Partition Numbers

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Date: Monday, April 8, 2019

1 Partition Numbers

The last combinatorial numbers that we have yet to look at in our twelvefold table are the partition numbers. We define the **partition number** $p(n)$ as the number of ways to write n as a sum of positive integers. For example, $p(5) = 7$ by enumeration:

$$\begin{aligned} 5 &= 1+1+1+1+1 = 1+1+1+2 = 1+2+2 = 1+1+3 \\ &= 1+4 = 2+3 \end{aligned}$$

Note the distinction between partitions and compositions of numbers - when doing compositions, we are counting the partitions $3, 1, 1$, $1, 3, 1$, and $1, 1, 3$ as independent ways to make compositions of 5, while these are all the same partition of 5.

The explicit formula for $p(n)$ is given by

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{k} A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n - \frac{1}{24}}} \sinh\left[\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right] \right)$$

where A_k is given by

$$A_k = \sum_{\gcd(m,k)=1, m<k} e^{\pi i (S(m,k) - 2nm/k}$$

derived by Ramanujan.

We can still state some results regarding partition numbers by comparing this to compositions. First, let $p_k(n)$ be the number of ways to partition n into k parts. Notice that the number of solutions to

$$x_1 + x_2 + \ldots x_k = n$$

is given by $\left(\binom{k}{n-k}\right)$, using the multiset notation, or also $\binom{n-1}{k-1}$. Each composition contributes at most $k!$ compositions, so the number of partitions on k elements, $p_k(n)$, multiplied by the number of ways to arrange k distinct elements is:

$$k! p_n(k) \geq \binom{n-1}{k-1}$$

so we have that we can bound $p_k(n)$ from below by

$$p_k(n) \geq \frac{1}{k!}\binom{n-1}{k-1}$$

2 Ferrers Graphs/Young Tableaux

We can write partitions of positive numbers in Ferrers graphs or Young tableaux - for example, here is a partition of 7 into 4, 2, and 1:

$$\begin{array}{llll} \circ & \circ & \circ & \circ \\ \circ & \circ & & \\ \circ & & & \end{array}$$

The transpose of each of these tableaux is still a partition, λ^T. Here is the transpose of the tableaux we saw above:

$$\begin{array}{lll} \circ & \circ & \circ \\ \circ & \circ & \\ \circ & & \\ \circ & & \end{array}$$

If a partition λ is *self-conjugate*, then $\lambda^T = \lambda$. Here's a theorem that we've kind of seen before:

Theorem 2.1. *The number of self-congruent partitions λ of n is the number of partitions of n into distinct odd-sized parts.*

Proof. Note that if we draw the Young tableaux for a self-conjugate partition, we can construct a partition of n into distinct odd numbers by taking the "outer" row/column (ie. the left most and the topmost row) and repeating on the remaining tableaux. If we repeat this process, this gives us a partition of n into odd numbers.

To go backwards, we can construct a self-conjugate partition by constructing the Young tableaux from the inside out, starting from the smallest odd number. ■

Another theorem of this nature:

Theorem 2.2. *The number of partitions of n into even parts is the same as the number of partitions of n into even multiplicity.*

Proof. Draw out a Young tableaux representing a partition of even multiplicity. Note that the columns are even-height, so the transpose of this partition will have even-length rows. This is easily reversible, so we have a bijection.

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Theorem 2.3.
$$p_4(n) = p_4(3n)$$

Proof. Consider a Young tableaux in a $4 \times n$ rectangle. The area taken up by the tableaux is n, and the area of the rectangle is $4n$, so the remaining area, when flipped upside down, is another Young tableaux for $3n$. This completes the proof.

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In general, for integers k, n:

$$p_k(n) = p_k((k-1)n)$$

Finally, a recursive formula:

$$p_k(n) = \sum_{i=1}^{k} p_i(n-k)$$

Proof. Consider chopping off the left-hand column of a Ferrers graph of n that has k rows, and then taking the transpose of the rows gives the recurrence for partitions of n over all possibilities.

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