More Permutations and Combinations

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Date: Day, Mon, Date Year

Problem 1. Let S be the set of all integers composed of digits in 1, 3, 5, 7 at most once. Find a) $|S|$ and b) $\sum_{x \in S} x$.

Define a **partition** of a set S is subsets $S_1, \dots S_k$ such that a) the $union_i$ $S_i = S$ - the sets "cover" S, b) $Si \cap Sj = \emptyset$, "pairwise disjoint", and c) $S_i \neq \emptyset$.

We partition S into S_1 the set of 1-digit numbers in the set, S_2 the set of 2-digit numbers in the set, etc. Notice that $|S_1| = 4$, $|S_2| = 4 \cdot 3 = 12$, etc.

for the sum, let $\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4$, where α_1 is the sum of all units digits in all numbers in S, α_2 is the sum of all tens digits in all numbers in S, etc. We compute α_1 by looking at its contributions from each of the sets. α_1 from S_1: 16, as it's just each of the units digits added together.

We can do it much more easily by considering the average of each of the sets S_i. Notice that we can pair each element x in S_i with another element $\overline{x} \in S_i$ such that $x + \overline{x} = 88 \dots 8$, where this number has i 8s. Therefore, we can just find $\alpha = 8\frac{|S_1|}{2} + 88\frac{|S_2|}{2} + 888\frac{|S_3|}{2} + 8888\frac{|S_4|}{2} = 117,856$ as before.

1 Recursive Definition of Permutations

Although we can easily define $P(n,r)$ as $\frac{n!}{(n-r)!}$, we will try to create a recursive definition for $P(n,r)$ in terms of $P(a,b)$ such that $a \leq n$ and $b \leq r$.

Consider a set S of n elements $\{s_1, s_2, \dots s_n\}$ and let r be given and $0 \leq r \leq n$ and let T be the set of all r-permutations of S. For example, if $r = 3$, then $T - \{s_1s_2s_3, s_3s_2s_1, s_4s_2s_1, s_5s_6s_1, \dots\}$.
We wish to find $|T|$. We may partition T into two sets, T_1 and T_2, where $t \in T_1 \implies S_1 \notin T$, or in other words, the set of all permutations that do not contain s_1. Similarly, we define T_2 as the set of t such that $S_1 \in t$. Notice that $|T_1|$ is simply $P(n-1, r)$, as we have to construct a permutation of r elements from the $n-1$ remaining elements. Also, $|T_2| = rP(n-1, r-1)$, as we have to order $r-1$ elements from the $n-1$ elements aside from s_1, and then pick the position of s_1 within the ordering in r ways. Therefore, we have the identity $P(n,r) = P(n-1,r) + rP(n-1,r-1)$. ■

2 Cyclic Permutations

Consider the set T of 3-permutations of the set $S = \{1, 2, 3, 4\}$. Therefore, $T = \{123, 132, 234, 214, \dots\}$. Recall that from defining an equivalence rela-

tion that ... combinations. We define the equivalence relation $x \cong y$ if x and y are cyclically equivalent - for example, $123 = 231 = 312$.
With this equivalence relation, consider the equation $123 \cong x$ where $x \in T$ - this has 3 solutions. We see from this that any sequence of length n is equivalent to n sequences (including itself).
Since these equivalence groups are pairwise disjoint, we can count the number of cyclic permutations of length r from a set of n elements, Q_r^n. The reasoning above directly shows that $Q_r^n = P_r^n / r$. For example, there are $(n-1)!$ ways to seat n people around a round table.

Problem 2. There are 3 girls and 5 boys sitting around a table. Find the number of ways that they can sit around the table so that

1. there are no restrictions
2. B_1 and G_1 cannot sit next to each other
3. No girls are adjacent to other girls.

Solution

1. $7!$
2. We can seat the 5 boys in $(5-1)!$ ways. In three of the remaining spaces between the boys, we have to put a girl in three of those spots. $2 \cdot 6!$

■

3 Combinations

Define the combination $C(n,r)$ be the number of ways to choose an (unordered) set of r elements from a set of n. We can define this in such a way that

$$C(n,r) =$$

We can construct a recursive definition for $C(n,r)$ as well. Let T be the set of subsets of a set $S = stuff$ of size r. We can find $|T|$ by partitioning it into T_1 and T_2, using a similar definition from before - let T_1 be the set of subsets of S that do not contain an arbitrary element s_1, and T_2 be the set of subsets of S that do contain s_1. Notice that $|T_1| = C(n-1, r)$, as we have to pick r elements from the other $n-1$ elements. We also see that

Problem 3. Given $2n$ tennis players, find the number of ways to arrange n (unordered) games.

Solution 1 Index your players from 1 to $2n$. We can match P_1 with another player in $2n-1$ ways. The next ■
Solution 2 We can pick a whole bunch of pairs. We can pick the first pair in $\binom{2n}{2}$ ways, the second pair in $\binom{2n-2}{2}$ ways, etc. until we are left with the lsat player. However, we have to de-order the games, as ■
Solution 3 We can permute all $2n$ players in $(2n)!$ ways. However, we've overcounted just as in Solution 2, as we've implicitly ordered the games ■

Problem 4.