The Lecture Title

Scribe: Your Name

Date: Day, Mon, Date Year

1 Burnside's Lemma

Given a set S and a permutation group G acting on S, the number of equivalence classes of S is given by

$$\frac{1}{|G|}\sum_{\pi\in G}|fix(\pi)|$$

where $fix(i)$ is the set of elements fixed by π.

For example, let's look at the square symmetry group D_4, and S be the 2-colors of the vertices of a square. S thus consists of elements that look like

Recall that G is the group $\{e, r_1, r_2, r_3, v, h, l, r\}$, where the index on r represents the number of 90-degree clockwise rotations. For every permutation in G, we construct a table that counts the number of elements in S that are left unchanged by each permutation. We start filling in as follows:

$\pi \in G$	Elements fixed
e	16
r_1	2
r_2	4
r_3	2
v	4
h	4
l	8
r	8
Total	48

By Burnside's, then, we have $\frac{1}{8}\cdot 48 = 6$. This is fairly easy to compute explicitly, but we can obviously also apply this to many more complicated problems.

Note that we may also interpret Burnside's as saying that the average size of $fix(\pi)$ is the number of equivalence classes, which somewhat leads to a proof of this (coming soon...)

Let's look at another example:

Problem C. onsider a bracelet with 5 beads, with each bead possibly being one of four colors (red, yellow, blue, white). Rotations of the bracelet are equivalent, but flipping the bracelet over gives a new coloring. How many colorings are there now?

Solution We construct a similar table:

$\pi \in G$	Elements fixed
e	4^5
r	4
r^2	4
r^3	4
r^4	4
Total	1040

By Burnside's, we get 208 distinct colorings. ∎

Problem S. ame problem as above, but now reflections are now the same.

Solution Add on this table:

$\pi \in G$	Elements fixed
f_1	64
f_2	64
f_3	64
f_4	64
f_5	64
Total	1360

By Burnside's, we get 136 distinct colorings. ∎

Problem H. ow many elements are in the group of rotations of a cube? How many 2-colorings are there of the faces? Of the vertices?

Solution Notice that we have 24 total rotations - rotations can put one of the six faces on top, and choosing one of the four adjacent faces to be the "front" face gives all possible rotations.

To break them down, we can look at what their axes are.

- There is one identity rotation. The cycle structure looks like $(U)(D)(F)(B)(L)(R)$.
- An axis through the face of a cube gives six different rotations by 90 degrees (cycle structure $(U)(D)(FLBR)$), and three different rotations by 180 degrees (cycle structure $(U)(D)(FB)(LR)$).
- An axis through a pair of opposite edges gives six different rotations (cycle structure $(UB)(DF)(LR)$).
- An axis through a pair of opposite vertices gives 8 different rotations, 2 for each space diagonal (cycle structure $(ULF)(BRD)$).

We now analyze how many colorings are fixed by these different types of rotations for each type. We notice that we can essentially assign every cycle to one color - so this gives 64 for the identity, 8 for 90-degree face rotations, 16 for 180-degree face rotations, 8 for edge rotations, 4 for vertex rotations, which gives a total of $64 \times 1 + 8 \times 6 + 16 \times 3 + 8 \times 6 + 8 \times 4 = 240$. By Burnside's, we have 10 possible colorings. ■