5. Find the number of ways of arranging the 26 letters in the English alphabet in a row such that there are exactly 5 letters between x and y.

6. Find the number of *odd* integers between 3000 and 8000 in which no digit is repeated.

7. Evaluate
$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!,$$
where $n \in \mathbf{N}$.

8. Evaluate
$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \cdots + \frac{n}{(n+1)!},$$
where $n \in \mathbf{N}$.

9. Prove that for each $n \in \mathbf{N}$,
$$(n+1)(n+2)\cdots(2n)$$
is divisible by 2^n. (Spanish Olympiad, 1985)

10. Find the number of common positive divisors of 10^{40} and 20^{30}.

11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:
(i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.

12. Show that for any $n \in \mathbf{N}$, the number of positive divisors of n^2 is always odd.

13. Show that the number of positive divisors of "$\underbrace{111\ldots1}_{1992}$" is even.

14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:
(i) $P_r^n = nP_{r-1}^{n-1}$,
(ii) $P_r^n = (n-r+1)P_{r-1}^n$,
(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,
(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,
(v) $P_r^{n+1} = r! + r(P_{r-1}^n + P_{r-1}^{n-1} + \cdots + P_{r-1}^r)$.

15. In a group of 15 students, 5 of them are female. If exactly 3 female students are to be selected, in how many ways can 9 students be chosen from the group
(i) to form a committee?
(ii) to take up 9 different posts in a committee?

16. Ten chairs have been arranged in a row. Seven students are to be seated in seven of them so that no two students share a common chair. Find the number of ways this can be done if no two empty chairs are adjacent.

17. Eight boxes are arranged in a row. In how many ways can five distinct balls be put into the boxes if each box can hold at most one ball and no two boxes without balls are adjacent?

18. A group of 20 students, including 3 particular girls and 4 particular boys, are to be lined up in two rows with 10 students each. In how many ways can this be done if the 3 particular girls must be in the front row while the 4 particular boys be in the back?

19. In how many ways can 7 boys and 2 girls be lined up in a row such that the girls must be separated by exactly 3 boys?

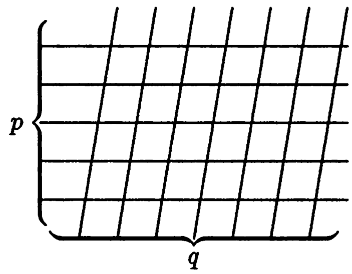
20. In a group of 15 students, 3 of them are female. If at least one female student is to be selected, in how many ways can 7 students be chosen from the group

    (i) to form a committee?

    (ii) to take up 7 different posts in a committee?

21. Find the number of $(m+n)$-digit binary sequences with m 0's and n 1's such that no two 1's are adjacent, where $n \le m+1$.

22. Two sets of parallel lines with p and q lines each are shown in the following diagram:

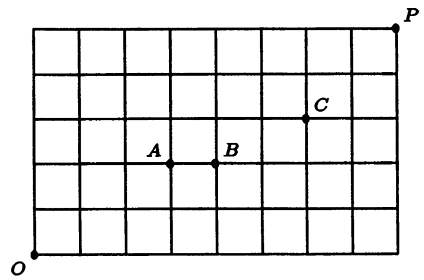


Find the number of parallelograms formed by the lines?

23. There are 10 girls and 15 boys in a junior class, and 4 girls and 10 boys in a senior class. A committee of 7 members is to be formed from these 2 classes. Find the number of ways this can be done if the committee must have exactly 4 senior students and exactly 5 boys.

24. A box contains 7 identical white balls and 5 identical black balls. They are to be drawn randomly, one at a time without replacement, until the box is empty. Find the probability that the 6th ball drawn is white, while before that exactly 3 black balls are drawn.

25. In each of the following cases, find the number of shortest routes from O to P in the street network shown below:



(i) The routes must pass through the junction A;

(ii) The routes must pass through the street AB;

(iii) The routes must pass through junctions A and C;

(iv) The street AB is closed.

26. Find the number of ways of forming a group of $2k$ people from n couples, where $k, n \in \mathbf{N}$ with $2k \leq n$, in each of the following cases:

(i) There are k couples in such a group;

(ii) No couples are included in such a group;

(iii) At least one couple is included in such a group;

(iv) Exactly two couples are included in such a group.

27. Let $S = \{1, 2, \ldots, n+1\}$ where $n \geq 2$, and let

$$T = \{(x, y, z) \in S^3 \mid x < z \quad \text{and} \quad y < z\}.$$

Show by counting $|T|$ in two different ways that

$$\sum_{k=1}^{n} k^2 = |T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

28. Consider the following set of points in the $x-y$ plane:

$$A = \{(a, b) \mid a, b \in \mathbf{Z},\ 0 \leq a \leq 9 \quad \text{and} \quad 0 \leq b \leq 5\}.$$

Find

(i) the number of rectangles whose vertices are points in A;

(ii) the number of squares whose vertices are points in A.

29. Fifteen points $P_1, P_2, \ldots, P_{15}$ are drawn in the plane in such a way that besides P_1, P_2, P_3, P_4, P_5 which are collinear, no other 3 points are collinear. Find

(i) the number of straight lines which pass through at least 2 of the 15 points;

(ii) the number of triangles whose vertices are 3 of the 15 points.

30. In each of the following 6-digit natural numbers:

$$333333, \ 225522, \ 118818, \ 707099,$$

every digit in the number appears at least twice. Find the number of such 6-digit natural numbers.

31. In each of the following 7-digit natural numbers:

$$1001011, \ 5550000, \ 3838383, \ 7777777,$$

every digit in the number appears at least 3 times. Find the number of such 7-digit natural numbers.

32. Let $X = \{1, 2, 3, ..., 1000\}$. Find the number of 2-element subsets $\{a, b\}$ of X such that the product $a \cdot b$ is divisible by 5.

33. Consider the following set of points in the $x - y$ plane:

$$A = \{(a, b) \mid a, b \in \mathbf{Z} \quad \text{and} \quad |a| + |b| \leq 2\}.$$

Find

(i) $|A|$;

(ii) the number of straight lines which pass through at least 2 points in A; and

(iii) the number of triangles whose vertices are points in A.

34. Let P be a convex n-gon, where $n \geq 6$. Find the number of triangles formed by any 3 vertices of P that are pairwise nonadjacent in P.

35. 6 boys and 5 girls are to be seated around a table. Find the number of ways that this can be done in each of the following cases:
    (i) There are no restrictions;
    (ii) No 2 girls are adjacent;
    (iii) All girls form a single block;
    (iv) A particular girl G is adjacent to two particular boys B_1 and B_2.

36. Show that the number of r-circular permutations of n distinct objects, where $1 \leq r \leq n$, is given by $\frac{n!}{(n-r)! \cdot r}$.

37. Let $k, n \in \mathbf{N}$. Show that the number of ways to seat kn people around k distinct tables such that there are n people in each table is given by $\frac{(kn)!}{n^k}$.

38. Let $r \in \mathbf{N}$ such that
$$\frac{1}{\binom{9}{r}} - \frac{1}{\binom{10}{r}} = \frac{11}{6\binom{11}{r}}.$$
Find the value of r.

39. Prove each of the following identities:
    (a) $\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$, where $n \geq r \geq 1$;
    (b) $\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$, where $n \geq r \geq 1$;
    (c) $\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$, where $n > r \geq 0$;
    (d) $\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$, where $n \geq m \geq r \geq 0$.

40. Prove the identity $\binom{n}{r} = \binom{n}{n-r}$ by (BP).

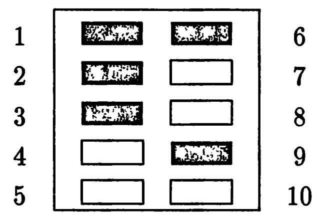
41. Let $X = \{1, 2, ..., n\}$, $\mathcal{A} = \{A \subseteq X \mid n \notin A\}$, and $\mathcal{B} = \{A \subseteq X \mid n \in A\}$. Show that $|\mathcal{A}| = |\mathcal{B}|$ by (BP).

42. Let $r, n \in \mathbf{N}$. Show that the product
$$(n+1)(n+2)\cdots(n+r)$$
of r consecutive positive integers is divisible by $r!$.

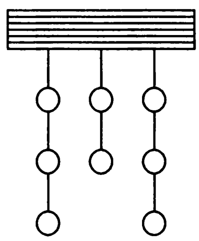
43. Let A be a set of kn elements, where $k, n \in \mathbf{N}$. A k-grouping of A is a partition of A into k-element subsets. Find the number of different k-groupings of A.

44. Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen – all choices of three being equally likely – and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator? (AIME, 1983/7) (Readers who wish to get more information about the AIME may write to Professor Walter E. Mientka, AMC Executive Director, Department of Mathematics & Statistics, University of Nebraska, Lincohn, NE 68588-0322, USA.)

45. One commercially available ten-button lock may be opened by depressing – in any order – the correct five buttons. The sample shown below has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow? (AIME, 1988/1)



46. In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured. A marksman is to break all eight targets according to the following rules: (1) The marksman first chooses a column from which a target is to be broken. (2) The marksman must then break the lowest remaining unbroken target in the chosen column. If these rules are followed, in how many different orders can the eight targets be broken? (AIME, 1990/8)



47. Using the numbers 1, 2, 3, 4, 5, we can form $5!(=120)$ 5-digit numbers in which the 5 digits are all distinct. If these numbers are listed in increasing order:

$$\underset{\text{1st}}{12345},\quad \underset{\text{2nd}}{12354},\quad \underset{\text{3rd}}{12435},\quad \ldots,\quad \underset{\text{120th}}{54321},$$

find (i) the position of the number 35421; (ii) the 100th number in the list.

48. The $P_3^4(=24)$ 3-permutations of the set $\{1,2,3,4\}$ can be arranged in the following way, called the lexicographic ordering:

$$123,\ 124,\ 132,\ 134,\ 142,\ 143,\ 213,\ 214,\ 231,\ 234,$$
$$241,\ 243,\ 312,\ \ldots,\ 431,\ 432.$$

Thus the 3-permutations "132" and "214" appear at the 3rd and 8th positions of the ordering respectively. There are $P_4^9(=3024)$ 4-permutations of the set $\{1,2,...,9\}$. What are the positions of the 4-permutations "4567" and "5182" in the corresponding lexicographic ordering of the 4 permutations of $\{1,2,...,9\}$?

49. The $\binom{5}{3}(=10)$ 3-element subsets of the set $\{1,2,3,4,5\}$ can be arranged in the following way, called the lexicographic ordering:

$$\{1,2,3\},\ \{1,2,4\},\ \{1,2,5\},\ \{1,3,4\},\ \{1,3,5\},\ \{1,4,5\},$$
$$\{2,3,4\},\ \{2,3,5\},\ \{2,4,5\},\ \{3,4,5\}.$$

Thus the subset $\{1,3,5\}$ appears at the 5th position of the ordering. There are $\binom{10}{4}$ 4-element subsets of the set $\{1,2,...,10\}$. What are the positions of the subsets $\{3,4,5,6\}$ and $\{3,5,7,9\}$ in the corresponding lexicographic ordering of the 4-element subsets of $\{1,2,...,10\}$?

50. Six scientists are working of a secret project. They wish to lock up the documents in a cabinet so that the cabinet can be opened when and only when three or more of the scientists are present. What is the smallest number of locks needed? What is the smallest number of keys each scientist must carry?

51. A 10-storey building is to be painted with some 4 different colours such that each storey is painted with one colour. It is not necessary that all 4 colours must be used. How many ways are there to paint the building if

(i) there are no other restrictions?

(ii) any 2 adjacent stories must be painted with different colours?

52. Find the number of all multi-subsets of $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \cdots, r_n \cdot a_n\}$.

53. Let $r, b \in \mathbf{N}$ with $r \leq n$. A permutation $x_1 x_2 \cdots x_{2n}$ of the set $\{1, 2, ..., 2n\}$ is said to have property $P(r)$ if $|x_i - x_{i+1}| = r$ for at least one i in $\{1, 2, ..., 2n-1\}$. Show that, for each n and r, there are more permutations with property $P(r)$ than without.

54. Prove by a combinatorial argument that each of the following numbers is always an integer for each $n \in \mathbf{N}$:
    (i) $\frac{(3n)!}{2^n \cdot 3^n}$,
    (ii) $\frac{(6n)!}{5^n \cdot 3^{2n} \cdot 2^{4n}}$,
    (iii) $\frac{(n^2)!}{(n!)^n}$,
    (iv) $\frac{(n!)!}{(n!)^{(n-1)!}}$.

55. Find the number of r-element multi-subsets of the multi-set

$$M = \{1 \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \ldots, \infty \cdot a_n\}.$$

56. Six distinct symbols are transmitted through a communication channel. A total of 18 blanks are to be inserted between the symbols with at least 2 blanks between every pair of symbols. In how many ways can the symbols and blanks be arranged?

57. In how many ways can the following 11 letters: A, B, C, D, E, F, X, X, X, Y, Y be arranged in a row so that every Y lies between two X's (not necessarily adjacent)?

58. Two n-digit integers (leading zero allowed) are said to be *equivalent* if one is a permutation of the other. For instance, 10075, 01057 and 00751 are equivalent 5-digit integers.
    (i) Find the number of 5-digit integers such that no two are equivalent.
    (ii) If the digits 5,7,9 can appear at most once, how many nonequivalent 5-digit integers are there?

59. How many 10-letter words are there using the letters a, b, c, d, e, f if
    (i) there are no restrictions?
    (ii) each vowel (a and e) appears 3 times and each consonant appears once?

(iii) the letters in the word appear in alphabetical order?

(iv) each letter occurs at least once and the letters in the word appear in alphabetical order?

60. Let $r, n, k \in \mathbf{N}$ such that $r \geq nk$. Find the number of ways of distributing r identical objects into n distinct boxes so that each box holds at least k objects.

61. Find the number of ways of arranging the 9 letters $r, s, t, u, v, w, x, y, z$ in a row so that y always lies between x and z (x and y, or y and z need not be adjacent in the row).

62. Three girls A, B and C, and nine boys are to be lined up in a row. In how many ways can this be done if B must lie between A and C, and A, B must be separated by exactly 4 boys?

63. Five girls and eleven boys are to be lined up in a row such that from left to right, the girls are in the order: G_1, G_2, G_3, G_4, G_5. In how many ways can this be done if G_1 and G_2 must be separated by at least 3 boys, and there is at most one boy between G_4 and G_5?

64. Given $r, n \in \mathbf{N}$ with $r \geq n$, let $L(r, n)$ denote the number of ways of distributing r distinct objects into n identical boxes so that no box is empty and the objects in each box are arranged in a row. Find $L(r, n)$ in terms of r and n.

65. Find the number of integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 60$$

in each of the following cases:

(i) $x_i \geq i - 1$ for each $i = 1, 2, \ldots, 6$;

(ii) $x_1 \geq 2$, $x_2 \geq 5$, $2 \leq x_3 \leq 7$, $x_4 \geq 1$, $x_5 \geq 3$ and $x_6 \geq 2$.

66. Find the number of integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 30$$

in each of the following cases:

(i) $x_i \geq 0$ for each $i = 1, 2, 3, 4$;

(ii) $2 \leq x_1 \leq 7$ and $x_i \geq 0$ for each $i = 2, 3, 4$;

(iii) $x_1 \geq -5$, $x_2 \geq -1$, $x_3 \geq 1$ and $x_4 \geq 2$.

67. Find the number of quadruples (w, x, y, z) of nonnegative integers which satisfy the inequality

$$w + x + y + z \leq 1992.$$

68. Find the number of nonnegative integer solutions to the equation:

$$5x_1 + x_2 + x_3 + x_4 = 14.$$

69. Find the number of nonnegative integer solutions to the equation:

$$rx_1 + x_2 + \cdots + x_n = kr,$$

where $k, r, n \in \mathbf{N}$.

70. Find the number of nonnegative integer solutions to the equation:

$$3x_1 + 5x_2 + x_3 + x_4 = 10.$$

71. Find the number of positive integer solutions to the equation:

$$(x_1 + x_2 + x_3)(y_1 + y_2 + y_3 + y_4) = 77.$$

72. Find the number of nonnegative integer solutions to the equation:

$$(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) = p,$$

where $n \in \mathbf{N}$ and p is a prime.

73. There are 5 ways to express "4" as a sum of 2 nonnegative integers in which the order counts:

$$4 = 4 + 0 = 3 + 1 = 2 + 2 = 1 + 3 = 0 + 4.$$

Given $r, n \in \mathbf{N}$, what is the number of ways to express r as a sum of n nonnegative integers in which the order counts?

74. There are 6 ways to express "5" as a sum of 3 positive integers in which the order counts:

$$5 = 3+1+1 = 2+2+1 = 2+1+2 = 1+3+1 = 1+2+2 = 1+1+3.$$

Given $r, n \in \mathbf{N}$ with $r \geq n$, what is the number of ways to express r as a sum of n positive integers in which the order counts?