of the set $A = \{1, 2, ..., r\}$. Thus, by definition, $S(r, n)$ counts the number of n-partitions of A, and therefore

$$\begin{aligned}\sum_{n=1}^{r} S(r,n) &= \text{ the number of partitions of } \{1,2,...,r\} \\ &= \text{ the number of equivalence relations on } \{1,2,...,r\}.\end{aligned}$$

The sum $\sum_{n=1}^{r} S(r,n)$, usually denoted by B_r, is called a *Bell number* after E.T. Bell (1883 – 1960). The first few Bell numbers are:

$$B_1 = 1,\ B_2 = 2,\ B_3 = 5,\ B_4 = 15,\ B_5 = 52,\ B_6 = 203, \ldots .$$

Exercise 1

1. Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, ..., 50\}$ such that
   (i) $|a - b| = 5$; (ii) $|a - b| \leq 5$.

2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if
   (i) there are no restrictions?
   (ii) the 5 girls must be together (forming a block)?
   (iii) no 2 girls are adjacent?
   (iv) between two particular boys A and B, there are no boys but exactly 3 girls?

3. m boys and n girls are to be arranged in a row, where $m, n \in \mathbf{N}$. Find the number of ways this can be done in each of the following cases:
   (i) There are no restrictions;
   (ii) No boys are adjacent ($m \leq n + 1$);
   (iii) The n girls form a single block;
   (iv) A particular boy and a particular girl must be adjacent.

4. How many 5-letter words can be formed using A, B, C, D, E, F, G, H, I, J,
   (i) if the letters in each word must be distinct?
   (ii) if, in addition, A, B, C, D, E, F can only occur as the first, third or fifth letters while the rest as the second or fourth letters?

5. Find the number of ways of arranging the 26 letters in the English alphabet in a row such that there are exactly 5 letters between x and y.

6. Find the number of *odd* integers between 3000 and 8000 in which no digit is repeated.

7. Evaluate
$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!,$$
where $n \in \mathbf{N}$.

8. Evaluate
$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \cdots + \frac{n}{(n+1)!},$$
where $n \in \mathbf{N}$.

9. Prove that for each $n \in \mathbf{N}$,
$$(n+1)(n+2)\cdots(2n)$$
is divisible by 2^n. (Spanish Olympiad, 1985)

10. Find the number of common positive divisors of 10^{40} and 20^{30}.

11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:

    (i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.

12. Show that for any $n \in \mathbf{N}$, the number of positive divisors of n^2 is always odd.

13. Show that the number of positive divisors of "$\underbrace{111\ldots1}_{1992}$" is even.

14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:

    (i) $P_r^n = nP_{r-1}^{n-1}$,

    (ii) $P_r^n = (n-r+1)P_{r-1}^n$,

    (iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,

    (iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,

    (v) $P_r^{n+1} = r! + r(P_{r-1}^n + P_{r-1}^{n-1} + \cdots + P_{r-1}^r)$.

15. In a group of 15 students, 5 of them are female. If exactly 3 female students are to be selected, in how many ways can 9 students be chosen from the group

    (i) to form a committee?

    (ii) to take up 9 different posts in a committee?