Solving Recurrence Equations

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1 First Example

Question: Given the equation $a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}$ and the base cases $a_0 = 2, a_1 = 3$, and $a_2 = 19$, find a generating function for the recurrence equation and a solution for any a_n.

1.1 Analyzing the Equation

To start let's look at the initial equation, $a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3}$. There are a few key things to note regarding the equation:

1) The function is a linear recurrence equation
How do we know this? Because the a_n's don't multiply with one another.

2) The function is homogeneous
How do we know this? Because there is no forcing function (e.g., n^2).

3) The function is a constant coefficient function
How do we know this? Because none of the a_n's have a coefficient that is some sort of variable.

We therefore call this a *linear, constant coefficient, homogeneous difference equation*.

1.2 Finding a Generating Function

Now that we know what kind of equation it is, we can start with a generating function:

$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$

From here, we can include the initial conditions stated above for $n = 0$, $n = 1$, and $n = 2$. This means that the expression in the summation will now go from $n = 3$ to infinity instead of starting from $n = 0$.

$g(x) = 2 + 3x + 19x^2 + \sum_{n=3}^{\infty} a_n x^n$

We can then also include the function that we have for a_n:

$g(x) = 2 + 3x + 19x^2 + \sum_{n=3}^{\infty}(a_{n-1} + 8a_{n-2} - 12a_{n-3})x^n$

Since the entire a_n function is being multiplied by x^n, we can distribute it appropriately:

$g(x) = 2 + 3x + 19x^2 + \sum_{n=3}^{\infty} a_{n-1}x^n + \sum_{n=3}^{\infty} 8a_{n-2}x^n - \sum_{n=3}^{\infty} 12a_{n-3}x^n$

Now, we'll need to make two manipulations. First, we want the subscript of each of the a_n's to match the power to which each x is being raised. For example, in $\sum_{n=3}^{\infty} a_{n-1}x^n$, we want the x to be raised to a power of $n-1$. To do this, we can factor out one x from the first summation, we can factor out $8x^2$ from the second summation, and we can factor out $12x^3$ from the third summation. Doing so gives us the following:

$g(x) = 2+3x+19x^2+x\sum_{n=3}^{\infty} a_{n-1}x^{n-1}+8x^2\sum_{n=3}^{\infty} a_{n-2}x^{n-2}-12x^3\sum_{n=3}^{\infty} a_{n-3}x^{n-3}$

The second manipulation involves normalizing the indices. What this means is that for each summation, the sums can be rewritten as some form of the original $\sum_{n=0}^{\infty} a_n x^n$ once we take out a few initial terms. $\sum_{n=3}^{\infty} a_{n-1}x^{n-1}$, for example, is equal to $\sum_{n=2}^{\infty} a_n x^n$. Once you do this for each summation, you end up with the following equation:

$g(x) = 2 + 3x + 19x^2 + x\sum_{n=2}^{\infty} a_n x^n + 8x^2\sum_{n=1}^{\infty} a_n x^n - 12x^3\sum_{n=0}^{\infty} a_n x^n$

Now that we have the new summations in terms of $a_n x^n$, they can all be rewritten using the original $g(x)$ function from the start of the problem and the three base cases:

$g(x) = 2 + 3x + 19x^2 + x(g(x) - a_0 - a_1 x) + 8x^2(g(x) - a_0) - 12x^3 g(x)$

Substituting the values we have for a_0 and a_1 into the equation, we get:

$g(x) = 2 + 3x + 19x^2 + x(g(x) - 2 - 3x) + 8x^2(g(x) - 2) - 12x^3 g(x)$

We can then move all the $g(x)$ terms to the left side of the equation, and once we simplify that equation, we obtain:

$g(x)(1 - x - 8x^2 - 12x^3) = 2 + x$

Dividing this equation by $(1 - x - 8x^2 - 12x^3)$ on both sides gives us our final equation for $g(x)$:

$$\boxed{g(x) = \frac{2+x}{1-x-8x^2-12x^3}}$$

****Note: Another way to obtain the denominator of the final $g(x)$ function is to look at the original a_n equation.** *By moving all the variables to one side of the equation, we obtain $a_n - a_{n-1} - 8a_{n-2} + 12a_{n-3} = 0$. This equation is knwon as the **Characteristic Equation of the Difference**. Each of the coefficients (1, 1, -8, and 12) correspond to the constant and coefficients found in the denominator of the final equation.*

1.3 Finding a Formula for a_n

With this new equation for $g(x)$, we can factor the denominator into $(1 + 3x)(1 - 2x)^2$ to determine an equation for a_n. More specifically, we can conduct *partial fraction decomposition*, meaning we'd rewrite the function as:

$$g(x) = \frac{2+x}{1-x-8x^2-12x^3} = \frac{A}{1+3x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}$$

To conduct partial fraction decomposition, we simply have to multiply each fraction on the right side by whatever it is missing in the denominator. Doing so allows us to simplify the equation to the following:

$$2 + x = A(1 - 2x)^2 + B(1 + 3x)(1 - 2x) + C(1 + 3x)$$

We can then solve for each variable by plugging in different values for x (e.g., when $x = -\frac{1}{3}$, $\frac{5}{3} = (\frac{5}{3})^2 A$, meaning that $A = \frac{3}{5}$). Doing this for each variable, gives us the following values:

$$A = \frac{3}{5}, B = \frac{2}{5}, C = 1$$

Plugging those values into the equation we have for $g(x)$ then gives us a new equation:

$$g(x) = \frac{3}{5}\left(\frac{1}{1+3x}\right) + \frac{2}{5}\left(\frac{1}{1-2x}\right) + \frac{1}{(1-2x)^2}$$

We can simplify these equations back into summation formulas using two basic principles:

1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ and 2) $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$

With these principles, we get the following equation:

$$g(x) = \frac{3}{5}(\sum_{n=0}^{\infty}(-3x)^n) + \frac{2}{5}(\sum_{n=0}^{\infty} 2x^n) + \sum_{n=0}^{\infty}(n+1)(2x)^n$$

We can now use this equation to achieve our final equation for a_n:

$$\boxed{a_n = \frac{3}{5}(-3)^n + \frac{2}{5}(2^n) + (n+1)2^n}$$

2 Second Example

Question: Given the equation $a_n = 6a_{n-1} - 9a_{n-2}$, the base cases $a_0 = 1$ and $a_1 = 9$, find a_n.

We can use the same steps from above to find a solution for this linear, constant coefficient, homogeneous difference equation:

$a_n = 6a_{n-1} - 9a_{n-2}$	Initial Function
$g(x) = \sum_{n=0}^{\infty} a_n x^n$	Initial Generating Function
$g(x) = 1 + 9x + \sum_{n=2}^{\infty}(6a_{n-1} - 9a_{n-2})x^n$	Substituting in base cases and initial a_n function
$g(x) = 1 + 9x + \sum_{n=2}^{\infty} 6a_{n-1}x^n - \sum_{n=2}^{\infty} 9a_{n-2}x^n$	Distributing the x^n
$g(x) = 1+9x+6x\sum_{n=2}^{\infty} 6a_{n-1}x^{n-1} - 9x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}$	Factoring out constants and xs
$g(x) = 1 + 9x + 6x\sum_{n=1}^{\infty} 6a_n x^n - 9x^2\sum_{n=0}^{\infty} a_n x^n$	Normalizing indices
$g(x) = 1+9x+6x(g(x)-a_0)-9x^2 g(x)$	Substituting initial generating function and base cases
$g(x)(1-6x-9x^2) = 1+9x-6x$	Simplifying and isolating $g(x)$

The final $g(x)$ then becomes:

$$\boxed{g(x) = \frac{1+3x}{1-6x-9x^2}}$$

Now, for a_n:

$g(x) = \frac{1+3x}{1-6x-9x^2}$	Final generating function
$1 - 6x - 9x^2 = (1 - 3x)^2$	Factorization of the denominator
$\frac{1+3x}{1-6x-9x^2} = \frac{A}{1-3x} + \frac{B}{(1-3x)^2}$	Partial Fraction Decomposition
$1 + 3x = A(1 - 3x) + B$	
If $x = 1/3 ... 2 = B$ If $x = 0 ... 1 = A + 2$, $A = -1$	
$g(x) = -\frac{1}{1-3x} + \frac{2}{(1-3x)^2}$	
$g(x) = -1 \sum_{n=0}^{\infty} (3x)^n + 2 \sum_{n=0}^{\infty} (n + 1)(3x)^n$	Substituting summation formulas
$a_n = (-1)3^n + 2(n + 1)3^n$	Finding a_n
$a_n = 3^n(-1 + 2n + 2)$	Simplifying

The final a_n then becomes:

$$\boxed{a_n = (2n + 1)3^n}$$