Expositions

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The rest of this lecture will proceed in a "choose your own adventure" format: For each of the first three **Core Ideas**, you will get to pick one of the two stories to talk about via democracy. These are notes to follow for every possible branch the talk may take.

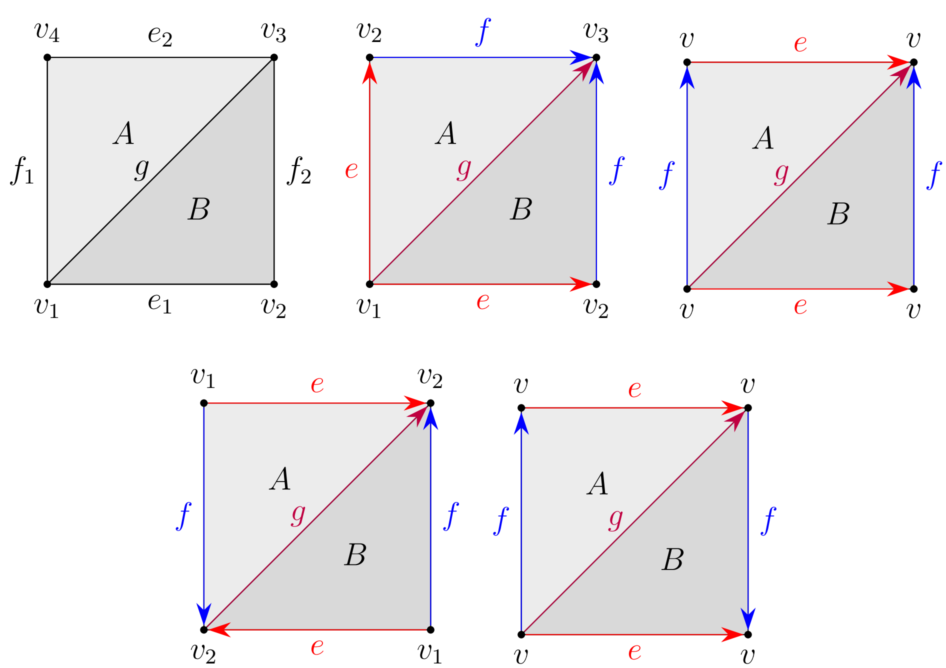
1 Connections

1.1 How to Tell Spaces Apart (Simplicial Homology)

From a study of planar graphs or polyhedra, one might recall the **Euler characteristic** – for a graph G with V vertices, E edges, and F faces (including the "outside" if necessary) we have $\chi(G) = V - E + F$. For planar graphs and polyhedral graphs we know that $\chi(G) = 2$.

Let's do it with what will look like roughly the same graph on paper, but we'll play around with gluing various edges together in various ways.

Exercise. Identify as many of the following shapes as you can. A correct answer will be "continuously deformable" into the pictures shown – they don't have to be exact matches! Vertices of the same color should also be thought of as being "glued together."



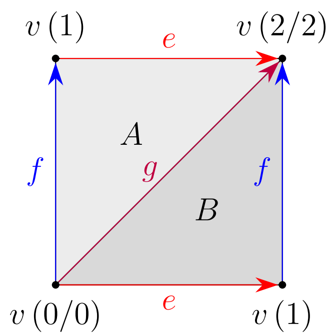
Keep these in mind! They will be our "prototypical examples" going forward.

How do you argue that the spaces that these graphs represent are *not isomorphic* to the normal plane $\mathbb{R}^2$? One way is with the Euler characteristics of all of these graph representations:

G	(A)	(B)	(C)	(D)	(E)
shape	$[0,1]^2$	S^2	$\mathbb{T}^2$	$\mathbb{R}P^2$	K
$\chi(G)$	2	2	0	1	0

Note that the Euler characteristic isn't quite enough to differentiate the surfaces depicted above – after all, we only have one number. Let's really use the structure of the graph now to analyze these surfaces at each dimension.

First, for every top-dimensional triangular (simplicial) face (here, 2-dimensional), let's order the vertices so that when we do our gluing, the orientation of the edges that are glued are consistent with the ordering of the vertices. As an example:



In general, we might have an n-dimensional **simplex** as a part of our space G, which we define as a continuous image of the canonical n-simplex $\Delta^n = \left\{\mathbf{x} \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i = 1, x_i \geq 0\right\}$ into G, which we also write in terms of its vertices $[v_0, v_1, \ldots, v_n]$. The relationship between dimensions is one of the **boundary** – for instance, when we take an n-dimensional simplex of our space, its boundary is a union of $(n-1)$-simplices. With the orientation on each simplex, we define the **boundary map** acting on an n-simplex σ to be the following formal sum of $n-1$-simplices:

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma\Big|_{[v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]}$$

The signs are maybe a surprising addition – but with this, one can consider the integer linear combinations of all of the k-dimensional simplices $C_k(G)$ for one of our spaces, and the boundary maps assemble the C_ks into a **chain complex**:

$$\cdots \to C_n(G) \xrightarrow{\partial_n} \cdots \to C_2(G) \xrightarrow{\partial_2} C_1(G) \xrightarrow{\partial_1} C_0(G) \xrightarrow{\partial_0} 0$$

where we extend the boundary maps to act linearly on integer linear combinations of simplices.

The key observation (due to Emmy Nöether and Mayer/Vietoris) – if the boundary of a suitable sum of simplices is 0, these simplices encircle a closed region in G. But, if such a sum of simplices is not the boundary of any simplex in G, then we have detected a hole within our space! Therefore, the linear subspace of C_n that encircles closed regions in G but ignoring any parts of it that actually do describe closed regions in G, gives us a subspace spanned by the holes of G! Interpreting this from the lens of linear algebra, we define the **homology groups**

$$\boxed{H_n(G) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}}$$

As a fact to make these well-defined, we need to know that $\boxed{\partial^2 = 0}$, which is a defining feature of a general chain complex (that I've omitted until now). Now we can basically use linear algebra to compute them, since a basis of these subspaces tells us exactly where the holes are in our space at any dimension we like! This is the construct that will allow us to make more granular comparisons between spaces:

G	S^2	$\mathbb{T}^2$	$\mathbb{R}P^2$	K
$H_0(G)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$H_1(G)$	0	$\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$H_2(G)$	$\mathbb{Z}$	$\mathbb{Z}$	0	0

Exercise. We can also recover the Euler characteristic from these groups! For a general topological space, the Euler characteristic is defined as $\chi(G) = \sum_{i=0}^{\infty}(-1)^i \dim_{\mathbb{Z}} H_i(G)$ (where the dimension here ignores any non-$\mathbb{Z}$ factors in the homology). Check this!

This construct isn't a panacea – there are spaces that homology cannot tell apart. For instance, consider the space constructed by gluing a sphere and two circles together all at the same point. The homology of a torus/doughnut is identical to this! Other tools have to come into play to tell these spaces apart (cohomology, homotopy). However, homology is still very powerful!

Idea 1.1

Representing general (topological) spaces as graphs is very powerful and fairly general, and allows one to study them with algebraic tools.

1.2 Word Problems (Geometric Group Theory)

Let's talk about **groups**! Formally, a group is a set G with a binary operation $m : G \times G \to G$ and inverses with respect to m, given by $i : G \to G$ such that m is associative and has an identity such that multiplying any element by its inverse gives the identity. Let's take a more constructive approach instead though:

Definition. Suppose S is a set of symbols, and let S^{-1} be the collection of formal symbols consisting of $\{s^{-1} : s \in S\}$. Let R be a collection of (non-empty) strings in S or S^{-1}, i.e.

$R \subseteq L(S \cup S^{-1})$. Two words $u, v \in L(S \cup S^{-1})$ are *equivalent* (giving an equivalence relation $\sim$) if there is a sequence of moves turning u into v such that each move either adds or removes a pair ss^{-1} or $s^{-1}s$ for $s \in S$, or adds or removes some $r \in R$.

Definition. The collection of equivalence classes of $L(S \cup S^{-1})/\sim$, where $\sim$ is generated by S and $R \subseteq L(S \cup S^{-1})$, is the group G with **presentation** $\langle S \mid R \rangle$ under concatenation and reversals/inversions. S is said to be the set of **generators** of G, and R is the set of **relations**.

Exercise. Identify as many of the following groups from their presentations as you can:

(a) $\langle a \mid \rangle$
(b) $\langle a \mid a^n \rangle$
(c) $\langle a, b \mid a^3, b^2, (ab)^2 \rangle$
(d) $\langle a, b \mid aba^{-1}b^{-1} \rangle$
(e) $\langle a, b \mid a^n, b^2, (ab)^2 \rangle$
(f) $\langle a, b \mid a^2, b^2 \rangle$

We can use the presentation of a group to construct its **Cayley graph**:

Definition. The **Cayley graph** of a group G (not necessarily finite!) with generators S is a graph $\mathrm{Cay}(G)$ with vertices g for every $g \in G$ and directed edges from g to gs for every $g \in G$, $s \in S$.

Be careful – the directions of the edges do matter! We'll describe the corresponding Cayley graphs for each of the above groups here:

(a) Directed line of vertices
(b) Directed n-cycle
(c) Two triangles oriented in opposite directions, connected by edges between corresponding vertices
(d) 2D square lattice of edges going up and to the right
(e) Two n-gons oriented oppositely, connected by edges between corresponding vertices
(f) Undirected line of alternating-label edges

Remark. One important application for Cayley graphs is that they are a natural space for a **group action** of G. Essentially, this means there is a natural way to "multiply" an element of the group G and any element of the set, and for the Cayley graph, we can simply consider left-multiplication by any group element as the action.

One is often concerned with whether an action is **faithful** or **free**, which are slightly different measures of the "injectivity" of the action. An action is faithful if there are no elements of the group that fix the entire set other than the identity, and an action is free if for all non-identity elements of the group, there are no fixed points of any of their actions. In particular, the action of G on its Cayley graph $\mathrm{Cay}(G)$ is free.

We now introduce a problem that will drive the rest of this section:

Problem (The Word Problem)**.** Given a group G with finite presentation $\langle S \mid R \rangle$ (i.e. S and R are both finite), for any word $w \in L(S \cup S^{-1})$, decide whether $[w]$ is the identity element e in a finite amount of time.

While on the way to think about solving this problem for general groups, we can study this problem in the Cayley graph of a group. Note that any word $w \in L(S \cup S^{-1})$ represents a path in the Cayley graph starting at the identity e, and a word that reduces to the identity is a loop at e, and vice versa. As such, it suffices to check that the Cayley graph is **constructible** in a finite amount of time – this doesn't necessarily mean that we have to be able to construct the entire Cayley graph, but instead that we should be able to construct the subgraph of every vertex that is reachable in at most n steps from e for arbitrarily large n.

For the examples above, it's clear that we can do this quite easily by looking at the Cayley graphs, and in fact many and most groups that we have a handle on do have word problems can be solved. However, this problem still has a negative answer in general:

Theorem 1.1 (Novikov-Boone)

The word problem for finitely presented groups is **undecidable** – in particular, there exists a finitely presented group G for which the word problem is undecidable.

And in fact, in connection to complexity theory, the following theorem gives us a hint for how to construct such a group:

Theorem 1.2 (Higman's Embedding Theorem)

Every finitely generated, **recursively presented** group H can be embedded as a subgroup of some finitely presented group G, i.e. H's set of relations is **recursively enumerable**.

Roughly, recursive enumerability is less restrictive than decidability, as an algorithm/Turing machine only needs to terminate running on any element of the set, not necessarily for all inputs. Using this theorem, we need only construct a recursively presented group with undecidable word problem to break the word problem in general!

Remark. One important class of examples are the **free groups** of rank n with n generators and no relations, whose Cayley graphs are infinite trees with degree $2n$ at every vertex (with half of the generators going in/out of each vertex). As such, it's very easy to solve the word problem in these groups because there is a pretty easy algorithm to decide if we can reduce any word to the identity.

A surprising fact about free groups – not only are they the only groups whose Cayley graphs are trees, they are the **only** groups that can act freely on trees! This fact is important to showing that every subgroup of a free group is free (the **Nielson-Schreier** theorem), which is actually quite difficult to show otherwise.

Idea 1.2

Groups can be studied not only by their own structure, but also by their encodings as graphs and their actions on those graphs.

(Enter break here, informal time for questions)

2 Generalizations

2.1 Doing Calculus Anywhere (Manifolds, Differential Forms)

Many of you might recall the Fundamental Theorem of Calculus:

Theorem (Fundamental Theorem of Calculus)**.** Let $f : U \to \mathbb{R}$ be continuous on an open interval U containing $[a, b]$, and $F : U \to \mathbb{R}$ an antiderivative of f on (a, b), i.e. $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

And some of you might recall the various "fundamental theorems" of multivariable calculus:

Theorem (Fund. Thm. of Line Integrals)**.** Let $\varphi : U \to \mathbb{R}$ be differentiable on an open subset $U \subseteq \mathbb{R}^3$ and $\gamma : [0, 1] \to U$ is a continuous curve in U. Then

$$\int_\gamma \nabla\varphi(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \varphi(\gamma(1)) - \varphi(\gamma(0)).$$

Theorem (Kelvin-Stokes Theorem)**.** Let $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field in an open region containing the smooth oriented surface S in $\mathbb{R}^3$, with boundary ∂S.

$$\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \oint_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Theorem (Green's Theorem)**.** Let C be a positively oriented, simple rectifiable closed curve in $\mathbb{R}^2$, bounding the region D. Let $L, M : \mathbb{R}^2 \to \mathbb{R}$ be C^1 on an open region containing D. Then

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

Theorem (Divergence Theorem)**.** Let V be a compact subset of $\mathbb{R}^3$ with piecewise smooth boundary ∂V, and let $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 vector field. Then

$$\iiint_V (\nabla \cdot \vec{\mathbf{F}})\,dV = \oiint_{\partial V} \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$$

What are all of these theorems saying? They are all saying if one integrates a **function** over the **boundary of a space**, this is equal to integrating the **derivative** of that **function** over the entire **space**.

How do we generalize this to arbitrary dimensions? Of course, we need a way to generalize "space", and likely also the notion of a derivative as it needs to encompass gradient, divergence, and curl – but what else? Looking at the proofs of these theorems, especially Stokes' Theorem, might show us how to do it. Here is the general sketch of the proofs for all of these theorems:

1. Cut up the bounded region we are integrating over into little boxes, over which we perform the integration locally.
2. Doing an analysis on the interior of each box with the "differentiated" integrand, we show that this is equal to the "undifferentiated" integrand integrated over the boundary of the box. This step usually requires us to have some notion of the tangent vector/plane to the region at the point we are concerned at, and often to combine this geometric information with the function being integrated
3. Piece the boxes back together, where the boundaries of the boxes in the interior cancel each other out.

This is, unsurprisingly, how the proof of the **generalized Stokes' Theorem** goes, kind of, and the machinery that it entails does essentially this. Let's just state the general theorem and we'll break down what everything is and why all of the above are just special cases of this. I've color-coded all of the different components of this:

Theorem 2.1 (Generalized Stokes' Theorem)

Let M be a k-dimensional **oriented smooth manifold-with-boundary** in $\mathbb{R}^n$, and give the boundary ∂M the **boundary orientation**. Let φ be a smooth $(k-1)$**-form** defined on an open set containing M. Then

$$\int_{\partial M} \varphi = \int_M \mathbf{d}\varphi.$$

Proof. (Sketch.) Break up M into parallelograms P_i that in the limit, which become vanishingly small, and also approximate the boundary ∂M. Then $\int_M \mathbf{d}\varphi = \sum_i \mathbf{d}\varphi = \sum_i \int_{\partial P_i} \varphi = \int_{\partial M} \varphi$, as desired. $\square$

Let's give a brief explanation for what (most of) everything is now:

What is a smooth manifold? What is its boundary? A **manifold** in general can be defined in various ways. A traditional abstract definition might look like:

Definition. A k-dimensional smooth manifold M is a topological space where every point $x \in M$ has a **chart** $\varphi_x : U_x \subseteq M \to V_x \subseteq \mathbb{R}^k$ that are homeomorphisms (isomorphisms of topological spaces), such that the **transition maps** $\varphi_{x_2} \circ \varphi_{x_1}^{-1}$ are smooth maps from $V_{x_1} \subseteq \mathbb{R}^k \to \mathbb{R}^k$ if $U_{x_1} \cap U_{x_2}$ is non-empty.

The general idea is that we want to describe a space that locally looks like Euclidean space, but could be bent/curved, which we describe with the charts. Taking the charts together, we want the places where those charts overlap to be compatible, and any smoothness that the transition maps have passes to the space itself.

It's more intuitively useful for us to instead appeal to an older definition of a k-dimensional smooth manifold in $\mathbb{R}^n$ as a level set of a smooth map. In particular, we can define a k-dimensional manifold with maps $F_i : U_i \subseteq \mathbb{R}^n \to \mathbb{R}^{n-k}$ where $M \cap U_i$ is the set of points where $F_i(\vec{\mathbf{x}}) = 0$, and as long as the F_is each define an **implicit function** of some $n-k$ coordinates in terms of the other k coordinates in $\mathbb{R}^n$. To make this true, the idea is that the map must behave like its derivative, which is a linear transformation $\mathbb{R}^n \to \mathbb{R}^{n-k}$, and if this map is onto, then the non-pivotal variables are the k active variables for which the other $n-k$ passive variables are functions of. This definition clearly identifies surfaces $z = f(x, y)$ or algebraic curves (i.e. $x^2 + y^2 - 1 = 0$) as manifolds, which look like examples of surfaces or curves you've seen.

To find a tangent space at each point, though, it's more useful to think about a parameterization of a k-dimensional manifold, which is a smooth map $\gamma : U \subseteq \mathbb{R}^k \to M$, where U is open in $\mathbb{R}^k$, and the span of the derivative of this map (at a point $\vec{\mathbf{x}}$) gives the **tangent space** to M at $\vec{\mathbf{x}}$, $T_{\vec{\mathbf{x}}}(M)$ (as long as the map is full-rank). This is a k-dimensional subspace, that, if translated to the point on the manifold, would be a linear approximation to the manifold.

Now, normally we integrate over bounded regions, and so we usually take a subset of the manifold to consider. Generally, the **boundary** of a subset of a space consists of all points such that, for any ball around them, contain points both in and not in the subset. In order to also be able to integrate over the boundary, it must also be a smooth manifold, and we want it to be "1-dimension lower" in some sense. The idea is that for a point that is on a part of the boundary that is "1-dimension lower", we can collapse a neighborhood of that point down to $\mathbb{R}$ in a unique way, whereas this is not necessarily true if our point looks like the corner of a square for instance. These points on the boundary where we have a unique way to determine the "inside" and "outside" of the k-manifold form a $(k-1)$-dimensional manifold, being the **smooth boundary** (ideally of finite volume), and everything else (the non-smooth bits) should have zero $(k-1)$-dimensional volume.

To ground our discussion, consider a surface (i.e. $z = f(x, y)$ in $\mathbb{R}^3$) which is a 2-manifold. Tangent spaces look like tangent planes (the tangent space is 2-dimensional), and when we look at the boundary of this surface, we usually are looking at a (piecewise) smooth curve, which is a 1-manifold. Any non-smooth points literally look like the corner of a square, which are visually not smooth, and in alignment with the definition, corners have "two directions" from which we could project a neighborhood of that point onto $\mathbb{R}$, whereas a part of the smooth boundary has only one way of doing so (i.e. with the normal vector pointing inwards).

What is an orientation in general? Remember that for surfaces, we always choose a normal vector to orient our surface, and for a curve, we pick a tangent vector to orient the direction of traversal. How does this translate to the general case?

For a manifold, what we do at every point is to assign any **ordered basis** of the tangent space either a $+1$ or a -1, and we do so in a way such that the sign of the determinant of the change of basis matrix between any two bases multiplies consistently with the choice of the sign we give any ordered basis. This basically splits half of the bases at every point into being "positive" (**direct**) and the other half as "negative" (**indirect**). Formally, an orientation at a point $\vec{\mathbf{x}}$ of M is a map $\Omega_{\vec{\mathbf{x}}} : B_{\vec{\mathbf{x}}}(M) \to \{\pm 1\}$, where $B_{\vec{\mathbf{x}}}(M)$ is the set of bases of the tangent space $T_{\vec{\mathbf{x}}}(M)$, satisfying the change of basis property described above.

As an example, for a curve, the tangent space is a line, for which one direction along the curve gives the "positive" direction, and the other direction gives the "negative" direction, which corresponds to the two directions in which one can traverse a curve, and an orientation corresponds to assigning a sign for each. For a surface, if we have a vector $\vec{\mathbf{n}}$ **transverse** to the surface (in any non-tangent direction, i.e. "upwards from the surface"), an orientation corresponds to looking at the cross product of the two basis vectors for the tangent plane in a given order, and looking at the sign of the dot product with $\vec{\mathbf{n}}$.

Now, if we can partition the bases of the tangent spaces into these halves at every point as we vary continuously along our manifold, the manifold is **orientable**. This is not necessarily possible in general – a standard example is a Möbius band, where an ordered basis can be translated continuously around the band in a full circle until the order of the basis and therefore the direction of the cross product flips, which gives a discontinuous map. Many manifolds are orientable, however. Concretely, if $B(M)$ is the collection of bases to all tangent spaces at all points of M indexed by the point that they are at, i.e. $\left\{(\vec{\mathbf{x}}, B) \in \mathbb{R}^{n(k+1)}\right\}$ for $\vec{\mathbf{x}} \in \mathbb{R}^n$, B a basis, then an orientation is a continuous map $\Omega : B(M) \to \{\pm 1\}$, where the restrictions to the basis elements each give an orientation of a tangent space.

It's important that a parameterization of the manifold respects a given orientation – a parameterization naturally gives a basis of the tangent space at every point via the partial derivatives in each coordinate, and we want to make sure that this basis is direct relative to that orientation.

Note that an orientation of a manifold-with-boundary naturally induces an orientation of its boundary. With our definition of the smooth boundary essentially defining an "inward" and "outward" at every point on the boundary, what we can do is to take an outward-pointing vector $\vec{\mathbf{v}}$ tangent to the manifold, and a basis of the tangent space to the boundary B, and orient the boundary using the orientation of the manifold Ω applied to the ordered basis $(\vec{\mathbf{v}}, B)$. In the case of a surface with a curve as its boundary, this results in the classic "right-hand-rule" choice for the orientation of the curve consistent with the normal to the surface.

What is a differential form? Differential forms, at their core, are locally linear functions of k vectors at a time. More specifically, a k-form on $\mathbb{R}^n$ is a multilinear, alternating function on k vectors from $\mathbb{R}^n$. Multilinear means that the function is linear in each of the k vectors, and alternating means that swapping any two of the arguments should flip the sign of the result. The purpose of a k-form is to act on the k tangent vectors forming the (ordered) basis of a tangent space of a k-manifold, where the ordering of the inputs matters since the tangent vectors are determined by a chosen orientation. Such k-forms are often written as a linear combination of **elementary forms** of the form $dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$, where the x_{i_j} are coordinates. To evaluate this form on k vectors, we take the $k \times k$ matrix whose rows are the corresponding coordinate in each of the k vectors in our input, and take the determinant – this may seem somewhat arbitrary, but the determinant happens to be the unique multilinear, alternating function on k vectors in $\mathbb{R}^k$, up to scaling, and so linear combinations of these functions give all possible constant k-forms, which we denote by $A_c^k(\mathbb{R}^n)$.

Of course, we don't just integrate constant functions – we technically are integrating k-form *fields*, which may vary from point to point in space, often in a continuous/differentiable manner. As such, a k-form field is a continuous map $\varphi : U \to A_c^k(\mathbb{R}^n)$ for some $U \subseteq \mathbb{R}^n$ open. We write the collection of k-form fields as $A^k(U)$, and for a $\varphi \in A^k(U)$, we can evaluate them at "parallelograms" of k vectors at a point $\vec{\mathbf{x}}$, i.e. $\varphi(P_{\vec{\mathbf{x}}}(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)) = \varphi(\vec{\mathbf{x}})(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$. In practice, we drop the word "field" and assume this to be understood, and we also write these as linear combinations of elementary forms that may have continous (often times smooth) functions of n variables as coefficients as well. In this sense, the integrand of a 1-variable integral is not merely a function $f(x)$ – it is a 1-form, $f\,dx$, so the dx to some degree carries semantic meaning!

At this point, it's best to just reframe an example from multivariable calculus in the language of differential forms:

Problem. Verify that $\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \oint_{\partial S} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ when $\vec{\mathbf{F}} = z^2\hat{\imath} - 3xy\hat{\jmath} + x^3y^3\hat{k}$ and S is the part of $z = 5 - x^2 - y^2$ lying above $z = 1$, oriented upwards.

Here, our manifold is the surface given by a function describing 1 coordinate in terms of 2 others in a continuous way, so we have a 2-manifold. We can simply use the given coordinates to get a parameterization $\gamma(s,t) = \left[\begin{smallmatrix} s \\ t \\ 5-s^2-t^2 \end{smallmatrix}\right]$, whose tangent vectors are $\left\{\left[\begin{smallmatrix} 1 \\ 0 \\ -2s \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ -2t \end{smallmatrix}\right]\right\}$, giving a basis of the tangent plane at every point. Note that the z-coordinate of the cross product of these vectors in this given basis is positive, so this ordering is direct relative to the orientation given by the upwards direction. The boundary is a circle parameterized by $\gamma(t) = \left[\begin{smallmatrix} 2\cos t \\ 2\sin t \\ 1 \end{smallmatrix}\right]$, and one can check that this parameterization is also consistent with the induced boundary orientation.

To translate this vector field into a 1-form that acts like the dot product when applied to this tangent vector, we can see that we can translate this as the 1-form $\varphi = z^2\,dx - 3xy\,dy + x^3y^3\,dz$, which allows us to recover the same parameterized integral on the right-hand side. The only thing we have to explain now is how to take the "derivative" of this 1-form, and matching the shape of Stokes' theorem, it appears that this "derivative" turns k-forms into $(k+1)$-forms.

What is this "derivative" acting on differential forms? The operation $\mathbf{d}$ is called the **exterior**

derivative, and it generalizes the notions of gradient, curl, and divergence for the suitable k-forms.

Formally, the exterior derivative is defined locally with the exact limit such that in the proof of Stokes' Theorem, everything works out, and is defined in the following way on a box B_V of $k+1$ vectors $V = \{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_{k+1}\}$, defined by $\{\sum_{i=1}^n t_i \vec{\mathbf{v}}_i : t_i \in [0,1]\}$, evaluated at a point $\vec{\mathbf{x}}$:

$$\mathbf{d}\varphi(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_{k+1}) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial(hB_V)} \varphi.$$

Critically, this has the properties that for a 0-form f (which is just a function $\mathbb{R}^n \to \mathbb{R}$, $\mathbf{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$, and for higher k-forms, $\mathbf{d}(f dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\mathbf{d}f) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. To make sense of this, the symbol $\wedge$ represents the **wedge product** on differential forms, and it has the property that $dx_i \wedge dx_i = 0$ for any i (since one would be taking a determinant with repeated rows). We can see verify this property when $k = 0$ quickly – note that $\mathbf{d}f(\vec{\mathbf{v}}) = \lim_{h\to 0} \frac{1}{h} \int_{[x, x+h\vec{\mathbf{v}}]} f = \lim_{h \to 0} \frac{1}{h}(f(\vec{\mathbf{x}} + h\vec{\mathbf{v}}) - f(\vec{\mathbf{x}}))$, which is the directional derivative of f at $\vec{\mathbf{x}}$ in the direction of $\vec{\mathbf{v}}$, which is computed exactly as stated.

Let's see this definition in action on our example above, and compare it to the curl $\nabla \times \vec{\mathbf{F}} = 3x^3y^2\hat{\imath} + (-3x^2y^3 + 2z)\hat{\jmath} - 3y\hat{k}$:

$$\begin{aligned}\mathbf{d}\varphi &= (2z\,dz) \wedge dx + (-3y\,dx - 3x\,dy) \wedge dy + (3x^2y^3\,dx + 3x^3y^2\,dy) \wedge dz \\ &= -3y\,dx \wedge dy + (3x^2y^3 - 2z)\,dx \wedge dz + 3x^3y^2\,dy \wedge dz\end{aligned}$$

Note that the 2-forms here will evaluate the components of the cross product in the normal-vector $d\vec{\mathbf{S}}$, and dot them with the appropriate components, and with the right signs. This completes our experimental verification that this formalism in fact recovers the same integrals to do in the end.

This formalism was rather complicated, but in general, the added machinery allows us to basically "refactor" the proofs of five theorems into the proof of one. Hopefully, this excursion also shows that the generalization employed wasn't random – the same key ingredients appeared in these very similar proofs, and looking for similar structures allowed us to see why the formalisms were defined in the way that they were.

Idea 2.1

Statements that are true for similar underlying reasons often can be abstracted with a more general framework.

2.2 A Case Study in Volume (Measure Theory)

So I hear you all know how to find volumes in geometry. Let's test that!
First, some warmups. What's the volume of:

- A three-dimensional ball of radius 1? $(\{\mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| \leq 1\})$ – $\frac{4}{3}\pi$

- A two-dimensional disk of radius 1? ($\{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| \leq 1\}$) – π
- The interval $[0, 1]$ in $\mathbb{R}$? – 1
- The singleton point $\{0\}$? – 0 maybe, or 1 with the counting measure?

Some subtle trickery I've done here: For each of the above, (I hope) I forced you to reinterpret the word "volume" into the correct dimension, based on the ambient space the set lives in. As such, the definition of "volume" you used to compute each of the above was **fundamentally different**. Keep this in mind!

Okay, now let's play a game called "$0, 1$, or not defined?" Please tell me if the volume of the following sets are 0, 1, or not defined:

(a) The two points $\{0, 1\}$.

(b) Fix any $n \in \mathbb{Z}^+$. The set $\left\{\frac{k}{n} : 0 \leq k \leq n, k \in \mathbb{Z}\right\}$ (fractions with denominator n).
These first few are definitely going to be zero in $\mathbb{R}$. They're finite sets of points.

(c) The converging sequence $\left\{\frac{1}{2^k} : k \in \mathbb{Z}_{\geq 0}\right\}$.
Still zero, even though it's infinite, and has a point of accumulation at 0.

(d) All dyadic fractions between 0 and 1, $\left\{\frac{m}{2^n} : m, n \in \mathbb{Z}\right\} \cap [0, 1]$ (fractions whose denominator is a power of 2).

(e) The rational numbers between 0 and 1, $\mathbb{Q} \cap [0, 1]$.
(Curious to see what people will say and if they will treat these differently.) From the Riemann integration point of view, both of these should be undefined, since the upper and lower sums are both 1 and 0, and so we don't have convergence. For the Lebesgue integral, these are both countable, so they are measure 0. Both are dense in $[0, 1]$ – tricky!

(f) Define a subset $C \subseteq [0, 1]$ as follows. Let $C_0 = [0, 1]$, and to define C_n inductively for $n \geq 1$ as a union of closed intervals, for each maximal closed interval in C_{n-1}, remove the middle third of the interval. Let $C = \bigcap_{k=0}^{\infty} C_k$. (This set is uncountable, unlike any of the examples up until this point.)
It doesn't matter – the integral is zero, because we can just track the amount of length removed. This set is nowhere dense (there are no open intervals contained in it) which might be a compelling reason for why this set, the Cantor set, has no volume...

(g) Define a similar subset $F \subseteq [0, 2]$ as follows. Let $F_0 = [0, 2]$. However, to generate F_1, remove the middle $\frac{1}{4}$ of each interval in F_0, and to generate F_2, remove the middle $\frac{1}{16}$ of each closed interval in F_1. In general, remove the middle $\frac{1}{4^i}$ of each interval from the intervals in F_{i-1} to get F_i. Again, take $F = \bigcap_{k=0}^{\infty} F_k$.
This actually has measure 1! However, like the above example, it too is nowhere dense, and no open interval is contained in it – tricky! This set is called a fat Cantor set.

(h) Consider $[0, 1]$ under the equivalence relation $\sim$ such that $x \sim y$ if $|x - y| \in \mathbb{Q}$. Partition

$[0, 1]$ into equivalence classes, and let V consist of one representative from each equivalence class.

This is called a Vitali set, and we've constructed it specifically so that the measure is not definable! To see this, suppose $\{q_1, q_2, \ldots\}$ *is an enumeration of the rationals in* $[-1, 1]$*. Then the translates* $V_i = V + q_i$ *have the same measure as* V*, and are all contained in* $[-1, 2]$*. Moreover, these sets are all disjoint and contain all elements of* $[0, 1]$*. If this set is measurable, it must have positive measure – but if it does, the measure of the union of all of the translates diverges, contradiction!*

Let's move on to playing a different game in different dimensions. As a warmup, let's consider the following questions:

- What's the 1-D volume of the sphere of radius $\frac{1}{2\sqrt{\pi}}$ in $\mathbb{R}^3$?
- What's the 2-D volume of the sphere of radius $\frac{1}{2\sqrt{\pi}}$ in $\mathbb{R}^3$?
- What's the 3-D volume of the sphere of radius $\frac{1}{2\sqrt{\pi}}$ in $\mathbb{R}^3$?

Exactly one of these for each shape should give an answer that's positive but not infinite – in this way, we can see that the number for which the d-dimensional volume of the set is positive and finite is also a good measure for the "dimension" of the set! Let's try this for some other shapes – for what d is the d-dimensional volume of these sets positve and finite?

(i) The Cantor set C constructed above in part (f) – $\frac{\ln 2}{\ln 3}$.
(ii) The Sierpinski triangle. Hopefully you know what this is? – $\frac{\ln 3}{\ln 2}$.
(iii) The Koch snowflake, made by putting together three Koch curves at their endpoints. For a Koch curve, start with a segment, trisect it, and replace the middle segment with the other two sides of an equilateral triangle. Repeat for every new segment ad infinitum. – $\frac{\ln 4}{\ln 3}$.
(iv) The Menger sponge. Start with a cube, subdivide it into 27 smaller cubes, and remove the "middle core" of cubes from each face. Repeat with each subcube ad infinitum. – $\frac{\ln 20}{\ln 3}$.

For all of these, use self-similarity and how one might decompose this object as a almost-disjoint sum of scaled copies of itself. Use also the fact that in d-dimensional volume, scaling by a factor of r scales the volume by a factor of r^d, which makes sense for integer dimensions! These, as it turns out, will not have integer dimension – they will have fractional dimension (hence the word **fractal**. This measure of dimension (when formalized with measures) is called the **Hausdorff dimension**.

Idea 2.2

Even a simple idea such as volume has a broad range of reinterpretations, redefinitions, and uses when applied to broader classes of objects!

3 Breaking and Repairing

3.1 I Forgot How to Factor (Algebraic Number Theory)

Let's solve some Diophantine equations in integers! In my opinion, the tricks used to solve them are generally fairly contrived, but hopefully we'll use methods that look a little more natural.

Problem. Find all solutions in integers to $y^2 = x^3 - 1$.

Solution. First, we rearrange to get $x^3 = y^2 + 1$. Note first that if y is odd, then x is even. Taking mod 4 makes $2 \equiv 0 \bmod 4$, contradiction, so y is even. From here, we'd like to do the following to solve the problem, being strong of mind and confident in our ability to manipulate complex numbers:

Factor the right-hand side as $x^3 = (y+i)(y-i)$. (Here, we're extending our numbers to be in $\mathbb{Z}[i]$, i.e. complex numbers but with integer coefficients for 1 and i, $\{a + bi : a, b \in \mathbb{Z}\}$. These are called the *Gaussian integers*.) We now claim that $y+i$ and $y-i$ have no nontrivial common divisors over $\mathbb{Z}[i]$.

Note that if $y+i$ and $y-i$ shared an irreducible common factor π (i.e. nothing divides it other than units), then $\pi \mid 2i = (1+i)^2$ (the difference). Then $\pi = 1+i$ since $1+i$ is irreducible, and therefore $y + i = (1+i)(a+bi)$, and by complex conjugation $y - i = (1-i)(a-bi)$ and so $y^2 + 1 = 2(a^2+b^2)$. But since y is even, this is a contradiction.

Now, since $y-i$ and $y+i$ share no common factors, then both $y-i$ and $y+i$ must be cubes. Then $y + i = (a+bi)^3 = (a^3 - 3ab^2) + (3a^2b - b^3)i$, so $b(3a^2 - b^2) = 1$ and $b \mid 1$. Doing casework, $b = 1$ gives no solutions, but $b = -1$ gives $a = 0$, so $y = 0$ and $x = 1$. Therefore $(1, 0)$ is the only solution. $\square$

Wow, amazing! So the key idea is:

Idea 3.1 (partial)

Working in generalizations of the integers is useful and works just as well as working in $\mathbb{Z}$.

Let's watch this technique destroy another Diophantine equation:

Problem. Find all solutions in integers to $y^2 = x^3 - 61$.

Solution. Same thing, right? Rearrange to get $x^3 = y^2 + 61$, and again note that if y is odd, then x is even, but then we get an impossibility mod 4, contradiction. Factor again, but this time, we are going to work in $\mathbb{Z}[\sqrt{-61}]$, which is a little stranger, but surely we can have $x^3 = (y + \sqrt{-61})(y - \sqrt{-61})$. Again, we claim that $y + \sqrt{-61}$ and $y - \sqrt{-61}$ have no common divisors over $\mathbb{Z}[\sqrt{-61}]$.

To see this, suppose there was an irreducible common factor $\pi = p + q\sqrt{-61}$ dividing both. Then it must divide the difference, being $2\sqrt{-61}$. Moreover, since $|\pi|^2$ divides $y^2 + 61$ which is odd, $|\pi|^2$ is odd, and it also divides $4 \cdot 61$. Therefore, $|\pi|^2 = 1, 61$, and assuming π is not a unit, then $|\pi|^2 = 61$. Then, $61 \mid y$ since $61 \mid y^2 + 61$, and from the original equation then $61 \mid x$, so then $61 = x^3 - y^2$ is divisible by 61^2, contradiction.

As such, these two factors share no common factors, so $y + \sqrt{-61}$ and $y - \sqrt{-61}$ are cubes. Then $y + \sqrt{-61} = (a + b\sqrt{-61})^3 = (a^3 - 3 \cdot 61ab^2) + (3a^2b - 61b^3)$, so $b(3a^2 - 61b^2) = 1$ and again $b = \pm 1$. But now, if $b = 1$, $3a^2 = 62$ and if $b = -1$, $3a^2 = 60$, neither of which give integer solutions. Therefore there are no solutions to this equation in integers. $\square$

At this point, I expect someone to ask a question and interrupt me, saying they have in fact found a solution... because $(x, y) = (5, \pm 8)$ works. So now math is broken and everything blows up because we have somehow proved something false! What is wrong here?

The thing that blows a hole in the whole argument is the following line from the proof:

As such, these two factors share no common divisors, so $y + \sqrt{-61}$ and $y - \sqrt{-61}$ are cubes.

Why is this false? The solution above actually provides a counterexample to this exact line of the argument, where $5^3 = (8 + \sqrt{-61})(8 - \sqrt{-61})$. This gives us two distinct factorizations of 125 into irreducibles in $\mathbb{Z}[\sqrt{-61}]$, so we have what kind of looks like a *violation* of the fundamental theorem of arithmetic here! However, this doesn't occur in our first example for $\mathbb{Z}[i]$. Both $\mathbb{Z}[\sqrt{-61}]$ and $\mathbb{Z}[i]$ are what are called **rings of integers** (for suitably chosen fields), but one is a **unique factorization domain** (a UFD) and the other is not.

Let's proceed to be a little more precise in our language in the act of generalizing our tactics. First, the fundamental property that makes the fundamental theorem of arithmetic tick (primality), and the thing we actually have, which isn't quite enough:

Definition. An element π is **prime** if for any a and b such that $\pi \mid ab$, then either $\pi \mid a$ or $\pi \mid b$.

Definition. An element π is **irreducible** if it is not invertible in the ring of integers (a **unit**) and it is not the product of two non-invertible elements.

In general, all prime elements are irreducible, but *not all irreducible elements are prime!* Here, 5 is irreducible, but $5 \mid (8 + \sqrt{-61})(8 - \sqrt{-61})$ without dividing either one! Now, let's go on to describe the setting in which we want to be working for our number theory to work:

Definition. A **unique factorization domain** (or UFD) is a ring where every element can be written as a product of irreducible elements and a unit, where these irreducible elements are unique up to ordering and multiplication by units.

Here, we see that $\mathbb{Z}[\sqrt{-61}]$ is not a UFD. For rings of integers that are not UFDs, Ernst Kummer initially introduced the idea of an "ideal number" that existed in rings that would allow one to conclude the same thing as if we had unique factorization. For instance, for the case of $\mathbb{Z}[\sqrt{-61}]$, Kummer would have introduced "ideal numbers" $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathbb{Z}[\sqrt{-61}]$ such that $5 = \mathfrak{p}\mathfrak{q}, 8 + \sqrt{-61} = \mathfrak{p}^3, 8 - \sqrt{-61} = \mathfrak{q}^3$, so that these two elements gave us exactly the result we wanted. This is kind of the right idea – today, it turns out "ideals" are not really thought of as numbers inside the ring, but are formulated as a kind of "invariant subspace":

Definition. An **ideal** I is an additively closed subset of a ring R such that for any $r \in R$, $rI = I$.

Rings of integers are examples of a broader class of rings called **Dedekind domains** which have nice properties surrounding their ideals – in particular, the analogue of "primeness" for ideals behaves how we want them to! Let's just state these properties:

Definition. An ideal $P \subset R$ is **prime** if for any $a, b \in R$ such that $ab \in P$, then either $a \in P$ or $b \in P$.

Theorem. Every nonzero ideal I factors uniquely as a product of prime ideals in a Dedekind domain, where the product of ideals I and J, written IJ, is defined as the closure of the set $\{ab : a \in I, b \in J\}$ under addition.

Armed with this knowledge, instead of our argument for the "relative primeness" of $y \pm \sqrt{-61}$, we instead argue that when we factor the ideals generated by $(y \pm \sqrt{-61})$ into prime ideals, they share no prime ideal factors. We then want to be able to have some notion of a "root" of an ideal, which seems kind of hard to work with at first glance. However, R-invariant subspaces carry their own structure amongst each other, and actually can be assembled into a group:

Definition. For a ring R, consider its field of fractions K. A **fractional ideal** $J \subseteq K$ is closed under addition and multiplication by R, and there exists some nonzero $r \in R$ such that $rJ \subseteq R$, i.e. there is a way to "clear the denominators" of every element of J that maps into R. These ideals form a group in a Dedekind domain.

Definition. An ideal I is **principal** if it consists of all the multiples of some element π in the ring.

Definition. For a ring of integers R with corresponding field of fractions K, the **class group** is the quotient group J_K/P_K, where J_K is the collection of fractional ideals, and P_K is the subgroup of principal ideals. The size of the class group is the **class number**.

To bring things back to a concrete setting, we actually can find that the class group of $\mathbb{Z}[\sqrt{-61}]$ is $\mathbb{Z}/6\mathbb{Z}$, and to take (for instance) a cube root of the principal ideal $(y + \sqrt{-61})$ which is a representative of the identity in the class group, it suffices to find an element of order 3 in this group (which there are only 2 of, and they are inverses of each other). More theory allows us to leverage these ideals along with information about the units in the ring (*Dirichlet's Unit Theorem*) to solve the problem completely.

In general, if a Dedekind domain is a UFD, all of its ideals are principal, so the class group is in some sense the "right" construction for capturing how much a ring of integers fails to satisfy unique factorization.

Idea 3.1

Working in generalizations of the integers is useful, but *there are many nice features about the integers that we generally take for granted, so things are more complicated!*

3.2 The Answer Is Not True (Analysis, Foundations)

We were going to have a true/false party, but I forgot to take the spoiler out of the section header... so to make it more interesting, **explain your answers**! Without further ado, let's begin:

These first few are about continuity and differentiability:

1. Every function that is continuous everywhere is also differentiable everywhere.
   Counter: absolute value. Lots of things, really!
2. The derivative of a function is always continuous.
   Counter: $x^2 \sin \frac{1}{x}$, 0 *at the origin breaks it! Check explicitly with the limit definition and you'll see this. Such functions are not* C^1.
3. If a function's derivative is finite everywhere, the derivative must be bounded everywhere.
   Counter: $x^2 \sin \frac{1}{x^2}$ *exists and is finite everywhere, but is unbounded in a neighborhood of 0 in* $[-1, 1]$
4. Every function that is continuous everywhere must be differentiable at some point.
   Counter: Weierstrass' function, which encompass the family of functions
   $$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$
   where $a \in (0, 1)$ *and* b *is an odd integer where* $ab > 1 + \frac{3}{2}\pi$.
5. If a continuous function has derivative zero almost everywhere, it is constant.
   Counter: The Cantor-Lebesgue function (devil's staircase), constructed first on the Cantor set C *by taking a number* $x \in C$, *rewriting it in a non-terminating ternary expansion, dividing the digits by 2, and re-interpreting it as a binary number. This function can then be completed with constant functions matching the left-hand value of any removed open interval.*
6. A function continuous on the irrationals must be continuous at some rational.
   Counter: Thomae's function, i.e. a function f *such that if* $x \in \mathbb{R}$ *is irrational then* $f(x) = 0$, *and if* $x = \frac{m}{n}$ *rational,* $f(x) = \frac{1}{n}$. *Thomae's function breaks a lot of integrability statements as well.*

For the multi enthusiasts, we have a couple for you:

7. A function $\mathbb{R}^2 \to \mathbb{R}$ that has first partial derivatives everywhere must be differentiable.
   Counter: You probably saw tons of these in multi! One example is $f(x, y) = \frac{x^2 y}{x^4+y^2}$ *away from the origin and 0 otherwise.*
8. For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, its mixed second-order partials are equal.
   Counter: This function is actually even C^1*, but this isn't enough – consider* $f(x, y) = \frac{xy(x^2-y^2)}{x^2+y^2}$ *away from the origin and 0 otherwise.*

Now, we move to integration. Some of these overlap with the "A Case Study in Volume" section above.

9. Every set that has a positive volume must contain some open interval.
   Counter: the fat Cantor set in "A Case Study in Volume," (g)
10. A function cannot be integrable if its set of points of discontinuity is dense in $\mathbb{R}$.
    Counter: Thomae's function, again!
11. Every bounded subset of $\mathbb{R}$ can be given a measure.
    Counter: A Vitali set, see "A Case Study in Volume," (h)
12. Regardless of your definition of the integral, the integral $\int_{-\infty}^{\infty} \frac{\sin x}{x}$ exists and is equal to π.
    Counter: this integral only exists as a limit of a Riemann integral (which is what the Cauchy principal value is doing, kind of, for the complex analysis veterans). This integral does not exist if we interpret this as a Lebesgue integral!
13. A function that integrates to 0 on any open interval is identically zero.
    Counter: Thomae's function again!

A couple about series:

14. For a function f that is positive and continuous on $x \geq 1$, $\int_1^\infty f(x)\,dx$ converges iff $\sum_{n=1}^\infty f(n)$ converges.
    Counter: Neither direction is true. For the forwards direction, define a function $g : [1, \infty) \to \mathbb{R}$ *where at each integer* $n > 1$*, we have a "sawtooth" with peak at* n *with height* $\frac{1}{n}$ *and width* $\frac{2}{n^2}$*, and zero elsewhere. Then* $f(x) = g(x) + \frac{1}{x^2}$ *is continuous but has convergent integral and divergent sum. Conversely, define a function* $h : [1, \infty) \to \mathbb{R}$ *where at each integer* $n > 1$*, we have a "valley" with trough at* n *where* $g(n) = 0$*, depth of 1, and width* $\frac{2}{n}$*, and let* $g = 1$ *elsewhere. Then if* $f(x) = g(x) + \frac{1}{x^2}$*, we have a convergent series but divergent integral.*
15. A function whose Taylor series at a point converges everywhere must converge to the function.
    Counter: $f = e^{-\frac{1}{x^2}}$ *for* $x \neq 0$*, and* 0 *at the origin. The Maclaurin series has a zero radius of convergence.*

Finally, some brain-benders:

16. Every Cartesian product of non-empty sets is non-empty.
17. Every vector space has a basis.
18. Every surjective function has a right inverse.
19. Every linear function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
20. Induction can only be done over finite sets, such as $\mathbb{N}$.

These are technically all considered be true, but only if we accept the **axiom of choice**:

Theorem 3.1 (Axiom of Choice)

Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a set of non-empty sets. There is a function $f : \mathcal{A} \to \bigcup_{i \in I} A_i$ such that $f(A_i)$ is an element of A_i for all $i \in I$.

It's not like people don't want these statements to be true – most people accept the axiom of choice, but it has gained notoriety and a number of models of set theory have been developed and investigated where this axiom is not true.

Depends, technically. It's true for finite-dimensional vector spaces, but it's only true for infinite-dimensional vector spaces if you accept the axiom of choice.

Idea 3.2

Broad statements that seem true in general may often have strange, pathological counterexamples that break them.