1. **(3 points)** How many numbers between 1 and 100,000,000 are not divisible by 2, 3, 5? Make sure you give the principle you used (1 point). No need to actually do the division. That is, feel free to write your answers in terms of things like $\lfloor 100000000/5 \rfloor$.

$100{,}000{,}000 = 10^8$

$$10^8 - \left[\left\lfloor \frac{10^8}{2} \right\rfloor + \left\lfloor \frac{10^8}{3} \right\rfloor + \left\lfloor \frac{10^8}{5} \right\rfloor - \left\lfloor \frac{10^8}{6} \right\rfloor - \left\lfloor \frac{10^8}{10} \right\rfloor - \left\lfloor \frac{10^8}{15} \right\rfloor + \left\lfloor \frac{10^8}{30} \right\rfloor\right]$$

inclusion-exclusion, or $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

2. **(7 points)** Prove the Binomial Theorem using Induction. That is, prove that

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Induction Hypothesis (1 point): $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Base Case (1 point):

$(x+y)^0 = 1 \quad \sum_{k=0}^{0} \binom{0}{k} x^{0-k} y^k = \binom{0}{0} x^0 y^0 = 1$

or

$(x+y)^1 = x+y \quad \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1$

$= x + y$

Inductive Step (5 points):

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

$$= (x+y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k+1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^{n-k+1} y^k$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n-k+1} y^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{n-k+1} y^k + y^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] x^{n-k+1} y^k + y^{n+1}$$

$$= x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n-k+1} y^k + y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n-k+1} y^k$$