Complex Analysis Notes

Bryan Lu

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0 Foreword

These are notes for the FA21 MATH 2230 study group for complex analysis over break. These are intended as a summary of the material presented in Stein-Shakarchi and Ahlfors, with supplements from Chapter V of *Advanced Mathematical Techniques for Scientists and Engineers* by Osborne.

Complex analysis is a wonderful subfield of analysis and is useful for a lot of math/physics majors! It's relatively independent from the study of real analysis, and arguably (in my opinion) it's a lot more fun and elegant than its real counterpart! :)

Some prerequisites – if you've taken MATH 2230 at Cornell, this should be accessible to you. Otherwise, the prerequisites are an interest in rigor, and some level of mathematical sophistication, i.e. some exposure to real analysis, particularly in regards to limits, continuity, and differentiation, and of course at least some familiarity with manipulating complex numbers. Some basic notions will be skimmed over here, as... well... it's really very similar to analysis in $\mathbb{R}^2$ in this respect so we skim over it with the intention of focusing on the highlights instead.

Many of the earlier diagrams in these notes are courtesy of Mathematica – it can make some pretty cool stuff! Unfortunately, as of the time of writing I temporarily cannot generate these plots. :(Later diagrams that are not computer generated are drawn in Inkscape and hopefully look pretty clean.

1 Welcome to Complex Land!

Analysis, of course, is a rigorous discipline that gives us a lot of the structure underlying calculus. However, the point of this first section is to be less concerned with the rigor and is more focused on introducing ourselves to some of the common actors.

1.1 The Stage: The Complex Numbers

Welcome to $\mathbb{C}$, the set of numbers over which we will assume to be working with in the entirety of these notes! We will take the following as our definition of $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b, \in \mathbb{R}, i^2 = -1\}$$

This will serves our purposes just fine. $\mathbb{R}$, of course, is the normal field of real numbers, which I hope we're all familiar with. If we were super concerned about rigor, this loose definition can be made into a more rigorous construction of $\mathbb{C}$, but we'll refrain from doing so here. We will assume some standard things about $\mathbb{R}$, that being that $\mathbb{R}$ is a field with a total ordering, and is complete (in particular, the completion of the rational numbers $\mathbb{Q}$ under the standard Euclidean metric).

We will endow the complex numbers with the standard operations of a field (addition and multiplication), and we'll assume you know how these operations work already, but just for the sake of defining it:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bd)i.$$

Every complex number $z = a + bi$ also has a *conjugate*, $\bar{z} = a - bi$. a is said to be the *real part* of z, $\mathrm{Re}(z)$, and b is the *imaginary part* of z, $\mathrm{Im}(z)$.

I would like to place the main focus of this section on endowing the complex numbers with a geometric interpretation as being similar to $\mathbb{R}^2$ by plotting them on the plane, like so:

This allows us to essentially interpret addition in $\mathbb{C}$ as essentially the same as vector addition in $\mathbb{R}^2$, except the imaginary part carries the second component. In this case, we see that we can also represent any complex number z in a polar form – in particular, in terms of its distance to the origin, and the angle it makes with the positive real axis. We can thus write

$$z = r(\cos\theta + i \sin\theta),$$

where r is a non-negative real number, and θ is any real number. We call r the *absolute value*, (or *modulus*, magnitude, or norm) of z, $|z|$, and θ the *argument* of z, $\arg z$.

We can calculate the modulus and argument similarly to how it's done when dealing with polar coordinates – if $z = a + bi$, $|z| = (a^2 + b^2)^{\frac{1}{2}}$, and $\tan(\arg z) = \frac{b}{a}$. We can also say that $|z|^2 = \bar{z} \cdot z$, as a useful corollary. Moreover, the norm satisfies the *triangle inequality*, so $|z + w| \leq |z| + |w|$, and as a corollary we also have the useful inequality $||z| - |w|| \leq |z - w|$.

The modulus will (like in real analysis) be used extensively later when we begin discussing the actual analysis. But for now, we'll discuss the argument further especially in relation to its behavior on $\mathbb{C}$. We first establish *Euler's Formula* (one of them) which establishes that

$$e^{i\theta} = \cos\theta + i \sin\theta = \mathrm{cis}\,\theta.$$

I personally strongly favor the exponential form over the cis form, unless I can't avoid it. I won't get into the weeds with what an appropriately rigorous proof of this looks like – if we take $\sin$ and $\cos$ to be defined based on geometry and the unit circle, this is a bit of a nontrivial task, but other definitions based on their (intended) Taylor series (or equivalently, as solutions to a certain differential equation) make this statement very clearly true. (Of course, such Taylor series have to be shown to converge and stuff – and also other nice properties you might want out of $\sin$ and $\cos$ are not so obvious, i.e. periodicity.) This is all very technical and we don't really care, it's the real analyst's job to figure it out.

Geometrically, this statement allows us to think of complex number multiplication in a nice way, if we choose to work in polar coordinates. If for two complex numbers z and w we have $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$, we have that $zw = r_1 r_2 e^{i(\theta_1+\theta_2)}$, i.e. the argument adds and the magnitudes multiply, by nature of the exponential properties.

Okay, but back to the argument. The argument of z is not a function, because of the periodicity in $\sin$ and $\cos$: $\operatorname{cis} 2\pi = e^{2i\pi} = 1$, so that means that if $z = re^{i\theta}$, z is also $re^{i(\theta+2\pi)}$. As stated here, the argument can take on infinitely many values. In order to make this a function, we let the *principal value* of the argument function be the value $\theta \in (-\pi, \pi]$ such that $z = re^{i\theta}$. **(This is not necessarily a standard choice! This is the choice that I am familiar with, however.)** We denote this argument θ as $\operatorname{Arg}(z)$, which makes this an actual function. This choice of the range of $\operatorname{Arg}(z)$ may seem somewhat arbitrary, and it will have **immense** consequences and implications for us down the line, but it also turns out the choice of how the range of $\operatorname{Arg}(z)$ is defined will not matter too much, and this choice will just be nicer for us to work with.

This issue with the argument carries over also to taking roots of complex numbers/taking a complex number to a non-integer power. For simplicity, let's work with the number $w = -1 + \sqrt{3}i = 2e^{i\frac{2\pi}{3}}$. Notice that we could square it, cube it, raise it to the nth power, etc. however we wish:

$$w^2 = \left(2e^{i\frac{2\pi}{3}}\right)^2 = 4e^{i\frac{4\pi}{3}} = -2 - 2i\sqrt{3}$$

$$w^3 = \left(2e^{i\frac{2\pi}{3}}\right)^3 = 8e^{2\pi i} = 8$$

$$w^n = \left(2e^{i\frac{2\pi}{3}}\right)^n = 2^n e^{i\frac{2\pi n}{3}}$$

In general, we get *De Moivre's Theorem* directly from the properties of exponentiation, which states for integer n and general θ:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

This should follow from considering the nth power of $z = re^{i\theta}$. What, then, is the issue if n is not an integer? Suppose we were to try to take the square root, i.e. raise the number $w = 2e^{i\frac{2\pi}{3}}$ to the $\frac{1}{2}$th power. We could do it this way, referring to the regular old square root on the reals for convenience:

$$w^{\frac{1}{2}} = \left(2e^{i\frac{2\pi}{3}}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} e^{i\frac{\pi}{3}} = \sqrt{2} + i\sqrt{6}$$

We could have also taken the square root this way:

$$w^{\frac{1}{2}} = \left(2e^{i\frac{2\pi}{3}}\right)^{\frac{1}{2}} = \left(2e^{i\frac{2\pi}{3}} \cdot e^{2\pi i}\right)^{\frac{1}{2}} = 2^{\frac{1}{2}} e^{i\frac{4\pi}{3}} = -\sqrt{2} - i\sqrt{6}$$

Again, we encounter the same problem that we encountered in the real numbers! *If we are simply asking "what number, when squared, gives w?" then we are asking for trouble, because there are two such numbers that satisfy this condition.* ***We have to make a choice.*** In the real line, we chose the "positive square root," and similarly here, we will choose the former ($\sqrt{2} + i\sqrt{6}$), even though "positive" is not a word that makes sense in $\mathbb{C}$.

You might ask where this problem was when we were taking cube roots and other odd roots on the real line – those functions didn't have branches that we had to choose from on $\mathbb{R}$. Not so fast. We can find all three cube roots of $\omega = i = e^{i\frac{\pi}{2}}$, where the other three roots can no longer hide from us in $\mathbb{C}$:

$$i^{\frac{1}{3}} \in \{e^{i\frac{\pi}{6}}, e^{i\frac{5\pi}{6}}, e^{i\frac{3\pi}{2}}\}$$

This issue is caused in parts by the nth *roots of unity* – i.e. the complex numbers when raised to the nth power, give 1. This allows us to have n distinct nth roots of any complex number z, for n integer.

The problem is even worse when we are raising numbers to a general power! Consider, for a moment, $i^{\sqrt{2}}$:

$$i^{\sqrt{2}} \in \left\{ \left(e^{i\left(\frac{\pi}{2}+2\pi n\right)} \right)^{\sqrt{2}}, n \in \mathbb{Z} \right\}$$

or even the classic i^i:

$$i^i \in \left\{ \left(e^{i\left(\frac{\pi}{2}+2\pi n\right)} \right)^{i}, n \in \mathbb{Z} \right\} = \left\{ e^{-\left(\frac{\pi}{2}+2\pi n\right)}, n \in \mathbb{Z} \right\}$$

Therefore, we have infinitely many numbers that are the $\sqrt{2}$th power of i, as we can never get the same number twice for two different integer n (if we could, this would imply $\sqrt{2}$ is rational, which it still isn't), and i^i pretty clearly can take on infinitely many real values. Thus, raising numbers to powers is not even *close* to a function, especially if it produces infinitely many possible values!

This is a horrific issue, especially since we should expect such a simple function to behave in not such a nasty way. The issue, again, stems from the fact that we had infinitely many choices for the argument, and multiplying by an arbitrary complex number power can sometimes cause these values to overlap, although frequently this does not happen. The fix is the same – we call the *principal root or power* or a complex number the one value we get when we write the number we are taking a power of with its principal argument. This fixes the value of the power function and gives us one and only one answer:

$$\sqrt[3]{i} = e^{i\frac{\pi}{6}} \quad i^{\sqrt{2}} = e^{i\frac{\pi\sqrt{2}}{2}} \quad i^i = e^{-\frac{\pi}{2}}$$

Our convention that we adopt – real numbers raised to real, non-integer powers will be treated like the principal roots, and complex numbers placed under radical signs will be implicitly assumed to be principal. Raising a general complex number enclosed in parentheses, however, to a non-integer power will produce the multitude of solutions that we see here.

This is a choice we have to make, and it will be one with fairly significant consequences. We will explore what we have done to the complex plane with the Argument once we have more tools to visualize this, but first we will discuss more of the common functions.

1.2 The Famous Players: Calculator Functions

Recall the exponential function e^z, which we can equivalently define over all $\mathbb{C}$ as the series $e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, which is convergent everywhere, but we're not really concerned with the underpinnings of this function from a qualitative analysis for right now. What's more important to us right now is how many other functions the exponential function gives rise to, in particular many of the common transcendental functions you may already be used to.

The first of these are $\sin$ and $\cos$ – these are real functions that can be generalized using Euler's Formula to complex numbers:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

These are consistent with the real-number definitions of $\sin$ and $\cos$, and extend this function to the entire complex plane. If we take, for example, $\sin(x+iy)$, for x, y real, we get (after applying this definition)

$$\begin{aligned} \sin(x+iy) &= \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} = \frac{e^{-y+ix}-e^{y-ix}}{2i} \\ &= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} \end{aligned}$$

$$= \sin x \left(\frac{e^y + e^{-y}}{2} \right) + i \cos x \left(\frac{e^y - e^{-y}}{2} \right)$$

This leads us to define very close relatives of the sine and cosine – the *hyperbolic sine and cosine*, denoted $\sinh$ and $\cosh$. We'll let

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

Essentially, it's the same definition as sine and cosine with exponential functions, removing any is that appear. This means that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

If we were to compare this to a use of the sine addition formula:

$$\sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

This allows us to conclude that

$$\cosh y = \cos(iy) \quad \sinh y = -i \sin(iy),$$

by comparing terms. Of course, we could have deduced this relationship directly from the definitions as well.

The hyperbolic sine and cosine behave very similarly to our regular trig functions, but with some properties flipped around. For example, consider $\cosh^2 x$ and $\sinh^2 x$ when added and subtracted (these are the squares of $\cosh, \sinh$):

$$\cosh^2 x - \sinh^2 x = 1 \quad \cosh^2 x + \sinh^2 x = \cosh 2x$$

If we define $\tanh x$, $\operatorname{sech} x$, $\operatorname{csch} x$, and $\coth x$ similarly to how their regular trigonometric counterparts are defined (i.e. $\frac{\sinh x}{\cosh x}, \frac{1}{\cosh x}, \frac{1}{\sinh x},$ and $\frac{\cosh x}{\sinh x}$, we see that this yields the very similar

$$1 - \tanh^2 x = \operatorname{sech}^2 x \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

These are all analogues of the ordinary Pythagorean identities. Recall the Pythagorean identity comes from a geometric interpretation of these functions – if we let $\cos \theta$ correspond to x, and $\sin \theta$ correspond to y, we get a nice parameterization of a unit circle. Similarly, letting x correspond to $\cosh t$ and y correspond to $\sinh t$ gives $x^2 - y^2 = 1$, a hyperbola! This is the reason for calling these functions hyperbolic in nature.

The derivatives of these functions behave a bit more nicely than the regular trig functions:

$$\frac{d}{dz} \sinh z = \cosh z \quad \frac{d}{dz} \cosh z = \sinh z$$

Note the absence of the minus sign on the derivative of $\cosh z$! It was present before because of the extra i that $\sin$ and $\cos$ had, but no more. One can derive the other corresponding derivatives of $\tanh z, \coth z, \operatorname{sech} z, \operatorname{csch} z$. Interestingly, all of them look the same as their trig counter parts, but $\operatorname{sech} z$ gets an extra minus sign... the price you pay, I suppose.

We can also talk about the inverses of these functions! Define $w = \log z$ as a number for which $e^w = z$. Notice I didn't say *the* number – there are an infinite number of them for every complex number! Even for reals, for this function we've extended to $\mathbb{C}$, something like $\log 2$ doesn't have a single value:

$$\log 2 = \log(2e^{2\pi i n}), n \in \mathbb{Z} \implies \log 2 \in \{\ln 2 + 2\pi i n, n \in ZZ\}$$

(Here, I distinguish between the real-valued $\ln x : \mathbb{R} \to \mathbb{R}$ and the complex $\log z$ for the logarithm, which is a distinction I will continue to use for clarity.)

This is a problem, which once again stems from the fact that numbers have infinitely many possible arguments, and that multiplying numbers by $e^{2\pi i n}$ doesn't change them. Again, to remedy this, we present the *principal log* function, $\operatorname{Log}(z)$, which forces the number to be written with a principal argument before the logarithm is taken.

Hence,

$$\operatorname{Log} 2 = \ln 2,$$

and everything is right in the world once more. Similarly, $\operatorname{Log} r = \ln r$ for all positive real r. We can also see the Log of other complex numbers:

$$\operatorname{Log}(1 + i\sqrt{3}) = \ln 2 + i\frac{\pi}{3} \quad \operatorname{Log}(4 - 4i) = \ln 4\sqrt{2} - i\frac{\pi}{4}$$

In general, see how we can write the Log function explicitly in its real and imaginary parts:

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg}(z)$$

As always, 0 is a problem point, as 0 doesn't have an Argument - it has infinitely many, even with the principal branch. This is an issue that even the Log cannot avoid, but we can only expect so much.

With this in mind, we can actually write down the inverse trigonometric functions explicitly in terms of the Log function. This is done by letting $w = \sin z$ (or other trig function), creating a quadratic in e^{iz}, and using the quadratic formula and picking the right branch based on consistency with the regular, real-number inverse trig functions. This gives us

$$\operatorname{Arcsin} z = -i \operatorname{Log}\left(iz + \sqrt{1 - z^2}\right) \quad \operatorname{Arccos} z = -i \operatorname{Log}\left(z + \sqrt{z^2 - 1}\right) \quad \operatorname{Arctan} z = \frac{i}{2} \operatorname{Log}\left(\frac{1 + iz}{1 - iz}\right),$$

where everything is done principally. This allows us (finally!) to take the Arcsin of numbers outside the real interval $[-1, 1]$, and instead take it of any complex number we want! In particular, $\operatorname{Arcsin} 2$ exists – it's just a complex number, which happens to be

$$\operatorname{Arcsin} 2 = -i \operatorname{Log}(2i + \sqrt{1 - 4}) = -i \operatorname{Log}\left(2 + \sqrt{3}i\right)$$

This, for the most part, addresses most of the functions on your handy-dandy scientific calculator that it has buttons for, in all of $\mathbb{C}$ – all stemming from the exponential function.

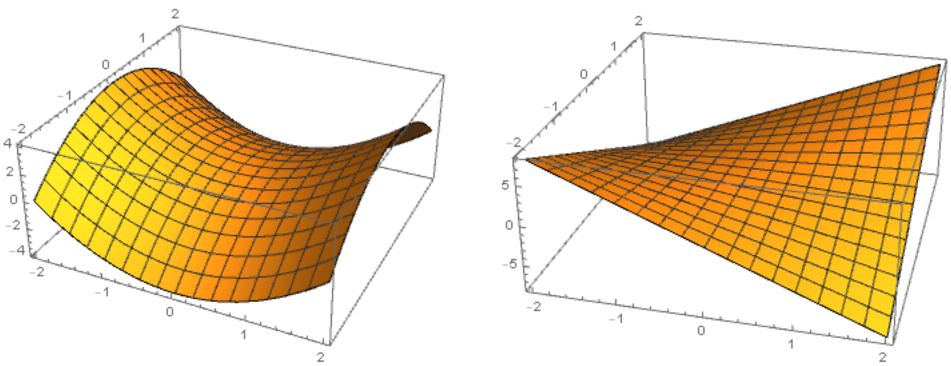
1.3 How (Not) to Graph In Four Dimensions

Now that we have the fundamentals of the complex number system under our belts, it's time to discuss functions on the complex numbers. All of the following are functions on the complex numbers:

$$f(z) = z^2 \quad f(x + iy) = 3y + 5ix \quad f(z) = \operatorname{Log} |z| + i \operatorname{Arg}(\bar{z}) \quad f(z) = \bar{z}^2 + 3z^2$$

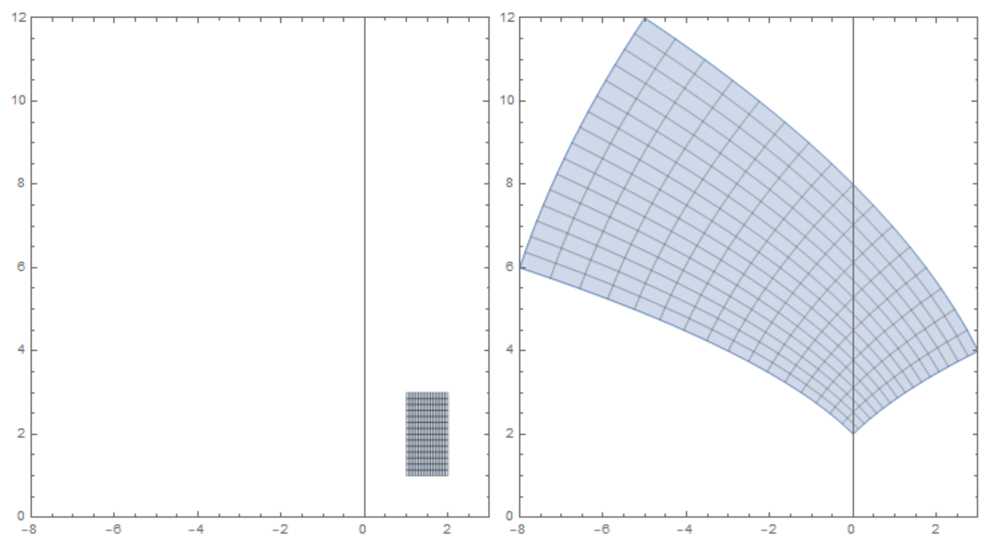
We would like a way to visualize all of these functions. The issue, however, that a traditional approach that shows the domain of a two-dimensional variable mapping to a two-dimensional output would require some four-dimensional graph paper, which unfortunately would not be useful on this two-dimensional page. We have to be more clever.

We got around this difficulty when visualizing functions of two variables by plotting the dependent variable on the z-axis and varying the other two variables along the x- and y-axes. From this, we might think to simply have the complex plane as our input, and graph the real and imaginary parts of the function separately as three-dimensional surfaces. Thus, for a function f, we split it into its real and imaginary parts, $u(x, y)$ and $v(x, y)$, respectively, both of which are a function of two variables. Below, we see a dissection of $f(z) = z^2 = u(x, y) + iv(x, y) = x^2 - y^2 + 2ixy$ into $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$:



These graphs reveal a lot about how nice u and v are, but not a lot about the actual function f itself. Sure, these will have their use/intuition when we investigate u and v, but we need to be even more clever to get a nice visual representation of f.

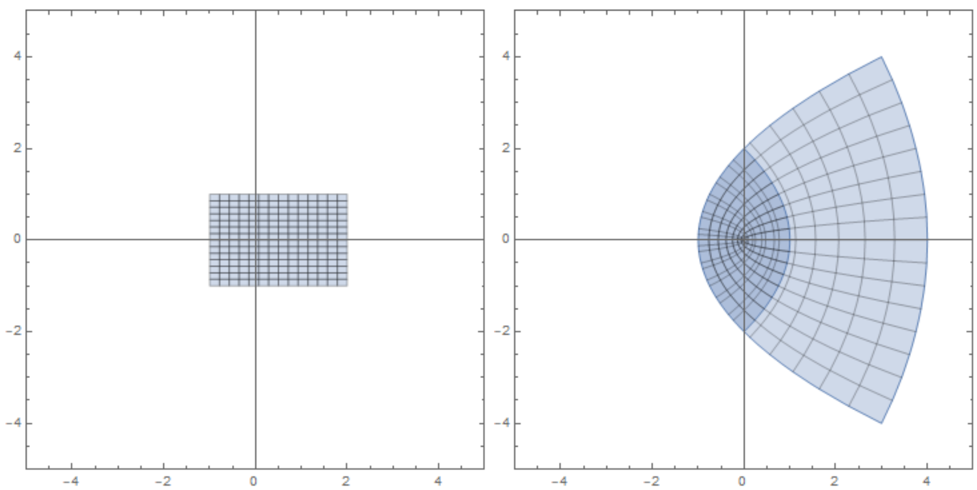
Another idea is to instead graph the domain and range of f separately – that is, feed in a selected subset of $\mathbb{C}$, and let Mathematica tell us where that maps to. That seems fairly reasonable! Below, we consider the image of the rectangle bounded by $1, 2, 1 + 3i, 2 + 3i$:



Here, we see more geometrically what z^2 is doing – it's doubling the arguments and rotating it, all the while the modulus is being squared and being pulled away from the origin. This captures the essence of the squaring function more completely than how our first method pulled the real and imaginary parts apart.

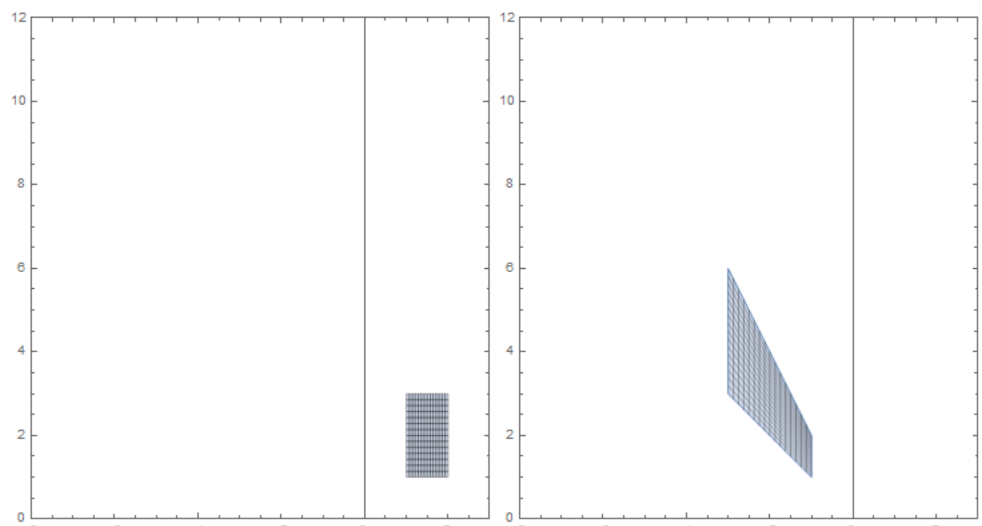
Notice also that I drew lines in my domain that represented lines of constant real/imaginary parts, and Mathematica graciously shows where these lines map to. While they don't remain straight, these curves still remain perpendicular to each other, which is a bit unexpected!

We go again, with the rectangle with vertices $-1 + i, -1 - i, 2 + i, 2 - i$:



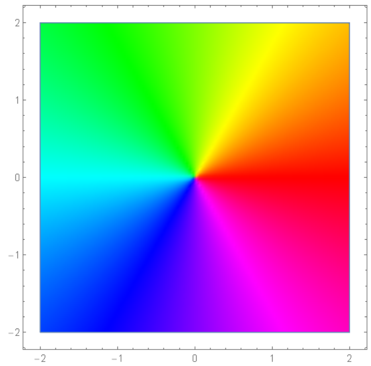
Let's see what happens if we were to walk around the image of the boundary of our domain in the output space. $2+i$ goes to $3+4i$, which corresponds to the top-right corner of the figure. We keep walking, keeping the interior to the left, until we reach the image of $-1+i=-2i$, past the other corner in the figure. We walk inside the figure, up to the other corner, and then overlap our own interior by walking to $3-4i$ and back up to the top-right. The orientation of the interior has still been preserved, and so has the angles between the lines, for the most part (except at 0). Functions which preserve angles and orientation in this manner are called *conformal*. We'll discuss more about conformal mappings later.

Here's a mapping that's decidedly *not* conformal – $f(x+iy)=-y+xyi$:



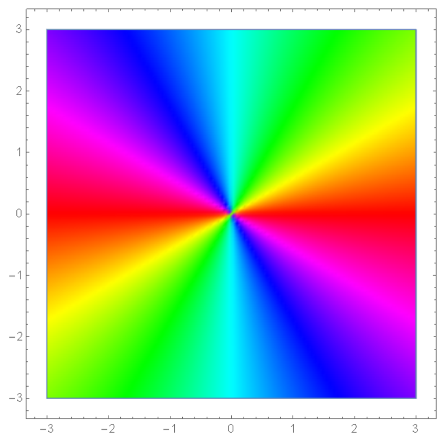
This, clearly, doesn't at all preserve angles by inspection, so not all functions $f(x+iy)$ with given $u(x,y)$ and $v(x,y)$ are conformal. We'll talk about what functions are and aren't conformal next time.

Finally, we turn to a fairly unorthodox, psychadelic method – color maps. Normally, to establish a sort of baseline for ourselves, we first "paint" the entire complex plane with colors, varying hues for different arguments and making the saturation/brightness of the colors closer to the origin. We're not as concerned with the modulus, so we're just going to color the plane with the rainbow like so:

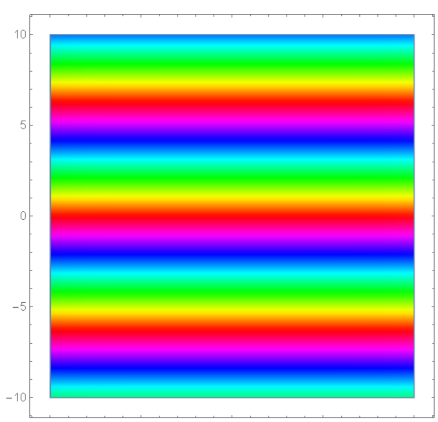


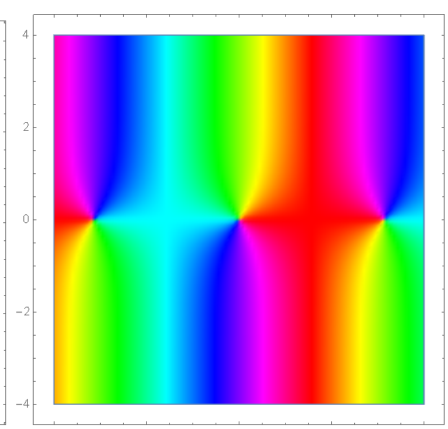
For reference, the positive real axis is red, the positive imaginary axis is light green, the negative real axis is a light aqua, and the negative imaginary axis is purple.

To read these maps, the color of each point of the plane shows what argument the point maps to. Here's a color map of $f(z) = z^2$:



Notice how the rainbow is repeated twice around the origin, as the we get to cycle through all of the different arguments twice. Here's e^z, and $\sin z$ for reference:





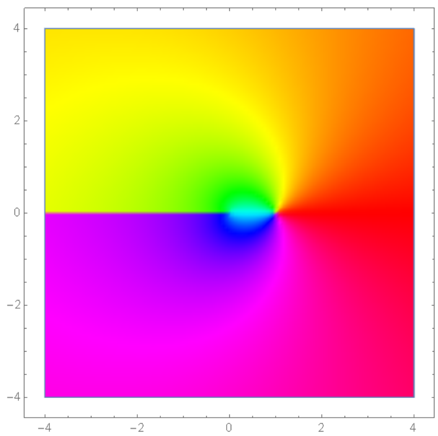
Note that the stripes on e^z come from the periodicity every 2π in $e^i r$ for r real, and we can see the regular periodicity in the $\sin$ function directly.

These color maps are going to be useful for visualizing discontinuities, which we will discuss later.

1.4 Tears in the Fabric: Branch Cuts and Points

Recall that in order to make the argument a function, we limited the range of the Argument such that $Arg(z) \in (-\pi, \pi]$. In a sense, we had cut off the other infinitely many values for the argument of a number. At the places where $\operatorname{Arg}(z) = \pi$, then, we get a discontinuity in the function all along the negative real axis, and at 0. This discontinuous "cut" in the function we call a *branch cut*[1]. Strange, strange things happen because of this choice, and we'll explore this now with the color maps.

Remember with the Log function, we wrote $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$. Since the Argument function has a branch cut all along the negative real axis, we should also see a branch cut all along the negative real axis here – and indeed, we do, looking at the color map of $\operatorname{Log}(z)$:

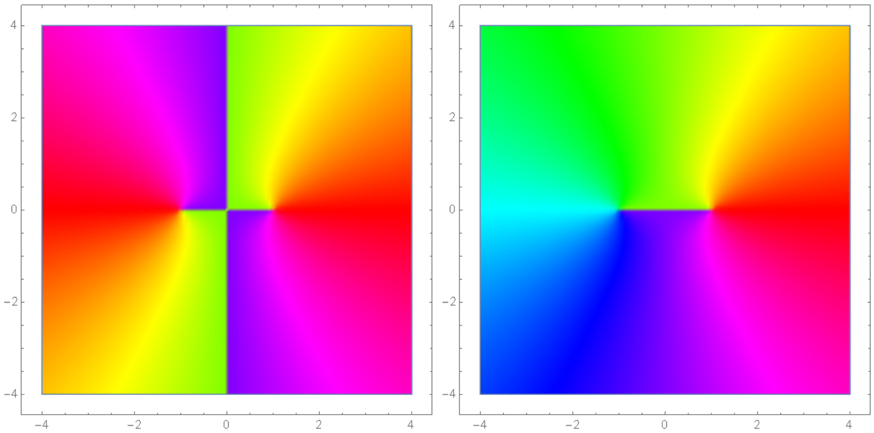


The branch cut here is displayed *very* prominently in the sharp discontinuity in color crossing the negative real axis. Cuts can sometimes start and end at *branch points*, and in this case, 0 is a branch point of $\operatorname{Log}(z)$[2]. One subtlety to note is that a branch point must be present in every branch of the multi-valued function – so some points that look like branch points as they are the endpoints of a branch cut, which can be very confusing. In general, a good way to identify possible branch points is to ask where one is taking a Log or a non-integer power of 0 or ∞, and *cuts always occur when one is taking a* Log *or non-integer powers of negative reals*.

We can use the color maps to look at the cuts in functions that look similar, but actually exhibit very different behaviors. For example, consider $\sqrt{z^2 - 1}$ and $z\sqrt{1 - \frac{1}{z^2}}$:

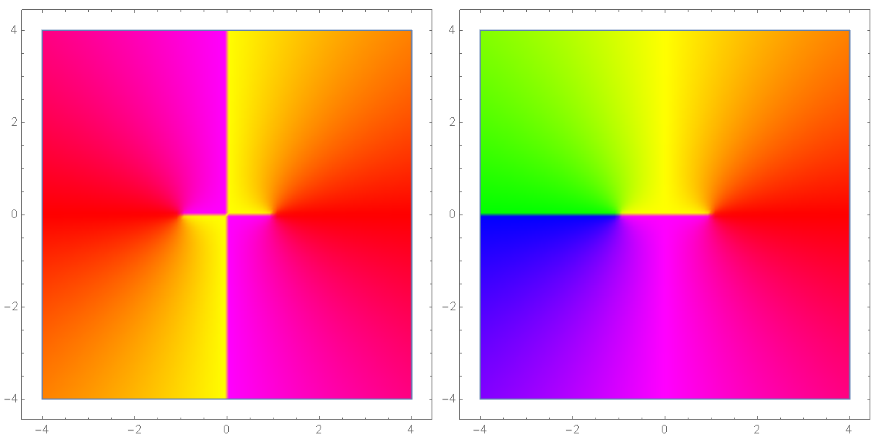
[1] In particular, the definition of a branch cut is a curve where the "flattening" of the multivalued function onto a single branch is discontinuous.

[2] The formal definition of a branch point is a point where winding an arbitrarily small loop around it once brings you to a different output value compared to the starting point, but it's a hard to define this super rigorously without some sort of topology and analysis machinery.



Here, both functions have branch points at ± 1, but the first appears to have a branch cut running through infinity along the imaginary axis. However, we've just identified a branch of the function $(z^2-1)^{\frac{1}{2}}$ where ∞ doesn't appear along any cuts, so it's actually an illusory branch point.

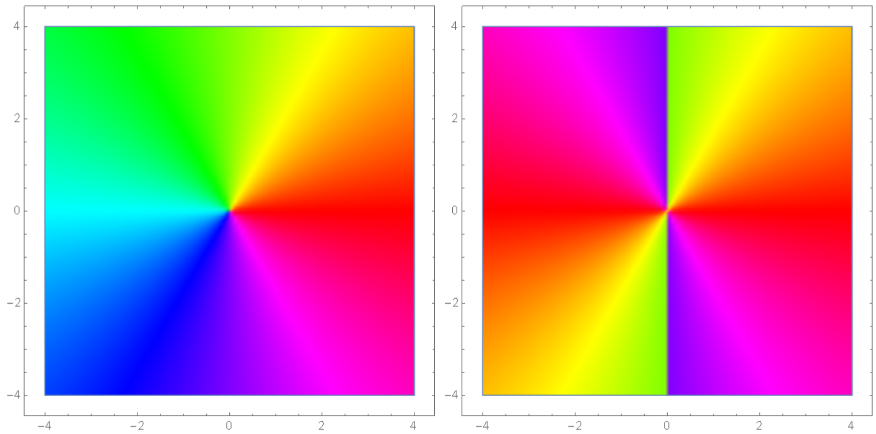
Contrast this with $\sqrt[3]{z^2-1}$ and $z^{\frac{2}{3}}\sqrt[3]{1-\frac{1}{z^2}}$:



Notice that here, ∞ is an actual branch point! Taking out the $z^{\frac{2}{3}}$[3] out of the cube root doesn't actually remove the point at infinity, as this function inherently has a branch cut along the negative real line. 0, however, is not a branch point of this function, as the first branch presented did not have it as a possible branch point. This manipulation has only made us move the cut through infinity in a strange manner.

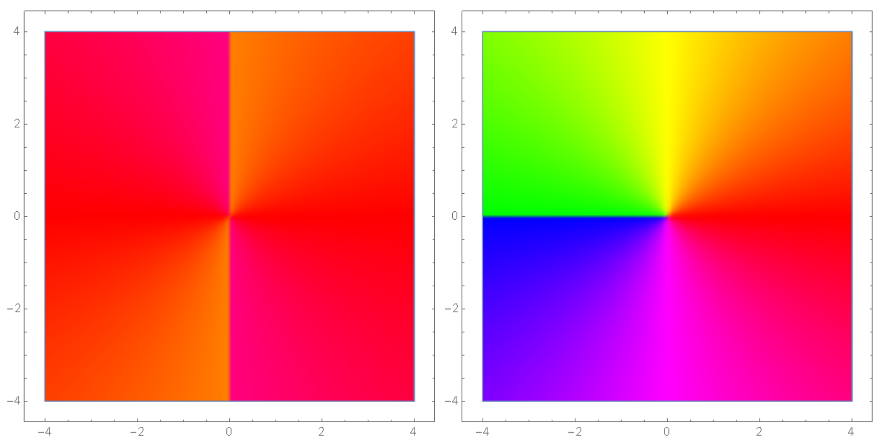
On that note, we can see that functions that really should be the same, such as $(\sqrt{z})^2$ and $\sqrt{z^2}$, have different branch cuts, meaning that they're different branches of z. Here they are, respectively:

[3]Normally, this would be multi-valued, but I mean the principal branch of this function here.



Notice the first function has no branch cuts anywhere – we call these kinds of functions *entire*, although that's not the proper definition of the term.

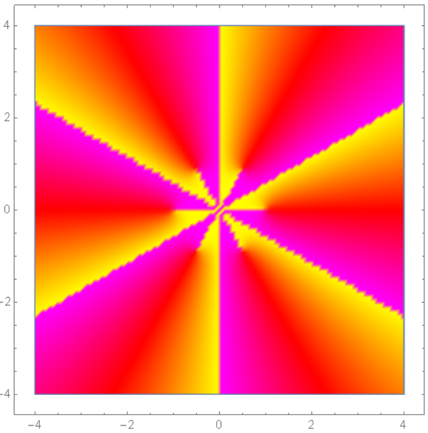
Similarly, $\sqrt[3]{z^2}$ and the principal branch of $z^{\frac{2}{3}}$ are different branches of the same function as well, evidenced by their branch cuts, and we show them here, respectively:



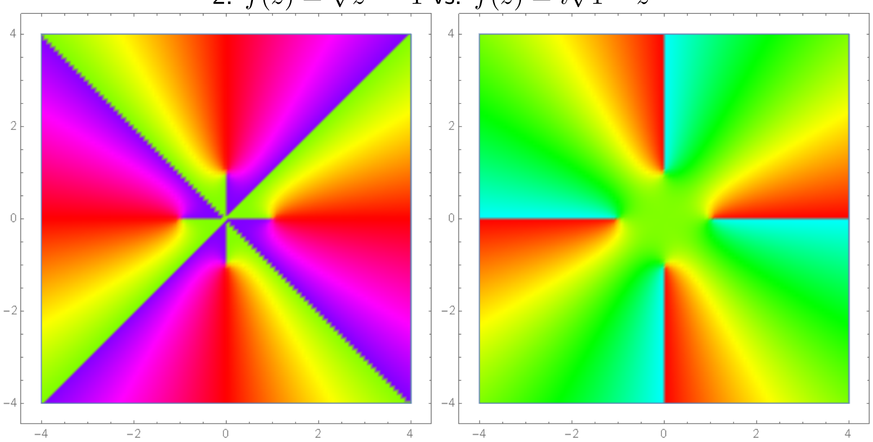
This is all very, very strange! This choice that we made in Section 1 has rent asunder what we think should be obviously true about these functions. Moving forward, whenever we see a function that has some sort of non-integer power, or has a Log, we should be careful to see where the branch cuts *actually* are, because they can be very, very tricky.

For your viewing pleasure, I present a few more pretty color plots – see if you can figure out for yourself why the branch cuts are in the places you see them! Feel free to come up with different branches of the multi-valued functions, if you can, that will eliminate illusory branch points as well. If a function looks multi-valued, assume I'm always using the principal branch.

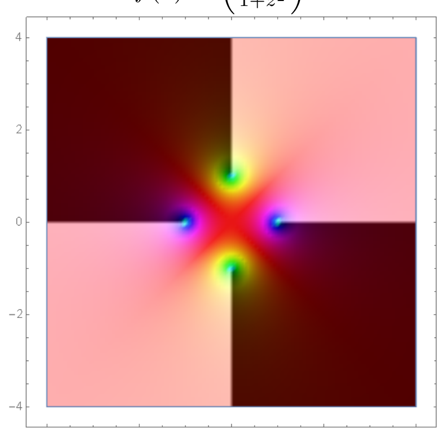
1. $f(z) = \sqrt[3]{z^6 - 1}$



2. $f(z) = \sqrt{z^4 - 1}$ vs. $f(z) = i\sqrt{1 - z^4}$



3. $f(z) = \left(\frac{1-z^2}{1+z^2}\right)^i$: [4]



Feel free to mess around with any other weird functions with branches on your own!

[4]This one has branch cuts due to the modulus, and not the argument, so I've added back the saturation/brightness functions to show this effect. Darker means closer to 0, brighter means closer to infinity.

2 The Best Functions

We begin our foray into complex analysis and start with a discussion of the smooth functions on $\mathbb{C}$. We'll soon see that these are actually incredibly nice!

2.1 Topology of the Complex Numbers

Before we start talking about analysis in $\mathbb{C}$, we should establish what the topology on $\mathbb{C}$ is. It's not at all unfamiliar – it's essentially just like $\mathbb{R}^2$! Remember that the norm $\|\cdot\|$ on $\mathbb{C}$ that we looked at in the last section gives $\mathbb{C}$ a metric space structure that looks almost exactly like the absolute value between points in $\mathbb{R}^2$. Recall:

Definition 1

A **metric space** is a space X with a **distance function** $d : X \times X \to \mathbb{R}$ such that the following three statements are true:

- For all $x, y \in X$, $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$.
- For all $x, y \in X$, $d(x, y) = d(y, x)$.
- For all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$.

Most of the analysis you have done so far has been in a metric space – all your work in $\mathbb{R}^n$ has been like this, and $\mathbb{C}$ is no different.

The following definitions should look familiar – again, it's literally just like $\mathbb{R}^2$!

Definition 2

An **open disk** of radius r around z_0 in $\mathbb{C}$ is the set of points $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Definition 3

A set $U \subseteq \mathbb{C}$ is **open** if for every $z \in U$, there exists some $r > 0$ in $\mathbb{R}$ such that $D_r(z_0) \subseteq U$.

(Note that the "in $\mathbb{R}$" is a little redundant, as $\mathbb{C}$ cannot be totally ordered – so if a total ordering is implied, we have should be assumed to be in $\mathbb{R}$.)

Definition 4

A set $V \subseteq \mathbb{C}$ is **closed** if its complement $\mathbb{C} - V$ is open.

We also establish limits of sequences in pretty much the same way:

Definition 5

A sequence of complex numbers $\{z_i\}_{i=1}^{\infty}$ **converges** to some $w \in \mathbb{C}$ iff

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} : \forall n > N, |z_n - w| < \varepsilon.$$

If this is true, then we write $\lim_{n\to\infty} z_n = w$.

It should be said that addition and multiplication of limits of sequences works exactly the same as in $\mathbb{R}^n$.

Definition 6

A point w is a **limit point** of some set $U \subseteq \mathbb{C}$ if there exists some sequence $\{z_n\}$ in U such that $\lim_{n\to\infty} z_n = w$.

Definition 7

The **closure** of a set U in $\mathbb{C}$, $\overline{U}$, is the union of U with all its limit points.

(As the name indicates, the closure of a set is closed – this follows the same proof from Hubbard, more or less.)

Definition 8

The **interior** of a set U, $\mathring{U}$, is the set of all points $z \in U$ such that there exists some $D_r(z) \subseteq U$.

Definition 9

The **boundary** of a set U is $\overline{U} - \mathring{U}$.

Compactness topologically usually means dealing with open covers, but in a metric space, closedness and boundedness is enough:

Definition 10

A set U is **compact** if any cover of U by open sets $\{U_\alpha\}$ such that $U \subseteq \bigcup_\alpha U_\alpha$ has a finite subset of some U_{α_i} for $1 \leq i \leq n$ for some finite n such that $U \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. In other words, any open cover of U has a finite subcover.

Equivalently, a compact set U in $\mathbb{C}$ is closed and bounded, just as in Hubbard. (These happen to be equivalent because we're in a metric space!)

Finally, we should recall that $\mathbb{C}$ is **complete**, just as $\mathbb{R}$ is. This means that for any **Cauchy sequence** $\{z_n\} \in \mathbb{C}$ (meaning that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $m, n > N$, $|z_m - z_n| < \varepsilon$) converges to some $z \in \mathbb{C}$.

Note that completeness is equivalent to the **nested compact set property**, which states that if we have a sequence of nested compact sets $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \subseteq \ldots$, such that the diameters of these sets $\operatorname{diam}(U_n) = \sup_{z,w\in U_n} |z - w|$ are a sequence converging to 0, then the intersection of all of these U_i is non-empty.

We will have a lot to say about **connectedness** as well. A set $U \subseteq \mathbb{C}$ is connected if for any two $z, w \in U$ that there is a **path** joining z and w in U. In particular, this means there exists a continuous map $\gamma : [0, 1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$. We will be very interested in paths, especially when we study integration.[5]

A special variety of a connected set is a **simply connected** set. A simply connected set can be thought of as a set that "has no holes in the middle", so that it's sort of convex. Rigorously, we'll say that U is simply connected if it and its complement, $\mathbb{C} - U$ are both connected.[6]

Okay, that's enough topology. Hopefully most of this is review from 2230!

[5] This is actually not the proper definition of "connected" – I've just given you the definition for "path-connected"ness instead, which is a little stronger but are equivalent in $\mathbb{C}$.

[6] Again, this actually isn't the full definition – the real definition requires a little bit of homotopy. Basically, we want all loops in the

2.2 Limits and Continuity

We now turn to a discussion of limits and continuity of functions in $\mathbb{C}$, but again, it's literally just like $\mathbb{R}^2$! But for completeness, we rehash the definitions here.

Definition 11

Let $U \subseteq \mathbb{C}$. A function $f : U \to \mathbb{C}$ has a **limit** w at z_0 if for all $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $z \in U$,

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - w| < \varepsilon.$$

We say then that $\lim_{z \to z_0} f(z) = w$.

Definition 12

If $U \subseteq \mathbb{C}$, and $f : U \to \mathbb{C}$, then f is **continuous** at z_0 if $f(z_0) = \lim_{z \to z_0} f(z)$.

A slight note here – Hubbard's definition of a limit is slightly different/non-standard. Hubbard's condition is instead that $|z - z_0| < \delta \Rightarrow |f(z) - w| < \varepsilon$, which essentially forces a limit to exist for f at a point z_0 if and only if the function f is just continuous at z_0. It's kinda nice, but it's a little strange, I guess.

Of course, sequential continuity is equivalent to continuity – if we have a sequence $\{z_n\}$ converging to z in $\mathbb{C}$, and $f : U \to \mathbb{C}$ where U is some subset of $\mathbb{C}$, $\lim_{n \to \infty} f(z_n) = f(z)$. Also, all the standard limit/continuity theorems for combining functions apply. Addition/multiplication/composition of limits behave as expected, and same with continuity.

2.3 Holomorphicity

We move to the notion of differentiability on $\mathbb{C}$. Differentiability

Definition 13

Let U be an open subset of $\mathbb{C}$. A function $f : U \to \mathbb{C}$ is **holomorphic** at $z_0 \in U$ if the limit $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. If it exists, the limit is denoted $f'(z_0)$, and is called the **derivative** of f at z_0.

If this holds at every point $z_0 \in U$, then f is said to be holomorphic on U.

Definition 14

If a function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, then f is said to be **entire**.

Naturally, this definition of the derivative follows the same rules as the real derivative – it's linear, and follows the product rule and chain rules, with the proofs being roughly the same as in $\mathbb{R}^2$.

Here's where it starts to get weird. Let's investigate f as a transformation from "$\mathbb{R}^2$" to "$\mathbb{R}^2$". In particular, if we think $f(x + iy) = u(x, y) + iv(x, y)$, then we have the following theorem about the real/imaginary parts of the function related to the real/imaginary parts of the input:

set to be contractible to a point, so that loops are homotopy-equivalent to a constant map. This means that you can continuously deform a loop and squish it down into a point. This is a pretty important notion in algebraic topology, but as a loose concept it will be really important here for us in complex analysis.

Theorem 1 (Cauchy-Riemann)

If $U \subseteq \mathbb{C}$ is open, then $f : U \to \mathbb{C}$ is holomorphic on U iff

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

everywhere in U. These equations are called the **Cauchy-Riemann equations** and can be used as a criterion for holomorphicity.

proof needed here

2.4 Power Series

One thing we will do extensively later on is talk about power series. Here, I'll briefly talk about power series and convergence and analyticity.

3 Contour Integration

Arguably, the most important parts of complex analysis address the integration of complex-valued functions along curves, especially holomorphic ones. We're going to use contour integrals to establish some deeper results about holomorphic functions, some of which might be extremely surprising!

3.1 What is a Contour Integral?

But first – what is a contour integral? If you're familiar with integrating along curves in $\mathbb{R}^2$, you won't be surprised to see that we do essentially the same thing in $\mathbb{C}$.

First, what kinds of curves can we integrate along? Generally, we want to be able to integrate along smooth paths in $\mathbb{C}$, but the definition of path we discussed in the previous section only has the "continuous" restriction. This means that if we were to stick only to paths, we would have to figure out how to deal with self-similar fractal curves and also space-filling Peano curves, and we really don't want to integrate along those. Stein-Shakarchi avoids the problem of dealing with this by essentially only considering curves that are okay for us integrate along to have only a finite number of non-smooth points, which is pretty much good enough for any reasonable purpose. We will actually go a smidge more general and talk instead about curves with finite arc length that can be approximated by line segments, which are called *rectifiable curves*.

Formally, we define a **rectifiable curve** in $\mathbb{C}$ as a continuous function $\gamma : [0,1] \to \mathbb{C}$ such that there is a continuous bijection $\varphi : [0,1] \to [0,1]$ such that $\varphi \circ \gamma$ is Lipschitz. The Lipschitz condition will be instrumental in showing that the length of such a curve will be finite when we integrate along it.[7] Really, we should be considering equivalence classes of continuous functions $\gamma : [0,1] \to \mathbb{C}$, such that γ is in the same equivalence class as some other continuous $\gamma' : [0,1] \to \mathbb{C}$ if we have a continuous bijection $\psi : [0,1] \to [0,1]$ such that $\gamma' = \psi \circ \gamma$. These equivalence classes are the actual curves C, which can be parameterized in possibly many ways (here, the γ and γ'). We also call these rectifiable curves **contours**.

Okay, so let's suppose we have a contour C as above, parameterized by some $\gamma : [0,1] \to \mathbb{C}$. Then, we define the **contour integral** of $\int_C f(z)\,dz$ some complex-valued function $f : \mathbb{C} \to \mathbb{C}$ along C by

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\gamma\left(\frac{k}{n}\right)\right) \cdot \left[\gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right)\right] = \int_0^1 f(\gamma(t))\gamma'(t)\,dt.$$

This should look really similar to the Riemann sum that you're used to, and you'll study this more in 2240 and get a glimpse as to why this real-valued integral converges when the curve is appropriately parameterized. (If you're concerned about the complex values of f, we could separate this into the real and imaginary parts and treat them separately, so this really reduces to the real-integral case.)

We can now compute integrals via parameterization explicitly, but this isn't really going to be useful for us to develop, and often exercises for this involve integrating non-holomorphic functions, which is really similar to just computing a line integral and isn't going to be useful for us (as this method will not be used to actually compute integrals, 99% of the time). Instead, let's start by developing some tools. Like in single-variable calculus, we have a **Fundamental Theorem of Contour Integrals**, which is a complex version of the Fundamental Theorem of Calculus:

Theorem 2

Let $f : U \to \mathbb{C}$ be a continuous function and $F : U \to \mathbb{C}$ be holomorphic such that $F'(z) = f(z)$. Then, for

[7] This source is what I'm referencing for this statement: Rectifiable curve

a curve γ that begins at z_1 and ends at z_2, we have that

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

The proof of this is essentially the same as that as the Fundamental Theorem of Calculus in $\mathbb{R}$.

We will also be very interested in integrating around closed loops in $\mathbb{C}$. A contour is **closed** if its endpoints are the same, i.e. if $\gamma : [0, 1] \to \mathbb{C}$ defines a contour C, then C is closed if $\gamma(0) = \gamma(1)$, and a contour that is not self-intersecting is called **simple**. An example of a simple closed contour is the unit circle centered at the origin, which may be oriented in one of two different ways. We call the counterclockwise orientation of any loop the **positive** orientation by convention, and the clockwise orientation of a loop the **negative** orientation. Integrating along a closed loop in opposite orientations gives the negative answer.

For instance, we can integrate the function $f(z) = \frac{1}{z}$ around the unit circle counterlockwise, along contour γ^+ parameterized by $\gamma^+ : [0, 2\pi] \to \mathbb{C}$, $\gamma^+(t) = e^{it}$ (we can easily rescale the parameterization to be on $[0, 1]$ if desired):

$$\oint_{\gamma^+} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = 2\pi i.$$

Or, we can integrate the same function clockwise around the unit circle, along contour γ^- parameterized by $\gamma^- : [0, 2\pi] \to \mathbb{C}$, $\gamma^-(t) = e^{-it}$:

$$\oint_{\gamma^-} \frac{1}{z}\,dz = \int_0^{2\pi} \frac{-ie^{-it}}{e^{-it}}\,dt = -2\pi i.$$

Note that we denote integrating around a closed contour with the $\oint$ integration symbol as opposed to the regular $\int$ sign, as the circle denotes that we're integrating around a loop.

We can apply the Fundamental Theorem of Calculus above for contour integrals to get a useful-ish result for closed contours:

Corollary 3

If $f : U \to \mathbb{C}$ is continuous and there exists some $F : U \to \mathbb{C}$ such that F is holomorphic on U and $F'(z) = f(z)$, then if C is a closed contour contained in U, then

$$\oint_C f(z)\,dz = 0.$$

Note that this result does NOT apply to our example above! If we wanted to construct the antiderivative of $\frac{1}{z}$ as $\operatorname{Log} z$, remember that the Log function has a branch cut along the negative real axis and a branch point at 0! There does not exist a domain on which this antiderivative is holomorphic that contains the domain, as the branch cut must cross the path.

3.2 Integral Bounds

Often, in the course of proofs or calculations we will need to construct bounds on integrals, so we will develop a couple of cool tricks here.

The most important of these is the **ML Lemma**:

Theorem 4 (ML Lemma)

Suppose C is a rectifiable contour with length L in some open $U \subseteq \mathbb{C}$, and let the maximum modulus of some continuous function $f : U \to \mathbb{C}$ on U be M. Then,

$$\left| \int_C f(z)\, dz \right| \le ML.$$

Note that the maximum exists as the contour is a compact subset of $\mathbb{C}$.

The proof of this is fairly straightforward, using the triangle inequality and the limit definition of the contour integral:

Proof.

$$\begin{aligned}
\left| \int_C f(z)\, dz \right| &= \left| \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\gamma\left(\frac{k}{n}\right)\right) \cdot \left[\gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right)\right] \right| \\
&\le \lim_{n\to\infty} \sum_{k=1}^{n} \left| f\left(\gamma\left(\frac{k}{n}\right)\right) \right| \cdot \left| \left[\gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right)\right] \right| \\
&\le M \lim_{n\to\infty} \sum_{k=1}^{n} \left| \left[\gamma\left(\frac{k}{n}\right) - \gamma\left(\frac{k-1}{n}\right)\right] \right| \\
&\le ML
\end{aligned}$$

□

This is an incredibly useful bound, and it's used in many proofs and computations.

Another bound that's useful mostly for computational purposes is **Jordan's Lemma**, which applies particularly to semicircular contours in the upper half-plane:

Theorem 5 (Jordan's Lemma)

Let C_R be the contour $\left\{Re^{it} : t \in [0, \pi]\right\}$ for $R > 0$ real, and let $f : \mathbb{C} \to \mathbb{C}$ be continuous on C_R with $|f(z)|$ bounded by M_R on C_R. Then,

$$\left| \int_{C_R} f(z)e^{iz}\, dz \right| \le M_R \pi.$$

Proof.

$$\begin{aligned}
\left| \int_{C_R} f(z)\, dz \right| &= \left| \int_0^{\pi} f(Re^{it})e^{iRe^{it}} \cdot iRe^{it}\, dt \right| = \left| \int_0^{\pi} f(Re^{it})e^{iR(\cos t + i \sin t)} \cdot iRe^{it}\, dt \right| \\
&\le \int_0^{\pi} \left| f(Re^{it})e^{iR\cos t} \cdot e^{-R\sin t} \cdot iRe^{it} \right| \, dt \\
&= R \int_0^{\pi} \left| f(Re^{it}) \right| \cdot \left| e^{-R\sin t} \right| \, dt \\
&\le M_R R \int_0^{\pi} \left| e^{-R\sin t} \right| \, dt = 2M_R R \int_0^{\frac{\pi}{2}} \left| e^{-R\sin t} \right| \, dt
\end{aligned}$$

It suffices to find a good bound for $e^{-R\sin t}$. Note that since $\sin t \ge \frac{2}{\pi}t$ on $[0, \frac{\pi}{2}]$, that therefore $e^{-R\sin t} \le e^{-\frac{2}{\pi}t}$. We can thus finish up the bounding:

$$\left| \int_{C_R} f(z)\, dz \right| \le 2M_R R \int_0^{\frac{\pi}{2}} e^{-R\sin t}\, dt \le 2M_R R \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi}t}\, dt = \frac{2M_R R\pi}{2R}\left(1 - e^{-R}\right) = M_R\pi\left(1 - e^{-R}\right) \le M_R\pi$$

□

3.3 The Cauchy-Goursat Theorem

The theorem that follows is possibly the most important theorem in complex analysis. It's the first of the three "miracles" of smooth functions that

Theorem 6 (Cauchy-Goursat)

Let U be an open and **simply connected** subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ be holomorphic on U. Then, for any simple closed contour C contained in U, we have that

$$\oint_C f(z)\, dz = 0.$$

This is a pretty hard theorem to prove rigorously... common attempts involve using Green's Theorem, but this assumes that f has to have continuous real and imaginary partials with respect to the real/imaginary parts of the inputs, which is... not necessarily known? It actually turns out this is a little circular, but it's not a bad method of verifying the theorem, I guess.

The proper way was created by Goursat, and his proof is the one I will sketch out here.

3.3.1 A Note on Homotopy

3.4 Integral Formulas

As a result of Cauchy-Goursat, we deduce the following result –

Theorem 7 (Cauchy's Integral Formula)

Let U be an open and simply connected subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ be holomorphic on U. Then, for any simple closed contour C and some z_0 in the interior of C, we have that

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz.$$

Theorem 8 (Cauchy's Integral Formula For Derivatives)

Let U be an open and simply connected subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ be holomorphic on U. Then, for any simple closed contour C and some z_0 in the interior of C, we have that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

3.5 Miracles (Consequences of CIF and Friends)

Theorem 9 (Taylor's Theorem)

Suppose $f : U \to \mathbb{C}$ is holomorphic on open set $U \subseteq \mathbb{C}$. Then, for any $z_0 \in U$, there exists an open disk D

whose closure is contained in U where f has a power series expansion centered at z_0:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{f^{(k)}(z_0)}{k!}.$$

This power series expansion converges to f for every point $z \in D$.

Corollary 10 (Infinite differentiability)

A function $f : U \to \mathbb{C}$ holomorphic in some open set $U \subseteq \mathbb{C}$ is infinitely differentiable on U.

As such, we have the corollary that holomorphic functions are therefore analytic, by Taylor's Theorem (the complex version). In particular, this means that holomorphic $\Leftrightarrow$ analytic, so we will use the words interchangably from here on out.

Theorem 11 (Liouville's Theorem)

A function that is entire and bounded in magnitude is constant.

Morera's Theorem is like a converse to Cauchy-Goursat – it tells us when a function is holomorphic based on its loop integrals:

Theorem 12 (Morera's Theorem)

Let $f : D \to \mathbb{C}$ be continuous in some open set $D \subseteq \mathbb{C}$ such that for every simple closed rectifiable contour C contained in D, $\oint_C f(z)\, dz = 0$. Then f is holomorphic on D.

The third miracle as advertised in Stein-Shakarchi is the idea of **analytic continuation**, where one may construct extensions of analytic functions.

Theorem 13 (Continuation of Zero)

Suppose f is a holomorphic function on some open connected $U \subseteq \mathbb{C}$ that vanishes on a sequence of distinct points $\{z_k\}_{k=1}^{\infty}$ such that $z_k \in U$. If this sequence $\{z_k\}$ has a limit point, then $f = 0$ on U.

Corollary 14 (Analytic Continuation)

Suppose $f, g : U \to \mathbb{C}$ are holomorphic on some open connected $U \subseteq \mathbb{C}$. Suppose $V \subseteq U$ is open and $f(z) = g(z)$ for all $z \in V$ (or, even some sequence of distinct points with limit point in U). Then, $f(z) = g(z)$ for all $z \in U$.

We can use this theorem to show the existence of or even construct analytic extensions of functions. Suppose we have holomorphic functions $F : U \to \mathbb{C}$ and $f : V \to \mathbb{C}$ where $V \subseteq U \subseteq \mathbb{C}$. If $F = f$ on V, then we know that F is a (unique) analytic continuation of f into U.

This is a strengthening of the idea that "a continuous function on a compact set obtains its maximum or minimum" that is true in $\mathbb{C}$. First, a lemma:

Lemma 15

Let U be an open neighborhood of a point $z_0 \in \mathbb{C}$ and f be holomorphic on U. Then, if $|f(z_0)| \geq |f(z)|$ for all $z \in U$, then f is constant on U.

Theorem 16 (Maximum Modulus Principle)

Let $f : U \to \mathbb{C}$ be holomorphic on an open set $U \subseteq \mathbb{C}$. f has no maximum modulus on U. Equivalently, if we have a closed subset $D \subseteq U$, then the maximum modulus of f on D must be achieved on the boundary of D.

4 Series and Singularities

4.1 Taylor Series

4.2 Laurent Series

4.3 Classification of Isolated Singularities

5 Residues and Applications

5.1 The Residue Formula

5.2 Evaluation of Real Integrals and Series

This is going to be a collection of examples that highlight the most common strategies for computing real integrals with complex analysis.

Example 17

Evaluate
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

You can do this with standard integration techniques, but let's see complex analysis at work.

Solution. Consider the closed semicircular contour Γ_R that runs from $-R$ to R on the real axis, where this contour closes in the upper half-plane along the semicircular arc C_R. Let $R > 1$. Let's consider the integral of $f(z) = \frac{1}{1+z^2}$ along Γ_R:
$$\oint_{\Gamma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz.$$
The loop integral is easy to evaluate, actually – we can even do it with Cauchy's integral formula. Let's let $g(z) = \frac{1}{z+i}$, and note that $g(z)$ is holomorphic on the interior of the contour Γ_R, so Cauchy's integral formula applies:
$$\oint_{\Gamma_R} f(z)\, dz = \oint_{\Gamma_R} \frac{g(z)}{z-i}\, dz = 2\pi i \cdot g(i) = 2\pi i \cdot \frac{1}{2i} = \pi.$$
This is true regardless of what R we pick.

It suffices to see what happens with the other integral, $\int_{C_R} f(z)\, dz$. Let's do some bounding with the ML Lemma. Since we're integrating along part of the circle $|z| = R$, note that using the Triangle Inequality, we get
$$\left|1+z^2\right| + |-1| \geq \left|z^2\right| \Rightarrow \left|1+z^2\right| \geq |z|^2 - 1 = R^2 - 1 \Rightarrow \frac{1}{R^2-1} \geq \frac{1}{|1+z^2|} = |f(z)|.$$
Then, we can use the ML Lemma on the integral along the arc:
$$\left| \int_{C_R} f(z)\, dz \right| \leq \frac{1}{R^2-1} \cdot 2\pi R = \frac{2\pi R}{R^2-1}.$$
Notice that as R gets arbitrarily large, the magnitude of this integral can be made to be smaller than any $\varepsilon > 0$, so as we take the limit as $R \to \infty$, this integral must go to zero, so we get that
$$\pi = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\, dx + 0 = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$$
as expected. $\square$

Of course, this is equivalent to evaluating the residue at i, but we're using low-power techniques just because we can.

Example 18

Evaluate
$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx.$$

Solution. We consider instead the integral of $f(z) = \frac{e^{iz}}{z^2+1}$ along the same closed semicircular contour Γ_R as above. Again, with Cauchy's Integral Formula, we have
$$\oint_{\Gamma_R} f(z)\, dz = \oint_{\Gamma_R} \frac{\frac{e^{iz}}{z+i}}{z-i}\, dz = 2\pi i \cdot \frac{e^{-1}}{2i} = \frac{\pi}{e}.$$

Now, we examine what happens when we break up this integral into the part along the real axis and the semicircular part. The real axis will eventually turn into what we want, so let's look at the modulus of the contour integral along the semicircular part. This calls for Jordan's Lemma:
$$\left| \int_{C_R} \frac{e^{iz}}{z^2+1}\, dz \right| \leq \frac{\pi}{R^2-1}$$
using the same bound on $\frac{1}{|z^2+1|}$ as above. Again, note that as R gets arbitrarily large, the magnitude of this integral tends to zero, so in the limit as $R \to \infty$, we have that
$$\oint_{\Gamma_R} f(z)\, dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+1}\, dx + \int_{C_R} \frac{e^{iz}}{z^2+1}\, dz \to \frac{\pi}{e} = \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx + 0.$$
Taking the real part of this last equation, we see that $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\, dx = \frac{\pi}{e}$, which is a rather exotic result! □

5.2.1 Integrals For You to Try

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5.3 The Argument Principle and Rouché's Theorem

5.3.1 That Proof of the Fundamental Theorem of Algebra in Hubbard, Done Properly

Hubbard's proof of the Fundamental Theorem of Algebra is quite jank, and it's because he essentially goes to try and prove Rouche's Theorem as an intermediate step using only real-analytical tools, which is kinda weird, not gonna lie! Here's the proof as it should be.