Basic Operations on Generating Functions

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Date: Tuesday, March 12, 2019

1 Practice with Generating Functions

Here's some practice turning sequences to generating functions:

1. $\langle 1, 1, 1, 1 \ldots \rangle = \frac{1}{1-x}$
2. $\langle 2, 2, 2, 2 \ldots \rangle = \frac{2}{1-x}$ by doubling above equation
3. $\langle 1, -1, 1, -1 \ldots \rangle = \frac{1}{1+x}$ by noticing the alternation of signs, so we can substitute $-x$ for x
4. $\langle 1, 0, 1, 0 \ldots \rangle = \frac{1}{1-x^2}$ by noticing we only want the even powers, so we can substitute x^2 for x
5. $\langle 0, 1, 0, 1 \ldots \rangle = \frac{x}{1-x^2}$ by noticing we can multiply the solution the last problem by x to shift the start of the sum.
6. $\langle 1, 2, 4, 8 \ldots \rangle = \frac{1}{1-2x}$ by noticing we can get powers of 2 by replacing x with $2x$
7. $\langle 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots \rangle = \frac{e^x - 1}{x}$ by noticing the similarity to the expansion of the exponential, but we have to shift it down by killing the first term and lowering the powers.
8. $\langle 1, 2, 3, 4, \ldots \rangle = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$ because taking a derivative multiplies each term by its index.
9. $\langle 2, 6, 12, 20, 30, \ldots \rangle = \frac{d^2}{dx^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}$ by applying the above trick twice.
10. $\langle 0, 1, 4, 9, 16, \ldots \rangle = ?$
    Recall that
    $$k^2 = 1(k+2)(k+1) - 3(k+1) + 1$$
    where the coefficients are Stirling numbers of the second kind. However, we already have expressions for the generating functions for the falling factori-

als:

$$\begin{aligned}(k+2)(k+1) &\rightarrow \frac{d^2}{dx^2}\frac{1}{1-x} = \frac{2}{(1-x)^3}\\(k+1) &\rightarrow \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}\\1 &\rightarrow \frac{1}{1-x}\end{aligned}$$

This gives us our answer $\frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} = \frac{x+x^2}{(1-x)^3}$

2 Multiplication of Generating Functions

Let

$$\begin{aligned}A(x) &= a_0x^0 + a_1x^1 + a_2x^2 + \ldots = \sum_{i\geq 0} a_i x_i\\B(x) &= b_0x^0 + b_1x^1 + b_2x^2 + \ldots = \sum_{i\geq 0} b_i x_i\end{aligned}$$

Let $C(x) = A(x) \cdot B(x)$. What is it's corresponding sequence?
We have to take the *Cauchy product*.

$$\begin{aligned}C(x) &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 \ldots\\C_n &= \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

Consider the two functions

$$\begin{aligned}A(x) &= \text{anything}\\B(x) &= 1 + x + x^2 + x^3 + \ldots\\C(x) = A(x)B(x) &= a_0 + (a_0 + a_1)x + \ldots = \sum_{i=0}^{k} a_i x^k\end{aligned}$$

Thus we arrive at the following identity:

Theorem 2.1. *If $A(x)$ be the ordinary generating function for $\langle a_0, a_1, a_2 \ldots\rangle$. Then, $A(x)/(1-x)$ for $\langle a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots\rangle$*

Let us now consider $A(x) = B(x) = \frac{1}{1-x}$ Recall the following theorem:

$$\sum \frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \left(\!\binom{n}{k}\!\right) x^k$$

$$\begin{aligned} C(x) &= \langle 1, 2, 3, 4, 5, \ldots \rangle \\ \frac{1}{(1-x)^2} &= \sum_{k=0}^{\infty} \left(\!\!\binom{2}{k}\!\!\right) x^k \\ &= \sum \binom{k+1}{k} x^k \\ &= \sum (k+1) x^k \end{aligned}$$

Now let us consider $B(x) \cdot C(x)$

$$\begin{aligned} \langle 1, 3, 6, \ldots \rangle &= \frac{1}{(1-x)^3} \\ &= \sum_{k=0}^{\infty} \left(\!\!\binom{3}{k}\!\!\right) x^k \\ &= \sum_{k=0}^{\infty} \binom{k+2}{2} x^k \\ &= \sum_{k=1}^{\infty} \frac{k(k+1)}{2} x^{k-1} \end{aligned}$$

This gives us the formula for the triangular numbers. We can go further and get the formula for the tetrahedral numbers.

$$\begin{aligned} D(x) &= \frac{x}{(1-x)^3} = 0 + x + 3x^2 + 6x^3 + 10x^4 + \ldots = \sum_{k=0}^{n} \frac{k(k+1)}{2} x^k \\ E(x) &= D(x) \cdot B(x) = \frac{x}{(1-x)^4} + x \cdot \sum_{k=0}^{\infty} \left(\!\!\binom{4}{k}\!\!\right) x^k \\ &= \sum_{k=0}^{\infty} \binom{k+3}{3} x^{k+1} = \sum_{k=1}^{\infty} \frac{k(k+1)(k+2)}{6} x^k \end{aligned}$$

Now it seems that, by extending this process of computing generating functions for hypertetrahedral numbers and equation coefficients.

$$\sum_{k=1}^{n} k^{\overline{m}} = \frac{n^{\overline{m+1}}}{m+1}$$

This is the power rule for $k^{\overline{m}}$ (finite integrals). For instance, $1^{\overline{4}} + 2^{\overline{4}} + 3^{\overline{4}} + 4^{\overline{4}} + 5^{\overline{4}} + 6^{\overline{4}} = \frac{6^{\overline{5}}}{5}$.

Problem 1. Find $\sum_{i=1}^{n} i^2$

Solution We can write $i^2 = i^{\overline{2}} - i^{\overline{1}}$, and we also know $\sum_{i=1}^{n} i^{\overline{1}} = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^{\overline{2}} = \frac{n(n+1)(n+2)}{3}$, so we can subtract and simplify, getting $\frac{n(n+1)(2n+1)}{6}$.

Solution Start by expanding the rising factorial in terms of power.

$$\sum i^3 = \sum i^{\overline{3}} - 3\sum i^2 - 2\sum i = \frac{n^{\overline{4}}}{4} - \frac{n(n+1)(2n+1)}{2} - n(n+1)$$