# The Lecture Title

Scribe: Your Name

**Date**: Day, Mon, Date Year

## 1 Undetermined Coefficients

Given $a_n + Aa_{n-1} + Ba_{n-2} = 0$, we can solve this homogeneous recurrence relation with a shorter method (with undetermined coefficients). We will find the solution to this recurrence using a

Notice that given this relation, we can notice that the following sum is still zero:

$$\sum_{n=0}^{\infty}(a_n + Aa_{n-1} + Ba_{n-2})x^n = 0$$

We can now simplify this in terms of the generating function $g(x)$ for $a_n$:

$$\implies (g(x) - a_0 - a_1x) + Ax(g(x) - a_0) + Bx^2g(x) = 0$$

We collect like terms:

$$g(x)(1 + Ax + Bx^2) = (a_1 + Aa_0)x + a_0 = Cx + D$$

This implies that we can write this generating function as

$$g(x) = \frac{Cx + D}{1 + Ax + Bx^2}$$

We can now use a partial fraction decomposition to decompose $1+Ax+Bx^2$ into a sum of reciprocals of (distinct) linear terms $1 - r_1x$, $1 - r_2x$:

$$g(x) = \frac{Cx + D}{1 + Ax + Bx^2} = \frac{E}{1 - r_1x} + \frac{F}{1 - r_2x}$$

where $(1-r_1x)(1-r_2x) = 1+Ax+Bx^2$. This decomposition directly leads to the explicit form

$$a_n = Er_1^n + Fr_2^n$$

Notice, however, that $(x - r_1)(x - r_2) = x^2 + Ax + B$, so instead we can use the following method to determine these exponents directly:

**Given $a_n + Aa_{n-1} + Ba_{n-2} = 0$:**

1. Solve $x^2 + Ax + B = 0 = (x - r_1)(x - r_2)$.

2. Write $a_n = C_1r_1^n + C_2r_2^n$.

3. Use initial conditions to solve for $C_1$, $C_2$.

**Problem 1.** Let $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 0, a_1 = 7$. Find an explicit formula for $a_n$.

**Solution** We write this recurrence relation as

$$a_n - 7a_{n-1} + 10a_{n-2} = 0$$

We can solve the quadratic $x^2 - 7x + 10 = 0$, which gives the roots 5 and 2. Therefore,

$$a_n = C_1 2^n + C_2 5^n$$

Plugging in our initial conditions gives

$$0 = C_1 + C_2 \quad 7 = 2C_1 + 5C_2$$

giving $C_1 = -\frac{7}{3}$, $C_2 = \frac{7}{3}$. Therefore,

$$a_n = \frac{7}{3}(5^n - 2^n)$$

■

What if we have now that $r_1 = r_2$? Recall that we had the form for our generating function above, but when we do the partial fraction decomposition, we must now have terms of the form

$$g(x) = \frac{Cx + D}{1 + Ax + Bx^2} = \frac{Cx + D}{(1 - rx)^2} = \frac{E}{1 - rx} + \frac{F}{(1 - rx)^2}$$

This yields the explicit form

$$a_n = Er^n + F(n + 1)r^n = E'r^n + F'nr^n$$

for some $E'$, $F'$. We can now do the same process as above if the roots have multiplicity greater than 1. For example, if a root $r$ has multiplicity 3, we will have terms in $a_n$ that look like

$$a_n = Er^n + Fnr^n + Gn^2r^n.$$

This takes a similar form for roots of greater multiplicity.

**Problem 2.** Find an explicit form for $a_n$: $a_n + 6a_{n-1} + 9a_{n-2} = 0$, $a_1 = 1$, $a_2 = 2$.

**Solution**

$$a_n = -\frac{8}{9}(-3)^n + \frac{5}{9}n(-3)^n$$

■

We will revisit the recurrence relation $a_n = 2a_{n-1} + n - 1$, and look at doing this relatively judiciously. By using the generating function, we have

$$g(x) = 1 + 2xg(x) + \sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n$$

$$\begin{aligned}
\rightarrow g(x)(1-2x) &= 1 + \sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n \\
&= 1 + \frac{x}{(1-x)^2} - \left(\frac{1}{1-x} - 1\right) \\
&= 2 + \frac{x}{(1-x)^2} - \frac{1}{1-x} \\
&= 2 + \frac{2x-1}{(1-x)^2} \\
\implies g(x) &= \frac{2}{1-2x} - \frac{1}{(1-x)^2}
\end{aligned}$$

which gives us the desired result much more quickly.