UNIVERSITY OF EDINBURGH
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CMSE11500 Stochastic Optimization

Individual Assignment

B223193

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Optimization of Whiskey Production and Distribution under Emissions Cap and Uncertain Demand

1 Introduction:

Chiwas, a whiskey producer, is planning to construct stills to meet the growing market demand in Scotland. To produce whiskey, two types of stills (column stills and distillers) are available in the market. Chiwas needs to optimize its production and distribution strategy to meet the demand from five different demand points while minimizing costs and reducing harmful emissions. The company is also bound by an emissions cap and a tax rate.

2 Approaches and Methodologies

The problem involves deciding the optimal number and type of whiskey stills to install, in order to satisfy the demand for whiskey at various demand points while minimizing cost and staying within an emissions cap. A stochastic optimization model, which takes into account the uncertainty and variability of data by modelling it as a probability distribution, can be utilized to solve this problem. Incorporating uncertainty into the model enables the acquisition of a more realistic and reliable solution that is less sensitive to changes in the input data. The utilization of a stochastic optimization model is a rational and effective approach to solving this problem, as it allows for a more robust, flexible, and balanced solution that considers the inherent uncertainty and variability in the data.

3 Assumptions and Data

Appendix 1.1 outlines the assumptions that define the scope and boundaries of the problem, providing a framework for constructing the stochastic optimization model. Appendix 1.2 lists the data, which is assumed based on the general rule of the differences between two types of stills and the uncertainties in different demand points. Although the provided data is not entirely precise or realistic, it aligns with the goal of the assignment to test and practice the use of a stochastic model.

4 Model

4.1 Deterministic model

The deterministic model presented in Appendix 1 minimizes the whiskey manufacturer's total cost, considering the constraints and variable domains defined. According to the result, the still installation decision should be to install three column stills and 15 distillers. The first four demand points should be selected, and 3479t, 3951t, 522.47t, 3440t should be allocated to these places, respectively. The total cost of this solution would reach £1,902,378.238.

Deterministic models lack uncertainty considerations. One of the primary underlying assumptions is that the data is known with certainty, meaning that the planner knows every aspect of the total cost over the whole period perfectly, which is rarely possible in reality.

4.2 Two-Stage Stochastic Programming Model

Therefore, *Two-Stage Stochastic Programming Model* (Appendix 2.1) for whiskey production has been developed to introduce uncertainty and probability into the model while considering both proactive and reactive decisions. Suppose new whiskey production data is available that suggests output capacity, demand, the fraction of reserved whiskey after transportation loss, harmful emissions generated, and the emission cap are within 25% of their expected values. There are no observable patterns, meaning that all uncertain data follows uniform distributions.

Scenario-based version

Let S^N be the set of N scenarios, Pr_s be the probability of scenario s, and a^s be the realization of the random variable $\tilde{a}$ in scenario s. To separate proactive decisions that are not easy to change with scenario realizations from reactive decisions that are adapted to different scenarios, while considering the probabilities of occurrence of scenarios, I introduced the versatile *Scenario-Based Two-Stage Stochastic Programming* reformulation of the deterministic model (Appendix 2.2).

SAA deterministic equivalent

The estimation of p_s can be a challenging task, especially when historical scenarios are not available to train the model. In such cases, the *Sample Average Approximation* (SAA) approach can be employed to address this issue, as presented in Appendix 2.3. Figure 1 illustrates that SAA stabilizes at N = 2500, and hence, this value is chosen for training the SAA model.

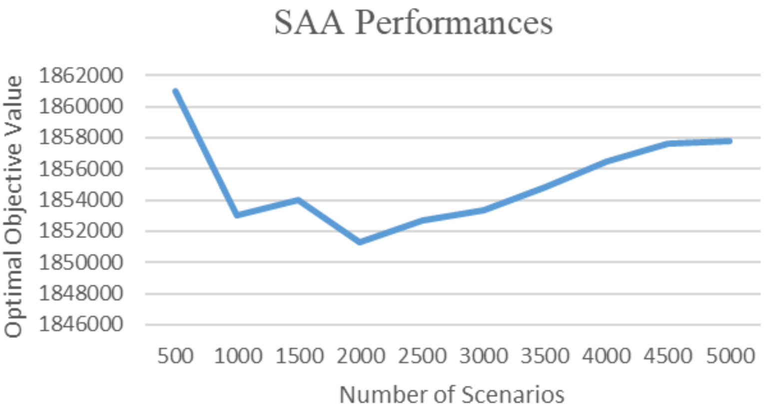

Figure 1: SAA Performances over 5000 Scenarios

Out-of-sample (OOS) tests are performed to evaluate the performance of the model on scenarios that it has not been trained on. To generate OOS data, a uniform distribution ranging from 0.75 to 1.25 of the original amounts is fitted.

Table 1: Results of the Deterministic Model and Stochastic Model

	Deterministic model	**Two-stage Stochastic Programming model**
installation solution	3 column stills and 15 distillers	3 column stills and 17 distillers
total cost	£1,902,378.238	£1,852,656.890

The stochastic installation solution proves to be more effective than the deterministic installation solution in terms of total operation cost, with a mean cost savings of £6,986.252. This finding suggests that stochastic optimization allows for a more flexible and adaptive decision-making process that can better account for uncertainty in the demand and emissions parameters. Moreover, the operation cost is separated for further analysis, including most of the total cost, but not the installation cost of the stills, which is considered a proactive decision.

A scenario-by-scenario comparison shows that the stochastic programming solution either has the same or lower unmet demand cost than the deterministic program in all OOS scenarios. The comparison of scenario-by-scenario results indicates that the stochastic programming model outperforms the deterministic model in all OOS scenarios, suggesting that the stochastic model is more robust and better suited to handling unexpected or extreme events.

Table 2: OOS performance comparisons on 1000 scenarios

	Deterministic model	**Two-stage Stochastic Programming model**
Mean of operation cost	£1,787,468.288	£1,780,482.036
Standard deviation of operation cost	£192,067.650	£8397.115
Installation cost	£28500	£31,900

4.3 Three-Stage Stochastic Programming Model

Regarding the *Three-Stage Stochastic Programming Model*, the loss of whiskey during transportation depends significantly on the size of the original production for different types of stills. Research has shown that the proportion of reserved whiskey after transportation loss is approximately between 1/1050 and 1/650 of the output capacity of each type. The *Three-Stage Stochastic Programming Model* is presented in Appendix 3.1. For the three-stage model, 50 scenarios were trained in Stage 1, and 50 scenarios were trained for each Stage 1 scenario, resulting in a total of 2500 scenario paths.

Table 3: Installation Solution of Different Models

	Deterministic model	**Two-stage Stochastic Programming model**	**Three-stage Stochastic Programming model**
installation solution	3 column stills and 15 distillers	3 column stills and 17 distillers	3 column stills and 17 distillers

Table 4: OOS performance comparisons on 1000 scenarios from a two-stage tree

	Deterministic model	**Two-stage Stochastic Programming model**	**Three-stage Stochastic Programming model**
Mean of operation cost	£1,787,468.288	£1,780,482.036	£1,862,358.533
Standard deviation of operation cost	£192,067.650	£8397.115	£4488.339
Installation cost	£28500	£31,900	£31,900

The three-stage stochastic programming model only outperforms the two-stage model in 12 out of 1000 OOS scenarios and the deterministic model in 21 out of 1000 OOS scenarios. It outperformed in the remaining ones. Therefore, it is not appropriate if uncertainty revelation follows the two-stage tree OOS.

Table 5: performance comparisons on 1000 scenarios from a three-stage tree

	Deterministic model	Two-stage Stochastic Programming model	Three-stage Stochastic Programming model
Mean of operation cost	£1,487,984.628	£1,487,725.459	£1,451,897.022
Standard deviation of operation cost	£290,521.016	£13,133.307	£188,033.666
Installation cost	£28,500	£31,900	£31,900

The three-stage stochastic programming model only outperforms the two-stage model in 12 out of 1000 OOS scenarios and the deterministic model in 21 out of 1000 OOS scenarios; it is outperformed in the remaining ones. Thus, it is not suitable if uncertainty revelation follows the two-stage tree OOS. On the three-stage tree, the three-stage stochastic programming model comprehensively outperforms both the deterministic and the two-stage models, highlighting the importance of properly modelling information revelation.

These findings suggest that decision-makers should consider the use of stochastic optimization models in similar situations where there is uncertainty in demand and cost parameters. Additionally, the choice of the appropriate installation solution should be based on the degree of uncertainty in the problem and the level of information revelation.

5 Conclusion and limitations

In summary, the results suggest that the installation of 3 column stills and 17 distillers is the optimal solution under the given assumptions and constraints, with a total operation cost of £1,780,482.036. In addition, the stochastic installation solution performs better than the deterministic installation solution in terms of total operation cost. However, it may not be appropriate if uncertainty revelation follows the two-stage tree out-of-sample. Moreover, the analysis presented in this paper is based on assumptions and approximations, and may not accurately reflect the complexities and uncertainties of real-world whiskey production processes. Further research may be necessary to validate the findings of this study and to explore alternative modelling approaches that may better capture the complexities and uncertainties of the whiskey production process.

Appendix 1 Deterministic model

Sets

I - Stills types (column stills, distillers) → whiskey

J - Set of demand points

Parameters (Data)

v_i - output capacity of still of type i ∈ I (t)

d_j – demand at demand point $j \in J$ (t)

l_i – the fraction of reserved whiskey after transportation loss of type i (different container)

u_j – unit cost of unmet demand

γ_i - harmful emissions per ton of whiskey generated from type i,

t - tax rate

f - available number of stills

b – lowest requirement

Γ - emissions cap

c_i^{set}- setup/installation cost of still of type i (£)

c_i^{gen} - unit generation cost from stills of type i ∈ I (£/t)

Decision variables

x_i - number of stills of type i ∈ I to install, integer

y_i – whiskey to generate from generators of type i ∈ I (t), positive

a_j – the amount of whiskey allocated to demand point $j \in J$, positive

Deterministic model

$\min \sum_{i \in I} (c_i^{set} x_i + \ c_i^{gen}\, y_i) + \ \sum_{j \in J} u_j \big(d_j - a_j\big) + t\ \sum_{i \in I} \gamma_i y_i$ - minimise total cost

s.t.

$\sum_{i \in I} x_i \le f$ – stills availability (1)

$x_i \ge b\ \forall i$ – lowest requirement (2)

$y_i \le v_i x_i\ \ \forall i \in I$ - output capacity (3)

$\sum_{j \in J} a_j \le \ \sum_{i \in I} l_i y_i$ – reserved to allocation (4)

$a_j \le d_j\ \ \forall j \in J$ – allocation ≤ demand (5)

$\sum_{i \in I} \gamma_i y_i \le \Gamma$ – emission (6)

$x_i \in Z_+, \forall i \in I,$ – non-negative integer-valued number of stills (7)

$y_i, a_j \ge 0\ \forall i \in I, j \in J$ - non-negative real-valued whiskey generation (8)

Description

The objective function includes the following terms:

- The total installation cost of all still types.
- The total generation cost of all still types.
- The total cost of unmet demand.
- The total tax paid for emissions.

Constraints:

1. There are f stills available in the market for each manufacturer to buy, the total number of stills cannot exceed the maximum supply.
2. Since the company needs to ensure diversity of equipment so that they can change according to the market situation, the minimum number of each type of still should be more than b.
3. The amount of whiskey produced should be no more than the output capacity generated by all the stills of different types.
4. The amount of whiskey allocated to each demand point cannot exceed the reserved amount after transportation loss.
5. The amount of whiskey allocated to each demand point cannot exceed the demand at that point.
6. The total amount of green gas generated from all still types cannot exceed the emissions cap.
7. The number of stills of each type to install must be an integer.
8. The amount of whiskey allocated to each demand point, the number of stills of each type to install, and the amount of whiskey generated from each still type must be non-negative.

Appendix 1.1 Assumptions

1. The demand for whiskey at each demand point is independent and identically distributed over time.

2. The emissions per ton of whiskey generated from each type of still, the tax rate on harmful emissions and the cost of installing and generating whiskey from each type of still are constant and known with certainty.
3. The available number of stills of each type is independent of the number of stills of other types.
4. The whiskey generated from each type of still is homogeneous and can be transported to any demand point without any quality degradation.
5. The amount of whiskey allocated to each demand point is a continuous variable.
6. The emissions cap is fixed and cannot be exceeded under any circumstances.

Appendix 1.2 Data

1. Output capacity:

The output capacity of a distiller is typically lower than that of a column still. This means that a distiller can produce less whiskey in a given amount of time than a column still. This parameter is represented by the parameter vi in the model.

2. Demand:

The demand at different demand points is independent of the still type used. This means that the demand parameter d_j in the model is the same for both distillers and column stills.

3. Fraction of reserved whiskey after transportation loss:

The fraction of reserved whiskey after transportation loss for a distiller is typically lower than that of a column still. This means that less whiskey is reserved for distribution after transportation loss for distillers than for column stills. This parameter is represented by the parameter li in the model.

4. Unit cost of unmet demand:

The unit cost of unmet demand is independent of the still type used. This means that the unit cost of unmet demand parameter u_j in the model is the same for both distillers and column stills.

5. Harmful emissions per ton of whiskey generated:

The harmful emissions per ton of whiskey generated from a distiller are typically higher than that of a column still. This means that distillers produce more harmful emissions per ton of whiskey generated than column stills. This parameter is represented by the parameter γi in the model.

6. Tax rate:

The tax rate is independent of the still type used. This means that the tax rate parameter t in the model is the same for both distillers and column stills.

7. Setup/installation cost:

The setup/installation cost of a distiller is typically lower than that of a column still. This means that installing a distiller is less expensive than installing a column still. This parameter is represented by the parameter c_i^{set} in the model.

8. Unit generation cost:

The unit generation cost of a distiller is typically higher than that of a column still. This means that generating whiskey using a distiller is more expensive than generating whiskey using a column still. This parameter is represented by the parameter ci^gen in the model.

In conclusion, there are differences between distillers and column stills at the data level for most parameters. Distillers typically have lower output capacity, reserve less whiskey for distribution after transportation loss, produce more harmful emissions per ton of whiskey generated, and have higher unit generation cost than column stills. However, distillers are typically less expensive to install than column stills. The demand and unit cost of unmet demand parameters are independent of the still type used.

	output capacity of type i	**proportion reserved whiskey after transportation loss**	**harmful emission per ton of whiskey generated from type i**	**installation cost of type i**	**unit generation cost of type i**
	v(i)	l(i)	gama(i)	cs(i)	cg(i)
Column Still	533	0.78	7.7	1000	70
Distiller	835	0.81	1.5	1700	73

	demand	**unit cost of unmet demand**
	d(j)	u(j)
de1	3479	98
de2	3951	96
de3	4559	93
de4	3440	100
de5	5045	92

	tax rate	**number of available stills in the market**	**number of lowest requirement**	**emission cap**
	t	f	b	Gamas
Scalar	26%	20	3	40000

Appendix 2.1 Two-stage Stochastic Programming model

$$\min \sum_{i \in I} c_i^{set} x_i + E\left[Q\left(x, \tilde{v}, \tilde{d}, \tilde{l}, \tilde{\gamma}, \tilde{\Gamma}\right)\right],$$

$s.t.\ \ \sum_{i \in I} x_i \leq f$

$x_i \geq b\ \forall i$

$x_i \in Z_+, \forall i \in I,$

where $Q(x, v, d, l, \gamma, \Gamma) = \min \sum_{i \in I} c_i^{gen} y_i + \sum_{j \in J} u_j\left(d_j - a_j\right) + t \sum_{i \in I} \gamma_i y_i$

$s.t.\ y_i \leq v_i x_i\ \ \forall i \in I$

$\sum_{j \in J} a_j \leq \sum_{i \in I} l_i y_i$

$a_j \leq d_j \;\; \forall j \in J$

$\sum_{i \in I} \gamma_i y_i \leq \Gamma$

$x_i \in Z_+ \; \forall i \in I, \; y_i, a_j \geq 0 \; \forall i \in I, j \in J$

Appendix 2.2 Scenario-based version

$$\min \sum_{i \in I} c_i^{set} x_i + \sum_{s \in S^N} Pr_s \, Q(\, x, v^s, \mathrm{d}^s, l^s, \gamma^s, \Gamma^s)$$

$s.t. \;\; \sum_{i \in I} x_i \leq f$

$x_i \geq b \; \forall i$

$x_i \in Z_+, \forall i \in I,$

where $Q(x, v, d, l, \gamma, \Gamma) = \min \sum_{i \in I} \; c_i^{gen} \, y_i + \sum_{j \in J} u_j \left(d_j - a_j\right) + t \; \sum_{i \in I} \gamma_i y_i$

$s.t. \; y_i \leq v_i x_i \;\; \forall i \in I$

$\sum_{j \in J} a_j \leq \sum_{i \in I} l_i y_i$

$a_j \leq d_j \;\; \forall j \in J$

$\sum_{i \in I} \gamma_i y_i \leq \Gamma$

$x_i \in Z_+, \forall i \in I, \; y_i, a_j \geq 0 \; \forall i \in I, j \in J$

Appendix 2.3 SAA deterministic equivalent

$$\min \sum_{i \in I} c_i^{set} x_i + \frac{1}{\mathrm{N}} \sum_{s \in S^N} (\sum_{i \in I} c_i^{gen} y_{is} + \sum_{j \in J} u_j \left(d_{js} - a_{js}\right) + t \sum_{i \in I} \gamma_{is} y_{is})$$

$s.t.\ \ \sum_{i \in I} x_i \leq f$

$x_i \geq b\ \forall i$

$y_{is} \leq v_{is} x_i\ \ \forall i \in I, s \in S^N$

$\sum_{j \in J} a_{js} \leq \sum_{i \in I} l_{is} y_{is}\ \ \forall s \in S^N$

$a_{js} \leq d_{js}\ \ \forall j \in J, s \in S^N$

$\sum_{i \in I} \gamma_{is} y_{is} \leq \Gamma_{\mathrm{s}}\ \ \forall s \in S^N$

$x_i \in Z_+, \forall i \in I,$

$y_{is}, a_{js} \geq 0\ \ \forall i \in I, j \in J, s \in S^N$

Appendix 3.1 Three-stage Stochastic Programming model

$$\min \sum_{i \in I} c_i^{set} x_i + E\left[Q_1\left(x, \tilde{v}, \tilde{d}, \tilde{\gamma}, \tilde{\Gamma}\right)\right]$$

$s.t.\ \ \sum_{i \in I} x_i \leq f$

$x_i \geq b\ \forall i$

$x_i \in Z_+, \forall i \in I,$

where $Q_1(x, v, d, \gamma, \Gamma) = \min \sum_{i \in I} c_i^{gen} y_i + t \sum_{i \in I} \gamma_i y_i + E\left[Q_2\left(x, \tilde{v}, \tilde{d}, \tilde{l}, \tilde{\gamma}, \tilde{\Gamma}\right) | (\tilde{v}, \tilde{d}, \tilde{\gamma}, \tilde{\Gamma})\right]$

$s.t.\ y_i \leq v_i x_i \ \ \forall i \in I$

$\sum_{i \in I} \gamma_i y_i \leq \Gamma$

$x_i \in Z_+ \ \ \forall i \in I, \ y_i \ \ \forall i \in I,$

$$where\ Q_2(x, v, d, l, \gamma, \Gamma) = min \sum_{j \in J} u_j\left(d_j - a_j\right)$$

$\sum_{j \in J} a_j \leq \sum_{i \in I} l_i y_i$

$a_j \leq d_j \ \ \forall j \in J$

$a_j \geq 0 \ \ \forall j \in J$

Appendix 3.2 Scenario-based three-stage Stochastic Programming model

$$\min \sum_{i \in I} c_i^{set} x_i + \sum_{s_1 \in S^{N_1}} \Pr_{s_1} Q_1(x, v^{s_1}, d^{s_1}, \gamma^{s_1}, \Gamma^{s_1})$$

$s.t.\ \ \sum_{i \in I} x_i \leq f$

$x_i \geq b \ \forall i$

$x_i \in Z_+, \forall i \in I,$

where $Q_1(x, v, d, \gamma, \Gamma) = \min \sum_{i \in I} c_i^{gen} y_i + t \sum_{i \in I} \gamma_i y_i +$

$\sum_{s_2 \in S^{N_{s_1}}} Pr_{s_2|s_1} Q_2(x, v^{s_1}, d^{s_1}, l^{s_2}, \gamma^{s_1}, \Gamma^{s_1})$

$s.t.\ y_i \leq v_i x_i \ \ \forall i \in I$

$\sum_{i \in I} \gamma_i y_i \leq \Gamma$

$x_i \in Z_+ \ \ \forall i \in I,\ y_i \ \ \forall i \in I,$

$$where\ Q_2(x, v, d, l, \gamma, \Gamma) = min \sum_{j \in J} u_j(d_j - a_j)$$

$\sum_{j \in J} a_j \leq \sum_{i \in I} l_i y_i$

$a_j \leq d_j \ \ \forall j \in J$

$a_j \geq 0 \ \ \forall j \in J$

Appendix 3.3 SAA three-stage Stochastic Programming model

$$\min \sum_{i \in I} c_i^{set} x_i + \sum_{s_1 \in S^{N_1}} [\frac{1}{N} (\sum_{i \in I} c_i^{gen} y_{is_1} + t \sum_{i \in I} \gamma_{is_1} y_{is_1}) + \frac{1}{\mathrm{N}^2} \sum_{s_2 \in S^{N_{s_1}}} \sum_{j \in J} u_j(d_{js_2} - a_{js_2})]$$

$s.t.\ \ \sum_{i \in I} x_i \leq f$

$x_i \geq b\ \forall i$

$y_{is_1} \leq v_{is_1} x_i \ \ \forall i \in I, s_1 \in S^{N_1}$

$\sum_{j \in J} a_{js_2} \leq \sum_{i \in I} l_{is_2} y_{is_1} \quad \forall s_2 \in S^{N_{s_1}},\ s_1 \in S^{N_1}$

$a_{js_2} \leq d_{js_2} \quad \forall j \in J, s_2 \in S^{N_2}$

$\sum_{i \in I} \gamma_{is_1} y_{is_1} \leq \Gamma_{s_1} \quad \forall\, s_1 \in S^{N_1}$

$x_i \in Z_+, \forall i \in I$

$y_{is_1}, a_{js_2} \geq 0 \quad \forall i \in I, j \in J, s_1 \in S^{N_1}, s_2 \in S^{N_2}$