How to Represent a Permutation

and why i drew boxes all summer

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Abstract

Representation theory describes a broad strategy of analyzing algebraic structures by understanding how they act on vector spaces. This allows us to employ the tools of linear algebra to make studying abstract algebraic objects more tractible. We will look at some basic tools in representation theory as applied to the symmetric group, and talk about connections to combinatorics. (Time-permitting, we may also look at applications of the theory to current research in algebraic combinatorics!)

pretext:

- ask: how much familiarity with linear algebra? today all of our vector spaces will be *complex* (algebraically closed, characteristic zero)
- ask: group theory?

1 Context

- what is a representation? why study representation theory?
- philosophy: studying the structure of objects by studying how they act on other things
- touches algebra (naturally), but also geometry and combinatorics
- acting on vector spaces is nice! in particular, we know how to study linear transformations on vector spaces (just matrices and linear algebra)

2 Basics

Definition. A (finite-dimensional) *representation* of a group G is:

- a group homomorphism $\rho : G \to \mathrm{GL}_n(\mathbb{C})$.
- a $\mathbb{C}[G]$-module M. (abelian group with action of $\mathbb{C}[G]$, much like a generalization of a vector space.)
  $\mathbb{C}[G]$ can be thought of the set of formal complex linear combinations of G, and you can think of this as a function $f : G \to \mathbb{C}$, where a linear combination corresponds to "$\sum_{g\in G} f(g)g$".

Example

Consider the dihedral group D_3 acting on the plane via rotations and reflections (can view it as $\mathbb{R}^2$ inside $\mathbb{C}^2$, really):

$$D_3 = \left\langle r, s \mid r^3 = s^2 = 1 \right\rangle$$

$$r = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \qquad s = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example

Look at all one-dimensional representations of D_3.

The trivial representation of D_3 – everything acts as identity! Also, thinking of $D_3 = S_3$, the sign representation

Also, the standard representation in matrices (embed this as acting on a space with 3 basis vectors)

Example

The regular representation of any finite group G – one basis element for every element in the group.

Definition. A *subrepresentation* of a representation:

- $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ is a subspace $W \subseteq \mathbb{C}^n$ preserved by $\rho(g)$ for all $g \in G$.
- M is a submodule of M.

Definition. A representation is *irreducible* if there are no proper subrepresentations (i.e. if ρ is a representation of G, then $\rho(g)$ has no nontrivial invariant subspaces across all g, and if M is a representation of G, it is a simple $\mathbb{C}[G]$-module.)

Very important theorems:

Theorem (Schur's Lemma)**.** Any intertwining map of irreducible representations is either the zero map (if these representations are not isomorphic) or a scalar multiple of the identity. $f : V \to W$, ρ_V, ρ_W reps, for any $g \in G$, $f \circ \rho_V(g) = \rho_W(g) \circ f$.

NOT true if the field is not algebraically closed!

Theorem (Maschke's Theorem (+ Schur's Lemma))**.** Any representation of G over $\mathbb{C}$ has a (unique) decomposition into a direct sum of irreducibles.

NOT true in fields with positive characteristic (or algebraically closed!)

3 Characters

Definition. For G a group and $\rho : G \to \mathrm{GL}(V)$ a representation of a group G on V, the *character* of ρ is the function $\chi_\rho : G \to \mathbb{C}$ given by $\chi(g) = \mathrm{tr}(\rho(g))$.

The theory of characters is very rich! skipping over a lot of proofs.

Let ρ be a representation of G over $\mathbb{C}^n$, and χ the associated character. Some facts about characters:

- $\chi(1) = n$, the dimension of the ambient space
- $\chi(g^{-1}) = \overline{\chi(g)}$
- for another representation σ, $\chi_\rho + \chi_\sigma = \chi_{\rho\oplus\sigma}$, and $\chi_\rho \cdot \chi_\sigma = \chi_{\rho\otimes\sigma}$.
- χ is constant on conjugacy classes

Definition. The *character table* of a group G is a square table with the rows being irreducible representations of G and the columns the conjugacy classes of G, and each entry the value of that irrep on that conjugacy class.

Example 3.1

Symmetric group $D_3 = S_3$ and S_4:

S_3	(1)	(12)	(123)
trivial, (3)	1	1	1
sign, (1,1,1)	1	-1	1
standard, (2,1)	2	0	-1

Definition. A *class function* on G is a complex-valued function on G which is constant on conjugacy classes. Note that characters are class functions, and inside the group ring (thought of as functions), these commute with all elements under multiplication.

Following from Schur's Lemma, irreducible characters are very very special:

Theorem. The set of class functions on G forms a complex vector space, on which we can define the following Hermitian inner product:

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g\in G} \overline{\varphi(g)}\psi(g).$$

Irreducible characters form an orthonormal basis for this vector space. In particular, if ρ_i, ρ_j are irreps of G, and χ_i and χ_j are the associated irreducible characters, then

$$\langle \chi_i, \chi_j \rangle = \begin{cases} 1 & \rho_i \cong \rho_j \\ 0 & \rho_i \ncong \rho_j \end{cases}$$

(check this by example)

Corollary. ρ is irreducible iff $\langle \chi, \chi \rangle = 1$.

Corollary. The number of irreps of G is equal to the number of conjugacy classes of G.

Corollary. Isomorphic representations have the same character.

Corollary. The columns of the character table are orthogonal under the standard inner product. Precisely, if s and t are in different conjugacy classes, then $\sum_{i=1}^{r} \chi_i(s)\overline{\chi_i(t)} = 0$, where this sum goes over all irreps.

Example

Character of the regular representation can be computed – $\chi_G(1) = |G|$, and $\chi_G(g) = 0$ for $g \neq 1 \in G$.

Projecting this in terms of the irreducibles χ_i, we have that

$$\langle \chi_i, \chi_G \rangle = \frac{1}{|G|}\overline{\chi_G(1)}\chi_i(1) = n_i$$

where n_i is the dimension of the irrep. Therefore, $\chi_G = \sum_{i=1}^{r} n_i \chi_i$ as we sum over all irreps, and by dimension counting we have that $|G| = \sum_{i=1}^{r} n_i^2$.

Let's use this information to figure out all of the irreps of S_4 by their characters.

- Take trivial and sign representation
- look at the standard representation, it's not irreducible but getting rid of $e_1 + e_2 + e_3 + e_4$, it then becomes so! (trivial rep)
- take the tensor product of this with the sign rep
- Work out the last irrep from dimension counting and then orthogonality (by columns is easiest)

S_4	(1)	(12)	(12)(34)	(123)	(1234)
trivial (4)	1	1	1	1	1
sign (1, 1, 1, 1)	1	-1	1	1	-1
standard (3, 1)	3	1	-1	0	-1
dual standard (2, 1, 1)	3	-1	-1	0	1
last (2, 2)	2	0	2	-1	0

4 Specht Modules

Note that a permutation $\sigma \in S_n$ can be decomposed into a product of cycles, and the conjugacy classes of S_n are indexed by partitions of n (into cycles). Therefore, the irreducible representations are also in bijection with the partitions of n – and in particular, we can construct them for every partition of n explicitly!

Fix your favorite partition $\lambda \vdash n$. Draw as a Young diagram.

- Look at every *standard Young tableau* of shape λ. Put the numbers 1-n in boxes of λ, increasing down and across.
- Consider the *tabloid* for every SYT t of shape λ (giving a tabloid $\{t\}$, where we take equivalence classes of rows.
- For every permutation σ of the numbers that preserves the columns of t, apply σ to $\{t\}$ and keep a running sum of $\operatorname{sgn}(\sigma)\,\{\sigma \cdot t\}$. This gives the *associated polytabloid* for t.

The complex vector space spanned by these associated polytabloids is called the *Specht module* corresponding to λ.

Example

$\lambda = (2, 2)$.

5 Symmetric Functions

Definition. A *symmetric function* f in n variables $x_1, \ldots, x_n$ is a polynomial in those variables such that for any permutation $\sigma \in S_n$, $f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

Several families:

$$p_k = \sum_{i=1}^{n} x_i^k$$

$$h_k = \sum_{l_1 + \ldots l_n = k, l_i \geq 0} x_1^{l_1} \ldots x_n^{l_n}$$

Special: *Schur polynomials*, a positive basis for products of symmetric functions.

$$s_\lambda = \sum_{T \in SSYT(\lambda)} x_T, \qquad x_T = \prod_{i \in T} x_i$$

Definition. For any class function on S_n, f, define $\operatorname{ch}(f) = \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma) p_{c(\sigma)}$, where $c(\sigma)$ is the partition that σ defines (its cycle type) and $p_\lambda = p_{\lambda_1} \ldots p_{\lambda_k}$

Theorem. The map ch from the ring of class functions of S_n to the symmetric functions is an isomorphism.

Manipulating representations of S_n is as easy as manipulating polynomials!

6 Why?

In algebraic combinatorics, it's not uncommon to study rings associated to certain combinatorial structures, which might admit a natural group action. (For instance, the Stanley-Reisner ring associated to a simplicial complex). One example that I care a lot about, a quotient of the Stanley-Reisner ring of the order complex of the Boolean lattice:

$$k[\Delta B_n] = \frac{k[x_F : F \in B_n]}{\langle x_P x_Q : P, Q \text{ incomparable}\rangle}$$

$$C(B_n) := \frac{k[\Delta B_n]}{\langle \Theta_{colorful}\rangle}, \Theta_{colorful} = \left\{ \sum_{|F|=i} x_F, 0 \leq k \leq n \right\}$$

When $k = \mathbb{C}$ (we usually want it to be a field) each graded component is a vector space, admitting an S_n-action $\sigma \cdot x_F = x_{\sigma(F)}$. This then makes each graded component an S_n-representation!

Different graded components of this family of rings are related to each other by restricting representations of S_n to the subgroup S_{n-1}. In the course of figuring this out we see that:

Theorem 6.1 (DHKLT 2023)

For λ/μ a ribbon tableau with n boxes,

$$s_{\lambda/\mu} \downarrow^{S_n}_{S_{n-1}} = \text{you know what it is}$$