The Lecture Title

Scribe: Your Name

Date: Day, Mon, Date Year

1 Catalan's Reprise

Recall we wanted to count the number of triangulations of a regular polygon with n sides - let T_n be the number of ways to triangulate a n-gon.

The recursive process is constructed by fixing an edge that must be in exactly one triangle, and then doing casework on which of the other $n-2$ vertices it must have. Suppose we label the vertices of the n-gon from 1 to n and have our fixed edge be the edge with vertices labeled 1 and n. If we pick vertex k to be the other vertex, we can triangulate the two other polygons, one in T_k and another in T_{n-k-1} ways. Therefore, we have the recurrence relation

$$T_n = \sum_{k=2}^{n-1} T_k T_{n-k+1}$$

Shifting the indices back by 2, it appears to be that C_n is the number of ways to triangulate an $n+2$-gon. To rigorously show this, we will have to manipulate the recurrence relation - and by doing this we see that

$$C_{n+1} = \sum_{k=1}^{n} C_k C_{n-k}$$

Fundamentally, we will see the Catalan numbers reappear when we can break up a problem into two smaller subproblems. This is not extremely obvious for the path problem, but we can consider the first point at which the path crosses the line $y = 1$ again?

2 Equivalence Relation Problems

An equivalence relation $\cong$ if it is symmetric, reflexive, and transitive. A general relation on a set A is a subset of the set $A \times A$. We can describe this relation as a table, called a **Cayley Table**. Here is a relation on 4 elements:

As another example, the "greater than" and "less than" relations are relations that

Problem H. ow many relations are there on a set of n elements? How many relations are reflexive? How many are reflexive and symmetric? How many are reflexive, symmetric, and transitive?

Solution There are clearly 2^{n^2} possible relations.

For a reflexive relation, we just fix the main diagonal. This gives 2^{n^2-n}.

For reflexive and symmetric relations, we not only must fix the main diagonal, but then out of any of the $\binom{n}{2}$

For reflexive, symmetric, and transitive relations, we must think of this differently.

The answer is $\sum_{k=1}^{n} \left\{ {n \atop k} \right\} = B_n$, the nth **Bell number**. ■