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(** HW2 **)
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In[1435]:=

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(*1*)
Remove["Global`*"]

r1[k_] := {x[k], y[k], z[k]};
r0 = {0, 0, 0};
r2[k_] := r1[k] - r0

δ[i_, j_] := If[i == j, 1, 0];

elem[i_, j_] := Sum[m[k] (δ[i, j] r2[k].r2[k] - r2[k][[i]] r2[k][[j]]), {k, 1, NPART}]

inertia := Table[elem[i, j], {i, 1, 3}, {j, 1, 3}];

m[1] = m1;
r1[1] = {x1, y1, z1};
NPART = 1;

prince = Eigensystem[inertia];

Print["If we have a single particle of mass ", m1, " located at r1=",
 r1[1], " our moment of inertia tensor looks like:\n", "I = ",
 inertia // MatrixForm, "\n Exactly as we would expect it to be!!!"]

Print["\nThe principle moments of the system are: \n", prince[[1]]]
Print["\nThe principle axes of the system are: \n", prince[[2]]]
```

If we have a single particle of mass m1 located at r1=
{x1, y1, z1} our moment of inertia tensor looks like:

$$I = \begin{pmatrix} m1\,(y1^2 + z1^2) & -m1\,x1\,y1 & -m1\,x1\,z1 \\ -m1\,x1\,y1 & m1\,(x1^2 + z1^2) & -m1\,y1\,z1 \\ -m1\,x1\,z1 & -m1\,y1\,z1 & m1\,(x1^2 + y1^2) \end{pmatrix}$$

Exactly as we would expect it to be!!!

The principle moments of the system are:

$$\{0,\ m1\,(x1^2 + y1^2 + z1^2),\ m1\,(x1^2 + y1^2 + z1^2)\}$$

The principle axes of the system are:

$$\left\{\left\{\frac{x1}{z1}, \frac{y1}{z1}, 1\right\}, \left\{-\frac{z1}{x1}, 0, 1\right\}, \left\{-\frac{y1}{x1}, 1, 0\right\}\right\}$$

```
(*2*)
Clear[θ, ϕ]
sph = {Sin[θ] Cos[ϕ], Sin[θ] Sin[ϕ], Cos[θ]};

NPART = 2;

m[2] = m2;

r1[1] = r1 * sph;
r1[2] = r2 * -sph;

prince = Eigensystem[inertia];

Print["The moment of inertia tensor for this system looks like:\n",
 "I = " , inertia // MatrixForm // Simplify]

Print["\nThe principle moments of the system are: \n", prince[[1]]]
Print["\nThe principle axes of the system are: \n", prince[[2]]]
```

The moment of inertia tensor for this system looks like:

$$I = \begin{pmatrix} (m1\ r1^2 + m2\ r2^2)\ (\text{Cos}[\theta]^2 + \text{Sin}[\theta]^2\ \text{Sin}[\phi]^2) & -(m1\ r1^2 + m2\ r2^2)\ \text{Cos}[\phi]\ \text{Sin}[\theta]^2\ \text{Sin}[\phi] & -(m\text{:} \\ -(m1\ r1^2 + m2\ r2^2)\ \text{Cos}[\phi]\ \text{Sin}[\theta]^2\ \text{Sin}[\phi] & (m1\ r1^2 + m2\ r2^2)\ (\text{Cos}[\theta]^2 + \text{Cos}[\phi]^2\ \text{Sin}[\theta]^2) & -(m\text{:} \\ -(m1\ r1^2 + m2\ r2^2)\ \text{Cos}[\theta]\ \text{Cos}[\phi]\ \text{Sin}[\theta] & -(m1\ r1^2 + m2\ r2^2)\ \text{Cos}[\theta]\ \text{Sin}[\theta]\ \text{Sin}[\phi] & \end{pmatrix}$$

The principle moments of the system are:

$\{m1\ r1^2 + m2\ r2^2,\ m1\ r1^2 + m2\ r2^2,\ 0\}$

The principle axes of the system are:

$\{\{-\text{Cos}[\phi]\ (\text{Csc}[\theta]\ \text{Sec}[\theta]\ \text{Sec}[\phi]^2 - \text{Tan}[\theta] - \text{Tan}[\theta]\ \text{Tan}[\phi]^2),\ 0,\ 1\},$
$\{-\text{Tan}[\phi],\ 1,\ 0\},\ \{\text{Cos}[\phi]\ \text{Tan}[\theta],\ \text{Sin}[\phi]\ \text{Tan}[\theta],\ 1\}\}$

```
(*3*)

Clear[θ, ϕ]
M = m[1] + m[2];

rcm = 1/M Table[m1 r1[1][[i]] + m2 r1[2][[i]], {i, 1, 3}];

r0 = rcm;

Print[
 "The moment of inertia tensor taken wrt the cm of the system is, in general:\n",
 "I = ", inertia // MatrixForm // Simplify]

θ = Pi/2;
ϕ = 0;

Print["When θ = π/2, ϕ = 0, the moment of inertia tensor becomes: \n",
  "I = ", inertia // MatrixForm // Simplify,
  "\n This matrix corresponds to the equivalent one body problem
    with reduced mass μ= m1 m2/(m1 + m2) rotating about the x-axis!!!"];
```

The moment of inertia tensor taken wrt the cm of the system is, in general:

$$I = \begin{pmatrix} \frac{\text{m1 m2 (r1+r2)}^2 (6+2\,\text{Cos}[2\theta]+\text{Cos}[2(\theta-\phi)]-2\,\text{Cos}[2\phi]+\text{Cos}[2(\theta+\phi)])}{8\,(\text{m1+m2})} & -\frac{\text{m1 m2 (r1+r2)}^2\,\text{Sin}[\theta]^2\,\text{Sin}[2\phi]}{2\,(\text{m1+m2})} \\ -\frac{\text{m1 m2 (r1+r2)}^2\,\text{Sin}[\theta]^2\,\text{Sin}[2\phi]}{2\,(\text{m1+m2})} & -\frac{\text{m1 m2 (r1+r2)}^2 (-6-2\,\text{Cos}[2\theta]+\text{Cos}[2(\theta-\phi)]-2\,\text{Cos}[2\phi]+\text{Cos}[2}{8\,(\text{m1+m2})} \\ -\frac{\text{m1 m2 (r1+r2)}^2\,\text{Cos}[\phi]\,\text{Sin}[2\theta]}{2\,(\text{m1+m2})} & -\frac{\text{m1 m2 (r1+r2)}^2\,\text{Sin}[2\theta]\,\text{Sin}[\phi]}{2\,(\text{m1+m2})} \end{pmatrix}$$

When $\theta = \frac{\pi}{2}$, $\phi = 0$, the moment of inertia tensor becomes:

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\text{m1 m2 (r1+r2)}^2}{\text{m1+m2}} & 0 \\ 0 & 0 & \frac{\text{m1 m2 (r1+r2)}^2}{\text{m1+m2}} \end{pmatrix}$$

This matrix corresponds to the equivalent one body problem with reduced mass $\mu = \frac{\text{m1 m2}}{\text{m1 + m2}}$ rotating about the x-axis!!!

```
In[1749]:= (*4*)
Remove["Global`*"]

mass = M;

δ[i_, j_] := If[i == j, 1, 0];

ρ[r_, ϕ_, θ_] = k ;

x = r Sin[θ] Cos[ϕ];
y = r Sin[θ] Sin[ϕ];
z = r Cos[θ];

dr = {r, a, b};
dθ = {θ, 0, Pi/2};
dϕ = {ϕ, 0, 2 Pi};
dV = r^2 Sin[θ];

k = k /. Solve[mass == Integrate[ρ[r, ϕ, θ] dV, dr, dϕ, dθ], k][[1]];

xcm = 1/M Integrate[ρ[r, ϕ, θ] x dV, dr, dϕ, dθ];
ycm = 1/M Integrate[ρ[r, ϕ, θ] y dV, dr, dϕ, dθ];
zcm = 1/M Integrate[ρ[r, ϕ, θ] z dV, dr, dϕ, dθ] ;

rcm = {xcm, ycm, zcm};

r1 = {x, y, z};
r0 = rcm;
r2 := r1 - r0;

elem[i_, j_] := Integrate[ρ[r, θ, ϕ] dV (δ[i, j] r2.r2 - r2[[i]] r2[[j]]), dr, dθ, dϕ]

inertia := Table[elem[i, j], {i, 1, 3}, {j, 1, 3}]

prince = Eigensystem[inertia];

Print["The moment of inertia tensor about the center of mass of the uniform
   hemisphere is: \n", "I_hemi = ", inertia // MatrixForm // Simplify]

Print["\nThe principle moments of the system are: \n", prince[[1]]]
Print["\nThe principle axes of the system are: \n", prince[[2]]]
```

The moment of inertia tensor about the center of mass of the uniform hemisphere is:

$$I_{hemi} = \left(\begin{array}{cc} \frac{(83\, a^6+166\, a^5\, b+249\, a^4\, b^2+204\, a^3\, b^3+249\, a^2\, b^4+166\, a\, b^5+83\, b^6)\, M}{320\, (a^2+a\, b+b^2)^2} & 0 \\ 0 & \frac{(83\, a^6+166\, a^5\, b+249\, a^4\, b^2+204\, a^3\, b^3+249\, a^2\, b^4+166\, a\, b^5+83\, b^6)\, M}{320\, (a^2+a\, b+b^2)^2} \\ 0 & 0 \end{array}\right.$$

The principle moments of the system are:

$$\Big\{\frac{2\, (a^4+a^3\, b+a^2\, b^2+a\, b^3+b^4)\, M}{5\, (a^2+a\, b+b^2)},$$
$$((83\, a^6+166\, a^5\, b+249\, a^4\, b^2+204\, a^3\, b^3+249\, a^2\, b^4+166\, a\, b^5+83\, b^6)\, M) \Big/ \left(320\, (a^2+a\, b+b^2)^2\right),$$
$$((83\, a^6+166\, a^5\, b+249\, a^4\, b^2+204\, a^3\, b^3+249\, a^2\, b^4+166\, a\, b^5+83\, b^6)\, M) \Big/ \left(320\, (a^2+a\, b+b^2)^2\right)\Big\}$$

The principle axes of the system are:

{{0, 0, 1}, {0, 1, 0}, {1, 0, 0}}

In[1704]:=

```
(*5*)

Clear[x, y, z, k]

ρ[x_, y_, z_] = k (x^2 + y^2) ;

dx = {x, 0, a};
dy = {y, 0, a};
dz = {z, 0, a};
dV = 1;

k = k /. Solve[mass == Integrate[ρ[x, y, z] dV, dx, dy, dz], k][[1]];

xcm = 1/M Integrate[ρ[x, y, z] x dV, dx, dy, dz];
ycm = 1/M Integrate[ρ[x, y, z] y dV, dx, dy, dz];
zcm = 1/M Integrate[ρ[x, y, z] z dV, dx, dy, dz] ;

rcm = {xcm, ycm, zcm};

r1 = {x, y, z};
r0 = rcm;
r2 := r1 - r0;

elem[i_, j_] := Integrate[ρ[x, y, z] dV (δ[i, j] r2.r2 - r2[[i]] r2[[j]]), dx, dy, dz]
inertia := Table[elem[i, j], {i, 1, 3}, {j, 1, 3}]

prince = Eigensystem[inertia];

Print["The moment of inertia tensor about the
   center of mass of a cube with density ρ[x,y,z]= ", ρ[x, y, z],
  " and side length a is: \n", "I_cube = ", inertia // MatrixForm // Simplify]

Print["\nThe principle moments of the system are: \n", prince[[1]]]
Print["\nThe principle axes of the system are: \n", prince[[2]]]
```

The moment of inertia tensor about the center of mass of a cube with density $\rho[x,y,z] = \frac{3 M (x^2 + y^2)}{2 a^5}$ and side length a is:

$$I_{cube} = \begin{pmatrix} \frac{51 a^2 M}{320} & \frac{a^2 M}{64} & 0 \\ \frac{a^2 M}{64} & \frac{51 a^2 M}{320} & 0 \\ 0 & 0 & \frac{73 a^2 M}{480} \end{pmatrix}$$

The principle moments of the system are:

$$\left\{\frac{7 a^2 M}{40}, \frac{73 a^2 M}{480}, \frac{23 a^2 M}{160}\right\}$$

The principle axes of the system are:

{{1, 1, 0}, {0, 0, 1}, {-1, 1, 0}}