The Lecture Title

Scribe: Your Name

Date: Day, Mon, Date Year

1 Recursive Problems and Non-Homogeneous Difference Equations

Problem 1. Draw n lines in a plane. What is the maximum number of subregions created?

Problem 2. How many sequences of $\{0, 1, 2, 3\}^n$ contain an even number of zeroes?

Solution 1 We can do this using recursion. Let a_n be the maximum number of subregions created. Notice that for $n = 1$, $a_n = 2$. In the general case, in order to maximize the number of regions, each new added line (the nth line) must intersect the other $n-1$ lines already present. This creates n new regions, the same as the number of "spaces" between the lines.

Altogether, this gives the recurrence

$$a_n = a_{n-1} + n$$

with our base case, can be solved to yield

$$a_n = \frac{n(n+1)}{2} + 1$$

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Solution Notice that if we truncate the last digit from a valid string, we can either have a string of length $n-1$ with an even number of zeroes or a string with an odd number of zeroes. Let o_n be the number of strings of length n with an odd number of zeroes, and

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How do we solve these recurrences explicitly (without silly tricks)?

Given $\sum_{i=0}^{n} c_i x_i = f$, a constant-coefficient linear recurrence equation that is non-homogeneous, because it has the "forcing function" f. We will solve this by looking at the particular and homogeneous equations. The **particular** solution looks like the following:

$$(p): \quad \sum_{i=0}^{n} c_i a_i^{(p)} = f$$

and the **homogeneous** solution looks like the following:

$$(h): \quad \sum_{i=0}^{n} c_i a_i^{(h)} = 0$$

We will find both $a_i^{(p)}$ and $a_i^{(h)}$ and finally give the answer $a_i = a_i^{(h)} + a_i^{(p)}$ and solve for initial conditions. Notice that $\sum c_i(a_i^{(h)} + a_i^{(p)}) = f$, as $\sum c_i a_i^{(h)} = 0$ - we just solve this with the initial conditions to yield the correct solution.

Problem 3. Solve $a_n + 2an - 1 = n + 3$, $a_0 = 3$.

Solution First, we solve the homogeneous equation:

$$a_n + 2a_{n-1} = 0 \implies x + 2 = 0, \quad x = -2$$

This gives $a_n^{(h)} = c(-2)^n$.

We now solve the particular equation using undetermined coefficients

$$a_n + 2a_{n-1} = n + 3$$

We will ansatz that $a_n = Bn + D$. If we do this, we then must have $a_{n-1} = Bn + D - B$. When plugging in we get:

$$Bn + D + 2Bn + 2D - 2B = n + 3$$

Equating like coefficients, we have

$$3Bn = n \quad 3D - 2B = 3$$

This implies that $B = \frac{1}{3}$, $D = \frac{11}{9}$, so $a_n^{(p)} = \frac{1}{3}n + \frac{11}{9}$. Altogether, we must have $a_n = \frac{1}{3}n + \frac{11}{9} + C(-2)^n$. Notice that this **always** satisfies the total recurrence, as this both satisfies the homogeneous equation and the particular equation. When we plug in $n = 0$, we get $C = \frac{16}{9}$. Therefore,

$$a_n = \frac{1}{3}n + \frac{11}{9} + \frac{16}{9}(-2)^n$$

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How do we know what to try? In general, choose something of the same type.

Solve these nonhomogeneous linear recurrences with this method:

Solution $a_n = 4^{n-1} + 2a_{n-1}$, $a_1 = 3$.

Homogeneous Solution:

$$a_n - 2a_{n-1} = 0 \implies x - 2 = 0, x = 2$$

$$\implies a_n^{(h)} = C2^n$$

Particular Solution:

$$a_n - 2a_{n-1} = 4^{n-1}$$

Ansatz $a_n = A4^n$.

$$A4^n - 2A4^{n-1} = 4^{n-1} \to 2A = 1 \to A = \frac{1}{2}$$

$$a_n = C2^n + \frac{1}{2}4^n$$

$$a_1 = 3 = 2C + 2 \implies C = \frac{1}{2}$$

$$a_n = \frac{1}{2}(4^n + 2^n)$$

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Solution $a_n = a_{n-1} + n$, $a_1 = 2$:
Homogeneous Solution:

$$a_n - a_{n-1} = 0 \implies x - 1 = 0, x = 1$$

$$\implies a_n^{(h)} = C$$

Particular Solution:

$$a_n = a_{n-1} + n$$

The ansatz $a_n = Cn + D$ does not work, but $a_n = Cn^2 + Dn + E$ works:

$$Cn^2 + Dn + E = C(n-1)^2 + D(n-1) + E + n$$

$$0 = -2Cn + C - D + n \to C = \frac{1}{2}, D = \frac{1}{2}$$

$$a_n = \frac{1}{2}n^2 + \frac{1}{2}n + C$$

$$a_1 = \frac{1}{2} + \frac{1}{2} + C = 2 \to C = 1$$

$$a_n = \frac{1}{2}n^2 + \frac{1}{2}n + 1 = \frac{n(n+1)}{2} + 1$$

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