1. **(5 points)** If if $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 2$ and $a_1 = 1$, Find the closed form for this linear homogeneous recurrence relation of order 2.

$a_n = 7a_{n-1} - 10a_{n-2} \rightarrow$ characteristic equation $r^2 = 7r - 10$, i.e. $r^2 - 7r + 10 = 0$.

Factoring, $r^2 - 7r + 10 = (r-2)(r-5)$, so the solution takes the form $a_n = \alpha_1 2^n + \alpha_2 5^n$.

Initial conditions: $a_0 = 2 = \alpha_1 2^0 + \alpha_2 5^0 = \alpha_1 + \alpha_2$ ①

$a_1 = 1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 5^1 = 2\alpha_1 + 5\alpha_2$. ②

Solve any way you like. I add the equations together:

$$\begin{array}{r} 2\alpha_1 + 5\alpha_2 = 1 \\ -2\alpha_1 - 2\alpha_2 = -4 \\ \hline 3\alpha_2 = -3 \end{array} \Rightarrow \alpha_2 = -1.$$

Substitute into eqn. ① to get $\alpha_1 - 1 = 2 \rightarrow \alpha_1 = 3$.

So the general solution is $\boxed{a_n = 3 \times 2^n - 5^n}$

2. **(5 points)** Prove your closed form is correct by induction on n.

Proof by induction that the recurrence $a_n = 7a_{n-1} - 10a_{n-2}$ with initial conditions $a_0 = 2$ and $a_1 = 1$ is solved by $a_n = 3 \cdot 2^n - 5^n$:

Base case: we need two because it's a second-order recurrence.

① $a_0 = 2 = 3 \cdot 2^0 - 5^0 = 3 - 1 = 2$ ✓

② $a_1 = 1 = 3 \cdot 2^1 - 5^1 = 6 - 5 = 1$ ✓

Inductive step: assume the inductive hypothesis that for some $k \in \mathbb{N}$, $k \geq 2$, $a_{k-1} = 3 \cdot 2^{k-1} - 5^{k-1}$ AND $a_k = 3 \cdot 2^k - 5^k$.

Want to prove that $a_{k+1} = 3 \cdot 2^{k+1} - 5^{k+1}$. Use the brain-dead induction recipe, so start with the LHS and use the recurrence to create a form that allows us to apply the IH:

$a_{k+1} = 7a_k - 10a_{k-1} = 7(3 \cdot 2^k - 5^k) - 10(3 \cdot 2^{k-1} - 5^{k-1})$ by IH.

$= 7 \cdot 3 \cdot 2^k - 5 \cdot 2 \cdot 3 \cdot 2^{k-1} - 7 \cdot 5^k + 2 \cdot 5 \cdot 5^{k-1}$

$= 7 \cdot 3 \cdot 2^k - 5 \cdot 3 \cdot 2^k - 7 \cdot 5^k + 2 \cdot 5^k$

$= (7-5) \cdot 3 \cdot 2^k - (7-2) \cdot 5^k$

$= 2 \cdot 3 \cdot 2^k - 5 \cdot 5^k$

$= 3 \cdot 2^{k+1} - 5^{k+1}$ ∎