匈牙利算法

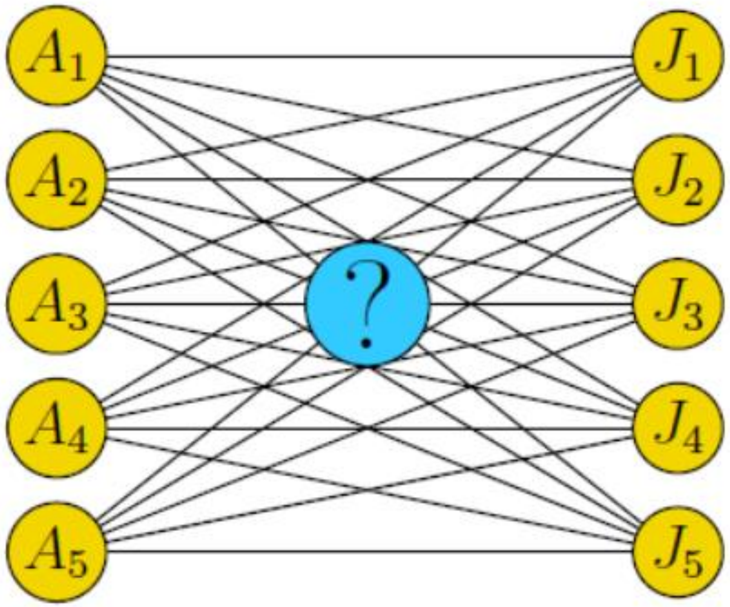
Hungarian Algorithm

Assignment Problem

分配问题（指派问题）

The assignment problem deals with assigning machines to tasks, workers to jobs, and so on.

The goal is to determine the optimum assignment that, for example, minimizes the total cost.



匈牙利算法是一种在多项式时间内求解任务分配问题的组合优化算法

Assignment problem :

A special class of transportation problem is called assignment problems in which:

1) $supply_i = demand_j = 1$

2) each decision variable is a binary decision varable (0,1)

3) all the constraints are in the form of equations (with "=" signs)

Assignment problem: Example

There are 4 cranes. Each crane must be allocated to one job.

The time required for each job is shown in the table.

Find the best assignment of cranes to the jobs so the total cost required for completing the jobs is minimized:

	Job1	Job2	Job3	Job4
crane1	16 x_{11}	2 x_{12}	3 x_{13}	7 x_{14}
crane2	5 x_{21}	13 x_{22}	7 x_{23}	5 x_{24}
crane3	8 x_{31}	6 x_{32}	5 x_{33}	9 x_{34}
crane4	3 X_{41}	4 X_{42}	5 X_{43}	11 X_{44}

$$\min Z = 16X_{11}+2X_{12}+3X_{13}+7X_{14}+5X_{21}+13X_{22}+7X_{23}+5X_{24}+8X_{31}+6X_{32}+5X_{33}+9X_{34}+3X_{41}+4X_{42}+5X_{43}+11X_{44}$$

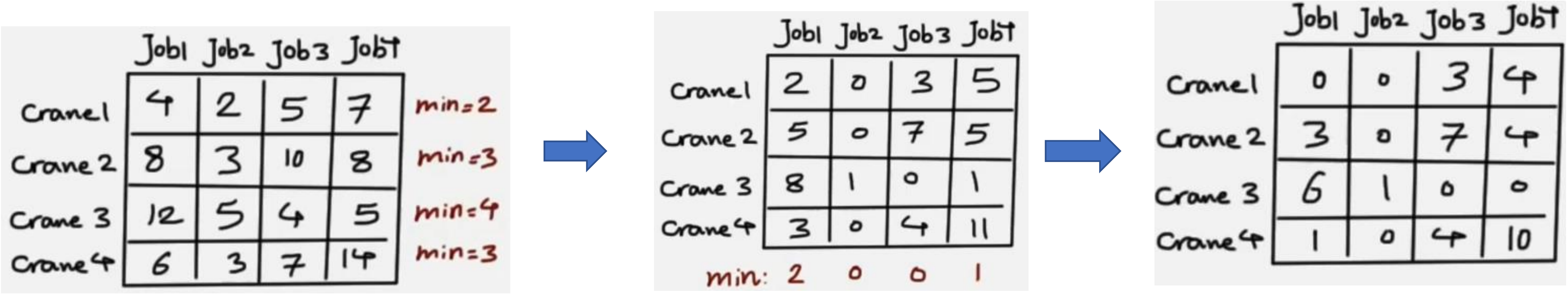
Cranes Constraints
$$\begin{cases} X_{11}+X_{12}+X_{13}+X_{14}=1 \\ X_{21}+X_{22}+X_{23}+X_{24}=1 \\ X_{31}+X_{32}+X_{33}+X_{34}=1 \\ X_{41}+X_{42}+X_{43}+X_{44}=1 \end{cases}$$

job Constraints
$$\begin{cases} X_{11}+X_{21}+X_{31}+X_{41}=1 \\ X_{12}+X_{22}+X_{23}+X_{24}=1 \\ X_{13}+X_{23}+X_{33}+X_{43}=1 \\ X_{14}+X_{24}+X_{34}+X_{44}=1 \end{cases}, \quad X_{ij}=\begin{cases} 1 & \text{if } i \text{ is allocated to } j \\ 0 & \text{otherwise} \end{cases}$$

算法步骤

1) **Row Reduction** : Find the minimum of each row, and subtract from each row the min value.

2) **Column Reduction**: Find the minimum of each column, and subtract from each column the min value.

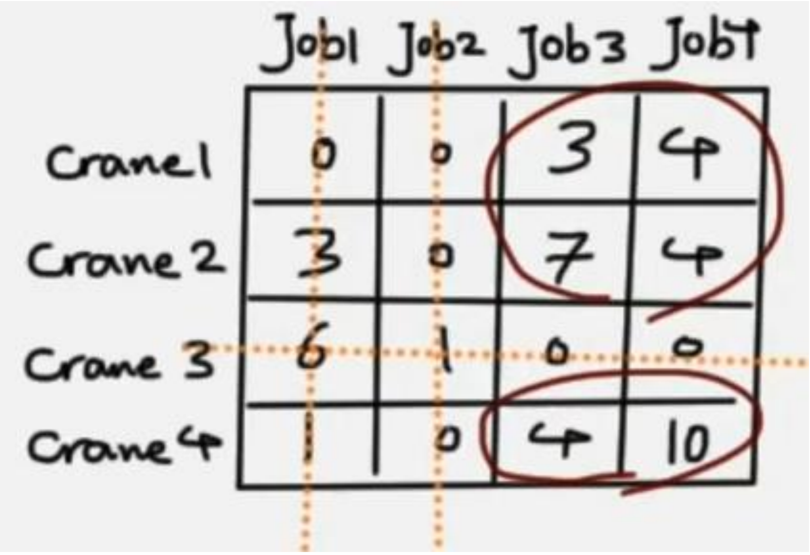


3） Cover all zeros with a minimum number of lines:

Find the min number of vertical/horizontal lines required to cover the zeros in the matrix.

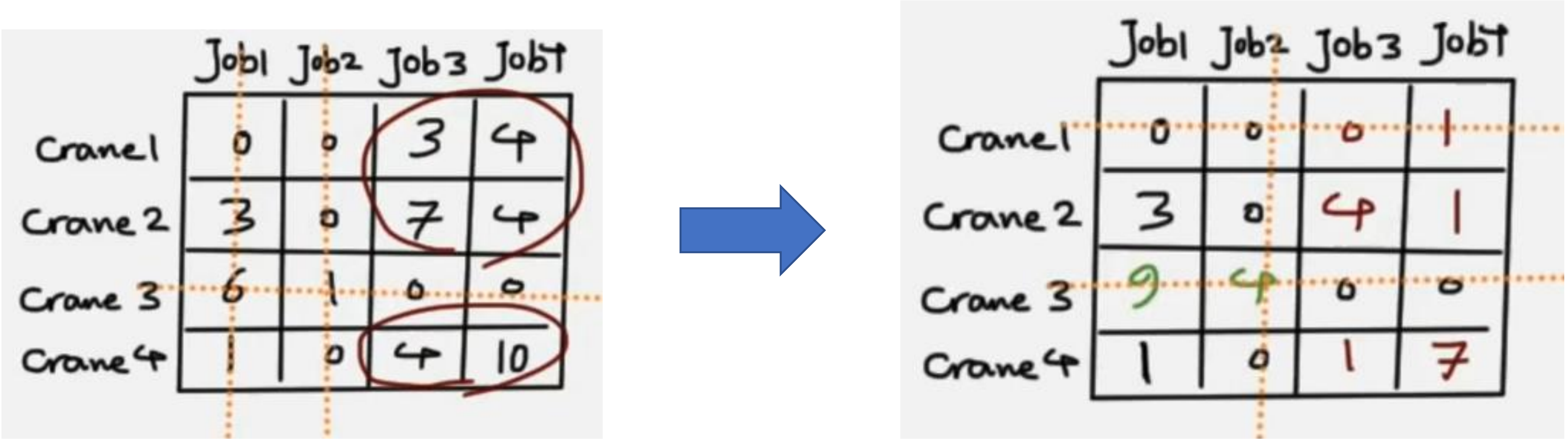
If the number of lines is equad to m, the optimal solution is available amony the covered zeros.

otherwise, proceed to Step four.



$3 \neq m = 4$

4) Create additional zeros: Find the min of uncovered values. Then subtract the min from all the uncovered values and add it to the corner points. Then go back to Step 3.



5) Making the final assignment

	Job1	Job2	Job3	Job4
Crane1	0	0	0	1
Crane 2	3	0	4	1
Crane 3	9	4	0	0
Crane 4	1	0	1	7

min = 1

	Job1	Job2	Job3	Job4
Crane1	0	1	0	1
Crane 2	2	0	3	0
Crane 3	9	5	0	0
Crane 4	0	0	0	6

minimum number of lines required=4=m So, we are at the optimal table.

Start the assignment from the row or column that has minimum number of zeros.

	Job1	Job2	Job3	Job4	
Crane1	0	1	0	1	2
Crane 2	2	0	3	0	2
Crane 3	9	5	0	0	2
Crane 4	0	0	0	6	~~3~~ ~~2~~ 1
	2	2	~~3~~ 2	~~2~~ 1	

This is the final assignment

	Job1	Job2	Job3	Job4
Crane1	0	1	0	1
Crane 2	2	0	3	0
Crane 3	9	5	0	0
Crane 4	0	0	0	6

$X_{11} = X_{22} = X_{34} = X_{43} = 1$

Find the cost associated to these assignments in the original cost table.

	Job1	Job2	Job3	Job4
Crane1	4	2	5	7
Crane 2	8	3	10	8
Crane 3	12	5	4	5
Crane 4	6	3	7	14

$Z^* = 4 + 3 + 5 + 7 = 19$

$\boxed{Z^* = 19}$