The Drawing Trick

and its implications in higher dimensions

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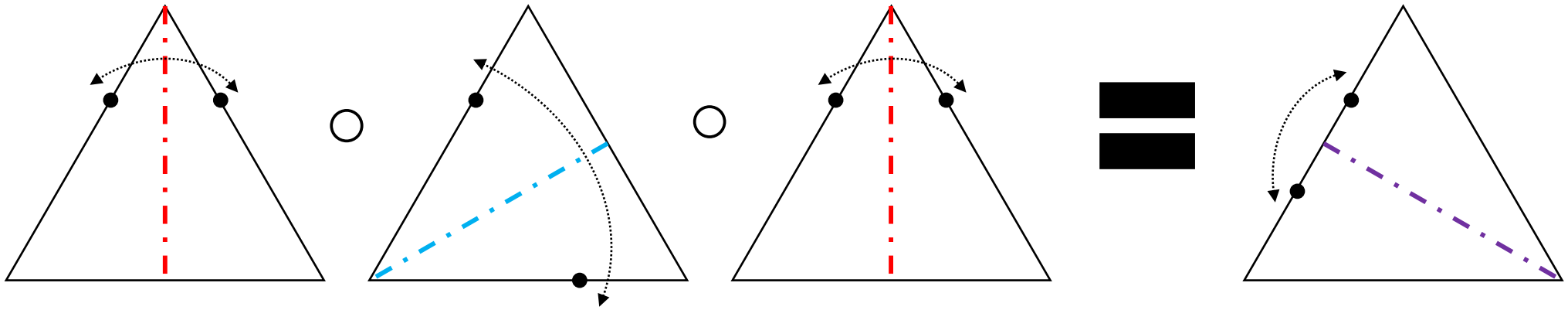
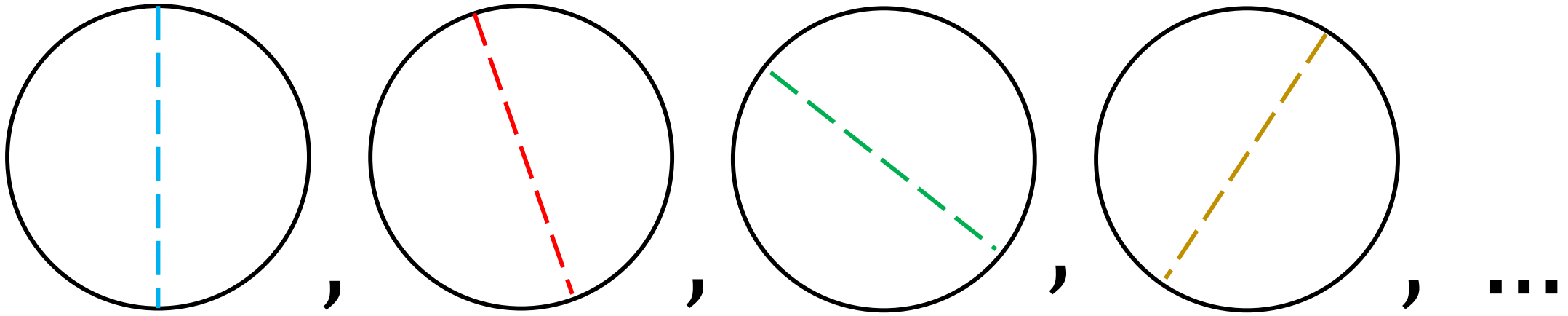
Background Info: Abstract Algebra

- Symmetry: a bijection from elements of a set to itself
- All mathematical objects (sets) can have some notion of symmetry
- When considering a set with structure (like a geometric object), a symmetry of that set should preserve that same structure



Background Info: Abstract Algebra

- Groups of symmetries like that of an equilateral triangle are special:
  - **Finite Presentation / Finitely Generated**
- For contrast, consider the symmetries of a circle



Background Info: Graphs and Symmetries

- Definition:
  - A graph is a set of vertices and edges which connect a pair of vertices
  - For connected vertices u and v, the pair (u, v) is the edge which connects them
- Graphs can have color and direction
  - Color is a way to partition the edges
  - A graph with direction redefines edges to be **ordered pairs** of vertices instead of pairs.

Background Info: Graphs and Symmetries

- For a graph Γ, a symmetry is defined by a bijection:

$$\varphi: \Gamma \to \Gamma$$
$$\varphi: (u, v) \mapsto \big(\varphi(u), \varphi(v)\big)$$

- ‘A symmetry preserves the connectedness of a graph’
- Additionally, symmetries of a graph with color must preserve color
- Symmetries of a directed graph must map directed edges to directed edges

Examples: Symmetries of graphs

Figure A

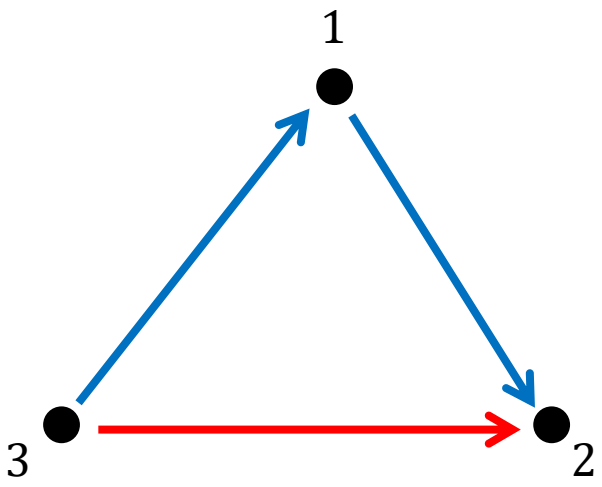
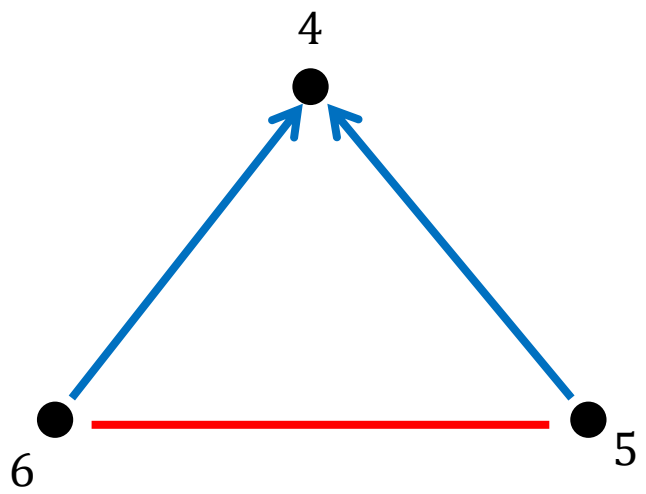
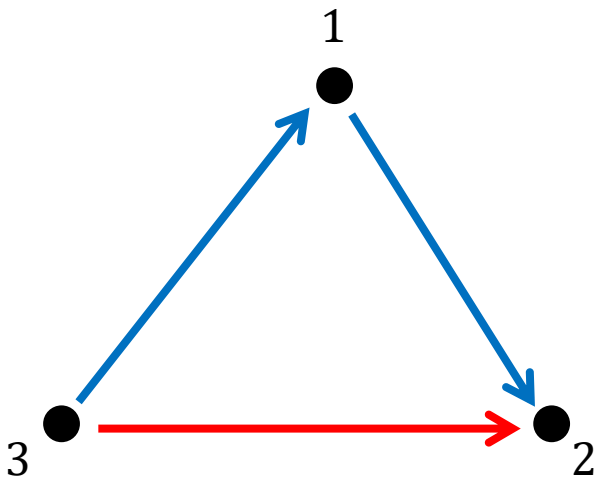


Figure B



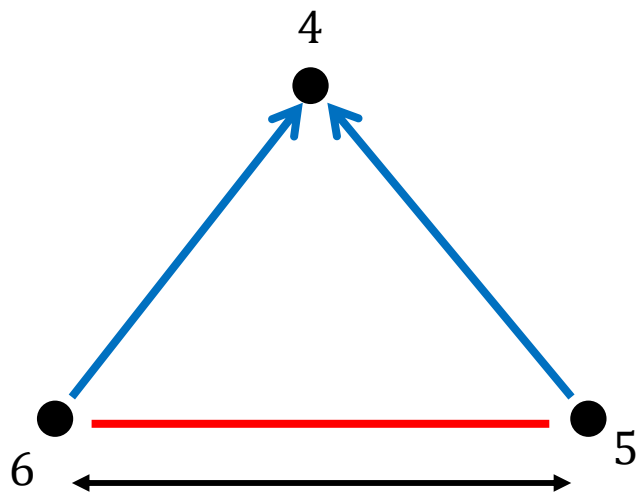
Examples: Symmetries of graphs

Figure A



No non-trivial symmetries

Figure B



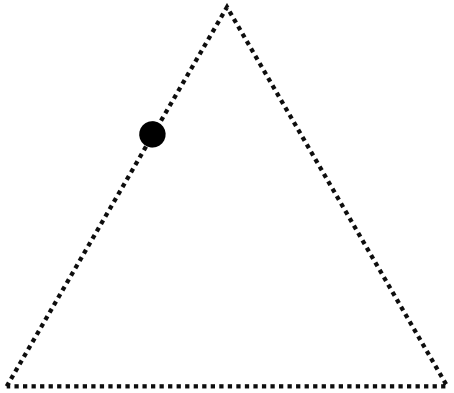
$\varphi: 5 \mapsto 6$
$\varphi: 6 \mapsto 5$
$\varphi: [5,4) \mapsto [6,4)$
$\varphi: [6,4) \mapsto [5,4)$

Main Topic: Cayley graphs and the drawing trick

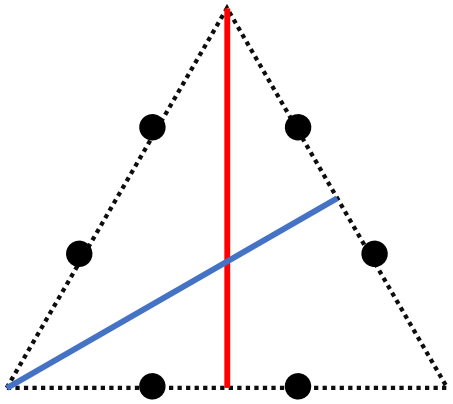
- Any group with **finite presentation** can be represented by the symmetries of a colored, directed graph
  - For a group G generated by a finite subset S, such a graph is called **the Cayley graph of G with respect to generators S**
- The drawing trick is an algorithm that constructs the Cayley graph of the group of symmetries for some geometric object, such as an equilateral triangle, which has a finite presentation.

The Drawing Trick on an Equilateral Triangle

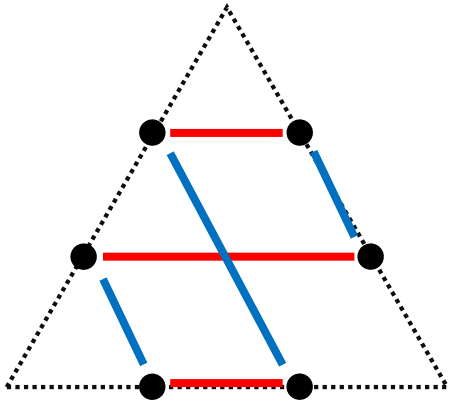
Start with a well chosen point:



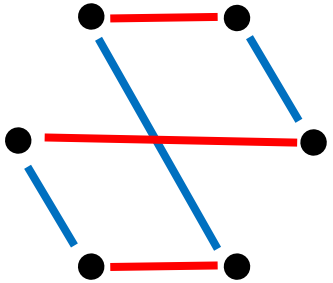
Repeatedly apply generating symmetries:



Connect the points together colored according to generator:

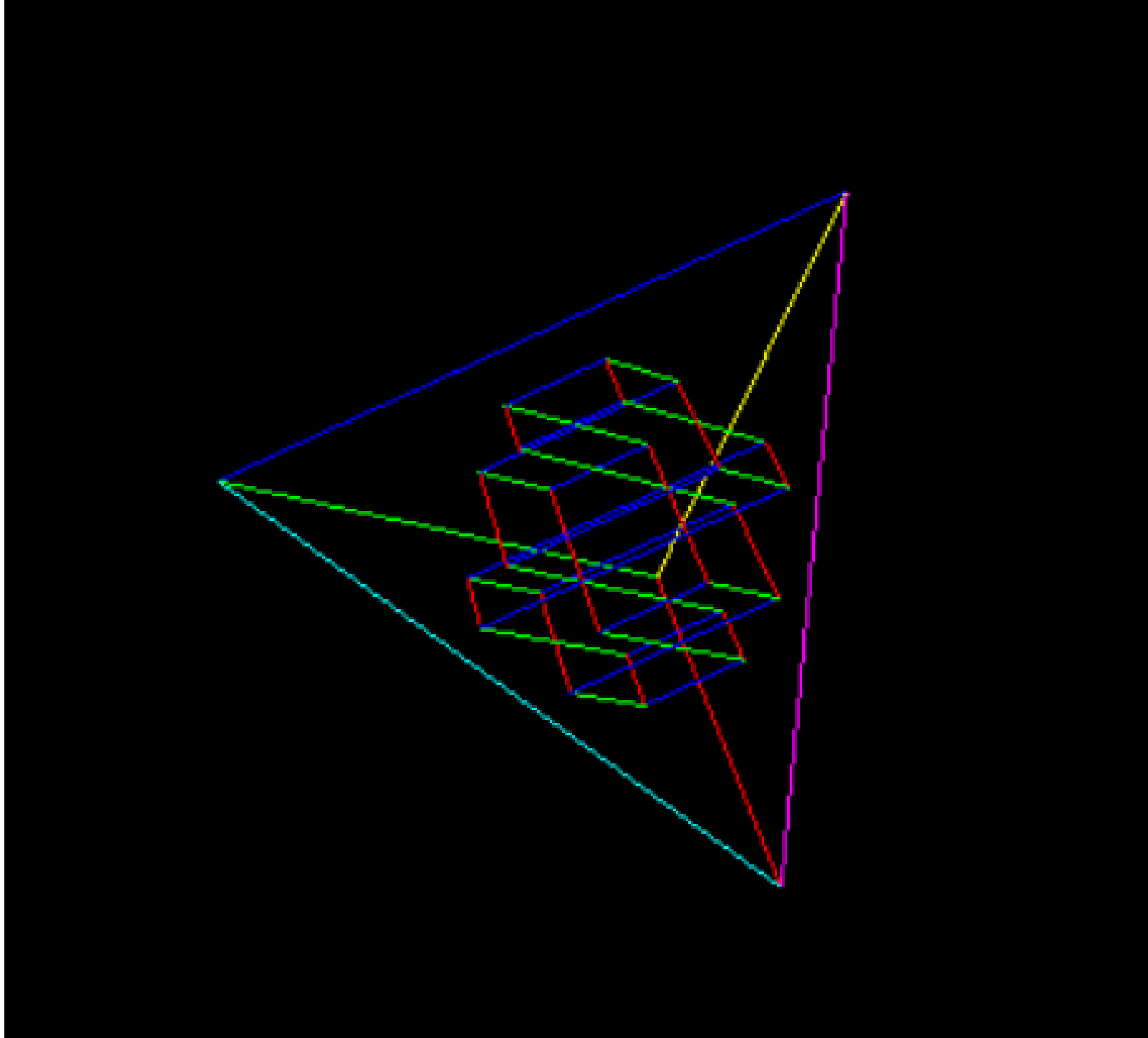


Final result:



The Drawing Trick works in 3 dimensions too!

- Just like an equilateral triangle, the symmetries of the regular tetrahedron are also finitely presented
- Here, the outer tetrahedron is shown. On the inside is the Cayley graph, constructed using the drawing trick!



References

Meier, John. *Groups, Graphs, and Trees: An Introductions to the Geometry of Infinite Groups*. London Mathematical Society Student Texts 73, Cambridge University Press, 2008.