Applications of Ordinary Generating Functions

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1 Revisiting an Old Problem

Problem 1. Given $\{a, b, c, d, e\}$, compute the number of ways pick a multiset of size 3 from these elements such that no item may appear more than 2 times.

We will do this using an ordinary generating function. Define an **ordinary generating function** $A(x)$ of a sequence $a_0, a_1, a_2, \dots$ as

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

Consider the sum

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2$$

We will evaluate this in two ways. First, consider the product $(1+x)^n \left(1 + \frac{1}{x}\right)^n$ and expand these using the binomial theorem:

$$\begin{aligned}
(1+x)^n \left(1 + \frac{1}{x}\right)^n &= \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) \left(\sum_{j=0}^{n} \binom{n}{j} x^{-j}\right) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \binom{n}{k} \binom{n}{j} x^{k-j}
\end{aligned}$$

Note that our desired sum is the coefficient of the constant term - ie. all the terms when $k = j$.

However, notice also that this product can be rearranged as

$$(1+x)^n \left(1 + \frac{1}{x}\right)^n = (1+x)^{2n} x^{-n} = x^{-n} \sum_{k=0}^{n} \binom{2n}{k} x^n$$

so the constant term is $\binom{2n}{n}$. Therefore, this sum is $\binom{2n}{n}$. ■

We can also show this with a clever combinatorial argument. Suppose we have n blue marbles and n green marbles, and we wish to pick n of these marbles, which can be done in $\binom{2n}{n}$ ways. However, we can also compute this by doing casework on the number of green marbles we pick. We could pick g green marbles, where g can range between 0 and n.

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Generating functions are based around the idea that algebra can do a lot of combinatorial problems for us. Consider first the *enumerator* polynomial:

$$(1+ax)(1+bx)(1+cx) = 1 + (a+b+c)x + (ab+bc+ac)x^2 + abcx^3$$

Notice that if we have sets A, B, C with sizes This isn't very useful for larger and larger sizes of sets - but what if we just wanted to count the number of possible subsets of size k from a set of n distinct elements? By our logic above, if we set $a = b = c$, we just get

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3 = \sum_{k=0} C_k x^k$$

where C_k is the number of subsets of size k. This directly gives us the identity

$$(1+x)^n =$$

Let's plug a few values of x into this expression. If we plug $x = 1$, we have:

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

If we plug in $x = -1$, we have:

$$0 = \sum_{n=0}^{n} \binom{n}{k} (-1)^k$$

or, by parity:

$$\sum_{keven} \binom{n}{k} = \sum_{kodd} \binom{n}{k}$$

If we take a derivative with respect to x of the original expression:

$$n(1+x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}$$

So if we plug in $x = 1$:

$$n2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}$$

We can also consider this combinatorially - consider the number of ways to pick a team of any size that has a designated captain.

What if we want to select k items from n, with each element chosen at most twice? We now have to change the way that our original enumerating function worked - notice that we must introduce a quadratic term in the $1+ax$ in a factor, a^2x^2, to pick two of any element from the set A. Repeating the same logic as above, we have the generating function for this problem

$$(1+x+x^2)^n.$$

Let us expand it using multinomial coefficients:

$$(1+x+x^2)^n = \sum_{i+j+k=n} 1^i x^j (x^2)^k \binom{n}{i,j,k}$$

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$, a *multinomial* coefficient. Therefore, we have the coefficient of x^{j+2k} as $\binom{n}{i,j,k}$, which, when written as $\sum C_r x^r$, is the number of ways to pick a multiset of size r with each element used up to two times. This allows us to do the first problem again from a different perspective:

Problem 1, again. Given $\{a,b,c,d,e\}$, compute the number of ways pick a multiset of size 3 from these elements such that no item may appear more than 2 times.

We can do this using a generating function - this equivalent to finding the coefficent of x^3 in the expansion of $(1+x+x^2)^5$. Notice that the only way to do this is to have $x^3 = (x)(x^2)^1$, or $x^3 = (x)^3$. This means that the number of ways to generate our multiset is $\binom{5}{3,0,2} + \binom{5}{1,1,3} = 10 + 20 = \boxed{30}$.

Problem F. rom 18 items, pick between 3 and 5 of each and select a multiset of size 79.

Solution This corresponds to finding the coefficient x^79 of $(x^3+x^4+x^5)^18$, which can be computed using some brute force to be ■

Given a set of n elements, choose a multiset of size r (with no restrictions). We know already that this is The corresponding generating function is

$$(1+x+x^2+\ldots+x^r+\ldots)^n$$

of which we want the coefficient of x^r. We don't have to have the coefficients after the x^r term, but the fact that we have an infinitude of terms after the x^r terms gives the following easy form for this generating function:

$$\frac{1}{(1-x)^n}$$

We can expand this binomially:

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \frac{-n^{\underline{k}}}{k!}(-x)^k = \sum_{k=0}^{\infty} \frac{n^{\overline{k}}}{k!} x^k$$

where $n^{\overline{k}}$ is the *rising factorial*, $n(n+1)(n+2)\ldots(n+k-1)$. This can be rearranged again to the form we already have seen:

$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k = \sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k$$

Problem 3. Given a set of n elements, choose a multiset of size r, but each element must be used at least once.

Solution Instead of starting each of our factors at 1, we must start each factor at x as each element must be chosen at least once. This gives us the generating function

$$(x + x^2 + \ldots)^n = \left(\frac{x}{1-x}\right)^n$$

We can expand:

$$\begin{aligned}
&= x^n \sum_{k=0}^{\infty} \frac{-n^{\underline{-k}}}{k!}(-x)^k \\
&= \sum_{k=0}^{\infty} \frac{n^{\overline{k}}}{k!} x^{k+n} \\
&= \sum_{r=n}^{\infty} \frac{n^{\overline{r-n}}}{(r-n)!} x^r \\
&= \sum_{r=n}^{\infty} \binom{r-1}{n-1} x^r = \sum_{r=n}^{\infty} \left(\!\binom{n}{r-n}\!\right) x^r
\end{aligned}$$

Therefore, we have the number of ways to do this as the coefficient of x^r, which is $\binom{r-1}{n-1} = \left(\!\binom{n}{r-n}\!\right)$. ■

Problem 4. Find the coefficient of x^23 in the expansion of $(1 + x^5 + x^9)^100$.

Solution The only way we can do this is by considering $23 = 2 \cdot 9 + 5$, so then

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