Orbit-Stabilizer Theorem

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1 Orbits and Stabilizer

Given $s \in S$ and G a permutation group acting on S.

Definition. *orbit(s)* = $\{\pi\ s \mid \pi \in G\}$

Definition. *stabilizer(s)* = $\{\pi \in G \mid \pi\ s = s\}$

Theorem 1.1. *Orbit-Stabilizer Theorem: For all $s \in S$ and G acting on S*
$|orbit(s)| \cdot |stabilizer(s)| = |G|$

stabilizer(s) is a subgroup of G, so by Lagrange's theorem, $|stabilizer(s)| \mid |G|$

Lemma 1.1. If $\pi = s$ and $\pi \in$ stabilizer(s), then $\pi^{-1}s = s$.

Proof.

$$\begin{aligned} s &= \pi s \\ \pi^{-1}s &= \pi^{-1}\pi s \\ \pi^{-1}s &= s \end{aligned}$$

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So now the orbit of S forms cosets of stabilizer(s) (there is a 1-1 corespondence). So $|\text{orbit}(s)| = \#$ of cosets of stabilizer(s). Applying Lagrange's theorme gives the orbit-stabilizer theorem.