Day 4: More Counting

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1 Partitions

A *partition* of a set S is a subset of $S_1, ...S_k$ such that

(i) $\bigcup_{i=1}^{k} S_i$. The subsets 'cover' the set S

(ii) $S_i \cap S_j = \varnothing$. The subsets are pairwise disjoint.

(iii) $S_i \neq \varnothing$

Problem 1. Let S be the set of all integers composed of digits in $\{1, 3, 5, 7\}$ at most one.

(i) Find $|S|$

(ii) $\sum_{x \in S} x$

Solution

(i) Let $S = S_1 \cup S_2 \cup S_3 \cup S_4$ where S_1 is the number of one digit numbers, S_2 is the number of two-digit numbers, and so on.

$$\begin{aligned}
|S_1| &= {}^4P_1 = 4 \\
|S_2| &= {}^4P_2 = 12 \\
|S_3| &= {}^4P_3 = 24 \\
|S_4| &= {}^4P_4 = 24 \\
|S| = |S_1| + |S_2| + |S_3| + |S_4| &= \boxed{64}
\end{aligned}$$

(ii) Let $\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4$ where α_1 is the sum of all units digits of all numbers in S, α_2 is the sum of all the tens digits of all the numbers, and so on. We will find the value of α_1 using the following:

$$\begin{aligned}
&S_1 \to s_1 = (1+3+5+7) \\
&S_2 \to s_2 = (1+3+5+7) \times (3) \\
&S_3 \to s_3 = (1+3+5+7) \times (3 \times 2) \\
&S_4 \to s_4 = (1+3+5+7) \times (3 \times 2 \times 1) \\
\hline
&\alpha_1 = 16 \times (1+3+6+6) = 256
\end{aligned}$$

Note that α_2 is the sum of the same values, excluding s_1 as S_1 is the set of only one digit numbers. α_3 is the sum of the same values as α_2, excluding s_2 as S_2 is the set of only two digit numbers, and so on.

$$\alpha_2 = \alpha_1 - s_1 = 240$$
$$\alpha_3 = \alpha_2 - s_2 = 192$$
$$\alpha_4 = \alpha_3 - s_3 = 96$$

Thus, $\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4 = \boxed{117,856}$

An easier solution is the following:

$$1+3+5+7 = (1+7)+(3+5) = 8(\frac{4}{2}) = 16$$
$$13 + ... + 75 = (13+75) + ... + (35+53) = 88(\frac{12}{2}) = 528$$

etc

Since each $x \in S_i$ pairs with $\bar{x} \in S_i$ to sum to $88...8$. We find $\alpha = 8\frac{|S_1|}{2} + 88\frac{|S_2|}{2} + 888\frac{|S_3|}{2} + 8888\frac{|S_4|}{2} = \boxed{117,856}$

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2 Cyclic Permutation

Consider the set T of 3 permutations of (s_1, s_2, s_3, s_4) or $(1,2,3,4)$. We know that $T = \{123, 132, 234, 214, ...\}$ and $|T| = P(4,3) = 24$.

We define $x \cong y \iff x + y$ are cyclically equivalently

Problem 1. Given $123 \cong x$, how many solutions are there for $x \in T$?

Solution The x values are 123, 231, 312, so there are $\boxed{3}$ solutions. Thus, we see any sequence of length $n \cong n$ sequences ■

Theorem 2.1. *If $Q(n,r)$ is the number of cyclic permutations of length r from a set of n elements, $Q(n,r) = \frac{P(n,r)}{r}$.*

Theorem 2.2. *There are (n-1)! ways to seat n people around a round table.*

Proof. Each ordering $\cong n$ orderings. Thus, $\frac{n!}{n} = (n-1)!$

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Problem 2. There are 5 boys and 3 girls seated around a round table.

(i) There are no restrictions.

(ii) B_1 and G_1 are not adjacent

(iii) No girls are adjacent to other girls

Solution

(i) Using theorem 2.1, there are $\boxed{7!}$ ways.

(ii) We first place B_1 in any of the 7 seats and set B_1 as our reference point. There are then 5 places for G_1 to sit not adjacent to B_1 and 6! ways for the remaining 6 people to sit. The total number of ways if $\boxed{6! \cdot 5}$. We can also consider the number of ways for B_1 and G_1 to sit next to each other, which is $2 \cdot 6!$. Subtracting that from arranging without restrictions, the total number of ways is $\boxed{7! - 2 \cdot 6!}$

(iii) We first arrange all the 5 boys, which is $4!$ ways. There are 5 spaces between each boy, so we can choose 3 of the seats and then arrange the 3 girls, $\binom{5}{3} \cdot 3!$. The total number of ways is $\boxed{4! \cdot \binom{5}{3} \cdot 3!}$

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3 Recursion

Exercise 1. Find the recursive definition of $P(n, r)$.

Solution We know that the closed form of $P(n, r) = \frac{n!}{(n-r)!}$. Our goal is to define $P(n, r) = f(P(< n, < r))$.
Let $S = \{s_1, ..., s_n\}$, r be given $0 \leq r \leq n$, and T be the set of all r-permutations of S. We can partition T into $T = T_1 \cup T_2$ where

$$t = T_1 \Leftrightarrow s_1 \notin t \text{ (no } s_1\text{)}$$
$$t = T_2 \Leftrightarrow s_1 \in t \text{ (yes } s_1\text{)}$$

We can find the $|T_1|$ in terms of $P(\leq n, \leq r)$

$$|T_1| = P(n-1, r)$$
$$|T_2| = r \cdot P(n-1, r-1)$$

For $|T_2|$, we can order $r - 1$ elements from $\{s_2, ..., s_n\}$ and place s_1 in any of the r locations. Thus, $\boxed{P(n, r) = P(n-1, r) + r \cdot P(n-1, r-1)}$ ■

Exercise 2. Find a recursive definition of $C(n, r)$

Solution Again, let T be the subset of $S = \{s_1, ..., s_n\}$ of size r.
To find $|T|$, let $T = T_1 \cup T_2$ where T_1 has no set containing s_1 and T_2 has every set containing s_1.

$|T_1| = C(n-1, r)$ we can choose r from $s_2, ...s_n$

$|T_2| = C(n-1, r-1)$ we choose $r-1$ from $s_2, ...s_n$ and add in s_1

Thus, $\boxed{C(n, r) = C(n-1, r) + C(n-1, r-1)}$ ■

Problem 1. Given $2n$ tennis players. How many ways are there to arrange n games/pairings?

Solution There are several solutions to this problem:

1. We can match P_1 with another $2n-1$ players. For the next player P_2 who hasn't been matched, we can choose $2n-3$ players, and so on. The solution is just $(2n-1)(2n-3)...(1) = \boxed{(2n-1)!!}$
2. We can choose each pair and divide by $n!$ to remove the ordering of the pairs. There are $\boxed{\dfrac{\binom{2n}{n}\binom{2n-2}{2}...\binom{2}{2}}{n!}}$ ways.
3. We can permute all $2n$ players and divide by $n!$ (the number of ways to order the game doesn't matter) and 2^n (the order of the partners doesn't matter). There are $\boxed{\dfrac{(2n)!}{n!2^n}}$

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