The Calkin-Wilf Tree

Scribe: Bryan Lu

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1 The Calkin-Wilf Tree

Problem 1. What is the 23rd rational positive number?

Solution We don't know - we need an ordering. ∎

We will introduce the **hyperbinary** numbers to do this. Numbers are written in the same way as traditional binary, but we allow 2s as digits.

Problem 2. How many ways are there to write n in hyperbinary, $b(n)$?

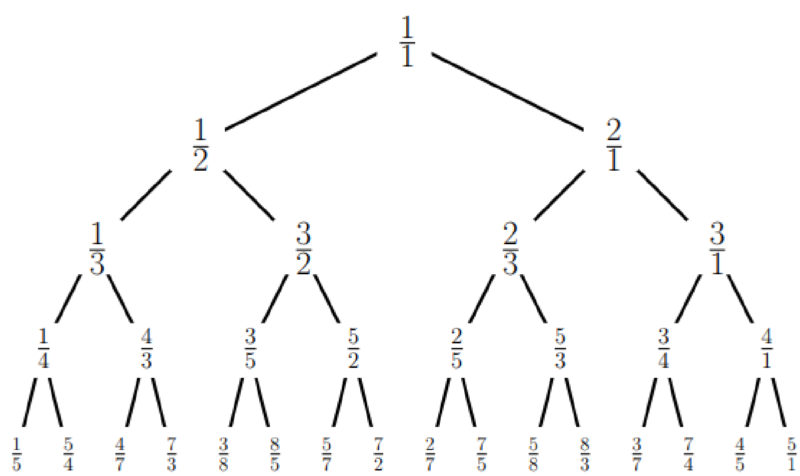
Solution We will construct a recurrence as follows. Consider if the number n is even or odd. If we have the odd integer $2n+1$, we must have 1 as its last digit. If we lop off the 1, the number of ways to write $2n+1$ in hyperbinary is clearly the number of ways to write n in hyperbinary. Thus, $b(2n+1) = b(n)$. This gives us the recurrence relations

$$b(2n+1) = b(n) \quad b(2n+2) = b(n+1) + b(n)$$

∎

Now, if we consider the sequence $\left\{\frac{b(n)}{b(n+1)}\right\}_{n=0}^{\infty}$, we claim that this sequence contains every rational in reduced form exactly once.

To show this, we consider constructing a complete binary tree such that if we run a breadth-first search on this tree, we would get this sequence in order. Here is a rendering of this tree for clarity:



Notice that by construction, the node with the number $\frac{b(n)}{b(n+1)}$ has the child $\frac{b(2n+1)}{b(2n+2)}$ to the left and the child $\frac{b(2n+2)}{b(2n+3)}$ to the right. From our recurrence relation, we have that these are equal to $\frac{b(n)}{b(n)+b(n+1)}$ and $\frac{b(n)+b(n+1)}{b(n+1)}$, respectively.

Therefore, for a node with the value $\frac{N}{D}$, it has the child $\frac{N}{N+D}$ to the left and the child $\frac{N+D}{D}$ to the right. We can show similarly that the parent of the node $\frac{x}{y}$ is $\frac{x}{y-x}$ if $y > x$ or $\frac{x-y}{y}$ if $x > y$.

With this structure in mind, we will prove the three following claims by contradiction.

Claim 1. $\gcd(x, y) = 1$ if $\frac{x}{y} \in T$, $\frac{x}{y} \neq 1$.

Proof. Assume by contradiction there exists some $\frac{x}{y}$ in the tree such that $\gcd(x, y) = k$ and without loss of generality $\frac{x}{y} = \frac{kx'}{ky'}$ is the node with the closest distance to the root. The parent of this node therefore either must be $\frac{kx'}{k(y'-x')}$ or $\frac{k(x'-y')}{ky'}$, but then the numerator and denominator have a common factor at least k, contradiction. Thus all $\frac{x}{y}$ in the tree are in lowest terms.

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Claim 2. If $\frac{x}{y} \in \mathbb{Z}$ and $\gcd(x, y) = 1$, $\frac{x}{y} \in T$.

Proof. Suppose by contradiction we can construct the set S of all rationals not in the tree. Let y be the minimum of the all the denominators in S, and x be the lowest numerator of all the numbers not in the tree that have the denominator y. Thus, $\frac{x}{y}$ can be said to be the "smallest" number not in the tree.

If $x > y$, then we can consider the number $\frac{x-y}{y}$. This can't be in the tree because its child $\frac{x}{y}$ is not in the tree. However, $x - y < x$, so $\frac{x}{y}$ is not the "smallest" member of the set S, contradiction.

Similarly, if $x < y$, we consider the number $\frac{x}{y-x}$, which can't be in the tree either, but $y - x < y$, contradiction.

Therefore, all rational numbers appear at least once in the tree.

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Claim 3. All rationals $\frac{x}{y}$, $x, y \in \mathbb{Z}$ appear exactly once $\frac{x}{y} \in T$.

Proof. Suppose by contradiction we can construct the set of S of all rationals that appear at least twice. By a similar method in the last proof, we take the "smallest" number in S. When considering the parent of this number in the tree, we note that this can either be $\frac{x}{y-x}$ or $\frac{x-y}{y}$, which must also appear at least twice. However, $x - y < x$ and $y - x < y$, so these parents are "smaller" than $\frac{x}{y}$, which is a contradiction.

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This gives us the following theorem now:

Theorem 1.1. *The sequence* $\left\{ \frac{b(n)}{b(n+1)} \right\}_{n=0}^{\infty}$ *includes every rational number exactly* *once in lowest terms.*

A cool way to find the nth number in this sequence - consider writing n in (regular) binary, running a run-length encoding of the binary string backwards (omitting the actual digits), and then constructing the continued fraction. This works very nicely :) - here is solution to our original problem by this ordering:

$$23 = 10111_2 \to \text{run-length encoding: } 311$$

$$\implies 3 + \frac{1}{1 + \frac{1}{1}} = 3 + \frac{1}{2} = \frac{7}{2}$$