Question. What does math/doing math mean to you?

- Why do you want to study math?
- Do you like math? Why?
- How could you learn about math outside of school/VMT?
- What does being "good" at math mean?
- How diverse is the math community?
- Does doing math make you "smarter?"
- Why should people study math?

What's Next?

math stories from beyond VMT

Bryan Lu

June 5, 2023

Your Landscape

Competition math categories:

- algebra (inequalities, FEs)
- geometry (mostly Euclidean)
- combinatorics
- number theory (elementary)

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This is **VERY FAR** from everything that's out there!

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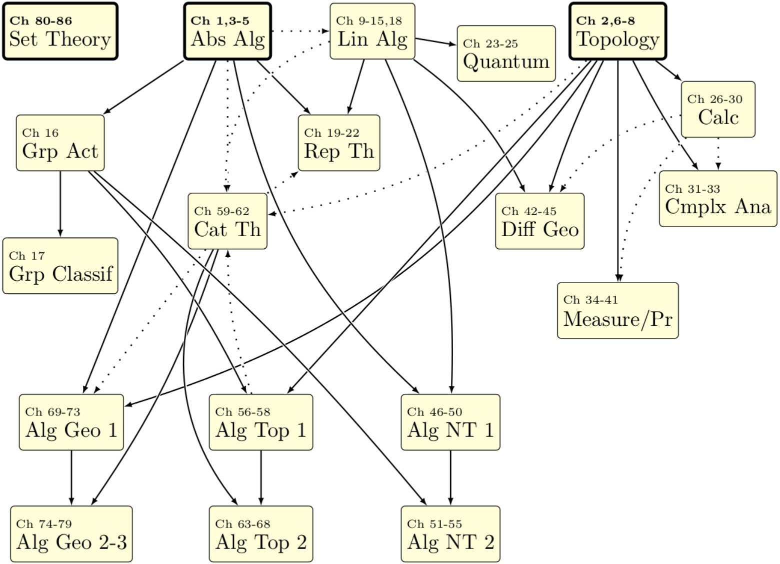
Broader fields (incomplete/ill-defined list; historically important):

- algebraic number theory
- analytic number theory
- dynamical systems
- Lie theory
- harmonic analysis
- operator theory
- computer science
- foundations
- game theory/economics
- algebraic geometry
- algebraic topology
- mathematical physics

Dependency Graph (*Napkin, Evan Chen*)



Myths/Misconceptions

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5. (there are probably tons more . . .)

Core Ideas

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the rest of this lecture: choose your own adventure

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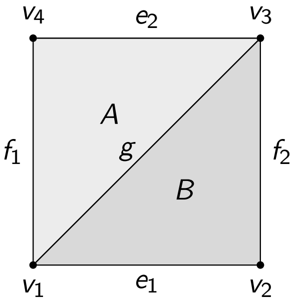
- crowd vote!! $\Rightarrow$ this talk could go in 8 possible ways

Idea 1 – Connections

- *How to Tell Spaces Apart (Simplicial Homology)* – We will use graphs as a model for describing various topological spaces (spheres, toruses, Klein bottles) and use an extension of the Euler characteristic with linear algebra to distinguish spaces from each other.
- (★) *Word Problems (Geometric Group Theory)* – We will take an alternate view of groups and study graphs induced by a group's structure, and use these graphs to explain why deciding whether an element in a group is the identity element is unsolvable.

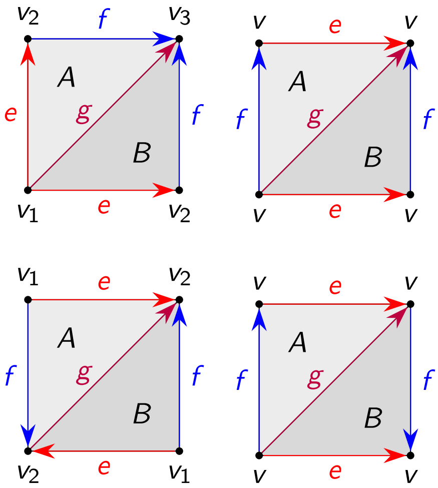
1.1 – Euler Characteristic

Euler characteristic of a graph G with V vertices, E edges, and F faces: $\chi(G) = V - E + F$.



$$\chi(G) = 2$$

1.1 – Example Spaces



1.1 – Chain Complexes

$$\sigma = [v_0, v_1, \ldots, v_n]$$

$$\Rightarrow \partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma\big|_{[v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]}$$

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$$\cdots \to C_n(G) \xrightarrow{\partial_n} \cdots \to C_2(G) \xrightarrow{\partial_2} C_1(G) \xrightarrow{\partial_1} C_0(G) \xrightarrow{\partial_0} 0$$

Holes have no boundary, but are not the boundary of any simplex.

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$$\boxed{\partial^2 = 0} \Rightarrow \boxed{H_n(G) = \frac{\quad \partial}{\partial \quad}}$$

1.1 – Homology Groups

G	S^2	$\mathbb{T}^2$	$\mathbb{R}P^2$	K
$H_0(G)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$H_1(G)$	0	$\mathbb{Z}^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
$H_2(G)$	$\mathbb{Z}$	$\mathbb{Z}$	0	0

1.1 – Identical Homology

$$\mathbb{T}^2 \qquad\qquad S^1 \vee S^1 \vee S^2$$

$$H_0 = \mathbb{Z} \quad H_1 = \mathbb{Z}^2 \quad H_2 = \mathbb{Z}$$

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$\langle S|R \rangle$ is the set of equivalence classes of $L(S \cup S^{-1})$ under $\sim$

1. $\langle a \mid \rangle$
2. $\langle a \mid a^n \rangle$
3. $\left\langle a, b \mid a^3, b^2, (ab)^2 \right\rangle$
4. $\left\langle a, b \mid aba^{-1}b^{-1} \right\rangle$
5. $\left\langle a, b \mid a^n, b^2, (ab)^2 \right\rangle$
6. $\left\langle a, b \mid a^2, b^2 \right\rangle$

1.2 – Cayley Graphs

Definition

The ***Cayley graph*** *of a group* $G = \langle S \mid R \rangle$, $\mathrm{Cay}(G)$, *has vertices* g *for every* $g \in G$, *and directed edges* $g \to gs$.

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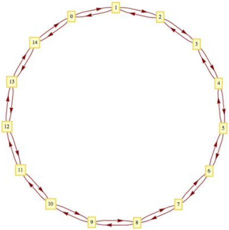

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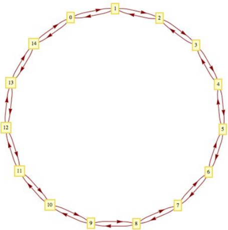

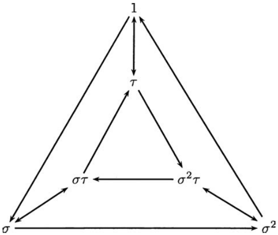

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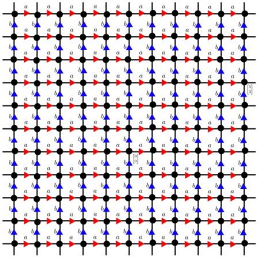

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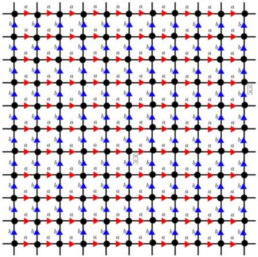

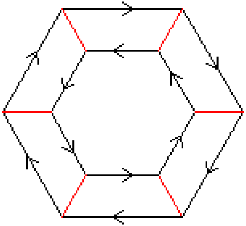

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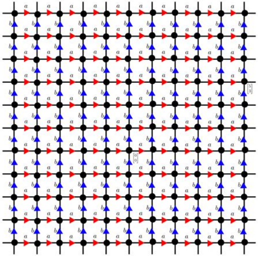
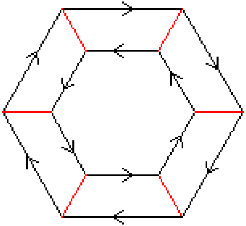
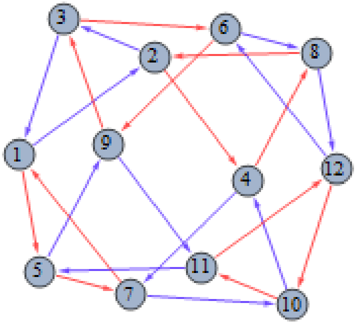




1.2 – The Word Problem

Problem (The Word Problem)

Given a group G with finite presentation $\langle S \mid R \rangle$ (i.e. S and R are both finite), for any word $w \in L(S \cup S^{-1})$, decide whether $[w]$ is the identity element e in a finite amount of time.



1.2 – Bad News

Theorem (Novikov-Boone)

The word problem for finitely presented groups is ***undecidable*** *– in particular, there exists a finitely presented group G for which the word problem is undecidable.*

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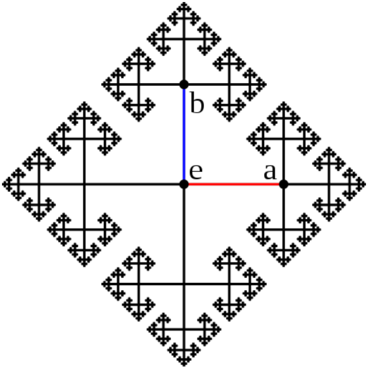
Theorem (Higman's Embedding Theorem)

*Every finitely generated, **recursively presented** group H can be embedded as a subgroup of some finitely presented group G, i.e. H's set of relations is **recursively enumerable**.*

1.2 – Free Groups

$$F_n = \langle x_1, \ldots, x_n \mid \rangle$$



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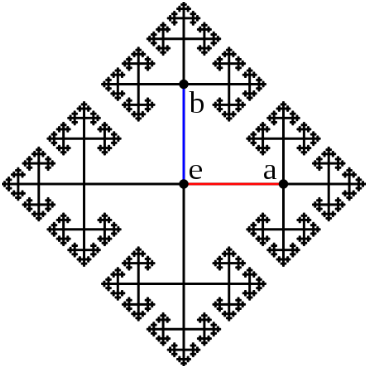

Theorem (Nielson-Schreier)

Every subgroup of a free group is free.

Idea 2 – Generalizations

- ($\bigstar$) *Doing Calculus Anywhere (Manifolds, Differential Forms)* – One of the staples of calculus is the Fundamental Theorem of Calculus, which has many further generalizations further afield in multivariable calculus. Let's see why these are actually all the same thing!
- *A Case Study in Volume (Measure Theory)* – So you think you know what volume is? Here we test your intuitive understanding of volume and use edge cases as a case study for generalizing what you might think of as "volume."

2.1 – Fundamental Theorems of (Multivariable) Calculus

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Theorem (Fund. Thm. of Line Integrals)

$$\int_\gamma \nabla\varphi(\vec{r}) \cdot d\vec{r} = \varphi(\gamma(1)) - \varphi(\gamma(0)).$$

Theorem (Green's Theorem)

$$\oint_C (L\,dx + M\,dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\,dx\,dy$$

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Theorem (Kelvin-Stokes' Theorem)

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

Theorem (Divergence Theorem)

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What do these theorems all have in common?

*Integrating a **function** over the **boundary of a space** is equal to integrating the **derivative** over the **whole space**.*

2.1 – Generalized Stokes' Theorem

Theorem (Generalized Stokes' Theorem)

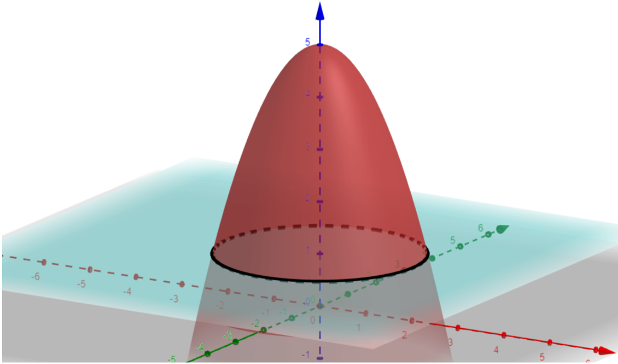
*Let M be a k-dimensional **oriented smooth manifold-with-boundary** in $\mathbb{R}^n$, and give the boundary ∂M the **boundary orientation**. Let φ be a smooth $(k-1)$-**form** defined on an open set containing M. Then*

$$\int_{\partial M} \varphi = \int_M \mathrm{d}\varphi.$$

2.1 – Example

Problem

Verify that Stokes' Theorem holds for the surface S *given by* $z = 5 - x^2 - y^2$ *above the plane* $z = 1$ *(oriented upwards), and vector field* $\vec{F} = z^2\hat{i} - 3xy\hat{j} + x^3y^3\hat{k}$.



2.2 – Warmups

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- The singleton point $\{0\}$?

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4. All dyadic fractions between 0 and 1, $\left\{\frac{m}{2^n} : m, n \in \mathbb{Z}\right\} \cap [0, 1]$ (fractions whose denominator is a power of 2).

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5. The rational numbers between 0 and 1, $\mathbb{Q} \cap [0, 1]$.

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7. Let $F_0 = [0, 2]$. To generate F_1, remove the middle $\frac{1}{4}$ of each closed interval in F_0. In general, remove the middle $\frac{1}{4^i}$ of each interval from the intervals in F_{i-1} to get F_i. Let $F = \bigcap_{k=0}^{\infty} F_k$.

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8. Consider $[0,1]$ under the equivalence relation $\sim$ such that $x \sim y$ if $|x-y| \in \mathbb{Q}$. Partition $[0,1]$ into equivalence classes, and let V consist of one representative from each equivalence class.

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Now, let's warm up for a different question:

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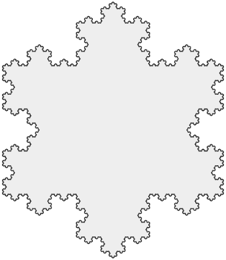
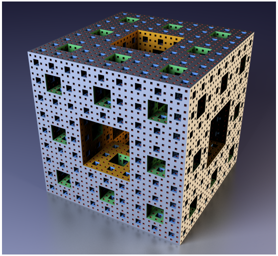
Note that for only one of these is the answer not 0 or ∞ – it's when the dimension is 2! We can actually define the dimension in this way...

2.2 – Dimensionality Exercises

For these spaces, compute the definition in this way, assuming there is some $d \in \mathbb{R}^+$ for which the volume is positive and finite (it's not going to be integral!):

1. The Cantor set C.
2. The Sierpinski triangle.
3. The Koch snowflake.
4. The Menger sponge.





Idea 3 – Breaking and Repairing

- *I Forgot How to Factor (Algebraic Number Theory)* – We will try solving some Diophantine equations in some creative ways, and realize that I don't really know how to prime-factorize things anymore. Can we fix that?
- *The Answer Is Not True (Analysis/Foundations)* – Because everyone loves the true/false section of a math exam, we're going to have a true/false party!
  Wait, I forgot to take the spoiler out of the section header... fine, to make it more interesting, be prepared to explain your answers!

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Proof. $\pi \mid y+i, y-i \Rightarrow \pi \mid 2i = (1+i)^2 \Rightarrow \pi = (1+i)$.

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Proof. $\pi \mid y + i, y - i \Rightarrow \pi \mid 2i = (1 + i)^2 \Rightarrow \pi = (1 + i)$. Then
$y + i = (1 + i)(a + bi), y - i = (1 - i)(a - bi) \Rightarrow y^2 + 1 = 2(a^2 + b^2)$.
Contradiction, $y^2 + 1$ is odd.

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$(y+i) = (a+bi)^3 = (a^3 - 3ab^2) + (3a^2b - b^3)$, so $b(3a^2 - b^2) = 1$, and $b = \pm 1$. If $b = 1$, no sol'ns, if $b = -1$, then $a = 0$, so $y = 0$, $x = 1$.

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$b = \pm 1$. If $b = 1$, no sol'ns, if $b = -1$, then $a = 0$, so $y = 0$, $x = 1$.
$\Rightarrow (x, y) = (1, 0)$ is the only solution.

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Proof. $\pi \mid y + \sqrt{-61}, y - \sqrt{-61} \Rightarrow \pi \mid 2\sqrt{-61}$. $|\pi|^2 \mid y^2 + 61$ odd so $|\pi|^2$ odd, and therefore $|\pi|^2 = 61$.

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$(y + \sqrt{-61}) = (a + b\sqrt{-61})^3 = (a^3 - 183ab^2) + (3a^2b - 61b^3)$, so $b(3a^2 - 61b^2) = 1$, and $b = \pm 1$.

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$(y + \sqrt{-61}) = (a + b\sqrt{-61})^3 = (a^3 - 183ab^2) + (3a^2b - 61b^3)$, so $b(3a^2 - 61b^2) = 1$, and $b = \pm 1$. In either case for b, no solutions.
There are no solutions in integers to this equation. Except...

3.1 – Formalizing Primes and Factoring

Definition

An element π is **prime** if for any a and b such that $\pi \mid ab$, then either $\pi \mid a$ or $\pi \mid b$.

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Definition

A **unique factorization domain** (or UFD) is a ring where every element can be uniquely written as a product of irreducible elements and a unit, up to ordering and multiplication by units.

3.1 – Ideals

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An **ideal** I is an additively closed subset of a ring R such that for any $r \in R$, $rI = I$.

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Definition

An ideal $P \subset R$ is **prime** if for any $a, b \in R$ such that $ab \in P$, then either $a \in P$ or $b \in P$.

Theorem

Every nonzero ideal I factors uniquely as a product of prime ideals in a Dedekind domain, where the product of ideals I and J, written IJ, is defined as the closure of the set $\{ab : a \in I, b \in J\}$ under addition.

3.1 – Class Group

Definition

For a ring R, consider its field of fractions K. A **fractional ideal** $J \subseteq K$ is closed under addition and multiplication by R, and there exists some nonzero $r \in R$ such that $rJ \subseteq R$. These ideals form a group in a Dedekind domain.

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Definition

For a ring of integers R with corresponding field of fractions K, the **class group** is the quotient group J_K/P_K, where J_K is the collection of fractional ideals, and P_K is the subgroup of principal ideals.

3.2 – True/False Party (Smoothness, Part I)

1. Every function that is continuous everywhere is also differentiable everywhere.
2. The derivative of a function is always continuous.
3. If a function's derivative is finite everywhere, the derivative must be bounded everywhere.
4. Every function that is continuous everywhere must be differentiable at some point.
5. If a continuous function has derivative zero almost everywhere, it is constant.
6. A function continuous on the irrationals must be continuous at some rational.

3.2 – True/False Party (Integration, Part II)

1. Every set that has a positive volume must contain some open interval.
2. A function cannot be integrable if its set of points of discontinuity is dense in $\mathbb{R}$.
3. Every bounded subset of $\mathbb{R}$ can be given a measure.
4. Regardless of your definition of the integral, the integral $\int_{-\infty}^{\infty} \frac{\sin x}{x}$ exists and is equal to π.
5. A function that integrates to 0 on any open interval is identically zero.

3.2 – True/False Party (Multivar/Series, Part III)

1. A function $\mathbb{R}^2 \to \mathbb{R}$ that has first partial derivatives everywhere must be differentiable.
2. For a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, its mixed second-order partials are equal.
3. For a function f that is positive and continuous on $x \geq 1$, $\int_1^\infty f(x)\,dx$ converges iff $\sum_{n=1}^\infty f(n)$ converges.
4. A function whose Taylor series at a point converges everywhere must converge to the function.

3.2 – True/False Party (General, Part IV)

1. Every Cartesian product of non-empty sets is non-empty.
2. Every vector space has a basis.
3. Every surjective function has a right inverse.
4. Every linear function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
5. Induction can only be done over finite sets, such as $\mathbb{N}$.

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Theorem (Axiom of Choice)

Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a set of non-empty sets. There is a function $f : \mathcal{A} \to \bigcup_{i \in I} A_i$ such that $f(A_i)$ is an element of A_i for all $i \in I$.