Structures and Exponential Generating Functions

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Date: Tuesday, April 2, 2019

1 Exponential Generating Functions

Definition. *A* structure *is a particular organization of the elements of a set or sets.*

For example, some structures include a set, a non-empty set, an even-sized set, a permutation, an increasing permutation, or a function from A into B

Definition. *The* exponential generating function *for a sequence* $f_0, f_1, f_2, f_3 \ldots$ *is*

$$F(x) = \sum_{k=0}^{\infty} f_n \frac{x^n}{n!}$$

Definition. $[n]$ *is the set* $1, 2, \ldots, n$, *or*the *set of* n *elements.*

Theorem 1.1. *If* f_n *counts the number of* F*-structures on* $[n]$*, then*

$$F(x) = \sum_{n=0}^{\infty} f_n \frac{x^n}{n!}$$

is the exponential generating function for the structure F*.*

Let's look at some examples of exponential generating functions.

- $\frac{1}{1-x}$ is the exponential generating function of $n!$
- e^{ax} is the exponential generating function of a^n

Exponential generating functions add as expected, which corresponds combinatorially to *OR*, or disjoint union. Multiplication of exponential generating function is more involved. Let $A(x) = \sum_p a_p \frac{x^p}{p!}$ and $B(x) = \sum_q a_q b_q \frac{x^q}{q!}$. Then

$$\begin{aligned} A(x)B(x) &= \sum_{p,q} a_p b_q \frac{x^{p+q}}{p!q!} \\ &= \sum_n \sum_p \frac{a_p b_{n-p} n!}{p!(n-p)!} \frac{x^n}{n!} \\ &= \sum_n \sum_p \binom{n}{p} a_p b_{n-p} \frac{x^n}{n!} \end{aligned}$$

Thus, if $C(x) = A(x)B(x) = \sum C_n \frac{x^n}{n!}$, then

$$C_n = \sum_p \binom{n}{p} a_p b_{n-p}$$

Recall that the Bell numbers obey recurrence

$$B_{n+1} = \sum \binom{n}{k} B_{n-k}$$

Let $b(x) = \sum B_n \frac{x^n}{n!}$ and let $a(x) = e^x$, the exponential generating function for $(1, 1, 1, \ldots)$. Plugging into the recurrence,

$$\begin{aligned} \sum_{n=0}^{\infty} B_{n+1} \frac{x^n}{n!} &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n}{k} B_{n-k} \frac{x^n}{n!} \\ b'(x) &= b(x)e^x \\ b(x) &= e^{e^x - 1} \end{aligned}$$

where we use the initial condition $b(0) = 1$. Now, notice

$$b(x) = \frac{1}{e} \sum_{k=0}^{\infty} \frac{(e^x)^k}{k!} = \frac{1}{e} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{k!} \frac{(kx)^n}{n!}$$

We can pull out our coefficient

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

2 Exponential Generating Functions for Structures

Let's consider some structures on sets. Consider some trivial structures:

- a set on $[n]$ is $f_n = 1$, so $F(x) = e^x$
- a non-empty set on $[n]$ is $f_n = 1 - \delta_{0n}$, so $F(x) = e^x - 1$
- an empty set on $[n]$ is $f_n = \delta_{0n}$, so $F(x) = 1$
- a singleton set on $[n]$ is $f_n = \delta_{1n}$, so $F(x) = x$
- a 2-element set on $[n]$ is $f_n = \delta_{2n}$, so $F(x) = \frac{1}{2}x^2$
- an even length set on $[n]$ has $F(x) = \cosh x$

- an odd length set on $[n]$ has $F(x) = \sinh x$

Theorem 2.1. *Addition Property: If G and H are exponential generating functions of structures, then $F = G + H$ is the exponential generating functions of the union of the structures*

To see the above theorem in action, notice that the exponential generating function for empty sets, 1, added to the exponential generating function for non-empty set, $e^x - 1$, gives the exponential generating function for all sets, e^x

Theorem 2.2. *Multiplication Property: Let g and h be structures on sets and let f be a gh structure. For set A, if we partition A into disjoint sets $A_1 \cup A_2$, put all elements in A_1 into a g-structure and A_2 into a h-structure, then the exponential generating function of f is $F(x) = G(X)H(x)$*

Example 1. Let us count subsets of $A = [n]$. Partition A into a set and another set (the complement). The number of ways to do this is given by the coefficient

$$f(x) = g(x)h(x) = e^x e^x = e^{2x} = \sum 2^n \frac{x^n}{n!}$$

Thus a set of size n has 2^n different subsets.

Example 2. Let us count non-empty subsets of $A = [n]$. Analogous to above, partition A in g-structure representing non-empty sets and h-structure representing sets. This gives

$$f(x) = g(x)h(x) = (e^x - 1)(e^x) = \sum_{n=0}^{\infty}(2^n - 1)\frac{x^n}{n!}$$

Thus a set of size n has $2^n - 1$ non-empty subsets.

Example 3. Let us count functions from $[n] \to [k]$. Let $A = A_1 \cup A_2 \cup \dots A_k$ where A_i is the preimage of i in $[n]$, so

$$f(x) = (e^x)^k = \sum_{n=0}^{\infty} k^n \frac{x^n}{n!}$$

Thus we have k^n functions from $[n] \to [k]$

Example 4. Let us count surjective functions from $[n] \to [k]$. By analogy to above problem, we have to let each of sets A_i be nonempty, so we have

$$f(x) = (e^x - 1)^k = \sum k! S(n,k)\frac{x^n}{n!}$$

where we appeal to the 12-fold table. Thus, the exponential generating function of the Stirling numbers of the second kind is

$$\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$$

Let's continue the derivation of the exponential generating function of the Stirling numbers of the second kind.

$$\begin{aligned}
\sum_{n=0}^{\infty} S(n,k)\frac{x^n}{n!} &= \frac{1}{k!}(e^x-1)^k \\
&= \frac{1}{k!}\sum_{t=0}^{k}\binom{k}{t}e^{tx}(-1)^{k-t} \\
&= \frac{1}{k!}\sum_{t=0}^{k}\binom{k}{t}\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}(-1)^{k-t} \\
&= \frac{1}{k!}\sum_{n=0}^{\infty}\left(\sum_{t=0}^{k}\binom{k}{t}(-1)^{k-t}t^n\right)\frac{x^n}{n!}
\end{aligned}$$

This gives

$$S(n,k) = \frac{1}{k!}\sum_{t=0}^{k}\binom{k}{t}(-1)^{k-t}t^n$$

which we had derived earlier by using Principle of Inclusion and Exclusion

Theorem 2.3. *Composition Property: Let g, h be structures on sets and let $f = g \circ h$, the structure by*

- *partitioning A into an arbitrary number of blocks $A_1 \cup A_2 \cup \ldots A_k$*
- *giving each A_i an h-structure*
- *and giving the partition itself a g-structure*

The exponential generating function obeys $F(x) = G(H(X))$

Example 5. Let us count the number of ways to partition a set. Each of our partitions of A must be non-empty sets, but the structure for the partitions is merely a set, so our exponential generating function is e^{e^x-1}. This gives again our exponential generating function for the Bell numbers.

Example 6. Let us count the number of ways to partition a set into subsets with at least two elements. Our exponential generating function is e^{e^x-1-x}

Example 7. Let us count the number of ways to partition a set into subsets with even number of elements. The exponential generating function is $e^{\cosh x}$