Analysis and Control of Systems Considering Signal Transmission Delays

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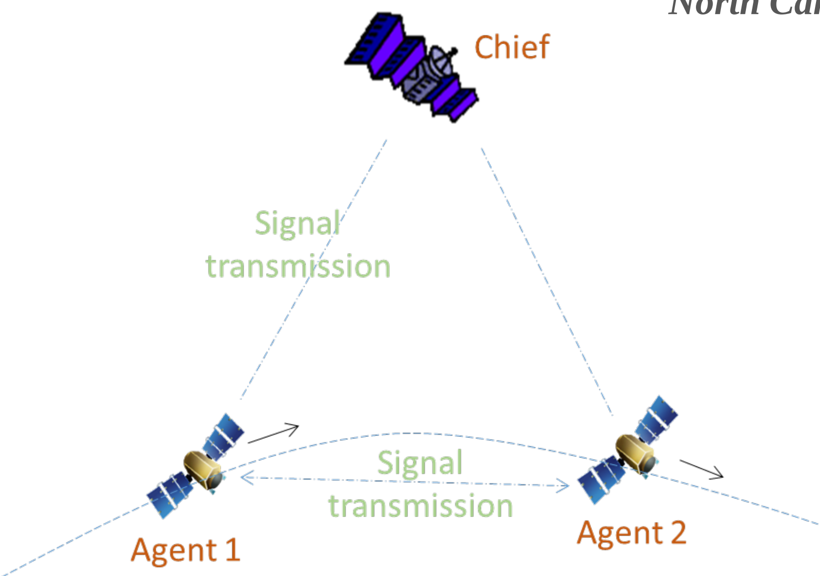
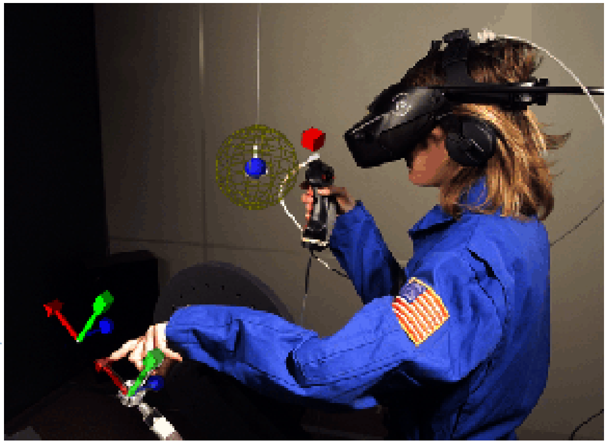
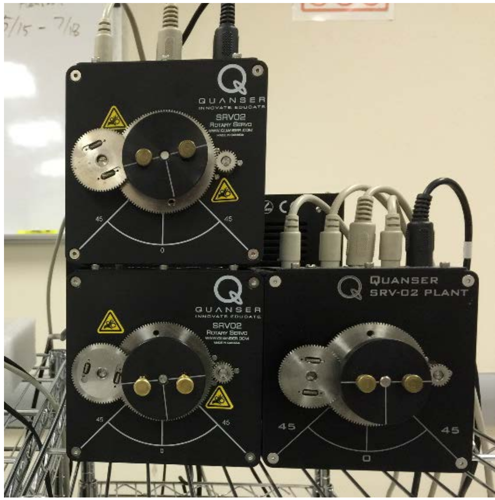


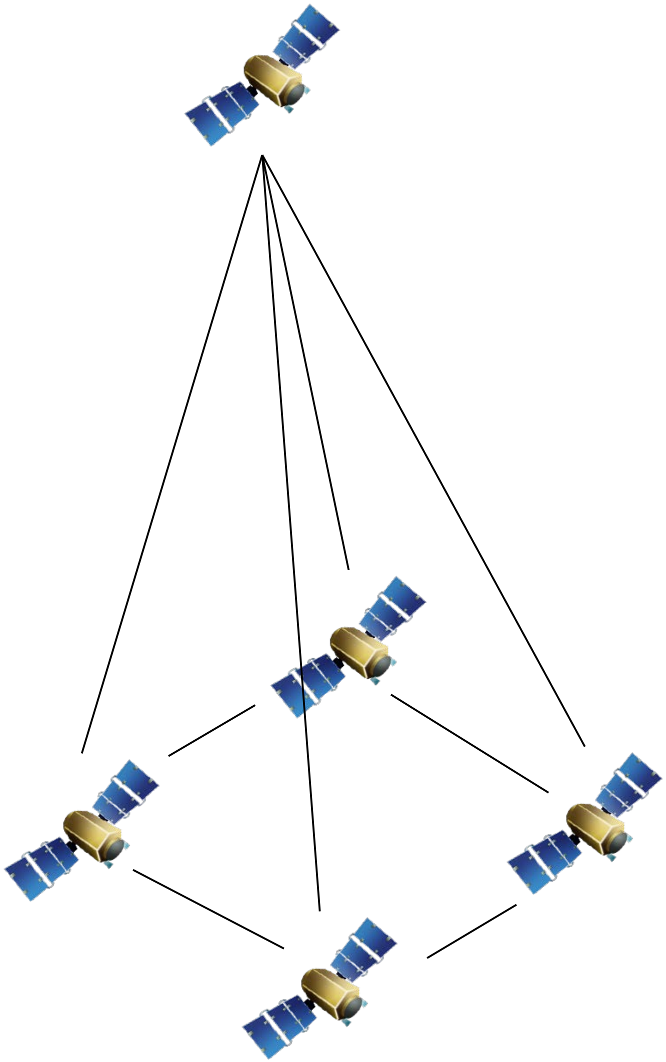




Outline

- Introduction
  - Motivations
  - Background
- Modeling of Delay Systems and Solution
- Methods for Analysis
- Methods for Control
- Concluding Remarks

Control for Formation Flying



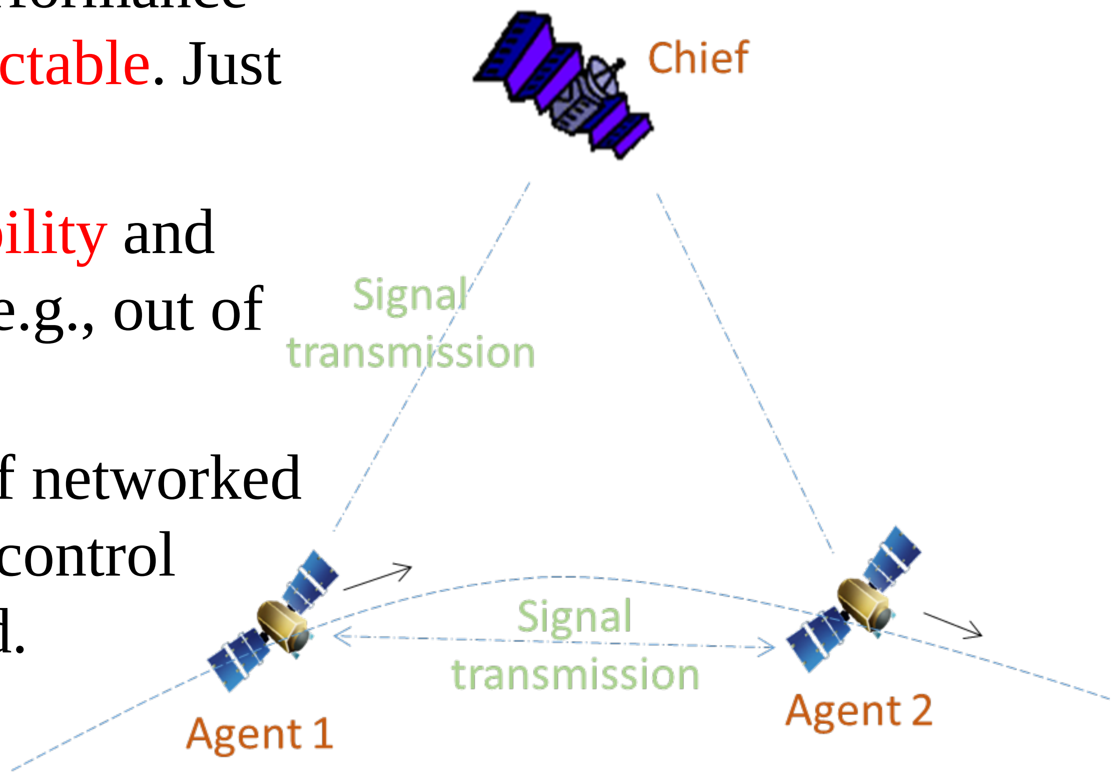
- Ever-increasing interest in ‘small’ satellites, such as micro-, nano- satellites,
- As a substitute for huge, complex and expensive monolithic satellites,
- Should be capable of maintaining inter-spacecraft distance accurately to operate as a virtual satellite with a large capability,
- Performance degradation that arises from the round-trip communication delay.

[Smith and Hadaegh, 2007]

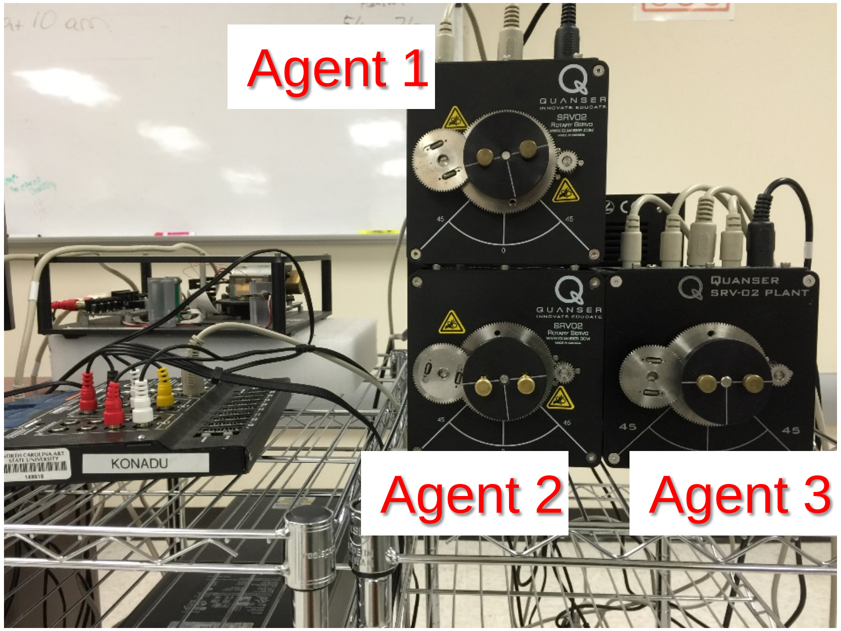
Networks of Agents: Intro

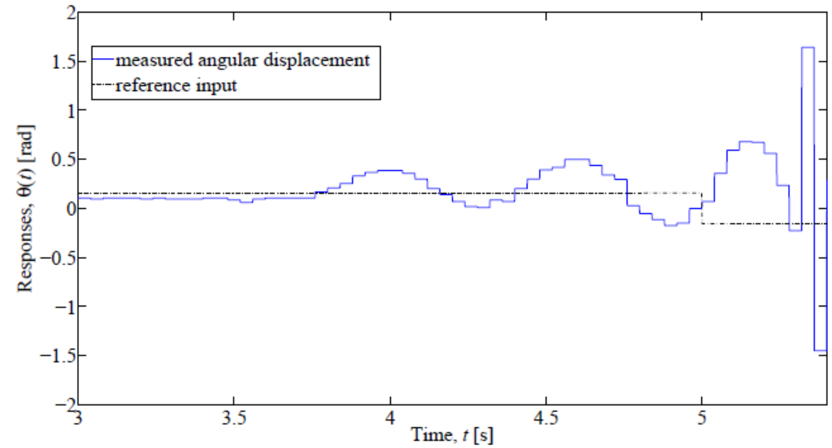
- Communications between agents often experience delays due to signal transmission, intermittent connectivity and link breakage.
- Their effects on system performance are not trivial but not predictable. Just known:
  - Delays can cause instability and unexpected behaviors (e.g., out of control).
- For reliable performance of networked multi -agent systems, new control strategy is critically needed.

$$\boldsymbol{\rho}(t) = \left| \mathbf{r}_{C,D}(t) - \mathbf{r}_{GPS}(t - t_d) \right|$$

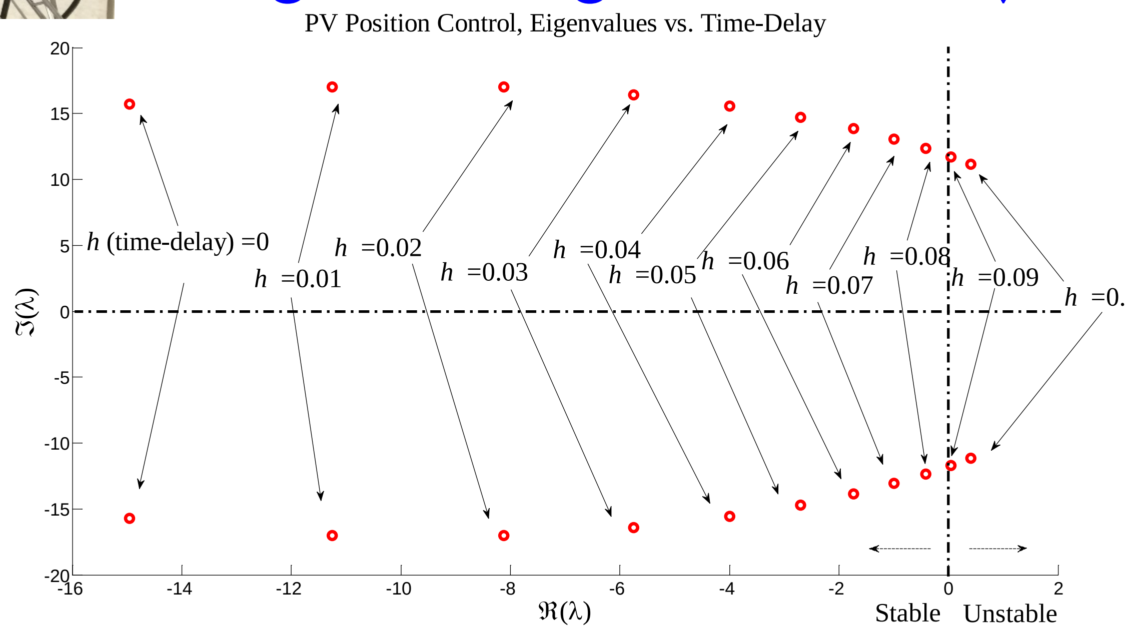


Effects of Time Delay on Synchronization





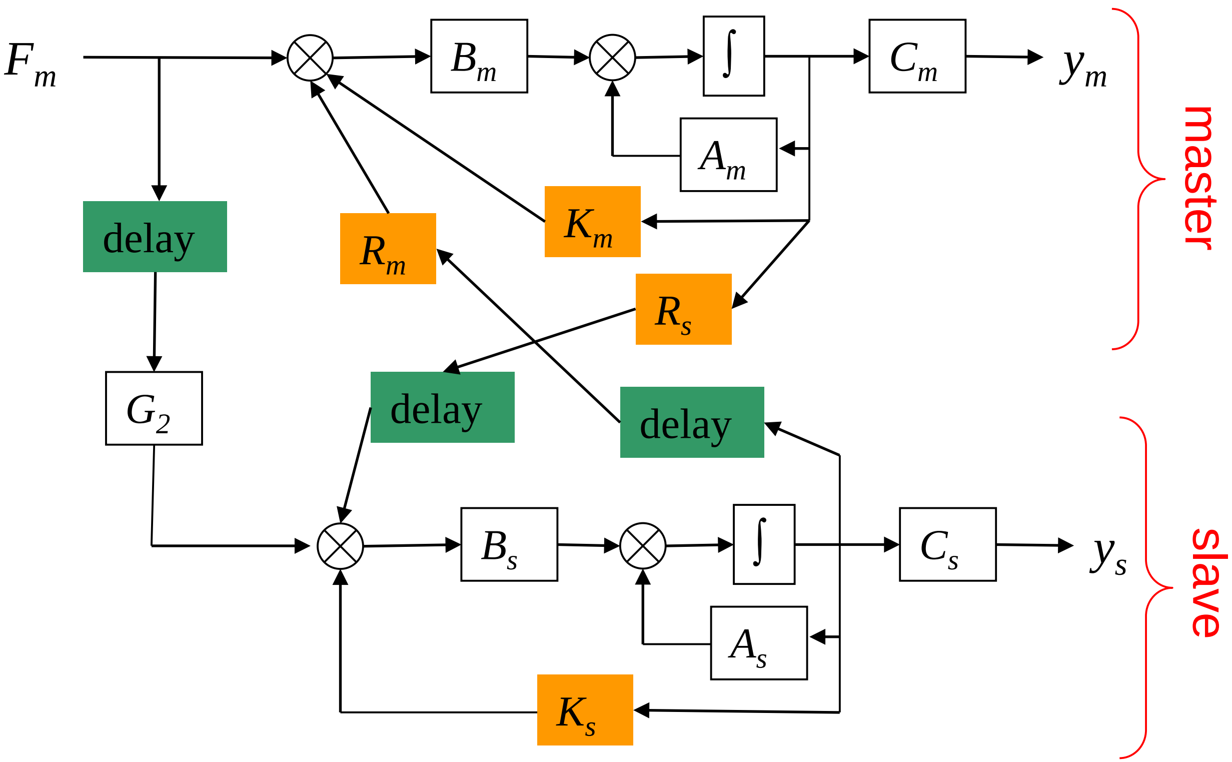
- As delay increases, the rightmost eigenvalues are ↓

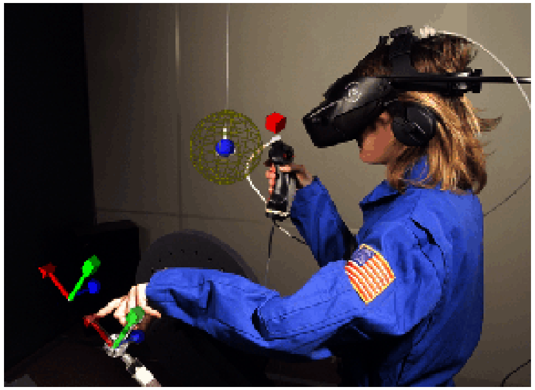


- Testbed for control of networked multi-agent systems

Tele-Operation

- Time Delay: information transmission between the local and remote environment (between “master” and “slave”).





from www.nasa.gov

* Padé approximation can’t meet requirement,

$\boxed{\mathrm{Re}\left(\lambda_{rightmost}\right) < -5}$

[Azorin, 04]

$$\begin{bmatrix}\dot{x}_s(t)\\ \dot{x}_m(t)\end{bmatrix} = \begin{bmatrix}A_s+B_sK_s & 0\\ 0 & A_m+B_mK_m\end{bmatrix}\begin{bmatrix}x_s(t)\\ x_m(t)\end{bmatrix} + \begin{bmatrix}0 & B_sR_s\\ B_mR_m & 0\end{bmatrix}\begin{bmatrix}x_s(t-T)\\ x_m(t-T)\end{bmatrix} + \begin{bmatrix}0\\ B_m\end{bmatrix}F_m(t) + \begin{bmatrix}B_sG_2\\ 0\end{bmatrix}F_m(t-T)$$

Delay Systems

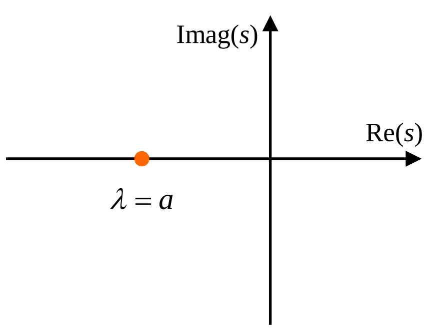
- Delays are inherent in many systems, e.g.
  - In engineering, biology, chemistry, economics, etc. [Niculescu, 2001]
  - Control loops: sensors, actuators, computational delays.

- What is challenging?
  - Delay operator leads to infinite spectrum due to

$$e^{-sh} = \sum_{k=0}^{\infty} \frac{(-sh)^k}{k!} = 1 - sh + \frac{1}{2!}(sh)^2 - \frac{1}{3!}(sh)^3 + \ldots$$

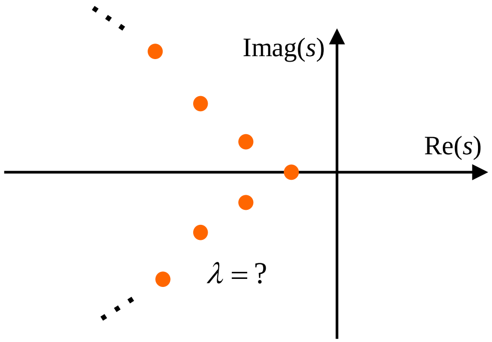
  - Difficulty in 1) determining stability, 2) designing controllers

$$\dot{x}(t) = ax(t)$$



Ordinary differential equations

$$\dot{x}(t) = ax(t) + \underbrace{a_d x(t-h)}_{\text{time delay}}$$



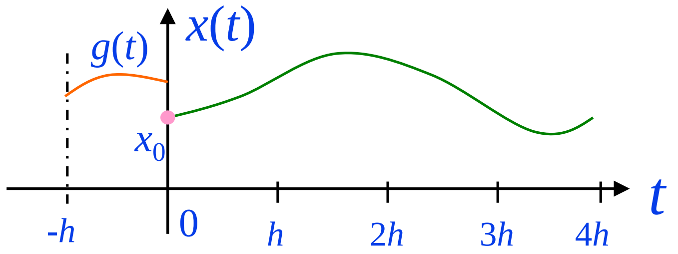
Delay differential equations

Delay Differential Equations (DDEs)

- History
  - 18th century: Laplace and Condorcet
  - Fundamental theory: Bellman [1963], Hale and Lunel [1993], etc.
  - Approximate, numerical, graphical methods

- Considered linear time-invariant systems with a single delay, *h*:

$$\begin{aligned}
&\dot{\mathbf{x}}(t) + \mathbf{A}\mathbf{x}(t) + \mathbf{A}_{\mathbf{d}}\mathbf{x}(t-h) = \mathbf{B}\mathbf{u}(t), && t > 0 \\
&\mathbf{x}(t) = \mathbf{g}(t), && t \in [-h, 0) \\
&\mathbf{x}(t) = \mathbf{x}_0, && t = 0
\end{aligned}$$



- This type of equations can represent those systems:
  - Tele-operation, formation flight, neural network, etc.
  - Machine tool chatter and wind turbines
  - Bio (HIV/HBV/HCV) dynamic model
  - Automotive powertrain systems control due to fluid transport

Existing Methods

- Representative current approaches:
  - Approximation, e.g., Padé approximation of the delay: $e^{-sh} \approx \dfrac{1-\frac{hs}{2}}{1+\frac{hs}{2}}$
  - Prediction-based methods, e.g., Smith predictor [Smith, 1958], finite spectrum assignment (FSA) [Manitius and Olbrot, 1979]
  - Bifurcation analysis, e.g., [Olgac et al., 1997] $s = \pm i v$
  - Numerical solutions, e.g., *dde23* in Matlab (Runge-Kutta)
  - Graphical methods, e.g., Nyquist [Desoer and Wu, 1968]
  - Lyapunov methods, e.g., LMI, ARE [Niculescu, 2001]
  - Great number of monographs devoted to this field of active research: [Richard, 2003; Yi et al., 2010]

Padé Approximation

$$e^{-sh}=\sum_{k=0}^{\infty}\frac{(-sh)^k}{k!}=1-sh+\frac{1}{2!}(sh)^2-\frac{1}{3!}(sh)^3+\ldots$$

$$\dot{x}(t)=ax(t)+\underbrace{a_d x(t-h)}_{\text{time-delay}}+kx(t)$$

The Padé approximation is

$$e^{-sh}=\frac{1-\frac{hs}{2}}{1+\frac{hs}{2}}$$

Then,

$$\underbrace{s-(a+k)-a_d e^{-sh}=0}_{\text{infinite dimensional}}\rightarrow\underbrace{s-(a+k)-a_d\frac{2-sh}{2+sh}=0}_{\text{finite dimensional}}$$

$$\rightarrow s(2+sh)-(a+k)(2+sh)-a_d(2-sh)=0$$

$$\rightarrow s^2h+s(2-ah-kh+a_dh)-2(a+k)-2a_d=0$$

DDE becomes a simple 2nd order ODE

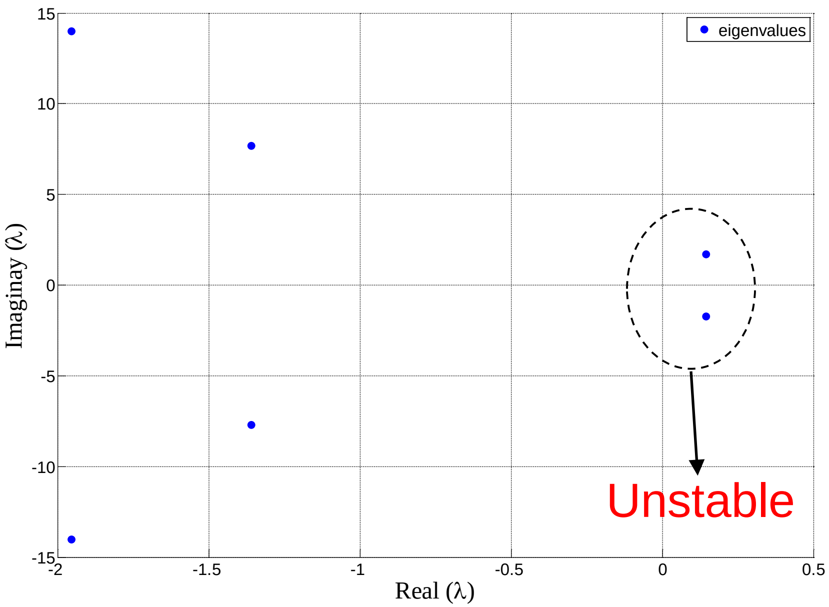
Padé Approximation

- With the parameters $a = 1, a_d = -2, h = 1$

$$s^2 h + s(2 - ah - kh + a_d h) - 2(a + k) - 2a_d = 0$$
$$\rightarrow s^2 + s(-1 - k) + 2 - 2k = 0$$

- For the value of k= -1.1, the above 2[nd] order equation has two stable poles, but the gain is applied to the original system and causes to instability.

But the system was still unstable



Rational Function Approximation

- Higher order Padé:

$$e^{-hs} \cong \frac{1 - \frac{hs}{2} + \frac{(hs)^2}{12}}{1 + \frac{hs}{2} + \frac{(hs)^2}{12}}$$

- Battle & Miralles (2000)

$$e^{-hs} \cong \frac{p(-hs)}{p(hs)}, \quad p(hs) = 6\pi^4 + \pi^4 hs + 6\pi^2 h^2 s^2 + h^3 s^3 \pi^2 - 24 h^3 s^3$$

- Inappropriate to be used in designing feedback control strategies, because the approximation methods cannot be free from errors.

Rational Function Approximation

- "Approximating the delay term by means of rational function by truncating infinite series, such as Padé approximation, constitutes a limitation in analysis and may lead to <u>unstable behaviors</u> of the true system."

e.g.) [1] J. P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *Automatica,* vol. 39, pp. 1667-1964, 2003.
[2] G. J. Silva and Datta, A. Bhattacharyya, S.P., "Controller design via pade approximation can lead to instability," in *Proceedings of the 40th IEEE Conference on Decision and Control* 2001, pp. 4733-4737.
[3] *PID controllers for time-delay systems* /Guillermo J. Silva, Aniruddha Datta, S.P. Bhattacharyya.. Boston : Birkhäuser, c2005
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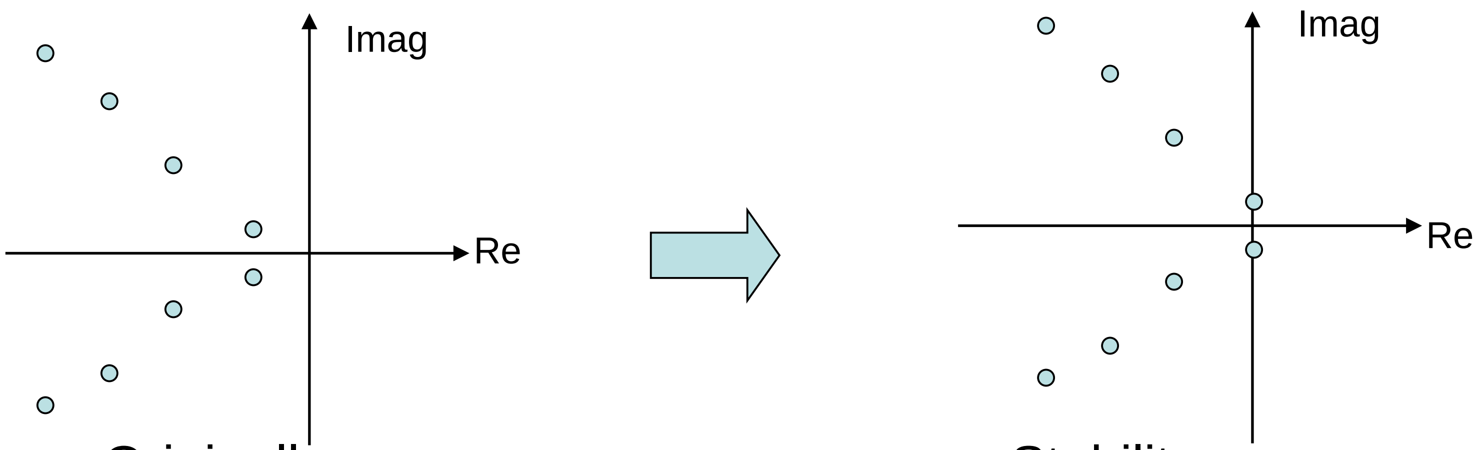
Bifurcation Analysis



Originally stable

Stability may change.

$$e^{-sh} = \sum_{k=0}^{\infty} \frac{(-sh)^k}{k!} = 1 - sh + \frac{1}{2!}(sh)^2 - \frac{1}{3!}(sh)^3 + \ldots$$

$$s = \pm iv$$

Bifurcation analysis

- Given a system of delay differential equation with stability for τ=0, and with the characteristic equation

$$\sum_{i=1}^{N} a_i \lambda^i + e^{-\lambda\tau} \sum_{i=1}^{M} b_i \lambda^i = 0$$

- If the Eq. has a pure imaginary root, *iv*,

$$P_1(iv) + P_2(iv)e^{-iv\tau} = 0$$

- Separated into its real and imaginary parts

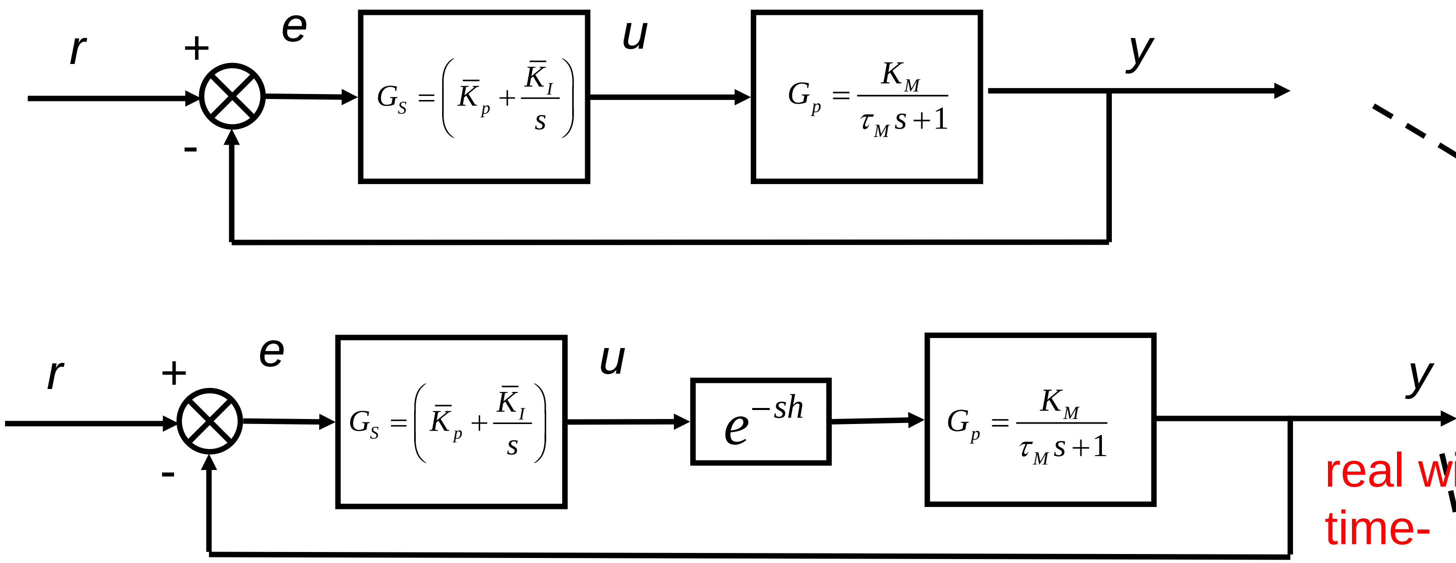
$$R_1(v) + iQ_1(v) + (R_2(v) + iQ_2(v))(\cos(v\tau) - i\sin(v\tau)) = 0$$

- So,

$$R_1(v) + R_2(v)\cos(v\tau) + Q_2(v)\sin(v\tau) = 0$$
$$Q_1(v) - R_2(v)\sin(v\tau) + Q_2(v)\cos(v\tau) = 0$$

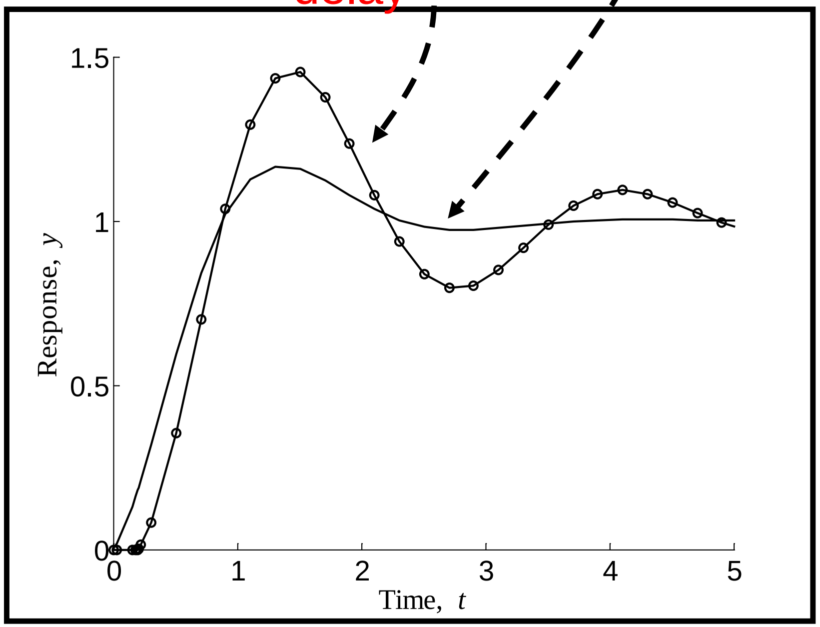
- If there is a solution, *v*, then stability may change.

Smith Predictor



desired

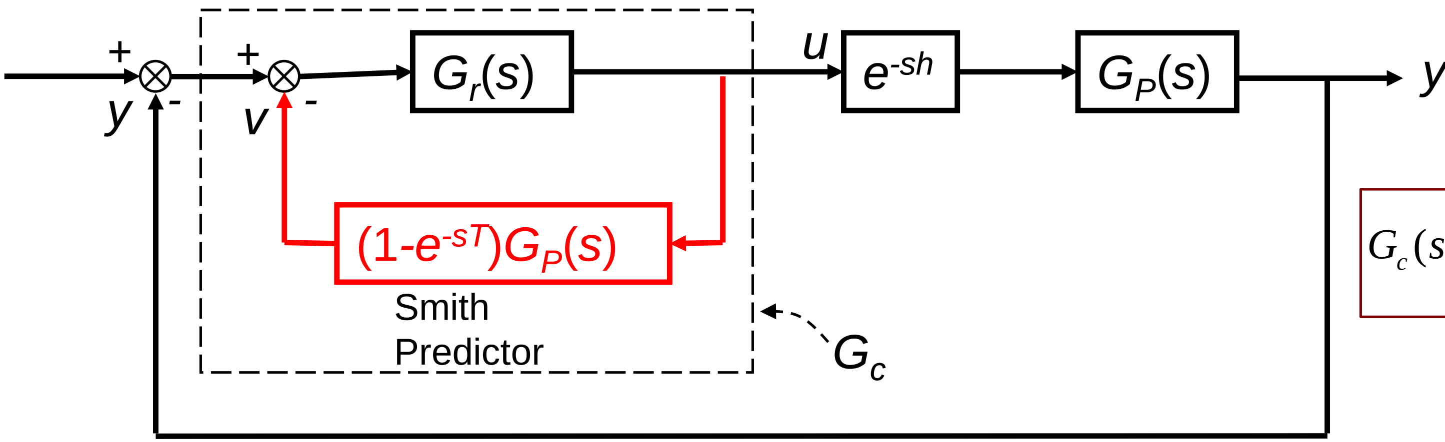
real with time-delay



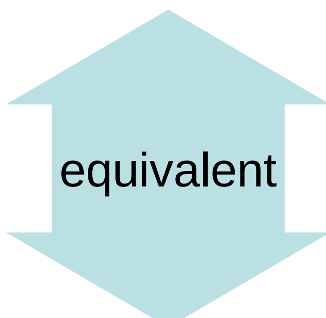
* Two responses show <u>disagreement</u> due to <u>delay</u>

where $\tau_M = 0.5; h = 0.2; \zeta = 0.5; \omega_n = 2.5; K_M = 1$

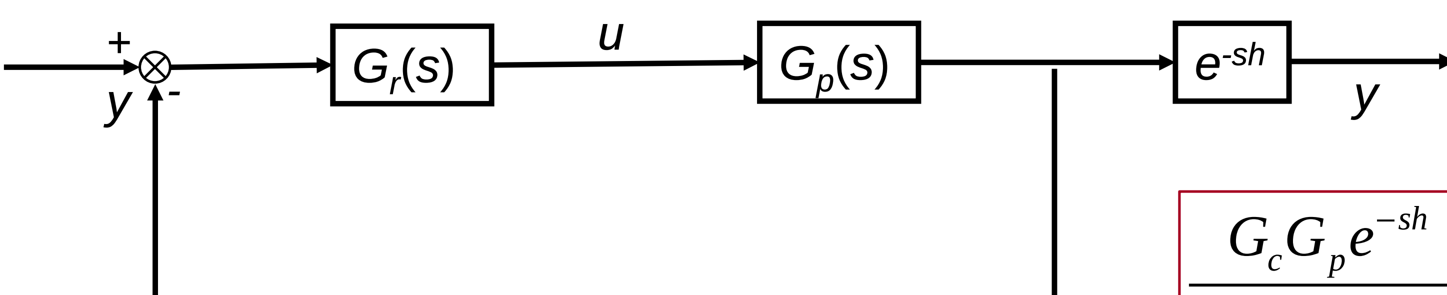
Smith Predictor: Basic Concept



$$G_c(s) = \frac{G_r}{1 + G_r G_p - G_r G_p e^{-sh}}$$

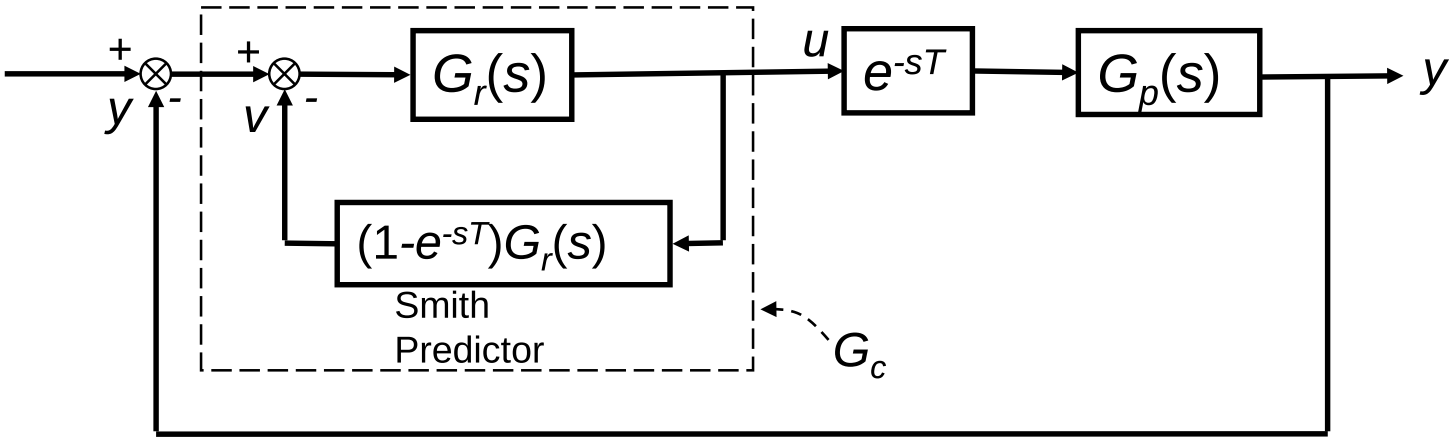


* Delay is moved <u>outside</u> the feedback loop.



$$\frac{G_c G_p e^{-sh}}{1 + G_c G_p e^{-sh}} = e^{-sh} \frac{G_r G_p}{1 + G_r G_p}$$

Smith Predictor: Basic concept (II)



*** Why it is a "predictor":**

$$
\begin{aligned}
y+v &= e^{-sT}G_p(s)u+\left(1-e^{-sT}\right)G_p(s)u \\
&= G_p(s)u \\
&= G_p(s)e^{-sT}u(s)\cdot e^{sT} \\
&= \underbrace{y\cdot e^{sT}}_{y(t+T)}
\end{aligned}
$$

(→ preceded response)

$$\frac{G_c G_p e^{-sT}}{1+G_c G_p e^{-sT}} = e^{-sT}\frac{G_r G_p}{1+G_r G_p}$$

$$\Downarrow$$

$$G_c(s) = \frac{G_r}{1+G_r G_p - G_r G_p e^{-sT}}$$

Discretizing

- Discretizing method
  - $\boxed{\text{Delay}: Z\left\{\delta(t-1)\right\} = \dfrac{1}{z} \Rightarrow \text{rational function of } z.}$
  - Can handle delay terms in a easy way,
  - If time step is small (for accuracy) and delay time, *h*, is large, then the dimension of the delayed system becomes high (n=100, 1000,….).
  - Need processes:

System in *t*-domain → System in *z*-domain → Design feedback controller in *z*-domain → System in *t*-domain

✓ Also, if delays are uncertain, or vary over time, analysis using discretization may not be possible.

Outline

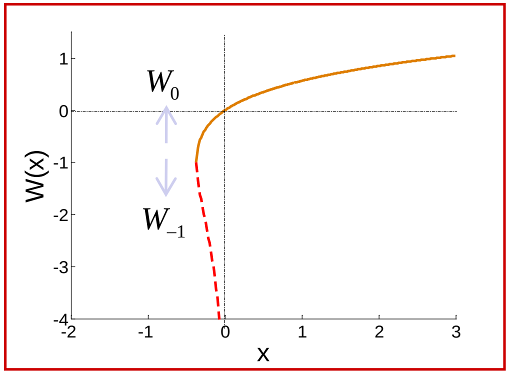
- Introduction
- Modeling of Delay Systems and Solution
  - Modeling
  - Solution to DDEs
- Methods for Analysis
- Methods for Control
- Concluding Remarks

Strategy

Ordinary differential equations	Delay differential equations
• Solution $\left(e.g.,\ \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0\right)$ ⬇ • Stability ⬇ • Controllability & Observability ⬇ • Design of feedback control (w/ observer) ⬇ • Transient response & robust control	• Solution (?) ?

The Lambert W Function

- Definition: $W_k(x)e^{W_k(x)} = x$ $(k = -\infty, \cdots, -1,\ 0,\ 1, \cdots,\ \infty)$



- Infinite number of branches

Principal branch $(k = 0)$

$$W_0(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n+1}}{n!} x^n$$

$(k \neq 0)$

$$W_k(x) = \ln_k(x) - \ln(\ln_k(x)) + \sum_{l=0}^{\infty}\sum_{m=1}^{\infty} c_{lm} \frac{(\ln(\ln_k(x)))^m}{(\ln_k(x))^{l+m}}$$

where, $c_{lm} = \frac{1}{m!}(-1)^l \begin{bmatrix} l+m \\ l+1 \end{bmatrix}$ (Stirling Cycle Numbers)

$$\ln_k(x) = \ln(x) + 2\pi i k$$

- Already embedded in MATLAB, Maple, Mathematica, etc.
- Contributions: Lambert [1758], Euler [1779], Corless et al., [1996], Asl and Ulsoy [2003]…
- Used to study jet fuel, combustion, enzyme, molecular forces.

Solution of DDEs (Free Scalar)

- Scalar (first-order) delay differential equation (DDE)

$$\boxed{\begin{aligned} &\dot{x}(t)+ax(t)+a_d x(t-h)=0, && \text{for } t>0 \\ &x(t)=g(t), && \text{for } t\in[-h,0) \\ &x(t)=x_0, && \text{for } t=0 \end{aligned}}$$

- Obtain a transcendental characteristic equation: $s+a+a_d e^{-sh}=0$

$\left(\text{multiplying } he^{h(s+a)}\right)$

$$h(s+a)e^{h(s+a)}=-a_d he^{ah}$$

$$W\left(-a_d he^{ah}\right)e^{W\left(-a_d he^{ah}\right)}=-a_d he^{ah}$$

$$\left(s+a\right)h=W\left(-a_d he^{ah}\right)$$

$$\boxed{W(x)e^{W(x)}=x}$$

Using the definition of the “Lambert W function “

$$s=\frac{1}{h}W\left(-a_d he^{ah}\right)-a$$

Roots in terms of the parameters, a, a_d, h !

* [Asl & Ulsoy, *ASME JDSMC*, 2003]

Example: Free Scalar DDEs

$$\dot{x}(t) + ax(t) + a_d x(t-h) = 0, \quad \text{for } t > 0$$
$$x(t) = g(t), \quad \text{for } t \in [-h, 0)$$
$$x(t) = x_0, \quad \text{for } t = 0$$

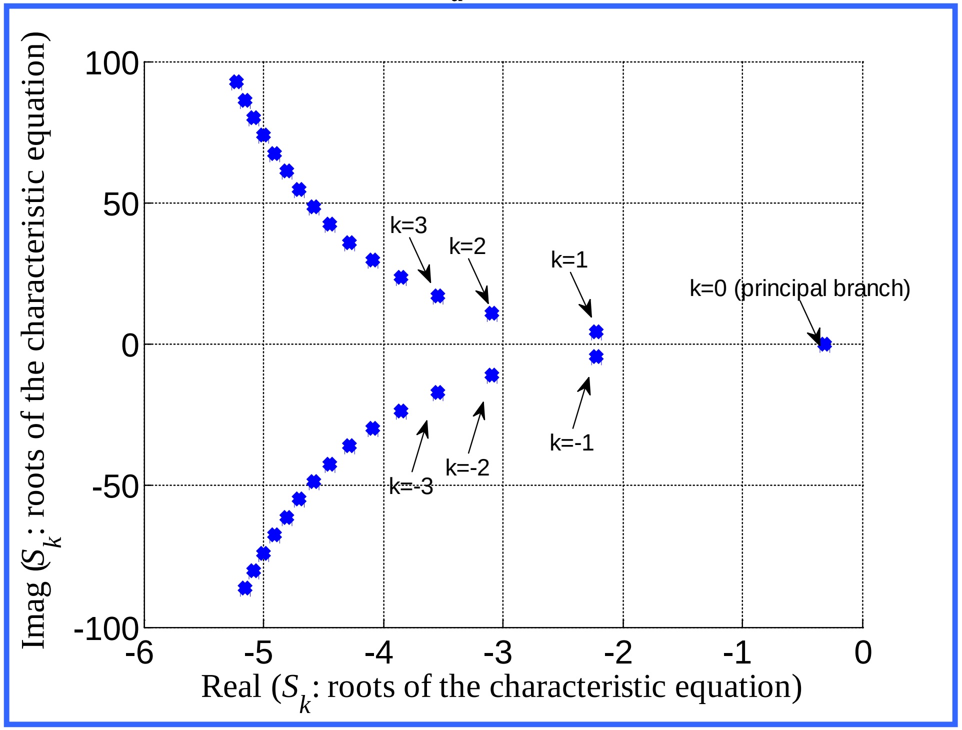
$$S_k = \frac{1}{h} W_k\left(-a_d h e^{ah}\right) - a$$

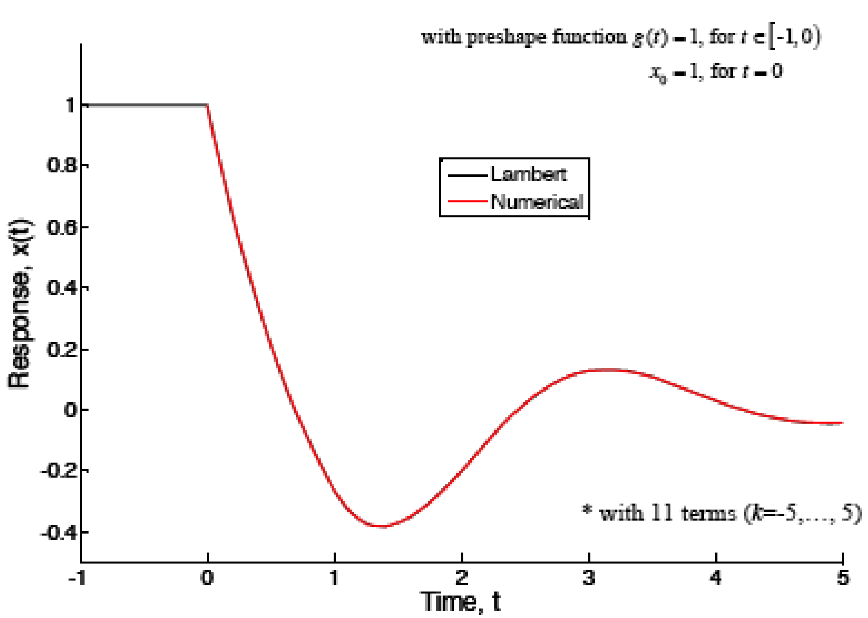
Poles

$$x(t) = \sum_{k=-\infty}^{\infty} e^{s_k t} C_k^I$$

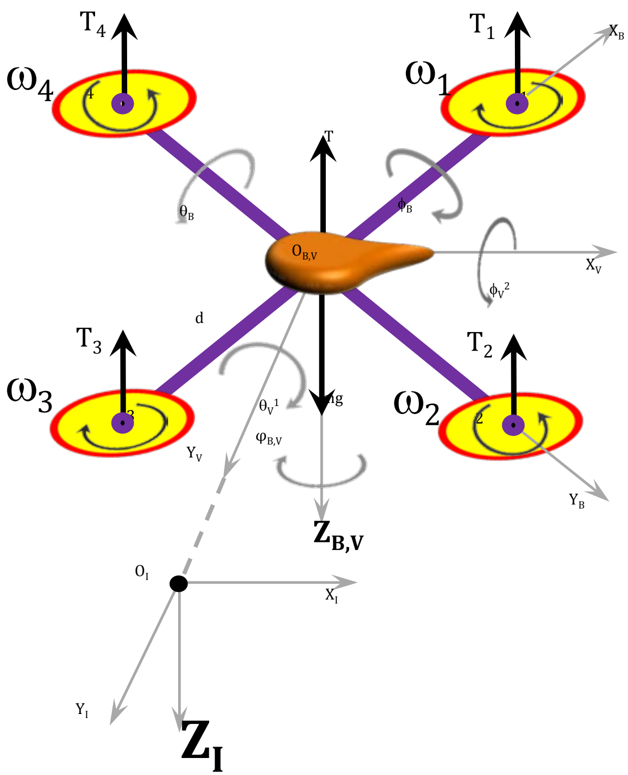
Analytical Solutions

ex) $a = 1$, $a_d = -0.5$, $h = 1$





Quadrotor Dynamics



- Total upward thrust, T, on the vehicle is

$$T = \sum_{i=1}^{i=4} T_i, \quad T_i = a\omega_i^2 \qquad a > 0$$

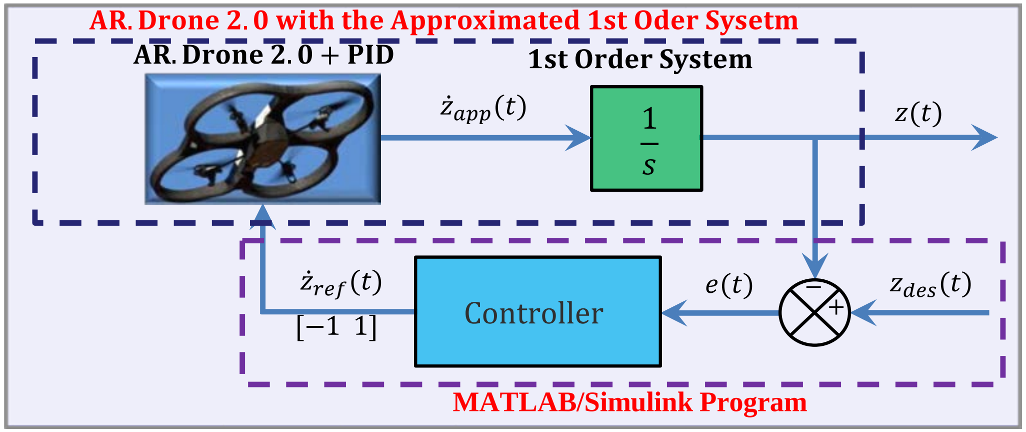
- Function of rotor speed, ω.
- The thrust constant, a, depends on the air density, the cube of the blade radius, etc.
- Taking the equation of motion only in the z-direction:

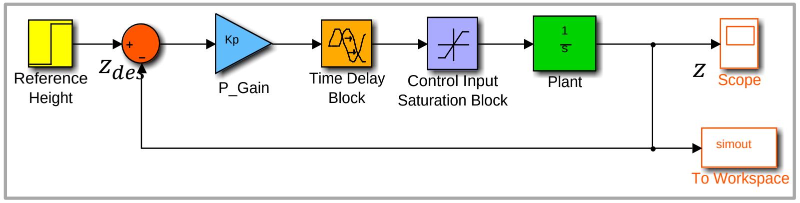
$$\ddot{z}(t) = g - \frac{4a\omega^2(\mathrm{t})}{m}$$

- Nonlinearity, noise, connection, disturbance, etc.

Control Overview

- Used ‘AR Drone Simulink Development Kit V1.1’ from Mathworks.com.
- Cascade control is beneficial only if the dynamics of the inner loop are fast compared to those of the outer loop.

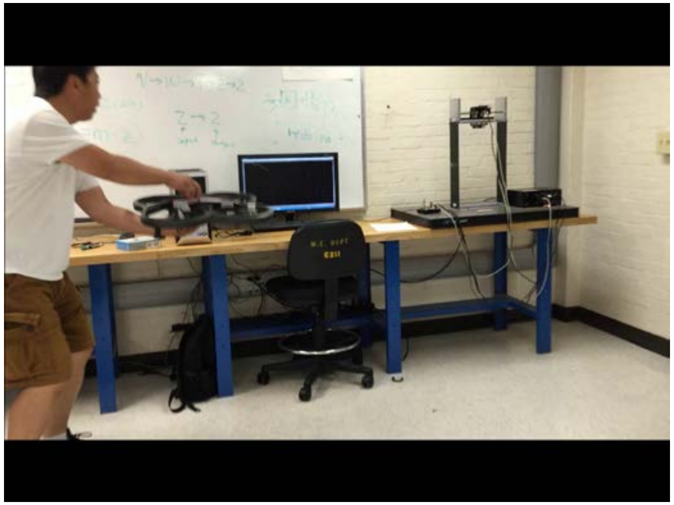


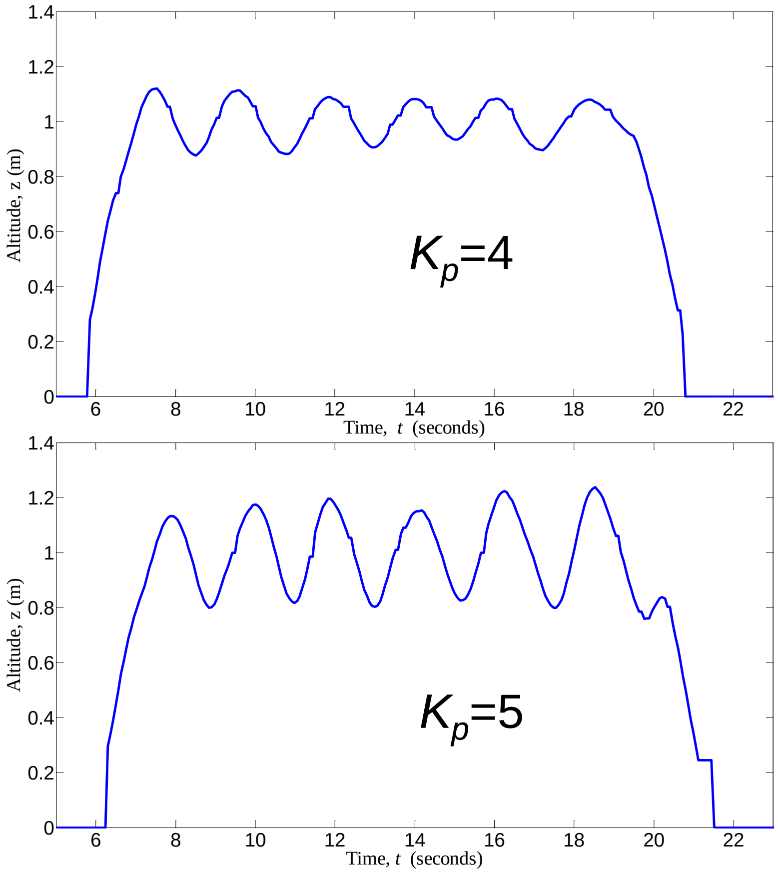


- For autonomous aerial robots, typical delay is around $0.4 \pm 0.2s$. Oscillations can emerge due to delays [Vasarhelyi, 2014].
- Large delays ($> 0.20s$) causes increased torque dramatically [Ailon, 2014].

Delay Observation

- If there is no delay, increase in the gain, K_p, does not destabilize the system.
- In receiving data,





- In order to design effective controllers (stability boundary), knowing the delay is helpful.

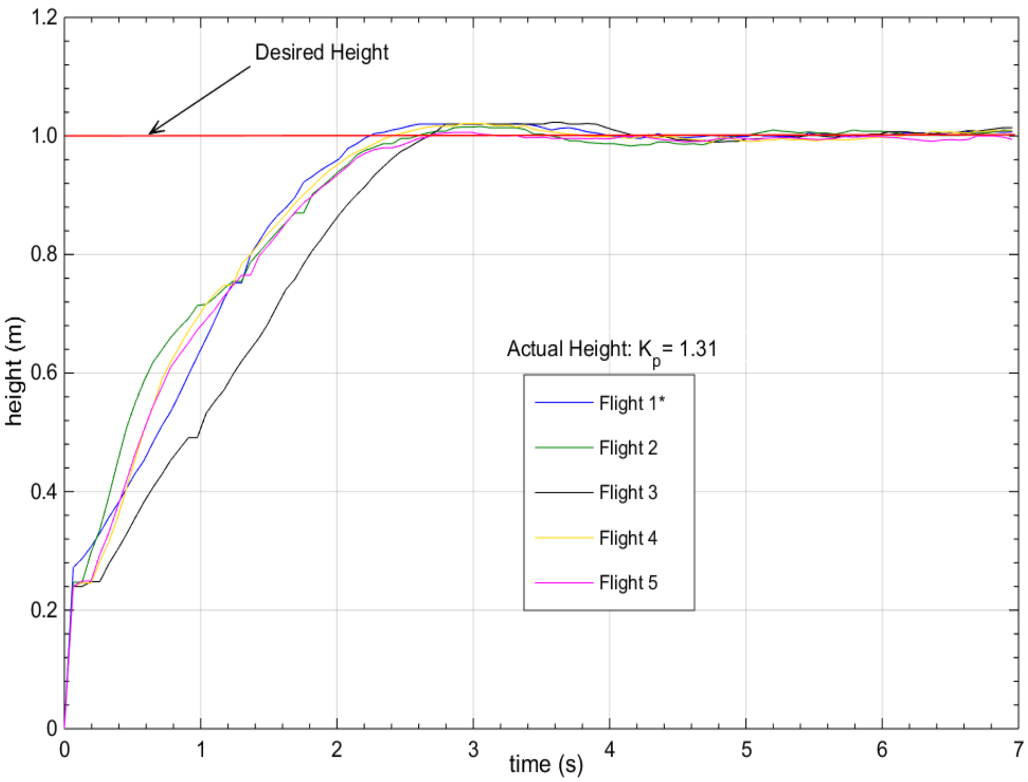
Time-Delay Estimation

- Delays are introduced by
  - Computation loads, actuation, and signal transmission/processing.
- Estimation is not straightforward
  - Even if considerable efforts have been made, there is no common approach. [Belkoura et. al. (2009)]
  - “Most of the existing methods for transfer function identification do not consider the process delay (or dead-time) or just assume knowledge of the delay” [Ljung (1985); Sagara & Zhao (1990)]
  - “None of the existing linear filter methods directly estimate the time delay except for some very special type of excitation” [Ahmed et al. (2006)]
- The delay knowledge can benefit [Belkoura and Richard (2006)]
  - Many of control techniques (e.g., Smith predictors, Finite Spectrum Assignment, etc.)

Methods for Estimation of Delays

- Finite-dimensional Chebyshev spectral continuous time approximation (CTA) [Torkamani and Butcher (2013)]
- Graphical methods [Mamat and Fleming (1995); Rangaiah and Krishnaswamy (1996); Ahmed et al. (2006)]
- Integral equation approach [Wang and Zhang (2001)]: integrate signals
- Discrete and Continuous ("more familiar to practicing control engineers" [Ahmed et al. (2006)])
- A cost function for a set of time delays in a certain range [Rao and Sivakumar (1976); Saha and Rao (1983)]
- Approximation: Padé approximation, the Laguerre expansion, etc. → additional parameters, errors
- Frequency-domain maximum likelihood [Pintelon and Biesen (1990)]

Altitude Response: Percent Overshoot



	Flight				
	1	2	3	4	5
M_o (%)	2.300*	2.290	2.300	2.270	<1

$$M_o = 100e^{\left(\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}\right)}$$

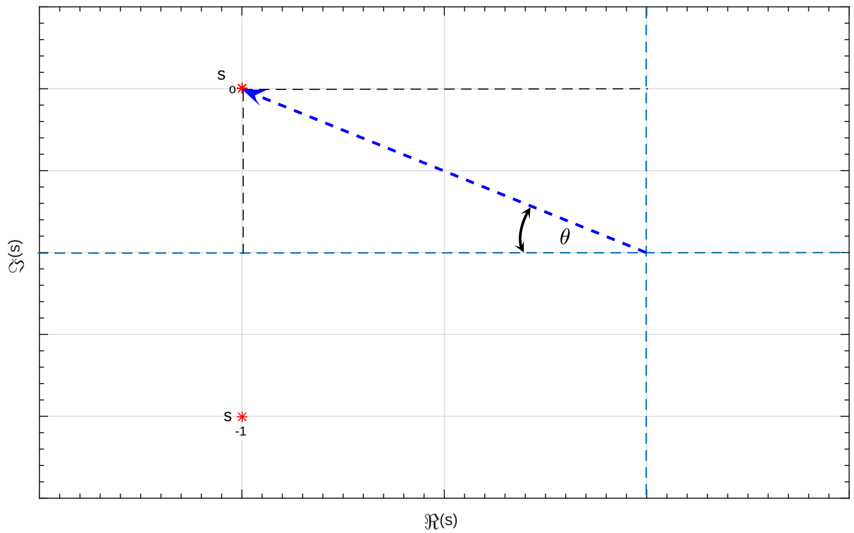
$$\zeta = 0.7684 = -\frac{\mathrm{Re}(s)}{\sqrt{\mathrm{Re}^2(s)+\mathrm{Im}^2(s)}}$$

$$s = \frac{1}{T_d}W\left(T_d a_1 e^{-a_o T_d}\right) + a_o$$

$$T_d = 0.3598 \approx 0.36s$$

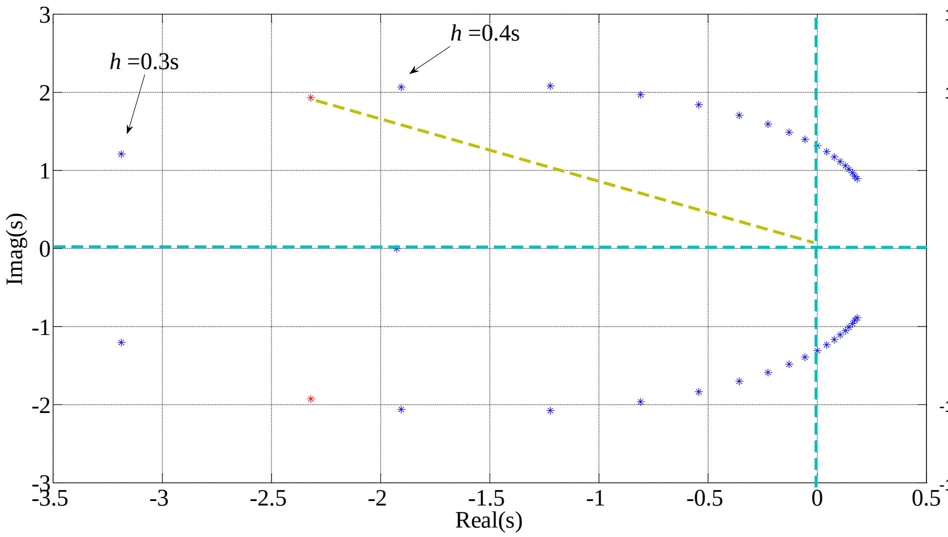
*When compared to Simulink, T_d=0.37s.

Locus of Rightmost Roots

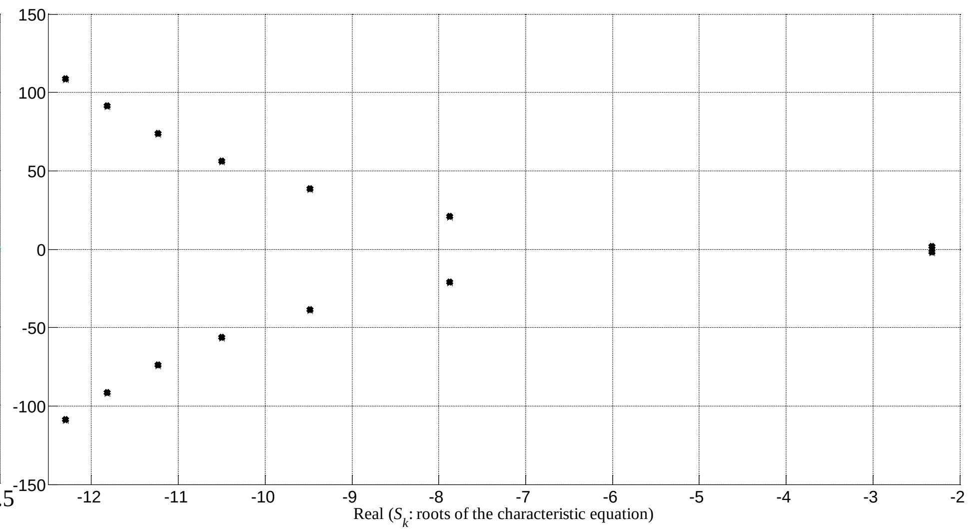


$$\xi = \cos\theta = \frac{|Re(s)|}{\sqrt{(Re(s))^2 + (Im(s))^2}}$$

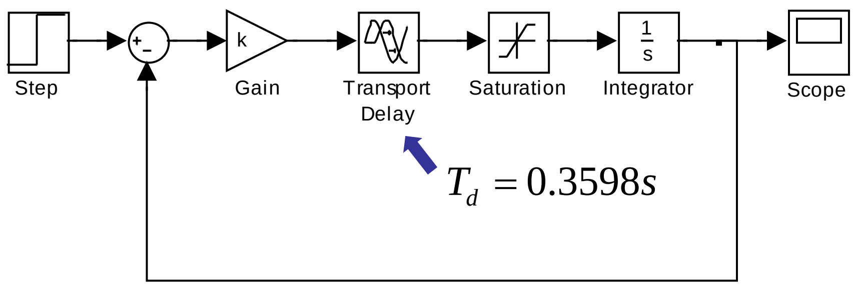
- As the time delay increases

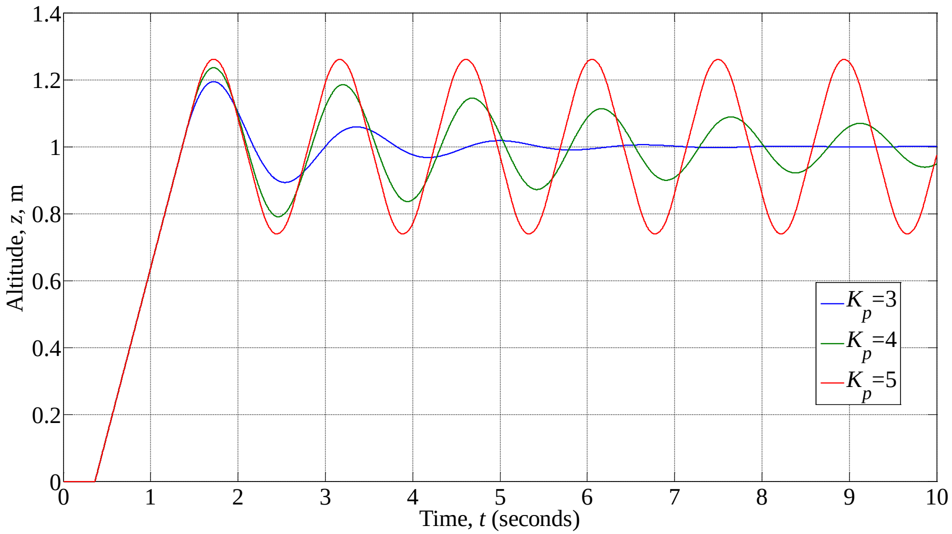


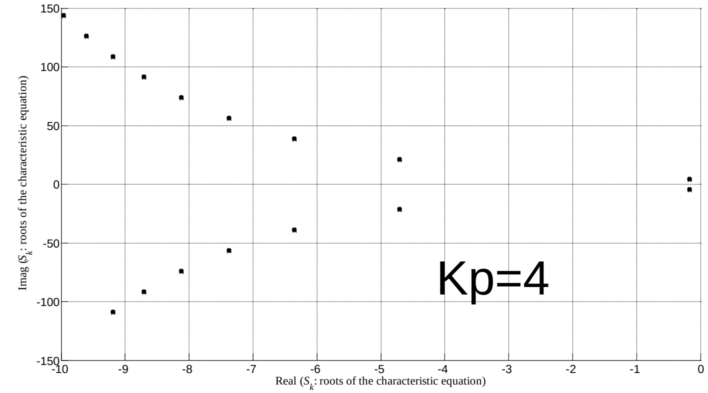
- Spectrum when $h = 0.3598s$

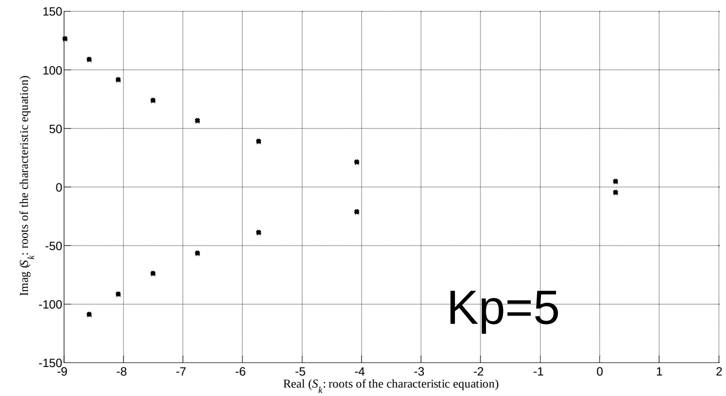


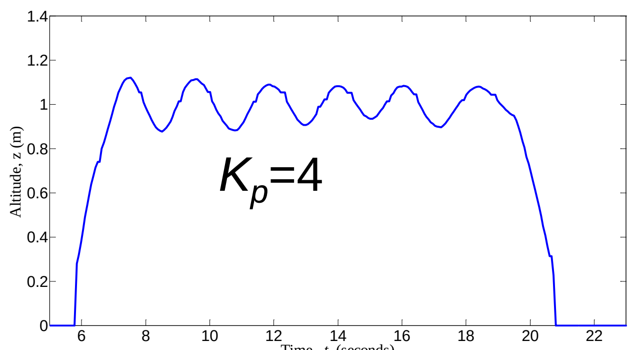
Stability Using the Estimated Delay

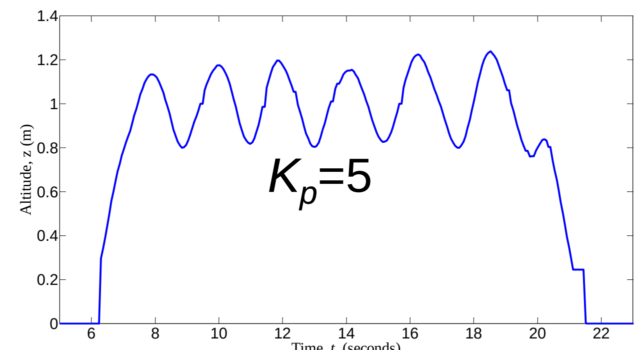












Non-homogeneous DDEs

Free solution
[Asl & Ulsoy, *JDSMC* 2003]

+

Forced Soln. Form
[Bellman *et al*. 1963]

$$x(t)=\int_0^t \Psi(t,\xi)bu(\xi)d\xi$$

Solution form

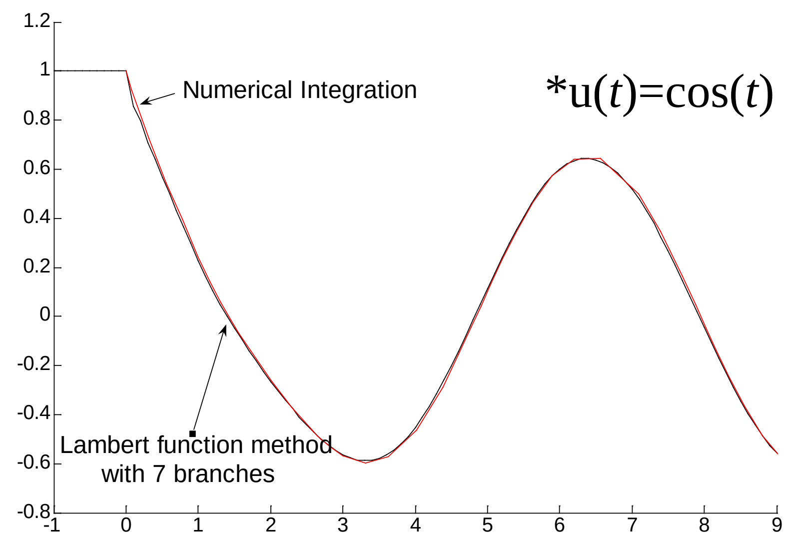
Condition

$$\begin{aligned}
&a)\quad \frac{\partial}{\partial \xi}\Psi(t,\xi)=a\Psi(t,\xi) && t-h\le \xi<t\\
&\qquad\qquad\qquad\quad =a\Psi(t,\xi)+a_d\Psi(t,\xi+T) && \xi<t-h\\
&b)\quad \Psi(t,t)=1\\
&c)\quad \Psi(t,\xi)=0 \qquad \text{for } \xi>t
\end{aligned}$$

External force

$$\begin{aligned}
&\dot{x}(t)+ax(t)+a_d x(t-h)=\boxed{bu(t)}, && t>0\\
&x(t)=g(t), && t\in[-h,0)\\
&x(t)=x_0, && t=0
\end{aligned}$$

$$x(t)=\underbrace{\sum_{k=-\infty}^{\infty} e^{S_k t}C_k^I}_{\substack{\text{free solution}\\ \text{(Asl and Ulsoy, 2003)}}}+\underbrace{\int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\xi)}C_k^N bu(\xi)d\xi}_{\substack{\text{forced solution}\\ \text{(Yi et al., 2006)}}}$$



*u(t)=cos(t)

*[Yi et. al., *CDC* 2006]

Generalized to Systems of DDE's

$$\boxed{\begin{aligned} &\dot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)+\mathbf{A_d}\mathbf{x}(t-h)=\mathbf{B}\mathbf{u}(t), && t>0\\ &\mathbf{x}(t)=\mathbf{g}(t), && t\in[-h,0)\\ &\mathbf{x}(t)=\mathbf{x}_0, && t=0 \end{aligned}}$$

$$^{*}\mathbf{x}(t)=\begin{Bmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t)\end{Bmatrix}, \mathbf{A}\in\mathbb{R}^{n\times n}, \mathbf{A_d}\in\mathbb{R}^{n\times n}$$

$$\underline{\text{Because } \mathbf{A}\times\mathbf{A_d}\neq\mathbf{A_d}\times\mathbf{A}\Rightarrow e^{\mathbf{A_d}h}e^{\mathbf{A}h}\neq e^{(\mathbf{A_d}+\mathbf{A})h}}$$

$$x(t)=\sum_{k=-\infty}^{\infty} e^{\left(\frac{1}{h}W_k(-a_d h e^{ah})-a\right)t} C_k^I$$

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty} e^{\left(\frac{1}{h}\mathbf{W}_k(-\mathbf{A_d}he^{\mathbf{A}h})-\mathbf{A}\right)t}\mathbf{C}_k^I$$

$$\mathbf{x}(t)=\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I=\sum_{k=-\infty}^{\infty} e^{\left(\frac{1}{h}\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q}_k)-\mathbf{A}\right)t}\mathbf{C}_k^I$$

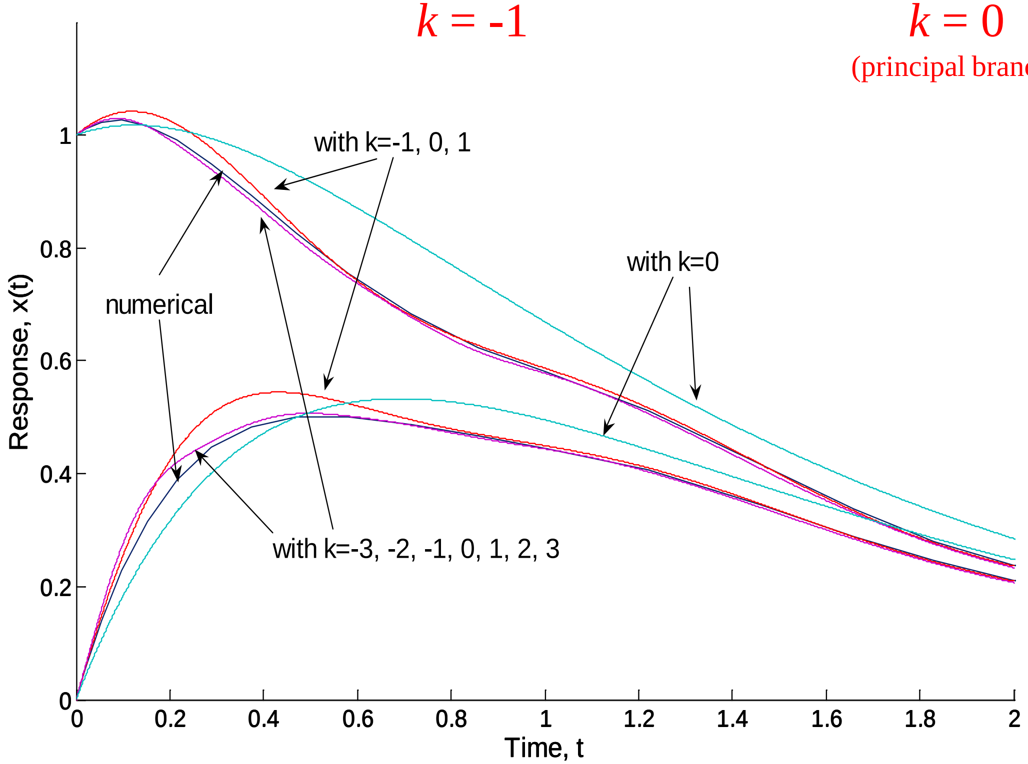
$$^{*}\left(\mathbf{Q}_k \quad \text{satisfies} \quad '\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q}_k)e^{\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q}_k)-\mathbf{A}h}=-\mathbf{A_d}h'\right)$$

$$\boxed{\mathbf{x}(t)=\underbrace{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I}_{\text{free}}+\underbrace{\int_0^t\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^N\mathbf{B}\mathbf{u}(\xi)\,d\xi}_{\text{forced}}}$$

Systems of DDEs: Example

$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{Bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -5 \end{bmatrix} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} + \begin{bmatrix} 1.66 & -0.697 \\ 0.93 & -0.330 \end{bmatrix} \begin{Bmatrix} x_1(t-1) \\ x_2(t-1) \end{Bmatrix}$$

$$\mathbf{x}(t) = \cdots + \underbrace{e^{\begin{bmatrix} -0.3499-4.9801i & -1.6253+0.1459i \\ 2.4174+0.1308i & -5.1048-4.5592i \end{bmatrix}t} \mathbf{C}_{-1}^I}_{k\,=\,-1} + \underbrace{e^{\begin{bmatrix} 0.3055 & -1.4150 \\ 2.1317 & -3.3015 \end{bmatrix}t} \mathbf{C}_0^I}_{k\,=\,0\ \text{(principal branch)}} + \underbrace{e^{\begin{bmatrix} -0.3499+4.9801i & -1.6253-0.1459i \\ 2.4174-0.1308i & -5.1048+4.5592i \end{bmatrix}t} \mathbf{C}_1^I}_{k\,=\,1} + \cdots$$



$$\vdots$$

$$\mathbf{C}_{-1}^I = \begin{Bmatrix} 1.3663+3.9491i \\ 3.2931+9.3999i \end{Bmatrix},$$

$$\mathbf{C}_{0}^I = \begin{Bmatrix} -1.7327+0.0000i \\ -6.5863+0.0000i \end{Bmatrix},$$

$$\mathbf{C}_{1}^I = \begin{Bmatrix} 1.3663-3.9491i \\ 3.2931-9.3999i \end{Bmatrix},$$

$$\vdots$$

Analogy to Systems of ODEs

ODEs	DDEs (Using Lambert W function)
Scalar case $\dot{x}(t)+ax(t)=bu(t),\ t>0$ $x(t)=x_0, \quad t=0$ $x(t)=e^{-at}x_0+\int_0^t e^{-a(t-\xi)}bu(\xi)d\xi$	$\dot{x}(t)+ax(t)+a_d x(t-h)=bu(t), \quad t>0$ $x(t)=g(t), \text{for } t\in[-h,0); x(t)=x_0,\ t=0$ $x(t)=\sum_{k=-\infty}^{\infty} e^{S_k t}C_k^I+\int_0^t \sum_{k=-\infty}^{\infty} e^{S_k(t-\xi)}C_k^N bu(\xi)d\xi$ where, $S_k=\frac{1}{h}W_k(-a_d h e^{ah})-a$
Matrix-Vector case $\dot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)=\mathbf{B}\mathbf{u}(t) \quad t>0$ $\mathbf{x}(t)=\mathbf{x}_0 \quad t=0$ $\mathbf{x}(t)=e^{-\mathbf{A}t}\mathbf{x}_0+\int_0^t e^{-\mathbf{A}(t-\xi)}\mathbf{B}\mathbf{u}(\xi)d\xi$	$\dot{\mathbf{x}}(t)+\mathbf{A}\mathbf{x}(t)+\mathbf{A_d}\mathbf{x}(t-h)=\mathbf{B}\mathbf{u}(t), \quad t>0$ $\mathbf{x}(t)=\mathbf{g}(t), \text{for } t\in[-h,0); \mathbf{x}(t)=\mathbf{x}_0,\ t=0$ $\mathbf{x}(t)=\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k t}\mathbf{C}_k^I+\int_0^t \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-\xi)}\mathbf{C}_k^N \mathbf{B}\mathbf{u}(\xi)d\xi$ where, $\mathbf{S}_k=\frac{1}{h}\mathbf{W}_k(-\mathbf{A_d}h\mathbf{Q}_k)-\mathbf{A}$

* Analogy between solution forms enables extension of control methods for systems of ODEs to DDEs.

Outline

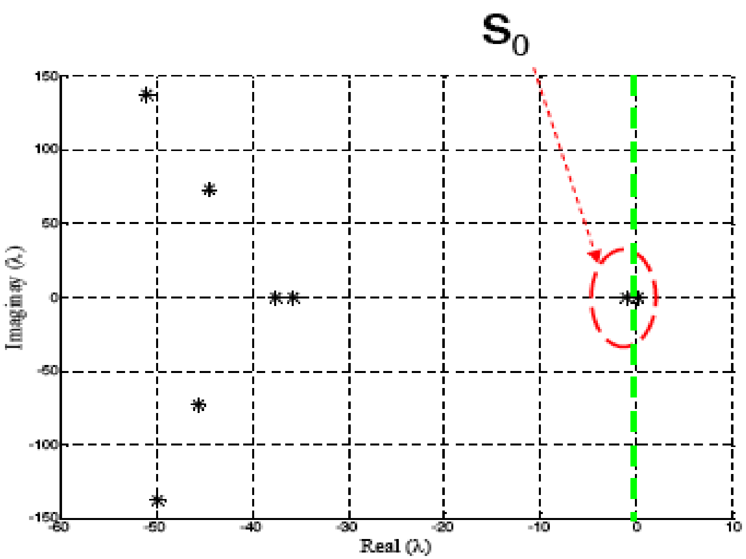
- Introduction
- Derivation of Solution to Delay Differential Eqs.
- Methods for Analysis
- Methods for Control
- Concluding Remarks

Stability Analysis

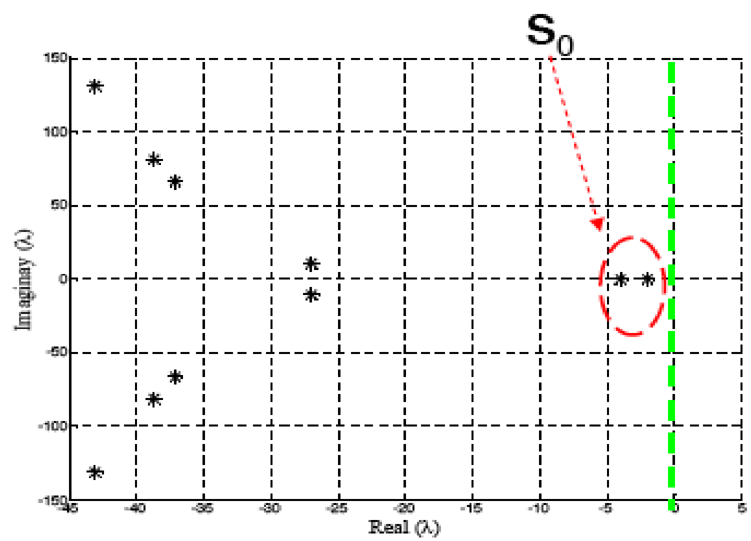
- DDEs have an infinite eigenspectrum: $\mathbf{S}_k$ for $k = -\infty, \cdots, -1,\ 0,\ 1, \cdots,\ \infty$
- Rightmost eigenvalues among them?

> If $\mathbf{A_d}$ has no repeated zero eigenvalues, then
> $\max\left[\mathrm{Re}\{\text{eigenvalues of } \mathbf{S}_0\}\right] \geq \mathrm{Re}\{\text{all other eigenvalues of } \mathbf{S}_k\}$

- Proven for scalar case, and some special systems of DDEs, but not for general cases.



Unstable



Stable

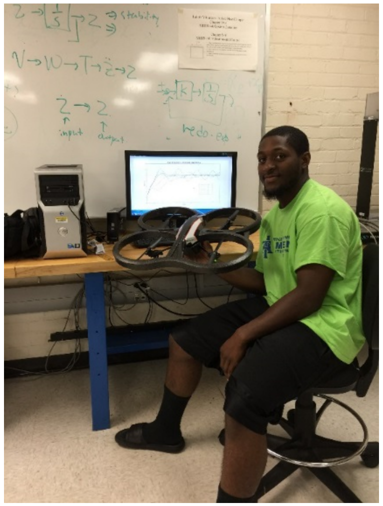
Controllability & Observability

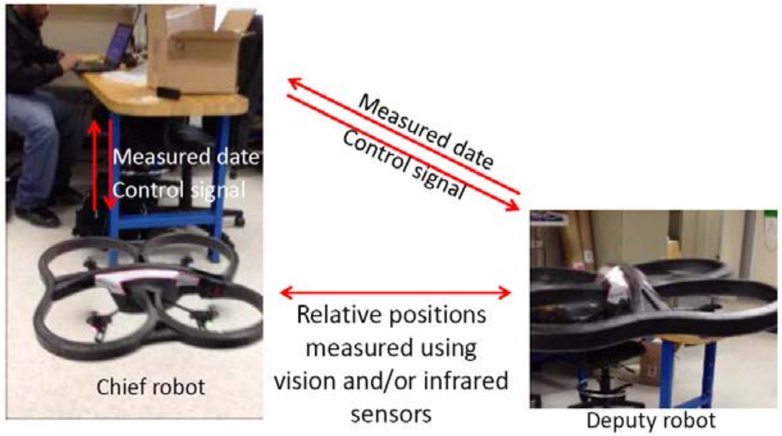
- Conditions for controllability & observability of DDEs
  - Using the developed solution form, and previous results by [Weiss, 1967]
  - Balanced realization based on Gramians

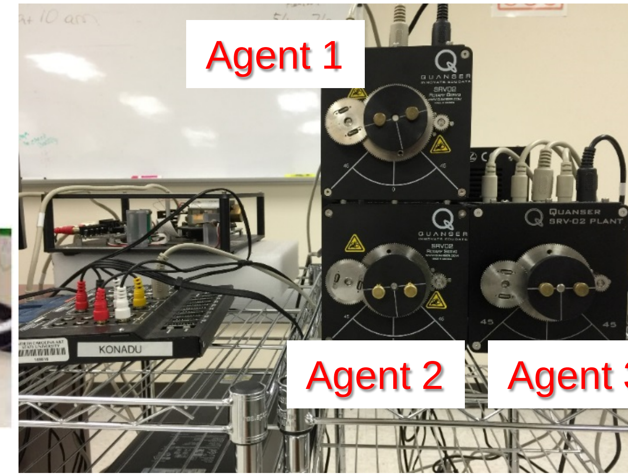
	ODEs	DDEs
	Controllability	Point-Wise Controllability
full rank:	$\mathcal{C}_{(0,t_1)} \equiv \int_0^{t_1} e^{\mathbf{A}(t_1-\xi)}\mathbf{B}\mathbf{B}^T\left\{e^{\mathbf{A}(t_1-\xi)}\right\}^T d\xi$	$\mathcal{C}_{(0,t_1)} \equiv \int_0^{t_1} \sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\mathbf{B}\mathbf{B}^T\left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t_1-\xi)}\mathbf{C}_k^N\right\}^T d\xi$
linearly independent rows:	$(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$	$\left(s\mathbf{I}-\mathbf{A}-\mathbf{A}_d e^{-sh}\right)^{-1}\mathbf{B}$
	$e^{\mathbf{A}(t-0)}\mathbf{B}$	$\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N\mathbf{B}$
	Observability	Point-Wise Observability
full rank:	$\mathcal{O}_{(0,t_1)} \equiv \int_0^{t_1} \left\{e^{\mathbf{A}(\xi-0)}\right\}^T\mathbf{C}^T\mathbf{C}e^{\mathbf{A}(\xi-0)}d\xi$	$\mathcal{O}_{(0,t_1)} \equiv \int_0^{t_1} \left\{\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N\right\}^T\mathbf{C}^T\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(\xi-0)}\mathbf{C}_k^N d\xi$
linearly independent columns:	$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}$	$\mathbf{C}\left(s\mathbf{I}-\mathbf{A}-\mathbf{A}_d e^{-sh}\right)^{-1}$
	$\mathbf{C}e^{\mathbf{A}(t-0)}$	$\mathbf{C}\sum_{k=-\infty}^{\infty} e^{\mathbf{S}_k(t-0)}\mathbf{C}_k^N$

*[Yi *et al.*, *IEEE TAC* 2008]

Networks of Agents: Lab Work







- Testbed for control of networked multi-agent systems

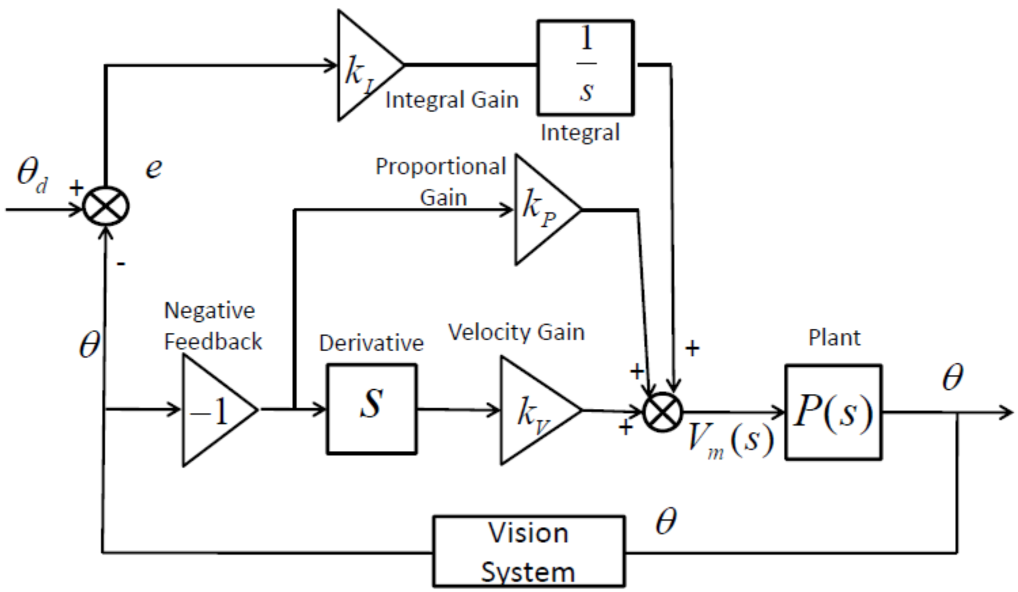
- Ground Robots by Dr. Robot for Autonomous Navigation

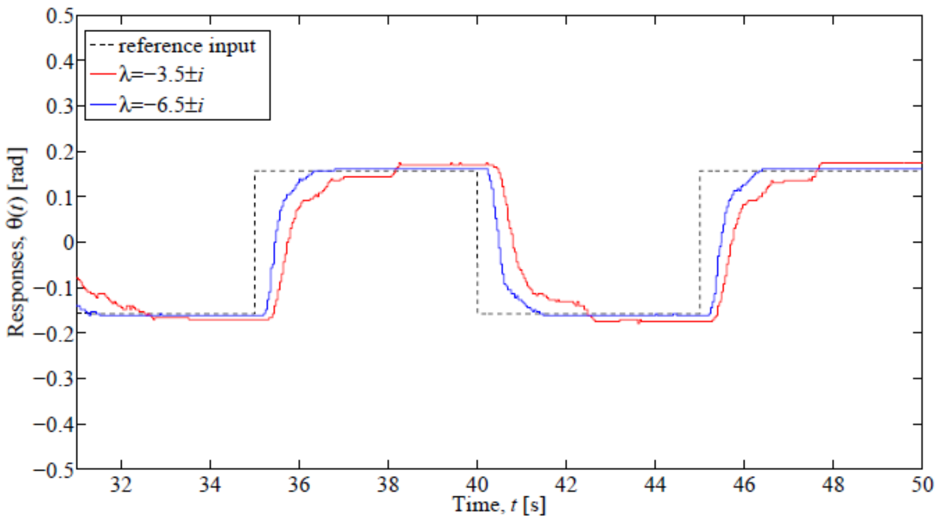
- <u>Chris Thomas</u> (Senior, ME) has been supported for autonomous control of AR Drone.

- <u>Myrielle Allen-Prince</u> (MS, ME) has been supported for networks of drones.

Motors: PV plus Integral Control

- Stable responses with good convergence.
- But non-zero steady state errors in results on the previous slide.
- Not in simulation or theory.
- Due to backlash, friction, and inductance in circuits.
- Need integral control →



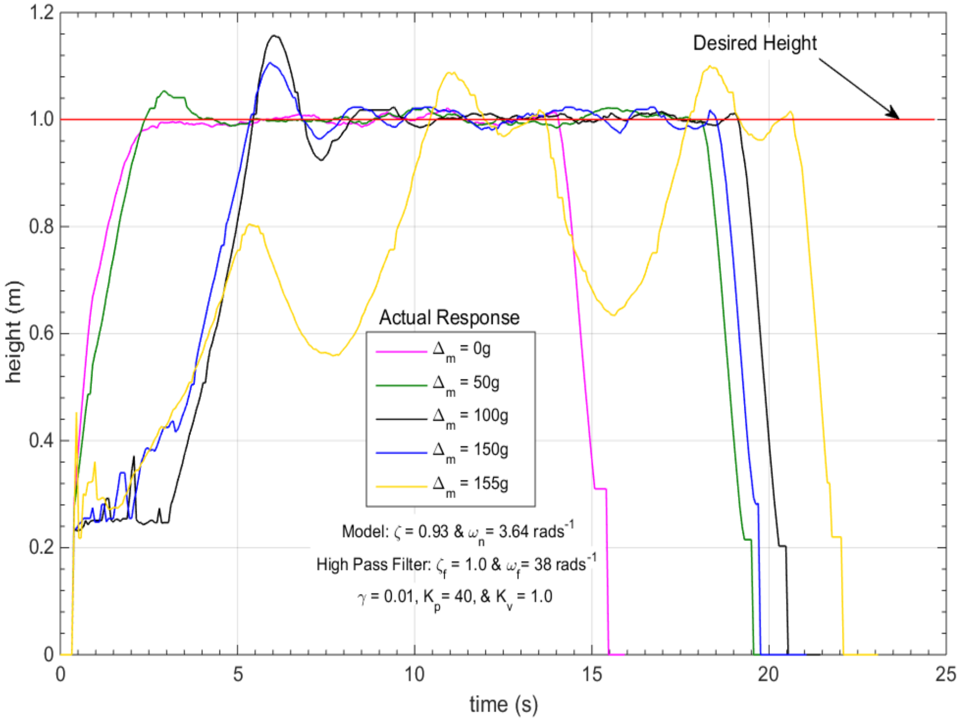


Video drone

Implementation on Drone

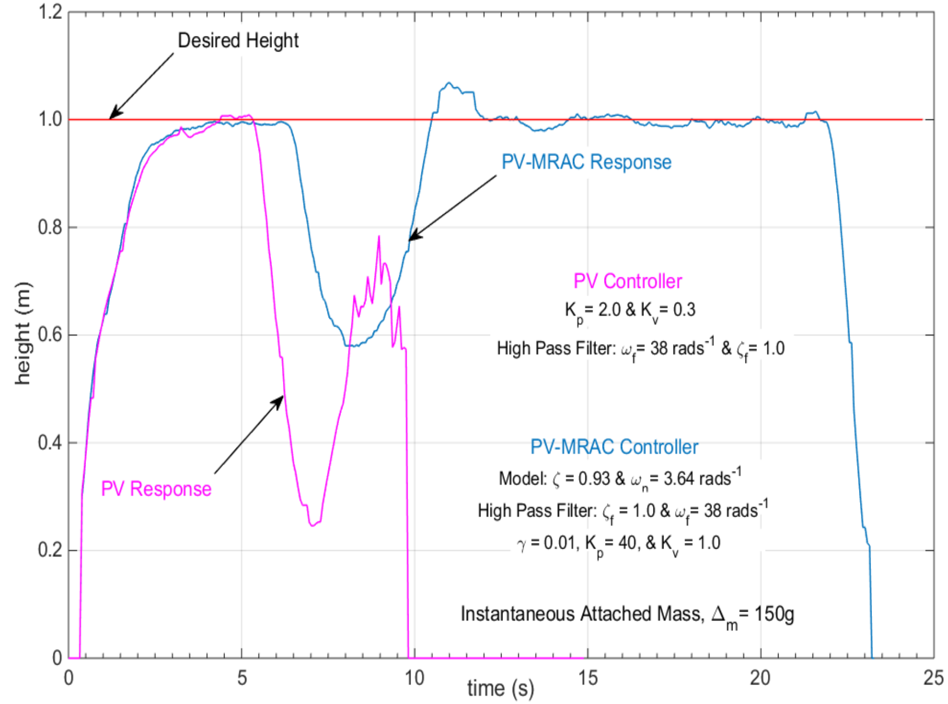
Effects of Disturbance Rejection: Payload

- PV-MRAC
  - o Stability bounds: 150g
  - o Instability: 155g



Experimented Altitude Responses: Varying Attached Mass on Top

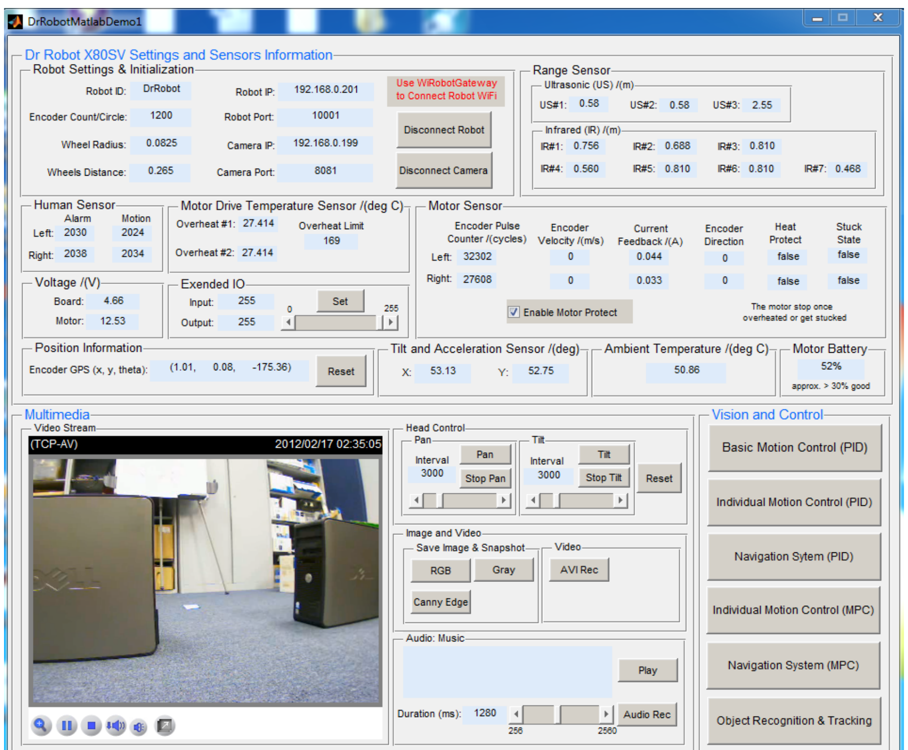
- Designed PV Vs PV-MRAC
  - o 100g: both performed well
  - o 150g:
    - ✓ Adaptive: safe operation & landing
    - ✓ PV: crash landing

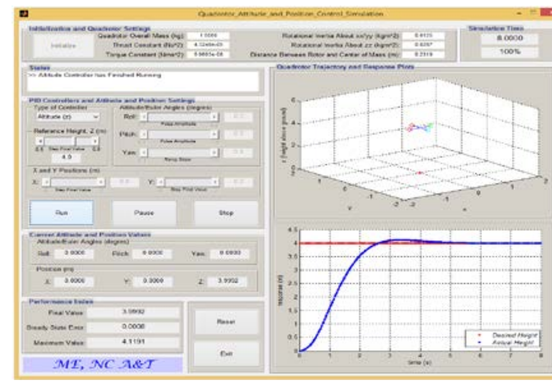


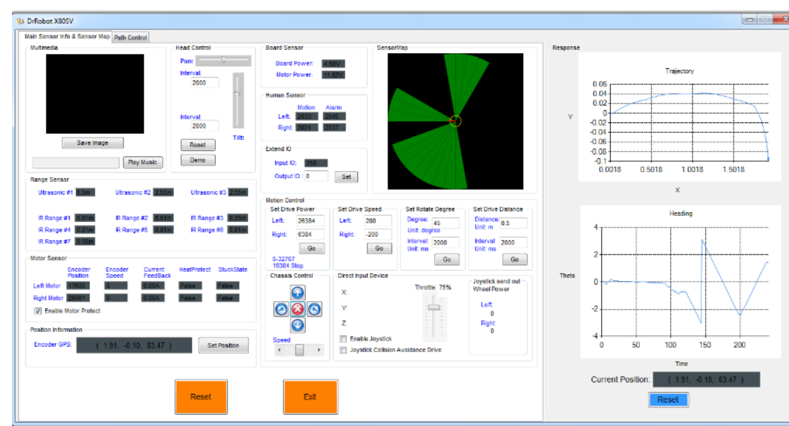
Experimented Altitude Responses: Instantaneous Mass, 150g

Networks of Agents: Simulation

- Software Package Development (being extended to multi-agent systems)
  - Real-time Control of Aerial and Ground Robots,
  - Graphic User Interface (GUI) for Simulation and Algorithm Implementation.







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Eigenvalue Assignment

Step 1. Select desired rightmost eigenvalues $\lambda_{i,desired}, \quad \text{for } i = 1,\ldots,n$

Step 2. "Linear" feedback controller

$$\mathbf{u}(t) = \underbrace{\mathbf{K}}_{\text{unknown}}\mathbf{x}(t) + \underbrace{\mathbf{K_d}}_{\text{unknown}}\mathbf{x}(t-h) \quad + \quad \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{A_d}\mathbf{x}(t-h) + \mathbf{B}\mathbf{u}(t)$$

$$\Downarrow$$

$$\dot{\mathbf{x}}(t) = (\underbrace{\mathbf{A} + \mathbf{BK}}_{\hat{\mathbf{A}}})\mathbf{x}(t) + (\underbrace{\mathbf{A_d} + \mathbf{BK_d}}_{\hat{\mathbf{A}}_\mathbf{d}})\mathbf{x}(t-h)$$ ← closed-loop system

Step 3. With the new coefficients,

$$\mathbf{W}_0(\hat{\mathbf{A}}_\mathbf{d} h\mathbf{Q}_0)e^{\mathbf{W}_0(\hat{\mathbf{A}}_\mathbf{d} h\mathbf{Q}_0)+\hat{\mathbf{A}}h} = \hat{\mathbf{A}}_\mathbf{d} h$$

$$\mathbf{S}_0 = \frac{1}{h}\mathbf{W}_0(\hat{\mathbf{A}}_\mathbf{d} h\mathbf{Q}_0) + \hat{\mathbf{A}}$$

Step 4. Equate the selected eigenvalues to those of the matrix $\mathbf{S}_0$ as (i.e., k = 0 branch only)

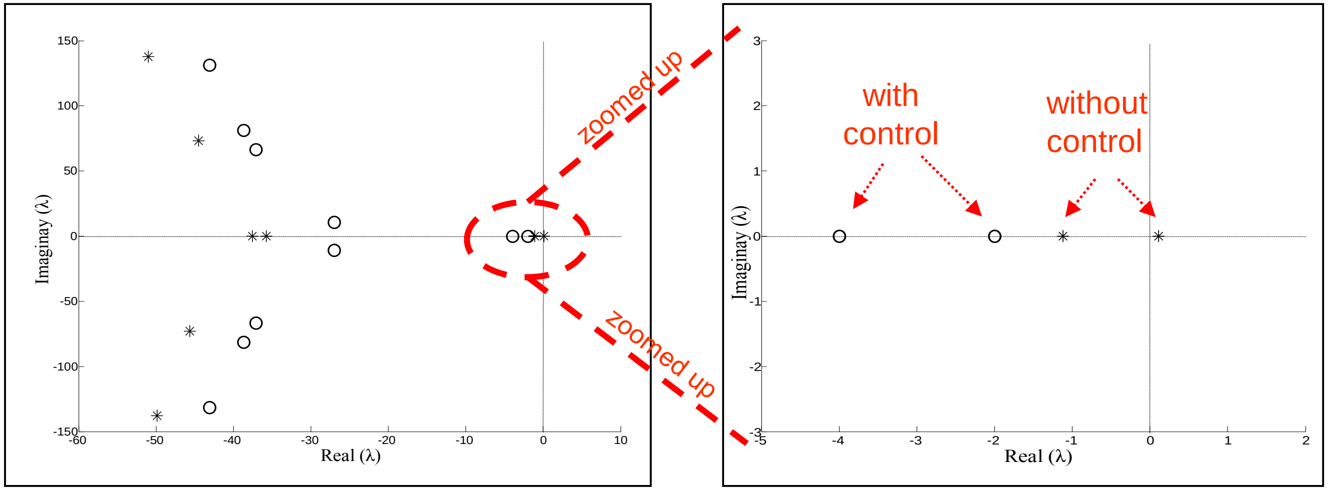
$$\boxed{\lambda_i(\mathbf{S}_0) = \lambda_{i,desired}, \quad \text{for } i = 1,\ldots,n}$$

Illustrative Example

- Example from [Yi et al, *Journal of Vibration and Control*]

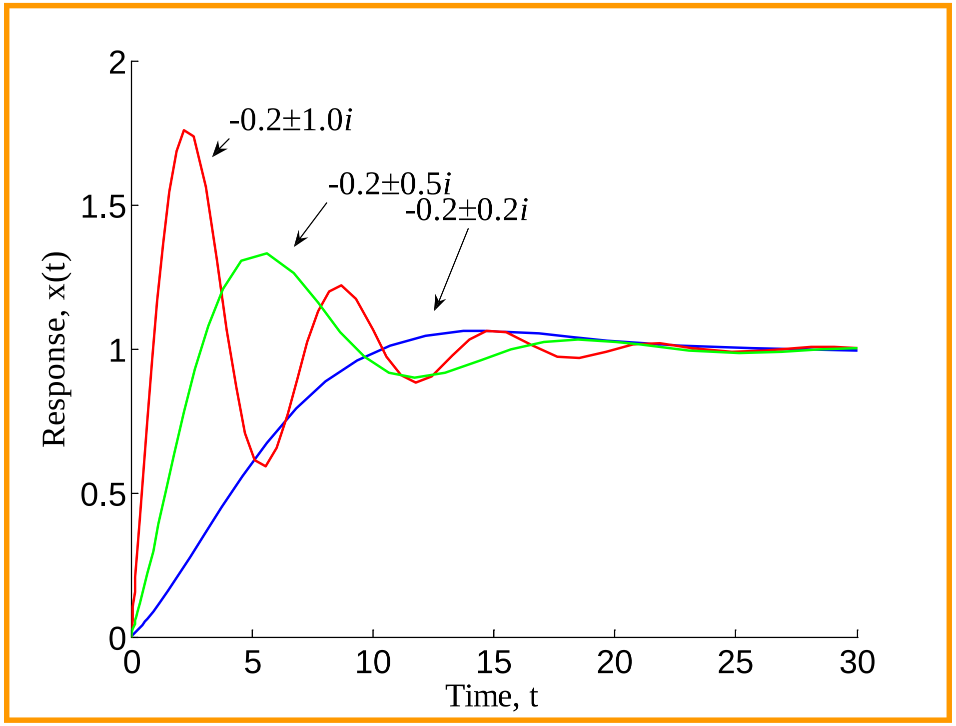
$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}\mathbf{x}(t-0.1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}(t) \quad \text{with} \quad \mathbf{u}(t) = \underbrace{\mathbf{K}}_{\text{unknown}}\mathbf{x}(t) + \underbrace{\mathbf{K_d}}_{\text{unknown}}\mathbf{x}(t-h)$$

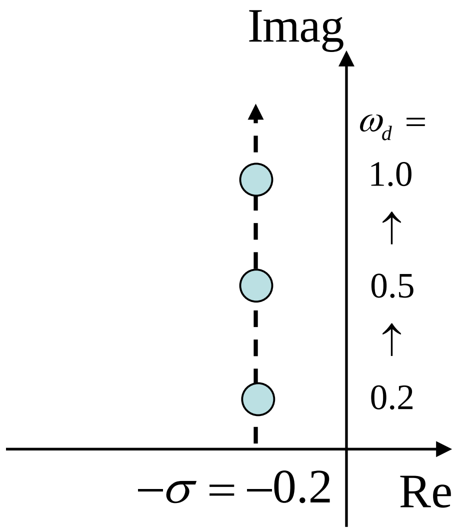
- Without feedback control
  - Rightmost eigenvalues: 0.1098 (unstable); –1.1183
- Desired: – 2 & – 4
- Resulting linear feedback: $\mathbf{u}(t) = \underbrace{\begin{bmatrix} -0.1687 & -3.6111 \end{bmatrix}}_{\mathbf{K}}\mathbf{x}(t) + \underbrace{\begin{bmatrix} 1.6231 & -0.9291 \end{bmatrix}}_{\mathbf{K_d}}\mathbf{x}(t-h)$



Transient Response

➢ Responses with different imaginary parts





➢ Assigning the dominant eigenvalues (real and imaginary parts) with a linear controller.

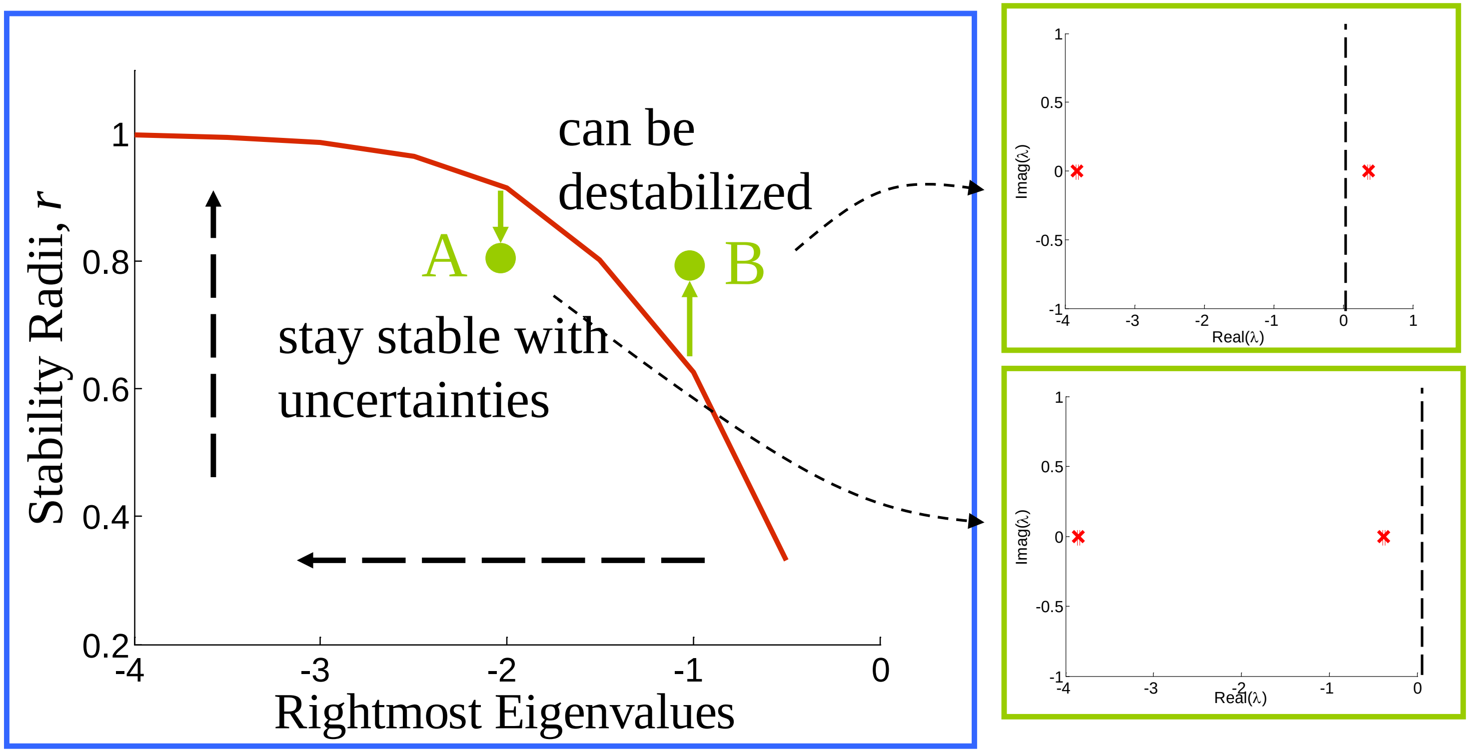
➢ Possible to meet time-domain specifications of DDEs following design guidelines for ODEs.

Rule of thumb

	t_r	t_s	M_p	t_p
$\sigma \uparrow$	$\downarrow$	$\downarrow$	$\downarrow$	fixed
$\omega_d \uparrow$	$\downarrow$	fixed	$\uparrow$	$\downarrow$
$\omega_n \uparrow$	$\downarrow$	$\downarrow$	fixed	$\downarrow$

* [Yi et al., ASME *JDSMC*]

Robust Control



Stable, but not robustly stable

Stable, and robustly stable

ISC

- Combined with the ‘stability radius’ concept with algorithms in [Hu and Davison, 2003].
- As the eigenvalue moves left, then the stability radius increases, which means, more “robust”.
- Comparing uncertainty and stability radius, one can choose the appropriate positions of the rightmost eigenvalues for robust stability.

* [Yi et al., ASME *JDSMC*]

Linear Feedback & Observer: Concept

- Feedback Controller

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{A}_{\mathbf{d}}\mathbf{x}_{\mathbf{d}} + \mathbf{B}\mathbf{u} \\
\mathbf{u} &= -\mathbf{K}\mathbf{x} + \mathbf{r} \\
\dot{\mathbf{x}} &= \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x} + \mathbf{A}_{\mathbf{d}}\mathbf{x}_{\mathbf{d}} + \mathbf{B}\mathbf{r}
\end{aligned}
$$

⇦ stabilize by choosing **K**.

- Observer (state estimator)

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{A}_{\mathbf{d}}\mathbf{x}_{\mathbf{d}} + \mathbf{B}\mathbf{u} \\
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{A}_{\mathbf{d}}\hat{\mathbf{x}}_{\mathbf{d}} + \mathbf{L}(\mathbf{y} - \mathbf{C}\mathbf{x}) + \mathbf{B}\mathbf{u} \\
\dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} &= \mathbf{A}\left(\mathbf{x} - \mathbf{x}\right) + \mathbf{A}_{\mathbf{d}}\left(\mathbf{x}_{\mathbf{d}} - \hat{\mathbf{x}}_{\mathbf{d}}\right) - \mathbf{L}(\mathbf{y} - \mathbf{C}\mathbf{x}) \\
\Rightarrow \dot{\mathbf{e}} &= \left(\mathbf{A} - \mathbf{L}\mathbf{C}\right)\mathbf{e} + \mathbf{A}_{\mathbf{d}}\mathbf{e}_{\mathbf{d}}
\end{aligned}
$$

⇦ stabilize by choosing **L**.

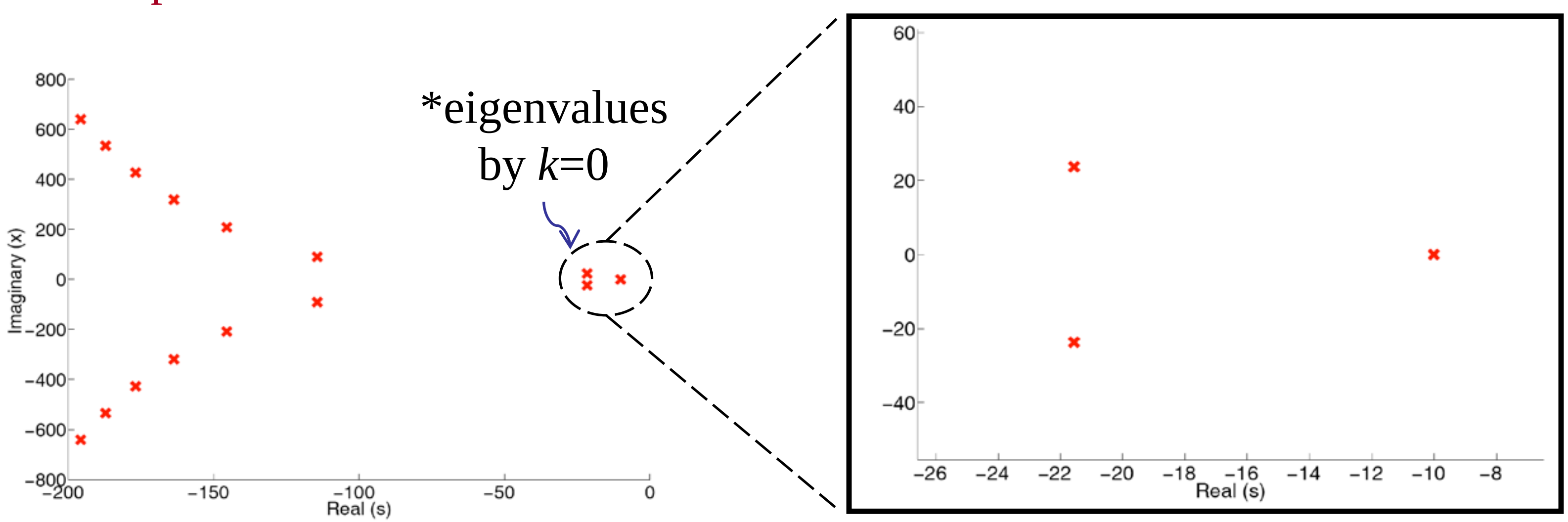
Diesel Engine: Feedback Control

- Diesel Engine: linearized system equation with a transport time delay at N=1500 RPM

$$\dot{\mathbf{x}}(t) = \underbrace{\begin{bmatrix} -27 & 3.6 & 6 \\ 9.6 & -12.5 & 0 \\ 0 & 9 & -5 \end{bmatrix}}_{\mathbf{A}} \mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 21 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{A_d}} \mathbf{x}(t - \underbrace{0.06}_{h}) + \underbrace{\begin{bmatrix} 0.26 & 0 \\ -0.9 & -0.8 \\ 0 & 0.18 \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{\mathbf{C}} \mathbf{x}(t)$$

- This system has an <u>unstable</u> eigenvalue at 0.9225 (> 0) without feedback
- With feedback control: $\boxed{\mathbf{u}(t) = \mathbf{K}\mathbf{x}(t)}$
- Desired position: -10



Diesel Engine: State Observer

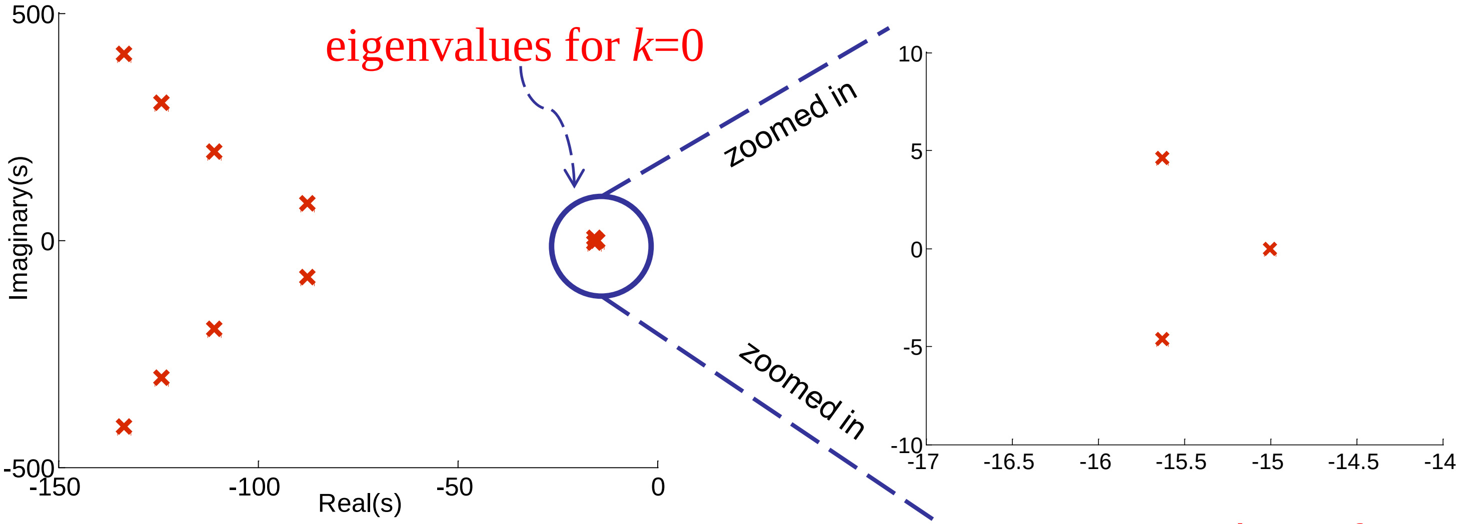
➢ Observer:

$$\mathbf{A}=\begin{bmatrix}-27 & 3.6 & 6\\ 9.6 & -12.5 & 0\\ 0 & 9 & -5\end{bmatrix},\mathbf{A_d}=\begin{bmatrix}0 & 0 & 0\\ 21 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\mathbf{C}=\begin{bmatrix}0 & 1 & 0\end{bmatrix},\mathbf{L}=\begin{bmatrix}L_1\\ L_2\\ L_3\end{bmatrix}$$

Unstable
(one positive real eigenvalue)

$$\tilde{\mathbf{A}}=\mathbf{A}-\mathbf{LC}=\begin{bmatrix}-27 & 3.6-L_1 & 6\\ 9.6 & -12.5-L_2 & 0\\ 0 & 9-L_3 & -5\end{bmatrix},\quad \mathbf{A_d}=\mathbf{A_d}=\begin{bmatrix}0 & 0 & 0\\ 21 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Goal:
Find stabilizing “L”

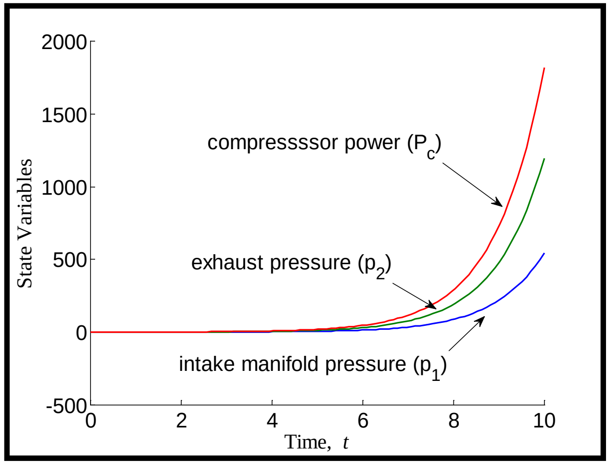


$$\mathbf{L}=\begin{bmatrix}6.4729\\ 9.5671\\ 16.0959\end{bmatrix}$$

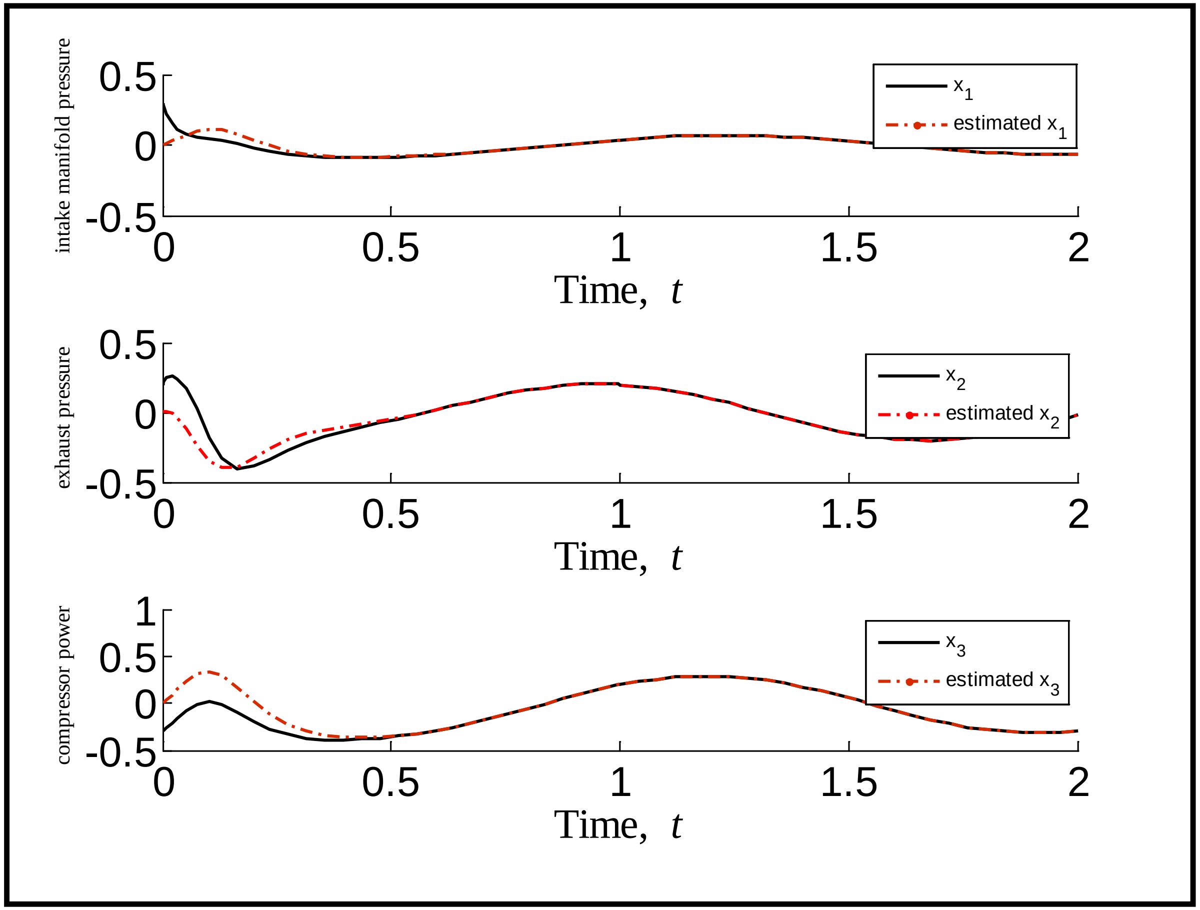
*1.5 times faster than feedback

Response of Controlled System

- Feedback Control with Observer:



↑ was unstable



* Reference inputs

r_1 : square wave with amplitude 0.5

r_2 : sine wave with amplitude 20 and frequency 0.7(Hz)

*[Yi et al., *JFI* 2010]

Outline

- Introduction
- Derivation of Solution to Delay Differential Eqs.
- Methods for Analysis
- Methods for Control
- Concluding Remarks

Summary and Contributions

Given a system of LTI DDEs with single delay *h*

1. Derive the solution (free & forced): [*DCDIS* 2007]

2. Determine stability of the system: [*MBE* 2007]

3. Check conditions for controllability and observability: [*IEEE TAC* 2008]

4. Design feedback controller via eigenvalue assignment: [*JVC* (in press)]

5. Transient response & Robust stability : [*ASME JDSMC* (in press)]

6. Observer-based feedback control: [*JFI* 2010]

Applications in engineering and biology

* For systems of ODEs, the above approach is standard.

* Using the Lambert W function-based approach we have extended this to DDEs.

Advantages

- Stability analysis
  - Accuracy compared to approximation approaches
  - 'How stable' compared to bifurcation analysis

- Feedback control w/ observer
  - Robustness compared to Smith predictor a
  - Ease of implementation compared to nonlinear methods (e.g., FSA, which has integrals in controllers)

- Controllability/observabiltiy
  - Quantitative information: 'How observable/controllable', and 'balanced realization' compared to algebraic conditions

Concluding Remarks & Future Work

- Still open problems, and long way to go.
- Trying to find an effective method, which is more intuitive and similar to ODEs
  - Focused on networked systems.
  - More challenges: noise, time-variance, disturbances, and nonlinearity.
- Used
  - Time-domain responses and
  - Analytical solutions in terms of Lambert W function.
- Extended stability analysis and control design.
- Multiple Drones
- Nonlinear controller: MRAC and Gain scheduling
- Path following
- Higher-order systems' delay estimation