Permutation Groups & Burnside's Lemma

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1 Permutation Groups

Let S be a set. Let G be a group of permutations, Π, acting on elements of S. Consider an element of

$$\Pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

where $\Pi_1(1) = 3$, $\Pi_1(4) = 5$, and so on. We can also write Π as a product of cycles, for example $\Pi_1 = (13)(2)(45)$. This is called the **cycle decomposition**.

Theorem 1.1. *Every permutation can be written as a product of cycles. This is unique up to rotations or the reordering of cycles.*

Example 1.1. $\Pi_1 = (2)(54)(13)$

Permutations can also be composed:

$$\Pi_1\Pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = e \rightarrow \therefore \Pi_1 = \Pi_1^{-1}$$

Thus, $G = \{e, \Pi_1\}$ is a permutation group since it satisfies the following 4 requirements: closure, identity existence, inverse existence, & associativity.

Example 1.2. Consider the permutation group Π_1 of square $ABCD$. The 4 permutations in the group of rotations $(closed)$:

e	$0°$	$ABCD$
Π_1	$90°$	$DABC$
Π_2	$180°$	$CDAB$
Π_3	$270°$	$BCDA$

This group is generated by the Π_1 or Π_3 turns $\rightarrow \Pi = \{\Pi_1, \Pi_2, \Pi_3, e\}$. It is also important to notice that S_4 is a **symmetric group** containing all 24 permutations of square $ABCD$.

Example 1.3. Consider all symmetries of $ABCD$: $\{e, r_{90}, r_{180}, r_{270}, V, H, L, R\}$, where V is a vertical reflection, H is a horizontal reflection, L is a left diagonal reflection, & R is a right diagonal reflection.

	e	r_{90}	r_{180}	r_{270}	V	H	L	R
e	e	r_{90}	r_{180}	r_{270}	V	H	L	R
r_{90}	r_{90}	r_{180}	r_{270}	e	R	L	V	H
r_{180}	r_{180}	r_{270}	e	r_{180}	H	V	R	L
r_{270}	r_{270}	e	r_{90}	r_{180}	L	R	H	V
V	V	L	H	R	e			
H	H	R	V	L		e		
L	L	H	R	V			e	
R	R	V	L	H				e

The leftmost column shows the first operation followed by the operation in the topmost row. This is known as the dihedral group on 4 elements $\to D_4$. Here $|D_4| = 8$, which generalizes to $|D_n| = 2n$. It is important to note that the dihedral group is a subgroup of the set of all possible permutations.

Example 1.4. Below is a table of decomposed operations for square $ABCD$, shown in terms of permutations:

e	e	
r_{90}	$(ADCB)$	
r_{180}	$(AC)\,(DB)$	r_{90}^2
r_{270}	$(ABCD)$	r_{90}^3
H	$(AD)\,(BC)$	
V	$(AB)\,(CD)$	
L	(BD)	
R	(AC)	