Concrete Math, Spring 2019, White

PS2. Due Thurs, Feb 21

Problems

1. In how many ways may we write n as a sum of an ordered list of k positive numbers? Such a list is called a *composition* of n into k parts.

2. What is the total number of compositions of n (into any number of parts).

3. A Grey Code is an ordering of n-digit binary strings such that each string differs from the previous in precisely one digit.

   (a) Write one Grey Code for $n = 4$

   (b) Prove by induction that Grey Codes exist for all $n \geq 4$

   (c) Prove that the number of even-sized subsets of an n-element set equals the number of odd-sized subsets of an n-element set.

4. A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example, (((()()))()) is balanced and ((()) and (()()))(() are not. How many balanced lists of n left and n right parentheses are there?

5. A tennis club has $4n$ members. To specify a doubles match, we choose two teams of two people. In how many ways may we arrange the members into doubles matches so that each player is in one doubles match? In how many ways may we do it if we specify in addition who serves first on each team?

6. A town has n streetlights running along the north side of main street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?

7. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green if we use green paint on an even number of them?

8. A boolean function $f : \{0,1\}^n \to 0,1$ is *self-dual* if replacing all 0s with 1s and 1s with 0s yields the same function. How many self-dual boolean functions are there as a function of n?

9. A boolean function $f : \{0,1\}^n \to 0,1$ is *symmetric* if any permutation applied to the digits in the domain yields the same function. *e.g.* $f(001) = f(010) = f(100)$. How many symmetric boolean functions are there as a function of n?

10. Prove $x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}$

11. Prove $(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha^{\underline{k}}}{k!} x^k$ for any real α.