The Lecture Title

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Date: Day, Mon, Date Year

1 Manipulating Generating Functions

Examples of turning sequences into functions:

1. $< 1, 1, 1, 1, \ldots >= 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$
2. $< 2, 2, 2, 2, \ldots >= 2 + 2x + 2x^2 + 2x^3 + \ldots = \frac{2}{1-x}$
3. $< 1, -1, 1, -1, \ldots >= 1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$
4. $< 1, 0, 1, 0, 1, \ldots >= 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}$
5. $< 0, 1, 0, 1, 0, \ldots >= x + x^3 + x^5 + \ldots = \frac{x}{1-x^2}$
6. $< 1, 2, 4, 8, 16, \ldots >= 1 + 2x + 4x^2 + 8x^3 + \ldots = \frac{1}{1-2x}$
7. $< 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \ldots >= 1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \ldots = \frac{e^x-1}{x}$
8. $< 1, 2, 3, 4, \ldots >= 1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$
9. $< 2, 6, 12, 20, 30, \ldots >= 2 + 6x + 12x^2 + 20x^3 + \ldots = \frac{d}{dx}(1 + 2x + 3x^2 + 4x^3 + \ldots) = \frac{d}{dx}\left(\frac{1}{(1-x)^2}\right) = \frac{2}{(1-x)^3}$
   Notice this is the sequence $\{(k+1)(k+2)\}$.
10. $< 0, 1, 4, 9, 16, \ldots >= x + 4x^2 + 9x^3 + 16x^4 + \ldots = \frac{2}{(1-x)^3} - 3\frac{1}{(1-x)^2} + \frac{1}{1-x} = \frac{x+x^2}{1-x^3}$
    Notice that $k^2 = 1(k+2)(k+1) - 3(k+1) + 1$, so we can just take a linear combination of the sequences we know.

2 Multiplying Generating Functions

Suppose we have

$$A(x) = a_0 + a_1 x + a_2 x^2 + \ldots = \sum_{k=0}^{\infty} a_k x^k$$

$$B(x) = b_0 + b_1x + b_2x^2 + \ldots = \sum_{k=0}^{\infty} b_k x^k$$

and $C(x) = A(x)B(x)$. How would we find c_n? Notice that we can only generate terms of x^n if we consider a_0b_n, a_1b_{n-1}, etc. Therefore, the coefficients are the Cauchy product of the coefficients:

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

so

$$C(x) = stuff.$$

Let's now consider a more specific example - if we let $B(x) = 1 + x + x^2 + x^3 + \ldots$, we will get $C(x) = A(x)B(x)$ as

$$C(x) = a_0x + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots + \sum_{i=0}^{k} x^k + \ldots.$$

Notice that leads us to conclude the following theorem:

Theorem 2.1. *If $A(x)$ is the ordinary generating function for the sequence $\{a_0, a_1, \ldots\}$, then $\frac{A(x)}{1-x}$ is the generating function for its partial sums.*

Let's see some applications of this now. Notice that we can very easily find the generating function for $< 1, 2, 3, 4, \ldots >$ by directly applying this result to $A(x) = \frac{1}{1-x}$:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$$

Notice the similarity we see with the generating function for the multiset coefficients:

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} \left(\binom{2}{k}\right) x^k = \sum_{k=0}^{\infty} \binom{2+k-1}{k} x^k = \sum_{k=0}^{\infty} \binom{k+1}{k} x^k = \sum_{k=0}^{\infty} (k+1)x^k$$

We can now get the generating function for the triangular numbers $D(x)$ if we take partial sums again:

$$\begin{aligned}
D(x) = B(x)C(x) &= \frac{1}{(1-x)^3} \\
&= \sum_{k=0}^{\infty} \left(\!\binom{3}{k}\!\right) x^k \\
&= \binom{3+k-1}{k} x^k \\
&= \sum_{k=0}^{\infty} \binom{k+2}{2} x^k \\
&= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)}{2} x^k = \sum_{k=1}^{\infty} \frac{k(k+1)}{2} x^{k-1}
\end{aligned}$$

Once more, to get the tetrahedral numbers

$$E(x) = 1 + 4x + 10x^2 + 20x^3 + 35x^4 + \dots$$

$$\begin{aligned}
E(x) = B(x) \cdot xD(x) &= \frac{x}{(1-x)^4} \\
&= \sum_{k=0}^{\infty} \left(\!\binom{4}{k}\!\right) x^{k+1} \\
&= \binom{4+k-1}{k} x^k \\
&= \sum_{k=0}^{\infty} \binom{k+3}{3} x^{k+1} \\
&= \sum_{k=0}^{\infty} \frac{(k+1)(k+2)(k+3)}{6} x^{k+1} = \sum_{k=1}^{\infty} \frac{k(k+1)(k+2))}{6} x^k
\end{aligned}$$

From here, it seems that

$$\sum_{k=1}^{n} k^{\overline{m}} = \frac{n^{\overline{m+1}}}{m+1}$$

just by considering summing the first n terms of the triangular sequence and comparing it to the coefficient of the tetrahedral sequence, and then generalizing the statement to arbitrary n. This is a "finite calculus" version of the power rule for integration.

Let's use this to find $\sum_{k=1}^{n} k^2$. Notice that we can write $k^2 = k(k+1) - k$, so we can just say

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} k(k+1) - \sum_{k=1}^{n} k \\ &= \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6}\end{aligned}$$

which is the correct formula that we have proved before.

Suppose we do this again, using the sum we have just shown:

$$\begin{aligned}\sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} k(k+1)(k+2) - 3\sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k \\ &= \frac{n^{\overline{4}}}{4} - 3\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} \\ &= \frac{n^2(n+1)^2}{4}\end{aligned}$$