9/1 12/12/2021

Homework

Bulat Jaclina-Jana

Exercise 51

a) $\int_0^{\infty} \frac{arctg(x)}{x^2+1} dx = ?$

$\int f(x) dx = F(x)$

$\int \frac{arctg(x)}{x^2+1} dx \Rightarrow u = arctg(x) \Rightarrow du = \frac{1}{x^2+1}$

$dx = (x^2+1) du$

$\Rightarrow \int \frac{arctg(x)}{x^2+1} dx = \int \frac{u}{u'} \cdot u' \cdot du = \int u du$

$\int u du = \frac{u^2}{2} + C \Rightarrow \int \frac{arctg(x)}{x^2+1} dx = \frac{arctg^2(x)}{2} + C$

$\Rightarrow \int f(x) dx = F(x) = \frac{arctg^2(x)}{2} + C \Rightarrow$

$\Rightarrow \int_0^{\infty} f(x) dx = \lim_{x \to \infty} \int_0^x \frac{arctg(x)}{x^2+1} dx =$

$= \lim_{x \to +\infty} \left(\frac{arctg^2(x)}{2} \right) \cancel{+C} = \frac{1}{2} \lim_{x \to +\infty} (arctg^2(x)) =$

$= \frac{1}{2} \left(\lim_{x \to +\infty} (arctg(x)) \right)^2$

$$\lim_{x \to +\infty} (\operatorname{arctg}(x)) = \frac{\pi}{2} \Rightarrow$$

$$\Rightarrow \frac{1}{2}\left(\lim_{x \to +\infty} (\operatorname{arctg}(x))\right)^2 = \frac{1}{2} \cdot \left(\frac{\pi}{2}\right)^2 = \frac{\pi^2}{2 \cdot 4} = \frac{\pi^2}{8}$$

b) $\int_{2}^{+\infty} \frac{x-1}{x^2+x+1} dx = ?$

$\int f(x) dx = F(x)$

$$\int \frac{x-1}{x^2+x+1} dx = \int \frac{2x-2}{2(x^2+x+1)} dx = \int \frac{2x-3+1}{2(x^2+x+1)} dx = \int \left(\frac{2x+1}{2(x^2+x+1)} - \frac{3}{2(x^2+x+1)} \right) dx =$$

$$= \frac{1}{2} \underset{(1)}{\int \frac{2x+1}{x^2+x+1} dx} - \frac{3}{2} \underset{(2)}{\int \frac{1}{x^2+x+1} dx}$$

(1) $\int \frac{2x+1}{x^2+x+1} dx \Rightarrow u = x^2+x+1 \Rightarrow du = (x^2+x+1)' = 2x+1$

$$dx = \frac{1}{2x+1} du$$

$$\Rightarrow \int \frac{2x+1}{x^2+x+1} dx = \int \frac{u'}{u} \cdot \frac{1}{u'} du = \int \frac{1}{u} du = \ln(u) + C =$$

$$= \ln(x^2+x+1) \quad (1)$$

(2) $\int \frac{1}{x^2+x+1} dx = \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx \Rightarrow f = \frac{2x+1}{\sqrt{3}} \Rightarrow du = \frac{2}{\sqrt{3}}$

$$dx = \frac{\sqrt{3}}{2} du$$

$$\Rightarrow \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx = \int \frac{\sqrt{3}}{2\left(\frac{3u^2}{4} + \frac{3}{4}\right)} du =$$

$$= \int \frac{\sqrt{3}}{2 \cdot \frac{3}{4}(u^2+1)} du = \frac{2}{\sqrt{3}} \int \frac{1}{u^2+1} du$$

$$\int \frac{1}{u^2+1} du = \operatorname{arctg}(u) + C \Rightarrow \frac{2}{\sqrt{3}} \int \frac{1}{u^2+1} du = \frac{2\operatorname{arctg}(u)}{\sqrt{3}} + C$$

$$\frac{2\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)}{\sqrt{3}} \quad (2)$$

$$(1),(2) \Rightarrow \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{3}{2}\int \frac{1}{x^2+x+1}\,dx =$$

$$= \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \int \frac{x-1}{x^2+x+1}\,dx = \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\int f(x) = F(x) = \frac{\ln(x^2+x+1)}{2} - \sqrt{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

c) $\int \ln^2(x)\,dx = ?$

$f(x) = \ln^2(x)$

$\int f(x)\,dx = F(x)$

$\int \ln^2(x)\,dx \Rightarrow \int fg' = fg - \int f'g$ (integrations by parts)

$f = \ln^2(x)$, $g' = 1 \Rightarrow f' = \frac{2\ln(x)}{x}$, $g = x \Rightarrow$

$\Rightarrow \int \ln^2(x)\,dx = x\ln^2(x) - \int \frac{2\ln(x)}{x} \cdot x\,dx =$

$= x\ln^2(x) - \int 2\ln(x)\,dx$

$\int 2\ln(x)\,dx = 2\int \ln(x)\,dx$ (*)

(*) $\int \ln(x)\,dx \Rightarrow \int uv' = u \cdot v - \int u' \cdot v$ (integration by parts)

$u = \ln(x)$, $v' = 1 \Rightarrow u' = \frac{1}{x}$, $v = x$

$\Rightarrow \int \ln(x)\,dx = x \cdot \ln(x) - \int \frac{1}{x} \cdot x\,dx =$

$= x \cdot \ln(x) - \int dx =$

$= x\ln(x) - x + C =$

$= x(\ln(x) - 1) + C$ (*)

$2\int \ln(x)\,dx = 2[x(\ln(x) - 1) + C] = 2x(\ln x - 1) + C$

$x\ln^2(x) - \int 2\ln(x)\,dx = x\ln^2(x) - 2x\ln(x) + 2x + C =$

$= x(\ln^2(x) - 2\ln(x) + 2) + C$

$\Rightarrow \int f(x)\,dx = F(x) = x(\ln^2(x) - 2\ln(x) + 2) + C$

$$\int_0^1 f(x)\,dx = x\left(\ln^2(x) - 2\ln(x) + 2\right)\Big|_0^1 =$$

$$= 2$$