Partition Numbers

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1 Partition Numbers

Definition. *Define the nth partition number, $p(n)$, be the number of ways to write n as a sum of positive integers.*

As an example,

$$\begin{aligned}
5 &= 5 \\
&= 4+1 \\
&= 3+2 \\
&= 3+1+1 \\
&= 2+2+1 \\
&= 2+1+1+1 \\
&= 1+1+1+1+1 \\
p(5) &= 7
\end{aligned}$$

Definition. *Let $p_k(n)$ count the number of ways to partition n into exactly k parts.*

Theorem 1.1.

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \frac{d}{dn} \left(\frac{1}{\sqrt{n-1/24}} \sinh\left[\frac{\pi}{k} \sqrt{\frac{2}{3}\left(n - \frac{1}{24}\right)} \right] \right)$$

where

$$A_k = \sum_{\substack{\gcd(m,k)=1 \\ m<k}} \exp\left(i\pi \left(S(m,k) - \frac{2mn}{k} \right) \right)$$

Proof. Trivial and left as an exercise to the reader. ■

Theorem 1.2.

$$p_k(n) \geq \frac{1}{k!} \binom{n-1}{k-1}$$

Proof. Let's look for solutions to

$$x_1 + x_2 + \ldots + x_k = n$$

By stars and bars, we there are $\left(\!\!\binom{k}{n-k}\!\!\right) = \binom{n-1}{k-1}$ solutions. On the other hand, each partition of k elements is associated with at most $k!$ compositions. Thus,

$$p_k(n) \geq \frac{1}{k!}\binom{n-1}{k-1}$$

■

A *Ferrers graph* or *Young Tableaux* is a way of drawing partitions graphically. Each row in the tableaux contains as many dots as the the magnitude of the corresponding part in the partition. Let the *transpose* of a partition λ into λ^T be the tableaux that you get by swapping rows with columns. A partition λ is *self-conjugate* if $\lambda = \lambda^T$

Theorem 1.3. *The number of self-conjugate partitions of n is equal to the number of partitions of n with distinct odd-sized parts.*

Proof. Any self-conjugate partition has symmetry about the main diagonal. If we select the top row and left-most column together to be one part, this gives an odd sized part. The remainder of the tableaux is still self-conjugate and we recursively get more odd-sized parts. Notice to reverse the proof, we need to have the odd-sized parts be distinct sizes, or else they won't stack correctly. ■

Theorem 1.4. *The number of partitions of n into even-sized parts is equal to the number of partitions of n where each part has even multiplicity.*

Proof. Take the transpose. ■

Theorem 1.5.

$$p_4(n) = p_4(3n)$$

Proof. Let λ be a partition of n with 4. Then construct λ^* by replacing each part with $n-4$. This gives a partition of $3n$ into 4 parts. ■

Trivially, the above proof generalizes to yield $p_k(n) = p_k((k-1)n)$.

Theorem 1.6.

$$p_k(n) = \sum_{i=1}^{k} p_i(n-k)$$

Proof. Consider pre-seeding each of the k parts with a 1. Then we have to partition $n-k$, allowing for empty parts; thus we have to sum over the allowed number of non-empty parts. This corresponds to excising the left-most column in the Young tableuax. ■