\partial}{\partial r}\left(\frac{2M}{r}-1\right) = \left(1+\frac{2M}{r}\right)^{-1}\left(\frac{M}{r^2}\right) \approx \left(1-\frac{2M}{r}\right)\left(\frac{M}{r^2}\right) \approx \frac{M}{r^2}$ (first-order) and $M \ll r$

$\Gamma^r_{\varphi\varphi} = -\frac{1}{2}(g_{rr})^{-1}\frac{g_{\varphi\varphi}}{\partial r} = -\frac{1}{2}\left(1+\frac{2M}{r}\right)^{-1}\frac{\partial}{\partial r}\left(r^2\sin^2\pi/2\right) = -\left(1+\frac{2M}{r}\right)^{-1} r \approx -\left(1-\frac{2M}{r}\right)r = 2M - r \approx -r$

Thus, for $r=r_0$, $\cancel{\frac{d^2r}{d\tau^2}}^{\,0} + \frac{M}{r_0^2}\left(\frac{dt}{d\tau}\right)^2 - r_0\left(\frac{d\varphi}{d\tau}\right)^2 = 0 \Rightarrow \frac{M}{r_0^2}dt^2 = r_0\, d\varphi^2$. $\frac{d\varphi}{dt} = \Omega$, so $\frac{M}{r_0^2} = r_0\Omega^2$, or

$\boxed{\Omega^2 = \frac{M}{r_0^3}}$. This is Kepler's 3rd Law (If you don't beleive me, consider a period $T = \frac{2\pi}{\Omega}$ and

add $G=1$ to get $\frac{4\pi^2}{T^2} = \frac{GM}{r_0^3} \Rightarrow \boxed{T^2 = \frac{4\pi^2 r_0^3}{GM}}$, which is a more conventional form of Kepler's 3rd Law

Also, this means $\boxed{\Omega(r_0) = \sqrt{\frac{M}{r_0^3}}}$