Exercise 1.13

Prove that $\mathrm{Fib}(n)$ is the closest integer to $\phi^n/\sqrt{5}$, where

$$\phi = \frac{1+\sqrt{5}}{2}$$

Hint: Let

$$\psi = \frac{1-\sqrt{5}}{2}$$

Use induction and the definition of the Fibonacci numbers to prove that

$$\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Theorem 1. The nth Fibonacci number, $\mathrm{Fib}(n)$, is equal to

$$\frac{\phi^n - \psi^n}{\sqrt{5}}$$

Proof. We prove by induction.

Base case.

- $\mathrm{Fib}(0) = \frac{0}{\sqrt{5}} = 0$
- $\mathrm{Fib}(1) = \frac{\phi-\psi}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$
- $\mathrm{Fib}(2) = \frac{\phi^2-\psi^2}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1$

Inductive step.

$$
\begin{aligned}
\mathrm{Fib}(n) &= \mathrm{Fib}(n-1) + \mathrm{Fib}(n-2) \\
&= \frac{\phi^{n-1} - \psi^{n-1}}{\sqrt{5}} + \frac{\phi^{n-2} - \psi^{n-2}}{\sqrt{5}} \\
&= \frac{\phi^{n-1} + \phi^{n-2} - (\psi^{n-1} + \psi^{n-2})}{\sqrt{5}} \\
&= \frac{\phi^{n-2}(\phi + 1) - \psi^{n-2}(\psi + 1)}{\sqrt{5}}
\end{aligned}
$$

But

$$
\begin{aligned}
\phi^{n-2}(\phi + 1) &= \phi^{n-2}\left(\frac{3+\sqrt{5}}{2}\right) \\
&= \phi^{n-2}\left(\frac{6+2\sqrt{5}}{4}\right) \\
&= \phi^{n-2}\left(\frac{1+2\sqrt{5}+\sqrt{5}^2}{4}\right) \\
&= \phi^{n-2}\phi^2 \\
&= \phi
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\psi^{n-2}(\psi + 1) &= \psi^{n-2}\left(\frac{3-\sqrt{5}}{2}\right) \\
&= \psi^{n-2}\left(\frac{6-2\sqrt{5}}{4}\right) \\
&= \psi^{n-2}\left(\frac{1-2\sqrt{5}+\sqrt{5}^2}{4}\right) \\
&= \psi^{n-2}\psi^2 \\
&= \psi
\end{aligned}
$$

Therefore,

$$
\mathrm{Fib}(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}
$$

Q.E.D.

Corollary 1.

$$\text{Fib}(n) = \left[\frac{\phi^n}{\sqrt{5}}\right]$$

Proof. We construct a chain of iffs:

$$\begin{aligned} \text{Fib}(n) &\Leftrightarrow \frac{\phi^n - \psi^n}{\sqrt{5}} \\ &\Leftrightarrow \frac{\phi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} \end{aligned}$$

Note that $\left|\frac{\psi^n}{\sqrt{5}}\right|$ will never exceed $\left|\frac{\psi}{\sqrt{5}}\right|$, since $\phi < 1$. By extension, $\left|\frac{\psi^n}{\sqrt{5}}\right| < \frac{1}{2}$.

By the definition of $[x]$, if x is an integer, $[x] = [x \pm r]$ if $r < 0.5$; therefore,

$$\frac{\phi^n}{\sqrt{5}} - \frac{\psi^n}{\sqrt{5}} \Leftrightarrow \left[\frac{\phi^n}{\sqrt{5}}\right]$$

Q.E.D.