Applications of Ordinary Generating Functions

Scribes: Maguire Papay and Justin Gou

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1 Applications of Ordinary Generating Functions

Example 1. Given $\{a, b, c, d, e\}$ pick a multiset of size 3 where no item may appear > 2 times.

Solution Simply count all possible ways, $\left(\!\binom{5}{3}\!\right)$, and subtract out the invalid ones, which is just the sets that have three of the same letter, thus there are 5 invalid sets. Evaluate to get 30 ways. ■

Example 2. 18 items, between 3 and 5 each, select a total of 79 items.

Definition. *Ordinary generating function of a sequence is $a_0, a_1, a_2, \ldots$ is*

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

Remember this is just a series with the coefficients of the k'th power of x being the k'th element in the series.

Consider:

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2$$

Evaluate two ways.

1. Expand $(1+x)^n(1+\frac{1}{x})^n$ (Note: This is not an obvious way to achieve this result, and to know to expand this to get the answer requires a lot of familiarity with the subject)

$$\begin{aligned}(1+x)^n(1+\frac{1}{x})^n &= \sum_{k=0}^{n}\binom{n}{k}x^k \cdot \sum_{j=0}^{n}\binom{n}{j}x^{-j} \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n}\binom{n}{k}\binom{n}{j}x^{k-j} \\ &= \sum_{k=0}^{n}\binom{n}{k}^2 + \sum C_k x^k\end{aligned}$$

Rewrite $(1+x)^n(1+\frac{1}{x})^n$ as $(1+x)^n(1+x)^n x^{-n}$

$$\begin{aligned}
(1+x)^n(1+x)^n x^{-n} &= (1+x)^{2n}x^{-n} \\
&= x^{-n}\sum_{k=0}^{2n}\binom{2n}{k}x^k \\
&= \sum_{k=0}^{2n}\binom{2n}{k}x^{k-n} \\
\text{Let } k = n \\
&= \binom{2n}{n} + \sum C_k x^k
\end{aligned}$$

Therefore

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$$

This conclusion is reached be equating the powers of x (in this case x^0) in both expansions. This technique is incredibly useful and will likely be leveraged in the future as well.

2. We have n blue marbles and n green marbles. Choose n total marbles.

$$\begin{aligned}
\binom{2n}{n} &= \overset{G}{\binom{n}{0}}\overset{B}{\binom{n}{n}} + \overset{G}{\binom{n}{1}}\overset{B}{\binom{n}{n-1}} + \overset{G}{\binom{n}{2}}\overset{B}{\binom{n}{n-2}} + \ldots \\
&= \sum_{k=0}^{n}\binom{n}{k}^2 \text{because} \binom{n}{r} = \binom{n}{n-r}
\end{aligned}$$

■

Consider.

$$(1+ax)(1+bx)(1+cx) = 1 + (a+b+c)x + (ab+bc+ac)x^2 + (abc)x^3$$

This is called the <u>enumerator</u> for subsets of $\{a, b, c\}$.

As you can see, the k'th power of x has all of the possible k length subsets of a, b, c in it.

By letting $a = b = c = 1$, then $(1+x^3) = 1 + 3x + 3x^2 + x^3 = \sum C_k x^k$ where C_k is the number of subsets of size k.

So $(1+x)^n = \sum \binom{n}{k} x^k$ is the Ordinary Generating Function (OGF) for binomial coefficients.
This is the same as choosing k items from a set of n where each item is chosen ≤ 1 times.

Equivalences

Given $(1+x)^n = \sum \binom{n}{k} x^k$
Plug in $x = 1$ to get

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Plug in $x = -1$ to get

$$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

$$\sum_{\substack{k=0 \\ k \text{ evens}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ odds}}}^{n} \binom{n}{k}$$

Note that the above relationship was also something we proved on the second pset theoretically using Grey Codes.

Take the derivative

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

$$n \cdot 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}$$

Now, let's select k items from n, each ≤ 2 times.

$$(1 + ax + a^2x^2)(1 + bx)(1 + cx)$$

By expanding this out, we would receive all possible subsets of $\{a, a, b, c\}$ (I suppose subsets isn't the correct word as sets are often thought to not have duplicates but I hope you get what we mean). Having an a^2 means that a was chosen twice. Having no a's in a term would mean that a was never chosen out of the objects. The power of x the combination accompanies is how many items we chose.

Mathematica will even expand this for us and we can see all of the combinations:
$a^2bcx^4 + a^2bx^3 + a^2cx^3 + a^2x^2 + abcx^3 + abx^2 + acx^2 + ax + bcx^2 + bx + cx + 1$

With this knowledge, just like earlier, we will take a, b, and c equal to 1 in the expression

$$(1 + ax + a^2x^2)(1 + bx + b^2x^2)(1 + cx + c^2x^2)$$

$$C_k x^k = (1 + x + x^2)^n = (1 + x + x^2)(1 + x + x^2)(1 + x + x^2)...$$

$$(1 + x + x^2)^n = \sum_{i+j+k=n} 1^i x^j (x^2)^k \binom{n}{i, j, k}$$

Recall.

$$\binom{n}{i, j, k} = \frac{n!}{i! \cdot j! \cdot k!}$$

So

$$(1 + x + x^2)^n = \sum_{r=0}^{n} C_r x^r$$

where the coefficient of x^{j+2k} is $\binom{n}{i,j,k}$

Let us now solve the first example problem with this knowledge. In order to be able to choose ≤ 2 of each object, we would represent that with $(1+ax+a^2x^2)$ or just $(1+x+x^2)$ after taking a=1. Since each object can be chosen like this, and there are n objects, we receive $(1 + x + x^2)^n$. Since we are only choosing 3 objects, we want the power of x to equal 3.

Example. Let $n = 5, k = 3,$ each ≤ 2 times. (Note: k refers to the overall power of x, not the actual k we are iterating over in the sum).

Solution Find the coefficient of x^3 in $(1 + x + x^2)^n$

$$x^3 = x^{3+2\cdot 0} \to j = 3, k = 0, i = 2$$
$$x^3 = x^{1+2\cdot 1} \to j = 1, n = 1, i = 3$$

$$\text{so the coefficient of } x^3 = \binom{5}{3, 0, 2} + \binom{5}{1, 1, 3} = \frac{5!}{3! \cdot 2!} + \frac{5!}{3!} = \frac{120}{12} + \frac{120}{6} = \boxed{30}$$

■

Going back to **Example 2**, where $n = 18, k = 79$, each between 3 and 5 times.

Solution Find the coefficient of x^{79} in $(x^3 + x^4 + x^5)^{18} = \sum_{k=0}^{n} C_k x^k$

Using Mathematica, we get 5895396. ■

Problem. Given a set of n elements, choose a multiset of size r.

Solution We have a notation for this: $\left(\!\binom{n}{r}\!\right)$ ■

Solution As an alternate solution, expand out $(1+x+...+x^r)^n$ and look for the coefficient of x^r. However, what happens if we don't stop at the x^r term in our initial expression, which although it has no meaning in the context of the problem (it would be like choosing $r+1$ of one item when we only want r total).

$$(1+x+...+x^r+...)^n = (\frac{1}{1-x})^n$$

This just comes from the Taylor series expansion of $\frac{1}{1-x}$.

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \frac{(-n)^{\underline{k}}}{k!}(-x)^k$$

Negatives cancel out in each term and by simplifying to a choose statement:

$$\frac{(-n)^{\underline{k}}}{k!}(-x)^k = \binom{n+k-1}{k}x^k = \left(\!\binom{n}{k}\!\right) x^k$$

This would make the coefficient of x^r equal to $\left(\!\binom{n}{r}\!\right)$, which agrees with our previous answer. ■

Problem. Given a set of n elements choose a multiset of size r, every element must be chosen at least once.

Solution

$$(x+x^2+...)^n = x^n \cdot \frac{1}{(1-x)^n}$$

$$\sum_{k=0}^{\infty} \frac{(n)^{\underline{k}}}{k!}(x)^{k+n} = \sum_{k=0}^{\infty} \left(\!\binom{n}{r}\!\right)(x)^{k+n}$$

For the exponent to equal r, k must equal $r-n$. Evaluating $\left(\!\binom{n}{r-n}\!\right)$ gives $\binom{r-1}{r-n}$ ■

Problem. Find C_k for $(1+x^5+x^9)^100$ for $k=23$

Solution

$$\sum_{i+j+k=100} (1)^i \cdot x^{5j+9k} \cdot \binom{n}{i,j,k}$$

The only combination of j and k that gives 23 is $j=1$ and $k=2$ giving us the answer $\binom{100}{2,1,97}$. ■