Exercise 2.9

The *width* of an interval is half of the difference between its upper and lower bounds. The width is a measure of the uncertainty of the number specified by the interval. For some arithmetic operations the width of the result of combining two intervals is a function only of the widths of the argument intervals, whereas for others the width of the combination is not a function of the widths of the argument intervals.

Question

Show that the width of the sum (or difference) of two intervals is a function only of the widths of the intervals being added (or subtracted).

Theorem.

$$\mathrm{Width}([a,b]+[c,d]) = f(\mathrm{Width}([a,b]), \mathrm{Width}([c,d]))$$

Proof. First, note that

$$\mathrm{Width}([a,b]) = \frac{b-a}{2} = w_{ab}$$

The width of $[a,b]+[c,d]$ is then

$$\begin{aligned}\mathrm{Width}([a,b]+[c,d]) &= \mathrm{Width}([a+c, b+d]) \\ &= \frac{(b+d)-(a+c)}{2} \\ &= \frac{b-a}{2} + \frac{d-c}{2} \\ &= w_{ab} + w_{cd}\end{aligned}$$

which is a function of the widths of the summands. A similar proof holds for subtraction. *Q.E.D*

Question 2

Give examples to show that this is not true for multiplication or division.

Let's give a general proof instead:

Theorem.

$$\text{Width}([a,b]\times[c,d]) \neq f(\text{Width}([a,b]),\text{Width}([c,d]))$$

Proof.

There are four possible values for the upper and lower intervals:

$$\begin{aligned} p_1 &= ac \\ p_2 &= ad \\ p_3 &= bc \\ p_4 &= bd \end{aligned}$$

Any pairwise combination of these four values (twelve pairs all in all) will be a function of at most one width:

Pair	Width	Pair	Width
w_{12}	aw_{cd}	w_{21}	$-aw_{cd}$
w_{13}	cw_{ab}	w_{31}	$-cw_{ab}$
w_{14}	$\frac{bd-ac}{2}$	w_{41}	$\frac{ac-bd}{2}$
w_{23}	$\frac{bc-ad}{2}$	w_{32}	$\frac{ad-bc}{2}$
w_{24}	dw_{ab}	w_{42}	$-dw_{ab}$
w_{34}	bw_{cd}	w_{43}	$-bw_{cd}$

Q.E.D