# Checking Proofs with Computers

## and the Curry-Howard Correspondence

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# A Little About Me

- HCSSiM 2019 alum
- Graduate Student in Math @ University of Washington
  - BA Math/Computer Science from Cornell University
- Combinatorics! 💙

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  - had a formal verification phase

# Context

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- Optimal sphere-packing (1998)
- “Happy ending problem” for hexagon – 17 points (2006)
- Solving any Rubik’s Cube – 20 moves (2010)
- Minimum number of clues in sudoku – 17 (2012)

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- The Feit-Thompson theorem about symsets of odd size (Rocq, 2012)
- Peter Scholze, condensed mathematics (Lean, 2021)
- Terence Tao, the Polynomial Freiman-Ruzsa conjecture (Lean, 2023)


Q: ⇒ ⇒ /?

Q: 🤔 ⇒ 💻 ⇒ ✅/❌?

A: ✨*programming languages*✨!

# Simply-Typed 🐑-da Calculus

# Syntax

Terms in the simply-typed -da calculus:

$$e ::= c \mid x \mid (x : t) \to e \mid e_1\, e_2$$

Some examples of terms:

$$\texttt{true},\ \texttt{false},\ *,\ x,\ y,\ (x) \to x,\ ((x) \to \texttt{true})\ *$$

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All terms have types:

$$t ::= T \mid t_1 \to t_2$$

Basic types include `bool`, `unit`, etc.

$$\texttt{true} : \texttt{bool} \quad \texttt{false} : \texttt{bool} \quad * : \texttt{unit} \quad 🐑(x) \to \texttt{true} : t \to \texttt{bool}$$

# Examples: Functions

## Example

- $e_1 = 🐑(x : t) \to x \ : \ ???$

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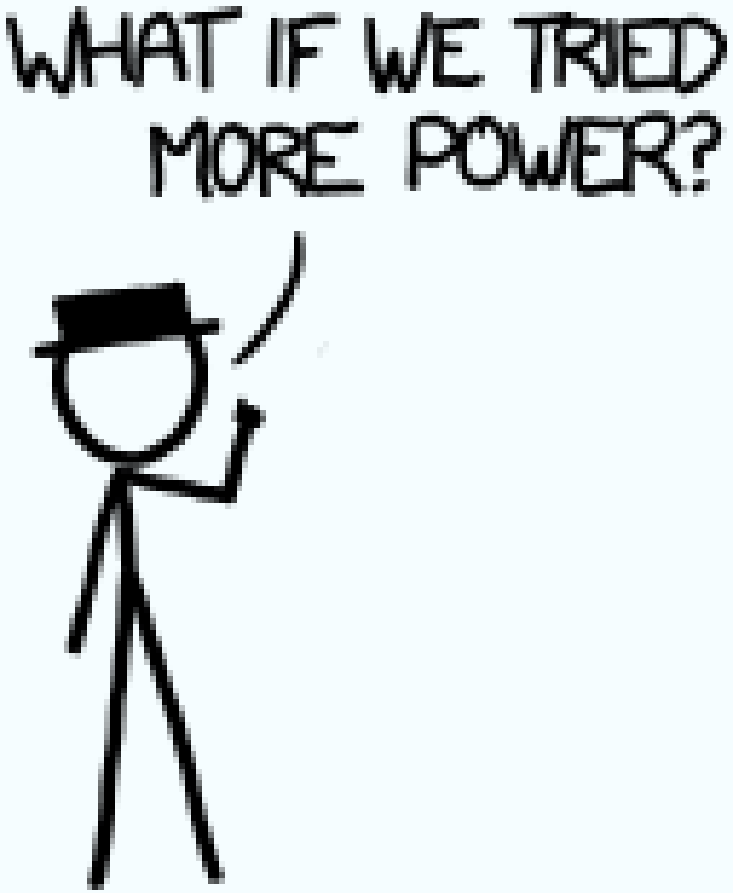
- $e_3 = \lambda(x, y) \to x \ y \ : \ (t_1 \to t_2) \to t_1 \to t_2$ evaluation function
  - $e_3 \ e_1 \ * = e_1 \ * = *$

# Exercise/Questions?

*What is the type of the following term?*

$$e_4 = \text{🐣}(x, y, z) \to x\ z\ (y\ z)$$


WHAT IF WE TRIED
MORE POWER?

# Let’s Talk About Types!

# The Natural Numbers

Introducing $\mathbb{N}$ with *inductive types*:

$$\begin{array}{l}\texttt{inductive}\ \mathbb{N} \\ \mid O : \mathbb{N} \\ \mid S : \mathbb{N} \to \mathbb{N}\end{array}$$

$O$ and $S$ are called *constructors* (also *introduction rules*).

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$$O \sim 0 \in \mathbb{N} \qquad S \sim \mathsf{succ} : \mathbb{N} \to \mathbb{N}$$
$$S\ O \sim \mathsf{succ}(0) = 1 \in \mathbb{N}$$

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is_zero : ℕ → bool =
  🐑(x : ℕ) → match x with
  | O ⇒ true
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```
add : ℕ → ℕ → ℕ =
  🐑(x, y) → match y with
  | O ⇒ x
  | S k ⇒ S (add x k)
```

# Data Structures

Inductive types allow us to model lists of numbers:

```
inductive natlist
| nil : natlist
| cons : 🐑(a : ℕ) → 🐑(t : natlist) → natlist
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**Example**

```
nil  ~  []
```

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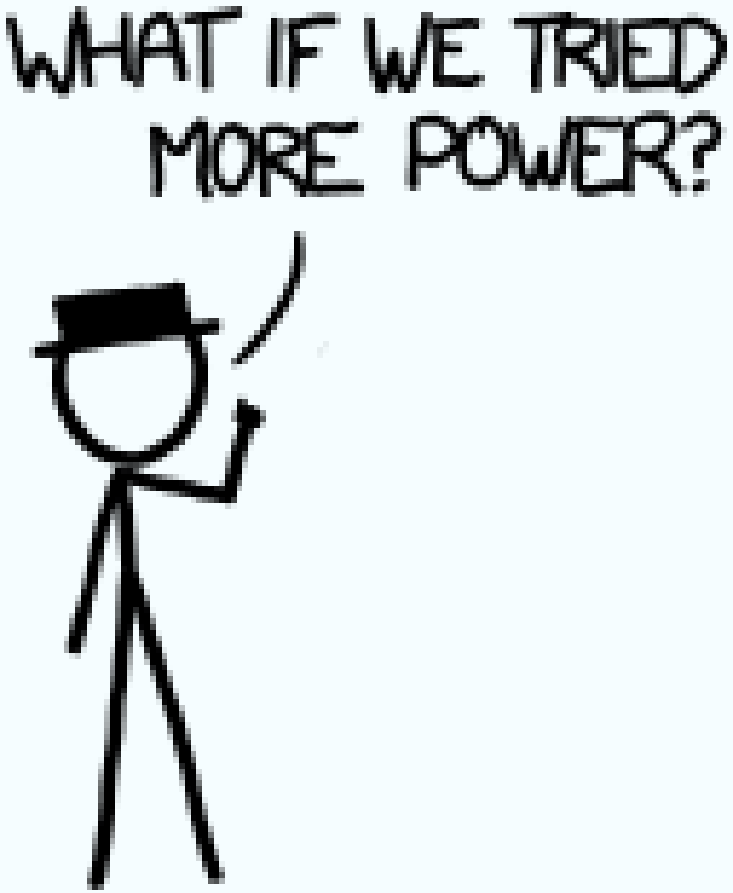
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# Combined Syntax

Combine rules for making terms vs. types into one rule:

$$t ::= x \mid T \mid \text{🐑}(x : t_1) \to t_2 \mid t_1\, t_2 \mid \mathbb{T}$$

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- Dependent function types
  - Functions whose type is $\text{🐑}(x : t_1) \to t_2$, where $t_2$ may contain $x$
  - "$t_1 \to t_2$" $= \text{🐑}(\_ : t_1) \to t_2$ ($t_2$ doesn't depend on $x$)

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- Introducing the *sort* $\mathbb{T}$, the "type" of types
  - $\texttt{bool} : \mathbb{T}$, $\texttt{unit} : \mathbb{T}$, $\mathbb{N} : \mathbb{T}$, etc.

# Application: Lists (again)

Recall our type for lists of $\mathbb{N}$s:

$$
\begin{aligned}
&\texttt{inductive natlist}\\
&\mid \texttt{nil} : \texttt{natlist}\\
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Why not build a list for *any* type?

$$\begin{array}{l}\texttt{inductive list} : \mathbb{T} \to \mathbb{T}\\ \mid \texttt{nil} : \textcolor{yellow}{\bullet}(\alpha : \mathbb{T}) \to \texttt{list}\ \alpha\\ \mid \texttt{cons} : \textcolor{yellow}{\bullet}(\alpha : \mathbb{T}) \to \textcolor{yellow}{\bullet}(a : \alpha) \to \textcolor{yellow}{\bullet}(t : \texttt{list}\ \alpha) \to \texttt{list}\ \alpha\end{array}$$

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- Types depending on types: type constructors
  - $\texttt{list}\ \alpha$ depends on the type $\alpha$.
- Types depending on terms: *dependent types*
  - ???

# Example: Type that Depends on a Term

Adapt our polymorphic list to record the length of the list!
Recall:

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Now consider:

$$\begin{aligned}&\texttt{inductive vector} : \mathbb{T} \to \mathbb{N} \to \mathbb{T}\\ &\mid \texttt{nil} : \textcolor{yellow}{\bullet}(\alpha : \mathbb{T}) \to \texttt{vector}\ \alpha\ O\\ &\mid \texttt{cons} : \textcolor{yellow}{\bullet}(\alpha : \mathbb{T}, a : \alpha, n : \mathbb{N}, t : \texttt{vector}\ \alpha\ n) \to \texttt{vector}\ \alpha\ (S\ n)\end{aligned}$$

# Examples: Inductive Types

Some important inductive dependent types:

» `inductive combine :` $(\alpha : \mathbb{T}) \to (\beta : \mathbb{T}) \to \mathbb{T}$
| `pair :` $\alpha \to \beta \to$ `combine` $\alpha\ \beta$

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  - $\texttt{pair}\ (S\ O)\ \texttt{true} : \texttt{combine}\ \mathbb{N}\ \texttt{bool}$

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- ❖ $\texttt{snd false} : \texttt{choose}\ \mathbb{N}\ \texttt{bool}$
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- `inductive none` : $\mathbb{T}$

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- `inductive none` : $\mathbb{T}$
  - There are NO terms of type `none`! (*uninhabited*)

# Revisiting Logic

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# The Correspondence

# The Curry-Howard Correspondence

*types* $\rightleftharpoons$ *propositions* *terms* $\rightleftharpoons$ *proofs*

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| ??? | $P \to Q$ | ??? | $f : p \mapsto q$ |
| ??? | true | ??? | $\exists$ |
| ??? | false | ??? | $\nexists$ |
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$\Downarrow$

checking proof correctness $\rightleftharpoons$ type-checking term

# Demo!

# Resources

- Rocq
  - Software Foundations
- Lean
  - The Natural Number Game
  - leanprover-community (mathlib), Xena Project
- Agda (less beginner-friendly)
  - HoTTEST Summer School 2022, HoTT Book
  - agda-unimath, TypeTopology
  - cubical Agda, 1Lab

# Secret Slides

# What's the type of a sort?

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**NO!** It's actually not consistent to allow $\mathbb{T} : \mathbb{T}$ – the argument of this is similar to "the barber who shaves everyone who doesn't shave themselves", i.e. Bertrand's paradox. Instead, we have *Girard's paradox* which breaks this construction.

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In fact, we actually need $\mathbb{T} : \mathbb{T}_1$, and then $\mathbb{T}_1 : \mathbb{T}_2$, etc. The $\mathbb{T}_i$s are called *type universes* and this is more of a technical point that's need to make things work, but often isn't relevant.

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We actually can just make an inductive type for this:

$$\texttt{inductive equal} : \prod(\alpha : \mathbb{T}) \to \alpha \to \alpha \to \mathbb{T}$$
$$\mid\ \texttt{refl} : \prod(\alpha : \mathbb{T}) \to \prod(a : \alpha) \to \texttt{equal}\ \alpha\ a\ a$$

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There is only one constructor of this type, and it represents the reflexivity of equality (where the two sides of the equality are equivalent).

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$$\begin{aligned}&\texttt{inductive leq} : (m : \mathbb{N}) \to (n : \mathbb{N}) \to \mathbb{T}\\ &\mid \texttt{leq-O} : (n : \mathbb{N}) \to \texttt{leq}\ O\ n\\ &\mid \texttt{leq-S} : (m : \mathbb{N}, n : \mathbb{N}, H : \texttt{leq}\ m\ n) \to \texttt{leq}\ (S\ m)\ (S\ n)\end{aligned}$$

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Here, we can get a proof of this proposition if we either generate it directly from the fact that all naturals are $\geq$ O, or by reducing successor functions on either side of the $\leq$ until we get the above case. This is again inductive!

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*Record types* accomplish this, bundling together the type for the underlying set and all necessary operations/properties. For example, here's a group:

$$
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&\texttt{struct group} : (\alpha : \mathbb{T}) \to \mathbb{T} \quad \{ \\
&\mid \quad \texttt{mul} : \alpha \to \alpha \to \alpha \\
&\mid \quad \texttt{id} : \alpha \\
&\mid \quad \texttt{inv} : \alpha \to \alpha \\
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One can also encode hierarchies of algebraic structures in types as well, e.g. a ring is a group + structure. Record types are also useful for encoding $\exists$ propositions.