Burnside's Lemma

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Date: 30, April, 2019

1 Burnside's Lemma

Lemma. Given a set S and a permutation group G acting on S, the number of equivalence classes of S is given by

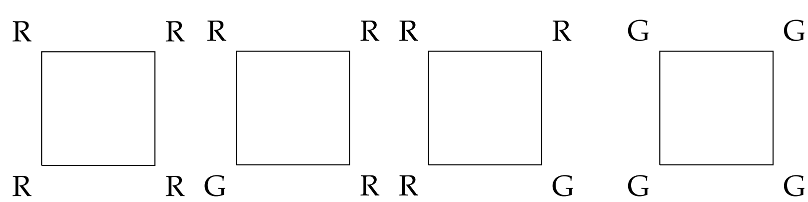
$$\frac{1}{\mid G \mid} \sum_{\pi \in G} \mid fix(\pi) \mid$$

NOTE: In the above, π is an operation on S, and $fix(\pi)$ is the set of elements in S fixed by π. So, for a given π, count the elements in S that do not change after that operation has been applied.

Example 1. The square, $G = D_4$, S = two colorings of vertices of a square. Recall that D_4 is the *dihedral group on four elements*. That is,

$$G = e, r_1, r_2, r_3, V, H, L, R$$

where e is the identity, each lowercase r represents a 90°-clockwise rotation applied the number of times indicated by the subscript, V is a vertical reflection, H horizontal, L across the left diagonal, and R across the right. Let us also say that the colors are red (R) and green (G).



We can use Burnside's Lemma to find the number of equivalence groups. First, let's make a table of all $\pi \in G$ and associated values for $\mid fix(\pi) \mid$.

π	
e	$2^4 = 16$
r_1	2 (all red, all green)
r_2	4 (diagonals, all red, all green)
r_3	2 (all red, all green)
V	4
H	4
L	8
R	8

Now, we can use Burnside's:

$$\frac{1}{\mid G \mid}\sum_{\pi \in G} \mid fix(\pi) \mid = \frac{16+16+16}{8} = \boxed{6}$$

So, there are six (6) equivalence groups for D_4.

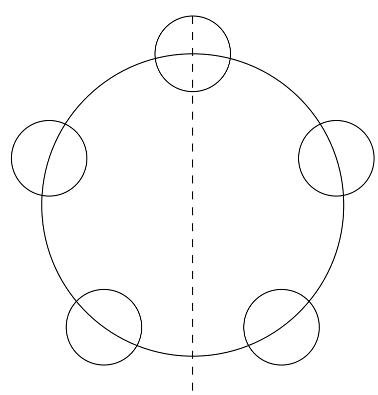
Example 2. Bracelet with 5 beads and colored with red, blue, white, and yellow. You can rotate but not flip the bracelet. How many distinct colorings are there?

Using Burnside's lemma:

π	
e	4^5
r_1	4
r_2	4
r_3	4
r_4	4

$$\frac{1}{\mid G \mid}\sum_{\pi \in G} \mid fix(\pi) \mid = \frac{4^5+4^2}{5} = 4^2 \times 13 = \boxed{208}$$

Example 3. Same as Example 2, but you can reflect.



The above diagram shows an example of such a reflection. In total, there will be five (5) such reflections, bringing the total number of operations on the bracelet to ten (10).

If we were to number the beads 1 through 5 clockwise from the top, the cycle notation for R_1, the illustrated reflection, would be (1)(2 5)(3 4). So, for all rotations R_i, $1 \le i \le 5$, $\mid fix(R_i) \mid = 4^3$.

Apply Burnside's Lemma again, we get

$$\frac{1}{\mid G \mid} \sum_{\pi \in G} \mid fix(\pi) \mid = \frac{4^5 + 4^2 + 5 \times 4^3}{5} = \frac{4^2 \times 85}{10} = \boxed{136}$$

Example 4. Let G be the number of rotations of a cube. Write the elements of G in cycle rotations.

A cube has six faces, so there are three pairs of opposite faces. For a spindle going through the centres of two opposite faces there are two possible 90 rotations, one going one way, one the other. This gives six 90 rotations. There is also a 180 rotation when the spindle goes through the centres of two opposite faces. This gives three further rotations.

A cube also has twelve edges, so there are six pairs of opposite edges. For a spindle going through the centres of two opposite edges, the only possibility is a rotation of 180. This gives six rotations, one for each pair of opposite edges.

Finally a cube also has eight corners, so there are four pairs of opposite corners. For a spindle going through two opposite corners, there are two possible rotations, both of 120, one going one way, one the other. This means there are eight 120 rotations, two for each pair of opposite corners.

We have counted 6+3+6+8 different rotations. Add the rotation by 0, which does nothing, and we have a group of $\boxed{24}$ rotations.