Infinitely Many Proofs of Infinitely Many Primes

(Special Thanks to Katherine Taylor, Forrest Hutchison, Emma Cardwell, Angela Zhao, HCSSiM 2019)

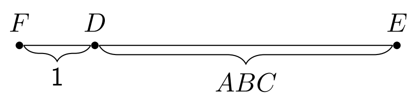
Bryan Lu

2 May 2022

1 Classic Number Theory Proofs

Proof by Euclid – All

Consider a finite list of primes A, B, C. Construct the line segment DE such that $DE = A \cdot B \cdot C$. Construct DF such that $DF = 1$ and D, E, F are collinear.



If EF is prime, we have found a larger prime. If EF is not prime, then it must divide some prime G. If G is any of the primes A, B, C, then it must also divide $DF = 1$. Since this isn't possible, G must be a prime not in our list. Thus, we have found a new prime, and there are an infinite number of primes. Proof count: 1.

Proof using Legendre's Formula (Whang, 2010) – Ari

Consider $n!$. Since prime factorization is unique, we may write $n! = \prod p^{N_p}$, where $N_p = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots$ which is the number of times p appears in $n!$. From this,

$$N_p \leq \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots$$

By geometric series,

$$N_p \leq \frac{n}{p-1}$$

$$\prod p^{N_p} \leq \prod p^{\frac{n}{p-1}}$$

Separately, we know that $\frac{n}{2}^{\frac{n}{2}} \leq n!$ (which can be shown by induction), so we write

$$\frac{n}{2}^{\frac{n}{2}} \leq (\prod p^{\frac{1}{p-1}})^n$$

$$\frac{n}{2}^{\frac{1}{2}} \leq \prod p^{\frac{1}{p-1}}$$

This can only be true for any n if there are infinitely many primes. Proof count: 4.

Proof (Folklore) – Amber

AFTSOC, let p be the largest prime. $\exists$ a prime q s.t. $q \mid 2^p - 1$. Consider the multiplicative group mod q $(\mathbb{Z}/q\mathbb{Z})^{\times}$. We know that $o(2) \mid p$, but since p is prime and $o(2) \neq 1$, $o(2) = p$.By Fermat's Little Theorem, $p \mid q - 1$, so $q > p$, which is a contradiction. So, there is no largest prime p and thus an infinite number of primes. Proof count: 6.

Proof using Combinatorics (the Incompressibility Method) – Milo

AFTSOC there are finitely many primes, $p_1, p_2, ..., p_n$. $\forall n \in \mathbb{Z}^+, n = p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$. There are n numbers between 1 and n. Each e_i can be at most $\log_2 n$, so there are $1 + \log_2 n$ options for each exponent. This gives at most $(1 + \log_2 n)^k$ possible prime factorizations for numbers between 1 and n. For ease of computation, consider $n = 2^{m-1}$. Then there are 2^{m-1} numbers between 1 and n, and $(1 + \log_2 2^{m-1})^k = m^k$ possible prime factorizations. For large enough m, there are not enough prime factorizations than there are numbers, a contradiction. Therefore, there are infinitely many primes. Proof count: 5.

Proof of Infinite Proofs by Paul

According to Bertrand's postulate, we are guaranteed to find at least one prime between n and $2n$ for any $n \in \mathbb{Z}^+$. Since we can start at any n and there are an infinite number of places to start at, we have an infinite number of primes! Proof count: $11 + \infty + \infty \cdot \infty + \infty$

Proof with Topology (Fürstenburg) – David

Call $U \subset \mathbb{Z}$ *nepo* if if $U = \emptyset$ or U is a union of sets of the form $S(a, b) = \{x \in \mathbb{Z} : x \equiv b \pmod a)$. In other words, every element of U is in an arithmetic progression.

Why is this a topology? Just need to check finite intersections of open sets is open.

Lemma 1: The intersection of finitely many nepo sets is nepo.

Proof: Let $U_1, U_2, ...U_n$ be nepo. Let $x \in U_1 \cap U_2 \cap ... \cap U_n$. Since $x \in U_1$, $x \in S(a_1, x) \subset U_1$. Since $x \in U_2$, $x \in S(a_2, x) \subset U_2$, and so on. Let $a = lcm(a_1, a_2, ...a_n)$. Then $S(a, x) \subset S(a_1, x) \subset U_1$, $S(a, x) \subset S(a_2, x) \subset U_2$ and so on. Now we have $S(a, x) \subset U_1 \cap U_2 \cap ... \cap U_n$, so $U_1 \cap U_2 \cap ... \cap U_n$ is nepo.

Consider $S(p, 0)$, the set of all integer multiples of p. Now consider $\{1, -1\}$, the only members of $\mathbb{Z}$ that aren't multiples of primes.

$\{-1, 1\}$ is open, but this is impossible.

Call a set $V \subset \mathbb{Z}$ *desolc* iff its complement is nepo.

Lemma 2: The union of finitely many desolc sets is desolc.

Proof: Let $V_1, V_2, ...V_n$ be desolc. $(V_1 \cup V_2 \cup ... \cup V_n)^c = V_1^c \cap V_2^c \cap ... \cap V_n^c$, which is the intersection of finitely many nepo sets and therefore nepo. So, its complement, or $V_1 \cup V_2 \cup ... \cup V_n$, is desolc.

Consider $S(p, 0)$, the set of all integer multiples of p. Now consider $\{1, -1\}$, the only members of $\mathbb{Z}$ that aren't multiples of primes. This set isn't nepo, so its complement can't be desolc. We can write this non-desolc set as $\mathbb{Z} \setminus \{1, -1\} = \bigcup_{p \text{ prime}} S(p, 0)$. But for a given p, $S(p, 0)$ is desolc. If there were finitely many primes, $\mathbb{Z} \setminus \{1, -1\}$ would be the union of finitely many desolc sets, so it would be desolc. Since we know that's not true, there must be infinitely many primes. Proof count: 7.

Proof with Analysis (Erdös) – Kelly

We will show that $\sum_p \frac{1}{p}$, where p ranges over all primes, diverges. Suppose that the series converges to some value. Then $\exists k$ such that

$$\frac{1}{p_{k+1}} + \frac{1}{p_{k+2}} + \frac{1}{p_{k+3}} + ... < \frac{1}{2}$$

Let's call all primes from p_{k+1} on up "big primes" and the primes from p_1 to p_k "small primes". Now we take some $n \in \mathbb{N}$ and count the number of numbers less than or equal n in two ways. First, just by counting, we get that there are n of them. Then, we'll count first the number of numbers less than n that only have small prime factors (call this total i), and add that to the number of numbers with at least one big prime factor (call this total j). Note that for any prime p the number of numbers $\leq n$ with p as a factor is $\lfloor \frac{n}{p} \rfloor$. Therefore, we know that

$$j \leq \frac{n}{p_{k+1}} + \frac{n}{p_{k+2}} + \frac{n}{p_{k+3}} + ... = n(\frac{1}{p_{k+1}} + \frac{1}{p_{k+2}} + \frac{1}{p_{k+3}} + ...) < \frac{n}{2}$$

Now we'll consider some $m \leq n$ with only small prime factors. We write m in the form w^2x, where x contains no squares as factors. This means none of the k possible prime factors can appear in x's factorization more than once, giving 2^k possibilities for the value of x. Since $w < \sqrt{n}$, we have $i \leq 2^k\sqrt{n}$. The square root function grows more slowly than $\frac{n}{2}$, so $i < \frac{n}{2}$ for sufficiently large n. Putting it all together, $i + j = \text{(number of numbers less than or equal to n)} < \frac{n}{2} + \frac{n}{2} = n$. This is a contradiction. Therefore, $\sum_p \frac{1}{p}$ diverges, so there must be infinitely many terms in the series, so there must be infinitely many primes. Proof count: 8.

Proof using the Prime Number Theorem

Well known that $\pi(x)$, the number of primes $\leq x$, grows asymptotically as $\frac{x}{\ln x}$.

Proof by Construction – Greg

We'll consider the series $F_0, F_1, ...$ such that $F_n = 2^{2^n} + 1$.
Lemma: $\prod_{i=0}^{n-1} F_i = F_n - 2$.
Proof: We will use induction. Assume FTSOC that the relation holds $\forall i$ s.t. $0 < i < n$. Then

$$\prod_{i=0}^{n} F_i = F_n \cdot \prod_{i=0}^{n-1} F_i = (2^{2^n} + 1) \cdot ((2^{2^n} + 1) - 2) = 2^{2^n} \cdot 2^{2^n} - 1 = F_{n+1} - 2$$

Now we wish to show that for any i, j s.t. $i \neq j$, F_i and F_j are relatively prime. WLOG, $j > i$. Let $d = gcd(F_i, F_j)$. Since $d|F_j$, $d|(\prod_{k=0}^{j-1} F_k + 2)$. We know $d| \prod_{k=0}^{j-1} F_k$ since $d|F_i$, so $d|2 \implies d = 2$ or $d = 1$. But all the F_n are odd, so $d = 1$. Now we must have an infinite sequence of relatively prime numbers, which is only possible if we have infinitely many primes at our disposal. Proof count: 10.

Proof of Infinite Proofs by Amber

Let's look at

$$17,\ 17 + 1,\ 17(17 + 1) + 1,\ 17(17 + 1)\Big(17(17 + 1) + 1\Big) + 1, ...$$

Each term is generated by multiplying the previous term by 17, then adding 1. Because all of the terms are relatively prime to each other, there must be a new prime for each new term in our list. We can extend this to all $n > 1, n \in \mathbb{N}$, and this gives us an infinite number of ways to show that there are an infinite number of primes:

$$n,\ n + 1,\ n(n + 1) + 1,\ n(n + 1)\Big(n(n + 1) + 1\Big) + 1, ...$$

We can further generalize this by considering any pair of numbers a, b where $\gcd(a, b) = 1$. We can construct a sequence of terms each relatively prime to the others by multiplying together all of the previous terms together:

$$a,\ a+b,\ a(a+b)+b,\ a(a+b)\Big(a(a+b)+b\Big)+b, ...$$

We've found an infinite number of ways to construct infinite sequences that prove there are an infinite number of primes. Proof count: $11 + \infty + \infty \cdot \infty$

Infinitude of Specific Classes of Primes

Proof (Folklore) – Jessica

We'll prove that there are infinitely many primes congruent to $3 \pmod 4$. Let $f_1, f_2, ...f_k$ be a list of primes of this form. Let $N = 4f_1f_2 \cdots f_k - 1$. Note that $N \equiv 3 \pmod 4$. Either N is prime or N is composite. If N is prime, we're done - we've found a prime $\equiv 3 \pmod 4$ that's not on our list. If N is composite, it must have some prime factor $\equiv 3 \pmod 4$, even though it's relatively prime to everything in our list. Therefore, we've found a new prime $\equiv 3 \pmod 4$. So, there are infinitely many primes. Proof count: 2.

Proof by Jessie

We will show that there are infintely many primes of the form $p = 4k + 1$. Let $N = (n!)^2 + 1$ for $n > 1$, and note that the smallest prime $p \mid N$ is some $p > n, p \neq 2$. Then,

$$N = 0 \text{ (mod } p)$$
$$(n!)^2 + 1 \equiv 0 \text{ (mod } p)$$
$$(n!)^2 \equiv -1 \text{ (mod } p)$$
$$(n!)^{p-1} \equiv (-1)^{\frac{p-1}{2}} \text{ (mod } p)$$

By Fermat's,

$$1 \equiv (-1)^{\frac{p-1}{2}} \text{ (mod } p)$$

Thus, $\frac{p-1}{2} = 2k$, so $p = 4k + 1$. This means we can generate a new prime for any n, so there are infinitely many primes. Proof count: 3.

Proof using Dirichlet's Theorem for Arithmetic Progressions

For any two positive coprime integers a and d, there are infinitely many primes of the form $a + nd$, where n is also a positive integer. In other words, there are infinitely many primes that are congruent to a mod d.

Case where $a = 1$:

Proof by Paul (2)

Counting by Cerruti and Murun:
Legendre Function:

$$\phi(x, y) = |\{n \mid n \leq x, n \text{ has no prime factors} \leq y\}|$$

Claim:

$$\pi(x) = \pi(\sqrt{x}) + \phi(x, \sqrt{x}) - 1.$$

Let $p_1, p_2, ..., p_k$ be all the primes $\leq y$. By PIE,

$$\phi(x,y) = x - \sum \left\lfloor \frac{x}{p_i} \right\rfloor + \sum \left\lfloor \frac{x}{p_i p_j} \right\rfloor + \cdots + (-1)^n \left\lfloor \frac{x}{p_1 p_2 \cdots p_k} \right\rfloor$$

If we let $N = p_1 p_2 \cdots p_k$, and this includes every prime, then

$$L = \phi(N^2, N) = N^2 - \sum \left\lfloor \frac{N^2}{p_i} \right\rfloor + \sum \left\lfloor \frac{N^2}{p_i p_j} \right\rfloor + \cdots + (-1)^n \left\lfloor \frac{N^2}{p_1 p_2 \cdots p_k} \right\rfloor = mN.$$

So every prime divides 1, which is a contradiction. Proof count: 11.

Proof of Infinite Proofs by Cailan:

Let's consider the Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n} s = \frac{1}{1s} + \frac{1}{2s} + \frac{1}{3s} + ... + \frac{1}{p_1^{e1} ... p_k^{ek} s} + ...$$

By Unique Prime Factorization, we can represent each n as a product of primes to the power of primes, $p_1^{e1} ... p_k^{ek}$. We can represent the entire sum as the product of many geometric sequences:

$$\zeta(s) = (1 + \frac{1}{2s} + \frac{1}{2^2 s} + ...)(1 + \frac{1}{3s} + \frac{1}{3^3 s} + ...)...(1 + \frac{1}{ps} + \frac{1}{p^p s} + ...) = \Pi_{p \text{ prime}}(1 - \frac{1}{p^s})^{-1}$$

Now, AFTSOC there are finitely many primes. For $s = 1$, the left hand side of the Riemann zeta function diverges. However, the right hand side is a bunch of geometric sequences, each with a common ratio $r = \frac{1}{p^s}$. The product of converging geometric sequences is finite, so the right hand side is finite. This contradicts the left hand side, which diverges, thus there are infinitely many primes.

Recall $\forall k \in \mathbb{Z}^+$ at $s = 2k$, where B is a Bernoulli number:

$$\Pi_{p \text{ prime}}(1 - \frac{1}{p^{2k}})^{-1} = \zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k!)}$$

AFTSOC that there are finitely many primes. Then, the left hand side is rational. However, π is transcendental, so the right hand side can't be rational. This is a contradiction, therefore there are infinitely many primes. Proof count: $11 + \infty$