# The Lecture Title

Scribe: Your Name

**Date**: Day, Mon, Date Year

## 1 Complex Roots and Bell Numbers

Consider writing $\frac{1}{1+x^2}$ as a series:

$$\frac{1}{1+x^2} = \sum(-x^2)^n = \sum(-1)^n x^{2n}$$

Notice that if $n$ is odd, the coefficient of $x^n$ is odd - otherwise, the coefficient is $(-1)^n$.

However, we could also decompose this into linear terms that are

Consider the recurrence relation $a_n = -a_{n-2}$, $a_0 = 0$, $a_1 = 1$. If we apply the technique we learned last class, we can consider the characteristic polynomial:

$$x^2 + 1 = 0 \implies x = \pm i$$

Therefore, $a_n = Ai^n + B(-i)^n$. Plugging in our initial conditions, we have

$$A + B = 0 \quad Ai - Bi = 1 \to A = -\frac{i}{2} \quad B = \frac{i}{2}$$

Altogether, this gives $a_n = -\frac{1}{2}i \cdot i^n + \frac{1}{2} \cdot i(-i)^n$. This is a really complex way to express a clearly periodic sequence of period 4 - why would we ever need to use complex numbers if we have functions (ie. $\sin$ and $\cos$) that can do this periodicity for us?

Let us backtrack and see if we can find a way to do this using trigonometric functions by applying DeMoivre's Theorem. Notice that the roots of the characteristic polynomial can be expressed as

$$r_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$r_2 = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

Notice that if we raise to the $n$th power, we have

$$a_n = Ar_1^n + Br_2^n = A\left(\cos\frac{\pi n}{2} + i\sin\frac{\pi n}{2}\right) + B\left(\cos\frac{\pi n}{2} - i\sin\frac{\pi n}{2}\right)$$

If we combine like terms, we have

$$a_n = (A+B)\cos\left(\frac{n\pi}{2}\right) + (A-B)\sin\left(\frac{n\pi}{2}\right)$$

Since all of the terms of the sequence are real, we must force $A$ and $B$ to have the same real part. In general, we can always just solve instead for the coefficients of $\sin$ and $\cos$ instead.

$$a_n = C\cos\left(\frac{n\pi}{2}\right) + D\sin\left(\frac{n\pi}{2}\right)$$

When we plug in:

$$a_0 = 0 = C\cos 0 \implies C = 0$$
$$a_1 = 1 = D\sin\frac{\pi}{2} \implies D = 1$$

which gives

$$a_n = \sin\frac{\pi n}{2}$$

We can use a similar method to find the roots of unity for $a_n = -a_{n-6}$.

Let's do this for $a_n = a_{n-1} - a_{n-2}, a_1 = 1, a_2 = 0$. Notice that the characteristic polynomial is

$$x^2 - x + 1 = 0 \implies x = \frac{1 \pm \sqrt{3}i}{2}$$

This is a 6th root of unity, so we can note that we can simply write this as a the linear combination:

$$a_n = A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}$$

We plug in our initial conditions:

$$a_1 = \frac{1}{2}A + \frac{3}{2}B = 1$$

$$a_2 = -\frac{1}{2}A + \frac{3}{2}B = 0$$

$$\implies A = 1, B = \frac{1}{\sqrt{3}}$$

This gives $a_n = \cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}}\sin\frac{n\pi}{3}$

## 2 Bell Numbers

Recall the **Bell number** $B_n$ is the number of non-empty partitions of a set of $n$ elements. By definition of the Stirling numbers, we have this is equal to

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

Recall also that $x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$. This is kind of close to the Bell numbers, but we can use this to get to an explicit formula for the Bell numbers.

Let $X$ be a random variable. Using linearity of expectation, we have that

$$E[X] = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} E[X^{\underline{k}}]$$

We are looking for a random variable $X$, then, such that $E[X^{\underline{k}}] = 1$ - this will give a computation for $B_n$.

It turns out that one exists! Let $X \sim \text{Poisson}(\lambda)$. The **Poisson distribution** is the limiting case of the binomial distribution. A diagram of the Poisson distribution is included below.

The probability distribution function for a Poissson distribution - if $X \sim \text{Poisson}(\lambda)$, then $P(x = k) = \frac{\lambda^k}{k!} e^{-\lambda}$. To prove this is valid, we have to show that:

$P(x = k) = \sum_{k=0}^{\infty} P(x = k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$

We now compute $E[X]$ for the Poisson distribution:

$$E[X] = \sum_{k=0}^{\infty} kP(x = k) = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda$$

We can compute the expected values we want using a moment-generating function, but we can write these out explicitly:

$$E[X^n] = \sum_{k=0}^{\infty} k^n P(x = k) = \sum_{k=0}^{\infty} k^n \frac{\lambda^k}{k!} e^{-\lambda}$$

To find $E[X^{\underline{k}}]$, we can instead look start with $E[t^X]$ and take successive derivatives (this is the **factorial generating function**):

$$E[t^X] = \sum_{k=0}^{\infty} t^k P(x = k) = \sum_{k=0}^{\infty} t^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(t\lambda)^k}{k!} = e^{-\lambda}(e^{t\lambda}) = e^{\lambda(t-1)}$$

Notice that if we take successive derivatives with respect to $t$, we have:

$$\frac{d^n}{dt^n} e^{\lambda(t-1)} = \lambda^n e^{\lambda(t-1)}$$

Differentiating the sum, we have that this is also

$$E[X^{\underline{n}} t^{X-n}] = e^{-\lambda} \sum_{k=0}^{\infty} \frac{k^{\underline{n}} t^{k-n} \lambda^k}{k!}$$

When plugging in $t = 1$ we obtain that

$$E[X^{\underline{n}}] = \lambda^n$$

If we let $X \sim \text{Poisson}(\lambda = 1)$, and thus $E[X^{\underline{n}}] = 1$, we can compute $B_n$:

$$E[X^n] = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!} = \sum_{k=0}^{n}\left\{ {n \atop k} \right\} = B_n$$

We can also construct a recursive relation for the Bell numbers using a combinatorial argument. Notice that the element $n+1$ of the set $\{1, 2, \ldots n, n+1\}$ can be put in a subset of any choice of $k$ other elements, and then partitioning the remaining $B_{n-k}$ elements. This gives the recursiion relation

$$B_{n+1} = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}$$