Application of Generating Functions

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Date: Day, Mon, Date Year

1 From the Review: Stirling

Problem 1. Prove
$$\sum_{j=0}^{n} {n \brace k} {n \brack k} = \delta_{nk}$$

Proof. We will prove this using coordinate transformations from the polynomial basis to the falling factorials, and vice versa. Note that:

$$x^n = \sum_{j=0}^{n} {n \brace k} x^{\underline{j}}$$

and

$$x^{\underline{j}} = \sum_{k=0}^{j} {j \brack k} x^k$$

Note that we can allow this last sum to go from 0 to n, which is fine because we are only adding zeros. This gives us:

$$x^n = \sum_{j=0}^{n} {n \brace k} \sum_{k=0}^{n} {j \brack k} x^k$$

Interchanging the sums freely, we can see that:

$$x^n = \sum_{k=0}^{n} \sum_{j=0}^{n} {n \brace k} {j \brack k} x^k$$

Notice that we **must** have that $x^k = x^n$ for the internal sum coefficient to be non-zero - otherwise, the sum is always zero. Therefore, we must have that:

$$\sum_{j=0}^{n} {n \brace k} {j \brack k} = \delta_{nk}$$

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2 Applications of Generating Functions

Problem 2. A food-counting problem - what is the number of ways to fill a bag with n pieces of fruit given:

- The number of apples is even.
- The number of bananas is a multiple of 5.
- The number of oranges is ≤ 4.
- The number of pears is ≤ 1.

Solution We will solve this with a generating function. Notice that we can express each of these conditions into a generating function:

$$(1+x^2+x^4+\ldots)(1+x^5+x^10+\ldots)(1+x+x^2+x^3+x^4)(1+x)$$

Notice that we can manipulate these as follows:

$$= \frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1-x^5}{1-x}(1+x) = \frac{1}{(1-x)^2}$$

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Problem F. ind the number of ways to make change for a dollar (in pennies, nickels, dimes, quarters, or half-dollars).

Solution This is the coefficient of x^{100} in

$$= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}$$

which happens to be 293. ■

Problem F. ind the number of ways to make change for a dollar if we have access to any denomination.

Solution This gives us the generating function for integer partitions:

$$g(x) = \frac{1}{(1-x)}\frac{1}{(1-x^2)}\frac{1}{(1-x^3)}\ldots = \Pi_{k>0}\frac{1}{1-x^k} = \sum_{r>0} P(r)x^r$$

where $P(r)$ is the number of integer partitions of r. This does not have a particularly pretty closed form. ■

Theorem 2.1. *The number of partitions of n into distinct parts (using every number exactly once) is equal to the number of partitions of n into odd parts (using only numbers that are odd).*

Proof. We will do this using generating functions. The generating function to partition a number into odd numbers is:

$$\frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\frac{1}{1-x^7}\cdots$$

The generating function for the number of ways to partition a number into distinct numbers is:

$$(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

If we manipulate the second generating function as follows:

$$=\frac{1-x^2}{1-x}\frac{1-x^4}{1-x^2}\frac{1-x^6}{1-x^3}\frac{1-x^8}{1-x^4}\cdots$$

we will see that all the terms of the form $(1-x^k)$ where k is even will cancel with the similar terms in the denominator, yielding the desired. ∎

3 Solving Recurrence Relations

We will apply generating functions to find explicit formulae for recurrence relations.

Problem G. iven the recurrence relation $a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}$, and the terms $a_0 = 2, a_1 = 3, a_2 = 19$, find an explicit formula for a_n and a generating function for the sequence.

Solution We will first construct the generating function:

$$g(x) = \sum_{n=0}^{\infty} a_n x^n = 2 + 3x + 19x^2 + \sum_{n=3}^{\infty} a_n x^n$$

Applying the recurrence relation:

$$\begin{aligned}
g(x) &= 2 + 3x + 19x^2 + \sum_{n=3}^{\infty}(a_{n-1} + 8a_{n-2} - 12a_{n-3})x^n \\
&= 2 + 3x + 19x^2 + x\sum_{n=3}^{\infty} a_{n-1}x^{n-1} + 8x^2\sum_{n=3}^{\infty} a_{n-2}x^{n-2} - 12x^3\sum_{n=3}^{\infty} a_{n-3}x^{n-3} \\
&= 2 + 3x + 19x^2 + x\sum_{n=2}^{\infty} a_n x^n + 8x^2\sum_{n=1}^{\infty} a_n x^n - 12x^3\sum_{n=0}^{\infty} a_n x^n \\
&= 2 + 3x + 19x^2 + x(g(x) - 2 - 3x) + 8x^2(g(x) - 2) + 12x^3 g(x)
\end{aligned}$$

This gives us, after some rearranging:

$$g(x) = \frac{2+x}{1-x-8x+12x^3}$$

We now have to do partial fractions to do this fully - breaking it down into

$$g(x) = \frac{A}{1+3x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

Using cover-up, we obtain $A = \frac{3}{5}, B = \frac{2}{5}, C = 1$. When we plug this in and expand using series:

$$\begin{aligned} g(x) &= \frac{\frac{3}{5}}{1+3x} + \frac{\frac{2}{5}}{1-2x} + \frac{1}{(1-2x)^2} \\ &= \frac{3}{5}\sum_{n=0}^{\infty}(-3x)^n + \frac{2}{5}\sum_{n=0}^{\infty}(2x)^n + \sum_{n=0}^{\infty}\left(\!\!\binom{2}{n}\!\!\right)(2x)^n \\ &= \frac{3}{5}\sum_{n=0}^{\infty}(-3x)^n + \frac{2}{5}\sum_{n=0}^{\infty}(2x)^n + \sum_{n=0}^{\infty}(n+1)(2x)^n \end{aligned}$$

We can now just read out the explicit formula:

$$a_n = \frac{3}{5}(-3)^n + \frac{2}{5}(2)^n + (n+1)(2)^n$$

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Problem F. ind a generating function and explicit formula for the sequence $a_0 = 1, a_1 = 9, a_n = 6a_{n-1} - 9a_{n-2}$.

Solution We find the generating function first:

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= 1 + 9x + \sum_{n=2}^{\infty}(6a_{n-1} - 9a_{n-2})x^n \\ &= \frac{1+3x}{1-6x+9x^2} \\ &= \frac{-1}{1-3x} + \frac{2}{(1-3x)^2} \end{aligned}$$

This directly gives us the explicit formula

$$a_n = -3^n + 2(n+1)3^n = (2n+1)3^n$$

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Problem F. ind a generating function and explicit formula for the sequence $a_0 = 0, a_1 = 6, a_n = -3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n$.

Solution Again, we find the generating function:

$$\begin{aligned}
g(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= 0 + 6x + \sum_{n=2}^{\infty} (-3a_{n-1} + 10a_{n-2} + 3 \cdot 2^n) x^n \\
&= 6x - 3xg(x) + 10x^2 g(x) + \frac{3}{1-2x} - 3 - 6x \\
&= \frac{6x}{(1-2x)^2(1+5x)} \\
&= \frac{-\frac{12}{49}}{1-2x} + \frac{\frac{6}{7}}{(1-2x)^2} + \frac{-\frac{30}{49}}{1+5x}
\end{aligned}$$

In the end, we have:

$$a_n = -\frac{12}{49} 2^n + \frac{6}{7}(n+1)2^n - \frac{30}{49}(-5)^n$$

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