Indeterminate Coefficients + Inhomogeneity

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1 Recursive Problems

Problem 1. We draw n lines in the plane. How many regions can we create?

Solution We can solve this with recursion. Obviously, $a_0 = 1$, but consider the a_{n-1} case. If we add another line, consider every place it can intersect. There are $n-1$ lines to intersect with, and if we have $n-1$ arbitrary lines, putting a line through them adds n regions to the total (imagine that the lines are all parallel since their intersections have nothing to do with the new line). Thus, we write $a_n = a_{n-1} + n$ with $a_0 = 1$, giving us

$$a_n = a_0 + \sum_{k=1}^{\infty} k = 1 + \frac{k}{k+1}2 = \boxed{\left(\!\!\binom{3}{n-1}\!\!\right) + 1}.$$

Problem 2. How many sequences of length n using $0-3$ contain an even number of zeroes?

Solution Since we're using sequences, we need to consider even and odd zeros. Construct two sequences O_n and E_n where $O_0 = 0$ and $E_0 = 1$. Then, we have $E_{n+1} = 3E_n + O_n$. We also have $O_n + E_n = 4^n$ since every sequence consists of an odd or even number of zeroes and there are four choices per digit in the sequence.

Thus, we write $E_{n+1} = 3E_n + (4^n - E_n) = 2E_n + 4^n$. If we instead write this as $E_n = 2E_{n-1} + 4^{n-1}$, we can solve for our generating function as follows:

$$(1-2x)g(x) = 1 + \frac{x}{1-4x} = \frac{1-3x}{1-4x}$$

$$\to g(x) = \frac{1-3x}{(1-2x)(1-4x)} = \frac{1/2}{1-2x} + \frac{1/2}{1-4x}$$

so we have $E_n = \boxed{\frac{2^n + 4^n}{2}} = 2^{2n-1} + 2^{n-1}$.

2 Extra Terms

Definition. *Say we have* $\sum_{i=0}^{n} c_i a_i = f$ *for some function* f*. Then* f *is the* ***forcing function*** *or* ***inhomogeneity*** *of the recursion.*

Definition. *If we have* $\sum_{i=0}^{n} c_i a_i = f$ *as before, then the solution to the equation* $\sum_{i=0}^{n} c_i a_i = 0$ *is called the* ***homogeneous*** *solution to the equation while the solution to the original equation is called the* ***particular*** *solution. Note that neither uses the initial conditions.*

We'll need both solutions to solve recurrences in general since the real solution will be a linear combination of what we have because initial conditions are annoying. Specifically, our particular solution cannot change, but we can always add a multiple of the homogeneous solution to get our final answer based on the initial equations. If $p(n)$ is our particular solution and $h(n)$ is our homogeneous, the solution will always be of the form $p(n) + c \cdot h(n)$ for some constant c.

Problem 3. Given $a_n + 2a_{n-1} = n + 3, a_0 = 3$, solve for a_n.

Solution The homogeneous solution is $(-2)^n$ (or a nonzero constant multiple of that) because our characteristic equation is $x^2 + 2x = 0$ and 0 clearly doesn't work.

Let's look for the particular solution now. Consider that the inhomogeneity is linear. We agree that $a_n = Bn + D$ by the **Method of Undetermined Coefficients,** which is essentially a way of "guessing" what the solution will look like. We have

$$a_n + 2a_{n-1} = Bn + D + 2(B(n-1) + D) = 3nB + 3D - 2B = n + 3,$$

so we have $B = \frac{1}{3}$ and $3D - 2B = 3D - \frac{2}{3} = 3 \to D = \frac{3+2/3}{3} = \frac{11}{9}$. Thus, our particular solution is $a_n = \frac{1}{3}n + \frac{11}{9}$.

Finally, let's combine the two. At $n = 0$, our particular says $\frac{11}{9}$, but our homogeneous says 1. Thus, we have $\frac{11}{9} + c \cdot 1 = 3$ since the solution is of the form $p(n) + c \cdot h(n)$. Thus, $c = \frac{16}{9}$. This gives us the solution

$$a_n = \boxed{\frac{1}{3}n + \frac{11}{9} + \frac{16}{9}(-2)^n}.$$

Let's now look at the original two problems again.

Problem 2 (revisit). Let $a_n = 4^{n-1} + 2a_{n-1}$. Solve for a_n.

Solution The homogeneous is clearly 2^n, but the particular is less clear-cut. Writing $a_n = c4^n$, we have $c(4^n - 2\cdot 4^{n-1}) = 4^{n-1}$ which gives $c = \frac{1}{2}$. If we then write $a_n = \frac{1}{2}4^n + d\cdot 2^n$ and use $a_1 = 3$, we have $a_1 = 2 + d\cdot 2 = 3 \to d = \frac{1}{2}$, so we get $a_n = \frac{4^n+2^n}{2}$ as before.

Problem 1 (revisit). Let $a_n = a_{n-1} + n$. Solve for a_n.

Solution The homogeneous for $a_n = a_{n-1} + n$ is literally 1. However, to find the particular here, the Method of Undetermined Coefficients says that we must use a quadratic since a linear term will cancel if we set $a_n - a_{n-1} = n$. (In more specific terms, if we write $a_n = Bn + D$, we find that $a_n - a_{n-1} = Bn + D - B(n-1) + D = B$, leaving us with no information.)

Thus, write $a_n = An^2 + Bn + C$. Using $a_0 = 1, a_1 = 2, a_2 = 4$, we have $C = 1, A+B+C = 2$, and $4A+2B+C = 4$. This gives $A = \frac{1}{2}, B = \frac{1}{2}, C = 1$. Thus, our final solution is $a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1 + d$, where d is the coefficient on the homogeneous solution. Taking $a_0 = 1$ again, we get $d = 0$, so the answer is just the particular: $a_n = \boxed{\dfrac{1}{2}n^2 + \dfrac{1}{2}n + 1}$ which equals our original answer.