Polya's Enumeration Theorem

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1 Burnside's & Orbit Stabilizer Theorem

Theorem 1 $|orb(s)| \times |stabilizer(s)| = |G|$

Proof Count fix points in the table $S \times G$ and $\pi(S) = s$ in two ways.

(i) $\sum_{\pi \in G} |fix(\pi)| = \sum_{s \in S} |stab(s)|$

(ii) So, $\frac{1}{|G|} \times \sum_{\pi \in G} |fix(G)| =$ Number of orbits

2 Polya's Enumeration Theorem

Definition Let X be a set of ("vertices") and S be a set of functions on X from X to C ("colors"). Let G be a group operating on X. We will say two elements of S are equivalent iff

(i) one element of S acting on x equals another acting on $\pi(x)$

Weight and Inventory

(i) The weight of an element x is $w(x)$

Ex: $w(R) = r, w(G) = g$

(ii) The weight of a function $W(f)$ for f changes x to C equals to $\Pi_{x \in X} w(f(x))$

(iii) The inventory of a set of functions is $\sum_{f \in S} W(f)$

Example: 2 colors in a square

(i) $w(R) = r, w(G) = g$

$W(c1) = r^4$, $W(c2) = r \times g^3$, etc.

(ii) Inventory of all $c16$ is $r^4 + 4r \times g^3 + 6 \times r^2 \times g^2 + 4g \times r^3 + g^4 = (r + g)^4$

Definition The cycle index polynomial of a permutation group $P_G(x_1, x_2, ..., x_k, ...) = \frac{1}{|G|} \times \sum_{\pi \in G} {x_1}^{b_1} * {x_2}^{b_2} * ... * {x_k}^{b_k} * ...$ where b_k is the number of cycles of length k in π.

Example: Cycle Index Polynomial for $D_4 = \frac{1}{8} \times ({x_1}^4 + 3{x_2}^2 + 2{x_1}^2 \times x_2 + 2x_4)$

3 Theorem(Polya)

The inventory of the equivalence classes of functions $f : x \to C$ under group G is given by

$$P_G(\sum w(x), \sum w^2(x), \sum w^3(x), \sum w^4(x), ...)$$

Examples

(i) The square/ D_4

Inventory= $P_G(r + g, r^2 + g^2, r^3 + g^3, r^4 + g^4)$

Since the cycle index poly for $D_4 = \frac{1}{8} \times ({x_1}^4 + 3{x_2}^2 + 2{x_1}^2 \times x_2 + 2x_4)$, plug in corresponding elements in the inventory to get

$g^4 + g^3r + 2g^2r^2 + gr^3 + r^4$

Corollary: The number of equivalence classes is $P_G(|C|, |C||C|, |C|, ...)$

(ii) Chemistry/ Tetrahedron

Consider a tetrahedron where each of the vertices can be any of 4 groups: methyl, ethyl, hydrogen, chlorine. This is basically coloring the vertices of a tetrahedron into 4 colors.

(i) Assume that tetrahedron is oriented so that one vertex points up and one points towards you. A is the leftmost, B is the closest, C is the rightmost, and D is the top.

e=(A)(B)(C)(D) : x_1^4

vert= (D)(ABC) : $4x_1x_3$

$vert^2$= (D)(ACB) : $4x_1x_3$

edge= (AB)(CD) : $3x_2^2$

(ii) So the polynomial is $\frac{x_1^4 + 8x_1x_3 + 3x_2^2}{12}$

(iii) Finally, plug in n for all x:

C(n)= $\frac{n^4 + 11n^2}{12}$