Generating Functions - The Catalan Numbers

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1 Unlabeled Probability

Problem 1. You throw two marbles into two buckets. Find the probability that they both land in the same bucket:

- Unlabeled into unlabeled: 1/2 (2 scenarios: marbles in separate buckets or together)
- Labeled into unlabeled: 1/2 (2 scenarios: marbles in separate buckets or together)
- Labeled into labeled: 1/2 (4 scenarios - 2 of which include both marbles in the same bucket)
- Unlabeled into labeled:

  Let's label buckets A and B. You have two indistinguishable marbles, and so there are three scenarios: marbles are in separate buckets, both marbles are in A, and both marbles in B. Therefore, you may incorrectly conclude that the probability is 2/3, when in reality it remains 1/2. The probability of marbles being in separate buckets is twice the odds that both marbles are in A.

  This is because **probability problems are ALWAYS labeled.**

  If you label the marbles 1 and 2, there are two ways for the marbles to be in separate buckets, each as probable as any of the other scenarios: 1-A and 2-B or 1-B and 2-A.

  When you are asked for the probability, always treat both the balls and urns, or marbles and buckets, as labeled into labeled.

2 Generating Function for Catalan Numbers

Recall:

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$$

$$C_0 = C_1 = 1$$

Or in other words: $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-1}$

Let's find a mathematical, instead of combinatorial, formula for C_n:

Define $f(x) = \sum_{n=0}^{\infty} C_n x^n$

Notice that $f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + ...$

$f(x)^2 = C_0^2 + (C_0 C_1 + C_1 C_0)x + (C_0 C_2 + C_1 C_1 + C_2 C_0)x^2 + ...$
$= C_1 + C_2 x + C_3 x^2 + C_4 x^3 + ...$

$xf(x)^2 = C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + ...$
$xf(x)^2 = f(x) - C_0 = f(x) - 1$
$xf(x)^2 - f(x) + 1 = 0$

Now, use the quadratic formula to get:
$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$

However, we need to express this in series notation, in order to directly know the nth coefficient.

Recall from p.20 of Unit 1, Section 3.2:

$(1-x)^{\frac{1}{2}} = \frac{\sum \frac{1}{2}^{\underline{k}}}{k!} x^k$
$= 1 - \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} \frac{1}{4^n} x^{n+1}$

So, $(1-4x)^{\frac{1}{2}} = 1 - \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} \frac{1}{4^n} x^{n+1}$
$(1-4x)^{\frac{1}{2}} = 1 - 2 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^{n+1}$

Based on what we previously found, $(1-4x)^{\frac{1}{2}} = 1 - 2 \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^{n+1}$, we know to choose the solution of $f(x) = \frac{1-\sqrt{1-4x}}{2x}$.

Now, we simplify, to get that the generating function for the Catalan Numbers is:

$$\boxed{\text{f(x)} = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^n}$$