Introduction to Counting

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1 Basic Notation

If A is a set, then we define

$\lvert A\rvert$ or $\#\{A\}$	represents the (finite) number of elements in the set, known as the *magnitdue*, *length*, *size*, or *cardinality* of that set
$A \cap B$	*intersection* (command in LaTeX is \cap)
$A \cup B$	*union* (command in LaTeX is \cup)
$\overline{A}$	*complement*, given by $\{x \mid x \notin A\}$
$A \setminus B$	*minus*, given by $\{x \mid x \in A, x \notin B\}$; sometimes written as $A - B$
$x \in A$	set inclusion (x is an *element* of A)
$A \subseteq B$	A is a *subset* of B ($x \in A \implies x \in B$)
$A \subsetneq B$	A is a *proper subset* of B ($A \subseteq B, A \neq B$)
$\varnothing$	empty set
2^A or $\mathcal{P}(A)$	*power set* of A: set of all subsets of A, including A and $\varnothing$

2 Basic Combinatorics

Theorem 2.1. $|2^A| = 2^{|A|}$

Proof. We begin with an example.

$$\mathcal{P}(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Notice that the number of subsets is indeed $2^3 = 8$.
Many problems in combinatorics are best solved by isomorphic counting; we rephrase the problem into something easier to count. Let us consider the binary functions on A, $f : A \to \{0,1\}$. Notice that each f can be uniquely represented with a binary string, which in turn represents a way to make a subset of A

$$001 \to f(1) = 0,\ f(2) = 0,\ f(3) = 1 \to \{3\}$$
$$110 \to f(1) = 1,\ f(2) = 1,\ f(3) = 0 \to \{1,2\}$$

Thus, $|2^A|$ is equinumerous to the number of binary numbers of length $|A|$, or $2^{|A|}$.

$\blacksquare$

Theorem 2.2. *If $A \subseteq B$ and $B \subseteq A$, then $A = B$*

Theorem 2.3 (Principle of Inclusion-Exclusion)**.** $|A \cup B| = |A| + |B| - |A \cap B|$

The Principle of Inclusion-Exclusion may be applied repeated to count the cardinality of unions of more than two sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |BC| - |AC| + |ABC|$$

Note that when it's clear what we're talking about, we can abbreviate $A \cap B$ as AB.

Problem. Prove that, for $m, n \in \mathbb{N}$ selected uniformly at random

$$P(\gcd(m, n) = 1) = \frac{6}{\pi^2}$$

Theorem 2.4 (Multiplication Theorem)**.** *Given sets $A_1, A_2, \ldots A_n$, the number of ways to select an element from each set is $|A_1||A_2| \ldots |A_n|$.*

Proof. Draw a multitree with root nodes in A_1 and let each node $x \in A_i$ have as its children A_{i+1}. As you traverse from root to leaf, each decision you make corresponds to a selection from that set, and the number of paths from root to leaf is $|A_1||A_2| \ldots |A_n|$.

■

Corollary. If $|A| = n$, we can select k of these elements, with repetition, in n^k ways.

Corollary. If $|A| = n$, we can select k of these elements, without repetition, in $n(n-1)(n-2)\ldots(n-k+1)$ ways.

We will notate the above expression as

$$n(n-1)(n-2)\ldots(n-k+1) = {}_nP_k = {}^nP_k = n^{\underline{k}}$$

The final notation will be the most commonly used in this class. This is called the "falling factorial". Similar, we can define a "rising factorial":

$$n^{\overline{k}} = (n)(n+1)\ldots(n+r-1)$$

Problem 1. How many passwords with only capital letters or digits contain 8, 9, 10 characters, barring repeated characters?
Answer: $36^{\underline{8}} + 36^{\underline{9}} + 36^{\underline{10}}$

Problem 2. How many passwords with only capital letters or digits containing at least 1 digit and 1 letter contain 8, 9, or 10 characters, barring repeated characters?
Answer: $(36^{\underline{8}} + 36^{\underline{9}} + 36^{\underline{10}}) - (26^{\underline{8}} + 26^{\underline{9}} + 26^{\underline{10}}) - (10^{\underline{8}} + 10^{\underline{9}} + 10^{\underline{10}})$

Problem 3. How many passwords with only capital letters or digits contain 8 characters and have capital letters in even-numbered spaces (the 0th position, the 2nd position, and so on), allowing for repitition of characters?
Answer: $5^4 * 36^4$

2.1 Quotient Sets

Problem. Let us count the number of permutations of "ABCD". There are 24 permutations of this string of length 4 (e.g. ABCD, ABDC, ...). There are also 24 permutations of length 3 (e.g. ABC, ABD, ...) Let define equivalence relationship $p_1 \cong p_2$ if they contain the same letters (e.g. ACB $\cong$ ABC). Define an *equivalence group* G to be such that $p_1, p_2 \in G \implies p_1 \cong p_2$. How many equivalence groups are there out of the 24 permutations of length 3?

Solution There are 4 equivalence groups. Note the following:

1. All equivalence groups are of the size same size, 3!
2. Different equivalence groups are disjoint
3. The set of all equivalence groups forms a partition of the set of all $4^{\underline{3}}$ permutations of length 3.

Thus, there are $4^{\underline{3}}/3!$ equivalence groups. ■

By noticing that equivalence groups are of size $r!$, we can now count *combinations*.

$$_nC_r = \frac{_nP_r}{r!}$$