Concrete Mathematics

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0 Introduction

0.1 Editing Notes

Notes on editing:

- Tried to be faithful as possible to the source – either by essentially including the tex source as input or by using Mathpix to generate an approximate source and modifying it to match.
- Tried to keep contributions from both authors if two notes existed on a day. sometimes (sorry) one of them is a little more coherent than the other so I made a semi-arbitrary choice (mostly in favor of coherency) but also kept the names of the other authors
- Mondays sometimes don't exist? Don't remember why to be honest... maybe no lectures happened those days?
- Some contributors' graduation years/personal information may be wrong for any number of reasons – want to honor people's current identities, preferred names, etc.
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Part I
Enumerations

Consider the following common variety of counting problem:

Classic Problem. Suppose you have b balls and u urns. How many ways are there to put the balls into the urns, if:

the balls are:	the urns are:	and there can be:
1. labeled	1. labeled	1. **no** restrictions on the balls in the urns
2. unlabeled	2. unlabeled	2. **at most** one ball per urn
		3. **at least** one ball per urn

This problem comes in 12 possible varieties! What happens when we have:

- 7 balls and 3 urns?
- 4 balls and 8 urns?

In time, we will be able to answer all 12 variants of each of these questions.

Remark. *This series of twelve problems for any b and u is called the* ***twelve-fold way****, posed by Gian-Carlo Rota.*

1 Basic Notation

If A is a set, then we define:

$\|A\|$ or $\#\{A\}$	represents the (finite) number of elements in the set, known as the *magnitdue*, *length*, *size*, or *cardinality* of that set
$A \cap B$	*intersection* (command in LaTeX is \cap)
$A \cup B$	*union* (command in LaTeX is \cup)
$\overline{A}$	*complement*, given by $\{x \mid x \notin A\}$
$A \setminus B$	*minus*, given by $\{x \mid x \in A, x \notin B\}$; sometimes written as $A - B$
$x \in A$	set inclusion (x is an *element* of A)
$A \subseteq B$	A is a *subset* of B ($x \in A \implies x \in B$)
$A \subsetneq B$	A is a *proper subset* of B ($A \subseteq B, A \neq B$)
$\varnothing$	empty set
2^A or $\mathcal{P}(A)$	*power set* of A: set of all subsets of A, including A and $\varnothing$

2 Counting Strategies

2.1 Basic Combinatorics

Theorem 2.1. $|2^A| = 2^{|A|}$

Proof. We begin with an example.

$$\mathcal{P}(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

Notice that the number of subsets is indeed $2^3 = 8$.
Many problems in combinatorics are best solved by isomorphic counting; we rephrase the problem into something easier to count. Let us consider the binary functions on A, $f : A \to \{0,1\}$. Notice that each f can be uniquely represented with a binary string, which in turn represents a way to make a subset of A

$$001 \to f(1) = 0,\ f(2) = 0,\ f(3) = 1 \to \{3\}$$
$$110 \to f(1) = 1,\ f(2) = 1,\ f(3) = 0 \to \{1,2\}$$

Thus, $|2^A|$ is equinumerous to the number of binary numbers of length $|A|$, or $2^{|A|}$.

$\blacksquare$

Theorem 2.2. *If $A \subseteq B$ and $B \subseteq A$, then $A = B$*

Theorem 2.3 (Principle of Inclusion-Exclusion)**.** $|A \cup B| = |A| + |B| - |A \cap B|$

The Principle of Inclusion-Exclusion may be applied repeated to count the cardinality of unions of more than two sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |BC| - |AC| + |ABC|$$

Note that when it's clear what we're talking about, we can abbreviate $A \cap B$ as AB.

Problem. Prove that, for $m, n \in \mathbb{N}$ selected uniformly at random

$$P(\gcd(m, n) = 1) = \frac{6}{\pi^2}$$

Theorem 2.4 (Multiplication Theorem)**.** *Given sets $A_1, A_2, \ldots A_n$, the number of ways to select an element from each set is $|A_1||A_2| \ldots |A_n|$.*

Proof. Draw a multitree with root nodes in A_1 and let each node $x \in A_i$ have as its children A_{i+1}. As you traverse from root to leaf, each decision you make corresponds to a selection from that set, and the number of paths from root to leaf is $|A_1||A_2| \ldots |A_n|$. ∎

Corollary. If $|A| = n$, we can select k of these elements, with repetition, in n^k ways.

Corollary. If $|A| = n$, we can select k of these elements, without repetition, in $n(n-1)(n-2) \ldots (n-k+1)$ ways.

We will notate the above expression as

$$n(n-1)(n-2) \ldots (n-k+1) = {}_nP_k = {}^nP_k = n^{\underline{k}}$$

The final notation will be the most commonly used in this class. This is called the "falling factorial". Similar, we can define a "rising factorial":

$$n^{\overline{k}} = (n)(n+1) \ldots (n+r-1)$$

Problem 1. How many passwords with only capital letters or digits contain 8, 9, 10 characters, barring repeated characters?
Answer: $36^{\underline{8}} + 36^{\underline{9}} + 36^{\underline{10}}$

Problem 2. How many passwords with only capital letters or digits containing at least 1 digit and 1 letter contain 8, 9, or 10 characters, barring repeated characters?
Answer: $(36^{\underline{8}} + 36^{\underline{9}} + 36^{\underline{10}}) - (26^{\underline{8}} + 26^{\underline{9}} + 26^{\underline{10}}) - (10^{\underline{8}} + 10^{\underline{9}} + 10^{\underline{10}})$

Problem 3. How many passwords with only capital letters or digits contain 8 characters and have capital letters in even-numbered spaces (the 0th position, the 2nd position, and so on), allowing for repitition of characters?
Answer: $5^4 * 36^4$

Problem 4. How many divisors does the number 496 have? What about the number 360?

Solution Consider the prime factorizations of these numbers:

$$496 = 31 \cdot 2^4 \qquad 360 = 5 \cdot 3^2 \cdot 2^3$$

A divisor of these numbers is created by choosing a possible number of prime factors that divide the original number. For example, a divisor of 496 can have either 0 or 1 factors of 31, and anywhere from 0 to 4 factors of 2. Then the number of divisors of 496 is $(1+1)(4+1) = 10$, and similarly, the number of divisors of 360 is $(1+1)(2+1)(3+1) = 24$, following from the Multiplication Theorem. ■

Problem 5. What are the sum of the divisors of 496 and 360? The "sum of all divisors" function is denoted $\sigma(n)$.

Solution Using a modification of the Multiplication Theorem, note that the multiplication operation encodes the idea of "all possible ways of combining two things." As such, if we consider the sum of all of the different prime powers that can appear in any divisor and multiply them together, we will create a term with every divisor, added together. For instance, the sum of the divisors of 496 is:

$$\sigma(496) = (1 + 31)(1 + 2 + 4 + 8 + 16) = 32\dot{3}1 = 992$$

and similarly

$$\sigma(360) = (1 + 5)(1 + 3 + 9)(1 + 2 + 4 + 8) = 6 \cdot 13 \cdot 15 = 1170.$$

■

Remark. *If we consider the sum of all proper divisors of a positive integer n ($\sigma(n) - n$), we can classify integers into three categories based on how $\sigma(n) - n$ compares to n:*

- *If $\sigma(n) - n < n$, then n is deficient.*
- *If $\sigma(n) - n > n$, then n is abundant.*
- *If $\sigma(n) - n = n$, then n is perfect.*

In the example above, note that 360 is abundant and 496 is perfect.

*Finding perfect numbers is actually an incredibly difficult problem in number theory – in fact, nobody even knows if there exists perfect numbers that are odd! In the even case, though, Euler showed that perfect numbers are of the form $2^{p-1}(2^p - 1)$ if $2^p - 1$ is prime (which requires p itself to be prime also). Primes of the form $2^p - 1$ are called **Mersenne primes**, and it is not even known if there are infinitely many of these primes! Only 51 Mersenne primes are known to exist as of 2023, so we only know of 51 perfect numbers.*

2.2 Quotient Sets

Problem. Let us count the number of permutations of "ABCD". There are 24 permutations of this string of length 4 (e.g. ABCD, ABDC, ...). There are also 24 permutations of length 3 (e.g. ABC, ABD, ...) Let define equivalence relationship $p_1 \cong p_2$ if they contain the same letters (e.g. ACB $\cong$ ABC). Define an *equivalence class C* to be such that $p_1, p_2 \in C \implies p_1 \cong p_2$. How many equivalence classes are there out of the 24 permutations of length 3?

Solution There are 4 equivalence classes. Note the following:

1. All equivalence classes are of the size same size, 3!
2. Different equivalence classes are disjoint
3. The set of all equivalence classes forms a partition of the set of all $4^{\underline{3}}$ permutations of length 3.

Thus, there are $4^{\underline{3}}/3!$ equivalence classes. ■

By noticing that equivalence classes are of size $r!$, we can now count *combinations*.

$${}_nC_r = \frac{{}_nP_r}{r!}$$

2.3 Partitions

A *partition* of a set S is a subset of $S_1, ...S_k$ such that

(i) $\bigcup_{i=1}^{k} S_i$. The subsets 'cover' the set S

(ii) $S_i \cap S_j = \varnothing$. The subsets are pairwise disjoint.

(iii) $S_i \neq \varnothing$

Problem 1. Let S be the set of all integers composed of digits in $\{1, 3, 5, 7\}$ at most one.

(i) Find $|S|$

(ii) $\sum_{x \in S} x$

Solution

(i) Let $S = S_1 \cup S_2 \cup S_3 \cup S_4$ where S_1 is the number of one digit numbers, S_2 is the number of two-digit numbers, and so on.

$$|S_1| = {}^4P_1 = 4$$
$$|S_2| = {}^4P_2 = 12$$
$$|S_3| = {}^4P_3 = 24$$
$$|S_4| = {}^4P_4 = 24$$
$$|S| = |S_1| + |S_2| + |S_3| + |S_4| = \boxed{64}$$

(ii) Let $\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4$ where α_1 is the sum of all units digits of all numbers in S, α_2 is the sum of all the tens digits of all the numbers, and so on. We will find the value of α_1 using the following:

$$\begin{aligned}
S_1 &\to s_1 = (1 + 3 + 5 + 7) \\
S_2 &\to s_2 = (1 + 3 + 5 + 7) \times (3) \\
S_3 &\to s_3 = (1 + 3 + 5 + 7) \times (3 \times 2) \\
S_4 &\to s_4 = (1 + 3 + 5 + 7) \times (3 \times 2 \times 1) \\
\hline
\alpha_1 &= 16 \times (1 + 3 + 6 + 6) = 256
\end{aligned}$$

Note that α_2 is the sum of the same values, excluding s_1 as S_1 is the set of only one digit numbers. α_3 is the sum of the same values as α_2, excluding s_2 as S_2 is the set of only two digit numbers, and so on.

$$\alpha_2 = \alpha_1 - s_1 = 240$$
$$\alpha_3 = \alpha_2 - s_2 = 192$$
$$\alpha_4 = \alpha_3 - s_3 = 96$$

Thus, $\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4 = \boxed{117,856}$

An easier solution is the following:

$$1 + 3 + 5 + 7 = (1 + 7) + (3 + 5) = 8(\frac{4}{2}) = 16$$
$$13 + ... + 75 = (13 + 75) + ... + (35 + 53) = 88(\frac{12}{2}) = 528$$

etc

Since each $x \in S_i$ pairs with $\bar{x} \in S_i$ to sum to $88...8$. We find $\alpha = 8\frac{|S_1|}{2} + 88\frac{|S_2|}{2} + 888\frac{|S_3|}{2} + 8888\frac{|S_4|}{2} = \boxed{117,856}$

■

2.4 Cyclic Permutation

Consider the set T of 3 permutations of (s_1, s_2, s_3, s_4) or $(1, 2, 3, 4)$. We know that $T = \{123, 132, 234, 214, ...\}$ and $|T| = P(4, 3) = 24$.

We define $x \cong y \iff x + y$ are cyclically equivalently

Problem 1. Given $123 \cong x$, how many solutions are there for $x \in T$?

Solution The x values are 123, 231, 312, so there are $\boxed{3}$ solutions. Thus, we see any sequence of length $n \cong n$ sequences ■

Theorem 2.5. *If $Q(n, r)$ is the number of cyclic permutations of length r from a set of n elements, $Q(n, r) = \frac{P(n,r)}{r}$.*

Theorem 2.6. *There are (n-1)! ways to seat n people around a round table.*

Proof. Each ordering $\cong n$ orderings. Thus, $\frac{n!}{n} = (n-1)!$ ■

Problem 2. There are 5 boys and 3 girls seated around a round table.

(i) There are no restrictions.

(ii) B_1 and G_1 are not adjacent

(iii) No girls are adjacent to other girls

Solution

(i) Using theorem 2.1, there are $\boxed{7!}$ ways.

(ii) We first place B_1 in any of the 7 seats and set B_1 as our reference point. There are then 5 places for G_1 to sit not adjacent to B_1 and 6! ways for the remaining 6 people to sit. The total number of ways if $\boxed{6! \cdot 5}$. We can also consider the number of ways for B_1 and G_1 to sit next to each other, which is $2 \cdot 6!$. Subtracting that from arranging without restrictions, the total number of ways is $\boxed{7! - 2 \cdot 6!}$

(iii) We first arrange all the 5 boys, which is $4!$ ways. There are 5 spaces between each boy, so we can choose 3 of the seats and then arrange the 3 girls, $\binom{5}{3} \cdot 3!$. The total number of ways is $\boxed{4! \cdot \binom{5}{3} \cdot 3!}$

■

One possible "bogus solution" to this last problem:
Solution Arrange the boys in a line in 5! ways, choose three of the boys to put girls to the left of, and arrange the girls in 3! ways. *We then divide by 8 in order to account for the cyclicity of the table.* ■
Recall, however, the **reason** we divided by 8 to begin with is to divide by the size of each of the equivalence groups in the original problem when arranging people around the table. There are **not** 8 elements in each of these equivalence groups - cyclical arrangements with a girl on the very right is not part of set that we counted. In fact, there are only 5.

2.5 Recursion

Another way to count a set is to recursively generate it from smaller cases. This is a very useful general strategy to compute a quantity.

Exercise 1. Find the recursive definition of $P(n,r)$.

Solution We know that the closed form of $P(n,r) = \frac{n!}{(n-r)!}$. Our goal is to define $P(n,r) = f(P(<n, <r))$.
Let $S = \{s_1, ..., s_n\}$, r be given $0 \le r \le n$, and T be the set of all r-permutations of S. We can partition T into $T = T_1 \cup T_2$ where

$$t = T_1 \Leftrightarrow s_1 \notin t \text{ (no } s_1)$$
$$t = T_2 \Leftrightarrow s_1 \in t \text{ (yes } s_1)$$

We can find the $|T_1|$ in terms of $P(\le n, \le r)$

$$|T_1| = P(n-1, r)$$
$$|T_2| = r \cdot P(n-1, r-1)$$

For $|T_2|$, we can order $r-1$ elements from $\{s_2, ..., s_n\}$ and place s_1 in any of the r locations. Thus, $\boxed{P(n,r) = P(n-1,r) + r \cdot P(n-1,r-1)}$
■

Exercise 2. Find a recursive definition of $C(n,r)$

Solution Again, let T be the subset of $S = \{s_1, ..., s_n\}$ of size r.
To find $|T|$, let $T = T_1 \cup T_2$ where T_1 has no set containing s_1 and T_2 has every set containing s_1.

$|T_1| = C(n-1, r)$ we can choose r from $s_2, ...s_n$
$|T_2| = C(n-1, r-1)$ we choose $r-1$ from $s_2, ...s_n$ and add in s_1

Thus, $\boxed{C(n,r) = C(n-1,r) + C(n-1,r-1)}$ ■

Problem 1. Given $2n$ tennis players. How many ways are there to arrange n games/pairings?

Solution There are several solutions to this problem:

1. We can match P_1 with another $2n-1$ players. For the next player P_2 who hasn't been matched, we can choose $2n-3$ players, and so on. The solution is just $(2n-1)(2n-3)...(1) = \boxed{(2n-1)!!}$
2. We can choose each pair and divide by $n!$ to remove the ordering of the pairs. There are $\boxed{\dfrac{\binom{2n}{n}\binom{2n-2}{2}...\binom{2}{2}}{n!}}$ ways.
3. We can permute all $2n$ players and divide by $n!$ (the number of ways to order the game doesn't matter) and 2^n (the order of the partners doesn't matter). There are $\boxed{\dfrac{(2n)!}{n!2^n}}$

■

2.6 Interlude: Points in the Plane

Here is a difficult but interesting problem from the International Mathematical Olympiad that uses the counting techniques we have encountered thus far, but in a creative manner:

Problem 1. (IMO 1989/3) Given $n, k \in \mathbb{Z}^+$, let S be a set of n points in the plane such that no three points of S are collinear, and for any point P of S there are at least k points of S equidistant from P. Prove that

$$k < \frac{1}{2} + \sqrt{2n}$$

Proof. Draw an edge between points P and Q if P and Q are one of the k points that are equidistant from another point O. We will count all such edges between all pairs of points. One way to do so is simply choose any two of the points as an upper bound, which can be done in $\binom{n}{2}$ ways.
The other way we can do so is by counting edges among the points equidistant from a point P. We can compute all pairs of edges between these equidistant points, which gives at least $\binom{k}{2}$ edges. Across all points, this gives at least $n\binom{k}{2}$. However, we have overcounted - notice that at a maximum, each edge can represent an intersection between two circles representing equidistant sets of points around each point. If this edge was the intersection of three

or more of these circles, this would give at least three points in S as collinear, which is forbidden. This means that we have overcounted by at most $\binom{n}{2}$. This method of counting may not have accounted for all the edges (i.e. it's a lower bound). Combining these two gives the inequality

$$\binom{n}{2} \geq n\binom{k}{2} - \binom{n}{2}$$

Expanding, we have:

$$\begin{aligned} 2 \cdot \frac{n(n-1)}{2} &\geq \frac{nk(k-1)}{2} \\ n - 1 &\geq \frac{k^2 - k}{2} \\ n \geq 1 + \frac{4k^2 - 4k}{8} &> \frac{4k^2 - 4k + 1}{8} \\ 8n &> (2k-1)^2 \\ 2\sqrt{2n} &> 2k - 1 \\ k &< \frac{1}{2} + \sqrt{2n} \end{aligned}$$

as desired.

■

3 Combinatorial Sequences

3.1 Binomial Coefficients

Recall we derived a recursive definition of combinations (ie. binomial coefficients) in the last lecture:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

This is also known as **Pascal's Identity**.
We now prove the famed **Binomial Theorem**:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

While proving this fact, we will extend the definition of the combination (or binomial coefficient) as follows:

$$\binom{n}{k} = \begin{cases} \frac{n^{\underline{k}}}{k!} & 0 \le k \le n \\ 0 & k > n, k < 0 \end{cases}$$

We will prove this in two ways - first by induction and then by a simple combinatorial argument.

Proof 1. First, let us consider the trivial base case for $n = 0$, for which $(x+y)^0 = 1$ which is easily true, as $\binom{0}{0} = 1$.
For the inductive step, we can consider $(x+y)^{n+1}$ as $(x+y)(x+y)^n$, and apply the Binomial Theorem to the $(x+y)^n$ term. We expand:

$$\begin{aligned}(x+y)(x+y)^n &= (x+y)\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k} \\ &= \sum_{k=0}^{n}\binom{n}{k}x^{k+1}y^{n-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k} \\ &= \sum_{k=1}^{n+1}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=0}^{n}\binom{n}{k}x^k y^{n+1-k}\end{aligned}$$

where we shift the indices of the first sum back so the exponents of the x and y terms match up. We will now extend the bounds of each of the sums by one in order to combine them together into one sum - this is legal, as the terms that we are adding are zero by our extended definition of the binomial coefficient.

$$\sum_{k=0}^{n+1}\binom{n}{k-1}x^k y^{n+1-k} + \sum_{k=0}^{n+1}\binom{n}{k}x^k y^{n+1-k}$$

Now we finish using Pascal's Identity:

$$\sum_{k=0}^{n+1}\left(\binom{n}{k-1} + \binom{n}{k}\right) x^k y^{n-k} = \sum_{k=0}^{n+1}\binom{n+1}{k}x^k y^{n-k}$$

■

Proof 2. We can write the product $(x+y)^n$ out explicitly:

$$(x+y)^n = \overbrace{(x+y)(x+y)\dots(x+y)}^{n}$$

Notice that the term $x^k y^{n-k}$ appears $\binom{n}{k}$ times, which immediately implies the desired as we sum over all possible k.

■

Reflection A. Notice how much cleaner the combinatorial argument was! We will appeal to such arguments again and again as the course progresses.

What if we want to expand something like $(1-x)^{\frac{1}{2}}$ binomially? This forces us to define binomial coefficients when the top argument is a real α - and furthermore, we have to define them in such a way so that it doesn't involve factorials of α (which aren't well defined). We can define as follows for $\binom{\alpha}{n}$ if n is an integer, but α can be any real:

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!} = \frac{\alpha^{\underline{n}}}{n!}$$

Once this is done, we can simply apply the Binomial Theorem, which turns out still to be true with this definition, except the sum does not terminate (left to the reader as an exercise):

$$\begin{aligned}(1-x)^{\frac{1}{2}} &= \sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}(-x)^k \\ &= 1 + \frac{\frac{1}{2}}{1!}(-x)^1 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-x)^3 + \dots \\ &\implies (1-x)^{\frac{1}{2}} = 1 - \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(n+1)!2^{n+1}}x^{n+1}\end{aligned}$$

where we define $(-1)!! = 1$.
Recall we had this double factorial last class, and in fact we showed that $(2n-1)!! = \frac{(2n)!}{2^n n!}$ by solving the same combinatorial problem of creating games in a tournament. Applying this, we obtain

$$\begin{aligned}(1-x)^{\frac{1}{2}} &= 1 - \sum_{n=0}^{\infty}\frac{(2n)!}{(n!)(n+1)!2^{n+1}2^n}x^{n+1} \\ &= 1 - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)(n+1)!4^n}x^{n+1} \\ &= 1 - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(2n)!}{(n!)^2(n+1)4^n}x^{n+1} \\ &= 1 - \frac{1}{2}\sum_{n=0}^{\infty}\binom{2n}{n}\frac{1}{n+1}\cdot\frac{1}{4^n}x^{n+1}\end{aligned}$$

Remark. *The term $\binom{2n}{n}\frac{1}{n+1}$ is the nth **Catalan number**. We will see these numbers again in the near future.*

3.2 Multisets and Problems with Repetition

We will now start to look at problems where we can have repetition of elements. Consider the following problems:

Problem 2. Given the elements of the set $\{a, b, c, d\}$, how many ways are there to construct a sequence of length 7 with repetition allowed?

Solution. For each element in the sequence, we can choose any element in the set, giving $\boxed{4^7}$ possibilities.

■

Problem 3. Given the multiset $\{3 \cdot a, 2 \cdot b, 5 \cdot c\}$, how many unique sequences are there that use all the letters once?

Solution. If we first label the as, bs, and cs so that they are distinguishable, we can arrange all the letters in $10!$ ways to begin with. However, since all of the as are indistinguishable, permuting the order of the labeled as produces an equivalent arrangement - so we must divide by a factor of $3!$. We must do the same thing for the bs and cs, giving $\boxed{\dfrac{10!}{3!2!5!}}$ sequences.

■

Problem 4. Given the multiset $\{\infty \cdot a, \infty \cdot b, \infty \cdot c, \infty \cdot d\}$, how many ways can one choose a sub-multiset of size 7?

Solution. Notice that we only care about the quantities of each of the letters a, b, c, d, so we can consider the equivalent problem of finding the number of positive-integer solutions to the equation $w + x + y + z = 7$. This can be counted by considering the isomorphic problem of counting binary strings of length 10 with three 1s. To see why this problem is the same as counting solutions to the equation, notice that for every such binary string, we can let the number of 0s to the left of the first 1 be the value of w, the number of 0s between the first and second 1s be the value of x, etc. This uniquely maps this binary string to a solution to this Diophantine equation, and similarly every solution to the equation gives a binary string. We can count the strings easily - there are $\boxed{\binom{10}{3}}$ of these strings, which is the answer to the original problem as well.

■

We can generalize this last problem: with the same reasoning, we find the number of ways to build a size-r multiset from n possible elements is $\binom{n-1+r}{n-1} = \binom{n-1+r}{r}$.
We now define some additional notation for these kinds of problems:

- We denote the number of multisets into a sequence as a *multinomial coefficient*:

$$\binom{n}{r_1, r_2, \ldots r_k} = \frac{n!}{r_1!r_2! \ldots r_k!}$$

  where $n = \sum_i r_i$.

- The number of multisets of length r constructed from n possible elements (colloquially known as the *stars and bars* problem) is denoted as

$$\left(\binom{n}{r}\right) = \binom{n-1+r}{n-1} = \binom{n-1+r}{r}$$

  For example, $\left(\binom{3}{7}\right) = \binom{3-1+7}{2} = \binom{9}{2} = 36$. Note that unlike binomial coeffients, n need not be less than r!

3.3 Partitions and Stirling Numbers

Problem 1. How many ways can a set of 4 elements be partitioned into 2 non-empty sets?

Solution: We perform casework. Here are the ways:

$$1\,|\,2\,3\,4 \quad 2\,|\,1\,3\,4 \quad 3\,|\,1\,2\,4 \quad 4\,|\,1\,2\,3$$

$$1\,2\,|\,3\,4 \quad 1\,3\,|\,2\,4 \quad 1\,4\,|\,2\,3$$

■

Definition. *The number of ways to partition a set of n elements into k non-empty subsets is the **Stirling Number of the second kind** $S(n,k)$ or* $\left\{ n \atop k \right\}$.

Let's construct a few base cases:

$$\begin{aligned} &S(1,1) = 1 \\ &S(n,1) = 1 \\ &S(n,n) = 1 \\ &S(0,0) = 1 \quad \text{(defined as such)} \\ &S(n,2) = 2^{n-1} - 1 \\ &S(n,n-1) = \binom{n}{2} \end{aligned}$$

The first four of these are fairly trivial to see, but the last two are not.

Proposition A. $S(n,2) = 2^{n-1} - 1$

Proof. Consider $A = \{0,1\}^n$. This will be the characteristic function of membership in the two sets S_0, S_1. Note that $|A| = 2^n$, but 1^n and 0^n are not valid partitions (since they have an empty set). So we have $|A| = 2^n - 2$ valid mappings. However, since we are double counting because the identity of the subsets are irrelevant, we divide by 2 to yield $2^{n-1} - 1$.

■

Proposition B. $S(n,n-1) = \binom{n}{2}$.

Proof. Choose two elements from the n that will be in the same set, and each of the rest of the elements must be in its own set.

■

With these base cases in mind, we can look at constructing a recursive formula for $S(n,k)$. We can start by looking at the partitions of an $n-1$-element set into $k-1$ elements - and the nth element must go into a new set by itself. Alternatively, we can consider $S(n-1,k)$, or the partitions of an $n-1$-element set into k-elements, and we must pick which one of the sets the last element must go into in k ways. This gives us the recursive formula

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

We can now begin computing values of the Stirling numbers of the second kind:

$n\backslash k$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	3	1	0	0	0
4	1	7	6	1	0	0
5	1	15	25	10	1	0
6	1	31	90	65	15	1

3.4 Revisiting Balls and Urns

We actually now have most of the information we need to understand the twelvefold way! If we have b balls and u urns, here are the number of ways to put the balls in the urns (subject to either no restrictions, or putting at least 1 in each/at most 1 in each):

B	U	$\varnothing$	≤ 1 per	≥ 1 per
labeled	labeled	u^b	$u^{\underline{b}}$	$u!\left\{ {b \atop u} \right\}$
unlabeled	labeled	$\left(\!\binom{u}{b}\!\right)$	$\binom{u}{b}$	$\left(\!\binom{u}{b-u}\!\right)$
labeled	unlabeled	$\sum_{n=1}^{u} \left\{ {b \atop n} \right\}$	$[b \leq u]$	$\left\{ {b \atop u} \right\}$
unlabeled	unlabeled	?	$[b \leq u]$	?

A few notes on some of these cases:

- unlabeled balls into labeled urns, ≥ 1 per: fill the urns with at least one ball each first, then do the multisets. This also has the name of a "composition of b into u parts."
- $[P]$ means 1 if P is true, and 0 otherwise. This appears for labeled balls unto unlabeled urns, ≤ 1 per urn – either we can do it, or there are no ways to do it. Same with unlabeled balls into unlabeled urns.

3.5 More on Stirling Numbers of the Second Kind

Recall one of the problems from the first day involving putting labeled balls into labeled urns, with the condition that there is at least one ball in each urn (but for consistency, we will use n nuggets and k kettles). For n nuggets and k kettles, we partitioned the n nuggets into k sets in $S(n,k)$ ways, and then labeling the sets in $k!$ ways, giving a total of $k! \cdot S(n,k)$, where $S(n,k)$ is a *Stirling number of the second kind*.

We can actually solve this problem in an alternative method is by using the Principle of Inclusion-Exclusion. We can first count the total number of ways we can place the nuggets into the kettles with no restrictions, and then remove the possibilities that have various numbers of empty kettles. Specifically, we subtract off the possibilities of placing n nuggets into (at most) $k-1$ kettles, and then adding back possibilities of placing n nuggets into $k-2$ kettles, as they have been subtracted off one extra time, etc. We can continue in this manner until we have added/subtracted back the possibility of n nuggets into zero kettles, which concludes all the possible cases.

Let's construct a specific expression for each of these cases. Notice that if we limit ourselves to j kettles to put the nuggets into, we only have j options for each nugget, so we only have j^n possibilities. We also have to account for the number of ways we can choose these kettles, which can be done in $\binom{k}{j}$ ways.

This argument leads to the following equivalent expression for the number of ways we can place these nuggets in the kettles:

$$k^n - \binom{k}{k-1}(k-1)^n + \binom{k}{k-2}(k-2)^n - \ldots + (-1)^k \binom{k}{0} 0^n = k!S(n,k)$$

or more compactly,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n = k!S(n,k).$$

This gives us the following closed form for the Stirling numbers of the second kind:

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

3.6 Stirling Numbers of the First Kind

With this (or the recurrence relation from last class) we can fill in the following table of Stirling numbers of the second kind (with blank entries taken to be 0). Notice how we can run the recurrence relation backwards to get the entries in the top left:

n, k	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7
-7	1														
-6	21	1													
-5	175	15	1												
-4	735	85	10	1											
-3	1624	225	35	6	1										
-2	1764	274	50	11	3	1									
-1	720	120	24	6	2	1	1								
0								1							
1									1						
2									1	1					
3									1	3	1				
4									1	7	6	1			
5									1	15	25	10	1		
6									1	31	90	65	15	1	
7									1	63	301	350	140	21	1

The numbers in the upper triangle are the **unsigned Stirling numbers of the first kind**, denoted $\left[{n \atop k}\right]$. Some authors may or may not multiply these numbers in the top left triangle by $(-1)^{n+k}$, giving the **signed Stirling numbers of the first kind** $s(n,k)$. We will use the following theorem (that we will prove later) as a "definition" for the unsigned Stirling numbers of the first kind:

Theorem 3.1 (Stirling Numbers of the First Kind)**.** *For $n, k \in \mathbb{Z}$, we have:*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{Bmatrix} -k \\ -n \end{Bmatrix}$$

This follows straight from the table that we developed, but we will develop these numbers more rigorously from a recursion relation and then prove this identity later. For now, we will investigate some interesting properties relating the Stirling numbers of the first and second kinds.

To begin, consider the expansions of the falling factorials:

$$\begin{aligned}
x^{\underline{1}} &= 1x \\
x^{\underline{2}} &= x(x-1) = 1x^2 - 1x \\
x^{\underline{3}} &= x(x-1)(x-2) = 1x^3 - 3x^2 + 2x \\
x^{\underline{4}} &= x(x-1)(x-2)(x-3) = 1x^4 - 6x^3 + 11x^2 - 6x \\
&\vdots
\end{aligned}$$

Notice that the coefficients of the polynomials are the (signed) Stirling numbers of the first kind! To be explicit, $(-1)^{n+k}\left[{n \atop k}\right]$ or $s(n,k)$ is the coefficient of the x^k term in the expansion of the falling factorial. We can also express polynomial terms x^n in terms of the falling factorials:

$$\begin{aligned}
x &= 1x^{\underline{1}} \\
x^2 &= 1x^{\underline{2}} + 1x^{\underline{1}} \\
x^3 &= 1x^{\underline{3}} + 3x^{\underline{2}} + 1x^{\underline{1}} \\
x^4 &= 1x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + 1x^{\underline{1}} \\
&\vdots
\end{aligned}$$

From this we see that the coefficient of $x^{\underline{k}}$ in the falling-factorial expansion of x^n is $S(n,k)$.

On a seemingly unrelated note, we can consider lower triangular $n \times n$ matrices built from these numbers, $S_1(n)$ and $S_2(n)$, such that for each entry in these matrices, $S_1(n)_{ij} = s(i,j)$ and $S_2(n)_{ij} = S(i,j)$. Below are $S_1(4)$ and $S_2(4)$:

$$S_1(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix} \quad S_2(4) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix}$$

Consider what happens when we take $S_1(4)S_2(4)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ -6 & 11 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 7 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

In general, the matrices $S_1(n)$ and $S_2(n)$ multiply to I_n, meaning that $S_1(n)$ and $S_2(n)$ are inverses of each other.

What do these two facts have to do with each other? At a deeper level, the Stirling numbers are the coefficients of the coordinate transformations between the basis of polynomial terms and basis of falling factorials. That is, we can use the Stirling numbers of the first kind to transform "polynomials" expressed in terms of the falling factorials into polynomials in the normal sense, and similarly we can use the Stirling numbers of the second kind to transform polynomials into corresponding "polynomials" in terms of falling factorials. The matrix discussion above reflects this property of the Stirling numbers - the triangular matrices $S_1(n)$ and $S_2(n)$ being inverses is no coincidence, as it shows how these transformations are inverses of each other.

This discussion gives us three identities involving S and s:

1. $$x^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$$

2. $$x^{\underline{n}} = \sum_{k=1}^{n} (-1)^{n+k} \left[{n \atop k} \right] x^k$$

3. $$\sum_{k=1}^{\max(n,m)} S(n,k)s(k,m) = \delta_{mn}$$

.

where δ_{mn} is the *Kronecker delta*, defined as follows:

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

3.7 Combinatorics of the Stirling Numbers of the First Kind

Let's look back at our recurrence relation for the Stirling numbers of the second kind:

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

Note the similarity between this and the other two recurrence relations we have seen thus far for permutations and combinations:

$$\text{Permutations: } P(n,k) = kP(n-1,k-1) + P(n-1,k)$$
$$\text{Combinations: } \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Let us construct another recurrence relation in a similar form to $f(n,k) = af(n-1,k-1) + b(n-1,k)$, and see what the function counts. Consider the recurrence relation

$$f(n,k) = f(n-1,k-1) + (n-1)f(n-1,k).$$

Proposition 1. The number of ways $f(n,k)$ that a set of n elements be arranged into k cyclic permutations follows the above recurrence relation.

Proof. Suppose we attempt to add an nth element to some set of cyclic permutations of $n-1$ elements. If this element forms its own cyclic permutation, this can be accomplished if the other elements are divided in $f(n-1,k-1)$ ways into these cyclic permutations. Otherwise, the nth element must go into an already-existing cyclic permutation, and we can arrange the $n-1$ elements into k cyclic permutations in $f(n-1,k)$ ways. We can insert the nth element directly after any of the existing $n-1$ elements in their cyclic permutations, for a total of $(n-1)f(n-1,k-1)$ ways. Since we are counting the same quantity, we must have $f(n-1,k-1) + (n-1)f(n-1,k) = f(n,k)$.

■

Remark. *We can write a permutation π in the following way (for 7 elements, for example)*

$$(14)(256)(37)$$

where (256) denotes a cyclic permutation that sends 2 to 5, 5 to 6, and 6 to 2. Similarly, we can see this permutation swaps 1 and 4, etc. In general, the following is true regarding ***any*** *permutation:*

Theorem 3.2. *Any permutation can be written as a product of cyclic permutations.*

We will not prove this rigorously, but we can sketch out a constructive proof by considering a permutation π of the set $\{1, 2, 3, \dots n\}$ as a one-to-one mapping of this set of n elements to itself. We can construct the cyclic permutations that comprise this permutation by "following the mapping for each element." To be clear, we can begin by considering $\pi(1) = c$, and then look for $\pi(c)$, etc. and at every step, we find the element that the previous element we found maps to. If this ever maps back to 1, we have a cyclic permutation - and if there are elements in the permutation that we haven't visited, we can repeat the process on the remaining elements starting at the lowest unvisited element. This is by no means a rigorous proof of the result, but it should give you an intuitive feel for why this is true.

We can now construct a few easy base cases for the function f:

$$f(1, 1) = 1$$
$$f(n, 1) = (n - 1)!$$

The last is a problem that we've already encountered before, and the first is pretty easy to see. Here is a table of small values of $f(n, k)$:

n, k	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	11	6	1	
5	24	50	35	10	1

Notice that $f(n, k)$ are the Stirling numbers of the first kind! In particular, this means that the Stirling numbers of the first kind count the number of ways a set of n elements can be arranged into k cyclic permutations.

Armed with this recursive definition (and the ability to run it backwards in a similar fashion), we can now show the theorem that we took to be as a definition, that is, $\left[{n \atop k}\right] = \left\{{-k \atop -n}\right\}$:

Proof. The statement is true whenever $k = 0$ or $n = 0$ - it can be easily shown that these are all 0, except when both are, in which case $\left[{0 \atop 0}\right] = \left\{{0 \atop 0}\right\} = 1$. Note that if we assume the statement is true for all $\left[{x \atop y}\right]$ such that $0 \leq x < n$ (employing the principle of strong induction), we have:

$$\begin{aligned}
\begin{bmatrix} n \\ k \end{bmatrix} &= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} \\
&= \begin{Bmatrix} 1-k \\ 1-n \end{Bmatrix} + (n-1)\begin{Bmatrix} -k \\ 1-n \end{Bmatrix} \\
&= \begin{Bmatrix} -k \\ -n \end{Bmatrix} + (1-n)\begin{Bmatrix} -k \\ 1-n \end{Bmatrix} + (n-1)\begin{Bmatrix} -k \\ 1-n \end{Bmatrix} = \begin{Bmatrix} -k \\ -n \end{Bmatrix}
\end{aligned}$$

where the last step follows from the recurrence relation for the Stirling numbers of the second kind. With a similar argument for the negative integers, by induction, the statement is therefore true for all $n, k \in \mathbb{Z}$.

■

With the remark above, we can also show the identity:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$$

Proof. The left hand side is adding up all the possible ways we can split a set of n elements into k cyclic permutations. From the remark, since every permutation can be written as one and only one product of some number of cyclic permutations from 0 to n, we have that this sum must be equal to the total number of permutations of n elements, or $n!$.

■

3.8 Triangulations (Catalan Numbers I)

Q: How many triangulations are there of a labeled hexagon? (This means that although two triangulations may be the same under rotation, which vertices they are attached to matter).
Let us first examine the general case of an n-gon and look at the behavior of the first few [1].

Figure 1: Triangulations for 4, 5, and 6 sided polygons.

[1] http://mathworld.wolfram.com/CatalanNumber.html

The rules for triangulation are only such that there are no crossings between the lines. Note: a triangulation of an n-gon uses n-3 line segments and results in n-2 interior triangles.
As we can see from Figure 1, the answer to our original question is 14, but let us instead try to come up with a general formula for $T_n =$ number of ways to $\triangle$ a n-gon.

Note: The following graphics will be done in Microsoft Paint. I'm sorry in advance.

Theorem 3.3.

$$T_n = \sum_{k=2}^{n-1} T_{n-(k-1)} \cdot T_{k-1}$$

Proof. WARNING: THIS WAS THE PROOF THAT WAS SHOW IN CLASS THAT HAD AN ERROR IN IT. I still think it is worth to look it over and see why it doesn't work. Just to clarify, although this proof does give the correct recursion, it is not correct. To my knowledge, it is nothing short of a miracle that this method accidentally over-counts and under-counts solutions in such a way to produce a correct recursion.

Figure 2: Recursion for a hexagon.

This hexagon could be broken up into 3 different shapes like this, which would result in $T_6 = T_3 \cdot T_5 + T_4 \cdot T_4 + T_5 \cdot T_3$. This obviously satisfies our original theorem, and all n-gons could be split like this, but this does over-count, because it does not account for when splitting up the remaining polygons you end up with the same triangulation. That is the reason why this proof does not work well for formulating the Catalan numbers.

∎

Proof. Instead of splitting the polygon with a line, which allows for over-counting, if it is instead split with a triangle, we are guaranteed that each solution found is unique, and from looking at the diagram below, we see that the sum of each of these will give us the correct number.

∎

Figure 3: Correct recursion for a hexagon.

3.9 Triangulating Polygons (Review of Catalan Number Proof)

Proof. The number of ways to triangulate a hexagon, T_6, can be found by recursion through dividing the interior of the hexagon with a triangle. After picking the triangle in one of 4 ways, the number of ways to triangulate the remaining area is either T_5T_2 or T_4T_3, depending on the triangle picked. This means we can say $T_6 = T_5T_2 + T_4T_3 + T_3T_4 + T_2T_5$. We can shift the indices here to get the recursive definition of Catalan numbers using the following:
$T_{n+1} = T_2T_n + T_3T_{n-1} + ...T_nT_2$
$T_{n+1} = \sum_{j=2}^{n} T_jT_{n+2-j}$
Let $T_{n+1} = C_n$
$T_{n+2} = \sum_{j=2}^{n+1} T_jT_{n+3-j}$
$C_n = \sum_{j=2}^{n+1} C_{j-2}C_{n+1-j}$
$C_{n+1} = \sum_{j=2}^{n+1} C_{j-2}C_{n+2-j}$
$C_{n+1} = \sum_{j=1}^{n} C_jC_{n-j}$ This is the recursive definition of Catalan numbers, so we can conclude that there are C_n ways to triangulate an n+2-gon.

∎

3.10 Paths (Catalan Numbers II)

Q: How many paths are there between the points (0, 0) and (2n, 0), only moving with steps of either northeast (up one right one) or southeast (down one right one).
(Note: This is analogous to the question of paths between (0, 0) and (n, n) with only vertical and horizontal steps except the algebra works better in this one. However, the pictures I can find are using this interpretation of the problem, so unless we want more Microsoft Paint pictures, these will do. If at any point you get confused between what we did in class and the pictures I'm showing, turn your head $45°$ to the left and it will make sense again.)
A: $\binom{2n}{n}$.

Proof. There are 2n moves, n of them must be up, n must be down, so choose which n will go southeast and we are done.

■

Figure 4: An example of a path.

Now we want to count all of the paths that do not cross the x-axis, as the one in Figure 4 does. I'm going to have to use paint for this as I cannot find pictures. Sorry again.

In order to do this, we first notice that all bad paths (ones that cross the x axis), touch the line y = -1 at least once. To count them, we will reflect everything BEFORE the first y = -1 touch along the line y = -1, and leave everything after it the same. (This is why we do it this way instead of diagonally because the reflection is much easier to visualize). This creates a mapping of all bad paths to something else. This is a one to one mapping, every bad path has a reflection, and every reflection has only 1 bad path associated with it. Now, by counting the reflections we also count bad paths. Since the bad paths start at y = -2, and need to reach (2n, 0), we have to choose n+1 up moves (note that this is because this will also result in one less down move, leaving us two above our initial position).
So bad paths = $\binom{2n}{n-1}$

This gives our total amount of good paths to be the following

$$P = \binom{2n}{n} - \binom{2n}{n+1}$$

$$\frac{(2n)!}{n! \cdot n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!}{n! \cdot n!} \cdot (1 - \frac{n}{n+1})$$
$$= \frac{1}{n+1}\binom{2n}{n}$$

Which is the Catalan numbers (note how by splitting up the original path problem into two smaller ones, we can also achieve the recursion relation for the Catalan numbers, like we did with the hexagons).

Figure 5: The transformation applied to all bad paths

4 Interlude: Stirling's Approximation

We intend to examine different ways that we may approximate the value of $n!$. The first of these methods is Stirling's Approximation.

Theorem 4.1. *(Stirling's Approximation)* $\lim_{n\to\infty} \frac{n!}{(\frac{n}{e})^n * \sqrt{2\pi n}} = 1$

This is the formal definition of Stirling's Approximation. The actual approximation that we are observing, would be the denominator of this fraction. Stirling's thinking behind this, is that as n increases in size, the ratio should eventually become equal to 1.
We won't however, be proving the formal definition of Stirling's Approximation. Instead, we want to find some sort of upper and lower bound for the value of $n!$. To do this, we will first create an informal definition for $n!$.

Informally speaking, $n! \sim \frac{n}{e}^n * \sqrt{2\pi n}$
We intend to prove that $n! = \frac{n}{e}^n * \delta\sqrt{n}$, where $\delta \cong \sqrt{2\pi}$

To do this proof, we also want to make note of an important summation remark:

Remark. $log(n!) = \sum_{i=1}^{n} log(i)$

Proof. To do the actual proof, we will consider the function $y = ln(x)$. From here we intend to make two Riemann-style approximations for the integral $\int_i^n ln(x)dx$. For our proof, both approximations will be trapezoidal approximations.

Approach one is two use a trapezoid whose bases are at the integers themselves.

Area of the trapezoids is $\frac{1}{2}\sum_{i=1}^{n-1}[ln(i) + ln(i+1)]$
$= \frac{1}{2}ln(1) + \sum_{i=2}^{n-1}[ln(i) + ln(i+1)] + \frac{1}{2}ln(n)$
$= ln(n!) - \frac{1}{2}ln(n)$

Approach two is two use a trapezoid whose bases are at the midpoints between integers. Notice that we don't care about the area before $x = \frac{1}{2}$ in our calculations.

Area = $ln(2) + ln(3) + ln(4) \cdots + ln(n-1) + \frac{1}{2}ln(n)$, which comes out to be the same as above. Notice that this is an overestimate however, and the previous approach was an underestimate.
$\therefore \int_{\frac{3}{2}}^{n} ln(x)dx + \frac{1}{2}ln(n) < ln(n!) < \int_{1}^{n} ln(x)dx + \frac{1}{2}ln(n)$

Given $\int ln(x)dx = xln(x) - x + C$

We get:
$(nln(n) - n) - (\frac{3}{2}ln(\frac{3}{2}) - \frac{3}{2}) + \frac{1}{2}ln(n) < ln(n!) < (nln(n) - n) - (ln(1) - 1) + \frac{1}{2}ln(n)$
$= (n + \frac{1}{2})ln(n) - n - \frac{3}{2}(ln(\frac{3}{2}) - 1) < ln(n!) < (n + \frac{1}{2})ln(n) - n + 1$

At this point, we have a range of error for $ln(n!)$.

We can define $ln(n!) = [(n + \frac{1}{2})ln(n) - n] + \delta_n$, where $\frac{3}{2}(1 - ln(\frac{3}{2})) < \delta_n < 1 \longrightarrow 0.891802 < \delta_n < 1$

Since $ln(n!) = (n + \frac{1}{2})ln(n) - n + \delta_n$,
$e^{ln(n!)} = e^{nln(n)}e^{\frac{1}{2}ln(n)}e^{-n}e^{\delta_n}$
$n! = n^n\sqrt(n)e^{-n}e^{\delta_n}$
$= (\frac{n}{e})^n\sqrt(n) * e^{\delta_n}$, where $e^{\delta_n} \in [2.439, 2.718]$. The value for $\sqrt{2\pi n} = 2.506$, which falls within our expected range of values.

$\blacksquare$

Now that we have an approximation for $n!$, we can look to write an expression that will give us the n^{th} Catalan number. The n^{th} Catalan number is given by the expression $\binom{2n}{n} * \frac{1}{n+1}$
$\binom{2n}{n} * \frac{1}{n+1} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{[(\frac{n}{e})^n * \sqrt{2\pi n}]^2}$
$= \frac{\frac{2n}{e}^{2n}}{\frac{n}{e}^{2n}} * \frac{\sqrt{4\pi n}}{2\pi n*(n+1)} = \frac{2^{2n} * \sqrt{\pi n}}{\pi n(n+1)} = \frac{4^n}{\sqrt{\pi n}(n+1)}$

4.1 Gamma Function

Gamma functions are an extension of factorials, such that the arguments are shifted down by one. In other words, $\Gamma(2) = 1!$, $\Gamma(3) = 2!$, etc. We intend to prove that $\Gamma(n+1) = n!$. To do this, we first begin by defining our gamma function.

Definition. $\Gamma(z+1) = \int_0^\infty x^z e^{-x}$

Now that we have defined our gamma function, we will attempt to prove that $\Gamma(n+1) = n!$.

Proof. First, we will integrate this function by parts:

- $u = x^7$, $du = zx^{z-1}$
- $v = -e^{-x}$, $dv = e^{-x}$

$\therefore \int_0^\infty x^z e^{-x} dx = [x^z(-e^{-x})]_0^\infty + \int_0^\infty zx^{z-1}e^{-x}dx$

Notice how when we evaluate the expression $\int_0^\infty x^z e^{-x}dx = [x^z(-e^{-x})]_0^\infty$, x^z becomes 0 when evaluated at $x = 0$, and $(-e^{-x})$ becomes zero when evaluated at $x = \infty$, meaning that this expression evaluates to just 0.

Therefore, we are left with just $\int_0^\infty zx^{z-1}e^{-x}dx$. We can redefine this as $z\int_0^\infty x^{z-1}e^{-x}dx = z\Gamma(z)$.

At this point, we have something that is similar to a factorial, but we can't consider our proof complete until we establish some base cases. We can begin by testing with $z = 1$. $\Gamma(1) = \int_0^\infty x^0 e^{-x}dx = \int_0^\infty e^{-x}dx = 1$
$\therefore \Gamma(1) = 1$
$\Gamma(2) = 1 * \Gamma(1) = 1$
$\Gamma(3) = 2 * \Gamma(2) = 3$
$\Gamma(4) = 3 * \Gamma(3) = 6$
$\therefore \Gamma(n+1) = n!$

∎ As we intended, we have now found a function that is similar to a factorial, withe the arguments shifted by one. However, we also want to observe how fractional factorials work. In our case, we'll observe $\Gamma(\frac{1}{2})$.

Proof. $\Gamma(\frac{1}{2}) = \int_0^\infty x^{\frac{-1}{2}} e^{-x}dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}$
Let $u = \sqrt{x}$, $u^2 = x$, and $du * 2u = dx$
Substituting u into the equation, we get $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-u^2}du$
Now suppose we define a variables v and K, such that
$K = \int_0^\infty e^{-u^2}du = \int_0^\infty e^{-v^2}dv$
$K^2 = \int_0^\infty e^{-u^2}du \int_0^\infty e^{-v^2}dv$
$= \int_0^\infty \int_0^\infty e^{-u^2}e^{-v^2}dudv = \int_0^\infty \int_0^\infty e^{-(u^2+v^2)}dudv$
In Cartesian coordinates, this integral would be difficult to solve.

However, we can convert to polar coordinates to solve this equation. Notice that because we are only using positive integers, θ is bounded within the first quadrant
$K^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} * r dr d\theta = \frac{\pi}{2} \int_0^{\infty} e^{-r^2} * r dr$
$= \frac{\pi}{2} * (\frac{-e^{-r^2}}{2})_0^{\infty} = \frac{\pi}{2}(\frac{1}{2} - 0) = \frac{\pi}{4}$
$\therefore K = \frac{\sqrt{\pi}}{2}$
$\therefore \Gamma(\frac{1}{2}) = 2K = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

■

5 Problems and Examples

5.1 Homework Problems (Problem Set 2)

Problem 1. In how many ways may we write n as a sum of an ordered list of k positive numbers? Such a list is called a composition of n into k parts.

Solution We can split n into n identical 1's. Then the problem becomes about dividing identical objects into k distinguishable piles with at least one in each pile to form ordered pairs of numbers which sum to n. That makes this stars and bars where we have k-1 bars (to divide the set into k-1 groups) and n-1 possible spaces (we want at least one 1 in each group). There are $\binom{n-1}{k-1}$ ways to do this. ■

Problem 2. What is the total number of compositions of n (into any number of parts).

Solution In this question, we have n-1 spaces between each 1 that we can make a "bar" that creates a new group. We have two groups that n-1 things can fall in, so our total number of ways to do that is 2^{n-1}. Alternatively, we could recognize that the total number of ways to do this is the sum of our answer from question one where k goes from 1 to n-1 ($\sum_{k=1}^{n-1} \binom{n-1}{k}$) matches the form of the sum in the binomial theorem ($(a+b)^j = \sum_{k=0}^{j} \binom{j}{k} a^{j-k} b^k$). In this case, a and b would both be 1 and j would be n-1, which would give us 2^{n-1}. ■

Problem 3. A Grey Code is an ordering of n-digit binary strings such that each string differs from the previous in precisely one digit.

Problem 3.1. Write one Grey Code for n=4.

1111, 1110, 1100, 1000, 0000, 0001, 0011, 0111, 0101, 1101, 1001, 1011, 1010, 0010, 0110, 0100.

Problem 3.2. Prove by induction that Grey Codes exist for all $n \geq 4$.

Let n=4 be the base case. Starting with a Grey Code for n=4, to create one for n=5, we can duplicate all the numbers in the Grey Code while keeping the same order and alternate between adding 1's and 0's in the front using the following pattern: 1, 0, 0, 1, 1, 0... This maintains the one-digit difference while listing all the 5-digit binary numbers (because each number listed is unique, since the list of 4-digit numbers was unique and either 0 or 1 is added only once to each number, and there are twice as many numbers in the list, just like there are twice as many 5 digit binary numbers as 4-digit ones). For example, using the earlier Grey Code, we get (11111, 01111, 01110, 11110...). This procedure can be repeated for 6-digit numbers, 7-digit numbers, and so on.

Problem 3.3. Prove that the number of even-sized subsets of an n-element set equals the number of odd-sized subsets of an n-element set.

The subsets of an n-element set can be mapped one-to-one to the strings in a Grey Code for n=n by having the 0's and 1's indicate whether or not to include each element of the original set. Since each string in a Grey Code differs from the previous string by exactly 1 digit, the strings must alternate between having even and odd numbers of 1's, indicating even or odd sized subsets. Since there are 2^n binary strings of length n, there are an even number of strings in each Grey Code, so the alternation between odd and even sized subsets must produce equal numbers of both.

Problem 4. A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example, (((()()))()) is balanced and ((()) and (()()))(() are not. How many balanced lists of n left and n right parentheses are there?

This is equivalent to the problem of counting the number of paths from (0, 0) to (2n, 0) using only northeast (1, 1) and southeast (1, -1) steps without crossing the x-axis. Adding left parentheses translates to moving northeast, and adding right parentheses translates to moving southeast. Their numbers must be equal in the end, and not crossing the x-axis ensures that the number of left parentheses is always at least as many as the number of right parentheses. To count the number of paths, we use complementary counting: there are $\binom{2n}{n}$ paths altogether, and $\binom{2n}{n+1}$ paths that cross the x-axis (which we can count by reflecting them across the line y=-1 to form paths from (0, -2) to (2n, 0)). This results in $\frac{\binom{2n}{n}}{n+1}$ acceptable paths and acceptable lists.

Problem 5. A tennis club has 4n members. To specify a doubles match, we choose two teams of two people. In how many ways may we arrange the members into doubles matches so that each player is in one doubles match? In how many ways may we do it if we specify in addition who serves first on each team?

We can start by arranging the 4n people into a line, which can be done in (4n)! ways. We can then assign them to courts by taking the first 4 people for the first court, the next 4 for the second, and so on. Since the order of the courts doesn't matter, we must divide by n!. Then, we must account for another overcounting issue: we have 8 different ways to specify the same set of 2 teams for each court. To account for this issue, we must divide by $2^{(3n)}$, since there are n courts. Altogether, this gives us $\frac{(4n)!}{n! \times 2^{3n}}$ ways.

Problem 6. A town has n streetlights running along the north side of main street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?

We can start by picking an even number of the n poles to be green: there are $\binom{n}{2k}$ ways to do this, where k ranges from 0 to $\lfloor n/2 \rfloor$ There are 3 ways to color each of the remaining n-2k poles, meaning 3^{n-2k} ways altogether. This means the total number of ways to color these streetlights is $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} * 3^{n-2k}$. We can get the closed form of this

by adding the binomial expansions of $(3+1)^n$ and $(3-1)^n$ and dividing by 2 to get the sum. This gives $2^{2n-1} + 2^{n-1}$.
Alternatively...
Preview of formulation 3:
Let E_n=number of ways to color n poles with an even number of greens, and O_n=number of ways to color n poles with an odd number of greens, so that E_n+O_n=the total number of ways to color n poles.$E_{n+1} = 3E_n + O_n$: Adding one more pole to a set of n poles such that we end up with an even number of green poles can be done in only one way if there are an odd number of green poles in the first set of n poles (the pole must be green) and can be done in 3 ways if there are an even number of green poles in the first n poles (the pole can be any color but green).

Problem 7. We have n identical ping pong balls. In how many ways may we paint them red, white, blue, and green if we use green paint on an even number of them?

There must be 2k balls painted green, where k ranges from 0 to $\lfloor n/2 \rfloor$ in order to ensure that the number of green balls is even. There are 3 ways to color each of the remaining n-2k balls. We can imagine this as a multiset problem, where there are n copies of each of the 3 colors. This means there are $\sum_{k=0}^{\lfloor n/2 \rfloor} \left(\!\binom{3}{n-2k}\!\right)$ ways to color the balls altogether.

Problem 8. A boolean function $f : (0,1)^n \rightarrow$0, 1 is self-dual if replacing all 0s with 1s and 1s with 0s yields the same function. How many self-dual boolean functions are there as a function of n?

We can regard the function's inputs to be in pairs, where switching the 0s and 1s of one member of the pair results in the other member. To ensure that the function is self-dual, both members of a pair must give the same result when inputted into f, meaning that there are $2^n/2$ distinct inputs. Since each input can be mapped to 0 or 1, this means the number of possible functions is $2^{2^{n-1}}$.

Problem 9. A boolean function $f : (0,1)^n \rightarrow$0, 1 is symmetric if any permutation applied to the digits in the domain yields the same function. e.g. f(001)=f(010)=f(100). How many symmetric boolean functions are there as a function of n?

Again, we can group the function inputs based on which must give the same output. There will now be n input groups based on how many 1s are in the input, since inputs with the same number of 1s are just permutations of each other. This means there are n+1 distinct inputs which can each be mapped to 0 or 1, so the number of possible functions is 2^{n+1}.

Problem 10. Prove x^n= $\sum_{k=0}^{n} S(n,k)x^{\underline{k}}$.

The number of ways to assign n labeled balls to k labeled urns is k!S(n, k) if we want at least one ball in each urn. To assign n balls to k urns out of x potential urns, we must assign at least one ball to k urns and no balls to the other x-k. There are k!S(n, k)*$\binom{x}{k}$, or $S(n,k)x^{\underline{k}}$ ways to do this. k ranges from 0 to n, since we wanted at least one of the n balls in each of the urns. Summing all the possible values of k up, we get $\sum_{k=0}^{n} S(n,k)x^{\underline{k}}$. This is equivalent to the left side of the equation, which corresponds to assigning n labeled balls to x labeled urns with no restrictions.

Problem 11. Prove that $(1+x)^{\alpha}$=$\sum_{k=0}^{\infty} \alpha^{\underline{k}}x^k/k!$ for any real α.

The Taylor series for 1+x centered around 0 is $(1+x)^{\alpha} + \alpha * x/1! + \alpha * (\alpha - 1) * x^2/2!$...which is equivalent to $\sum_{k=0}^{\infty} \alpha^{\underline{k}}x^k/k!$

5.2 Equivalence Relations (Bell Numbers)

A relation on set A is a subset of AxA=(a, a), (a, b)...(b, a)...(d, d). An equivalence relation is defined as any relation that is symmetric, reflexive, and transitive.
Definitions:
Reflexive - a relates to a, or must relate to itself (e.g., greater than is not a reflexive relationship because a number cannot be greater than itself).
Transitive - if a relates to b and b relates to c, then a relates to c (e.g., greater than is symmetric because if a $>$ b and b $>$ c, then a $>$ c).
Symmetric - if a relates to b, then b relates to a (e.g., greater than is not symmetric because if a $>$ b, b cannot be greater than a).
Cayely tables represent relations as a table. An example can be found below:

	A	B	C	D
A	X			
B	X	X		
C				
D				X

This table says that a relates to a, b relates to a, b relates to b, and d relates to d. As an example, this table isn't reflexive, symmetric, or transitive. The following revised table makes the relation symmetric.

	A	B	C	D
A	X			
B	X	X		
C			X	
D				X

The table still isn't transitive or symmetric, though. The next table is symmetric and reflexive, but not transitive.

	A	B	C	D
A	X	X		
B	X	X	X	
C		X	X	
D				X

The following table is all three:

	A	B	C	D
A	X	X	X	
B	X	X	X	
C	X	X	X	
D				X

How can we count how many possible relations are reflexive; reflexive and symmetric; and reflexive, symmetric, and transitive (given n rows and columns)?

Problem 1. How can we count the number of reflexive relations?

All reflexive relations must have the diagonal filled in. Without the diagonal, there are $n^2 - n$ spaces that can be filled in or left empty. Because each space can go two ways, our final answer is 2^{n^2-n}.

Problem 2. How can we count the number of reflexive and symmetric relations?

We have to start with the diagonal we had last time in order to find this next answer. For the relation to be symmetric, if a box on one side of the diagonal is filled in, then the reflection of that box over the diagonal must be filled on. That means that we have the freedom to fill in boxes on only one side of the diagonal, or $\frac{n^2-n}{2}$ boxes. Incidentally, $\frac{n^2-n}{2}$ is also $\binom{n}{2}$, which makes our final answer either $2^{\frac{n^2-n}{2}}$ or $2^{\binom{n}{2}}$.

Problem 3. How can we count the number of true equivalence relations?

We can no longer consider the diagonal method we considered earlier. Now, we must realize that to be transitive, we must create subsets of elements that all relate to each other. For instance, if we had elements a, b, c, d, e, and f, we could separate the group into two subsets: b, c, e, f and a, d. If every element in that set related to every other element in that set (including itself), we would have a true equivalence relation. This example is shown in a Cayley table below.

	A	B	C	D	E	F
A	X			X		
B		X	X		X	X
C		X	X		X	X
D	X			X		
E		X	X		X	X
F		X	X		X	X

This is an example of a true equivalence relation. But how can we count that? We need to find the total number of ways to partition n elements, which is $\sum_{k=1}^{n} S(n,k)$ or a set of numbers we call the Bell numbers (B_n).

5.3 Trapezoid Problem

Problem 1. We have n rows of triangles as shown below. How many parallelograms can be formed in this figure?

Solution There can be three orientations of parallelograms, shown below, which by symmetry must have the same number of parallelograms. Thus, we can count one and multiply that number by three.

We can observe that a parallelograms has four lines that can be extended out. If we extend them out to one layer lower than the triangle, we can notice that the lines intersect with the bottom layer to find four points. Each one of the parallelograms corresponds to a unique selection of the four points, and vice versa.

Thus, for each of the $n+2$ points on the level below the last one, we can choose four of them to get a unique parallelogram. Thus, in conclusion, the solution is

$$3 \cdot \binom{n+2}{4}$$

■

5.4 Hat Problem

Problem 2. n mathematicians walk into a bar. They each remove their hat and toss it in a pile as they arrive. Several hours later, they leave one by one, grabbing a hat at random to face the brutal March wind. What is the probability that none of the mathematicians receive their own hat?

Solution This problem is equivalent to finding the number of **Derangement** of a set of size n, up to a factor of $n!$. The derangement number is notated as D_n, or $!n$.
We can approach this by counting the number of ways to get the desired outcome via complementary counting, and then to divide it through by the total number of arrangements to get the probability, as each configuration is equally likely. First, to count the number of ways in total that there can be reordered. This is just n!. However,

in this, we have counted the number of ways that at least 0 mathematicians receive their hat. Next, we must subtract out the cases that have at least 1 mathematician getting back his or her hat. We can count this by choosing which mathematician gets his or her hat back, and then ordering the rest. Thus, the term is $\binom{n}{1}(n-1)!$. However, this overcounts those configurations for which more than one mathematician receives their hat back; we can fix this problem by using the principle of inclusion/exclusion, to get that the total number of derangements is

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! + \cdots + (-1)^n \binom{n}{n}(n-n)!.$$

This approaches $\frac{n!}{e}$ for sufficiently large n, as $e^x = 1 + x + x^2/2! + \cdots$. Thus, by dividing the nth derangement number by $n!$, we get the probability to be

$$1 - \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{(-1)^n}{n!} \approx e^{-1}$$

for sufficiently large n. ■

5.5 "Coupon Collector"-esqe Problem

Problem 3. You have a bag containing x numbered marbles. Draw n marbles with replacement, where $n \leq x$. What is the probability that you drew exactly k distinct marbles?

Solution To solve this, we can consider sequences of draws. First, we must determine the total number of "events" or valid sequences. There are x^n such sequences. Next, we must pick k of the x distinct marbles to be seen, introducing a factor of $\binom{x}{k}$. Next, we must partition the sequence of n draws into k subsets, each subset representing the draws that got one distinct value of the marble, introducing a factor of $S(n,k)$. And finally, we must assign each marble to a subset, introducing a factor of $k!$. Thus, the probability is

$$\frac{k!\binom{x}{n}S(n,k)}{x^n}.$$

Example I. f there are 10 distinct marbles, we Draw 6, and see 3, one possible outcome is, if the number j represent the jth draw,

$$C : \{1,4\}, A : \{2,3,5\}, B : \{6\}$$

Simplifying the probability a small amount, we get it to be

$$\frac{x^{\underline{k}} S(n,k)}{x^n}.$$

This also turns out to be a way to prove that $x^n = \sum_{k=1}^{n} x^{\underline{k}} S(n,k)$, as the sum of the probabilities of all valid outcomes must be one. ■

Part II
Generating Functions

Part III
Groups and Group Actions

Part IV
Coda: Computational Complexity