Concrete 2019, Test 1 Review Problems

The problems posted on the website already are just a bunch of counting problems. The problems below focus more proofs, induction, recursion and Stirling numbers in the ways they have appeared in class. If I think of more good problems I'll post them as well. This is **not** an assignment. Completion is optional.

1. Let $NC(n, k)$ be defined as the number of ways of selecting k non-consecutive integers from a set of n consecutive integers.
   (a) What are the bounds on k in terms of n? (We'll include $k = 0$ as a base case. There's only one way to select an empty sequence.)
   (b) What are some simple base cases and their evaluations?
   (c) Write a recursive formula for $NC(n, k)$ as a function of smaller values of n, k
   (d) Using the previous results, make an "NC triangle" and compute the first 10 rows, starting with $NC(2, 0) = 1$ as the apex.
   (e) Now define $NC(n) = \sum_k NC(n, k)$ as the number of ways of selecting any non-consecutive subsequence of n consecutive integers. Write a recursive function for $NC(n)$ and check it against the values in your triangle.
   (f) Now write a closed, form, non-recursive formula for $NC(n, k)$. You may deduce this from the triangle, and/or by counting subsequences of $n - k + 1$ consecutive integers. (Show a bijection from this problem to $NC(n, k)$).
   (g) Finally, look at the row sums of your triangle the definition of $NC(n)$ and deduce a cool identity.

2. Let $S^3(n, k)$ be defined as the number of ways of partitioning a set of n distinct labeled items into exactly k subsets, with each subset containing at least 2 distinct items. (The Stirling number of the 'third kind??' is not a real thing...)
   (a) Establish bounds on k and reasonable base cases for $S^3(n, k)$
   (b) Derive a recursive formula for $S^3(n, k)$ and make the first few rows of the S^3 triangle
   (c) Find a different, direct formula for $S^3(n, n/2)$ when n is even, and numerically verify both formulas give the same answer for $S^3(6, 3)$

3. Prove the identity
$$\sum_{t=0}^{k} \binom{m}{k}\binom{n}{t-k} = \binom{m+n}{t}$$
by modifying a combinatorial argument made in class.

4. Prove $\sum_{j=0}^{n} S(n,j)s(j,k) = \delta_{n,k}$, where s is the *signed* Stirling number of the first kind and δ is the Kronecker delta. (This might be too hard. At least set it up.)
5. Evaluate $\sum_{k=0}^{n} k^2 \binom{n}{k}$
6. The sum of the factors of 6 is $1+2+3+6 = 12$. Prove that whenever $m = (2^{n-1})(2^n-1)$ and $2^n - 1$ is prime, then the sum of the factors of m is $2m$.
7. Prove that p always divides evenly into $\binom{p}{k}$ whenever p is prime and $0 < k < p$.
8. In how many ways may 10 people be seated at 3 circular tables so that no table is empty? Tables are labeled. Seats are not. People are not clones.
9. Given the definition of Catalan numbers $C_n = C_0C_{n-1}+C_1C_{n-2}+\cdots+C_{n-1}C_0$; $C_0 = 1$, let $C(x)$ be the ordinary generating function of C_n. Prove $xC(x)^2 = C(x)-1$. (The test does not cover generating functions. This is more of an algebra-of-Catalan-numbers problem)
10. Know the recursive definitions of P, C, S, s and non-recursive/direct definitions when given.
11. Understand thoroughly the first 9 cells of the "12fold table" and be able to apply or generalize.
12. Know the relationships between Stirling numbers, falling/rising factorials and powers of x.
13. Know the outline of the Stirling proof for $n!$. So – what integral are we estimating? What are the upper and lower bounds derived from (which trapezoidal approximations). And what bounds does this eventually give us?
14. Since 13 is unlucky, prove that $\sum_{n>0} \frac{1}{n}$ diverges.