Lecture Week 4

Random number generation and Monte Carlo methods with Python

week 1-3 | week 4

$$\text{data} \xrightarrow{\text{statistics}} \text{model} \xrightarrow[\text{comp. mathematics}]{\text{simulation}} \text{approximation}$$

Recall on probability theory

- $(\Omega, \mathcal{A}, P)$ probability space
  - Ω ↳ set
  - $\mathcal{A}$ ↳ σ-algebra
  - P ↳ probability measure

- $X : \Omega \to \mathbb{R}$ random variable

  $\mathcal{B}(\mathbb{R}) = \sigma([a,b],\ a<b)$ Borel σ-algebra

  X is $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable, ie. $\forall B \in \mathcal{B}(\mathbb{R}) : \{\omega \in \Omega,\ X(\omega) \in B\} \in \mathcal{A}$

- $P_X(B) = P(\{\omega \in \Omega,\ X(\omega) \in B\})$ image measure $B \in \mathcal{B}(\mathbb{R})$

- g density of X

  $$P_X(B) = \int_B g(x)\,dx \quad , \quad B \in \mathcal{B}(\mathbb{R})$$

- $\mathbb{E}[X]$ expectation of X with

  $$\mathbb{E}[X] = \int_\Omega X\,dP = \int_{-\infty}^{\infty} x\,g(x)\,dx$$

Monte Carlo methods

Strong law of large numbers

- $(U^{(m)}, m \in \mathbb{N})$ sequence of independent $\mathcal{U}([0,1])$-distributed random variables

  strong law of large numbers
  $$\frac{1}{M} \sum_{i=1}^{M} f(U^{(i)}) \longrightarrow \mathbb{E}[f(U)] = \vartheta$$
- therefore for M sufficiently large
  $$\frac{1}{M} \sum_{i=1}^{M} f(U^{(i)}) \approx \mathbb{E}[f(U)]$$

Algorithm I

- Simulate $\frac{1}{M} \sum_{i=1}^{M} f(U^{(i)})$
- given : $M \in \mathbb{N}$ $\quad f: \mathbb{R} \to \mathbb{R}$

```
ϑ ← 0
for k = 0,..., M-1 do
    generate U ~ U([0,1))

    ϑ ← ϑ + f(U)
end for
return ϑ/M
```

Strong law of large numbers

- $\mathbb{E}[\mathbb{1}_A(U_1, U_2)] = \vartheta$ $\qquad A = \{(x,y) \in [0,1]^2, \ y \leq f(x)\}$
- $((U_1^{(m)}, U_2^{(m)}), m \in \mathbb{N})$ sequence of independent $\mathcal{U}([0,1)) \times \mathcal{U}([0,1))$ distributed random variables

  strong law of large numbers
  $$\frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_A (U_1^{(m)}, U_2^{(m)}) \longrightarrow \mathbb{E}[\mathbb{1}_A(U_1, U_2)] = \vartheta$$

- therefore for M sufficiently large

$$\frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_A\left(U_1^{(m)}, U_2^{(m)}\right) \approx \mathbb{E}[f(U)]$$

Algorithm II (Hit-or-Miss)

- Simulate $\frac{1}{M}\sum_{m=1}^{M} \mathbb{1}_A\left(U_1^{(m)}, U_2^{(m)}\right)$
- given: $M \in \mathbb{N}$, $f : \mathbb{R} \to \mathbb{R}$

```
count ← 0
for  m = 0, ..., M-1 do
     generate  U_1, U_2 ~ U([0,1])
     if  U_2 ≤ f(U_1)  then
           count ← count + 1
     end if
end for
return  count / M
```

Speed of convergence

$$\mathbb{E}\left[\left[\mathbb{E}[f(U)] - \frac{1}{M}\sum_{m=1}^{M} f(U^{(m)})\right]^2\right]^{1/2}$$

$$= \mathbb{E}\left[\left(\frac{1}{M}\sum_{m=1}^{M}\left(\mathbb{E}[f(U)] - f(U^{(m)})\right)\right)^2\right]^{1/2}$$

$$= \frac{1}{M}\,\mathbb{E}\left[\left(\sum_{m=1}^{M}\left(f(U^{(m)}) - \mathbb{E}[f(U^{(m)})]\right)\right)^2\right]^{1/2}$$

$$= \frac{1}{M}\,\mathrm{Var}\left[\sum_{m=1}^{M} f(U^{(m)})\right]^{1/2}$$

$$= \frac{1}{M}\left(\sum_{m=1}^{M} \mathrm{Var}\left[f(U^{(m)})\right]\right)^{1/2}$$

$$= \frac{1}{M}\left(M\, \mathrm{Var}[f(U)]\right)^{1/2}$$

$$= \frac{1}{\sqrt{M}}\left(\mathrm{Var}[f(U)]\right)^{1/2}$$