Properties of n!

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1 Stirling's Approximation

We intend to examine different ways that we may approximate the value of $n!$. The first of these methods is Stirling's Approximation.

Theorem 1.1. *(Stirling's Approximation)* $\lim_{n\to\infty} \frac{n!}{(\frac{n}{e})^n * \sqrt{2\pi n}} = 1$

This is the formal definition of Stirling's Approximation. The actual approximation that we are observing, would be the denominator of this fraction. Stirling's thinking behind this, is that as n increases in size, the ratio should eventually become equal to 1.
We won't however, be proving the formal definition of Stirling's Approximation. Instead, we want to find some sort of upper and lower bound for the value of $n!$. To do this, we will first create an informal definition for $n!$.

Informally speaking, $n! \sim \binom{n}{e}^n * \sqrt{2\pi n}$
We intend to prove that $n! = \binom{n}{e}^n * \delta\sqrt{n}$, where $\delta \cong \sqrt{2\pi}$

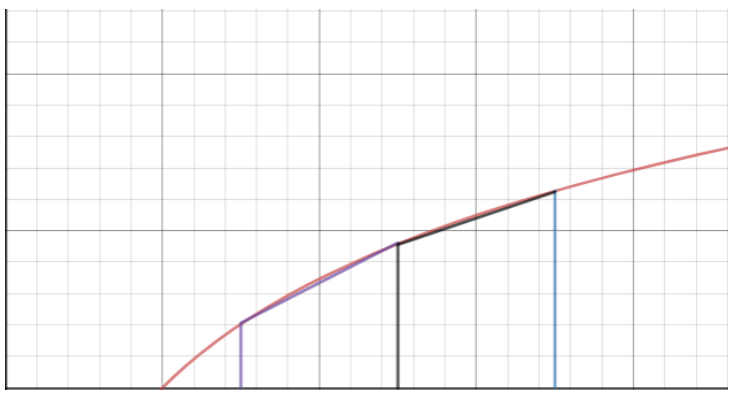
To do this proof, we also want to make note of an important summation remark:

Remark. $log(n!) = \sum_{i=1}^{n} log(i)$

Proof. To do the actual proof, we will consider the function $y = ln(x)$. From here we intend to make two Riemann-style approximations for the integral $\int_i^n ln(x)dx$. For our proof, both approximations will be trapezoidal approximations.

Approach one is two use a trapezoid whose bases are at the integers themselves.

Area of the trapezoids is $\frac{1}{2}\sum_{i=1}^{n-1}[ln(i)+ln(i+1)]$
$=\frac{1}{2}ln(1)+\sum_{i=2}^{n-1}[ln(i)+ln(i+1)]+\frac{1}{2}ln(n)$
$=ln(n!)-\frac{1}{2}ln(n)$



Approach two is two use a trapezoid whose bases are at the midpoints between integers. Notice that we don't care about the area before $x=\frac{1}{2}$ in our calculations.

Area = $ln(2)+ln(3)+ln(4)\cdots+ln(n-1)+\frac{1}{2}ln(n)$, which comes out to be the same as above. Notice that this is an overestimate however, and the previous approach was an underestimate.
$\therefore \int_{\frac{3}{2}}^{n} ln(x)dx+\frac{1}{2}ln(n)<ln(n!)<\int_{1}^{n} ln(x)dx+\frac{1}{2}ln(n)$

Given $\int ln(x)dx = xln(x)-x+C$

We get:
$(nln(n)-n)-(\frac{3}{2}ln(\frac{3}{2})-\frac{3}{2})+\frac{1}{2}ln(n)<ln(n!)<(nln(n)-n)-(ln(1)-1)+\frac{1}{2}ln(n)$
$=(n+\frac{1}{2})ln(n)-n-\frac{3}{2}(ln(\frac{3}{2})-1)<ln(n!)<(n+\frac{1}{2})ln(n)-n+1$

At this point, we have a range of error for $ln(n!)$.

We can define $ln(n!) = [(n + \frac{1}{2})ln(n) - n] + \delta_n$, where $\frac{3}{2}(1 - ln(\frac{3}{2})) < \delta_n < 1 \longrightarrow 0.891802 < \delta_n < 1$

Since $ln(n!) = (n + \frac{1}{2})ln(n) - n + \delta_n$,
$e^{ln(n!)} = e^{nln(n)}e^{\frac{1}{2}ln(n)}e^{-n}e^{\delta_n}$
$n! = n^n\sqrt(n)e^{-n}e^{\delta_n}$
$= (\frac{n}{e})^n\sqrt(n) * e^{\delta_n}$, where $e^{\delta_n} \in [2.439, 2.718]$. The value for $\sqrt{2\pi n} = 2.506$, which falls within our expected range of values.

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Now that we have an approximation for $n!$, we can look to write an expression that will give us the n^{th} Catalan number. The n^{th} Catalan number is given by the expression $\binom{2n}{n} * \frac{1}{n+1}$
$\binom{2n}{n} * \frac{1}{n+1} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{[(\frac{n}{e})^n * \sqrt{2\pi n}]^2}$
$= \frac{\frac{2n}{e}^{2n}}{\frac{n}{e}^{2n}} * \frac{\sqrt{4\pi n}}{2\pi n*(n+1)} = \frac{2^{2n} * \sqrt{\pi n}}{\pi n(n+1)} = \frac{4^n}{\sqrt{\pi n}(n+1)}$

2 Gamma Function

Gamma functions are an extension of factorials, such that the arguments are shifted down by one. In other words, $\Gamma(2) = 1!$, $\Gamma(3) = 2!$, etc. We intend to prove that $\Gamma(n + 1) = n!$. To do this, we first begin by defining our gamma function.

Definition. $\Gamma(z + 1) = \int_0^\infty x^z e^{-x}$

Now that we have defined our gamma function, we will attempt to prove that $\Gamma(n + 1) = n!$.

Proof. First, we will integrate this function by parts:

- $u = x^7$, $du = zx^{z-1}$
- $v = -e^{-x}$, $dv = e^{-x}$

$\therefore \int_0^\infty x^z e^{-x}dx = [x^z(-e^{-x})]_0^\infty + \int_0^\infty zx^{z-1}e^{-x}dx$

Notice how when we evaluate the expression $\int_0^\infty x^z e^{-x}dx = [x^z(-e^{-x})]_0^\infty$, xz becomes 0 when evaluated at $x = 0$, and $(-e^{-x})$ becomes zero when evaluated at $x = \infty$, meaning that this expression evaluates to just 0.

Therefore, we are left with just $\int_0^\infty zx^{z-1}e^{-x}dx$. We can redefine this as $z\int_0^\infty x^{z-1}e^{-x}dx = z\Gamma(z)$.

At this point, we have something that is similar to a factorial, but we can't consider our proof complete until we establish some base cases. We can begin by testing with $z = 1$. $\Gamma(1) = \int_0^\infty x^0 e^{-x}dx = \int_0^\infty e^{-x}dx = 1$
$\therefore \Gamma(1) = 1$
$\Gamma(2) = 1 * \Gamma(1) = 1$
$\Gamma(3) = 2 * \Gamma(2) = 3$
$\Gamma(4) = 3 * \Gamma(3) = 6$
$\therefore \Gamma(n+1) = n!$

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As we intended, we have now found a function that is similar to a factorial, withe the arguments shifted by one. However, we also want to observe how fractional factorials work. In our case, we'll observe $\Gamma(\frac{1}{2})$.

Proof. $\Gamma(\frac{1}{2}) = \int_0^\infty x^{\frac{-1}{2}} e^{-x}dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}$
Let $u = \sqrt{x}$, $u^2 = x$, and $du * 2u = dx$
Substituting u into the equation, we get $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}du$
Now suppose we define a variables v and K, such that
$K = \int_0^\infty e^{-u^2}du = \int_0^\infty e^{-v^2}dv$
$K^2 = \int_0^\infty e^{-u^2}du \int_0^\infty e^{-v^2}dv$
$= \int_0^\infty \int_0^\infty e^{-u^2}e^{-v^2}dudv = \int_0^\infty \int_0^\infty e^{-(u^2+v^2)}dudv$
In Cartesian coordinates, this integral would be difficult to solve. However, we can convert to polar coordinates to solve this equation. Notice that because we are only using positive integers, θ is bounded within the first quadrant
$K^2 = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} * rdrd\theta = \frac{\pi}{2}\int_0^\infty e^{-r^2} * rdr$
$= \frac{\pi}{2} * (\frac{-e^{-r^2}}{2})_0^\infty = \frac{\pi}{2}(\frac{1}{2} - 0) = \frac{\pi}{4}$
$\therefore K = \frac{\sqrt{\pi}}{2}$
$\therefore \Gamma(\frac{1}{2}) = 2K = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

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