Polya's Enumeration Theorem

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Polya's Theorem is a generalization of Burnside's Lemma

1 Proof of Burnside's Lemma

First, recall Burnside's Lemma:

Lemma 1.1. *The number of equivalence classes in S under the action of permutation group G can be calculated to be*

$$\frac{1}{|G|}\sum_{\pi \in G} |fix(\pi)|$$

Recall also the Orbit-Stabilizer Theorem:

Theorem 1.2. *For any $s \in S$ with associated permutation group G,*

$$|Orb(s)| \cdot |stab(s)| = |G|$$

Now, let us count the the orbits/equivalence classes of the vertices of the square under the dihedral group D_4, using the nomenclature for colorings established in the "Orbit Stablizer Theorem Quiz." We see the orbits to be

$$\{c_1\}, \{c_2, c_3, c_4, c_5\}, \{c_6, c_7, c_8, c_9\}, \{c_{10}, c_{11}\}, \{c_{12}, c_{13}, c_{14}, c_{15}\}, \{c_{16}\}$$

There are 6 equivalence classes.

Proof. Count the fixed points in the table $S \times G \to S$. $\pi(s) = s$ in two ways. Using the intuition of the table, we see that

$$\sum_{\pi \in G} |fix(\pi)| = \sum_{s \in S} |stab(s)|,$$

which, via the Orbit-Stabilizer theorem we see to be equal to

$$\sum_{s \in S} \frac{|G|}{|orb(s)|}.$$

Partitioning the elements of S into their respective orbits, we can rewrite this as

$$\sum_{O_i \in Orbits} \sum_{s \in O_i} \frac{|G|}{|orb(s)|} = |G| \sum_{O_i \in Orbits} \sum_{s \in O_i} \frac{1}{|O_i|} = |G| \sum_{O_i \in Orbits} |O_i||O_i|$$

$$= |G| \sum_{O_i \in Orbits} 1 = |G| \cdot \#orbits.$$

Dividing both sides by $|G|$, we see

$$\frac{1}{|G|} \sum_{\pi \in G} |fix(\pi)| = \#orbits.$$

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2 Polya's Theorem

Definition. *Let X be a set ("vertices") and S be a set of functions $X \to C$("colors"). Let G be a group operating on X. We will say $f_i, f_j \in S$ are **equivalent,** or that $f_i \cong f_j$ iff*

$$\exists \pi \in G | f_i(x) = f_j(\pi(x)) \forall x \in X.$$

Example 2.1.

$$\begin{matrix} G & R \\ R & R \end{matrix} \cong \begin{matrix} R & R \\ G & R \end{matrix}$$

because H turns c_2 into c_{15}. Explicitly,

$$c_{15}(H(1)) = c_{15}(4) = G$$

$$c_{15}(H(2)) = c_{15}(3) = R$$

$$c_{15}(H(3)) = c_{15}(2) = R$$

$$c_{15}(H(4)) = c_{15}(1) = R$$

and

$$C_{12}(1) = G$$

$$c_{12}(2) = R$$

$$c_{12}(3) = R$$

$$c_{12}(4) = R$$

Definition. *The weight of the colors is some function $w : C \to \mathcal{F}$ for some field $\mathcal{F}$.*

Definition. *The weight $W(f)$ of a function $f : X \to C$ is defined to be*

$$\prod_{x \in X} w(f(x)).$$

Definition. *The inventory of a set S of functions is*

$$\sum_{f\in\mathcal{S}} W(f).$$

Example 2.2. *Take when we 2-color a square.*

$$w(R) = r$$

$$w(G) = g$$

$$W(c_1) = r^4$$

$$W(c_2) = rg^3$$

$$W(c_4) = r^2g^2$$

The inventory of $c_1, \cdots, c_{16}$ is

$$r^4 + 4rg^3 + 6r^2g^2 + 4rg^3 + g^4 = (r+g)^4.$$

Definition. *The cycle index (polynomial) of a permutation group*

$$P_G(x_1, \cdots, x_k, \cdots = \frac{1}{|G|}\sum_{\pi\in G} x_1^{b_1}x_2^{b_2}x_3^{b_3}\cdots x_k^{b_k}\cdots$$

where b_k is the number of cycles of length k in π.

Example 2.3. D_4 :

$$e : (1)(2)(3)(4) \to x_1^4$$

$$r_{90} : (2143) \to x_4^1$$

$$r_{180} : (24)(13) \to x_2^2$$

$$r_{270} \to x_4^1$$

$$H : (14)(23) \to x_2^2$$

$$V : (12)(34) \to x_2^2$$

$$L : (24)(1)(3) \to x_1^2x_2$$

$$R : (13)(2)(4) \to x_1^2x_2$$

So, the cycle index polynomial for D_4 is

$$\frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2x_2 + 2x_4).$$

Now, to state Polya's theorem:

Theorem 2.1. *The inventory of the equivalence classes of functions $f : X \to C$ under the action of permutation group G is given by*

$$P_G(\sum w(x), \sum w^2(c), \cdots, \sum w^k(c), \cdots)$$

Example 2.4. *For the square under D_4, the inventory is given to be*

$$P_G(r+g, r^2+g^2, r^3+g^3, r^4+g^4)$$

$$= \frac{(r+g)^4 + 3(r^2+g^2)^2 + 2(r+g)^2(r^2+g^2) + 2(r^4+r^4)}{8}$$

$$= g^4 + g^3r + 2g^2r^2 + gr^3 + r^4$$

Corollary 2.1.1. *The number of equivalence classes of functions $f : X \to C$ under the action of permutation group G is*

$$P_G(|C|, |C|, |C|, |C|, \cdots, |C|$$

Proof. Plug in 1 for $w(c)$. So, $w(c) = 1$ for each color:

$$w(R) = 1,$$

$$w(G) = 1.$$

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Example 2.5. *The cycle index polynomial of the symmetries of the cube is:*

$$P_G(x_1, x_2, x_3, x_4) = \frac{1}{24}(x_1^6 + 3x_1^2x_2^2 + 6x_2^3 + 6x_1^2x_4 + 8x_3^2).$$

Plugging $|C| = n$ into the polynomial, we get the number of equivalence classes of the colorings of the cube to be equal to

$$C(n) = \frac{1}{24}(n^6 + 6n^3 + 3n^4 + 6n^3 + 8n^2) = \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}$$

Take this example from chemistry:

Example 2.6.

$$\begin{array}{ccccc} & & \mathrm{R} & & \\ & & | & & \\ \mathrm{R} & — & \mathrm{C} & — & \mathrm{R} \\ & & | & & \\ & & \mathrm{R} & & \end{array} \qquad R = \begin{cases} CH_3 & methyl \\ C_2H_5 & ethyl \\ H & hydrogen \\ Cl & chlorine \end{cases}$$

Find the a) number of molecules, and b) the ones containing ≥ 1 *solitary H atom. It is a chemical fact that these types of molecules in fact are not planer, but actually tetrahedral. The symmetry group of the tetrahedron has these categories of elements, written in cycle notation:*

$$e : (1)(2)(3)(4) : 1$$

$$r_v : (1)(234) : 4$$

$$r_v^2 : (1, 243) : 4$$

$$r_3 : (12)(34) : 3$$

We can use these to find the cycle index polynomial to be

$$\frac{x_1^4 + 8x_1x_3 + 3x_2^2}{12}.$$

So $C(n) = \frac{n^4+11n^2}{12}$. *We then see that* $C(4) = 36$ *and* $C(3) = 15$. *Using complementary counting we can find the answer to b) to be* $C(4) - C(3) = 21$.