The Lecture Title

Scribe: Your Name

Date: Day, Mon, Date Year

1 Generating Function for Catalan Numbers

Recall the recursive definition of the Catalan numbers with base cases $C_0 = C_1 = 1$, and recursion

$$C_n = \sum_{k=0}^{n-1} C_i C_{n-1-i}$$

Let's find an explicit form for C_n by finding the generating function. Let $f(x) = \sum_{n=0}^{\infty} C_n$ - or in expanded form,

$$f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

Let's go ahead and try to square this, collecting the terms of the same degree by looking at the Cauchy products of Catalan numbers:

$$f(x)^2 = C_0^2 + (C_0 C_1 + C_1 C_2)x + (C_0 C_2 + C_1 C_1 + C_2 C_0)x^2 + \dots$$

Applying the Catalan recursion, we have:

$$f(x)^2 = C_1 + C_2 x + C_3 x^2 + \dots \implies x f(x)^2 = C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

This right hand side is almost $f(x)$! We now have the functional equation

$$x f(x)^2 = f(x) - 1$$

By the quadratic formula, we can find that

$$f(x) = \frac{1 \pm \sqrt{1 - 4x}}{2x}$$

We will eventually pick the form that satisfies these base cases above. For now, let us expand this binomially to see which case we would like. Notice first we have to expand $(1 - 4x)^{\frac{1}{2}}$, which we have partially done before:

$$\begin{aligned}
(1 - 4x)^{\frac{1}{2}} &= \sum_{n=0}^{\infty} \frac{\binom{\frac{1}{2}}{k}}{k!} x^k \\
&= 1 - \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} \frac{1}{4^n} (4x)^{n+1} \\
&= 1 - 2 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^{n+1}
\end{aligned}$$

Notice that in order to get the 1 cancelling out in the numerator, we have to pick the negative solution to avoid a divergence. Thus, we have

$$\begin{aligned} f(x) &= \frac{1}{2x}\left(1-\left(1-2\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{n+1}x^{n+1}\right)\right) \\ &= \frac{1}{2x}\left(\right) \end{aligned}$$