The Lecture Title

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Date: Day, Mon, Date Year

1 Permutation Groups and Burnside's Lemma

Definition. Let S be a set and G be a group of permutations π, acting on elements on S. Then G is a **permutation group**.

What exactly do we mean by a permutation? A permutation refers essentially to the following mapping of elements:

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

Notice that $\pi_1(1) = 3$ and $\pi_1(4) = 5$, etc. We also showed in Unit 1 that every permutation can be written as a product of cycles. π_1 can thus be decomposed into cycles and written in the following way:

$$\pi_1 = (13)(2)(45)$$

This is unique up to rotations and reordering of the cycles - so the following is the same permutation:

$$\pi_1 = (2)(13)(54)$$

Permutations can be composed - notice that if we apply π_1 to itself we have

$$\pi_1\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e$$

so $\pi_1 = \pi_1^{-1}$. Thus, $G = \{e, \pi_1\}$ is a permutation group (if we check all four necessary properties).

Another example of a permutation group - the group of rotations of a square. First, define the **symmetric group** S_n as all of the $n!$ permutations of n elements, so S_4 is essentially all permutations of the elements A, B, C, D.

The group of rotations of a square can be pretty clearly to be every $90°$ rotation of the square (which we will call the set $\{e, \pi_1, \pi_2, \pi_3\}$ where $e = \pi_0$ is the identity and a rotation of $90i$ degrees clockwise is π_i.

What about the group of all rotations of a square? In addition to $e, r_{90}, r_{180}, r_{270}$ (or π_1, π_2, π_3, whatever), we have the reflections H, V, L, R, where H is a reflection about a horizontal axis, V is a reflection about a vertical axis, L is a reflection about a main diagonal emanating from the top-left corner, and similarly for R. Below is the Cauchy table for the group:

	e	r_{90}	r_{180}	r_{270}	V	H	L	R
e	e	r_{90}	r_{180}	r_{270}	V	H	L	R

This group has a name - called D_4, or the dihedral group on four elements. It turns out that $|D_4| = 8$, and it turns out that $|D_n| = 2n$.

What is one $\pi \in S_4$ that is not in D_4?

Notice that we can write all the elements of D_4 as products of cycles: