The Lecture Title

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Date: Day, Mon, Date Year

1 Polya's Enumeration Theorem

Polya's Enumeration Theorem is a generalization of Burnside's Lemma.

We will first prove Burnside's Lemma. Recall the statement of Burnside's Lemma: for a set S and a permutation group G, then the number of equivalence classes of S under G is given by

$$\frac{1}{|G|}\sum_{\pi \in G}|fix(\pi)|$$

where $fix(\pi)$ is the set of elements that are fixed by π.

Recall that we proved last time the Orbit Stabilizer Theorem: For any element $s \in S$ and the permutation group G, we have that

$$|orbit(s)||stabilizer(s)| = |G|.$$

Recall also that $orbit(s)$ is the set of all other elements in the set S that permutations on s can reach, and the stabilizer group $stabilizer(s)$ is the group of permutations acting on s that leave it fixed.

First Proof of Burnside's. We want to count equivalence classes there are, so essentially we are looking at *how many* orbits there are.

Let's do this by considering the fixed points in the table $S \times G \to S$. We will do this in two ways. The first way is to simply do it naively, adding up the number of ways that each π in G fixes (down in columns). This gives the expression

$$\sum_{\pi \in G}|fix(\pi)|.$$

If we add across in rows, however, we similarly get the sum

$$\sum_{s \in S}|stab(s)|.$$

By the Orbit-Stabilizer Theorem, this is

$$\sum_{s \in S}\frac{|G|}{|orbit(s)|}.$$

We can now break up S into distinct orbits and sum over the orbits $O_1, O_2, \ldots O_k$. We can rewrite this as a double sum:

$$\sum_{O_i \text{ orbits}} \sum_{s \in O_i} \frac{|G|}{|orbit(s)|}.$$

What is the size of $orbit(s)$ if $s \in O_i$? Well, it has to be $|O_i|$:

$$|G| \sum_{O_i \text{ orbits}} \sum_{s \in O_i} \frac{1}{|O_i|}.$$

The inner sum can be evaluated by noting that if we're adding up $\frac{1}{|O_i|}$ for $|O_i|$ elements, this has to be 1:

$$|G| \sum_{O_i \text{ orbits}} 1$$

This is therefore $|G|$(the # of orbits), so the number of orbits or the number of equivalence classes is given by Burnside's statement, as desired.

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Let's build up to Polya's Enumeration Theorem through a series of definitions.

Let X be a set (of vertices) and S is a set of functions on X, $X \to C$, where C is another set (of colors). Let G be a group operating on X. We will say that if $f_i, f_j \in S$ are *equivalent*, ie. $(f_i \cong f_2)$ if and only if $f_i(x) = f_j(\pi(x))$ for all $x \in X$ for some $\pi \in G$.

This is fairly abstract, but our functions f_i are essentially colorings of objects, and all we're saying that two colorings are the same if they transform to the same thing under an element in the group.

Example: c_{12} and c_{15} in the table, and test it for all elements under the permutation H.

We define the *weight* of an element $c \in C$ as $w(c)$, and the *weight* of a function $f : X \to C$, $W(f)$, is the product of the weights in its range, namely

$$\prod_{x \in X} w(f(x))$$

where the weight of both functions and elements map to an abelian group and a field.

The *inventory* of a set of functions is

$$\sum_{f_i} W(f_i)$$

To be concrete, suppose if we consider the two coloring of a square with red and green, and let $w(R) = r$ and $w(G) = g$, then from the table, we can see that coloring c_1 has weight r^4 and coloring c_2 has weight rg^3, etc.

For example, what is the inventory of all the two-colorings of a square, $c_1, c_2, \ldots c_{16}$? By enumeration, this is $r^4+4r^3g+6r^2g^2+4rg^3+g^4 = (r+g)^4$. This makes sense from a generating functions perspective, as each of the four vertices can be assigned either the weight r or g.

Under the action of a group, **Polya's Enumeration Theorem** will tell us the correct coefficients of this generating function, ie. the correct number colorings that are independent of each other under a group.

We have to introduce one last definition - let the *cycle index (polynomial)* of a permutation group G be

$$P_G(x_1, x_2, \ldots, x_k, \ldots) = \frac{1}{|G|} \sum_{\pi \in G} x_1^{b_1} x_2^{b_2} \ldots x_k^{b_k} \ldots$$

where b_k is the number of cycles of length k in π and the subscript of x_i is the size of the cycle.

For example, in the dihedral group D_4, ie. rotations and reflections of a square, we have the following correspondences:

$$e \to x_1^4 \quad r_{90} \to x_4^1 \quad r_{180} \to x_2^2 \quad r_{270} \to x_4^1$$

$$H \to x_2^2 \quad V \to x_2^2 \quad L \to x_1^2 x_2 \quad R \to x_1^2 x_2$$

Thus, the cycle index is

$$\frac{1}{8}(x_1^4 + 3x_2^2 + 2x_1^2 x_2 + 2x_4)$$

Finally, we have Polya's Enumeration Theorem, wrapped in these definitions:

Theorem 1.1. *The inventory of the equivalence classes of functions $f : X \to C$ under a permutation group G is given by*

$$P_G\left(\sum w(c), \sum w^2(c), \sum w^3(c), \ldots, \sum w^k(c), \ldots\right)$$

where the sums are over C.

If we apply this to the two-colorings, we get the desired

$$r^4 + r^3 g + 2r^2 g^2 + rg^3 + g^4.$$

And a simple corollary: The number of equivalence classes is just given by letting each of the weights be 1, or letting each of the inputs in P_G be $|C|$.

As an example, let's consider the vertices in a cube...

The cycle index is

$$\overline{24}$$

This gives 10.

Polya was notorious for using this in chemistry to count the number of molecules with differing functional groups that were different but rotationally, the same. One example - consider a methane center and the possibilities of a methyl group, ethyl group, a hydrogen, and a chlorine group. These give 4 "colors" for the 4 vertices of this tetrahedron, so the number of distinct molecules is the number of distinct 4-colorings of the vertices of a tetrahedron.

To do this, we find the cycle index for a tetrahedron:

This gives 36.

What if we mandate that the molecule has to have one molecule:

This gives 21.