Partitions of a Set and Stirling Numbers

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1 Partitions of a Set

Problem 1. How many ways can a set of 4 elements be partitioned into 2 non-empty sets?

Solution The possible sets are:

{A}, {BCD}	{AB}, {CD}
{B}, {ACD}	{AC}, {BD}
{C}, {ABD}	{AD}, {BC}
{D}, {ABC}	

Therefore, there are $\boxed{7}$ total ways. ∎

Definition. *The number of ways to partition a set of n elements into k nonempty subsets is the Stirling number of the second kind. Stirling numbers of the second kind are denoted by* $S(n, k)$ *or* $\left\{ {a \atop b} \right\}$.

2 Stirling Numbers

Preliminary/Base Cases:

S(1,1) = 1	S(n,1) = 1
S(n,n) = 1	S(0,0) = 1

Claim: $S(n, 2) = 2^{n-1} - 1$

Proof. Consider: A = $\{0,1\}^n$. $|A| = 2^n$ This will be the *characteristic function* of membership in the two subsets S_0, S_1. Now 1^n and 0^n are not valid partitions because they leave an empty set. So we now have $|A| - 2 = 2^n - 2$ mappings. But, we're double counting because order in a subset is irrelevant. So our count is $\frac{2^n-2}{2} = \boxed{2^{n-1} - 1}$. ∎

Claim: $S(n, n-1) = \binom{n}{2}$

Proof. Choose 2 elements from n, put them in a set, and put every other element in a singleton set. ∎

Definition. *The recursive definition of Stirling Numbers:*

$S(n,k) = S(n-1,k-1) + S(n-1,k)\cdot k$ *This is because it includes the two cases where element n is by either itself,* $\{n\}$ *(* $S(n-1,k-1)$*), or in a pre-existing set, of which there are* k*, giving us* $S(n-1,k)\cdot k$*.*

Stirling Numbers of the Second Kind, S(n, k):

n \ k	1	2	3	4	5	6	7
1	1	0	0	0	0	0	0
2	1	1	0	0	0	0	0
3	1	3	1	0	0	0	0
4	1	7	6	1	0	0	0
5	1	15	25	10	1	0	0
6	1	31	90	65	15	1	0
7	1	63	301	350	140	21	1

Balls and Urns

Balls and Urns	No restrictions	≤ 1 per urn	≥ 1 per urn
labeled into labeled	U^B	$U^{\underline{B}}$	$U! * S(B,U)$
unlabeled into labeled	$\left(\!\binom{U}{B}\!\right)$	$\binom{U}{B}$	$\left(\!\binom{U}{B-U}\!\right)$
labeled into unlabeled	$\sum_{n=1}^{U} S(B,n)$	[B≤U]	$S(B,U)$