# The Lecture Title

Scribe: Your Name

**Date**: Day, Mon, Date Year

## 1 Partitions and Stirling Numbers

**Problem 1.** How many ways can a set of 4 elements be partitioned into 2 non-empty sets?

*Solution:* casework

■

Defintion: the number of ways to partition a set of $n$ elements into $k$ non-empty subsets is the **Stirling Number of the second kind** $S(n,k)$ or

Let's construct a few base cases:

$$
\begin{aligned}
&S(1,1) = 1 \\
&S(n,1) = 1 \\
&S(n,n) = 1 \\
&S(0,0) = 1 \quad \text{(defined as such)} \\
&S(n,2) =
\end{aligned}
$$

The first four of these are fairly trivial to see, but the last two are not.

**Problem Claim:.** $S(n,2) = 2^{n-1} - 1$

*Proof.* Consider $A = \{0,1\}^n$. This will be the characteristic function of membership in the two sets $S_0$, $S_1$. Note that $|A| = 2^n$, but $1^n$ and $0^n$ are not valid partitions (since they have an empty set). So we have $|A| = 2^n - 2$ valid mappings. However, since we are double counting because the identity of the subsets are irrelevant, we divide by 2 to yield $2^{n-1} - 1$. ■

**Problem Claim: .** $S(n, n-1) = \binom{n}{2}$.

*Proof.* Choose two elements from the $n$ that will be in the same set, and each of the rest of the elements must be in its own set.

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With these base cases in mind, we can look at constructing a recursive formula for $S(n,k)$. We can start by looking at the partitions of an $n-1$-element set into $k-1$ elements - and the $n$th element must go into a new set by itself. Alternatively, we can consider $S(n-1,k)$, or the partitions of an

$n-1$-element set into $k$-elements, and we must pick which one of the sets the last element must go into in $k$ ways. This gives us the recursive formula

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

We can now begin computing values of the Stirling numbers of the second kind:

| B | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| labeled | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| unlabeled | | | | | | | |
| labeled | | | | | | | |
| unlabeled | | | | | | | |

$1 - - - - - - 11 - - - - - 131 - - - - 1761 - - - 11525101 - - 1319065151 -$ $163301350140211$ etc.

## 2 Revisiting Balls and Urns

| B | U | no restrictions | $\leq 1$ per urn | $\geq 1$ per urn |
|---|---|---|---|---|
| labeled | labeled | | | |
| unlabeled | labeled | | | |
| labeled | unlabeled | | | |
| unlabeled | unlabeled | | | |

$L- > L$ $U- > L$, leq 1 - choose B of the U urns (U B) $U- > L$, no restrictions - multiset problem, ((U, B)) $U- > L$, $\geq 1$ - fill the urns with balls, and then same as before - ((U, B-U)) $L- > U$, geq 1 - literally the Stirling numbers of the second kind $L- > U$, leq 1 - either 1 or 0 $[B \leq U]$ only 1 if $B \leq U$. $L- > U$, no restrictions - sum of Stirling numbers $S(B,n)$ from $n = 1toU$, and if we go off the triangle it is 0 $L- > L$, geq 1 - partition the elements, and then label the subsets $U!S(B,U)$.

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