# Applying Polya's Theorem: Counting Unlabled Graphs

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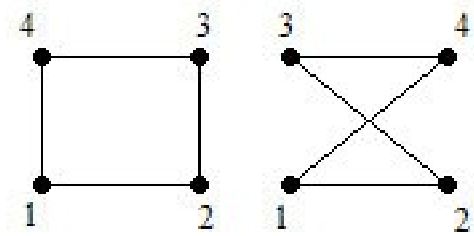
**Definition.** *A graph is a pair of sets (V,E) where $V$ is the set of vertices and $E \subseteq V \times V$*

**Definition.** *Two graphs are isomorphic if there is a map $\pi : (V, E) \to (V_2, E_2)$ such that $\pi(E) \equiv \{(\pi(v_1), \pi(v_2)) \forall (v_1, v_2) \in V\} = E_2$ and $\pi(V) = V_2$.*




**Example 0.1.** *These two graphs are isomorphic under the transformation $\pi = (1)(2)(34)$.*

**Exercise 1.** Apply Polya's theorem to count all graphs for n=3, 4

**Example 0.2.** *find the cycle index for $S_3$*

| $\pi$ | monomial | Action on edges |
|---|---|---|
| (1)(2)(3) | $x_1^3$ | $(\overline{12})(\overline{13})(\overline{23})$ |
| (1)(23) | $x_1 x_2$ | $(\overline{1213})(\overline{23})$ |
| (2)(13) | $x_1 x_2$ | $(\overline{2123})(\overline{13})$ |
| (3)(12) | $x_1 x_2$ | $(\overline{3132})(\overline{12})$ |
| (123) | $x_3$ | $(\overline{122331})$ |
| (132) | $x_3$ | $(\overline{133221})$ |

$$P_{S_3^{(2)}} = \frac{1}{6}(x_1^3 + 3x_1x_2 + 2x_3)$$

To enumerate, let the weight of an edge be $r$, and the weight of a lack of edge be $b$, Thus, plugging in in accordance with the enumeration theorem,

$$P_{S_3} = \frac{1}{6}((r+b)^3 + 3(r+b)(r^2+b^2) + 2(r^3+b^3)).$$

Because we don't care about lack of edges, let $b = 1$. Then this simplifies to

$$r^3 + r^2 + r + 1.$$

Next, let $r = 1$ so $x_i = 2$.

$$P_3 = \frac{1}{6}(2^3 + 3 \cdot 4 + 2 \cdot 2) = 4.$$

Now, let's do this for $S_4$ :

| Number | form of $\pi$ | $S_4^{(2)}$ monomial | Action on edges |
|---|---|---|---|
| $\binom{4}{0}(1-1)!$ | (1)(2)(3)(4) | $x_1^6$ | $(\overline{12})(\overline{13})(\overline{14})(\overline{23})(\overline{24})(\overline{34})$ |
| $\binom{4}{2}(2-1)!$ | (1)(2)(34) | $x_1^2x_2^2$ | $(\overline{12})(\overline{1314})(\overline{2324})(\overline{34})$ |
| $\binom{4}{3}(3-1)!$ | (1)(234) | $x_3^2$ | $(\overline{121314})(\overline{233442})$ |
| $\binom{4}{2}\cdot\frac{1}{2}$ | (12)(34) | $x_1^2x_2^2$ | $(\overline{12})(\overline{2314})(\overline{2413})(\overline{34})$ |
| $(4-1)!$ | (1234) | $x_2x_4$ | $(\overline{12233441})(\overline{1324})$ |

Thus,

$$P_{S_4^{(2)}} = \frac{1}{24}(x_1^6 + 9x_1^2 + 8x_3^2 + 6x_2x_4).$$

Then, to enumerate, we let the weight of no edge be $1$, and the weight of an edge be $x$, and get

$$x^6 + x^5 + 2x^4 + 3x^3 + 2x^2 + x + 1.$$

When we let $x = 1$, this simplifies to $11$.
Also covered in class was this same procedure for $n = 5$.