Abstract

This paper assumes the reader has at least an elementary understanding of algebraic groups, and familiarity with the group of permutations of n objects, commonly depicted as S_n or SYM_n. It is also assumed that the reader is comfortable with concepts in linear algebra such as linear transformations, vector subspaces, and orthogonal projections. In the first section, I begin by describing the geometry of a family of shapes called regular simplexes, and how to find the cartesian coordinates of their vertices when they are centered at the origin. Some key properties of regular simplexes are described and utilized in the later sections of this paper.

In the next section, I give a brief overview of the definition of a graph, while emphasizing the ideas in graph theory which are important for the topics explored in this paper. Those topics are used in the proceeding section, where I connect them back to the previous construction of the vertices of a regular simplex. In that section, connections to the permutation group are explored, and I make use of linear transformations by describing reflections across subspaces whose bases are given by vectors pointing to the vertices of the simplex. The purpose of that section is to give a formal description of the symmetries of a regular simplex, using the language of linear transformations. Finally, I give a brief description of a concept in geometric group theory called a Cayley graph. I incorporate aspects from all the previous sections to give a formal construction of the Cayley graph for the permutation group, with respect to a particular generating set.

Introduction

This paper serves as an elaborate solution to an exercise found in *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups*, by John Meier. Exercise 1.48 states "Give a description of the Cayley graph of SYM_n, with respect to the generating set $S = \{(12), (23), \ldots (n\text{-}1\ n)\}$, for all $n \geq 5$" (Meier, 25). For context, this exercise follows a description of the Cayley graph for SYM_4 with respect to the same generating set. Here, SYM_n refers to the group of permutations of n objects. The author describes the geometry of the graph by its relationship with a regular tetrahedron. I had previously devised a method of constructing the vertices of a regular tetrahedron for another personal project, so I was immediately inspired to begin writing a solution to this exercise that utilized my own formal definition for the vertices of a regular tetrahedron.

I began by expanding my method for constructing the vertices of a regular tetrahedron to instead construct the vertices of the more general regular n-simplex, which is written about further in the first section of this paper: **What is a regular simplex?** In Meier's text, he gives an informal description of reflection-symmetries of the regular tetrahedron. This works well with his informal depiction of a regular tetrahedron, but in this paper, my formal construction of a regular n-simplex yields insight into a formal definition of how these reflections are represented. I define the vertices of the regular n-simplex to be vectors in $\mathbb{R}^n$, and the reflections to be specific linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, so they can be represented by square matrices. This development took the most amount of effort because, at first, I hadn't seen the need to have a formal definition in the first place. I had hypothesized that these reflections were linear transformations, but I was struggling to come up with a convenient way to represent them until I came up with the method discussed in the section titled **Permuting the vertices of a regular Simplex**. It was this development that allowed me to connect everything back to the Cayley graphs.

In addition to this paper, I have also written scripts in python which visualize the topics explored in this paper. One of the scripts shows an interactive colored tetrahedron, and a 3-dimensional Cayley graph of its symmetries, with respect to the generating set described previously. The other script shows an interactive 3-dimensional projection of a hypertetrahedron, and a 4-dimensional Cayley graph of its symmetries. That script might seem a little overwhelming because there are 120 vertices on the graph, and hundreds of edges connecting them. Not to mention that it's actually a 4-dimensional object being projected into a 3-dimensional space that's being rendered by a graphics engine to give a 2-dimensional representation of it onto a computer screen. Despite the information-overload, it's quite beautiful in its organization. The edges are colored, and all edges of the same color run parallel to each other. The graph is connected into a single object, where you can find similar patterns to the what you see in the lower-dimensional Cayley graph.

Symmetries of the Regular Simplex

What is a regular simplex?

Regular simplexes are a family of geometric objects in n dimensions. Each one of them has the unique property of having n+1 vertices in n-dimensional space which are equidistant from each other. In 2-dimensional space, the so called regular 2-simplex takes the familiar form of an equilateral triangle. In 3-dimensional space, the regular 3-simplex takes the form of a tetrahedron, otherwise known as a pyramid, each face of which is itself an equilateral triangle. Note that if you choose any 3 vertices of out of the 4 points on a tetrahedron, they form one of the 4 faces of the tetrahedron. The regular 4-simplex is known as a hypertetrahedron. The faces of a 3-dimensional object have a 4-dimensional analogy called the cells, each of which is itself a 3-dimensional object just as the faces of a 3-dimensional object have 2 dimensions. In a hypertetrahedron, each cell is itself a regular tetrahedron. Any 4 vertices chosen out of the 5 points on a hypertetrahedron form a cell. In turn, any 3 vertices chosen out of the 5 will be a face of one of the cells, thus forming an equilateral triangle. This pattern continues for the regular n-simplex.

The construction of a regular n-simplex can be a little tricky, but it follows intuitively from the pattern which connects a regular n-simplex to a regular (n+1)-simplex. First, think about all the vertices of a regular n-simplex as a set of vectors on the unit sphere in $\mathbb{R}^n$. In a simplex centered at the origin, all vectors pointing to the vertices have the same magnitude, and the sum of the vectors is **0**, this fact can be used to construct vertices of a regular n-simplex. Second, think about how the coordinates of the vertices of a regular 1-simplex can reveal the coordinates of a regular 2-simplex. There are only 2 points on the unit sphere in $\mathbb{R}$: (-1) and (1), so these are the coordinates of a regular 1-simplex. Transform these points into the vectors (-1, 0) and (1,0) in $\mathbb{R}^2$, and then scale them by $\sqrt{1-(\frac{1}{2})^2}$ and add the vector $(0, -\frac{1}{2})$ to each of them to get $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$, a pair of vectors on the unit sphere in $\mathbb{R}^2$ whose sum is (0,-1). Thus, the third point of the regular 2-simplex is (0,1). Voila! You are left with the vertices of an equilateral triangle, the coordinates of which can now be used to reveal the coordinates of the vertices of a regular 3-simplex, and so forth. To do so, you must similarly transform the vectors into $\mathbb{R}^3$, and scale them by $\sqrt{1-(\frac{1}{3})^2}$, so that all three will fall on the unit sphere after the vector

$(0, 0, -\frac{1}{3})$ is added. The result will be a set of 3 vectors on the unit sphere in $\mathbb{R}^3$ whose sum is (0,0,-1). Thus, the fourth point of the regular 3-simplex is (0,0,1).

This pattern continues, and allows you to construct the vertices of any regular (n+1)-simplex inductively from the coordinates of the vertices of a regular n-simplex. First, for each vector $(x_1, x_2, \ldots, x_n)$ in a regular n-simplex, transform it to $(x_1, x_2, \ldots, x_n, 0)$, and scale it by $\sqrt{1-(\frac{1}{n+1})^2}$. Then add the vector $(0, 0, \ldots, -\frac{1}{n+1})$ to each of the n vectors, and the $(n+1)^{\text{th}}$ vector is simply (0, 0, …, 1). All that's left is to connect edges, faces, and cells appropriately for each combination of 2, 3, 4, etc. vertices respectively.

One criticism of this method is that, while all vertices of all the regular simplexes have the same norm, the distance between two vertices changes for different dimensions. As a result, the equilateral triangle of the regular 2-simplex is a different size than the equilateral triangles on the faces of the regular 3-simplex. This, understandably, can seem like a nuisance, but simply scaling all vectors by the desired factor after the vertices have been constructed will solve this issue.

The simplex graphs

A graph is a set of vertices, also called nodes, and of edges, each consisting of a pair of vertices. The most common way to depict a graph is to plot out points on a plane which correspond to the vertices of the graph, and then connect the points with lines which correspond with the edges. But graphs don't have to be drawn on a plane, they simply consist of a set, and could be represented in 3-dimensional space, for example. There is a special family of graphs called complete graphs. Complete graphs are graphs consisting of n vertices, each connected by an edge to all other vertices. A complete graph with n vertices is referred to as K_n. If you take an n-simplex, and form a set of all the vertices and edges of the simplex, then by definition you get K_{n+1}, since all vertices in a simplex are connected. When represented as vertices and edges of a simplex, the complete graph K_n can also be called the (n-1)-simplex graph.

In graph theory there is an important notion of automorphisms for a graph. An automorphism is a bijection from a graph to itself which preserves how edges and vertices are connected. So for a graph G, if $\varphi: G \rightarrow G$ is an automorphism, then for every edge in G consisting of the pair of vertices (u,v), $\varphi(u,v)=(\varphi(u), \varphi(v))$. Under composition of functions, automorphisms of a graph form a group, with the identity being the identity automorphism which sends every vertex and edge to itself. For the complete graph K_n, the automorphism group is isomorphic to the group of permutations of n. Indeed, since any pair of vertices in K_n are connected by an edge, it is trivial to permute them and preserve the connectedness of the graph. As such, one can represent the permutation group S_n with the geometric symmetries of a regular simplex.

Permuting the vertices of a regular simplex

One of the simplest ways to generate the permutation group S_n is to use the generating set {(12), (23), …, (n-1 n)}. By arbitrarily labeling the vertices of the (n-1)-simplex graph with the labels {1, 2, ..., n}, it is easily shown that the image of the generating set under an isomorphism would be the set of automorphisms which exchange the vertices labelled 1 and 2, 2 and 3, etc. and the corresponding edges. This leads to an interesting result when you apply this to the vertices of a regular simplex constructed previously.

Consider the regular 2-simplex: an equilateral triangle whose vertices and edges represent K_3. By labeling the vertices with {1,2,3}, observe how the transpositions (12) and (23) in S_3 act on the triangle by reflecting about the line which connects the origin with the vertices 3 and 1 respectively. Here, the lines are subspaces in $\mathbb{R}^2$ spanned by the vector that is left unchanged by the transposition acting on the triangle. This is no coincidence; as stated before the sum of the vectors pointing to the vertices of any regular simplex centered at the origin is $\mathbf{0}$. Therefore, in a partition of the set of vectors into two subsets, the vector sum of one subset is the inverse of the vector sum of the other. Therefore, in a set of vectors pointing to vertices of a regular n-simplex V, $\forall$ v, w $\in$V, $\frac{v+w}{2} = \sum_{u \in V'} \frac{-u}{2}$, where V'=V \{v,w}. So, the midpoint of any two vertices is a linear combination of the other vertices. Furthermore, $\forall$ v,w $\in$V the line passing through v and w given by $\{v + s(v-w) \ | s \in \mathbb{R}\}$ is orthogonal to the subspace spanned by V \{v,w}. Therefore, the projection onto the subspace spanned by V \{v,w} of v and w is the midpoint between v and w, so the reflection across the subspace of v is w, and vice versa. Thus, if the vectors v and w represent the vertices labeled 1 and 2, then the transposition (12) acting on the simplex is represented by the reflection across the subspace spanned by V \{v,w}. The reason that it's important to frame the vertices of a simplex this way is that a reflection over a subspace is a linear transformation, so it can be expressed by left multiplication of a matrix. For some subspace W, if the P_w is the matrix for orthogonal projection onto W, then the matrix ($2P_w$ - I) reflects across W.

Let $V = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ be a set of vectors pointing to the vertices of a regular n-simplex centered at the origin, and let each vector v_i point to the vertex labeled *i*. For any pair of vertices *i* and *j*, let the subspace spanned by $V \setminus \{v_i, v_j\}$ be called (*ij*), indicative of its relationship with the transposition of the same name. Then $P_{(ij)}$ is the orthogonal projection matrix onto (*ij*), and $(2P_{(ij)} - I)$ reflects across (*ij*). Since all of $V \setminus \{v_i, v_j\}$ is contained in the subspace (*ij*), they are unchanged when multiplied on the left by $(2P_{(ij)} - I)$, while v_i and v_j are switched when multiplied on the left by $(2P_{(ij)} - I)$. Therefore, the transposition (*ij*) acts on the vertices of a regular simplex by left multiplying the matrix $(2P_{(ij)} - I)$, thus the reflection matrix represents the transposition itself.

Now, let $R = \bigcup_{k=1}^{n} \{2P_{(k\ k+1)} - I\}$. R is the set of all reflections which represent a transposition of the vertices labeled *k* and *k*+1 for $k \in \{1, 2, \ldots, n\}$. Since the symmetries of a regular n-simplex are isomorphic to the permutation group S_{n+1}, and the set of transpositions {(12), (23), …, (n n+1)} generates S_{n+1}, it follows that R is a generating set for the group of symmetries of the regular n-simplex. So, every single symmetry in the group can be represented as a linear transformation in n-space, in particular some composition of reflections over subspaces. This means that the matrices for each symmetry of a regular (n-1)-simplex form a group isomorphic to S_n, under matrix multiplication.

Cayley graphs for S_n

A Cayley graph is another special kind of graph which represents a finitely generated group. A Cayley graph is directed, meaning that edges are stylized with an arrow, and consist of an ordered pair of vertices instead of a plain pair. If a group G has a finite generating set $S = \{s_1, s_2, \ldots, s_n\}$, then a Cayley graph Γ for G generated by S is constructed as follows: For each element of G, there is one vertex in Γ. For each vertex representing the element g, $\forall\ i \in \{1, 2, \ldots, n\}$ there is a directed edge which starts at the vertex representing g and ends at the vertex representing gs_i. Cayley graphs are also colored, so there is a distinction between the edge which connects the vertex g to gs_i and gs_j for generators s_i and s_j. The colorization also puts the edges which connect g to gs_i and h to hs_i into the same category, since they have the same color. Cayley graphs are constructed this way so that left multiplication by any element $h \in G$ is represented by a symmetry of Γ which sends every vertex representing the element g, to the vertex representing hg, and which sends every directed edge representing the pair (g_1, g_2) to

(hg_1, hg_2). In a sense, the element g itself is represented by this symmetry of Γ. In fact, G is isomorphic to the entire symmetry group of Γ.

The Cayley graph for S_n with respect to the generating set discussed above is, in my opinion, the most beautiful result of the prior construction for the vertices of a regular (n-1)-simplex. To construct the Cayley graph for S_n, first consider the group of symmetries for a regular (n-1)-simplex. This group is homomorphic to the group of linear transformations from $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-1}$, by the homomorphism which transforms each symmetry to the linear transformation which produces that symmetry on a regular (n-1)-simplex centered at the origin. The matrices which correspond to these linear transformations are helpful for computing a concrete example. This homomorphism is from S_n to the group $GL_{n-1}(\mathbb{R})$, and is a group action on $\mathbb{R}^{n-1}$, where $GL_{n-1}(\mathbb{R})$ is the implied group of symmetries for $\mathbb{R}^{n-1}$ as a mathematical object. The image of S_n under the homomorphism is a subgroup of $GL_{n-1}(\mathbb{R})$. Since it's a group action, the orbit-stabilizer theorem applies.

In the Cayley graph for S_n, there must be exactly one vertex which represents the identity symmetry. The orbit of this vertex must be the same size as S_n, so the stabilizer must be trivial. One way to pick such a point in $\mathbb{R}^{n-1}$ is to consider the union of all subspaces which are being reflected across by one of the generators, and to pick a point that is not in that union. The orbit for the chosen vertex will be all the vertices for the Cayley graph. Normally a Cayley graph is directed, but the generating set only contains transpositions, so all the generators are their own inverses. As such, a vertex g in the Cayley graph would have an edge connecting g to gs for a generator s, but the vertex gs would have an edge connecting gs to gss = g. Since the connection, while directed, is always going to be bidirectional anyway, there's no reason to stylize the edges with directions in the first place. So, for the construction of the Cayley graph, once you have all the vertices, all that's left is to connect them to their reflections over each subspace which is considered for the generating set of reflections, and color the edges according to which multiplication they represent. Incidentally, all edges of the same color by this construction will be parallel, and can be thought to trace the reflection itself. This is what makes the action of S_n on $\mathbb{R}^{n-1}$ so beautiful.