Method of Undetermined Coefficients

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1 General Setup

Given a recursive function defined as $a_n + Aa_{n-1} + Ba_{n-2} = 0$, we can see that $\sum_{n=2}^{\infty}(a_n + Aa_{n-1} + Ba_{n-2})x^n = 0$.

If we decompose this summation, we notice that, if we define $g(x) = \sum_{n=0}^{\infty} a_n x^n$ to be the OGF of the recurrence relationship:

$$\sum_{n=2}^{\infty} a_n x^n = g(x) - a_0 - a_1 x$$
$$\sum_{n=2}^{\infty} Aa_{n-1} x^n = Ax(g(x) - a_0)$$
$$\sum_{n=2}^{\infty} Ba_{n-2} x^n = Bx^2 g(x)$$

Therefore, $(g(x) - a_0 - a_1x) + Ax(g(x) - a_0) + Bx^2 g(x) = 0$. Rearranging and factoring $g(x)$ out of the expression, we get $(1 + Ax + Bx^2)g(x) = (a_1 + Aa_0)x + a_0 = Cx + D$ for appropriately-defined constants C and D.

Then, if we define r_1 and r_2 such that $(1 - r_1x)(1 - r_2x) = 1 + Ax + Bx^2$:

$$g(x) = \frac{Cx + D}{1 + Ax + Bx^2} = \frac{E}{1 - r_1x} + \frac{F}{1 - r_2x}$$

2 Case One: $r_1 \neq r_2$

If we rearrange the equation $(1 - r_1x)(1 - r_2x) = 1 + Ax + Bx^2$ (divide both sides by x^2 and replace $1/x$ with x), we obtain $(x - r_1)(x - r_2) = x^2 + Ax + Bx$. Then, using this result, we can rewrite $g(x) = E\sum_{n=0}^{\infty} {r_1}^n x^n + F\sum_{n=0}^{\infty} {r_2}^n x^n$. Thus, because we defined $g(x)$ as the OGF of the recurrence relationship, the generalized $a_n = E{r_1}^n + F{r_2}^n$, where we can solve for the arbitrary E and F.

Remark. *As a general method, we can follow the below steps to determine a generalized closed form for a recursion given by $a_n + Aa_{n-1} + Ba_{n-2} = 0$:*

Step 1: *Solve $(x - r_1)(x - r_2) = x^2 + Ax + Bx$.*

Step 2: *Write $a_n = C_1{r_1}^n + C_2{r_2}^n$ (which correspond to E and F from before).*

Step 3: *Use initial conditions (most often given as a_0 and a_1) to solve C_1 and C_2.*

Example 2.1. *Determine a closed form for a_n given the recursion $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 0$, $a_1 = 7$.*

Solution:
Step 1: Moving everything to the LHS, we get $a_n - 7a_{n-1} + 10a_{n-2} = 0$. Then, $x^2 - 7x + 10 = (x-5)(x-2) = 0$. Therefore, $r_1 = 5$ and $r_2 = 2$

Step 2: We can write the general $a_n = C_1 5^n + C_2 2^n$

Step 3: We will now plug in the two base cases:
$a_0 = 0 = C_1 + C_2$
$a_1 = 7 = 5C_1 + 2C_2$

Step 4: Solving, we obtain $C_1 = \frac{7}{3}$ and $C_2 = -\frac{7}{3}$

Step 5: Putting this all together, we obtain $\boxed{a_n = \frac{7}{3}5^n - \frac{7}{3}2^n}$

3 Case Two: $r_1 = r_2 = r$

Taking the definition of r_1 and r_2 such that $(1-r_1x)(1-r_2x) = 1+Ax+Bx^2$:

$$g(x) = \frac{Cx+D}{1+Ax+Bx^2} = \frac{E}{1-r_1x} + \frac{F}{1-r_2x}$$

we replace r_1 and r_2 with r since they are all equal, resulting in:

$$\begin{aligned} g(x) &= \frac{Cx+D}{1+Ax+Bx^2} = \frac{Cx+D}{(1-rx)^2} = \frac{E}{1-rx} + \frac{F}{(1-rx)^2} \\ &= E\sum_{n=0}^{\infty} r^n x^n + F\sum_{n=0}^{\infty} \left(\!\!\binom{2}{n}\!\!\right) r^n x^n \end{aligned}$$

thus giving us:

$$\begin{aligned} a_n &= Er^n + F(n+1)r^n \\ &= \boxed{E'r^n + F'nr^n} \end{aligned}$$

which is the Standard Form for a quadratic with a repeated group.

Remark. *If r_i has multiplicity of 3 or more $\rightarrow a_n = Er^n + Fnr^n + Gn^2r^n + \ldots$*

Example 3.1. *Determine a closed form for a_n given the recursion $a_n = -6a_{n-1} - 9a_{n-2}$; $a_1 = 1$, $a_2 = 2$.*

Remark. *Initial conditions do not always have to be a_0 and a_1.*

Solution:
Step 1: Moving everything to the LHS, we get $a_n + 6a_{n-1} + 9a_{n-2} = 0$. Then, $x^2 + 6x + 9 = (x+3)(x+3) = 0$. Therefore $r_1 = r_2 = -3$

Step 2: We can write the general $a_n = C_1(-3)^n + C_2 n(-3)^n$

Step 3: We will now plug in the two base cases:
$a_1 = 1 = 3C_1 + 3C_2$
$a_2 = 2 = 9C_1 + 18C_2$

Step 4: Solving, we obtain $C_1 = \frac{-8}{9}$ and $C_2 = \frac{5}{9}$

Step 5: Putting this all together, we obtain $\boxed{a_n = \frac{-8}{9}(-3)^n + \frac{5}{9}n(-3)^n}$

4 Recurrence Problems Classwork

Problem 1: $a_n = 2a_{n-1} + n - 1$; $a_0 = 1$

Solution:

$$\begin{aligned}
g(x) &= 1 + 2xg(x) + \sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n \\
g(x)(1-2x) &= 1 + \sum_{n=1}^{\infty} nx^n - \sum_{n=1}^{\infty} x^n \\
&= 1 + \frac{x}{(1-x)^2} - (\frac{1}{1-x} - 1) \\
&= 2 + \frac{x}{(1-x)^2} - \frac{1}{(1-x)} \\
&= 2 + \frac{2x-1}{(1-x)^2} \\
g(x) &= \frac{2}{1-2x} - \frac{x}{(1-x)^2} \\
&\rightarrow \boxed{a_n = 2(2)^n - \left(\!\!\binom{2}{n}\!\!\right)}
\end{aligned}$$