Permutation Groups and Burnside's Lemma

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1 Permutation Groups

Let S be a set. Let G be a group of permutations, π, acting on elements of S. Then, G is a "permutation group".

Example 1.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

$$\pi_1(1) = 3,\ \pi_1(4) = 5, \text{etc.}$$

We can write π_1 as a product of cycles, as we have done in the past.

$$\pi_1 = (1\ 3)(2)(4\ 5)$$

This is the "cycle decomposition" of π_1.

Theorem 1.1. *Every permutation can be written as a product of cycles.*

These cycles are unique up to rotations and re-ordering cycles.
I.E. $\pi_1 = (2)(5\ 4)(1\ 3)$
Permutations can also be composed

$$\pi_1\pi_1 = \pi_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e$$

So $\pi_1 = \pi_1^{-1}$
Thus, $G = \{e, \pi_1\}$ is a permutation group.

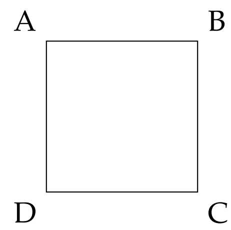
A) Closed

B) Identity

C) Inverses

D) Associativity

Example 2. The group of rotations of a square.

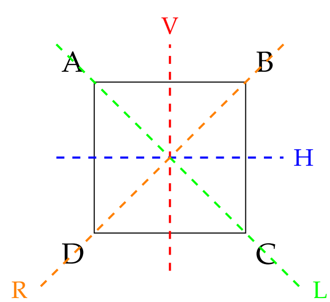


Let S_4 be the "symmetric group", which is all 24 permutations of $ABCD$. Using only rotations, we can only find 4 of the permutations, as shown here:

$$\left[\begin{array}{c|cccc} r_0 & \text{A} & \text{B} & \text{C} & \text{D} \\ r_{90} & \text{D} & \text{A} & \text{B} & \text{C} \\ r_{180} & \text{C} & \text{D} & \text{A} & \text{B} \\ r_{270} & \text{B} & \text{C} & \text{D} & \text{A} \end{array}\right]$$

This group can be written as $G = \{e, \pi_1 = 90°, \pi_2 = 180°, \pi_3 = 270°\}$
This group can also be generated by π_1 and π_3.
$< \pi_1 >= \{\pi_1, \pi_2, \pi_3, e\}$

Example 3. The group of all symmetries of a square (rotations and flips)



Let's construct a Cayley table!

2nd\1st	e	r_{90}	r_{180}	r_{270}	V	H	L	R
e	e	r_{90}	r_{180}	r_{270}	V	H	L	R
r_{90}	r_{90}	r_{180}	r_{270}	e	R	L	V	H
r_{180}	r_{180}	r_{270}	e	r_{90}	H	V	R	L
r_{270}	r_{270}	e	r_{90}	r_{180}	L	R	H	V
V	V	L	H	R	e	r_{180}	r_{90}	r_{270}
H	H	R	V	L	r_{180}	e	r_{270}	r_{90}
L	L	H	R	V	r_{270}	r_{90}	e	r_{180}
R	R	V	L	H	r_{90}	r_{270}	r_{180}	e

D_4 is the dihedral group on 4 elements. We found that $|D_4| = 8$

Theorem 1.2. $|D_n| = 2n$

Question 1. What is <u>one</u> $\pi \in S_4;\ \pi \notin D_4$?

$$\pi = \begin{pmatrix} A & B & C & D \\ B & C & A & D \end{pmatrix}$$

Let's write out the cycle decomposition for each of the transformations of the square.

$$\begin{aligned}
e &= e \\
r_{90} &= (A\ D\ C\ B) \\
r_{180} &= (A\ C)(D\ B) \\
r_{270} &= (A\ B\ C\ D) \\
H &= (A\ D)(B\ C) \\
V &= (A\ B)(C\ D) \\
L &= (B\ D)(A\ C) \\
R &= (A\ C)(B\ D)
\end{aligned}$$