# The UMAP Journal

Vol. 25, No. 3

# Subscription Rates for 2004 Calendar Year: Volume 25

## Membership Plus

Individuals subscribe to *The UMAP Journal* through COMAP's Membership Plus. This subscription also includes print copies of our annual collection *UMAP Modules: Tools for Teaching*, our organizational newsletter *Consortium*, on-line membership that allows members to download and reproduce COMAP materials, and a 10% discount on all COMAP purchases.

| | | |
|---|---|---|
| **(Domestic)** | **#2420** | **$90** |
| **(Outside U.S.)** | **#2421** | **$105** |

## Institutional Plus Membership

Institutions can subscribe to the *Journal* through either Institutional Plus Membership, Regular Institutional Membership, or a Library Subscription. Institutional Plus Members receive two print copies of each of the quarterly issues of *The UMAP Journal*, our annual collection *UMAP Modules: Tools for Teaching*, our organizational newsletter *Consortium*, on-line membership that allows members to download and reproduce COMAP materials, and a 10% discount on all COMAP purchases.

| | | |
|---|---|---|
| **(Domestic)** | **#2470** | **$415** |
| **(Outside U.S.)** | **#2471** | **$435** |

## Institutional Membership

Regular Institutional members receive print copies of *The UMAP Journal*, our annual collection *UMAP Modules: Tools for Teaching*, our organizational newsletter *Consortium*, and a 10% discount on all COMAP purchases.

| | | |
|---|---|---|
| **(Domestic)** | **#2440** | **$180** |
| **(Outside U.S.)** | **#2441** | **$200** |

## Web Membership

Web membership does not provide print materials. Web members can download and reproduce COMAP materials, and receive a 10% discount on all COMAP purchases.

| | | |
|---|---|---|
| **(Domestic)** | **#2410** | **$39** |
| **(Outside U.S.)** | **#2410** | **$39** |

To order, send a check or money order to COMAP, or call toll-free
**1-800-77-COMAP** (1-800-772-6627).

*The UMAP Journal* is published quarterly by the Consortium for Mathematics and Its Applications (COMAP), Inc., Suite 210, 57 Bedford Street, Lexington, MA, 02420, in cooperation with the American Mathematical Association of Two-Year Colleges (AMATYC), the Mathematical Association of America (MAA), the National Council of Teachers of Mathematics (NCTM), the American Statistical Association (ASA), the Society for Industrial and Applied Mathematics (SIAM), and The Institute for Operations Research and the Management Sciences (INFORMS). The Journal acquaints readers with a wide variety of professional applications of the mathematical sciences and provides a forum for the discussion of new directions in mathematical education (ISSN 0197-3622).

Periodical rate postage paid at Boston, MA and at additional mailing offices.

**Send address changes to: info@comap.com**

COMAP, Inc. 57 Bedford Street, Suite 210, Lexington, MA 02420

Vol. 25, No. 3 2004

# Table of Contents

# Publisher's Editorial

## The Good Fight

Solomon A. Garfunkel
Executive Director
COMAP, Inc.
57 Bedford St., Suite 210
Lexington, MA 02420
s.garfunkel@mail.comap.com

The MCM issue of *The UMAP Journal* has historically been an opportunity for me to reflect on the year at COMAP and discuss with you many of the new projects underway. I am, with your indulgence, going to diverge from that tradition this year. Perhaps it is the coming (as I write this) election, but this has been an extremely political year. And that politics is having its effect on all of us involved in mathematics education. I have found myself as a consequence writing pieces that are more "polemical" in nature—defending our beliefs and our work within our community and without. So, with no apologies, here are two short essays that reflect my thoughts on the "good fight."

## Mathematical Breadth for All

The discussion (debate, war) about differentiating the curriculum for students with the perceived ability to go on in mathematics vs. the rest usually misses the crucial point of breadth. It is in many ways ironic that the "mathematics for all" movement has succeeded in infusing the secondary school mathematics curriculum with many important ideas and concepts that the "better" students simply do not see. A great deal of effort has gone into the creation of materials intended to show all students the usefulness of mathematics, through the use of contemporary applications and the processes of mathematical modeling. In many cases, this means teaching more discrete mathematics such as graph theory, game theory, social choice theory, and operations research.

Henry Pollak is fond of saying that there are four reasons for students to learn mathematics—to help them as they enter the world of work, to make them more knowledgeable citizens, to help them in their daily lives, and to

*The UMAP Journal* 24 (3) (2003) 185–188. 

have them appreciate the beauty and power of the discipline. Clearly, for the mathematically talented, we focus on the last, while for more average or less motivated students, we (hopefully) stress the first three. I believe that this is a terrible mistake and that we are paying a terrible price.

It is no secret that there has been a worldwide decline in mathematics majors. In the U.S., the half-life of students in mathematics courses, from 10th grade to the Ph.D., remains one year. In other words, if we look at the students enrolled in 11th-grade math courses, there are approximately half as many as were enrolled the previous year in 10th-grade math courses, and so on to the Ph.D. I argue that while we are doing a much better job in showing average students the importance and relevance of mathematics in their lives, we are simultaneously discouraging our brightest students from continuing their mathematical studies.

I believe that the reasons for this are clear. We assiduously avoid showing the mathematical elite the utility of our subject and its relevance to their daily lives, career choices, and role in society. There is some mythical linear sequence of courses from birth to the Ph.D., which we feel that they must take. Many of theses courses are highly technical, providing practice in skills necessary only for the next course. But we have ample proof that this delayed gratification simply does not work. Our best students are leaving mathematics for what they perceive as more relevant and rewarding fields, such as biology and finance.

Even if one accepts the notion that we should have a differentiated curriculum, based on ability, it is patently absurd to avoid showing our best and brightest students the power and utility of our subject. It isn't a horse race. Students don't have to take advanced calculus or point set topology by the age of 17, no matter how talented they might be. We need to show our students, at every ability level, the breadth of our discipline and the breadth of its applications. By not doing so, we only invite the disaffection we see.

# Now It's Personal

I should preface my comments by saying that I am first and foremost a curriculum developer. For the past 30 years, I have worked to produce curriculum materials that attempt to teach mathematics through contemporary applications and modeling. COMAP has produced literally thousands of modules from primary school through university-level mathematics, as well as several high school and tertiary texts, television and Web-based courses.

There is, however, in many countries, a feeling that we have created sufficient new curricula over the past several years and that before we create more, we need to look hard at what we have done and whether we have made a difference in student achievement. At first blush this makes perfectly good sense, but the devil is truly in the details. Everyone is aware that student achievement is affected by several factors, not the least of which is teacher preparation and performance. And with the new curricula, teacher training and staff development consistently lag woefully behind. In part, this is due to the enormity of the task, and in large part, to the enormity of the expense involved in doing staff development "right."

But most of this discussion is beside the point. What we have today is a call for research. We have politicians and colleagues saying that before we develop new materials we must learn what works. We must experiment (almost in the medical sense) in order to be certain that what we teach from now on has a sound body of research behind it. While I realize that this analysis sounds extreme, I assure you that at least in U.S. educational circles it is a reality. Moreover, this reality is being played out by real politicians who decide where educational funds will be spent. Sadly, this dichotomy can also be seen within the discipline of mathematics education. Most of us, of a certain age, came to mathematics education through other pathways—as mathematical researchers, or as university or secondary-school mathematics faculty. There was not yet a discipline of mathematics education, few Ph.D.s, no direct career path, and few journals to publish our work. We were in the truest sense self-taught. We learned what works by working.

I would argue that there are examples / models / problems that are beneficial on their face. These problems illustrate key aspects of the modeling process, can be set in contemporary and inherently interesting contexts, and permit us to teach and / or reinforce important mathematical concepts and skills. I believe that their introduction into the curriculum should not wait for double-blind experiments with control groups, based on a theoretical framework, evaluated through statistical techniques valid in a 95% confidence interval.

That mathematics education is now a respected discipline is, of course, a good thing. More and more talented young people are entering the field and more and more journals and international meetings give them respected outlets for their work. But, I fear that we are losing the best part of our past. Much of that past does not rest in journals, but in ourselves. Anyone who was fortunate enough to view the tape that Henry Pollak made for the ICMI study conference in Dortmund [2004], or to listen to the stirring words of Ubi D'Ambrosio at ICME-10 [2004], or go to any talk by Claudi Alsina (see [2001a; 2001b] will understand what I mean.

These giants may or may not describe their work in the vernacular of the day. They may or may not explicate a theoretical framework, or reference a standards document for content or an educational statistics journal for a methodology. But the quality of their ideas is a thing to be treasured. Yes, we must describe our work in ways that can be replicated. Yes, we must conduct real research to establish whether our ideas as implemented make a positive difference in student performance. Yes, we must publish our work in respectable journals, reviewed by our peers. But in the same way that we understand analogous truths about mathematics research, we must not lose sight of the art of mathematics education. Just as with mathematics, there is beauty and elegance here. We must continue to make room for those who would strike out in new ways—try new content, new applications, new technologies.

# References

Alsina, Claudi. 2001a. A guided tour of daily geometry. *The UMAP Journal* 21 (1): 3–14.

________. 2001b. Mathematics and comfort. *The UMAP Journal* 21 (1): 97–115.

D'Ambrosio, Ubiratan. 2004. Diffusion and popularization of science and mathematics. International Study Group on the Relations between History and Pedagogy of Mathematics, affiliated to the International Commission on Mathematical Instruction (ICMI). 10th International Congress on Mathematical Education (ICME-10). Copenhagen, Denmark.

Pollak, Henry. 2004. A history of the teaching of mathematical modeling in the U.S.A. Plenary Lecture, International Commission on Mathematical Instruction (ICMI) Study Conference 14: Applications and Modeling in Mathematics Education. Dortmund, Germany: University of Dortmund.

# About the Author

Sol Garfunkel received his Ph.D. in mathematical logic from the University of Wisconsin in 1967. He was at Cornell University and at the University of Connecticut at Storrs for 11 years and has dedicated the last 25 years to research and development efforts in mathematics education. He has been the Executive Director of COMAP since its inception in 1980.

He has directed a wide variety of projects, including UMAP (Undergraduate Mathematics and Its Applications Project), which led to the founding of this *Journal*, and HiMAP (High School Mathematics and Its Applications Project), both funded by the NSF. For Annenberg/CPB, he directed three telecourse projects: *For All Practical Purposes* (in which he also appeared as the on-camera host), *Against All Odds: Inside Statistics* (still showing on late-night TV in New York!), and *In Simplest Terms: College Algebra*. He is currently co-director of the Applications Reform in Secondary Education (ARISE) project, a comprehensive curriculum development project for secondary school mathematics.

# Modeling Forum

## Results of the 2004 Mathematical Contest in Modeling

Frank Giordano, MCM Director
Naval Postgraduate School
1 University Circle
Monterey, CA 93943-5000
`frgiorda@nps.navy.mil`

## Introduction

A total of 600 teams of undergraduates, from 253 institutions and 346 departments in 11 countries, spent the second weekend in February working on applied mathematics problems in the 20th Mathematical Contest in Modeling (MCM).

The 2004 MCM began at 8:00 P.M. EST on Thursday, Feb. 5 and ended at 8:00 P.M. EST on Monday, Feb. 9. During that time, teams of up to three undergraduates were to research and submit an optimal solution for one of two open-ended modeling problems. Students registered, obtained contest materials, downloaded the problems at the appropriate time, and entered completion data through COMAP'S MCM Website. After a weekend of hard work, solution papers were sent to COMAP on Monday. The top papers appear in this issue of *The UMAP Journal*.

In addition, this year, on the 20th anniversary of the founding of the MCM by Ben Fusaro, COMAP announces a new annual award for MCM papers. Typically, among the final papers from which the Outstanding ones are selected is a paper that is especially creative but contains a flaw that prevents it from attaining the Outstanding designation. In accord with Ben's wishes, the award will recognize such teams.

Results and winning papers from the first 19 contests were published in special issues of *Mathematical Modeling* (1985–1987) and *The UMAP Journal* (1985–2003). The 1994 volume of *Tools for Teaching*, commemorating the tenth anniversary of the contest, contains all of the 20 problems used in the first 10 years of

*The UMAP Journal* 25 (3) (2004) 189–214. 

the contest and a winning paper for each. Limited quantities of that volume and of the special MCM issues of the *Journal* for the last few years are available from COMAP.

This year's Problem A asked teams to develop a model to address the issue of the uniqueness of human thumbprints. Problem B asked teams to propose and test schemes for a Quick Pass system in an amusement park that allows customers to decrease their time spent waiting in line for the park rides.

In addition to the MCM, COMAP also sponsors the Interdisciplinary Contest in Modeling (ICM) and the High School Mathematical Contest in Modeling (HiMCM). The ICM, which runs concurrently with MCM, offers a modeling problem involving concepts in operations research, information science, and interdisciplinary issues in security and safety. Results of this year's ICM are on the COMAP Website at `http://www.comap.com/undergraduate/contests`; results and Outstanding papers appeared in Vol. 25 (2004), No. 2. The HiMCM offers high school students a modeling opportunity similar to the MCM. Further details about the HiMCM are at `http://www.comap.com/highschool/contests` .

# Problem A: Are Fingerprints Unique?

It is a commonplace belief that the thumbprint of every human who has ever lived is different.

Develop and analyze a model that will allow you to assess the probability that this is true.

Compare the odds (that you found in this problem) of misidentification by fingerprint evidence against the odds of misidentification by DNA evidence.

# Problem B: A Faster Quick Pass System

"Quick Pass" systems are increasingly appearing to reduce people's time waiting in line, whether it is at tollbooths, amusement parks, or elsewhere.

Consider the design of a Quick Pass system for an amusement park. The amusement park has experimented by offering Quick Passes for several popular rides as a test. The idea is that for certain popular rides you can go to a kiosk near that ride and insert your daily park entrance ticket, and out will come a slip that states that you can return to that ride at a specific time later. For example, you insert your daily park entrance ticket at 1:15 P.M., and the Quick Pass states that you can come back between 3:30 and 4:30 P.M. when you can use your slip to enter a second, and presumably much shorter, line that will get you to the ride faster. To prevent people from obtaining Quick Passes for several rides at once, the Quick Pass machines allow you to have only one active Quick Pass at a time.

You have been hired as one of several competing consultants to improve the operation of Quick Pass. Customers have been complaining about some anomalies in the test system. For example, customers observed that in one instance Quick Passes were being offered for a return time as long as 4 hours later. A short time later on the same ride, the Quick Passes were given for times only an hour or so later. In some instances, the lines for people with Quick Passes are nearly as long and slow as the regular lines.

The problem then is to propose and test schemes for issuing Quick Passes in order to increase people's enjoyment of the amusement park. Part of the problem is to determine what criteria to use in evaluating alternative schemes. Include in your report a nontechnical summary for amusement park executives who must choose between alternatives from competing consultants.

# The Results

The solution papers were coded at COMAP headquarters so that names and affiliations of the authors would be unknown to the judges. Each paper was then read preliminarily by two "triage" judges at either Appalachian State University (Fingerprints Problem) or at the National Security Agency (Quick Pass Problem). At the triage stage, the summary and overall organization are the basis for judging a paper. If the judges' scores diverged for a paper, the judges conferred; if they still did not agree on a score, a third judge evaluated the paper.

This year, again an additional Regional Judging site was created at the U.S. Military Academy to support the growing number of contest submissions.

Final judging took place at Harvey Mudd College, Claremont, California. The judges classified the papers as follows:

| | Outstanding | Meritorious | Honorable Mention | Successful Participation | Total |
|---|---|---|---|---|---|
| Fingerprints Problem | 3 | 24 | 50 | 126 | 203 |
| Quick Pass Problem | 4 | 38 | 109 | 246 | 397 |
| | 7 | 62 | 159 | 372 | 600 |

The seven papers that the judges designated as Outstanding appear in this special issue of *The UMAP Journal*, together with commentaries. We list those teams and the Meritorious teams (and advisors) below; the list of all participating schools, advisors, and results is in the **Appendix**.

## Outstanding Teams

| Institution and Advisor | Team Members |
|---|---|
| **Fingerprints Papers** | |
| "The Myth of 'The Myth of Fingerprints'" | |
| Harvey Mudd College | Steven G. Avery |
| Claremont, CA | Eric Thomas Harley |
| Jon Jacobsen | Eric J. Malm |
| "Can't Quite Put Our Finger On It" | |
| University College Cork | Séamus Ó Ceallaigh |
| Cork, Ireland | Alva Sheeley |
| James J. Grannell | Aidan Crangle |
| "Not Such a Small Whorl After All" | |
| University of Colorado at Boulder | Brian Camley |
| Boulder, CO | Pascal Getreuer |
| Anne M. Dougherty | Bradley Klingenberg |
| **Quick Pass Papers** | |
| "A Myopic Aggregate-Decision Model for Reservation Systems in Amusement Parks" | |
| Harvard University | Ivan Corwin |
| Cambridge, MA | Sheel Ganatra |
| Clifford H. Taubes | Nikita Rozenblyum |
| "Theme-Park Queueing Systems" | |
| Merton College, University of Oxford | Alexander V. Frolkin |
| Oxford, U.K. | Frderick D.W. van der Wyck |
| Ulrike Tillmann | Stephen Burgess |
| "Developing Improved Algorithms for QuickPass Systems" | |
| University of Colorado | Moorea L. Brega |
| Boulder, CO | Alejandro L. Cantarero |
| Bengt Fornberg | Corry L. Lee |

"KalmanQueue: An Adaptive Approach to Virtual Queueing"
University of Washington
Seattle, WA
James Allen Morrow

Tracy Clark Lovejoy
Aleksandr Yakovlevitch Aravkin
Casey Schneider-Mizell

## Meritorious Teams

**Fingerprints Papers** (24 teams)
Bethel College, St. Paul, MN (William M. Kinney)
Central Washington University, Ellensburg, WA (Stuart F. Boersma)
Chongqing University, Chongqing, China (Li Zhiliang)
Cornell University, Ithaca, NY (Alexander Vladimirsky)
Dalian University of Technology, Dalian, Liaoning, China (We Mingfeng)
Dalian University, Dalian, Liaoning, China (Tan Xinxin)
Donghua University, Shanghai, Shanghai, China (Ding Yongsheng)
Duke University, Durham, NC (William G. Mitchener)
Gettysburg College, Gettysburg, PA (Peter T. Otto)
Kansas State University, Manhattan, KS (Fosskorten N. Auckly)
Luther College, Decorah, IA (Reginald, D. Laursen)
MIT, Cambridge, MA (Martin Z. Bazant)
Northwestern Politechnical University, Xi'an, Shaanxi, China (Peng Guohua)
Olin College of Engineering, Needham, MA (Burt S. Tilley)
Shanghai Jiaotong University,(William G. Mitchener)
Rensselaer Polytechnic Institute, Troy, NY (Peter R. Kramer)
Simpson College, Indianola, IA (Murphy Waggoner)
Tsinghua University, Beijing, China (Hu Zhiming)
University College Cork, Cork, Ireland (James J. Grannell)
University of Colorado at Boulder, Boulder, CO (Anne M. Dougherty)
University of Delaware, Newark, DE (Louis F. Rossi)
University of South Carolina Aiken, Aiken, SC (Thomas F. Reid)
University of Washington, Seattle, WA, (Rekha R. Thomas)
Zhejiang University, Hangzhou, Zhejiang, China (Yong He)

**Quick Pass Papers** (38 teams)
Beijing Forestry University, Beijing, China (Gao Mengning)
Bloomsburg University, Bloomsburg, PA (Kevin K. Ferland)
Carroll College, Helena, MT (Marilyn S. Schendel)
Civil Aviation University of China, Tianjin, China (Nie Runtu)
The College of Wooster, Wooster, OH (Charles R. Hampton)
Colorado College, Colorado Springs, CO (Jane M. McDougall)
Concordia College New York, Bronxville, NY (John F. Loase)
Davidson College, Davidson, NC (Dennis R. Appleyard)
Duke University, Durham, NC (William G. Mitchener)
Grand View College, Des Moines, IA (Sergio Loch)
Greenville College, Greenville, IL (George R. Peters)
Harbin Institute of Technology, Harbin, Heilongjiang, China (Liu Kean) (two teams)

Harvey Mudd College, Claremont, CA (Ran Libeskind-Hadas)
Kansas State University, Manhattan, KS (Fosskorten N. Auckly)
Loyola College, Baltimore, MD (Christos A. Xenophontos)
MIT, Cambridge, MA (Martin Z. Bazant)
Nanjing University of Science and Technology, Nanjing, Jiangsu, China (Zhao Peibiao)
Nankai University, Tianjin, China (Yang Qingzhi)
North China Electric Power University, Baoding, Hebei, China (Shi HuiFeng)
Rensselaer Polytechnic Institute, Troy, NY (Peter R. Kramer)
Salisbury University, Salisbury, MD (Joseph W. Howard)
Shanghai Jiaotong University, Shanghai, China (Song Baorui) (two teams)
Simpson College, Indianola, IA (Werner S. Kolln)
Southeast Missouri State University, Cape Girardeau, MO (Robert W. Sheets)
University of California, Berkeley, CA (Lawrence C. Evans)
United States Military Academy, West Point, NY (J. Scott Billie)
University College Cork, Cork, Ireland (Patrick Fitzpatrick)
University of Massachusetts Lowell, Lowell, MA (James Graham-Eagle)
University of Pittsburgh, Pittsburgh, PA (Jonathan E. Rubin)
University of Saskatchewan, Saskatoon, SK, Canada (James A. Brooke)
University of Trier, Trier, Germany (Volker H. Schulz)
University of Washington, Seattle, WA (James Allen Morrow)
Wake Forest University, Winston Salem, NC (Miaohua Jiang) (two teams)
Wake Forest University, Winston Salem, NC (Robert Plemmons)
Wartburg College, Waverly, IA (Brian J. Birgen)

# Awards and Contributions

Each participating MCM advisor and team member received a certificate signed by the Contest Director and the appropriate Head Judge.

INFORMS, the Institute for Operations Research and the Management Sciences, recognized the teams from Harvey Mudd College (Fingerprints Problem) and Merton College, Oxford University (Quick Pass Problem) as INFORMS Outstanding teams and provided the following recognition:

- a letter of congratulations from the current president of INFORMS to each team member and to the faculty advisor;
- a check in the amount of $300 to each team member;
- a bronze plaque for display at the team's institution, commemorating their achievement;
- individual certificates for team members and faculty advisor as a personal commemoration of this achievement;
- a one-year student membership in INFORMS for each team member, which includes their choice of a professional journal plus the *OR/MS Today* periodical and the INFORMS society newsletter.

- a one-year subscription access to the COMAP modeling materials Website for the faculty advisor.

The Society for Industrial and Applied Mathematics (SIAM) designated one Outstanding team from each problem as a SIAM Winner. The teams were from Harvey Mudd College (Fingerprints Problem) and University of Colorado at Boulder (Quick Pass Problem). Each of the team members was awarded a $300 cash prize and the teams received partial expenses to present their results in a special Minisymposium at the SIAM Annual Meeting in Portland, OR in July. Their schools were given a framed hand-lettered certificate in gold leaf.

The Mathematical Association of America (MAA) designated one Outstanding team from each problem as an MAA Winner. The teams were from University of Colorado at Boulder (Fingerprints Problem) and Harvard University (Quick Pass Problem). With partial travel support from the MAA, both teams presented their solutions at a special session of the MAA Mathfest in Providence, RI in August. Each team member was presented a certificate by Richard S. Neal, Co-Chair of the MAA Committee on Undergraduate Student Activities and Chapters.

## New: The Ben Fusaro Award

Two Meritorious papers were selected for the Ben Fusaro Award, named for the Founding Director of the MCM and awarded for the first time this year. It recognizes teams for an especially creative approach to the contest problem. The Ben Fusaro Award teams were from Central Washington University (Fingerprints Problem) and MIT (Quick Pass Problem). Each team received a plaque from COMAP.

### Background

The Ben Fusaro Award is created to recognize technical papers that demonstrate an exemplary modeling effort for the MCM. These papers are well-written; their modeling approach and the reasoning for adopting such an approach are clearly communicated; and their analysis, results, and conclusions are measured and appropriate within the context of the problem.

### Award Committee

The award committee consists of two judges for each problem: the Triage Head Judge for the problem, plus a COMAP-sponsored judge currently serving in a two- or four-year college who has at least one prior year of experience in final judging for the MCM.

## Eligibility

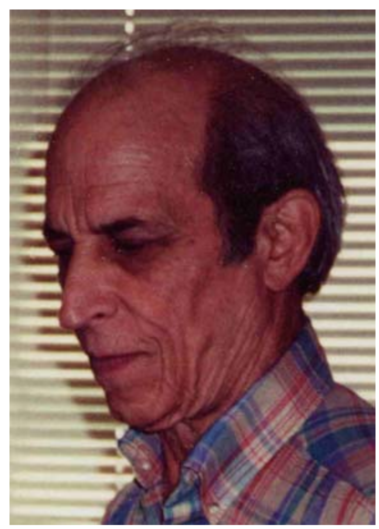
To compete for the Ben Fusaro Award, a paper must be considered as Meritorious or Outstanding by the problem judges and normally be among those remaining for final discussion prior to the identification of Outstanding papers. Ideally, a paper selected to receive the Ben Fusaro Award should not be one of those already selected for recognition by one of the professional societies.

## Selection Process

Following the final discussion round, the award committee will select one paper from each problem that best demonstrates the following characteristics:

- The paper presents high-quality application of the complete modeling process as represented by the 100-point-scale elements developed for the final rounds of judging.
- The team has demonstrated noteworthy originality and creativity in their modeling effort to solve the problem as given.
- The paper is well-written with a clear exposition, and a pleasure to read.

# Ben Fusaro



Ben was the founder of the MCM and its director for the first seven years. He has a B.A. from Swarthmore College, an M.A. from Columbia University (analysis), a Ph.D. from the University of Maryland (partial differential equations), and most recently (1990) an M.A. from the University of Maryland (computer science ).

He taught at several other colleges and universities before going to Salisbury State in 1974, where he served as chair of the Mathematics and Computer Science Dept. 1974–82 and received the Distinguished Faculty Award in 1992. Ben was NSF Lecturer at New Mexico Highlands University and at the University of Oklahoma, Fulbright Professor at National Taiwan Normal University, and visiting professor at the U.S. Military Academy at West Point. He has taught most undergraduate mathematics courses, plus graduate courses in integral equations, partial differential equations, and mathematical modeling.

In recent years, Ben has been a major exponent of environmental mathematics, a topic on which he has presented several minicourses.

# Judging

*Director*
Frank R. Giordano, Naval Postgraduate School, Monterey, CA

*Associate Directors*
Robert L. Borrelli, Mathematics Dept., Harvey Mudd College, Claremont, CA
Patrick J. Driscoll, Dept. of Systems Engineering, U.S. Military Academy, West Point, NY

*Contest Coordinator*
Kevin Darcy, COMAP Inc., Lexington, MA

**Fingerprints Problem**

*Head Judge*
Marvin S. Keener, Executive Vice-President, Oklahoma State University, Stillwater, OK (MAA)

*Associate Judges*
William C. Bauldry, Chair, Dept. of Mathematical Sciences, Appalachian State University, Boone, NC (Triage)
Kelly Black, Mathematics Dept., University of New Hampshire, Durham, NH (SIAM)
Lisette De Pillis, Mathematics Dept., Harvey Mudd College, Claremont, CA
J. Douglas Faires, Youngstown State University, Youngstown, OH (MAA)
Ben Fusaro, Mathematics Dept., Florida State University, Tallahassee, FL (SIAM)
Mario Juncosa, RAND Corporation, Santa Monica, CA (retired)
Deborah P. Levinson, Hewlett-Packard Company, Colorado Springs, CO
Michael Moody, Olin College of Engineering, Needham, MA
John L. Scharf, Mathematics Dept., Carroll College, Helena, MT
Dan Solow, Mathematics Dept., Case Western Reserve University, Cleveland, OH (INFORMS)
Michael Tortorella, Dept. of Industrial and Systems Engineering, Rutgers University, Piscataway, NJ
Richard Douglas West, Francis Marion University, Florence, SC
Daniel Zwillinger, Newton, MA

**Quick Pass Problem**

*Head Judge*
Maynard Thompson, Mathematics Dept., University of Indiana, Bloomington, IN

*Associate Judges*
Peter Anspach, National Security Agency, Ft. Meade, MD (Triage)

Karen D. Bolinger, Mathematics Dept., Clarion University of Pennsylvania, Clarion PA
James Case, Baltimore, MD (SIAM)
William P. Fox, Mathematics Dept., Francis Marion University, Florence, SC (MAA)
Jerry Griggs, Mathematics Dept., University of South Carolina, Columbia, SC
John Kobza, Mathematics Dept., Texas Tech University, Lubbock, TX (INFORMS)
Veena Mendiratta, Lucent Technologies, Naperville, IL
Don Miller, Mathematics Dept., St. Mary's College, Notre Dame, IN (SIAM)
Kathleen M. Shannon, Dept. of Mathematics and Computer Science, Salisbury University, Salisbury, MD
Marie Vanisko, Dept. of Mathematics, California State University, Stanislaus, CA (MAA)

**Regional Judging Session**

*Head Judge*
Patrick J. Driscoll, Dept. of Systems Engineering

*Associate Judges*
Darrall Henderson, Dept. of Mathematical Sciences
Steven Henderson, Dept. of Systems Engineering
Steven Horton, Dept. of Mathematical Sciences
Michael Jaye, Dept. of Mathematical Sciences
—all of the U.S. Military Academy, West Point, NY

**Triage Sessions:**

**Fingerprints Problem**

*Head Triage Judge*
William C. Bauldry, Chair

*Associate Judges*
Terry Anderson,
Mark Ginn,
Jeff Hirst,
Rick Klima,
Katie Mawhinney,
and
Vickie Williams
—all from Dept. of Math'l Sciences, Appalachian State University, Boone, NC

**Quick Pass Problem**

*Head Triage Judge*
Peter Anspach, National Security Agency (NSA), Ft. Meade, MD

*Associate Judges*
Dean McCullough, High Performance Technologies, Inc.
Robert L. Ward (retired)
Blair Kelly,
Craig Orr,
Brian Pilz,
Eric Schram,
and other members of NSA.

**Fusaro Award Committee**

**Fingerprints Problem:**
William C. Bauldry, Chair, Dept. of Mathematical Sciences,
Appalachian State University, Boone, NC
Michael Moody, Olin College of Engineering, Needham, MA

**Quick Pass Problem:**
Peter Anspach, National Security Agency, Ft. Meade, MD
Kathleen M. Shannon, Dept. of Mathematics and Computer Science,
Salisbury University, Salisbury, MD

# Sources of the Problems

The Fingerprints Problem was contributed by Michael Tortorella (Dept. of Industrial and Systems Engineering, Rutgers University, Piscataway, NJ).

The Quick Pass Problem was contributed by Jerry Griggs (Mathematics Dept., University of South Carolina, Columbia, SC).

# Acknowledgments

Major funding for the MCM is provided by the National Security Agency and by COMAP. We thank Dr. Gene Berg of NSA for his coordinating efforts. Additional support is provided by the Institute for Operations Research and the Management Sciences (INFORMS), the Society for Industrial and Applied Mathematics (SIAM), and the Mathematical Association of America (MAA). We are indebted to these organizations for providing judges and prizes.

We thank the MCM judges and MCM Board members for their valuable and unflagging efforts. Harvey Mudd College, its Mathematics Dept. staff, and Prof. Borrelli were gracious hosts to the judges.

# Cautions

*To the reader of research journals:*

Usually a published paper has been presented to an audience, shown to colleagues, rewritten, checked by referees, revised, and edited by a journal editor. Each of the student papers here is the result of undergraduates working on a problem over a weekend; allowing substantial revision by the authors could give a false impression of accomplishment. So these papers are essentially au naturel. Editing (and sometimes substantial cutting) has taken place: Minor errors have been corrected, wording has been altered for clarity or economy, and style has been adjusted to that of *The UMAP Journal*. Please peruse these student efforts in that context.

*To the potential MCM Advisor:*

It might be overpowering to encounter such output from a weekend of work by a small team of undergraduates, but these solution papers are highly atypical. A team that prepares and participates will have an enriching learning experience, independent of what any other team does.

---

COMAP's Mathematical Contest in Modeling and Interdisciplinary Contest in Modeling are the only international modeling contests in which students work in teams. Centering its educational philosophy on mathematical modeling, COMAP uses mathematical tools to explore real-world problems. It serves the educational community as well as the world of work by preparing students to become better-informed and better-prepared citizens.

# Appendix: Successful Participants

KEY:
P = Successful Participation
H = Honorable Mention
M = Meritorious
O = Outstanding (published in this special issue)

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| ALASKA | | | | |
| U. of Alaska | Fairbanks | Jill R. Faudree | P | H |
| ARIZONA | | | | |
| Northern Arizona U. | Flagstaff | Terence R. Blows | P | |
| ARKANSAS | | | | |
| Hendrix College | Conway | Duff Gordon Campbell | | H |
| CALIFORNIA | | | | |
| California Baptist U. | Riverside | Catherine Kong | | P |
| Calif. Poly. State U. | San Luis Obispo | Jonathan E. Shapiro | | P,P |
| Calif. State Poly. U. | Pomona | Hale, Mihaila, and Switkes | | P,P |
| Calif. State U. | Seaside | Hongde Hu | P | P |
| Calif. State U. | Bakersfield | Maureen E. Rush | P | |
| Calif. State U. | Northridge | Ali Zakeri | | P |
| Calif. State U. | Turlock | Brian Jue | | P |
| Christian Heritage C. | El Cajon | Tibor F. Szarvas | | P |
| Harvey Mudd Coll. | Claremont | Jon Jacobsen | O | |
| | | Hank Krieger | | H |
| (CS) | | Ran Libeskind-Hadas | | M,H |
| UC Berkeley | Berkeley | Lawrence C. Evans | | M |
| U. of San Diego | San Diego | Jeffrey H. Wright | P | P |
| (CS) | | Diane Hoffoss | | P |
| COLORADO | | | | |
| Colorado College | Colorado Springs | Jane M. McDougall | | M |
| Colo. State U. | Pueblo | Bruce N. Lundberg | | H |
| Regis University | Denver | Jim Seibert | P | P |
| U.S. Air Force Acad. | USAF Academy | James S. Rolf | | P |
| U. of Colorado | Boulder | Anne M. Dougherty | M | |
| | | Bengt Fornberg | | O |
| | | Michael H. Ritzwoller | | H,H |
| | Denver | William L. Briggs | | P |
| | Colorado Springs | Radu C. Cascaval | P | |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| CONNECTICUT | | | | |
| Sacred Heart University | Fairfield | Peter Loth | | P |
| | | Hema Gopalakrishnan | | P |
| Southern Connecticut State U. | New Haven | Ross B. Gingrich | | P |
| Yale University (Stat) | New Haven | Andrew R. Barron | | P |
| DELAWARE | | | | |
| University of Delaware | Newark | Louis F. Rossi | M | |
| FLORIDA | | | | |
| Embry-Riddle Aeronautical U. | Daytona Beach | Greg S. Spradlin | | H,P |
| Jacksonville University | Jacksonville | Robert A. Hollister | | P,P |
| Stetson University | DeLand | Lisa O. Coulter | | P |
| University of Central Florida (Phys) | Orlando | Costas J. Efthimiou | P | |
| GEORGIA | | | | |
| Georgia Institute of Tech. (Eng) | Atlanta | Bernard Kippelen | H | |
| Georgia Southern University | Statesboro | Laurene V. Fausett | | P,P |
| State University of West Georgia | Carrollton | Scott Gordon | H | |
| IDAHO | | | | |
| Boise State University | Boise | Jodi L. Mead | | P |
| ILLINOIS | | | | |
| Greenville College | Greenville | George R, Peters | | M |
| Illinois Wesleyan University | Bloomington | Zahia Drici | | H |
| Monmouth College (Phys) | Monmouth | Christopher G. Fasano | | P |
| Northern Illinois University | DeKalb | Ying C. Kwong | P | |
| Wheaton College | Wheaton | Paul Isihara | | P |
| INDIANA | | | | |
| Earlham College | Richmond | Michael Bee Jackson | | P |
| | | Timothy J. McLarnan | | P |
| (Phys) | | Mihir Sejpal | | H,P |
| Franklin College | Franklin | John P. Boardman | | P |
| Goshen College | Goshen | David Housman | | H,P |
| Rose-Hulman Institute of Tech. | Terre Haute | David J. Rader | H | P |
| | | Cary Laxer | | H |
| Saint Mary's College (CS) | Notre Dame | Joanne R. Snow | P | H |
| IOWA | | | | |
| Grand View College | Des Moines | Sergio Loch | H | M |
| Grinnell College | Grinnell | Marc Chamberland | | H,H |
| (Phys) | | Jason Zimba | | P |
| Luther College | Decorah | Reginald D. Laursen | M | P |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Mt. Mercy College | Cedar Rapids | K.R. Knopp | | P |
| Simpson College | Indianola | Murphy Waggoner | M,H | |
| (Chm) | | Werner S. Kolln | P | M |
| Wartburg College | Waverly | Brian J. Birgen | | M |
| KANSAS | | | | |
| Emporia State University | Emporia | Brian D. Hollenbeck | P | |
| Kansas State University | Manhattan | Fosskorten N. Auckly | M | M |
| KENTUCKY | | | | |
| Asbury College | Wilmore | Duk Lee | | H |
| | | Ken P. Rietz | | H |
| Northern Kentucky Univ. | Highland Heights | Gail Mackin | P | P |
| (Phys) | | Sharmanthie Fernando | | P |
| Thomas More College | Crestview Hills | Robert M. Riehemann | | H |
| MAINE | | | | |
| Colby College | Waterville | Jan E. Holly | | H,P |
| MARYLAND | | | | |
| Hood College | Frederick | Betty Mayfield | P | |
| | | Frederick Kimber Tysdal | P | |
| Loyola College | Baltimore | Christos A. Xenophontos | | M,P |
| Mount St. Mary's College | Emmitsburg | Fred Portier | | P,P |
| (Sci) | | Robert Richman | | P |
| Salisbury University | Salisbury | Mike J. Bardzell | | P |
| | | Steven Hetzler | | P |
| (Phys) | | Joseph W. Howard | | M |
| Towson University | Towson | Mike O'Leary | | P |
| Villa Julie College | Stevenson | Eileen C. McGraw | | P |
| Washington College | Chestertown | Eugene P. Hamilton | | P,P |
| MASSACHUSETTS | | | | |
| Boston University | Boston | Glen R. Hall | P | |
| Harvard University | Cambridge | Clifford H. Taubes | | O |
| MIT | Cambridge | Martin Z. Bazant | M | M |
| Olin College of Engineering | Needham | Burt S. Tilley | M | |
| | | John Geddes | | P |
| Simon's Rock College | Great Barrington | Allen B. Altman | P | |
| | | Michael Bergman | | P,P |
| U. of Massachusetts | Lowell | James Graham-Eagle | | M,P |
| Western New England Coll. | Springfield | Lorna B. Hanes | P,P | |
| Worcester Polytechnic Inst. | Worcester | Suzanne L. Weekes | | P |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| MICHIGAN | | | | |
| Eastern Michigan Univ. | Ypsilanti | Christopher E. Hee | | P,P |
| Lawrence Technological U. | Southfield | Ruth G. Favro | | P,P |
| (Sci) | | Valentina Tobos | ,PP | |
| Siena Heights University | Adrian | Toni Carroll | | P |
| | | Pamela K. Warton | P,P | |
| Univ. of Michigan (Phys) | Ann Arbor | James D. Wells | | H |
| MINNESOTA | | | | |
| Bemidji State University | Bemidji | Colleen G. Livingston | | P |
| Bethel College | St. Paul | William M. Kinney | M | |
| College of Saint Benedict / Saint John's University | Collegeville | Robert J. Hesse | | H |
| Macalester College | St. Paul | Daniel T. Kaplan | | H,H |
| MISSOURI | | | | |
| Northwest Missouri State U. | Maryville | Russell N. Euler | | P |
| Saint Louis University | St. Louis | James E. Dowdy | P | |
| (CS) | | Dennis J. Bouvier | | P |
| Southeast Missouri State U. | Cape Girardeau | Robert W. Sheets | | M |
| Truman State University | Kirksville | Steve J. Smith | H | H |
| Washington Univ. (Eng) | St. Louis | Hiro Mukai | | H |
| MONTANA | | | | |
| Carroll College | Helena | Holly S. Zullo | | H |
| | | Kelly Slater Cline | | H |
| (Sci) | | Marilyn S. Schendel | P | M |
| NEBRASKA | | | | |
| Hastings College | Hastings | David B. Cooke | | P |
| NEW JERSEY | | | | |
| Rowan University | Glassboro | Hieu D. Nguyen | | P |
| NEW MEXICO | | | | |
| New Mexico Tech | Socorro | William D. Stone | H | |
| NEW YORK | | | | |
| Bard College | Annandale-on-Hudson | Lauren L. Rose | P | |
| (CS) | | Robert W. McGrail | | P,P |
| Colgate University | Hamilton | Warren Weckesser | | P |
| Concordia Coll. New York | Bronxville | John F. Loase | | M,H |
| | | Eric Friedman | | H,P |
| Cornell University | Ithaca | Alexander Vladimirsky | M,H | |
| (Eng) | | Eric Friedman | | H,P |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Marist College | Poughkeepsie | Tracey B. McGrail | | P |
| Nazareth College | Rochester | Daniel Birmajer | | H |
| Rensselaer Polytechnic Institute | Troy | Peter R. Kramer | M | M |
| Roberts Wesleyan College | Rochester | Gary L. Raduns | P | |
| United States Military Academy | West Point | J. Scott Billie | | M |
| (Eng) | | Gregory S. Parnell | P | |
| (Ctr for Teaching Excellence) | | A. David Trubatch | P | |
| Westchester Community College | Valhalla | Marvin L. Littman | | P |
| | | Janine Epps | | H |
| NORTH CAROLINA | | | | |
| Appalachian State University | Boone | Eric S. Marland | | P,P |
| | | Anthony G. Calamai | | P |
| Davidson College | Davidson | Laurie J. Heyer | | H,P |
| | | Dennis R. Appleyard | | M |
| Duke University | Durham | William G. Mitchener | M | M |
| | | Owen Astrachan | | H,H |
| Meredith College | Raleigh | Cammey E. Cole | | P |
| North Carolina School of Science and Mathematics | Durham | Dot Doyle | H | |
| North Carolina State University | ”Raleigh | Jeffrey S. Scroggs | | P |
| Wake Forest University | Winston Salem | Miaohua Jiang | | M,M |
| Robert Plemmons | | M | | |
| Western Carolina University | Cullowhee | Erin K. McNelis | | P |
| OHIO | | | | |
| College of Wooster | Wooster | Charles R. Hampton | | M |
| Malone College | Canton | David W. Hahn | | H,P |
| Miami University | Oxford | Stephen E. Wright | | P,P |
| University of Dayton | Dayton | Youssef N. Raffoul | H | |
| Wright State University | Dayton | Thomas P. Svobodny | | P |
| Youngstown State University | Youngstown | Angela Spalsbury | P | P |
| (Eng) | | Scott Martin | P | H |
| OKLAHOMA | | | | |
| Southeastern Oklahoma State U. | Durant | Brett M. Elliott | | P |
| OREGON | | | | |
| Eastern Oregon University | La Grande | David L. Allen | | H |
| Lewis and Clark College | Portland | Robert W. Owens | | H,H |
| (Econ) | | Clifford Bekar | | P,P |
| Linfield College | McMinnville | Jennifer A. Nordstrom | | P |
| Pacific University | Forest Grove | Chris C. Lane | H | P |
| Southern Oregon University | Ashland | Kemble R. Yates | | P |
| Willamette University | Salem | Liz A. Stanhope | H | |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| PENNSYLVANIA | | | | |
| Bloomsburg University | Bloomsburg | Kevin K. Ferland | | M |
| Bucknell University (Phys) | Lewisburg | Sally Koutsoliotas | | H |
| Clarion University | Clarion | Jon Beal | | P |
| Gettysburg College | Gettysburg | Peter T. Otto | M | |
| (Eng) | | Sharon L. Stephenson | | P |
| Juniata College | Huntingdon | John F. Bukowski | H,P | |
| University of Pittsburgh | Pittsburgh | Jonathan E. Rubin | | M,H |
| Villanova University | Villanova | Bruce Pollack-Johnson | | P |
| Westminster College | New Wilmington | Barbara T. Faires | P | P |
| RHODE ISLAND | | | | |
| Rhode Island College | Providence | David L. Abrahamson | P | P |
| SOUTH CAROLINA | | | | |
| Benedict College | Columbia | Balaji Iyangar | | P |
| Midlands Technical College | Columbia | John R. Long | P,P | |
| University of South Carolina | Aiken | Thomas F. Reid | M | P |
| York Technical College | Rock Hill | Frank W. Caldwell | P | |
| SOUTH DAKOTA | | | | |
| SD School of Mines & Tech. | Rapid City | Kyle Riley | P | H |
| TENNESSEE | | | | |
| Austin Peay State University | Clarksville | Nell K. Rayburn | P,P | |
| Belmont University | Nashville | Andrew J. Miller | | P |
| TEXAS | | | | |
| Austin College | Sherman | John H. Jaroma | | P,P |
| Baylor University | Waco | Frank H. Mathis | | P |
| Liberty Christian School | Denton | Laura A. Duncan | P | |
| (Technology) | | Bryan Lee Bunselmeyer | P | P |
| Trinity University | San Antonio | Allen G. Holder | P | |
| | | Diane Saphire | | H |
| (Econ) | | Jorge G. Gonzalez | | H |
| (Phys) | | Robert Laird | | P |
| VERMONT | | | | |
| Johnson State College | Johnson | Glenn D. Sproul | P | |
| VIRGINIA | | | | |
| Maggie Walker Governor's Schl | Richmond | John A. Barnes | | H,P |
| (Sci) | | Harold Houghton | H | H |
| Roanoke College | Salem | Jeffrey L. Spielman | | P |
| University of Richmond | Richmond | Kathy W. Hoke | | H,P |
| Virginia Western Comm. Coll. | Roanoke | Ruth A. Sherman | | P,P |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| WASHINGTON | | | | |
| Central Washington University | Ellensburg | Stuart F. Boersma | M | |
| Heritage College | Toppenish | Richard W. Swearingen | | H,P |
| Pacific Lutheran University | Tacoma | Daniel J. Heath | P | |
| University of Puget Sound | Tacoma | Michael S. Casey | H | P |
| (CS) | | John E. Riegsecker | | P |
| University of Washington | Seattle | James Allen Morrow | | O,M |
| | | Rekha R. Thomas | M | P |
| Western Washington University | Bellingham | Tjalling J. Ypma | P | |
| WISCONSIN | | | | |
| Edgewood College | Madison | Steven B. Post | | P |
| Northland College | Ashland | William M. Long | P | |
| University of Wisconsin | River Falls | Kathy A. Tomlinson | | P |
| AUSTRALIA | | | | |
| University of New South Wales | Sydney | James W. Franklin | P,P | |
| CANADA | | | | |
| McGill University | Montreal | Antony R. Humphries | H | |
| University of Saskatchewan | Saskatoon | James A. Brooke | P | M |
| (CS) | | Raj Srinivasan | H | |
| (Phys) | | Kaori Tanaka | H | |
| University of Western Ontario | London | Martin H. Mueser | H | H |
| York University | Toronto | Huaxiong Huang | P | |
| | | Huaiping Zhu | | P |
| CHINA | | | | |
| Anhui | | | | |
| Anhui University | Hefei | He Zehui | | P |
| | | Wang Xuejun | | P |
| | | Wang Jihui | | P |
| (CS) | | Zhang Quanbing | | P |
| (Phys) | | Chen Mingsheng | | P |
| Anhui University of Tech. and Science | Hong Kong | Sun Hongyi | | P |
| | | Yang Xubing | | P |
| | | Wang Chuanyu | P | |
| Hefei University of Technology | Hefei | Shi Lei | H | |
| | | Liu Fanglin | | H |
| | | Huang Youdu | | P |
| | | Chen Hua | | P |
| U. of Science and Tech. of China (Phys) | Hefei | Le Xuli | | P |
| (Stat) | | Bin Su | | H |
| (Eng) | | Yao Xie | | H |
| (Modern Phys) | | Tao Zhou | | H |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Beijing | | | | |
| Beihang University | Beijing | Peng Linping | P | |
| | | Liu Hongying | P,P | |
| | | Wu Sanxing | | P |
| (Phys) | | Zhang Yan Jia | | P |
| Beijing Forestry University | Beijing | Hongjun Li | P | |
| (Bio) | | Gao Mengning | | M,H |
| Beijing Institute of Technology | Beijing | Li Bingzhao | | H |
| | | Chen Yihong | P | P |
| | | Cui Xiaodi | H | |
| Beijing Jiaotong University | Beijing | Zhang Shangli | | P,P |
| (CS) | | Liu Minghui | | P |
| (Chm) | | Bing Tuan | | P |
| (Phys) | | Wang Bingtuan | P | P |
| Beijing Materials Institute | Beijing | Tian De Liang | | P,P |
| | | Cheng Xiao Hong | P | P |
| Beijing Normal University | Beijing | Liu Laifu | | H,P |
| | | Qing He | | P,P |
| | | Lu Zijuan | P | |
| Beijing University of Chemical Tech. | Beijing | Liu Hui | | H |
| | | Jiang Guangfeng | H | |
| (Chem Eng) | | Huang Jinyang | P | |
| (Chem) | | Liu Damin | P | |
| Beijing U. of Posts and Telecomm. (Sci) | Beijing | He Zuguo | H | |
| (Sci) | | Sun Hongxiang | H | |
| (CS) | | Ding Jinkou | | H |
| (CS) | | Wang Xiaoxia | | P |
| Beijing University of Technology (Sci) | Beijing | Yi Xue | P | |
| (Sci) | | Guo Enli | | H,P |
| (Info) | | Yang Shi Lin | | H |
| (CS) | | Deng Mike | P | |
| China Agriculture University | Beijing | Liu Junfeng | | H,P |
| Peking University | Beijing | Liu Xufeng | H | H |
| | | Deng Minghua | H,H | |
| (Phys) | | Zheng Hanqing | | P |
| Tsinghua University | Beijing | Ye Jun | H,H | |
| | | Hu Zhiming | M,H | |
| | | Jiang Qiyuan | | P |
| Chongqing | | | | |
| Chongqing University | Chongqing | Gong Qu | | P |
| | | Li Chuandong | | P |
| | | He Renbin | | P |
| (Chm) | | Li Zhiliang | M | |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Chongqing U. of Posts and Telecomm. | Chongqing | Yang Chun-de | | H |
| (CS) | | Zheng Ji-ming | | H |
| Guangdong | | | | |
| Guangzhou University | Guangzhou | Feng Yongping | P | |
| | | Liang Dahong | P | |
| (Econ) | | Fu RongLin | P,P | |
| Jinan University | GuangZhou | Hu Daiqiang | | P |
| (CS) | | Luo Shizhuang | | P |
| (CS) | | Zhang Chuanlin | P | |
| (Phys) | | Ye Shiqi | | P |
| South China University of Technology | Guangzhou | Liu Shenquan | | H,P |
| (Phys) | | Liang Manfa | | P |
| (Sci) | | Tao Zhisui | | H |
| South-China Normal University | Guangzhou | Wang Henggeng | H,H | |
| (Phys) | | Liu Xiuxiang | P | |
| Sun Yat-Sen University | Guangzhou | Jiang XiaoLong | P | |
| | | Feng GuoCan | | P |
| (CS) | | Li CaiWei | | P |
| (Geology) | | Yuan ZhuoJian | | H |
| Guangxi | | | | |
| Guangxi University | Nanning | Lu Yuejin | | P,P |
| Hebei | | | | |
| North China Electric Power U. | Baoding | Zhang Po | | P |
| | | Gu Gendai | | H |
| (CS) | | Shi HuiFeng | | M |
| (Eng) | | Ma Xinshun | | P |
| Heilongjiang | | | | |
| Harbin Engineering University | Harbin | Zhang Xiaowei | | P.P |
| | | Gao Zhenbin | P | |
| | | Yu Fei | P | |
| (Sci) | | Shen Jihongn | | P,P |
| Harbin Institute of Technology | Harbin | Hong Ge | H | P |
| | | Zhang Chiping | P | H |
| | | Liu Kean | M,M | |
| (Sci) | | Shang Shouting | P | |
| Harbin University of Science and Tech. | Harbin | Chen Dongyan | P | |
| | | Wang Shuzhong | | P |
| Jiamusi University | Jiamusi | Bai FengShan | P | P |
| (Eng) | | Fan Wei | P | P |
| Northeast Agricultural University | Harbin | Ge Huiline | P | |
| | | Wu Qiufeng | | P |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Harbin Normal University | Harbin | Xu Mingyue | | P,P |
| | | Yin Hongcai | | P,P |
| (CS) | | Yao Huanmin | | P,P |
| (CS) | | Zhang Yanying | | P |
| (CS) | | Zeng Weiliang | | P |
| Hubei | | | | |
| China University of Geosciences (CS) | Wuhan | Cai ZhiHua | | P |
| Huazhong University of Sci. & Tech. | Wuhan | Ye Ying | H | |
| | | He Nanzhong | | P |
| | | Wang Yizhi | H | P |
| (Chm) | | Guan Wenchao | | P |
| (Eng) | | Wang Yongji | P | |
| (Civil Eng) | | Jiang Yi Sheng | | P |
| Wuhan University | Wuhan | Deng Aijiao | H | |
| | | Hu Yuanming | P | |
| Wuhan University of Technology | Wuhan | Peng Si jun | | P |
| | | Wang Zhan qing | | P |
| | | He Lang | | P |
| (Stat) | | Li Yuguang | | H |
| Hunan | | | | |
| Central South University (CS) | Changsha | Hou Muzhou | P | |
| (CS) | | Zhang Hongyan | | P |
| (Stat) | | Yi Kunnan | | H,P |
| National University of Defence Tech. | Changsha | Mao Ziyang | | P,P |
| | | Wu Mengda | P | H |
| (Econ) | | Cheng Lizhi | H,P | |
| Jiangsu | | | | |
| China University of Mining and Tech. | Xuzhou | Zhou Shengwu | P | |
| | | Zhang Xingyong | | P |
| (Educational Admin.) | | Wu Zongxiang | | H |
| (Educational Admin.) | | Zhu Kaiyong | | H |
| Commanding Inst. of Engineer Corps, People's Liberation Army | Xuzhou | Wang Chuanwei | H,P | |
| Jiang Su University, Nonlinear Inst. Ctr | ZhenJiang | Yi Min Li | | H |
| Nanjing Normal University | Nanjing | Fu Shitai | P | P |
| | | Chen Xin | | P,P |
| Nanjing U. of Posts and Telecomm. | Nanjing | Qiu ZhongHua | | H |
| | | Xu LiWei | | H |
| (CS) | | Ming He | | P |
| (Optical Info. Tech.) | | Yang Zhenhua | P | |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Nanjing U. of Science & Technology | Nanjing | Xu Chungen | | P |
| | | Zhao Peibiao | | M |
| (Econ) | | Liu Liwei | | P |
| (Econ) | | Zhang Zhengjun | | P |
| Xuzhou Institute of Technology | Xuzhou | Li Subei | P | |
| | | Jiang Yingzi | H | |
| Southeast University | Nanjing | Zhang Zhi-qiang | P | H |
| | | He Dan | P | H |
| Jiangxi | | | | |
| Nanchang University (Sci) | Nanchang | Tao Chen | P,P | |
| | | Liao Chuanrong | | P |
| Jilin | | | | |
| Jilin University | Changchun City | Zou Yongkui | H | |
| | | Huang Qingdao | | P |
| | | Fang Peichen | | P |
| (Eng) | | Pei Yongchen | | P |
| Northeast China Institute of Electric Power Engineering | Jilin City | Guo Xinchen | | P |
| | | Chang Zhiwen | | P |
| Liaoning | | | | |
| Dalian Coll. of Chemical Engineering | Dalian | Yu Haiyan | | H |
| Dalian Nationalities University | Dalian Dev. Zone | Li Xiaoniu | P | |
| | | Zhang Hengbo | | H |
| Dalian University | Dalian | Tan Xinxin | M | P |
| (CS) | | Gang Jiatai | | H,H |
| Dalian University of Technology | Dalian | He Mingfeng | M | |
| | | Zhao Lizhong | | P |
| | | Yu Hongquan | | P |
| | | Yi Wang | | P |
| Northeastern University (CS) | Shenyang | Hao Peifeng | P | H |
| (Phys) | Shenyang | He Xuehong | P | P |
| (Eng) | Shenyang | Jing Yuanwei | P | |
| (Sci) | Shenyang | Sun Ping | | H,P |
| Shenyang Pharmaceutical U. (Basic Courses) | Shenyang | Xiang Rongwu | P | P |
| Shaanxi | | | | |
| North China U. of Science and Tech. | Taiyuan | Xue Ya Kui | | H |
| | | Yang Ming | | P |
| | | Lei Ying Jie | P | |
| | | Bi Yong | P | |
| Northwest University | Xi'an | Wang Liantang | P | |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Northwestern Polytechnical U. | Xi'an | Peng Guohua | M | |
| | | Nie Yufeng | | P |
| (Chm) | | Liu Xiaodong | | H |
| (Chm) | | Zhang Shenggui | | P |
| Xi'an Communication Institute | Xi'an | Yan Jiang | | P |
| | | Yang Dongsheng | P | |
| | | Guo Li | P,P | |
| Xi'an Jiaotong University | Xi'an | Dai Yonghong | | P,P |
| | | Zhou Yicang | | H,H |
| | | He Xiaoliang He | P | P |
| Xidian University (Comm. Eng) | Xi'an | Zhang Zhuokui | H,P | |
| (CS) | | Ye Ji-Min | | P |
| (Sci) | | Zhou Shuisheng | | P |
| Shandong | | | | |
| Shandong University | Jinan | Huang Shuxiang | | P |
| | | Ma Jianhua | P | |
| | | Fu Guohua | | P |
| (CS) | | Wang Qian | P | |
| (Econ) | | Zhang Zhaoping | | P |
| (Phys) | | He Xiqing | P | |
| University of Petroleum of China | Dongying | Wang Ziting | | P |
| Shanghai | | | | |
| Donghua University (CS) | Shanghai | Chen Jingchao | H | |
| (Phys) | | He Guoxing | P | |
| (Econ) | | Ding Yongsheng | M | |
| (Eng) | | Guo Zhaoxia | P | |
| East China Univ. of Science and Technology | Shanghai | Su Chunjie | | P |
| | | Liu Zhaohui | | P |
| (Phys) | | Lu Yuanhong | H | |
| (Phys) | | Qin Yan | P | |
| Fudan University | Shanghai | Cai Zhijie | P | |
| | | Cao Yuan | P | |
| Jiading No. 1 Senior High School | Shanghai | Xu Rong | | H |
| | | Xie Xilin | | P |
| Shanghai Foreign Language School | Shanghai | Pan Liqun | | H,H |
| | | Sun Yu | | H |
| Shanghai International Studies University | | | | |
| Shanghai Foreign Language School | Shanghai | Chen Lieda | | H |
| Shanghai Jiaotong University | Shanghai | Song Baorui | | M,M |
| | | Huang Jianguo | M | |
| | | Zhou Guobiao | P | |
| Tianjin Normal University | Tianjin | Li Jianquan | | H |
| | | Li Baoyi | | H |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| Tianjin Polytechnic University | Tianjin | Fan Liya | | H |
| | | Su Yongfu | P | |
| Tianjin University (CS) | Tianjin | Song Zhanjie | | H |
| (Comm. Eng) | | Liang Fengzhen | H | |
| (Comm. Eng) | | Lin Dan | | H |
| (Software Eng) | | Zhang Fengling | H | |
| Shanghai Normal University | Shanghai | Guo Shenghuan | | P |
| | | Shi Yongbin | | P |
| | | Zhang Jizhou | | P |
| (CS) | | Liu Rongguan | | P |
| Shanghai University of Finance & Economics | Shanghai | Fan Zhang | | H |
| | | Xie Bing | P | |
| Shanghai Yucai High School (Sci) | Shanghai | Li Zhengtai | | P |
| Sichuan | | | | |
| Southwest Transportation U. (Eng) | E'mei | Han Yang | P | P |
| Southwestern U. of Finance and Econ. (Econ) | Chengdu | Sun Yunlong | | H |
| Univ. of Elec. Sci. & Tech. | Chengdu | Gao Qing | P | H |
| (CS) | | Li Mingqi | P | P |
| Tianjin | | | | |
| Civil Aviation University of China | | | | |
| (Air Traffic Mgmnt) | Tianjin | Nie Runtu | | M |
| (Air Traffic Mgmnt) | | Gao Yongxin | | P |
| (Sci) | | He Songnian | | P |
| Nankai University | Tianjin | Chen Wanyi | | H |
| | | Yang Qingzhi | | M |
| (Phys) | | Huang Wuqun | | P |
| Zhejiang | | | | |
| Hangzhou University of Commerce | Hangzhou | Ding Zhengzhong | | H,P |
| (CS) | | Hua Jiukun | P,P | |
| Zhejiang University | Hangzhou | He Yong | M | P |
| | | Tan Zhiyi | H | |
| | | Yang Qifan | | P,P |
| City College (CS) | Hangzhou | Zhao Yanan | P,P | |
| (CS) | | Huang Waibin | | P |
| (CS) | | Dong Ting | | P |
| Zhejiang University of Finance and Econ. | Hangzhou | Luo Ji | | H |
| | | Wang Fulai | | H |
| FINLAND | | | | |
| Mathematical High School of Helsinki | Helsinki | Ville Virtanen | P | P |
| Päivölä College | Tarttila | Esa I. Lappi | | P |

| INSTITUTION | CITY | ADVISOR | A | B |
|---|---|---|---|---|
| GERMANY | | | | |
| University of Trier | Trier | Volker H. Schulz | | M,P |
| HONG KONG | | | | |
| Hong Kong Baptist University | Kowloon | Chong Sze Tong | H | |
| | | Wai Chee Shiu | | H |
| INDONESIA | | | | |
| Institut Teknologi Bandung | Bandung | Edy Soewono | P | |
| | | Rieske Hadianti | | P |
| IRELAND | | | | |
| National University of Ireland, Galway | Galway | Niall Madden | | P,P |
| University College Cork | Cork | Patrick Fitzpatrick | | M |
| | | James J. Grannell | O,M | |
| | | Andrew Usher | | P |
| University College Dublin | Dublin | Maria Meehan | | H,P |
| (Phys) | | Peter Duffy | | H |
| SOUTH AFRICA | | | | |
| University of Stellenbosch | Stellenbosch | Jan H. van Vuuren | H,H | |
| UNITED KINGDOM | | | | |
| Dulwich College | London | Jeremy Lord | | P |
| Merton College, Oxford University | Oxford | Ulrike Tillmann | | O |

## **Abbreviations for Organizational Unit Types** (in parentheses in the listings)

| | | |
|---|---|---|
| (none) | Mathematics | M; Applied M; Computing M; M and Computer Science; M and Computational Science; Computing M; M and Information Science; M and Statistics; M, Computer Science, and Statistics; M, Computer Science, and Physics; Mathematical Sciences; Applied Mathematical and Computational Sciences; Natural Science and M; M and Systems Science; Applied M and Physics |
| Bio | Biology | B; B Science and Biotechnology |
| Chm | Chemistry | C; Applied C; C and Physics; C, Chemical Engineering, and Applied C |
| CS | Computer | C Science; C and Computing Science; C Science and Technology; C Science and Software Engineering; Software Engineering; Artificial Intelligence; Automation; Computing Machinery; Science and Technology of Computers |
| CS | Information | I Science; I and Computation Science; I and Calculation Science; I Science and Computation; I and Computer Science; I and Computing Science; I Engineering |
| Econ | Economics | E; E Mathematics; Financial Mathematics; Financial Mathematics and Statistics; Management; Business Management; Management Science and Engineering |
| Eng | Engineering | Civil E; Electrical Eng; Electronic E; Electrical and Computer E; Electrical E and Information Science; Electrical E and Systems E; Communications E; Civil, Environmental, and Chemical E; Propulsion E; Machinery and E; Control Science and E; Operations Research and Industrial E; Automatic Control |
| Phys | Physics | P; Applied P; Mathematical P; Modern P; P and Engineering P; P and Geology; Mechanics; Electronics |
| Sci | Science | S; Natural S; Applied S; Integrated S |
| Stat | Statistics | S; S and Finance; Mathematical S |

For team advisors from China, we have endeavored to list family name first.

# The Myth of "The Myth of Fingerprints"

Steven G. Amery
Eric Thomas Harley
Eric J. Malm
Harvey Mudd College
Claremont, CA

Advisor: Jon Jacobsen

## Summary

For over a century, fingerprints have been an undisputed personal identifier. Recent court rulings have sparked interest in verifying uniqueness of fingerprints.

We seek to determine precisely the probability of duplicate fingerprints. Our model of fingerprint structure must achieve the following objectives:

- **Topological structure** in the print, determined by the overall flow of ridges and valleys, should be described accurately.
- **Fine detail**, in the form of ridge bifurcations and terminations, must also be characterized accurately.
- **Intrinsic uncertainties**, in our ability to reproduce and measure fingerprint data, must be considered.
- **Definite probabilities** for specified fingerprint configurations must be calculated.

We place special emphasis on meeting the modeling criteria established by Stoney and Thornton [1986] in their assessment of prior fingerprint models.

We apply our model to the conditions encountered in forensic science, to determine the legitimacy of current methodology. We also compare the accuracies of DNA and fingerprint evidence.

Our model predicts uniqueness of prints throughout human history. Furthermore, fingerprint evidence can be as valid as DNA evidence, if not more so, although both depend on the quality of the forensic material recovered.

*The UMAP Journal* 25 (3) (2004) 215–230. 

# Introduction

## What is a Fingerprint?

A fingerprint is a two-dimensional pattern created by the friction ridges on a human finger [Beeton 2002]. Such ridges are believed to form in the embryo and to persist unchanged through life. The physical ridge structure appears to depend chaotically on factors such as genetic makeup and embryonic fluid flow [Prabhakar 2001]. When a finger is pressed onto a surface, the friction ridges transfer to it (via skin oil, ink, or blood) a representation of their structure.

Fingerprints have three levels of detail [Beeton 2002]:

1. Overall ridge flow and scarring patterns, insufficient for discrimination.
2. Bifurcations, terminations, and other discontinuities of ridges. The pairwise locations and orientations of the up to 60 such features in a full print, called *minutiae*, provide for detailed comparison [Pankanti et al. 2002].
3. The width of the ridges, the placement of pores, and other intraridge features. Such detail is frequently missing from all but the best of fingerprints.

## Fingerprints as Evidence

The first two levels have been used to match suspects with crime scenes, and fingerprint evidence was long used without major challenge in U.S. courts [OnIn.com 2003]. In 1993, however, in *Daubert v. Merrill Dow Pharmaceutical*, the U.S. Supreme Court set standards for "scientific" evidence [Wayman 2000]:

1. The theory or technique has been or can be tested.
2. The theory or technique has been subjected to peer review or publication.
3. The existence and maintenance of standards controlling use of the technique.
4. General acceptance of the technique in the scientific community.
5. A known potential rate of error.

Since then, there have been challenges to fingerprint evidence.

## Individuality of Fingerprints

Francis Galton [1892] divides a fingerprint into squares with a side length of six ridge periods and estimates that he can recreate the ridge structure of a missing square with probability $\frac{1}{2}$. Assuming independence of squares and introducing additional factors, he concludes that the probability of a given fingerprint occurring is $1.45 \times 10^{-11}$. Pearson refines Galton's model and finds a probability of $1.09 \times 10^{-41}$ [Stoney and Thornton 1986].

Osterburg [1977] extends Galton's approach by dividing a fingerprint into cells that can each contain one of 12 minutia types. Based on independence among cells and observed frequencies of minutiae, he finds the probability of a configuration to be $1.33 \times 10^{-27}$. Sclove [1979] extends Osterburg's model to dependencies among cells and multiple minutiae in a single cell.

Stoney and Thornton [1986] charge that these models fail to consider key issues completely: the topological information in level-one detail; minutiae location, orientation, and type; normal variations in fingerprints; and number of positions considered. We try to correct some of these omissions.

# Our Model: Assumptions and Constraints

## Assumptions

- **Fingerprints are persistent:** they remain the same throughout a person's lifetime. Galton [1892] established this fact, in recent times verified from the processes of development of dermal tissues [Beeton 2002].

- **Fingerprints are of the highest possible quality,** without damage from abrasion and injury.

- **The pattern of ridges has some degree of continuity and flow.**

- **The ridge structure of a fingerprint is in one of five categories:** Arch, Left Loop, Right Loop, Tented Arch, or Whorl, employed in the automatic classification system of Cappelli et al. [1999] (derived from those of the FBI and Watson and Wilson [1992]). Each category has a characteristic ridge flow topology, which we break into homogeneous domains of approximately unidirectional flow. While Cappelli et al. [1999] raise the issue of "unclassifiable" prints, and they and Marcialis et al. [2001] confuse classes of ridge structures, we assume that such ambiguities stem from poor print quality.

- **Fingerprints may further be distinguished by the location and orientation of minutiae relative to local ridge flow.** Stoney and Thornton [1986] argue that the ridges define a natural coordinate system, so the location of a minutia can be specified with a ridge number and a linear measure along that ridge. Finally, minutiae have one of two equally likely orientations along a ridge.

- **Each minutia can be classified as a bifurcation, a termination, or a dot** (**Figure 1**) [Pankanti et al. 2002; Stoney and Thornton 1986]. Though Galton [1892] identifies 10 minutia structures and others find 13 [Osterburg et al. 1977], we can ignore these further structures (which are compositions of the basic three) because of their low frequency [Osterburg et al. 1977].

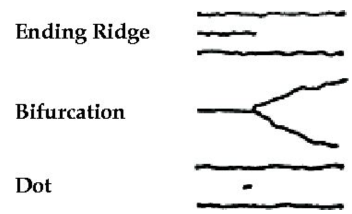



**Figure 1.** The three basic minutiae types (from Galton [1892]). We refer to ending ridges as *terminations*.

- **A ridge structure produces an unambiguous fingerprint, up to some level of resolution.** A ridge structure can vary in print representations primarily in geometric data, such as ridge spacing, curvature, and location of minutiae [Stoney and Thornton 1986]. Topological data—ridge counts, minutiae orientation, and ordering—are robust to such variation and are replicated consistently.

  A more serious consideration is connective ambiguities, such as when a given physical minutia is represented sometimes as a bifurcation and sometimes as a termination. But our highest-quality assumption dictates that such ambiguity arise only where the physical structure itself is ambiguous.

- **Location and orientation of minutiae relative to each other are independent,** though Stoney and Thornton [1986] find some dependency and Sclove [1979] model such dependency in a Markov process.

- **Ridge widths are uniform throughout the print and among different prints, and ridge detail such as pores and edge shapes is not significant.** While ridge detail is potentially useful, we have little data about types and frequencies.

- **Frequencies of ridge structure classes and configurations and minutiae types do not change appreciably with time.** We need this invariance for our model's probabilities to apply throughout human history.

## Constraints Implied by Assumptions

- Our model must consider ridge structure, relative position, orientation, and type of minutiae.

- Locations of minutiae must be specified only to within some uncertainty dependent on the inherent uncertainty in feature representation.

# Model Formulation

We examine a hierarchy of probabilities:

- that the given class of ridge structure occurs,

- that the ridge structure occurs in the specified configuration of ridge flow regions, and
- that minutiae are distributed as specified throughout the regions.

We further break this last probability down into a composition of the following region-specific probabilities:

- that a region contains the specified number of minutiae,
- that the minutiae in this region follow the specified configuration, and
- that the minutiae occur with the specified types and orientations.

## Probability of Ridge Structure Class

To each of the five classes of ridge structures (Arches, Left and Right Loops, Tented Arches, and Whorls), we associate a probability of occurrence ($\nu_A$, $\nu_L$, $\nu_R$, $\nu_T$, $\nu_W$), which we estimate from observed frequency in the population.

## Probability of Ridge Structure Configuration

Each print is partitioned into regions in which the overall flow is relatively unidirectional, and the class of the print is determined from five prototypical masks characteristic of ridge-structure classes (**Figure 2**) [Cappelli et al. 1999]. The variations of flow region structure within each class then depend on parameters for the class. For example, the ridge structure of a Loop print can be determined from the locations of the triangular singularity and the core of the loop (**Figure 3**). To determine the probability of a particular region configuration, we determine the probability that the associated parameters occur.

Because of uncertainty in the parameters, we discretize the parameter space at the fundamental resolution limit $\delta_1$ (subscript indicates feature level). We use independent Gaussian distributions about the mean values of the parameters.

We now detail the parameter spaces for each ridge-structure class. The use of the prototypes requires an $X \times Y$ region within the print.

### Arch

The parameters for the Arch consist of the Cartesian coordinates $(x, y)$ of the lower corner of the left region, the height $h$ of the central corridor, and the four angles $\theta_1, \theta_2, \theta_3, \theta_4$ at the inner corners of the left and right regions. We consider as fixed the width $b$ of the central corridor. The ratio of the resolution limit $\delta_1$ to the mean length of a typical segment determines the uncertainty in the angular measurement of that segment.

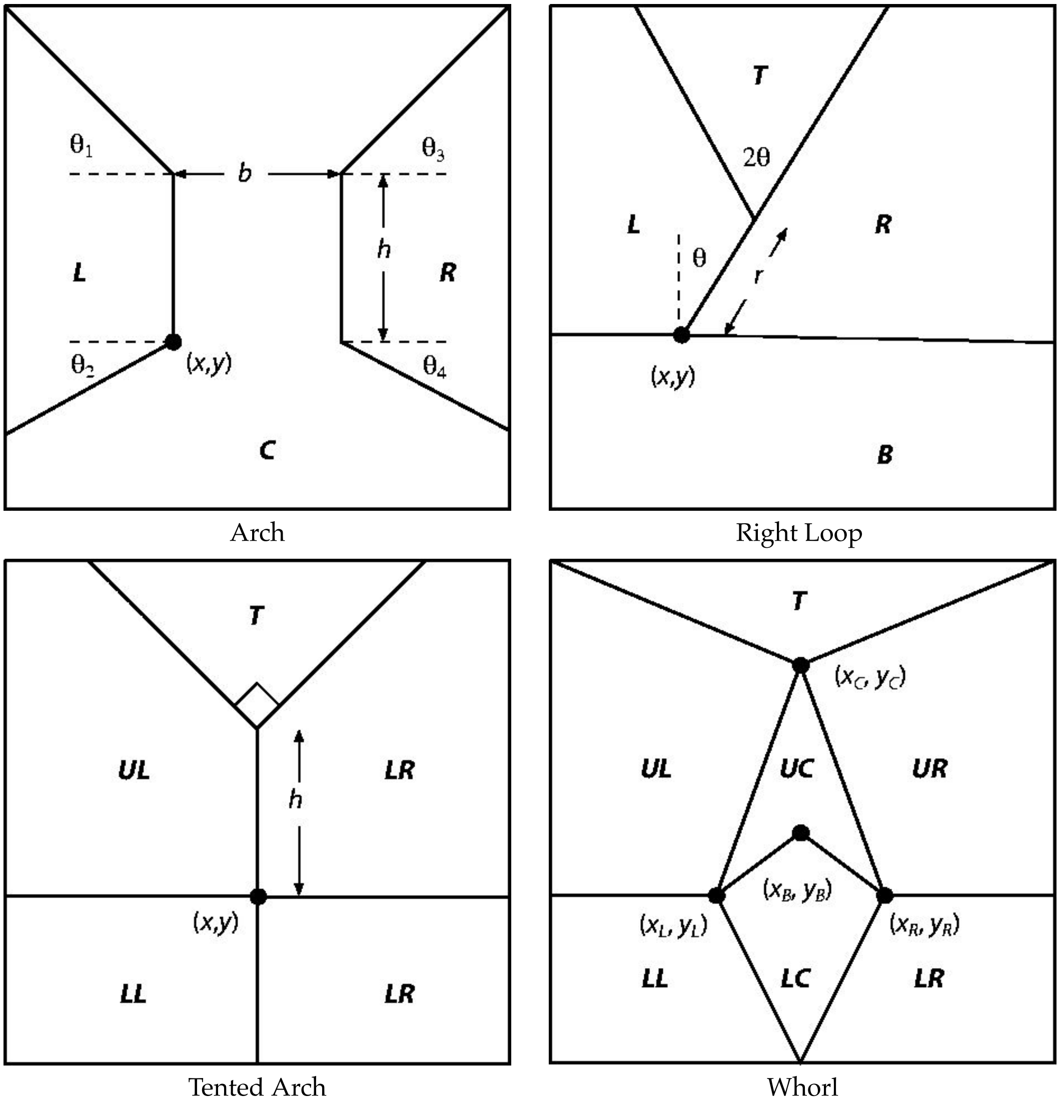



**Figure 2.** The prototypical region structures and parameters for each ridge structure class, derived from the masks in Cappelli et al. [1999].

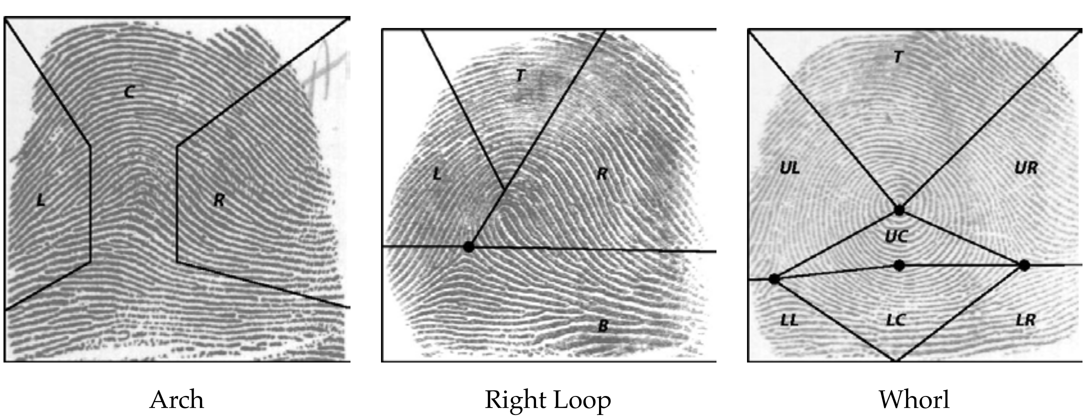



**Figure 3.** The prototypical region structures applied to an Arch, a Right Loop, and a Whorl.

### Loops, Left and Right

Since Left and Right Loops are identical except for a horizontal reflection, we use the same parameter space for both classes. The two principal features are the position $(x, y)$ of the triangular singularity outside the loop and the distance $r$ and angle $\theta$ of the core of the loop relative to this singularity.

### Tented Arch

The major structure is the arch itself; the parameters are the position $(x, y)$ of the base of the arch and the height $h$ of the arch.

### Whorl

The Whorl structure has four major features: the center of the whorl, $(x_C, y_C)$; the base of the whorl, $(x_B, y_B)$; and the triangular singularities to the left and right of the base of the whorl, at $(x_L, y_:)$ and $(x_R, y_R)$. We assume that the center and the base lie between the two singularities, so that $x_L \leq x_C$ and $x_B \leq x_R$, and that the base lies above the singularities, so that $y_B \geq y_L$ and $y_B \geq y_R$.

## Probabilities of Intraregion Minutiae Distribution

Since the geometry of a region is uniquely determined by the configuration parameters, we can divide each unidirectional flow region into parallel ridges. We can represent the ridge structure of the region as a list of ridge lengths.

We assume a fundamental limit $\delta_2$ to resolution of the position of minutiae along a ridge and divide a ridge into cells of length $\delta_2$, in each of which we find at most one minutia. The probability $P_{TC}(n, l, k)$ that the $n$th ridge in the partition, with length $l$, has a particular configuration of $k$ minutiae is

$$P_{TC}(n, l, k, \dots) = P_p(n, k, l) P_c(n, k, l) P_{to}(\{k_i, p_i, o_i\}),$$

where $P_p$ is the probability that $k$ minutiae occur on this ridge, $P_c$ the probability that these $k$ minutiae are configured in the specified pattern on the ridge, and $P_{to}$ the probability that these minutiae are of the specified types and orientations, indexed by $i$ and occurring with type probability $p_i$ and orientation probability $o_i$.

### Probability of Minutiae Number

Under the assumption that minutiae occur at uniform rates along a ridge, we expect a binomial distribution for the number of minutiae on the ridge. Denote the linear minutiae density on ridge $n$ by $\lambda(n)$. The probability that a minutia occurs in a given cell of length $\delta_2$ is $\delta_2 \lambda(n)$. Thus, the probability that $k$ minutiae occur is

$$P_p(n, k, l, \lambda) = \binom{l/\delta_2}{k} (\delta_2 \lambda)^k (1 - \delta_2 \lambda)^{l/\delta_2 - k}.$$

### Probability of Minutiae Configuration

Assuming that all configurations of $k$ minutiae are equally likely along the ridge, the probability of the specified configuration is

$$P_c(n,k,l) = \frac{1}{\binom{l/\delta_2}{k}}.$$

### Probability of Minutiae Type and Orientation

The probability that minutiae occur with specified types and orientations is

$$P_{to}(\{k_i, p_i, o_i\}) = \prod_i p_i^{k_i} o_i^{k_i}.$$

Applying our assumption that the only level-two features are bifurcations, terminations, and dots, and that orientations are equally likely and independent along the ridge, this expression reduces to

$$P_{to} = p_b^{k_b} p_t^{k_t} p_d^{k_d} \, \frac{1}{2^{k_b+k_t}},$$

with $k_b + k_t + k_d = k$. Then the total probability for the ridge configuration is

$$P_{TC}(n,l,k,\lambda,\{k_i,p_i,o_i\}) = (\delta_2\lambda)^k (1-\delta_2\lambda)^{l/\delta_2 - k} p_b^{k_b} p_t^{k_t} p_d^{k_d} \, \frac{1}{2^{k_b+k_t}}.$$

The total probability that minutiae are configured as specified through the entire print is then product of the $P_{TC}$s for all ridges in all domains, since we assume that ridges develop minutiae independently.

Applying the assumption that $\lambda$ and other factors do not depend on $n$ and are hence uniform throughout the print, we can collapse these multiplicative factors to an expression for the configuration probability of the entire print:

$$P_{TC}^{\text{global}} = (\delta_2\lambda)^K (1-\delta_2\lambda)^{L/\delta_2 - K} p_b^{K_b} p_t^{K_t} p_d^{K_d} \, \frac{1}{2^{K_b+K_t}}.$$

Here $K$ is the total number of minutiae in the print, $K_i$ the number of type $i$, and $L$ is the total linear length of the ridge structure in the print. The length $L$ is determined only by the total area $XY$ of the print and the average ridge width $w$ and is therefore independent of the topological structure of the print.

# Parameter Estimation

For parameters in our model, we use published values and estimates based on the NIST-4 database [Watson and Wilson 1992].

## Level-One Parameters

All lengths are in millimeters (mm); angles are in radians or in degrees.

- **Level-one spatial resolution limit** $\delta_1$: Cappelli et al. [1999] discretize images into a $28 \times 30$ grid to determine level-one detail. From these grid dimensions, the physical dimensions of fingerprints, and the assumption of an uncertainty of three blocks for any level-one feature, we estimate $\delta_1 = 1.5$.

- **Level-one angular resolution limit** $\delta_\theta$: Taking $X/2 = 5.4$ (determined below) as a typical length scale, we have $\delta_\theta = \delta_1/5.4 = 0.279$ radians.

- **Ridge structure class frequencies** $\nu_A$**,** $\nu_L$**,** $\nu_R$**,** $\nu_T$**, and** $\nu_W$: We use the estimates in Prabhakar [2001] (**Table 1**).

**Table 1.** Relative frequencies of ridge structure classes (from Prabhakar [2001]).

| $\nu_A$ | $\nu_L$ | $\nu_R$ | $\nu_T$ | $\nu_W$ |
|---|---|---|---|---|
| 0.0616 | 0.1703 | 0.3648 | 0.0779 | 0.3252 |

- **Thumbprint width** $X$ **and height** $Y$: Examining thumbprints from the NIST-4 database and comparing them with the area given by Pankanti et al. [2002], we conclude that a width that covers the majority of thumbprints is 212 pixels in the 500 dpi images, a physical length of $10.8$ mm. Similarly, $Y = 16.2$ mm.

- **Arch parameters** $(x, y)$**,** $h$**,** $b$**,** $\theta_1$**,** $\theta_2$**,** $\theta_3$**, and** $\theta_4$: We restrict the parameter space for $(x, y)$ to the lower half of the thumbprint with horizontal margins of length $b$. We estimate $b = 2.5$ from examination of Arch fingerprints in the NIST database and from Cappelli et al. [1999]. This estimate places $x \in (0, 8.3)$ and $y \in (0, 5.6)$. The mean for $(x, y)$, which we need to describe the distribution of $(x, y)$, is then $(4.2, 2.8)$. We estimate that $x$ and $y$ both have a standard deviation of $0.7$. We assume that $\theta_1$, ... ,$\theta_4$ are all between $0°$ and $45°$ with mean $22.5°$ and standard deviation $5.13°$.

- **Loop parameters** $(x, y)$**,** $\theta$**, and** $r$: For a left loop (a right loop is reflection of this), $(x, y)$ must lie in the bottom left quadrant and the mean coordinate pair is $(2.7, 2.8)$. Additionally, we restrict $\theta$ to lie between $15°$ and $75°$, which allows us to estimate the mean $\theta$ as $45°$ with a standard deviation of $15°$. We estimate that $r$ must be greater than $0$ and less than $9.6$.

- **Tented arch parameters** $(x, y)$ **and** $h$: Along the $y$ direction, we restrict the bottom of the arch $(x, y)$ to lie in the bottom half of the thumbprint. We further estimate that $x$ lies in the middle two-thirds of $X$. These assumptions yield $x \in (1.8, 9)$ and $y \in (0, 8.1)$. Assuming a symmetric distribution of $(x, y)$ yields $(x, y) = (5.4, 2.8)$ with a standard deviation of $0.7$ in both directions. Logically, we place $h$ between $0$ and $Y/2 = 8.1$. Again, assuming a symmetric distribution in this parameter space and a standard deviation of one-eighth the maximum value yields $h = 4.05 \pm 1.02$.

- **Whorl parameters** $(x_L, y_L)$**,** $(x_C, y_C)$**,** $(x_R, y_R)$**, and** $(x_B, y_B)$: We expect $(x_L, y_L)$ to be in the bottom left quadrant for all but the most extreme examples and similarly $(x_R, y_R)$ to lie in the bottom right quadrant. We place $(x_B, y_B)$ between $x = X/4$ and $x = 3X/4$ and $y = 0$ and $y = 2Y/3$. The topmost point, $(x_C, y_C)$, we place in the top half of the thumbprint. We again put the average values in the center of their restricted areas.

**Table 2** summarizes the estimates for these four classes of ridge structures.

**Table 2.** Parameter range estimates for the ridge structure classes.
All lengths in millimeters (mm), angles in degrees.

| Arch parameter ranges | |
|---|---|
| $(x, y)$ | $(4.2, 2.8) \pm (0.7, 0.7)$ |
| $h$ | $4.05 \pm 0.7$ |
| $b$ | $2.5 \pm 0$ |
| $\theta_1$–$\theta_4$ | $22.5° \pm 5.13°$ |
| **Loop parameter ranges** | |
| $(x, y)$ | $(2.7, 2.8) \pm (0.7, 0.7)$ |
| $\theta$ | $45° \pm 15°$ |
| $r$ | $4.58 \pm 0.7$ |
| **Tented Arch Parameter Ranges** | |
| $(x, y)$ | $(5.4, 2.8) \pm (0.7, 0.7)$ |
| $h$ | $4.05 \pm 1.02$ |
| **Whorl parameter ranges** | |
| $(x_L, y_L)$ | $(2.7, 4.1) \pm (0.7, 0.7)$ |
| $(x_C, y_C)$ | $(5.4, 12.2) \pm (0.7, 0.7)$ |
| $(x_R, y_R)$ | $(8.1, 4.1) \pm (0.7, 0.7)$ |
| $(x_B, y_B)$ | $(5.4, 4.1) \pm (0.7, 0.7)$ |

# Level-Two Parameters

- **Level-two spatial resolution limit** $\delta_2$: We estimate $\delta_2$ by $r_0$, the spatial uncertainty of minutiae location in two dimensions [Pankanti et al. 2002], and propose $\delta_2 = 1$ for best-case calculations.

- **Relative minutiae type frequencies** $p_d$**,** $p_b$**, and** $p_t$: Almost every compound minutia can be broken into a combination of bifurcations and terminations separated spatially. Counting these compound minutiae appropriately, we determine the relative minutiae frequencies in **Table 3**.

- **Ridge period** $w$: We use $0.463$ mm/ridge for the ridge period, the distance from the middle of one ridge to the middle of an adjacent one [Stoney and Thornton 1986].

**Table 3.** Frequencies of simple minutiae types (from Osterburg et al. [1977]).

| $p_b$ | $p_t$ | $p_d$ |
|---|---|---|
| 0.356 | 0.581 | 0.0629 |

- **Mean number of minutiae per print** $\mu$: Under ideal circumstances, we discern 40 to 60 minutiae on a print [Pankanti et al. 2002]; we take $\mu = 50 \pm 10$.
- **Linear minutiae density** $\lambda$: We calculate $\lambda$ by dividing the average number of minutiae per a thumbprint $\mu$ by the total ridge length of a thumbprint $XY/w$. Under ideal conditions, this gives $\lambda = 0.13 \pm 0.03$ minutiae/mm. In practice, we may have $\lambda = 0.05 \pm 0.03$ minutiae/mm [Pankanti et al. 2002].

Finally, we estimate that there have been 100 billion humans [Haub 1995].

# Model Analysis and Testing

Let the probability that a print has a configuration $x$ be $p_c(x)$. Assuming that fingerprint patterns are distributed independently, the probability that two prints match is $p_c^2(x)$. The sum of these probabilities over the configuration space is the total probability that some match occurs.

The probabilities associated with the two levels of detail are determined independently, so the total occurrence probability factors into $p_{c1}(x_1)p_{c2}(x_2)$. Denoting the level-one configuration subspace as $C_1$ and the level-two subspace as $C_2$, the total probability of the prints matching is

$$p = \sum_{i \in C_1} \sum_{j \in C_2} [p_{c1}(i) p_{c2}(j)]^2 = \left( \sum_{i \in C_1} p_{c1}^2(i) \right) \left( \sum_{j \in C_2} p_{c2}^2(j) \right) = p_1 p_2.$$

## Level-One Detail Matching

We restrict each parameter to a region of parameter space in which we expect to find it and assume that it is uniformly distributed there. This approximation is enough to estimate order of magnitude, which suffices for our analysis. Then

$$p_{c1}(i) = \frac{\nu_i}{\left( \prod_{j \in V(i)} \frac{L_j}{\delta_1} \right)}, \tag{1}$$

where $L_j$ is the range of parameter $j$ in $V(i)$, the set of parameters for a type-$i$ ridge structure. For **(1)** to be accurate, we should make any $L_j$ corresponding to angular parameters the product of the angle range with our typical length of 5.4 mm. The product is simply the total number of compartments in the parameter space, since we assume a uniform distribution in that range. Calculating $p_{c1}(i)$ for each ridge structure type, and summing squares, we find the probability that two thumbprints have the same overall ridge structure:

$$p_1 = \sum_{i \in C_1} p_{c1}^2(i) = .00044. \tag{2}$$

## Level-Two Detail Matching

If we disregard the infrequent dot minutiae, we obtain the probability

$$p_{c2}(j) = (\delta_2\lambda)^k (1-\delta_2\lambda)^{C-K} p_b^{k_b} p_t^{k-k_b} \frac{1}{2^k}$$

for a configuration $j$ with $k$ minutiae, $k_b$ of which are bifurcations (and the rest ridges), placed in $C = XY/w\delta_2$ cells. If we simplify minutia-type frequencies to $p_b = p_t = 1/2$, and note that there are

$$\binom{C}{k}\binom{k}{k_b}2^k$$

ways to configure $j$ given $k$ and $k_b$, the total probability of a match becomes

$$\begin{aligned} p_2 &= \sum_{k=0}^{C}\sum_{k_b=0}^{k}\left((\delta_2\lambda)^k(1-\delta_2\lambda)^{C-K}\frac{1}{4^k}\right)\binom{C}{k}\binom{k}{k_b}2^k \\ &= \sum_{k=0}^{C}(\delta_2\lambda)^{2k}(1-\delta_2\lambda)^{2(C-k)}\frac{1}{4^k}\binom{C}{k} \\ &= \left(\frac{5(\delta_2\lambda)^2 - 8\delta_2\lambda + 4}{4}\right)^C. \end{aligned}$$

Match probabilities for $\lambda = 0.13 \pm 0.03/\text{mm}$, $\delta_2 = 1$ mm, and $C = 250$ to $400$ cells range from $2.9 \times 10^{-23}$ to $9.8 \times 10^{-60}$; probabilities for the more realistic values $\lambda = 0.05 \pm 0.03/\text{mm}$, $\delta_2 = 2$–$3$ mm, and $C = 100$ to $250$ cells range from $3.7 \times 10^{-5}$ to $1.7 \times 10^{-47}$.

## Historical Uniqueness of Fingerprints

Denote the probability of a match of any two left thumbprints in the history of the human race by $p$ and the world total population by $N$. The probability of at least one match among $\binom{N}{2}$ thumbprints is

$$P = 1 - (1-p)^{\binom{N}{2}}.$$

**Figure 4a** illustrates the probability of at least one match for $N = 10^{11}$, while **Figure 4b** shows a log-log plot of the probability for very small $p$-values. Since even conservative parameter values in the ideal case give $p \ll 10^{-30}$, our model solidly establishes uniqueness of fingerprints through history.

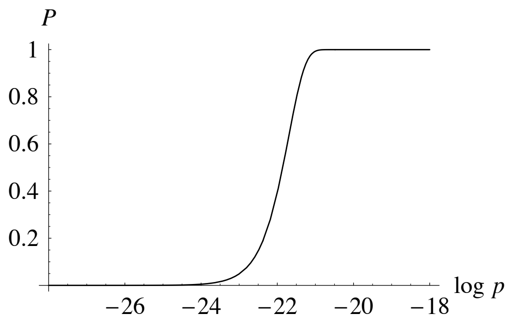



**Figure 4a.** For $N = 10^{11}$, probability of at least one thumbprint match through history.

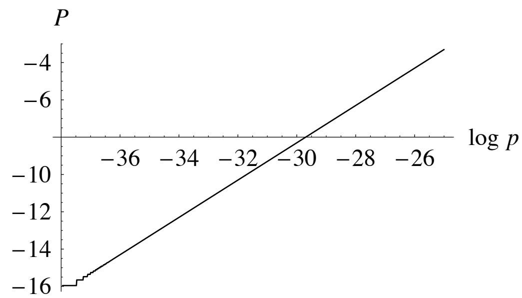



**Figure 4b.** Log-log plot of probability.

# Strengths and Weaknesses of the Model

## Strengths

- **Topological coordinate system**: We take topological considerations into account, as demanded by Stoney and Thornton [1986].
- **Incorporation of ridge structure detail**: We use this in addition to the minutiae detail that is the primary focus of most other models.
- **Integration of nonuniform distributions**: We allow for more-complex distributions of the ridge structure parameters, such as Gaussian distributions for singularity locations, and we consider that distribution of minutiae along ridges may depend on the location of the ridge in the overall structure.
- **Accurate representation of minutia type and orientation**: We follow models such as those developed by Roxburgh and Stoney in emphasizing the bidirectional orientation of minutiae along ridges, and we further consider the type of minutiae present as well as their location and orientation. Cruder models of minutiae structure [Osterburg et al. 1977; Pankanti et al. 2002] neglect some of this information.
- **Flexibility in parameter ranges**: We test a range of parameters in both ideal and practical scenarios and find that the model behaves as expected.

## Weaknesses

- **Ambiguous prints, smearing, or partial matches**: We assume that ambiguities in prints reflect ambiguities in physical structure and are not introduced by the printing. This is certainly not the case for actual fingerprints.
- **Domain discontinuities**: We have no guarantee of continuity between regions of flow; continuity requirements may affect the level-one matching probabilities significantly.

- **Nonuniform minutia distribution**: We assume that the distribution of minutiae along a ridge is uniform. However, models should account for variations in minutiae density and clustering of minutiae [Stoney and Thornton 1986]. Although we have a mechanism for varying this distribution, we have no data on what the distribution should be.
- **Left/right orientation distribution**: We assume that the distribution of minutiae orientation is independent and uniform throughout the print. Amy notes, however, that the preferential divergence or convergence of ridges in a particular direction can lead to an excess of minutiae with a particular orientation [Stoney and Thornton 1986].
- **Level-three information**: We neglect level-three information, such as pores and edge shapes, because of uncertainty about its reproducibility in prints.

# Comparison with DNA Methods

## DNA Fingerprinting

The genetic material in living organisms consists of deoxyribonucleic acid (DNA), a macromolecule in the shape of a double helix with nitrogen-base "rungs" connecting the two helices. The configurations of these nitrogen bases encode the genetic information for each organism and are unique to the organism (except for identical twins and other cases in which an organism splits into multiple separate organisms).

Direct comparison of base-pair sequences for two individuals is infeasible, so scientists sequence patterns in a person's DNA called *variable number tandem repeats (VNTR)*, sections of the genome with no apparent genetic function.

## Comparison of Traditional and DNA Fingerprinting

While level-two data is often limited by print quality, we expect level-one detail to remain relatively unchanged unless significant sections of the print are obscured or absent. We use $p_1 = 10^{-3}$ from **(2)**, allowing for a conservative loss of seven-eighths of the level-one information. Multiplying by this level-one factor $10^{-3}$, all but the three worst probabilities are less than $10^{-9}$.

DNA fingerprinting has its flaws: False positives can arise from mishandling samples, but the frequency is difficult to estimate. The probability of two different patterns exhibiting the same VNTR by chance varies between $10^{-2}$ and $10^{-4}$, depending on the VNTR [Roeder 1994; Woodworth 2001]. The total probability of an individual's DNA pattern occurring by chance is computed under the assumption that the VNTRs are independent, which has been verified for the ten most commonly used VNTRs [Lambert et al. 1995].

# Results and Conclusions

We present a model that determines whether fingerprints are unique. We consider both the topological structure of a fingerprint and the fine detail present in the individual ridges. We compute probabilities that suggest that fingerprints are reasonably unique among all humans who have lived.

Fingerprint evidence compares well with DNA evidence in forensic settings. Our model predicts that with even a reasonably small fingerprint area and number of features, the probability that a match between a latent print and a suspect's print occurs by chance is less than $10^{-9}$. Both DNA evidence with few VNTRs and fingerprints of poor quality with few features can give inconclusive results, resulting in uncertainty beyond a reasonable doubt.

# References

Adams, Julian. 2002. DNA fingerprinting, genetics and crime: DNA testing and the courtroom. `http://www.fathom.com/courese/21701758/`.

Beeton, Mary. 2002. Scientific methodology and the friction ridge identification process. *Identification Canada* 25 (3) (September 2002). Reprinted in *Georgia Forensic News* 32 (3) (November 2002) 1, 6–8. `http://www.ridgesandfurrows.homestead.com/files/Scientific_Methodology.pdf`.

Cappelli, Raffaele, Alessandra Lumini, Dario Maio, and Davide Maltoni. 1999. Fingerprint classification by directional image partitioning. *IEEE Transactions on Pattern Analysis and Machine Intelligence* 21 (5): 402–421.

Epstein, Robert. 2002. Fingerprints meet *Daubert*: The myth of fingerprint "science" is revealed. *Southern California Law Review* 75: 605–658.

Galton, Francis. 1892. *Finger Prints*. London: Macmillan.

Haub, Carl. 1995. How many people have ever lived on Earth? *Population Today* 23 (2) (February) 4–5. Reprinted. 30 (8) (November / December 2002) 4–5. `http://www.prb.org/Content/ContentGroups/PTarticle/Oct-Dec02/How_Many_People_Have_Ever_Lived_on_Earth_.htm`.

Jeffreys, A.J., V. Wilson, and S.L. Thein. 1995. Individual-specific fingerprints of human DNA. *Nature* 316: 76–79.

Lambert, J.A., J.K. Scranage, and I.W. Evett. 1995. Large scale database experiments to assess the significance of matching DNA profiles. *International Journal of Legal Medicine* 108: 8–13.

Marcialis, Gian Luca, Fabio Roli, and Paolo Frasconi. 2001. Fingerprint classification by combination of flat and structural approaches. In *International*

*Conference on Audio- and Video-Based Biometric Person Authentication AVBPA01* (June 2001, Halmstad, Sweden), edited by J. Bigun and F. Smeraldi. *Springer Lecture Notes in Computer Science* 2091: 241–246.

National Center for State Courts (NCSC). 2004. `http://www.ncsconline.org/` .

German, Ed. 2003. The history of fingerprints. `http://onin.com/fp/fphistory.html` .

Osterburg, James W., T. Parthasarathy, T.E.S. Ragvahan, and Stanley L. Sclove. 1977. Development of a mathematical formula for the calculation of fingerprint probabilities based on individual characteristics. *Journal of the American Statistical Association* 72: 772–778.

Pankanti, S., S. Prabhakar, and A. Jain. 2002. On the individuality of fingerprints. *IEEE Transaction on Pattern Analysis and Machine Intelligence*: 1010–1025. `http://citeseer.ist.psu.edu/472091.html` .

Prabhakar, Salil. 2001. Fingerprint matching and classification using a filterbank. Ph.D. thesis. East Lansing, Michigan: Michigan State University.

Roeder, K. 1994. DNA fingerprinting: a review of the controversy. *Statistical Science* 9: 222–247.

Sclove, Stanley L. 1979. The occurrence of fingerprint characteristics as a two-dimensional process. *Journal of the American Staistical Association* 74: 588–595.

Stoney, David A., and John I. Thornton. 1986. A critical analysis of quantitative fingerprint individuality models. *Journal of Forensic Sciences* 31 (4): 1187–1216.

Thompson, William. 2003. Examiner bias. `http://www.bioforensics.com/conference/ExaminerBias/` .

Watson, C.I., and C.L. Wilson. 1992. *NIST Special Database 4, Fingerprint Database*. U.S. National Institute of Standards and Technology.

Wayman, James L. 2000. Daubert hearing on fingerprinting: When bad science leads to good law: The disturbing irony of the *Daubert* hearing in the case of *U.S. v. Byron C. Mitchell*. `http://www.engr.sjsu.edu/biometrics/publications_daubert.html` .

Woodworth, George. 2001. Probability in court—DNA fingerprinting. `http://www.stat.uiowa.edu/~gwoodwor/statsoc` .

# Can't Quite Put Our Finger On It

Séamus Ó Ceallaigh
Alva Sheeley
Aidan Crangle
University College Cork
Cork, Ireland

Advisor: James J. Grannell

## Summary

There are two main paths to identification through fingerprints.

- Global analysis relies on the specific arrangement and characteristics of the ridges of a print.
- Local analysis assumes that the individuality of a print is based on the position and orientation of the two basic types of minutiae.

We subdivide a print into a grid of square cells, consider the distribution of ridge features, and calculate probabilities from combinatorial analysis. We make predictions by refining parameters of the model, such as the number of minutiae required for a positive match between two generic prints, and the size of a cell in the main grid. We compare our results to previous studies and discuss the relation to DNA profiling. The simplicity of our model is its key strength.

We conclude that it is extremely unlikely that any two randomly selected people have, or have ever had, the same set of fingerprints.

Despite the apparently simplistic nature of fingerprinting, it is vastly more reliable in identification than a DNA comparison test.

## Introduction

In recent years, the scientific basis of fingerprint analysis has been questioned, in the U.S. Supreme Court ruling in the Daubert case that the reliability of expert scientific testimony must be established along the following five criteria:

---

*The UMAP Journal* 25 (3) (2004) 231–244. 

1. whether the particular technique or methodology in question has been subject to a statistical hypothesis testing;
2. whether its error rate has been established;
3. whether the standards controlling the technique's operation exist and have been maintained;
4. whether it has been peer-reviewed and published; and
5. whether it has a general widespread acceptance.

Our model tries to address the first two issues. We aim to produce a probabilistic method to measure the uniqueness of a particular print.

# Assumptions

- A thumbprint is defined globally by ridge patterns and locally by a distribution of *minutiae*, which we refer to also as *features*.
- The area of interest typical thumbprint is a 20 mm $\times$ 20 mm square grid.
- There are two significant types of minutiae, the bifurcation and the ridge ending: all other minutiae are compositions of these [Osterburg et al. 1977].
- The probability of a minutia occurring in a grid box is .234 [Osterburg 1977].
- The orientation of the minutiae was not taken into account by Osterburg; we assign a minutiae one of eight angles, from 0° to 157.5°, in steps of 22.5°.
- When comparing two prints, we know one print arbitrarily well.
- The number of people who have ever lived is $1.064 \times 10^{11}$ [Haub 1995].

# The Model

## Global Analysis

### Ridge Patterns And Orientation Fields

Global analysis concerns ridge patterns, which distinguish prints into six main pattern groups: Arch, Tented Arch, Left Loop, Right Loop, Twin Loop and Whorl. Each pattern is determined by an orientation field, which may have specific stationary points, known as the the *delta* and the *core*. If a print contains 0 or 1 delta points and 0 or 1 core points, then it is classified as Lasso, and Wirbel otherwise.

The Lasso class consists of arch, tented arch, right loop, and left loop.

- If the fingerprint has 0 delta points or 0 core points, then it is an arch.
- Otherwise, if the core point and the delta point are aligned in the vertical direction, then the fingerprint is an arch if the length between the core point and the delta point is less than 2.5 mm and a tented arch otherwise.
- Otherwise, if the core point is to the right of the delta point, the fingerprint is a right loop.
- Otherwise, the fingerprint is a left loop.

The Wirbel class consists of the whorl and the twin loop classes:

- If there are exactly two core points and exactly two delta points, then the fingerprint is a whorl if the two core points are aligned horizontally and a twin loop otherwise.
- Otherwise, the fingerprint is a whorl.

The main aim of global analysis is a vector field or orientation field to the ridge lines of a fingerprint. We must find suitable parameters for such functions that give rise to the different classes of ridge pattern. The most basic pattern without stationary points is the arch **Figure 1**, modeled by the simple system

$$\frac{dx}{dt} = \mu y, \qquad \frac{dy}{dt} = -\nu,$$

with parameters $\mu$ and $\nu$. The orientation fields for other ridge patterns are more complex, so the bulk of our model is directed at the print's local features.

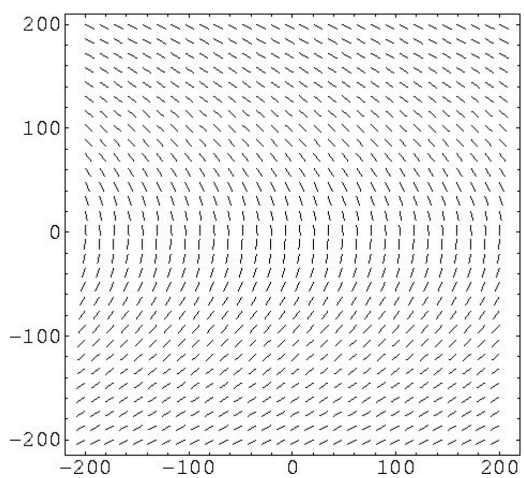



**Figure 1.** Arch orientation field.

# Local Analysis

## Estimates

For an initial estimate of the probability of any two people having the same thumbprint, we must consider:

- the total number of people who have ever lived, estimated to be $1.064 \times 10^{11}$ [Haub 1995] (that is, about 6% of all people are alive right now); and
- the total number of possible thumbprints that can be classified as "different."

To decide whether or not one thumbprint is the same as another, one must first decide on what exactly is a thumbprint. In our model, we take a typical area of the print as 20 mm $\times$ 20 mm, which is large enough to encompass the area of interest on any print. We divide this area into boxes 1 mm on a side, thus giving 400 boxes, each of area 1 mm$^2$. In principle, each box can be examined to determine whether or not it contains a minutia.

Minutiae are the features on a thumbprint that are used by almost all identification techniques to distinguish between prints. There are from 10 [Galton 1892] to 16 [Optel Ltd. 2003] different types of minutiae, but they are all composed of two fundamental types: ridge endings and ridge bifurcations. In our analyses, we consider only these two types. Also, if there are two minutiae in the same cell, it is impossible to resolve them separately.

Say that there are $i$ resolvable features on the print. The number of ways that we can insert these $i$ features into the 400 spaces in the grid is

$$\binom{400}{i}.$$

Remembering that there are two possibilities for each feature, the total number of combinations is then

$$\binom{400}{i} 2^i.$$

What is the value of $i$? The probability that any box contains a feature we take as .234 [Osterburg et al. 1977]. Then $i$ has a binomial distribution:

$$P(i = x) = \binom{400}{x} (.234)^x (1 - .234)^{400-x},$$

with mean $\mu \approx 94$ and standard deviation $\sigma \approx 8$, so that the average number of cells containing features is 94.

Thus, the total number of possible thumbprints is

$$N = \sum_{i=0}^{400} \binom{400}{i} 2^i.$$

The binomial distribution for $i$, however, is concentrated mainly in the region $\mu - \sigma < i < \mu + \sigma$, or $94 - 8 < i < 94 + 8$. To be conservative, we consider only this range of numbers of minutiae; thus, there are approximately

$$N \approx \sum_{i=86}^{102} \binom{400}{i} 2^i \approx 1.19 \times 10^{128}$$

different thumbprints "available" for any actual thumb to hold. So very roughly,

$$P(\text{two people ever having the same thumbprint}) = \frac{1.064 \times 10^{11}}{1.19 \times 10^{128}} \sim 10^{-117}.$$

## Comparison

This figure is the (approximate) probability that there have ever been two people who have had the same thumbprint. How might we take into account the chance that, when compared, two prints will be *judged* to be the same? To do this, we consider two hypothetical prints:

- A *control* print: an ideal known print, in which all $i$ features are seen.
- A *sample* print: a print with more features $i$ than the $n$ available for comparison.

For two prints that are compared in a realistic circumstance, there will be at least $(i-n)$ features that are not included in the comparison. These features, in theory, could be in any combination of positions in the grid. The main question is: How many prints have $n < i$ features corresponding to a match? In other words, how many different ways can the remaining $(i-n)$ features be inserted into the grid and still produce a match with the control print? Knowing that, we can estimate how likely it is that two thumbprints not actually the same will match.

## Incorrect Matching

We have $(i-n)$ features to distribute among $(400-n)$ grid elements. The number of different ways to do this is, by previous reasoning,

$$\sum_{86}^{102} \binom{400-n}{i-n} 2^{i-n}.$$

In criminal proceedings, a matching number of minutiae of anything from 8 [Collins 1992] to 15 [Vacca 2002] are accepted as conclusive proof of identification. Our model predicts that for $n = 12$, the total number of thumbprints that could have the same set of matching minutiae while not being the same print is $N = 1.3 \times 10^{117}$. But expressed as a fraction of the *total* number of possible prints, the probability of the print being one of these, if it selected from them, is

$$P(\text{false match}) = \frac{1.3 \times 10^{117}}{1.19 \times 10^{128}} \approx 1.09 \times 10^{-11}. \quad \textbf{(1)}$$

This is an extremely low probability.

## Varying Parameters

The result **(1)** depends on the parameters, which can be varied according to circumstance and also as a way of refining the model:

- $p$: the probability of finding a feature in a grid cell. We take $p = .234$ [Osterburg et al. 1977]. Others [Thai 2003; Kingston 1964; Stoney and Thornton 1987; Dankmeijer et al. 1980 ] give values in the range $.19 < p < .25$.
- $N$: the number of cells in the grid. If there are more cells, on average, then more features will be observed, since $p(\text{feature})$ for a cell remains the same.
- $n$: the number of minutiae that one takes for comparison. We take $n = 12$.
- $i$: a variable, determined by $p$ and $N$, that gives reasonable bounds for the summation.
- $F$: the number of different features that can appear in a grid cell. In our initial estimate, we take $F = 2$.
- $L, A$: the length of a side, and the area, of a grid cell. We take $L \approx 1$ mm, the average distance between features [Thai 2003].

If we wish to examine a thumbprint more closely, we should consider smaller and smaller areas of the print. It is not meaningful, however, to take $L$ less than $\sim 0.1$ mm, since this is the typical ridge width.

## The Dependence on $L$

We rework the model, taking the width of the generic grid cell to be 0.5 mm. Taking the overall area of the print to be the same, there are now 1,600 grid cells to consider, each with the same probability of having a feature. The binomial expression for $i$, the number of features observed on the whole print, is now

$$P(i = x) = \binom{1600}{x} (.234)^x (1 - .234)^{1600-x},$$

with mean $\mu = 374$ and standard deviation $\sigma = 17$. Thus, the region of relevance when summing is now $374 - 17 = 357 < i < 391 = 374 + 17$. This means that when the thumbprint is examined on a scale half that of the initial, about four times as many minutiae will be observed. Intuitively, the likelihood of a false match will decrease, since there are more possibilities for the number of prints:

$$N_{1600} = \sum_{i=357}^{391} \binom{1600}{i} 2^i \approx 3.5 \times 10^{502}.$$

The probability that any of the 100 billion people who have ever lived have had the same thumbprint is

$$P(\text{2 people ever having the same thumbprint}) = \frac{1.064 \times 10^{11}}{3.5 \times 10^{502}} \sim 10^{-492}.$$

We now determine the number of ways in which, when a certain number $i$ of minutiae are selected for comparison, the remaining minutiae can be arranged. Following the same logic as before, this figure is

$$\sum_{357}^{391} \binom{1600-n}{i-n} 2^{i-n},$$

which evaluates to $3.4 \times 10^{491}$ for $n = 12$. The probability that any two compared thumbprints, judged to be identical by the standards of comparison, are actually different is therefore

$$P(\text{false match} \mid n = 12, N = 1600) = \frac{3.4 \times 10^{491}}{3.5 \times 10^{502}} \approx 10^{-11}.$$

This, interestingly, is not much greater than the probability for the previous estimate. The result is not therefore acutely dependent on the value of $L$, nor, by association, on the number $N$ of grid cells. That said, it easy to examine details in any print to a scale of 0.5 mm.

## The Dependence on $p$

The probability of a feature in a cell is, as of yet, a purely empirical figure. The formation of fingerprints, and their associated characteristics, is known: The foetus, at about 6.5 weeks, grows eleven "volar pads"—pouches on various locations of the hand [Anonymous 2001]. These shrink at about 11 weeks; and when they are gone, beneath where they lay are fingerprints. However, the mechanism of formation of the specific features is unknown. Genetic influences are present, but the environment is crucial also, evidenced by the fact that identical twins—who have the same DNA genotype—do *not* have the same fingerprints.

There is no way yet determined of predicting the frequency of occurrence of any type of minutia on the print of a particular person. Previous studies, cited in **Table 1**, show variation about $\sim 0.2$ minutiae/mm$^2$. It is not unreasonable to propose that the density depends on the print classification (i.e., whorl, loop, arch, etc.).

The range is $.204 < p < .246$. For 1,600 boxes, we have

$$P(\text{false match}) \approx 1.89 \times 10^{-12} \qquad \text{for } p = .204,$$
$$P(\text{false match}) \approx 1.78 \times 10^{-11} \qquad \text{for } p = .246.$$

The variation of $p$ changes the final prediction by no more than an order of magnitude.

**Table 1.**
The multiplication table of $D_{10}$.

| Source | Number of prints | Mean density (minutiae/mm$^2$) |
|---|---|---|
| Osterburg et al. [1977] | | .234 |
| Dankmeijer et al. [1980] | 1,000 | .19 |
| Stoney and Thornton [1987] | 412 | .223 |
| Kingston [1964] | 100 | .246 |
| Thai [2003] | 30 | .204 |

## The Dependence on $F$

In out initial estimate, we take the number $F$ of degrees of freedom of a feature in a print to be two: either a ridge ending or a ridge bifurcation. However, one can also consider the *orientation* of a feature. Each minutia lies on a ridge, which has a well-defined direction. We discretized this variable to one of eight possible directions, angles from 0° to 157.5°. Thus each feature, instead of having 2 degrees of freedom, now has 16.

The probability of a false match, taking 1,600 grid cells and a probability of occurrence $p = .234$, is now

$$P(\text{false match}) = \frac{\sum_{i=357}^{391} \binom{1600-n}{i-n} 16^{i-n}}{\sum_{i=357}^{391} \binom{400}{i} 16^{i}}.$$

Taking $n = 12$ as before, we find that

$$P(\text{false match}) \approx 2.6 \times 10^{-22}.$$

This is an astonishingly smaller probability than the previous estimate of $10^{-11}$. The orientation of a feature is no more difficult to determine in practice than its nature, so including it in the comparison process is a great improvement in efficacy with a modest increase in effort.

## The Dependence on $n$

It is crucial to determine how many matching minutiae are necessary for a positive comparison. We have taken $n = 12$ in the preceding analyses; it is instructive to consider the variation of the probability of a false match with $n$. The graphs in **Figure 2** show that the probability falls off sharply, even as $n$ increases beyond 1. A value of $n \approx 5$ is quite sufficient.

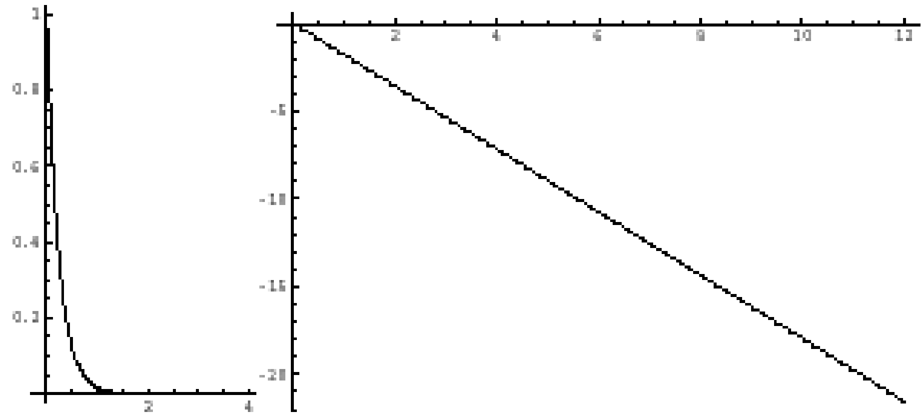


**Figure 2.** Probability of false match (linear and logarithmic scales) vs. $n$.

# Conclusion

## The Model

Our model, in the most general form, is

$$P(\text{false match}) = \frac{\sum_{i=\mu-\sigma}^{\mu+\sigma} \binom{N-n}{i-n} F^{i-n}}{\sum_{i=\mu-\sigma}^{\mu+\sigma} \binom{N}{i} F^{i}},$$

where

- $F$ is the number of degrees of freedom and $N$ is the number of grid cells,
- $\mu$ and $\sigma$ are the mean and standard deviation of the binomial distribution determined by $N$ and $p$,
- $p$ is the probability of there being a feature in a cell, and
- $n$ is the number of minutiae being used to make a comparison.

The preliminary result returned from our model, for $n = 12$ (a typical threshold for positive identification in many countries), is $P \approx 10^{-11}$. Further refinement of the parameters reduces this to $P \approx 7 \times 10^{-22}$. We conclude that 12 is a very reasonable comparison criterion, and that $n = 5$ or $6$ is quite damning for any suspect so compared.

Thus, we conclude that to a very high degree of certainty, not only that no two people, now living or having ever lived in the past, have had the same thumbprint, but also that there is a vanishingly small chance that two prints are even close enough to be confused, given a small fraction of minutiae from their patterns to compare.

## DNA Analysis

DNA identity testing is based on aspects of the DNA patterns called *loci*. For a 100 % match, the FBI [Thompson et al. n.d.] recommends that 13 loci be used. Using STR (Short Tandem Repeat) markers ensures that the inheritance profile at one location does not influence the inheritance at other locations. Each loci has two alleles, so 26 alleles must match. The FBI says that the possibility of a false match is $2.60 \times 10^{-9}$ while other sources quote between $10^{-9}$ and $10^{-12}$.

For two people chosen at random, the probability of a match based on the four most frequently analyzed alleles is between $1 \times 10^{-5}$ and $1 \times 10^{-8}$. This is significantly higher than our estimated probability or a match for thumbprints. Hence, thumbprinting remains the most accurate form of biometric security known.

# Strengths and Weaknesses

## Strengths of the Model

- **Simplicity.** Our model is based on easily understood principles and simply expressed assumptions.
- **Realistic assumptions.**
- **Parameters.** The parameters in the model, such as the size of the grid-box, the total area, and the number of minutiae needed to match two thumbprints, can be easily varied.
- **Degrees of freedom.** In specifying two different kinds of possible minutiae and 8 orientation ranges for each one, the number of degrees of freedom is 16, greatly increasing the number of possible configurations of thumbprints and so minimising the probability of misidentification. Other studies [Osterburg et al. 1977; Galton 1892] do not take into account the orientation of the minutiae. By discretizing the directions of the features, we again keep the model simple.
- **Corroboration.** The probabilities returned by our model tie in with those given by previous studies by experts in the field (**Table 2**).

**Table 2.**

Comparison probabilities of studies.

| | |
|---|---|
| Galton [1892] | $1.45 \times 10^{-11}$ |
| Osterburg et al. [1977] | $1.33 \times 10^{-27}$ |
| Stoney and Thornton [1987] | $3.5 \times 10^{-26}$ |
| our model | $10^{-11}$ to $10^{-22}$ |

## Weaknesses of the Model

- **Multiple entries.** We assumed that in any given grid-box only one minutia can be present, which is sufficiently accurate for most types of minutiae. For example, both the bridge (consisting of two ridge bifurcations) and the spur (consisting of a ridge bifurcation and a ridge ending) have been defined [Osterburg et al. 1977] as being less than 2 mm in length. Thus if the bridge or spur is more than 0.707 mm in length, their constituent endings and bifurcations appear in different boxes and are counted as two separate minutiae.

  However, for minutiae consisting of ridge endings and ridge bifurcations in very close proximity, there is a chance that each will not be caught in a different box. An example is a dot. The distance between the two ridge endings that make up a dot is so small that it is unlikely that our model would catch these two occurrences of ridge endings in different boxes. A dot has been defined [Osterburg et al. 1977] as being large enough to encompass one pore, whose size ranges from 0.088 mm to 0.22 mm [Roddy and Stosz 1997]. Therefore, the two ridge endings will not appear in different boxes but will instead be misidentified as a single minutia.

- **Independence of minutia occurrence.** We assume that the placement of a minutia is completely unrelated to the placement of any others. This is not quite the case; there is a slight tendency for minutiae *not* to occur in direct proximity to each other.

- **Global analysis.** The overall ridge pattern of a thumbprint is entirely distinctive in its own right. We have not quantified this factor in our model.

# Appendix: Classification

## Minutiae

The ridges in a fingerprint or thumbprint form various patterns, those patterns being called *minutiae*. Ten different types are shown in **Figure A1**.

- **Ridge Ending.** A ridge ending occurs when a ridge ends abruptly. We define the orientation of a ridge ending as the direction the ridge came from.

- **Bifurcation.** A bifurcation is formed when two different ridges merge. We define the orientation as being the direction in which the merged ridge came from.

- **Island.** An island is a short ridge, comprised of two ridge endings whose orientations are in opposing directions. Two ridge endings occurring in neighbouring boxes with opposite configuration indicate the presence of an island.

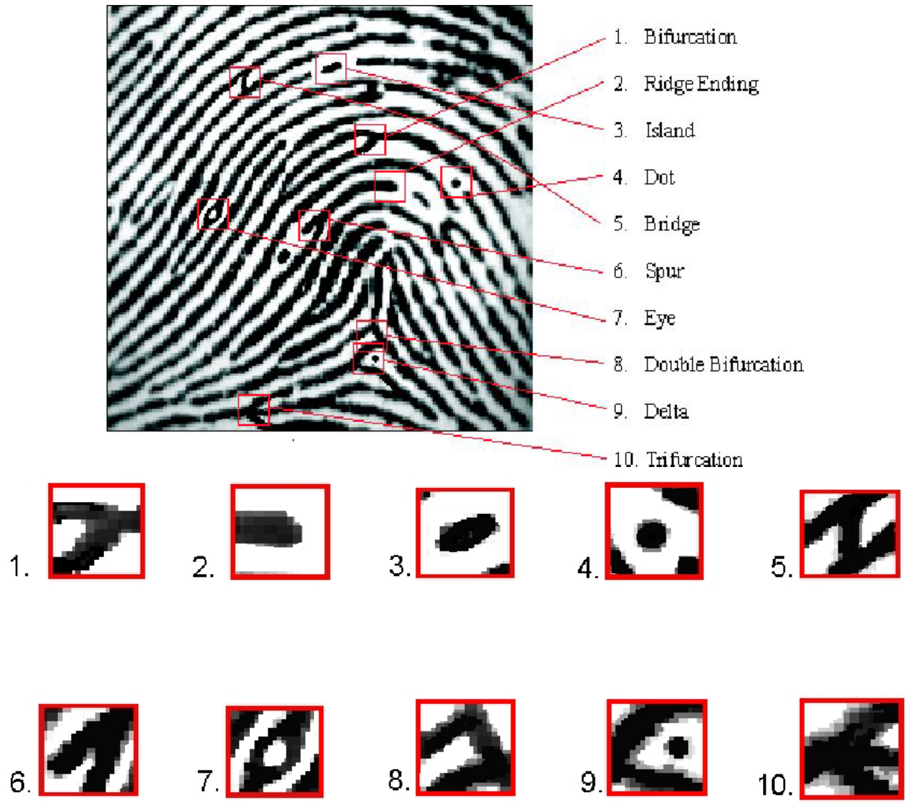



**Figure A1.** Ten types of minutia.

- **Dot.** A dot is an island but on a smaller scale.
- **Bridge.** A bridge is when a ridge branches out and merges with another ridge within a short region. It is composed of two bifurcations.
- **Spur.** A spur is when a ridge branches out and does not merge with another ridge. It is composed of one bifurcation and one ridge ending.
- **Eye.** An eye is formed by a ridge branching out into two ridges, and then recombining again a short distance later. It consists of two bifurcations.
- **Double Bifurcation.** As the name suggests, this type of minutia contains two bifurcations in succession.
- **Delta.** This type of minutia is composed of a dot and bifurcation, where the dot is between the merging ridges.
- **Trifurcation.** A trifurcation is a ridge that splits into 3 separate branches. It can be thought of as two bifurcations occurring in the same place.

## Ridge Patterns

**Figure A2** shows examples of the different classifications of a fingerprint, including the presence of cores and delta points.

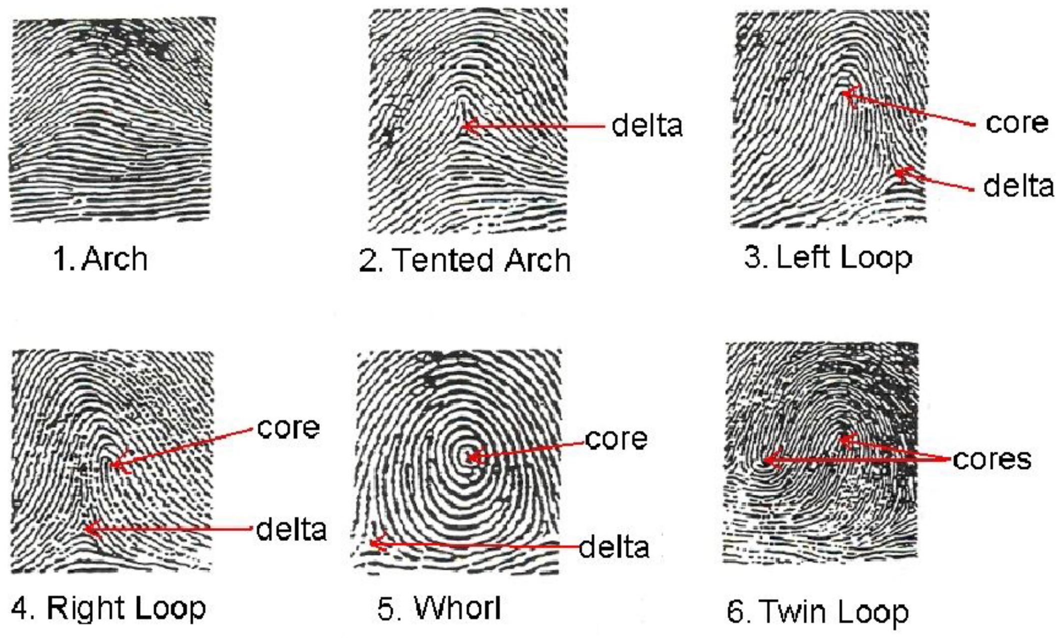



**Figure A2.** Features in a fingerprint.

# References

Anonymous. 2001. Friction skin growth. `http://www.ridgesandfurrows.homestead.com/Friction_Skin_Growth.html` .

Ballan, Meltem, F. Ayhan Sakarya, and Brian L. Evans. 1997. A fingerprint classification technique using directional images. In *Proceedings of the IEEE Asilomar Conference on Signals, Systems, and Computers* (Nov. 3–5, 1997, Pacific Grove, CA), vol. 1, 101–104. `http://www.ece.utexas.edu/~bevans/papers/1997/fingerprints/` .

Beeton, Mary. 2002. Scientific methodology and the friction ridge identification process. *Identification Canada* 25 (3) (September 2002). Reprinted in *Georgia Forensic News* 32 (3) (November 2002): 1, 6–8. `http://www.ridgesandfurrows.homestead.com/files/Scientific_Methodology.pdf` .

Collins, Martin W. "Marty". 1992. Realizing the full value of latent prints. *California Identification Digest* (March 1992): 4–11. `http://www.latent-prints.com/realizing_the_full_value_of_late.htm` .

Dankmeijer, J., J.M. Waltman, and A.G. de Wilde. 1980. Biological foundations for forensic identification based on fingerprints. *Acta Morphologica Nederlando-scandinavica* 18 (1) (March 1980): 67–83. `http://www.ncbi.nlm.nih.gov/entrez/query.fcgi?cmd=Retrieve&db=PubMed&list_uids=7395553&dopt=Citation` .

Galton, F. 1892. *Finger Prints*. London: Macmillan.

Haub, C. 1995. How many people have ever lived on Earth? *Population Today* 23 (2) (February): 4–5. Reprinted. 30 (8) (November/December 2002): 4–5. `http://www.prb.org/Content/ContentGroups/PTarticle/Oct-Dec02/How_Many_People_Have_Ever_Lived_on_Earth_.htm` .

Kingston, C.R. 1964. Probabilistic analysis of partial fingerprint patterns. D.Crim. dissertation, University of California, Berkeley.

Lindsay, Don. 2002. Don Lindsay archive. `http://www.don-lindsay-archive.org/creation/definitions.htm` .

Nilsson, Kenneth, and Josef Bigun. 2003. Localization of corresponding points in fingerprints by complex filtering. *Pattern Recognition Letters* 24 (13) (September 2003): 2135–2144. `www.hh.se/staff/josef/public/publications/nilsson03prl.pdf` .

Optel Ltd. 2003. New numerical methods of fingerprints' recognition based on mathematical description of arrangement of dermatoglyphics and creation of minutiea [sic]. `http://www.optel.pl/software/english/method.htm` .

Osterburg, James W., T. Parthasarathy, T.E.S. Raghavan and Stanley L. Sclove. 1977. Development of a mathematical formula for the calculation of fingerprint probabilities based on individual characteristics. *Journal of the American Statistical Association* 72: 772–778.

Pankanti, Sharath, Salil Prabhakar, and Anil K. Jain. 2001. On the individuality of fingerprints. `http://citeseer.ist.psu.edu/472091.html` .

Roddy, A.R., and J.D. Stosz. 1997. Fingerprint features—Statistical analysis and system performances estimates. *Proceedings of the IEEE* 85 (9): 1390–1421. `http://citeseer.ist.psu.edu/roddy99fingerprint.html` .

Sclove, Stanley L. 1980. The occurrence of fingerprint characteristics as a two-dimensional process. *Communications in Statistics* (A) 9: 675–695. `http://tigger.uic.edu/~slsclove/papers.htm` .

Stoney, David A., and John I. Thornton. 1987. A systematic study of epidermal ridge minutiae. *Journal of Forensic Sciences*.

Thai, Raymond. 2003. Fingerprint image enhancement and minutiae extraction. Honours thesis, School of Computer Science and Software Engineering, University of Western Australia. `http://www.csse.uwa.edu.au/~pk/studentprojects/raymondthai/` .

Vacca, J. 2002. Biometric security solutions. Excerpt from *Identity Theft*. 2002. Indianapolis, IN: Prentice Hall PTR.`http://www.informit.com/articles/article.asp?p=29801&redir=1` .

Thompson, William C., Simon Ford, Travis Doom, Michael Raymer, and Dan Krane. n.d. Evaluating forensic DNA evidence: Essential elements of a competent defense review. `www.cs.wright.edu/itri/EVENTS/SUMMER-INST-2003/SIAC03-Krane2.PDF` .

# Not Such a Small Whorl After All

Brian Camley
Pascal Getreuer
Bradley Klingenberg
University of Colorado at Boulder
Boulder, CO

Advisor: Anne M. Dougherty

## Summary

Fingerprint identification depends on the assumption that a person's fingerprints are unique. We assess the truth of this assumption by calculating the total number of distinct fingerprints.

We assume accurate fingerprints (ignoring procedural error) that are defined by 12 points of detail or *minutiae*. The number of distinct fingerprints depends also on the number of potential positions of these minutiae. Two historical methods and a geometric analysis estimate there to be 1,400 positions, a figure confirmed by our algorithm for counting ridges in a fingerprint.

We create two models to estimate the number of unique fingerprints:

- One model computes fingerprints as arrangements in minutiae;
- the other extrapolates the number of fingerprints from the Shannon entropy of the information that defines a fingerprint.

These two models agree to within an order of magnitude that there are $5 \times 10^{33}$ unique fingerprints, a compelling validation of our general approach.

To handle the large number of fingerprints, we implement an approximation for the calculation of probabilities. Given a cumulative world population of 120 billion [Catton 2000], the probability of two people ever having the same fingerprint is $1.4 \times 10^{-6}$.

The probability of two humans living today sharing a fingerprint is $3.5 \times 10^{-15}$, which suggests that fingerprints are a theoretically more reliable method of identification than DNA analysis, which has a false positive probability of $10^{-9}$. None of these calculations take into account procedural errors.

*The UMAP Journal* 25 (3) (2004) 245–258. 

# Introduction

The possibility of duplicate fingerprints, or fingerprints likely to be mistaken for each other, has led to recent criticism of fingerprints as a means of identification [Pankanti et al. 2001]. The key problem is:

*How many distinct fingerprints are there?*

We approach this problem with two general methods:

- "building" a fingerprint from the ground up, using different models; and
- using the information content of a fingerprint.

# Assumptions

- **Fingerprint matching is done by comparing minutiae.** Comparing small features (minutiae) such as ridge endings and bifurcations within a fingerprint is a typical method (used by the FBI) for recognition of identity, and it provides very good accuracy [Andrew 2002; Hrechak and McHugh 1990; Jain et al. 2001].
- **Two fingerprints with the same minutiae configuration are identical.**
- **The distribution of minutiae within a fingerprint is uniform.** In fact, the minutiae are over-dispersed, but the uniform distribution is a good estimate for their locations [Stoney 1988].
- **Minutiae are statistically independent of one another.** This assumption is justified by studies of fingerprint individuality (Galton and Henry) that assume independence [Stoney 1988; Stoney and Thornton 1986].
- **Minutiae are either directed along the flow of the ridge of a fingerprint, or against it, that is, there are only two possible directions.** Attempting to measure more than two directions is very difficult [Stoney and Thornton 1986].
- **There is only one type of minutia, bifurcation.** We make this assumption to simplify the problem and to avoid dealing with minutiae (i.e., dots) that have no direction.
- **There are no errors in collection—we are dealing with "true" fingerprints.** The greatest source of error in fingerprint identification is not in recognition but in training of employees and the condition of equipment [Fickes 2003]. In addition, determining the minutiae of a fingerprint is nontrivial.

# Model 1: Designing from the Ground Up

## The Worst-Possible Case

*There are no more than $10^{35,000}$ possible fingerprints.*

The FBI standard for storing and comparing fingerprint data uses 500 dpi (250,000 pixels per square inch), with 8 bits per pixel and an average size of 1.5 square inches per fingerprint [Aboufadel and Schlicker 1999]. If the image is stored as a bitmap, there are $250,000 \times 1.5 \times 8 = 3 \times 10^6$ bits of information in a fingerprint. This implies an absolute maximum of $2^{3,000,000}$ possible fingerprint images without a pixel-for-pixel match.

However, the FBI does not store images in bitmapped form but instead uses the Cohen-Daubechies-Feauveau 9/7 or "Spline 9/7" wavelet for compression, by a factor of 26:1 with the thresholding used by the FBI. So there can be only $3 \times 10^6 \div 26 = 1.15 \times 10^5$ bits of information within a stored fingerprint. We compressed several typical fingerprints [Bio-Lab 2000] to about 26:1 using our implementation of the wavelet. Comparison of edges in the original and compressed images shows that information about the minutiae is lost at 52:1 and higher levels of compression (**Figure 1**).

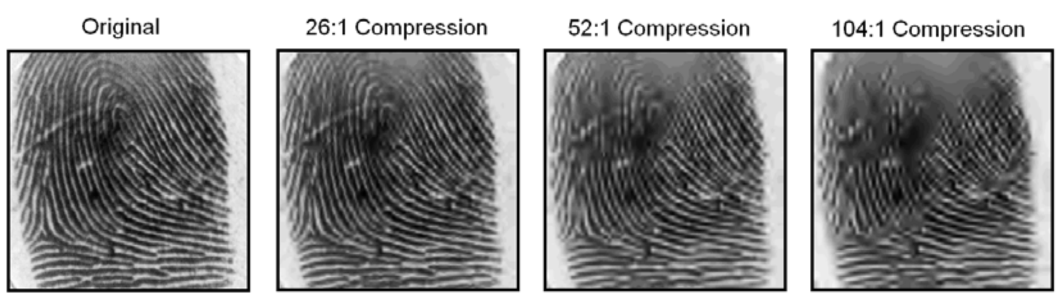



**Figure 1.** CDF 9/7 wavelet compression. Minutiae detail, such as bifurcations, is lost at compression greater than 26:1.

These results give the maximum number of fingerprint images as $2^{1.15\times10^5} \approx 10^{35,000}$.

## Limited Space for Minutiae

In the previous subsection, we didn't make any assumptions about the image—it could have been a picture of a moose. If minutiae are limited to $L$ physical locations, with at most one per location, the number of distinct fingerprints determined by $n$ minutiae is $\binom{L}{n}$.

# The Lower Bound for Total Fingerprints

Minutiae always occur on ridges, so it makes sense to represent the fingerprint as a set of ridges. We consider a typical fingerprint to be a set of 20 concentric ellipses. We assume that there are no minutiae closer than 5° away from each other—in other words, they are reasonably separated. Essentially, we are assuming that the minutiae are more or less uniformly distributed. These assumptions are equivalent to restricting minutiae to the intersections of 20 ellipses and 72 equally spaced radial lines (a simplified version of this is seen in **Figure 2**). This is similar to the empirical Roxburgh model [Stoney and Thornton 1986].

There are therefore $20 \times 72 = 1{,}440$ possible locations for minutiae.


**Figure 2.** Potential minutiae locations are on intersections of concentric ellipses and radial lines; 1,440 locations (far right) represent a fingerprint.

What number of minutiae should we choose? A typical fingerprint has 30 to 40 minutiae, but not all of these are significant. In fact, even as few as 6 minutiae may be important [Bhowmick et al. n.d.].

The number of distinct fingerprints that can be created by arrangements of 6 minutiae in 1,440 possible locations is :

$$\binom{1440}{6} = \frac{1440!}{(6!)(1440-6)!} \approx 1.23 \times 10^{16}.$$

This is a lower bound on the total number of distinguishable fingerprints.

# Improving the Estimate

Though in some cases there are only 6 useful minutiae, typically there are about 30 to 40 minutiae in a fingerprint [Stoney and Thornton 1986]. If all of these were used for identification, and there were still only 1,440 possible locations for minutiae, then the value for the number of total fingerprints would be

$$\binom{1440}{35} \approx 2.23 \times 10^{70}.$$

However, the criteria for identity vary from about 10 to 16 matching features [Stoney and Thornton 1986], implying that using 35 features to define a fingerprint overestimates the number of fingerprints. We assume that 12 features are required to match a fingerprint (the FBI's "quality assurance" standard [Duffy 2002]). Then the number of distinguishable fingerprints is

$$\binom{1440}{12} \approx 1.59 \times 10^{29}.$$

## Accounting for Direction of Minutiae

Allowing two orientations (with or against the flow of the ridge) for a minutia doubles the number of possible placements and increases the number of fingerprints to

$$\binom{2880}{12} \approx 6.64 \times 10^{32}.$$

Hence, the number of fingerprints is bounded between $1.23 \times 10^{16}$ and $2.23 \times 10^{70}$ but is most likely around $6.64 \times 10^{32}$.

We now narrow this range by using information theory.

# Model 2: Information Theory

## Clarification of Assumptions

We phrase our original assumptions in a new way:

- A single fingerprint contains $n$ minutiae.
- A fingerprint can be effectively mapped as $n$ minutiae falling into the squares of an $x \times x$ grid (at most one minutia per square). There are $x^2$ possible locations for the $n$ minutiae, so $x = \sqrt{L}$, where $L$ is the number of locations for minutiae.

## Derivation of the Entropy of a Fingerprint

We can visualize a fingerprint as an $x$ by $x$ grid. If there is a minutia in a space, we mark it with an X; if not, we leave it blank (**Figure 3**).

We treat the two mutually exclusive states, minutia and non-minutia, as the elements of a two-letter alphabet, $a$. An $x$ by $x$ arrangement of this alphabet represents a fingerprint. Shannon's classic first-order equation gives the entropy of the alphabet:

$$H = -\sum_{i=1}^{m} P(a_i) \log_2 P(a_i), \qquad \textbf{(1)}$$

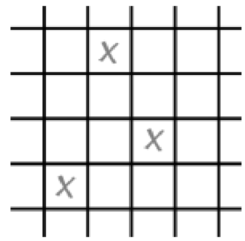


**Figure 3.** A section of a possible fingerprint configuration.

where $P(a_i)$ is the independent probability of the state $i$ occurring in the fingerprint and $m$ is the length of the alphabet [Shannon 1948].

For us, $m = 2$. The respective probabilities of each state (minutia or non-minutia) occurring in the alphabet are:

$$P(a_1) = \frac{\text{minutiae}}{\text{number of spaces}} = \frac{n}{x^2}, \qquad P(a_2) = \frac{\text{non-minutiae}}{\text{number of spaces}} = \frac{x^2 - n}{x^2}.$$

Substituting back into **(1)**, we are left with the following value for $H$:

$$H = -\left[\frac{n}{x^2}\log_2\left(\frac{n}{x^2}\right) + \frac{x^2 - n}{x^2}\log_2\left(\frac{x^2 - n}{x^2}\right)\right].$$

# The Number of Fingerprints Based on Entropy

Because entropy is a measure of the minimum number of bits required to represent each element of the grid, it can be used to determine the total representative requirement of any fingerprint:

$$\text{bits required for fingerprint} = H \times \text{size of fingerprint} = Hx^2.$$

There are $Hx^2$ bits of information in a fingerprint, so there should be $2^{Hx^2}$ possible fingerprints. However, in our definition of the grid, we ignored the direction of minutiae. Each minutia has two possible directions, resulting in a total of $2^n$ possible directional configurations.

Combining these numbers, we find $2^{Hx^2} \times 2^n = 2^{Hx^2+n}$ possible fingerprints.

Using the values established earlier ($n = 12$, $L = x^2 = 1{,}440$), we get

$$H = -\left[\frac{12}{38^2}\log_2\left(\frac{12}{38^2}\right) + \frac{38^2 - 12}{38^2}\log_2\left(\frac{38^2 - 12}{38^2}\right)\right] \approx 0.07,$$

which leads to

$$H \times x^2 \approx 0.07 \times 38^2 \approx 100$$

and therefore $2^{Hx^2+n} = 2^{100+12} \approx 5.19 \times 10^{33}$ fingerprints.

# Consistency of the Two Models

Experimentation with different values for $L$ and $n$—not only for reasonable ranges of $L$ (500–2,500) and $n$ (6–18) but also for truly ridiculous numbers—indicates that the values from the combinatorial model and from the information theory agree to within an order of magnitude.

# How Many Minutiae Locations Are There?

The physical dimensions of a minutia confirm the estimate of $L \approx 1{,}400$ in two different ways.

## The Kingston Method

A visual inspection of a $300 \times 300$ pixel fingerprint image [Bio-Lab 2000] reveals that a minutia can be contained in a $9 \times 9$ pixel square (**Figure 4**). This would suggest that we can put a maximum of

$$\frac{300 \times 300}{9 \times 9} \approx 1{,}100$$

minutiae into one image. This method for estimating the number of possible minutiae locations in a fingerprint recalls the Kingston method, which calculates this value based on the area occupied by a minutia [Stoney and Thornton 1986]. This value for $L$ confirms our initial geometric estimate.

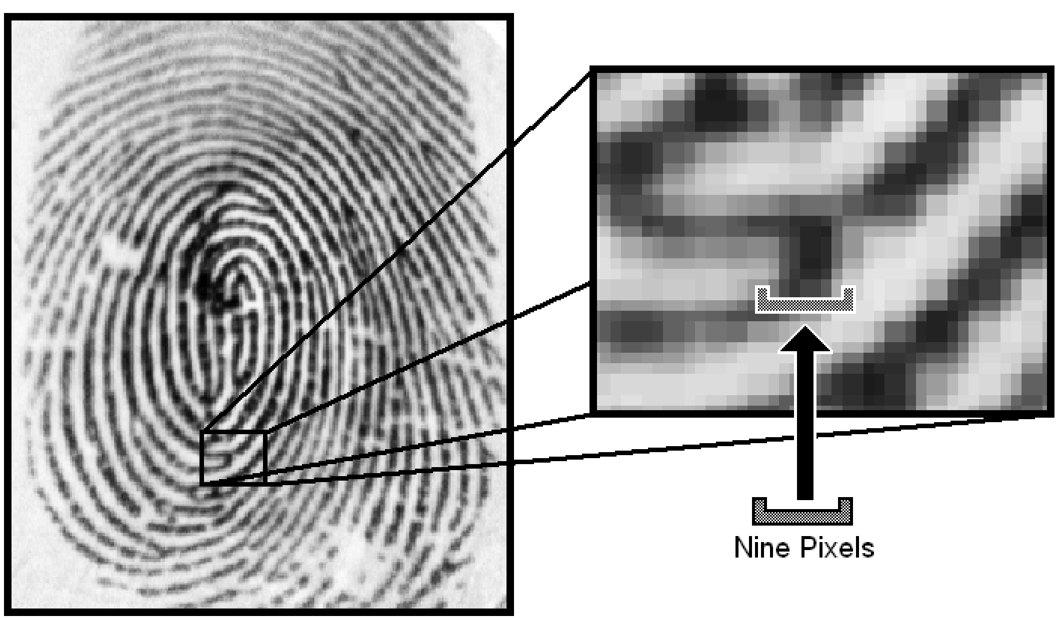



**Figure 4.** A minutia can typically be contained in a 9-by-9 pixel area.

## The Amy Method

Amy's method [Stoney and Thornton 1986] calculates the number of potential minutiae positions $L$ as

$$L = (\mathcal{I} - \iota + 1)^2,$$

where $\mathcal{I}$ is the number of ridge intervals on a side of a known fingerprint (see **Figure 5**). The studies of Roxburgh established the value of $\iota$, the size of a minutia, to be $\sqrt{5/2}$ ridge intervals [Stoney and Thornton 1986].

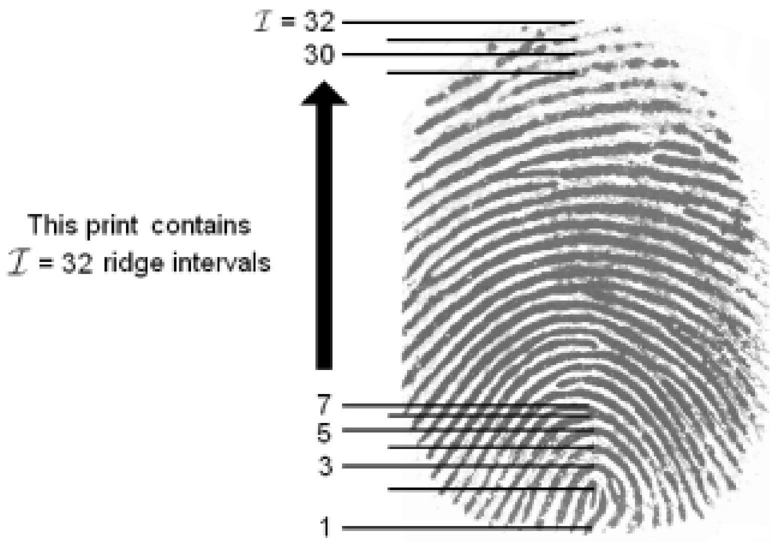



**Figure 5.** The number of ridge intervals on a fingerprint.

The average value of $\mathcal{I}$ is 38 for a typical fingerprint, from our ridge-counting algorithm (see the **Appendix**). We calculate the number of potential minutiae locations as

$$L = \left(38 - \sqrt{\frac{5}{2}} + 1\right)^2 \approx 1{,}400.$$

This value is almost exactly the number that we predicted using only the initial geometry of the fingerprint!

The geometry of fingerprint images and Amy's method confirm, in two unrelated ways, our prediction of $L = 1{,}440$.

# Estimating the Odds of Duplicate Prints

## General Method

Select a random person from a group of $N$. The probability that a second person selected at random has a different fingerprint from the first is $(f-1)/f$, where $f$ is the total number of fingerprints. Generalizing, as in the classic birthday coincidence problem, the probability $P(N)$ of picking $N$ unique fingerprints (that is, no duplication) is

$$P(N) = \prod_{i=1}^{N-1} \left( \frac{f-i}{f} \right) = \frac{1}{f^{N-1}} (f-1)(f-2)(f-3)\cdots(f-N+1). \qquad \textbf{(2)}$$

We calculated the probability of a duplication this by writing a C program, using arbitrary precision arithmetic to deal with the fact that $f$ is very large. However, this calculation requires a lot of time for large $N$, so it is useful to have an easy-to-calculate approximation.

## Approximation

We express **(2)** as

$$P(N) = \frac{1}{f^{N-1}} \left( f^{N-1} + c_1 f^{N-2} + c_2 f^{N-3} + \cdots + c_{N-1} \right),$$

where the $c_i$ are integer coefficients of the powers of $f$. We can then write

$$P(N) = 1 + \frac{c_1}{f} + \frac{c_2}{f^2} + \cdots + \frac{c_{N-1}}{f^{N-1}}.$$

Since $f$ is large, $f^{-2} \ll f^{-1}$ and we can discard everything except for the first term, as long as $N$ is not of the same order as $f$, getting

$$P(N) \approx 1 + \frac{c_1}{f}.$$

Now, what is the coefficient $c_1$? The product $(f-1)(f-2)\cdots(f-N)$ is a sum of terms created by choosing either $f$ or the number from each binomial. A term with $f^{N-1}$ occurs when $f$ is chosen for each binomial except one. There are $N$ ways to do so, resulting in the coefficients $-1, -2, \ldots, -N$. Therefore,

$$c_1 = (-1) + (-2) + \cdots + (-N) = \frac{-(N^2+N)}{2}.$$

Hence

$$P(N) \approx 1 - \frac{N^2+N}{2f}.$$

The probability of a fingerprint duplication is

$$1 - P(N) = \frac{N^2+N}{2f}.$$

**Table 1.**
Probability of a fingerprint coincidence, for various numbers of fingerprints and population sizes.

| $N$ | Number of fingerprints | | |
|---|---|---|---|
| | $1.23 \times 10^{16}$ | $6.64 \times 10^{32}$ | $5.19 \times 10^{33}$ |
| $10^5$ | $4 \times 10^{-7}$ | $10^{-22}$ | $10^{-24}$ |
| $10^6$ | $4 \times 10^{-5}$ | $10^{-21}$ | $10^{-22}$ |
| $10^7$ | $4 \times 10^{-3}$ | $10^{-19}$ | $10^{-20}$ |
| $10^8$ | 0.334 | $10^{-22}$ | $10^{-18}$ |
| $10^9$ | 0.999 | $10^{-22}$ | $10^{-16}$ |
| $6 \times 10^9$ (current world) | 1 | $10^{-22}$ | $3 \times 10^{-15}$ |
| $120 \times 10^9$ (cumulative world) | 1 | $10^{-22}$ | $1.4 \times 10^{-6}$ |

# Odds of Misidentification

**Table 1** gives the probability of a fingerprint coincidence for various population sizes, for each of our estimates of the number of fingerprints.

Based on either our best value ($6.64 \times 10^{32}$) for the number of different prints, or the information theory estimate ($5.19 \times 10^{33}$), there is essentially no chance of duplicating a fingerprint.

# Conclusions

We use the basic geometry of a fingerprint and the known distribution of minutiae to determine that there are about 1,440 possible minutiae locations in a fingerprint. We confirm this by studying minutiae in digitized fingerprints.

Using this value, we calculate the number of possible distinct fingerprints given a certain number of minutiae to be used for identification. We choose the FBI "quality assurance" standard of 12 minutiae [Duffy 2002], which results in $6.64 \times 10^{32}$ possible fingerprints, once minutiae direction is taken into account. This number was confirmed by using the amount of information in a model of a fingerprint, which estimated $5.19 \times 10^{33}$ fingerprints using 12 minutiae.

If only six minutiae are used to determine a fingerprint, and thus there are only $1.23 \times 10^{16}$ fingerprints, then the probability for a duplication in one billion people approaches unity. In fact, even in only 100 million people (the order of the FBI's fingerprint database), there would be a reasonable probability (.33) of a fingerprint duplication.

However, using 12 minutiae, the probability of a fingerprint duplication in six billion people is only $3 \times 10^{-15}$; the probability of a duplication among the 120 billion people who have ever lived is only $1.39 \times 10^{-6}$. Therefore, there is little risk of mistaken identity based on 12 minutiae, given perfect fingerprints. In the real world, fingerprints are not perfect, and the largest sources of error are from mishandling and errors in the process [Fickes 2003].

DNA analysis has a theoretical probability of false positives on the order of $10^{-9}$, though this figure is often disputed [Thompson et al. 2003]. Fingerprint identification is thus theoretically more reliable than DNA testing.

# Strengths and Weaknesses of the Models

## Strengths

- **Agreement of the models.** The same general number of possible minutiae locations is calculated in three completely separate ways (by assuming uniform distribution, by looking at the size of a minutia, and by counting ridges and using Amy's method). This consistency suggests that our result is reasonable. In addition, our two vastly different models (combinatorial and information theory) produce consistent numbers.

## Weaknesses

- **Assumptions about the minutiae.** We assume that there is only one type of minutia, when in fact there are at least three (bifurcations, endpoints, and dots) [Stoney and Thornton 1986]. Some of these are orientable and others are not, but we assume that all minutiae have two directions. In addition, though our assumption of uniform distribution is borne out by study of real fingerprints, correlations between minutiae could strongly skew the number of possible locations.

- **The models work only in an ideal setting.** Many factors can create errors in a fingerprint, such as dirt, operator error, and mechanical breakdowns. Our models do not address this and only set a maximum theoretical accuracy for fingerprinting.

# Appendix: Ridge-Counting Algorithm

Our ridge-counting algorithm, which estimates the number of concentric ridges in an image, supplies useful empirical results to support and validate our theoretical work. The steps of the algorithm are:

- edge detection and thresholding,
- selecting the ridge core location, and
- counting maximum number of ridges around the core.

## Edge-Detection and Thresholding

Given an image $X(x, y)$, we use edge detection filters $f_x$ and $f_y$to estimate the image gradient. These are directional derivatives of the general Gaussian function (**Figure 6**).

$$f_x(x, y) = -x \exp\left(-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right), \qquad f_y(x, y) = -y \exp\left(-\frac{y^2}{2\sigma_y^2} - \frac{x^2}{2\sigma_x^2}\right).$$

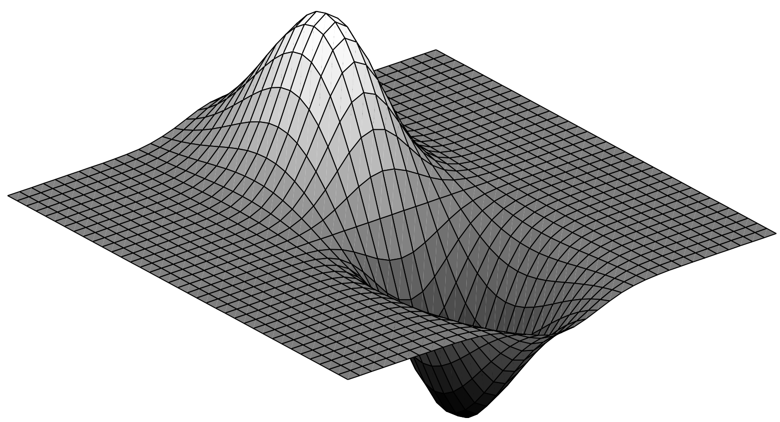


**Figure 6.** Edge detector filter.

With $f_x$ operating horizontally and $f_y$ vertically, the magnitude of the image gradient is

$$\nabla X(x, y) \approx \sqrt{[(X * f_x)(x, y)]^2 + [(X * f_y)(x, y)]^2}.$$

Thresholding the gradient yields a binarized image $E$ of the edges,

$$E(x, y) = \begin{cases} 1, & \text{if } \nabla X(x, y) > \alpha; \\ 0, & \text{if } \nabla X(x, y) \leq \alpha, \end{cases} \qquad (\min \nabla X < \alpha < \max \nabla X)$$

where $\alpha$ is the threshold level.

## Selecting the Ridge Core Location

Most fingerprints are essentially concentric curves around a central part of the print, the ridge core. To find the number of ridges, one can begin at the core and move outwards, counting the ridges crossed in the path. Unfortunately, automatically determining the core location is difficult. One method to do this estimates the direction of the ridges through the directional field [Chan et al. 2004]. In this method, the image is segmented into $w \times w$-pixel blocks and a least-squares orientation method finds the smoothed orientation field $O(x, y)$. The value $\sin(O(x, y))$ reflects the local ridge direction and indicates the core, which is where the direction curves fastest.

It is very easy to locate the core of a fingerprint by eye; and small deviations in the location of the core, like those expected from human error, do not make large differences in the measured number of ridges.

## Maximum Number of Ridges around the Core

The next step is to use the $E(x, y)$ image to count the ridges, beginning at the core and counting the number of changes between 1 and 0. A typical edge is counted twice and a typical ridge has two edges; so dividing the count by 4 estimates the number of ridges. However, moving along only one path may miss the shorter ridges at the edge of the print or ridges with noisy edges. Instead, we count along many directions and use the highest count (**Figure 5**).

## Results from Implementation

We applied the algorithm to 80 optically-scanned fingerprint images from Bio-Lab [2000]. The last nine images were too noisy for the edge detection and were discarded. The distribution of the remainder is roughly symmetrical about a mean of 38.0 ridges, with a standard deviation of 5.3.

# References

Aboufadel, Edward, and Steven Schlicker. 1999. *Discovering Wavelets*. 1st ed. John Wiley and Sons.

Bolle, Ruud M., Andrew W. Senior, Nalini K. Ratha, and Sharath Pankanti. 2002. Fingerprint minutiae: A constructive definition. In *Proceedings ECCV Workshop on Biometrics*. http://citeseer.ist.psu.edu/591141.html .

Bhowmick, P., A. Bishnu, B.B. Bhattacharya, M.K. Kundu, C.A. Murthy, and T. Acharya. n.d. Determination of minutiae scores for fingerprint image applications. http://citeseer.ist.psu.edu/554334.html .

Bio-Lab, Univerity of Bologna. 2001. FVC 2000: Fingerprint Verification Competition. http://bias.csr.unibo.it/fvc2000/ .

Brualdi, Richard A. 1999. *Introductory Combinatorics*. 3rd ed. Upper Saddle River, NJ: Prentice Hall.

Catton, William R., Jr. 2000. Worse than foreseen by Malthus (even if the living do not outnumber the dead). http://desip.igc.org/malthus/Catton.html .

Chan, K.C., Y.S. Moon, and P.S. Cheng. 2004. Fast fingerprint verification using subregions of fingerprint images. *IEEE Transactions on Circuits and Systems* *for Video Technology* 14 (1) (January 2004) 95–101. http://ieeexplore.ieee.org/xpl/abs_free.jsp?arNumber=1262035 .

Cotton, R.W., and C.J. Word. 2003. Commentary on Thompson et al. [2003]. *Journal of Forensic Sciences* 48 (5): 1200.

Duffy, S.P. 2002. Experts may no longer testify that fingerprints "match." *The Legal Intelligencer* (9 January 2002). http://www.truthinjustice.org/print-match.com .

Fickes, Michael. 2003. Dirt: A fingerprint's weakest link. *Access Control and Security Systems* (1 February 2003). http://govtsecurity.securitysolutions.com/ar/security_dirt_fingerprints_weakest/ .

Hrechak, A.K., and J. McHugh. 1990. Automated fingerprint recognition using structural matching. *Pattern Recognition* 23: 893–904.

Jain, Anil, Arun Ross, and Salil Prabhakar. 2001. Fingerprint matching using minutiae and texture features. In *Proceedings of the International Conference on Image Processing (ICIP)* (Thessaloniki, Greece), 282–285. http://citeseer.ist.psu.edu/jain01fingerprint.html .

Pankanti, Sharath, Salil Prabhakar, and Anil K. Jain. 2001. On the individuality of fingerprints. *IEEE Transaction on Pattern Analysis and Machine Intelligence*: 1010–1025. http://citeseer.ist.psu.edu/472091.html .

Shannon, C.E. 1948. A mathematical theory of communication. *Bell System Technical Journal* 27: 379–423, 623–656.

Stoney, D. A. 1988. Distribution of epidermal ridge minutiae. *American Journal of Physical Anthropology* 77: 367–376.

———, and J.I. Thornton. 1986. A critical analysis of quantitative fingerprint individuality models. *Journal of Forensic Sciences* 31: 1187–1216.

Thompson, W.C., F. Taroni, and C.G.G. Aitken. 2003. How the probability of a false positive affects the value of DNA evidence. *Journal of Forensic Sciences* 48 (1): 47–54.

See Cotton and Word [2003] for a commentary.

# Rule of Thumb: Prints Beat DNA

Seth Miller
Dustin Mixon
Jonathan Pickett
Central Washington University
Ellensburg, WA

Advisor: Stuart F. Boersma

## Summary

We distinguish thumbprints by their global features (those that can be easily detected by the naked eye): the thumbprint's pattern, the ridge count, and the type lines. With this model, the probability that two thumbprints have the same global characteristics is at least $5.02 \times 10^4$.

In a second model, we pay more attention to local features (those not easily detected by the naked eye), in particular the thumbprint's minutiae. We look at the number of minutiae, their location, and the direction in which their corresponding furrow endings point. With this stronger model, the probability that two thumbprints have the same local characteristics is approximately $6.41 \times 10^{-143}$.

With both models, we estimate the probability that every thumbprint in the history of mankind is unique to be virtually zero (though this contradicts what our logic dictates).

Since identical twins share the same DNA pattern, the probability that every DNA strand in the history of mankind is unique must be zero. Given two DNA strands taken at random from people the probability that they are similar is approximately $8.29 \times 10^{-14}$. Since this probability is significantly more than the analogous probability for local thumbprint features, we conclude that DNA testing has a higher rate of misidentification.

*The UMAP Journal* 25 (3) (2004) 259. 

# Judge's Commentary: The Outstanding Fingerprints Papers

Michael Tortorella
Dept. of Industrial and Systems Engineering
Rutgers University
Piscataway, NJ
mtortore@rci.rutgers.edu

## Introduction

The brief statement of this problem hid many layers of complication. Teams were challenged to find the mathematical kernel of a problem of interest to anthropologists, forensic scientists, attorneys and judges, and just plain folks. In essence, the problem reduces to

> *Estimate the probability that two humans who have ever lived have the same fingerprint.*

After the MCM was over, the *Wall Street Journal* carried an article entitled "Despite its reputation, fingerprint evidence isn't really infallible" [Begley 2004]. The uncritical acceptance of fingerprint evidence that was common in the past is undergoing new scrutiny, and our examination of the question in the MCM was timely indeed, if unplanned.

## The Issues

### Philosophical Questions

The problem seems innocuous enough; but you get very quickly into some deep—even philosophical—questions that have to be addressed in modeling assumptions. For example, exactly how finely does nature model the

*The UMAP Journal* 25 (3) (2004) 261–266. 

real numbers? In mathematics, between every two real numbers there is another one; if between any two positions in physical space there is another one (distinguishable—by whom?—from the other two), then a homotopy between any two distinct fingerprints—however "fingerprint" is defined—produces an infinite number of distinct fingerprints. So the probability requested could reasonably be asserted to be zero, even if we say translations and rotations of a given fingerprint are not different from the original. Hence, a purely mathematical approach to the problem is not very interesting. On the other hand, the number of people $n$ who have ever lived is finite, so we may find the answer "zero" unsatisfying.

## A Simple Model

Here is a simple model that takes a next step: Assume that fingerprints (the actual skin patterns) are assigned at birth, at random from a pool of potential fingerprints. If we assume that the pool contains $N >> n$ elements and selection is made on an equally-likely basis, then the probability that there are no two fingerprints alike is the solution to a birthday problem with $n$ people and $N$ birthdays; namely, the probability of no match (denoted $Q_1 = 1 - P_2$ in Weisstein [2004]) is given by

$$P(\text{no match}) = \frac{N}{N}\frac{N-1}{N}\cdots\frac{N-n+1}{N} = \frac{N!}{n!N^n} \approx e^{-n(n-1)/2N},$$

which for fixed $n$ is asymptotic to 1 as $N \to \infty$[1].

This is about the simplest model that one could devise for this problem, and teams should use these sorts of simple models as a baseline against which to assess other more complicated efforts. One thing that we learn from this model is that in effect, all the additional definitions for "fingerprint" serve essentially to shrink the pool of possible "fingerprints," that is, restrict $N$ so that there is a chance that the probability of no match will be less than 1.

## Reinterpreting the Question

In fingerprint analysis, a human being, either with the unaided eye or with some tool(s), judges two fingerprints to be "identical." So a reasonable interpretation of the relevant question could be:

> *Determine the probability that there have never been two identical fingerprints, given the capability of the technology used to determine "different."*

This is a little more interesting a question. Different assumptions can reasonably be made concerning this capability, which lead to different models and, usually, different answers.

---

[1]The condition that $N >> n$ is not idle: When $N = 2n$, the probability of no match is approximately $e^{-(n-1)/4}$, which is actually quite small for large $n$. But we already knew this from the "standard" birthday problem learned in Probability 101.

## But First You Have to Define . . .

The first step in developing a model based on this question is to *define*:

- "fingerprint,"
- the probability space in which this experiment is conducted, and
- "distinct."

"Distinct" depends on who's looking; or, to put it more conventionally, resolution matters. All this is by way of scope delineation, so that when an answer is arrived at, the domain in which the answer is valid will be clear.

The definition of "fingerprint" is wrapped up in the definition of the probability space, because most teams made assumptions about the minimum spacing between ridges that could possibly occur. This assumption is based on empirical evidence (at least from humans who have been alive in the last century or so) and is the first step down a road leading to consideration of only a finite number of potential fingerprints.

Additional assumptions of this nature included restriction of the mathematical model to the six common types of fingerprint patterns (loops, arches, whorls, etc.) and a few variations.

# The Importance of Interpretation

As always, interpretation is the key to success in modeling problems. The first key was to understand that the word "fingerprint," in addition to its usual semantic or prose usage, must be given a *mathematical meaning* in the context of a model. Successful papers began by providing a mathematical definition of "fingerprint," for example, as a rectangular area, 2 cm by 3 cm, containing alternating ridges and valleys arranged in one of 6 global patterns (arch, tented arch, left loop, right loop, whorl, and twin loop). Alternatively, one may distinguish between the fingerprint as a physical or biological entity on the body and a fingerprint as an image made on paper or other surface by a deliberate or accidental process. Any of these can lead to reasonable solutions but the modeler's choice should be made clear.

Once that is accomplished, it begins to be possible to talk in quantitative terms about how two fingerprints may be distinguished. Most papers adopted the FBI criteria concerning the number and location of minutiae as their differentiating method. A minutia is a local feature of the fingerprint, for example, the end of a ridge line or an isolated very short ridge of approximately the same length and width. Again, the standard FBI categorization of minutiae was most often used. A grid of some size (typically 1 mm on a side) is imposed on the fingerprint and the presence or absence of a minutia in a grid square is recorded (at most one minutia per square is permitted). Some papers noted that the size of the grid square should be approximately equal to, or slightly smaller than, the typical size of a minutia so that the possibility of more than one minutia in a grid square is minimized. The feature-resolving capability of the instrument used to view the fingerprint also matters, because if infinite resolution is possible, then all fingerprints will look different. In fact, this observation implies that one can pose this problem

- "theoretically," treating the "fingerprint" as a mathematical construct and using only properties of the real numbers, etc., to form a solution; or
- "practically," where the aspects of detectability of differences by human or machine methods are central.

A good solution to this problem, like that of the paper from the team at University College Cork, treats both aspects and their interplay.

# At Last, a Model

Even with these few assumptions, a model is possible: the total number of possible distinct fingerprints implied is $2^{600} \approx 10^{181}$. The number of people who have ever lived is about $1.06 \times 10^{11}$; so, assuming that all $2^{600}$ patterns are equally likely, the probability that no two persons who have ever lived have the same fingerprint is approximately $1 - 10^{-159}/2$ (this latter computed from the "birthday problem" with $1.06 \times 10^{1}1$ people and $2^{600}$ possible "birthdays," a point that many teams missed). The University College Cork team handled this approach about as well as could be.

It is easy to poke holes in this model. Empirically, it is clear that

- not every grid square has the same probability of containing a minutia,
- stochastic independence of the presence or absence of minutiae from grid square to grid square is not reasonable, and
- there are several different types of minutiae.

Many teams overcame these objections by adding to the basic model assumptions comprehending several types of minutiae and various other refinements based on empirical observations of physical characteristics, such as ridge width, interridge distance, and frequency of occurrence of different types of minutiae in a grid square. For example, both the papers from Harvey Mudd College and University College Cork introduced orientation of minutiae as another distinguishing characteristic (although the University College Cork paper does not follow through on this additional detail, giving this the feeling of a dead end). In all cases, though, when such assumptions based on empirical observation are introduced, the modeler should attempt to bound the answers using a range of possible reasonable values for the inputs because sampling error could affect the assumptions. Once could argue that sampling error should be negligible in drawing inferences from a database containing millions of records, like most fingerprint databases, but most teams did not address this issue in any way.

Finally, the problem asks for comparison of the computed probability with the probability of misidentification by DNA evidence, a topic much in the public eye in the last decade. Some teams ignored this requirement. Others quoted popular anecdotes concerning the DNA misidentification probability. In the latter case, teams would be advised to bolster their contentions with at least one legitimate citation.

# As Always, Advice

- **Make your paper easy to read.** That means, at least, number the pages and the equations, check your spelling, and double-space the text (or at least use a font size large enough for easy readability). All three Outstanding papers shown here did a good job with this.

- Good organization will not make up for poor results, but poor organization can easily overwhelm good results and make them hard to dig out. **Organize the paper into sections corresponding to the parts of the problem.**

- **Define all terms** that a reader might find ambiguous; in particular, any term used in the model that also has a common prose meaning should be carefully considered. The paper from University College Cork in particular does a very thorough job with this.

- **Complete all the requirements of the problem.** If the problem statement says certain broad topics are required, begin by making an outline based on those requirements.

- **Read the problem statement carefully,** looking for key words implying actions: design, analyze, compare, etc. (imperative verbs). These are keys to the sections your paper ought to contain.

- **Address sensitivity to assumptions as well as the strengths and weaknesses of the model.** That means that these topics should be covered separately in sections of their own.

- When you do strengths and weaknesses, or sensitivity analysis, **go back to your list of assumptions and make sure that each one is addressed.** This is your own built-in checklist aiding completeness; use it.

- **Your summary should state the results that you obtained, not just what you did.** Keeping the reader in suspense is a good technique in a novel, but it simply frustrates judges who typically read dozens of papers in a weekend. The University of Colorado paper has an excellent summary: crisp, concise, and thorough.

- **Use high-quality references.** Papers in peer-reviewed journals, book, and government Websites are preferred to individuals' websites. Note also that it is not sufficient to copy, summarize, or otherwise recast existing literature; judges want to see *your* ideas. It's okay to build on the literature, but there must be an obvious contribution from the team.
- **Verify as much as you can.** For example, the total population of the earth should be readily verifiable. Make whatever sanity checks are possible: is answer you get larger than the number of atoms in the known universe? If it is, should it be?
- **Finally, an outstanding paper usually does more than is asked.** For example, the University of Colorado team created two different models to attack the problem and compared the results from each approach; the reasonably good agreement they obtained showed that either
  - they were on the right track, or
  - they were victims of very bad luck in that both of the methods that they tried gave nearly the same bad answers!

# Reference

Begley, Sharon. 2004. Despite its reputation, fingerprint evidence isn't really infallible. *Wall Street Journal* (4 June 2004) B1.

Weisstein, Eric. 2004. Birthday problem. `http://mathworld.wolfram.com/BirthdayProblem.html` .

# About the Author

Mike Tortorella is a Research Professor in the Department of Industrial and Systems Engineering at Rutgers, the State University of New Jersey, and the Managing Director of Assured Networks, LLC. He retired from Bell Laboratories as a Distinguished Member of the Technical Staff after 26 years of service. He holds the Ph.D. degree in mathematics from Purdue University. His current research interests include stochastic flow networks, information quality and service reliability, and numerical methods in applied probability. Mike has been a judge at the MCM since 1993 and has particularly enjoyed the MCM problems that have a practical flavor of mathematics and society. Mike enjoys amateur radio, the piano, and cycling; he is a founding member of the Zaftig Guys in Spandex road cycling team.

# Practitioner's Commentary: The Outstanding Fingerprints Papers

Mary Beeton
A.F.I.S. Technician
Durham Regional Police Service
Oshawa, Ontario
Canada
`mbeeton@drps.ca`

## Introduction

I read with great interest the three Outstanding papers. A mathematical model to assess the requested probability, that the thumbprint of every human who has ever lived is different, is beneficial to the science of identification from fingerprints.

I offer a viewpoint from the forensic discipline of friction ridge identification. As an A.F.I.S. (Automated Fingerprint Identification System) Technician, I examine more than 125,000 fingerprints each year. On a daily basis, I determine if "friction ridge skin impressions" (also known as *latent prints* or *latent marks*) collected at crime scenes came from the same source as a known "rolled" inked fingerprint (also known as an *exemplar*). Mathematical models, such as those suggested by the Outstanding teams, could help resolve the hypothesis that a "small distorted fragment" of a friction ridge skin impression came from the same source as a known "rolled" inked fingerprint.

## Uniqueness of Fingerprints

Over 100 years of research in embryology, genetics, biology, and anatomy have documented the extensive genetic, biological, and random environmental occurrences that take place during fetal growth; all this work supports the premise that friction skin is unique, as does statistical research. Despite the oversimplification in each model from unrealistic assumptions such as

- minutiae occur at uniform rates along a particular ridge,
- only two types of minutiae exist,
- all fingerprints are perfect rolled prints, and
- ridge widths are consistent, so that pores and edge shapes are not significant,

each of the teams' conclusions support fingerprint uniqueness. Given the time constraint in the contest, oversimplification was likely unavoidable.

Most legal professionals, scholars and scientists support the view that "nature never repeats itself," and most do not dispute fingerprint uniqueness.

# Challenges of Fingerprint Identification

Why is it, then, that the scientific reliability of friction ridge identification is frequently challenged in the courtroom? In 1993, a U.S. Supreme Court, in a ruling for the civil case *Daubert v. Merrell Dow Pharmaceuticals, Inc.*, 509 U.S. 579 (1993), outlined specific "Daubert" criteria to be used by trial judges in assessing the admissibility of scientific evidence. While the Court's ruling is not necessarily binding on the individual states, many states have adopted the "Daubert" standard or the federal rules on which it was based. In 1999, the Assistant Federal Defender for the State of Pennsylvania was the first to challenge the admissibility of fingerprint evidence based on his own interpretation of the Court's ruling and the five "Daubert" criteria. Robert Epstein, defense counsel for Byron Mitchell on trial for armed robbery, posed the question:

> Is there a scientific basis for a fingerprint examiner to make an identification, of absolute certainty, from a small distorted latent fingerprint fragment, revealing only a small number of basic ridge characteristics such as nine characteristics identified by the FBI examiner at Mr. Mitchell's first trial?

Friction ridge identifications usually involve smaller, frequently distorted friction ridge skin impressions that are compared to a usually larger rolled inked impression collected under a controlled environment. Therefore, it is necessary for the friction ridge identification specialist to analyze the latent print and assess if sufficient quantity and quality of unique friction ridge features can be observed. If sufficient information exists, the latent print is deemed to be "identifiable." Dr. Christophe Champod, co-author of a new book on fingerprint science [Champod et al. 2004] states that, "The number of studies devoted to *partial* marks, taking into account realistic features and effects such as *pressure distortion* and *clarity* is very limited. There is huge room for improvement here."

A probability model, based on a statistical analysis of the frequency or rarity of all types of friction ridge features and modified to account for different types of distortion, may possibly be used to quantify the apparent "sufficient observed uniqueness" in a latent print and hence help support a conclusion of a positive fingerprint identification or "individualization." However, given our knowledge of the morphology of friction ridge skin, statistical analysis may never encompass all of the significant friction ridge features that can be observed in the latent print by the friction ridge identification specialist and applied to the identification process. Nevertheless, it is certainly an endeavour worthy of considerable attention by the scientific community.

# Certainty

A key phrase in Robert Epstein's statement is "of absolute certainty." How can Friction Ridge Identification Specialists do what they say they can do with absolute certainty? Dr. Champod argues that the opinion of positive identification made by the friction ridge identification specialist "is based on inductive reasoning" and, therefore, "must be probabilistic" [Champod et al. 2004]. However, others suggest that the probability is so small that it can be disregarded and hence the latent print examiner's conclusion of 100% certainty is acceptable. Even Dr. Champod supports the view that individualization cannot be achieved through statistics, "But statistics can do no more than provide a probability. It is for others to decide on whether that probability is small enough to conclude identity of source" [Champod et al. 2004].

Dr. Champod further suggests that

> ... the benefits from statistics applied to the fingerprint identification field will include a way to assess the statistical value of marks declared insufficient for identification. A model should allow probabilities to be assigned to partial marks, e.g., assessing the chance of finding another finger showing a limited number of matching features.

Unfortunately, there are inherent risks in bringing a probability model out of academia and into a courtroom. In one DNA "match" case that he discusses, Andre Moenssens [2000] states, "The odds of the arrestee's DNA being wrongly matched against that of the crime scene were said to be one in 37 million." Moenssens believes that it is a common misunderstanding among lawyers, judges, lay persons and police that when a DNA "match" is reported with odds of one in 37 million that a like match in the DNA pattern exists once in 37 million people. This is clearly a misunderstanding of the statistics used by the experts. Monenssens continues by adding, "According to DNA scientist Keith Inman... it should be understood that the calculated frequency is an estimate, and can be off by an order of magnitude in either direction."

# Crucial Considerations

Correct interpretation of friction ridge features is critical to the friction ridge identification process. Whether or not these features are recognized, ignored, or given any significance can be seriously affected by any distortion present in the fingerprint. Another factor that should be considered is that the majority of crime scene fingerprints contain only 20% of the information found in a rolled inked fingerprint. Therefore, the inability to accurately recognize friction ridge features or interpret them correctly may be detrimental to the reliability of the probability model. Unless the probability model accurately accounts for the effects of distortion in a crime scene print, the application of such a model could be detrimental to the judge or jury's assessment of the value of the fingerprint evidence.

In my opinion, the conclusions of the Outstanding papers support the underlying premise that fingerprints are unique. Time constraints may have prevented a more detailed examination of comparing the probability of a fingerprint misidentification to a misidentification by DNA evidence. Unfortunately, the assumptions used in the models preclude the significance of distortion and lack of clarity in the crime scene print. Other factors, such as the subjective nature of latent print analysis by specialists with varying levels of training, knowledge, and experience, also need to be examined in assessing the odds of a fingerprint misidentification.

# Conclusion

I agree with Dr. Champod that "Statistical data, even gathered through myopic models, can only help the discipline work toward more reliable and transparent methods of assessing evidential value." I highly recommend that anyone interested in pursuing further the application of statistical analysis and probability models to friction ridge identification read his book [Champod et al. 2004].

I applaud the efforts of all this year's modeling teams in considering the important problems involved in the complex statistical analysis of friction ridge features and the advancement of the science of friction ridge identification.

# References

Beeton, Mary. 2004. Ridges and furrows. `http://www.ridgesandfurrows.homestead.com/`.

Champod, Christophe. 2002. The Weekly Detail (7 January 2002) (weekly electronic newsletter at Kasey [2004]).

————, Chris Lennard, Pierre Margot, and Milutin Stoilovic. 2004. *Fingerprints and Other Ridge Skin Impressions*. Boca Raton, FL: CRC Press.

Moenssens, Andre A. 2000. Mistaken DNA identification? What does it mean? Updated 20 October 2000. `http://www.forensic-evidence.com/site/EVID/EL_DNAerror.html` .

Wertheim, Kasey. 2004. Complete latent print examination. . . a site FOR latent print examiners, BY latent print examiners. `http://clpex.com` .

## About the Author

Mary Beeton was first introduced to the applied science of friction ridge identification through her training and education as an Automated Fingerprint Identification System Technician with the Durham Regional Police Service in Ontario, Canada. Her Website "Ridges and Furrows" is the culmination of many hours spent researching topics relating to the forensic discipline of friction ridge identification. Ms. Beeton frequently gives presentations on A.F.I.S., the history of friction skin identification, fingerprint patterns, and digit determination to police officers as part of their advanced training. Ms. Beeton is currently President of the Canadian Identification Society (C.I.S.), an organization with approximately 900 members from Canada, the United States, and other countries worldwide. The C.I.S. encourages forensic identification specialists to share their knowledge and experience and supports continuing research in all areas of forensic science.

# Editor's Commentary: Fingerprint Identification

Paul J. Campbell
Mathematics and Computer Science
Beloit College
700 College St.
Beloit, WI 53511
campbell@beloit.edu

## Introduction

Some problems from COMAP's Mathematical Contest in Modeling (MCM) and the Interdisciplinary Contest in Modeling (ICM) have arisen in very specific current situations, and it was not clear that specific ideas from the solution papers could have any immediate further application beyond the original setting. I am thinking here of MCM problems such as the

- Emergency Facilities Location Problem (1986),
- Parking Lot Problem (1987),
- Midge Classification Problem (1989),
- Helix Intersection Problem (1995),
- Velociraptor Problem (1997),
- Lawful Capacity Problem (1999),
- Bicycle Wheel Problem (2001),
- Wind and Waterspray Problem (2002),
- Gamma Knife Treatment Problem (2003), and
- Stunt Person Problem (2003).

*The UMAP Journal* 25 (3) (2004) 273–280. 

The same is true of some of the ICM problems, such as the Zebra Mussel Problem (2001) and the Scrub Lizard Problem (2002).

Other contest problems have arisen from situations that society faces chronically but have no urgency, yet the solution papers provide valuable ideas that could be put into practice. Here I include the

- Salt Storage Problem (1987),
- College Salaries Problem (1995),
- Contest Judging Problem (1996),
- Discussion Groups Problem (1997),
- Grade Inflation Problem (1998), and
- Quick Pass Problem (2004).

Finally, some problems have touched on issues of immediate concern, and the solution papers offer important insights:

- Emergency Power-Restoration Problem (1992),
- Asteroid Impact Problem (1999),
- Hurricane Evacuation Problem (2001)—eminently relevant in multiple-hurricane season of 2004,
- Airline Overbooking Problem (2002),
- IT Security Problem (ICM 2003), and
- Airport Security Problem (2004).

Perhaps no problem has been as aptly timed, however, as the Fingerprints Problem of this year's MCM [Giordano 2004].

# Previous Developments

The Outstanding papers for the Fingerprints Problem [Amery et al. 2004; Camley et al. 2004; O'Ceallaigh et al. 2004] and the commentaries by contest judge Michael Tortorella [2004] and practitioner Mary Beeton [2004] note the recent questioning in U.S. courts of the reliability of fingerprint evidence. That questioning took place after the U.S. Supreme Court set forth standards for admissibility of scientific testimony and evidence, the so-called *Daubert* criteria:

1. that "the theory or technique" is one that "can be (and has been) tested";
2. that "the theory or technique has been subjected to peer review and publication";

3. "in the case of a particular scientific technique, the court ordinarily should consider the known or potential rate of error ... and the existence and maintenance of standards controlling the technique's operation"; and

4. "general acceptance" in the "scientific community."

509 (1993) U.S. at 593–594.

The primary recent decisions about fingerprint evidence were made by Justice J. Pollak of the U.S. District Court for the Eastern District of Pennsylvania. In his first ruling, he agreed to the uniqueness and permanence of fingerprints and to allow the government to present evidence comparing latent prints and exemplars (in the terminology of Beeton [2004]) but disallowed any testimony that "a particular latent print is—or is not—the print of a particular person" [Pollak 2002, 49]. He found that only the fourth *Daubert* criterion was fulfilled ("general acceptance within the American fingerprint examiner community"), and that the difficulty with the *Daubert* criteria arises at the point that a fingerprint specialist uses subjective judgment and criteria to assert that two prints came from the same person [2002, 42–44].

Judge Pollak granted a subsequent hearing to reconsider his ruling and subsequently reversed his own decision, allowing such testimony. His change of mind resulted from being convinced by evidence presented that fingerprint identification does satisfy the "peer review" criterion of *Daubert* and also the "rate of error" / "standards" criterion ("there is no evidence that the error rate of certified FBI fingerprint examiners is unacceptably high" [2004, 36]). (Perhaps he would have a different opinion after reading the subsequent revelations by Heath [2004].) However, he still regarded the testing criterion as not met. Nevertheless, "to postpone present in-court utilization of this 'bedrock forensic identifier' pending ... research would be to make the best the enemy of the good" [2004, 49–50].

# Developments Since the Contest

As the contest papers and the commentaries point out, fingerprint identification is not solely a scientific enterprise but takes place in an environment where human error can prevail. Various recent events pointedly identify some such sources of error.

DNA testing is subject to similar errors; and with DNA evidence, even further questions can be raised, about possible contamination and the interpretation of the "odds" offered by DNA analysts (see Wood [1991] for discussion of the latter).

## Appeals Court Ruling

In an appeals ruling on a different case, the court found fingerprinting "testable" (though not completely tested), the error rate very low (though not "precisely quantified"), but standards to be lacking. Nevertheless, it found in the case at hand (*U.S. v. Byron Mitchell*) that most factors in the *Daubert* principles supported admitting the government's latent fingerprint evidence [Barry et al. 2004].

## Mistake

Stephen Cowans, convicted in 1998 of shooting a police officer on the basis of a fingerprint match from a glass at the crime scene, was freed from prison in February 2004; reanalysis of the latent print showed that it did not match his prints [Mnookin 2004]. Fraud? Incompetence? Just plain error?

## Misfiling

Rene Ramon Sanchez was accused in an immigration court of being Leo Rosario and was arrested three times for Rosario's crimes, spending two months in custody. The reason: Sanchez's prints matched Rosario's. And they did match, perfectly; at least, they matched the prints that were on Sanchez's fingerprint card on record. That was because when police had fingerprinted Sanchez earlier on another charge (later dropped), they put Sanchez's prints on a card with the name and data for Rosario. Finally, the authorities compared *photos* of the two. The aggrieved Sanchez says that he has never received an apology from any of the authorities involved [Weiser 2004].

## Dueling Experts

Brandon Mayfield, a lawyer in Portland, OR, was arrested and jailed in connection with the bombings of trains in Spain in April 2004. The basis was discovery of a fingerprint on a bag of detonators at the bomb scene, which three FBI fingerprint examiners concluded was a match to Mayfield's: a "100% positive identification."

The FBI turned out also to be "100% wrong." Despite contentions all along by Spanish fingerprint experts that the match was "conclusively negative," the FBI maintained its position for five weeks. In a meeting of American and Spanish experts, the Americans maintained that the prints had 15 "Galton points" in common, while the Spaniards said there were only 7. (No specific minimum number of common "points" is required for an identification in the protocol used by the FBI.)

Only after the Spaniards matched the print to Ouhnane Daoud, an Algerian, did the FBI admit that theirs had been a faulty match; subsequently, despite the match by the Spaniards, the FBI claimed that the print was unusable in the first place (i.e., the latent print was of poor quality).

The Spaniards later expressed surprised at the FBI's singleminded pursuit of Mayfield, who had converted to Islam and had represented in a custody case an individual who was also a defendant in a terrorism case. "It seemed as though they [the FBI] had something against him, and they wanted to involve us." However, according to FBI authorities, the fingerprint examiners who made the mistaken match did not know Mayfield's name or anything about him [Kershaw 2004]. (Kershaw's article includes photos of the latent print and of Brandon Mayfield's; images are available at German [2004].)

> This is going to kill prosecutors for years every time they introduce a fingerprint ID by the FBI. The defense will be saying "is this a 100 percent match like the Mayfield case?"
>
> —U.S. Senate aide [Kershaw 2004, A13]

# Philosophical Questions but Practical Implications

As Tortorella [2004] notes, this MCM problem raises philosophical questions.

## No Sound Statistical Foundation?

One question that Tortorella mentions is about how to assign probabilities to the sample space of modeled fingerprints.

The calculations in the Outstanding papers admit assuming independence of features from one area of a fingerprint to another (thus enabling their multiplication of probabilities), despite obvious local dependence (ridges cross multiple cells).

However, more dangerously, implicit in the papers' calculations of the probability of a match is the assumption of a uniform distribution: that all the many fingerprints are equally (un)likely. As Mnookin [2004] puts it:

> Fingerprinting . . . currently lacks any valid statistical foundation. . . . The important question is how often two people might have fingerprints sufficiently similar that a competent examiner could believe they came from the same person. This problem is accentuated when analyzing a partial print. . . . How often might one part of someone's fingerprint strongly resemble part of someone else's print? No good data on this question exist.
>
> The growing size of computer fingerprint databases makes this issue still more acute. As a database grows in size, the probability that a number of people will have strikingly similar prints also grows. . . .
>
> The FBI called the resemblance between Mayfield and Daoud's prints "remarkable." What is truly remarkable is that we simply do not know how often different people's prints may significantly resemble one another, or how good examiners are at distinguishing between such prints.

## Is Science Certain . . . Enough?

A second key issue is the status of scientific truth and of evidence obtained by technical means. In May 2004, Gov. Mitt Romney proposed a death-penalty statute for Massachusetts (the state does not currently have a death penalty). A death sentence would require "conclusive scientific evidence" of guilt.

Romney's proposal highlights the philosophical question:

> *Can scientific evidence yield certainty? Should scientific evidence be regarded as more reliable than other evidence? Is it more reliable?*

Those who believe that scientific evidence is more reliable need to confront that today's science may be tomorrow's alchemy—"today's certainty is tomorrow's question mark" [Daley 2004]. The last few years have seen DNA analysis bring many cases of wrongful conviction to light; but many of those wrongful convictions were based primarily on the best "science" of the time, including microscopic analysis of hairs and also fingerprints.

> What we say in forensic science is the more certain the scientist is, the less reliable the scientist is . . . . [O]ur society can easily be taken in by science, and that is worrisome.
>
> —James Starrs, Prof. of Law and Forensic Science,
> George Washington University [Daley 2004]

# Subjectivity

Fingerprint experts maintain, and the FBI agrees, that in the final analysis declaring a match of two fingerprints is a subjective decision, made by a human being based on training, experience, and all of the circumstances involved in the comparison. But what science is without subjective decisions, at some level? Judge Pollak found that the techniques in fingerprint identification have not been subjected to sufficient testing—so it is high time that the work be done to put fingerprinting on as unimpeachable a scientific basis as it deserves!—but nevertheless he was willing to consider such matching as "scientific" evidence.

However, caution about subjectivity is in order, and not just in the realm of fingerprint matching:

> [F]ingerprints are valuable forensic evidence. . . . But when the evaluation of that data rests on a because-I said-so analysis, the door is wide open for injustice. And as Brandon Mayfield's case amply demonstrates, taking the government's say-so as definitive simply isn't enough. And when pseudoscience is turned loose in the context of the war on terror, the results may well terrify.
>
> —David Feige [2004]

# References

Amery, Steven G, Eric Thomas Harley, and Eric J. Malm 2004. The myth of "the myth of fingerprints." *The UMAP Journal* 25 (3): 215–230.

Barry, Becker, and Greenberg, Circuit Judges. 2004. Precedential. 29 April 2004. United States Court of Appeals for the Third Circuit. *United States of America v. Byron Mitchell*. No. 02-2859. `caselaw.findlaw.com/data2/circs/3rd/022859p.pdf` .

Beeton, Mary. 2004. Practitioner's commentary: The outstanding fingerprints papers. *The UMAP Journal* 25 (3): 267–272.

Camley, Brian, Pascal Getreuer, and Bradley Klingenberg. 2004. Not such a small whorl after all. *The UMAP Journal* 25 (3): 245–258.

Daley, Beth. 2004. Foolproof forensics. *Boston Globe* (8 June 2004). `http://www.boston.com/news/globe/health_science/articles/2004/06/08/foolproof_forensics` .

Ellement, John. 2004. Scientist rebuts reliance on fingerprints. *Boston Globe* (18 May 2004). `http://www.boston.com/news/local/massachusetts/articles/2004/05/18/scientist_rebuts_reliance_on_fingerprints` .

Feige, David. 2004. The inexact science of fingerprint analysis. *Slate* (27 May 2004). `http://slate.msn.com/id/2101379/` .

German, Ed. Problem idents: Madrid erroneous identification. Last updated 23 September 2004. `http://onin.com/fp/problemidents.html#madrid` .

Giordano, Frank. 2004. Results of the 2004 Mathematical Contest in Modeling. *The UMAP Journal* 25 (3): 189–214.

Heath, David. 2004. Bungled fingerprints expose problems at FBI. *Seattle Times* (7 June 2004). `http://seattletimes.nwsource.com/html/localnews/2001949987_fingerprints07m.html` .

Kershaw, Sarah. 2004. Spain and U.S. at odds on mistaken terror arrest. *New York Times* (5 June 2004) (National Edition): A1, A13.

Mnookin, Jennifer L. 2004. A blow to the credibility of fingerprint evidence. *Boston Globe* (2 February 2004). `http://www.boston.com/news/globe/editorial_opinion/oped/articles/2004/02/02/a_blow_to_the_credibility_of_fingerprint_evidence` .

________. The Achilles' heel of fingerprints. *The Washington Post* (29 May 2004): A27. `http://www.washingtonpost.com/wp-dyn/articles/A64711-2004May28.html` .

O'Ceallaigh, Seamus, Alva Sheeley, and Aidan Crangle. 2004. Can't quite put our finger on it. *The UMAP Journal* 25 (3): 231–244.

Pollak, J. 2002a. Opinion. 7 January 2002. United States District Court for the Eastern District of Pennsylvania. *United States of America v. Carlos Ivan Llera Plaza, Wilfredo Martinez Acosta, and Victor Rodriguez*. Cr. No. 98-362-10, 11, 12. http://www.dartmouth.edu/~chance/chance_news/for_chance_news/ChanceNews12.05/Pollak.pdf .

________. 2002b. Opinion [reversal]. 13 March 2002. http://www.dartmouth.edu/~chance/chance_news/for_chance_news/ChanceNews12.05/PollakReverse.pdf .

Snell, J. Laurie. 2003. The controversy of fingerprints in the courts. http://www.dartmouth.edu/~chance/chance_news/recent_news/chance_news_12.05.html#item11 .

Tortorella, Michael. 2004. Judge's commentary: The outstanding fingerprints papers. *The UMAP Journal* 25 (3): 261–265.

Weiser, Benjamin. 2004. Can prints lie? Yes, man finds to his dismay. *New York Times* (31 May 2004) (National Edition): A1, A17.

Wood, John B. Paternity probability: An inappropriate artifact [with commentaries]. *The UMAP Journal* 12 (1): 7–42.

# About the Author

Paul Campbell graduated summa cum laude from the University of Dayton and received an M.S. in algebra and a Ph.D. in mathematical logic from Cornell University. He has been at Beloit College since 1977, where he served as Director of Academic Computing from 1987 to 1990. He is Reviews Editor for *Mathematics Magazine* and has been editor of *The UMAP Journal* since 1984.

# A Myopic Aggregate-Decision Model for Reservation Systems in Amusement Parks

Ivan Corwin
Sheel Ganatra
Nikita Rozenblyum
Harvard University
Cambridge, MA

Advisor: Clifford H. Taubes

## Summary

We address the problem of optimizing amusement park enjoyment through distributing QuickPasses (QP), reservation slips that ideally allow an individual to spend less time waiting in line. After realistically considering the lack of knowledge faced by individuals and assuming a rational utility-oriented human-decision model and normally-distributed ride preferences, we develop our Aggregate-Decision Model, a statistical model of waiting lines at an amusement park that is based entirely on the utility preferences of the aggregate.

We identify in this model general methods in determining aggregate behavior and net aggregate utility and use these methods, along with complex but versatile QP accounting and allocation systems to develop the Aggregate-Decision QuickPass Model. We develop criteria for judging QP schemes based on a total utility measure and a fairness measure, both of which the Aggregate-Decision QuickPass Model is able to predict. Varying the levels of individual knowledge, the QP line-serving rates, the ability to cancel one's QP, and the QP allocation routines, we obtain a variety of different schemes and test them using real life data from Six Flags: Magic Mountain as a case study. We conclude that the scheme in which individuals are able to cancel their QPs, know the time for which a QP will be issued, and are allocated to the earliest QP spot available provides park-goers with the greatest total utility while keeping unfairness levels relatively low.

*The UMAP Journal* 25 (3) (2004) 281–300. 

# Introduction

As the number of park-goers increases, so do the waiting lines. In a QP system, rather than standing in a regular line, people can opt for a ticket to come back later and join a presumably faster line to the ride. People may hold only one active ticket at any given time.

We develop a means to evaluate QP systems. We first develop a working economic understanding of myopic human decision-making in amusement parks. Applying this to all park-goers, we develop our Aggregate-Decision model, which predicts the statistical behavior of groups faced with queueing choices. We then include QP lines and develop the Aggregate-Decision Quick-Pass model, which describes large-group statistical decisions about joining a regular line or obtaining a QP ticket. We test various QP distribution schemes and compare them on the criteria of maximizing utility while maintaining an acceptable level or fairness.

# Definitions and Key Terms

- An *amusement park* is a collection of $n$ rides $R_1, \ldots, R_n$ associated with a number $P_T$, representing the total population of the park (people in the park who are either looking, waiting, or are on a ride).

- For the $i$th ride, $l_i$ is the number of people in line to get on the ride and $k_i$ is the rate (persons/min) at which the line moves.

- The *fluid population* $P_F$ of an amusement park is the number of people actively looking for a ride.

- The *utility* of a ride is measured by how long people are willing to stand in line for it, given that the alternative provides them with zero utility. Individuals have utilities $t_i$, while the utilities for each ride have distributions $\mathcal{T}_i$ with expected values $\mu_i$ and standard deviations $\sigma_i$.

- The *preference* that an individual has for a ride is normalized to $r_i = t_i / \sum_k t_k$.

- The *popularity* of a ride $R_i$ is determined by its popularity rating $\rho_i = \mu_i / \sum_{k=1}^{n} \mu_k$ (we use here a first-order approximation of $E[\mathcal{T}_i / \sum_{k=1}^{n} \mathcal{T}_k]$ [Brown 2001]).

- A *QuickPass system* is a line-management scheme that allows a person to obtain a ticket for return later to a presumably faster QuickPass line.

- A QuickPass is *live* when it can be used by a holder to gain access to the QuickPass line.

- A QuickPass is *active* from the time of issue to the end of the time interval in which it is live.

**Table 1.**
Symbol table.

| Symbol | Definition | Units |
|---|---|---|
| | **Variables** | |
| $l_i$ | Number of people waiting in the regular line for $R_i$ (such that $l_{i,QP} + l_{i,\text{NOQP}} = l_i$) | people |
| $l_{i,QP}$ | Number of people waiting in the regular line for $R_i$ with an active QuickPass | people |
| $l_{i,\text{NOQP}}$ | Number of people waiting in the regular line for $R_i$ without an active QuickPass | people |
| $q_i$ | Number of people waiting in QP line for $R_i$ | people |
| $w_{i,ex}$ | Expected free waiting time | min |
| $k_i$ | Regular line speed (in non-QP model equal to $c_i$) | people/min |
| $d_i$ | QP line speed | people/min |
| $t_i$ | Measure of the utility for $R_i$ | min |
| $\vec{t}$ | Collection of $t_i$ | vector |
| $r_i$ | Preference for $R_i$ an individual has based on the $t_i$ | unitless |
| $\nu$ | Impatience measure for individual based on $t_i$ | unitless |
| $\mathcal{T}_i$ | Random variable representing based on distribution for $t_i$ about $\mu_i$ with standard deviation $\sigma_i$ | min |
| $\vec{\mathcal{T}}$ | Collection of $\mathcal{T}_i$ | vector |
| $\rho_i$ | Aggregate popularity for $R_i$ based on the $\mu_i$ | unitless |
| $\chi$ | Expected impatience measure based on $\mu_i$ | unitless |
| $U_{j,i}$ | Expected net utility provided by $R_j$, gauged with variables from $R_i$ | min |
| $U_{j,i}^{qp}$ | Expected net utility provided by $R_j$ (including QP waiting), gauged with variables from $R_i$ | min |
| $U_{k,s}^{qp}$ | Expected utility provided by $R_k$ (including QP waiting) for people with QP for $R_k$ which becomes live between $sI$ and $(s+1)I$ time steps, gauged with variables from $R_k$ | min |
| $P_F$ | Fluid population of the park (such that $P_{f,QP} + P_{f,\text{NOQP}} = P_F$) | people |
| $P_{F,QP}$ | Part of fluid population with an active QuickPass | people |
| $P_{F,\text{NOQP}}$ | Part of fluid population without an active QuickPass | people |
| $QP_{i,t}$ | Number of people with a QuickPass for $R_i$ which becomes live in $t$ time steps | people |
| $QP_i$ | Number of people with a QuickPass for $R_i$ which becomes live in any number of steps | people |
| $\phi_i$ | Preference distribution function | function |
| | **Constants** | |
| $n$ | Number of rides | unitless |
| $m$ | Number of QuickPass machines | unitless |
| $R_i$ | Ride | unitless |
| $\mu_i$ | Expected value of $t_i$ over the population | min |
| $\sigma_i$ | Standard deviation of $t_i$ over the population | min |
| $c_i$ | Serving rate of $R_i$ | people/min |
| $P_T$ | Total park population | people |
| $e$ | Ratio of the disutility for free waiting to line waiting | unitless |
| $\kappa$ | Ratio of the time for free waiting to regular line waiting | unitless |
| $I$ | Length of interval for which a QuickPass is issued | min |
| Cancel | Whether active QuickPasses can be canceled | boolean |
| DisplayTime | Whether QuickPass kiosk displays to the public when the next QuickPass issued with become live | boolean |
| $\delta_{i,s}$ | Maximum number of QuickPasses issued for $R_i$ in interval $s$ | unitless |

- A *time interval* is the period of time over which a QuickPass is live.
- The *free waiting time* is the time from when a QuickPass becomes active to when the QuickPass becomes live.
- The *net utility* of a ride is the utility gained by taking the ride added to the disutility associated with waiting for it, plus (in the case of a QuickPass) the disutility associated with free waiting time.

# General Assumptions

## Time Stepping

- The park runs continuously for a fixed amount of time (such as a working day), split into discrete time steps of length $\tau$. We use $\tau = 5$ min.
- In one time step, a person can either take one action or make one decision: move to a ride to consider it (we assume that all rides are an equal distance apart and people can on average cover that distance in time $\tau$) or actually be on a ride. While considering a ride, a person can decide to get in line, get a QP, or wander on to another ride in one time step $\tau$.
- A person in line will not leave before completing the ride.
- The serving rate $k_i$ of a ride is constant and independent of time and the length of the waiting line.
- Every ride takes one time step to run and lets out a batch of served people at the end of the time step in accordance with $k_i$. Thus, the number of people let out per time step is $\tau k_i$.
- People cannot trade QPs. (This assumption simplifies analysis.) Even if trading were allowed, it seems practically unfeasible. For a different perspective on reservation trading, see Prado and Wurman [2002].
- The park population size "ramps up" to its target size over a small number of time steps and then stays constant. The ramp-up is consistent with arrivals at the park; it follows an exponential distribution (**Figure 1**). After reaching its maximum, the park population stays constant until the end of the popular period. In reality, a ramp-down period follows; but as populations dwindle and lines shorten, QPs play a lesser role.

## Individual Behavior

- All things are measured in utilities, and individuals seek to maximize their utilities. We measure all utilities in terms of time. Thus, the utility $t_i$ for

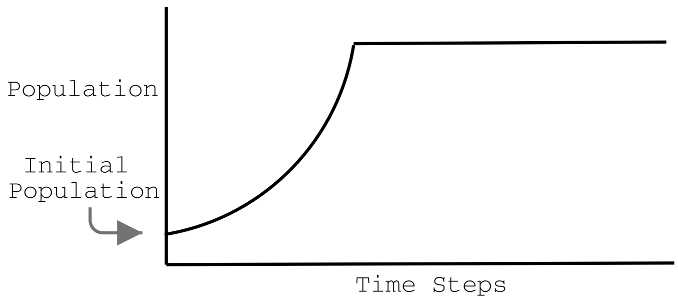



**Figure 1.** Population levels as a function of time.

taking a particular ride is measured by the length of time that it is worth waiting for the ride, given that all alternatives provide zero utility. We also measure the disutilities of waiting, as the total waiting time.

- An individual's enjoyment of a ride is fixed, not affected by waiting time.
- Disutility is linearly proportional to time waiting in a line.
- Individuals are *myopic*: They know information only about the ride where they are and thus determine their expectations for other rides based on their own preferences $r_i$ and the line-serving rates $c_i$, $d_i$, and $k_i$. This is reasonable because in reality rides are a significant distance apart.
- An individual can immediately and accurately gauge how long a line is and its serving rate.
- Each individual knows $r_i$, the individual's own preferences.

## Aggregates

- The population's preference distribution $\phi_i$ for riding each $R_i$ is normal. This is reasonable, because for large populations the central limit theorem [Weisstein 2004] applies—the time that a person is willing to wait for a ride is a function of many random variables. Moreover, we assume further that in any subset of the population, the distribution $\phi_i$ applies with equal validity. This is reasonable if the population (and the aggregate) is sufficiently large. Thus, it makes sense to discuss the random normally-distributed preference variables $\mathcal{T}_i$ of an aggregate. The actual specifics of the distribution, such as $\mu_i = E[\mathcal{T}_i]$ and $\sigma_i$, can be estimated empirically, e.g., by taking surveys.
- The random preference variables $\mathcal{T}_i$ are independent of one other: Each person considers different rides independently. In reality, preferences might be correlated based on type of ride, age, and enjoyment of amusement parks in general, as well as other factors; but since there are many other factors within each ride, this assumption is reasonable.

- The population preference distributions $\phi_i$ are temporally invariant. That is, aggregate preferences do not change with time and with ride experience; in essence, there is no aggregate memory. An individual's preference for a ride is likely a function of the number of visits [Prado and Wurman 2002], but we can either assume that this function is the constant function (that people have unchanging utility functions) or that preference changes over time cancel out in the distribution. If one person prefers a ride less after riding it, another will prefer it more.

- The popularity $\rho_i$ of a ride corresponds to the fraction of the fluid population that goes to and considers $R_i$ at any given time step. Effectively this is the fraction of the population for whom $R_i$ is their favorite ride. This assumption is a result of defining $\rho_i$ as a first-order approximate to $E[\mathcal{T}_i/\sum_{k=1}^{n}\mathcal{T}_k]$.

# The Aggregate-Decision Model

## Expectations of Our Model

- After the ramp-up period, the marginal utility for each time step for small-population amusement parks will be greater than for larger amusement parks. In large parks, there is more crowding and thus longer lines and more disutility at each time step.

- Increasing the sum of the $\mu_i$s should increase the cumulative utility over the course of the day.

- Increasing the popularity of a given ride should increase its line length.

- At a small park, people tend disproportionately toward the most popular rides. This expectation is suggested by the second-order expansion for expected value [Brown 2001]. At a large park, people tend toward popular rides less than expected, because of increased disutility from waiting in longer lines.

## Individual Behavior with No QuickPass

Let $T = (t_1, \ldots, t_n)$. We define $\nu = 1/\sum_k t_k$ and $r_i = t_i\nu$. The $r_i$ measure the individual's preference for a ride over the alternatives. We call $\nu$ the *impatience measure*, since multiplying by it normalizes utility (willingness to wait) and thus neutralizes differences in patience. We assume that people seek rides that they prefer most. Thus, a person considers the ride with the highest $r_i$.

An individual's net utility from a ride is the utility that the ride provides minus the disutility from waiting. As before, let $k_i$ be the rate at which the line moves and $l_i$ the length of the line; the approximate waiting time is then $l_i/k_i$. Thus, for ride $R_i$, a person's utility is

$$U_i(t_1, \ldots, t_n) = t_i - \frac{l_i}{k_i}.$$

Since individuals cannot know the lengths and speeds of the other rides, they must estimate the utilities of those rides. Let $U_{j,i}$ be the utility that the individual estimates for $R_j$, using variables from $R_i$ (we assume that the individual is considering $R_i$). The person estimates $k_j$ and $l_j$ from their preferences towards $R_j$ and information about $R_i$. In our model, a person assumes that the population has the same preferences as their own; that is, the number of people at a ride is proportional to the individual's preference for that ride, so that the person estimates that $l_j = r_j P_T$. Furthermore, the person predicts the speed of the lines to be roughly be the same. Thus, an individual at $R_i$ would reason that

$$U_{j,i}(t_1, \ldots, t_n) = t_j - \frac{r_j P_T}{k_i} = t_j - \frac{P_T}{k_i} \frac{t_j}{\sum_k t_k}.$$

Then the person, comparing the utility from $R_i$ with the expected utilities of the other rides, stays at $R_i$ if $U_i$ exceeds all the other expected utilities $U_{j,i}$. Thus, a person joins line for the ride $R_i$ if $U_i \geq U_{j,i}$ for all $j \neq i$.

## Aggregate Behavior

In dealing with an aggregate, randomly selected members considering ride $R_i$ have as preference the random normally-distributed variables $\mathcal{T}_i$ instead of $t_i$, the case for the individual. Thus, because the utility functions $U_i$, and $U_{j,i}$ are functions of the $t_i$, they induce the following utility distribution variables on the aggregate population:

$$\mathcal{U}_i(\mathcal{T}_1, \ldots, \mathcal{T}_n) = \mathcal{T}_i - l_i/k_i, \qquad \mathcal{U}_{j,i}(\mathcal{T}_1, \ldots, \mathcal{T}_n) = \mathcal{T}_i - \frac{\mathcal{T}_i}{\sum_j \mathcal{T}_j} \frac{P_T}{k_i}. \qquad \textbf{(1)}$$

## The Formal Model

We develop an iterative process for determining how lines and utilities change as a function of time.

In our model, $P_T$ (and $P_F$) "ramps up" until approximately 20 5-min time steps are completed and the park is at full capacity. At each time step, $\rho_i P_F$ people consider entering the line for ride $R_i$. (This quantity may not be an integer; at our final calculation, we round.) The aggregate population considering $R_i$ has the utility distributions given by **(1)**.

An individual stays if $U_i \geq U_{j,i}$ for all $j \neq i$. Since the $\mathcal{T}_i$ are normal, the probability distribution function for each $\mathcal{T}_i$ is

$$\phi_i(t_i) = \frac{1}{\sigma_i \sqrt{2\pi}} e^{-(x-\mu_i)^2/2\sigma_i^2}.$$

We would like to find the probability that $\mathcal{U}_i(\mathcal{T}_1, \ldots, \mathcal{T}_n) \geq \mathcal{U}_{j,i}(\mathcal{T}_1, \ldots, \mathcal{T}_n)$ for all $i, j$. Define the domain $\Omega \subset \mathbb{R}^n$ as follows:

$$\Omega = \{(t_1, \ldots, t_n) \in \mathbb{R}^n : U_i(t_1, \ldots, t_n) \geq U_{j,i}(t_1, \ldots, t_n) \text{ for all } j \neq i\}.$$

Then the probability that a person prefers ride $R_i$ to all other rides is

$$\tilde{P} = P(\mathcal{U}_i \geq \mathcal{U}_{j,i} \text{ for all } j) = \int_\Omega \phi(\vec{t})d\vec{t},$$

where the distribution function $\phi(t_1, \ldots, t_n) = \prod_{i=1}^n \phi_i(t_i)$ because the $\phi_i$ are independent of one another. So the number of people who get in line is the rounded value of the product of this probability and the number of people, or $\lfloor \tilde{P}\rho_i P_F \rfloor$. Since $\Omega$ may be a complicated domain, direct integration is impossible. We compute this integral numerically, using a variant of the Monte Carlo method. The average utility gained for these people is

$$\bar{U} = \langle U_i \rangle = \int_\Omega U_i(\vec{t})\phi(\vec{t})d\vec{t},$$

and the total utility gained is the product of this with the (rounded) number who get in line, or $\bar{U}\lfloor \tilde{P}\rho_i P_F \rfloor$. In a similar manner, we calculate the variance

$$\sigma^2 = \langle U_i^2 \rangle - \langle U_i \rangle.$$

The model proceeds by adjusting the line counts, removing back into the fluid population people who do not enter the line, and increasing the total (and fluid) populations if in the ramp-up period. **Figure 2** gives a flowchart.

## Model Validation

We programmed our model into a computer simulation. We simulate three different sized parks, with different $\mu_i$s (thus creating different preference distributions and measures of impatience). Then we process the resulting distributions of people and the cumulative utility throughout the day. We find that our model meets our expectations for a basic human-behavior model.

# The Aggregate-Decision QuickPass Model

We add QP machines at $R_1, \ldots, R_m$. QPs are given out for a constant time interval of $I$ units of our time step: A QP becomes live in one of the time intervals $[0, \tau I]$, $[\tau I, 2\tau I]$, etc.

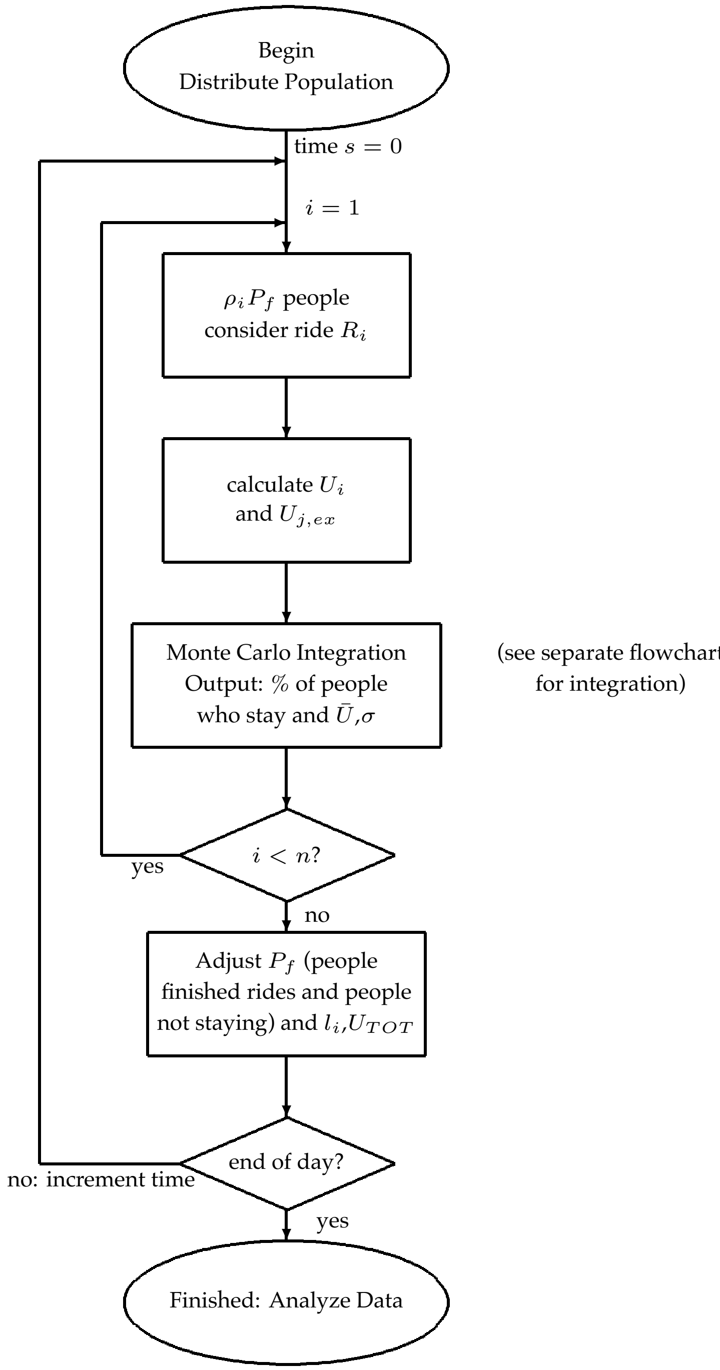



**Figure 2.** A schematic depicting the steps occurring in the Aggregate-Decision Model.

## Additional Definitions

- A QP *expires* if
  - the individual does not enter the QP line during the QP's time interval (the individual *forfeits* the QP).
  - The individual accepts another QP (in models that allow doing so).

Since our framework presents a different decision model for QP holders vs. non-QP holders, we track the population of QP holders.

- Let $q_i$ be population of the QP line at ride $R_i$. Let $d_i$ be the rate at which $R_i$ draws from $q_i$ (people/min) and let $c_i$ similarly be the rate at which $l_i$ shortens—clearly, $d_i + c_i = k_i$.
- $l_{i,QP}, l_{i,\text{NOQP}}$ are the number of QP- and non-QP-holders in the line to $R_i$, respectively.
- Similarly, $P_{F,\text{QP}}$, $P_{F,\text{NOQP}}$ are the fluid populations of QP- and non-QP-holders, respectively.
- We do not track individuals but we track the QPs handed out for each time interval. Let $\text{QP}_{i,s}$ be the number of QP users with QPs for ride $R_i$ at the $s$th time interval $[sI, (s+1)I]$. The $\text{QP}_{i,s}$ can decrease through forfeiting and increase according to the *allocation routine* of the QP scheme.

## Assumptions about the QuickPass System

We assume that $\text{QP}_{i,s}$ is uniformly distributed throughout all lines and rides; that is, for any line, the people in that line with a QP are uniformly distributed throughout the line. However, the proportion of people with a QP can vary from ride to ride.

There is a limit $\delta$ for the total number of QPs for a ride.

## Formal Development of the Model

The model relies on examining the populations with and without QPs, determining the proportion of each who take certain actions, and updating the populations and line counts. **Figure 3** presents the intuitive summary of how our model works. We describe the model illustrating two different scheme factors: the case where QP holders may cancel their QP for another QP (the cancel model), and the case where they cannot (and must wait until their QP expires to obtain a new QP).

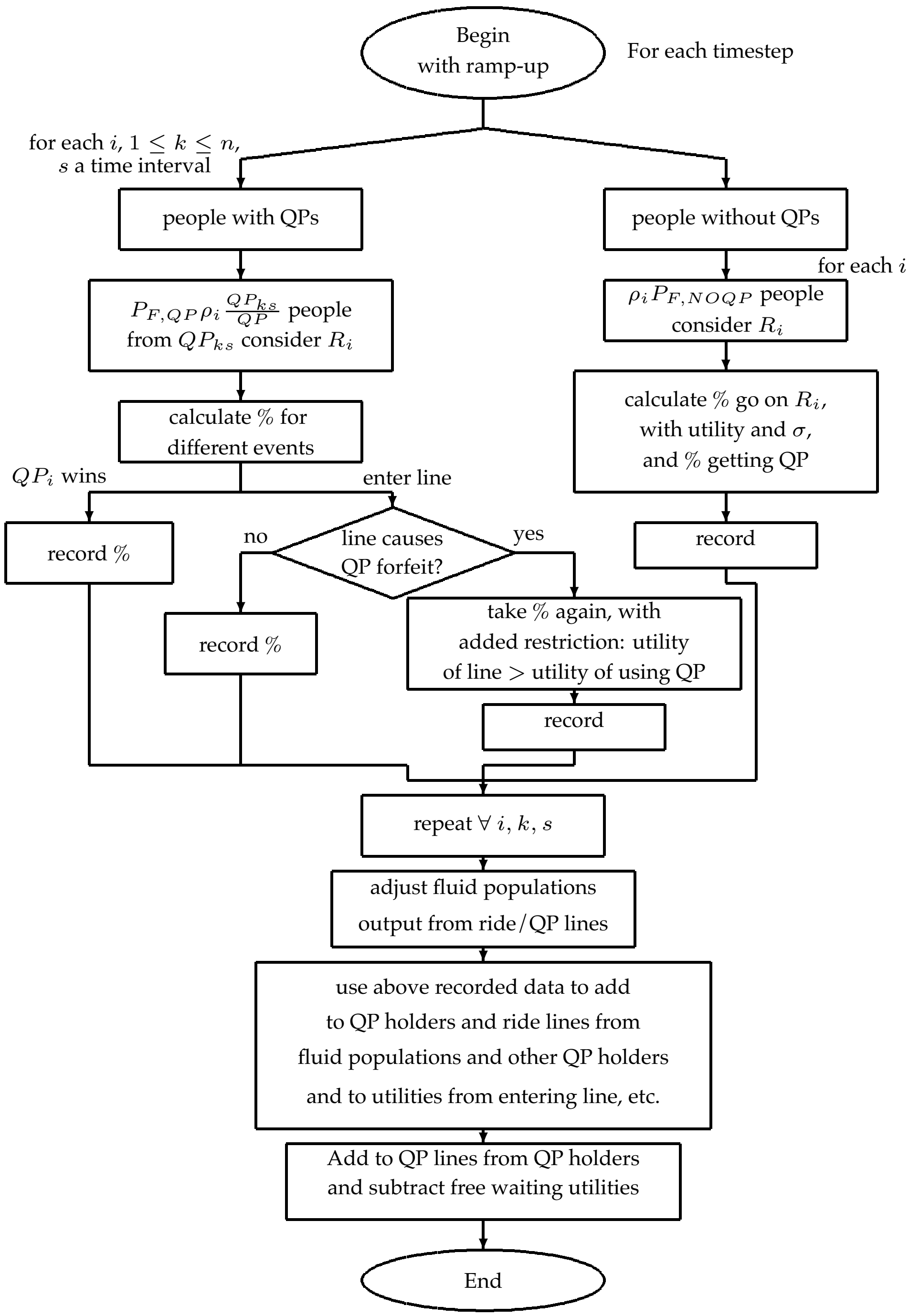



**Figure 3.** A flowchart detailing the Aggregate-Decision QuickPass Model.

## Aggregates without QuickPasses

An individual without a QP must decide not only which ride to go to but also whether to get a QP for some ride, hence will examine the expected utility from staying at a particular ride vs. the expected utilities of other rides—just as in the non-QP model. However, with QPs, an individual also compares that with the expected utilities of obtainable QPs.

As in the non-QP model, the individual estimates the utility of ride $j$ as

$$\mathcal{U}_{ji}(\mathcal{T}_1, \ldots, \mathcal{T}_n) = \mathcal{T}_j - \frac{\mathcal{T}_j P_T}{c_i \sum_k \mathcal{T}_k}.$$

Now the person compares these utilities to the utility provided by a QP. Because the person cannot go on the ride immediately, there is a disutility proportional to the wait. We assume that the constant of proportionality $e$ is approximately the same for all individuals. Further, an individual assumes that the length of the QP line later will be approximately the current length. Therefore, the utility for a person getting a QP at ride $R_i$ (if available) is

$$\mathcal{U}_{ii}^{qp}(\mathcal{T}_1, \ldots, \mathcal{T}_n) = \mathcal{T}_i - \frac{q_i}{d_i} - e w_i,$$

where $w_i$ is the time of the wait on the QP ticket. If there is free waiting-time clairvoyance, the individual will know $w_i$ in advance of getting the ticket; if not, and the person does not know what $w_i$ will be, the person approximates $w_i$ as proportional to the current length $l_i$ of the line. This is reasonable if we regard the $l_i$ and $w_i$ as correlated with the popularity of the ride. Thus, a person not knowing $w_i$ approximates it as $\kappa l_i / c_i$, for some constant of proportionality $\kappa$; for rides other than $R_i$, the approximation is $\kappa P_T r_j / c_i$. Thus, we have

$$\mathcal{U}_{ii}^{qp} = \mathcal{T}_i - \frac{q_i}{d_i} - e\kappa l_i / c_i, \qquad \mathcal{U}_{ji}^{qp} = \mathcal{T}_j - \frac{q_i}{d_i} - \frac{\mathcal{T}_j}{\sum_k \mathcal{T}_k} \frac{e\kappa P_T}{c_i}.$$

So an individual decides to go on ride $R_i$ if the utility $U_{ii}$ exceeds each $U_{ji}$ and $U_{ji}^{qp}$. Similarly, a person gets a QP at ride $R_i$ if $U_{ii}^{qp}$ exceeds everything else (that is, if $U_{ii}$ exists—if not, then the individual cannot opt for a QP). To find the proportions of the aggregate that make these decisions, we must integrate over the domains $\Omega_0, \Omega_1 \subset \mathbb{R}^n$ in which $U_{ii}$ is greater than all alternatives and $U_{ii}^{qp}$ is greater than all alternatives, respectively. Similarly, we calculate the average utility gained by entering the line.

## Aggregates with QuickPasses

People who already hold QPs provide an additional complication. As **Figure 3** demonstrates, a QP-holder considering a ride can forfeit the QP and enter a line, simply enter the line, obtain a new QP (in our cancel model), or do nothing. Because the decision to forfeit depends on the time interval in which the user holds the QP, we must split up our aggregates into proportions corresponding to each QP$_{i,s}$. Since each QP$_{i,s}$ is uniformly distributed throughout the population of QP-holders, which in turn is uniformly distributed in line and in the fluid populations, the subset of $QP_{i,s}$ that is fluid is $P_{F,\mathrm{QP}}\mathrm{QP}_{i,s}/\mathrm{QP}_{TOT}$; hence, we can say as before that $\rho_i P_{F,\mathrm{QP}}\mathrm{QP}_{i,s}/\mathrm{QP}_{TOT}$ people consider ride $R_i$.

Then these people consider the expected and actual utilities of obtaining a QP for each ride and for entering any ride line. Via integration, we calculate the proportion of people who choose another QP, with the added comparison against the remaining utility of the existing QP $U^{qp}_{k,i,s}$, which only accounts for the remaining free waiting time (the lost time is sunk).

We calculate the proportion of people who enter a line (and the respective average utility) as the previous section, with one caveat: If members of the QP population would forfeit their QPs by entering a line, then we must also ensure that the utility that they would gain by entering line is greater than the remaining utility of their existing QP $U^{qp}_{k,i,s}$—an additional constraint on the domain of integration.

## Adjustments to Populations, Lines, and Utility

- At every ride, we must move people from lines and QP lines into the fluid population $P_{F,\mathrm{NOQP}}$ and $P_{F,\mathrm{QP}}$. To account for the fact that our lines $l_i$ and $q_i$ may have fewer individuals than the rates at which they are drawn from $c_i\tau$ and $d_i\tau$, respectively, we create a function that incorporates checking whether ($q_i$ or $d_i\tau$) and ($l_i$ or $c_i\tau$), respectively, is the minimum. Let $x_i = \min(q_i, d_i\tau)$, $y_i = \min(l_i, c_i\tau)$, $\eta_i$ = the number of people who leave from $l_i$, and $\zeta_i$ = the number who leave from $q_i$. Then

$$\eta_i = \min\left(c_i\tau\,\frac{x_i}{d_i\tau} + k_i\tau\,\frac{d_i\tau - x_i}{d_i\tau},\ \ l_i\right),$$
$$\zeta_i = \min\left(d_i\tau\,\frac{y_i}{c_i\tau} + k_i\tau\,\frac{c_i\tau - y_i}{c_i\tau},\ \ q_i\right).$$

Then from this ride, the new line and fluid population lengths become:

$$\begin{aligned}
P_{F,\mathrm{NOQP}} &\leftarrow P_{F,\mathrm{NOQP}} + \lfloor \frac{l_{i,\mathrm{NOQP}}}{l_i}\eta_i \rfloor + \lceil \zeta_i \rceil \\
P_{F,\mathrm{QP}} &\leftarrow P_{F,\mathrm{QP}} + \lceil \frac{l_{i,\mathrm{QP}}}{l_i}\eta_i \rceil \\
l_{i,\mathrm{NOQP}} &\leftarrow l_{i,\mathrm{NOQP}} - \lfloor \frac{l_{i,\mathrm{NOQP}}}{l_i}\eta_i \rfloor \\
l_{i,\mathrm{QP}} &\leftarrow l_{i,\mathrm{NOQP}} - \lceil \frac{l_{i,\mathrm{QP}}}{l_i}\eta_i \rceil \\
q_i &\leftarrow q_i - \lceil \zeta_i \rceil.
\end{aligned}$$

To see why this is so (up to our rounding), suppose that $q_i < d_i\tau$ (the argument for $l_i < c_i\tau$ is symmetrical). Then, since

$$d_i\tau\frac{y_i}{c_i\tau} + k_i\tau\frac{c_i\tau - y_i}{c_i\tau} > d_i\tau$$

(because $k_i \geq d_i$), note that $\eta_i = q_i$, and $\zeta_i$ is as we would like, because $R_i$ takes people from $l_i$ with rate $c_i$ until $q_i$ is depleted, after which it takes from $l_i$ at the full rate $k_i$—that is, unless this number is less than $l_i$.

- For all of the QP populations free-waiting another unit, subtract a unit of utility $e\tau$.
- Once this process has been completed, for each ride $R_i$, for the population $\text{QP}_{i,s}$ where $s$ is an interval that is currently live, move a certain number of people into the QP line $q_i$ from $R_i$. This number need not be the entire population $\text{QP}_{i,s}$ but rather a random number predetermined by our model such that after the live period is over, the entire population has entered the QP line. This certainly corresponds to real life, in which the arrival time of humans in queues is erratic and can affect the dynamics of the queue. For each collection of individuals added to the queue, add the remaining utility of their QP—essentially $(t_i - q_i/d_i)$.
- Multiply each of the average utilities by the number of people entering the line, and add, to get approximate total net utility, and add these total net utilities to the total utility.
- Use proportions calculated and the aggregate size to determine the number of people entering a given line $l_i$, from the fluid members of each $\text{QP}_{i,s}$, (and consequently $P_{F,\text{NOQP}}$—these are added to $l_{i,\text{QP}}$), and from the non-QP fluid populations (which are added to $l_{i,\text{NOQP}}$). If the fluid members of $\text{QP}_{i,s}$ entering the line are forfeiting their QPs, add them to $l_{i,\text{NOQP}}$ instead, and remove that number of members from $\text{QP}_{i,s}$.
- Finally, use the proportions calculated to determine how many people are given a new QP, whether from the fluid non-QP population or from another QP population $\text{QP}_{j,s}$. Distribute this number among the $\text{QP}_i$ using the *assignment routine* of the scheme.

# QuickPass Schemes

We propose alternatives to the four factors below:

- **Free-Waiting-Time Clairvoyance** We allow people to know the free waiting time prior to getting a QP ticket, so they can better gauge whether to get a QP or to wait in line for the ride.
- **Cancellation Flag** People can get a new QP while a previous one is active; doing so deactivates the old QP.
- **Service Protocol** The QP line is served at a rate proportional to the length of the QP line, instead of at a constant rate.

- **Assignment Routine** A QP is *front-end-loaded* if the time interval on the ticket is the closest time interval with an available spot open (whether from cancellation or otherwise). Such loading is efficient but leads to anomalies such as two people getting QPs within minutes but the second person having a shorter wait. A QP is *queue-loaded* if it goes to the next time interval that is not fully populated and assigns a person to that time. This scheme is fair but does not take into account cancellations.

# Case Study

To test the schemes, we study the amusement park Six Flags Magic Mountain, in Los Angeles, CA. Even though Magic Mountain does not use the Fastlane technology (an electronic modified version of QP), many other Six Flags parks of comparable size and type do [Six Flags Theme Parks 2004]. We estimate $R_i$, $k_i$, $\mu_i$, $\sigma_i$, and $P_T$.

Magic Mountain has six rides with long waits during the busiest time, 3:00–4:00 [Ahmadi 1997]; we give these rides QP lines. We do not give QP lines to two other rides with medium-length lines, and we combine the other 20 rides into two generic rides with minimal utility, high $k_i$ (serving rates), $\mu_i = 5$, and $c_i = 50$. We calculate all $k_i$ data (except for FreeFall) from the Roller Coaster DataBase [Marden 2004]; Freefall's $k_i$ is calculated from the New Jersey Six Flags Freefall [Six Flags Great Adventure 2004]. Meanwhile, we estimate the expected utilities $\mu_i$ from the values for wait-line times between 3:00–4:00 [Ahmadi 1997]. Lastly, to introduce reasonable variation, we estimate the $\sigma_i$ to be one-fifth of the $\mu_i$. **Table 2** summarizes the values.

**Table 2.**

Line speeds and expected utility values for Case Study rides. The first group are lines with QP lines and the second group are without such lines.

| Ride | Name | $k_i$ (people/min) | $\mu_i$ (min) | $\sigma_i$ (min) |
|---|---|---|---|---|
| **QP** | | | | |
| $R_1$ | Ninja | 26.67 | 48 | 9.6 |
| $R_2$ | Colossus | 43.33 | 80 | 16 |
| $R_3$ | Flashback | 18.33 | 60 | 12 |
| $R_4$ | FreeFall | 3.75 | 72 | 14.4 |
| $R_5$ | Psyclone | 20 | 46 | 9.2 |
| $R_6$ | Viper | 28.33 | 54 | 10.8 |
| **Not QP** | | | | |
| $R_7$ | Goldrusher | 29.17 | 25 | 5 |
| $R_8$ | Revolution | 6.67 | 35 | 7 |
| $R_9$ | Generic Ride 1 | 50 | 5 | 1 |
| $R_{10}$ | Generic Ride 2 | 50 | 5 | 1 |

We also must estimate $P_T$. Magic Mountain has daily attendance from 9,000 to 35,000 [Ahmadi 1997]; we estimate an expected $P_T$ of 20,000 people/day. From surveys, we estimate $e = 0.2$ (ratio of the disutility for free waiting to line waiting) and $\kappa = 3$ (ratio of the time for free waiting to regular line waiting). We assume that $\delta = d_i \tau I$ (maximum number of QPs issued for $R_i$ in an interval), which is the number of people that the QP line can handle in one time interval.

## Criteria for Judging Schemes

- **Enjoyment:** Measured in minutes, this quantity is the net total utility. A QP system should enhance total utility.
- **Fairness:** Fairness plays a major role in people's satisfaction [Larson [1987]. In a fair system, a person obtaining a QP before someone else should be serviced first. We keep track of all relevant data by recording the allocation of QPs to different time intervals for each time step.

## Representative Schemes and Simulation Results

We choose five representative schemes (plus a control case based on the Aggregate-Decision Model), to which we apply our two criteria for success. The various features of the test schemes are indicated in **Table 3**.

**Table 3.**
Overview of properties of test schemes.

| Scheme # | 1 | 2 | 3 | 4 | 5 | Control |
|---|---|---|---|---|---|---|
| Waiting-time clairvoyance | No | No | No | Yes | Yes | Control |
| Cancellation allowed | No | No | Yes | No | Yes | Control |
| Service protocol (A: Const. Am't., P: Const. Prop'n) | A | P | A | P | P | Control |
| Assignment routine (Q: Queue, FE: Front-end) | Q | Q | FE | FE | FE | Control |

# Conclusions and Extensions of the Problem

## Solving the Problem

In **Figure 4** and in **Table 4** we give the results, in terms of our criteria of enjoyment and fairness, for the five representative schemes plus control case.

In enjoyment, Scheme 5 out-performs all of the other schemes (**Figure 4**). This result meshes with our expectations that increased knowledge and choices, as well as a more-efficient service protocol and assignment routine, result in higher utility.

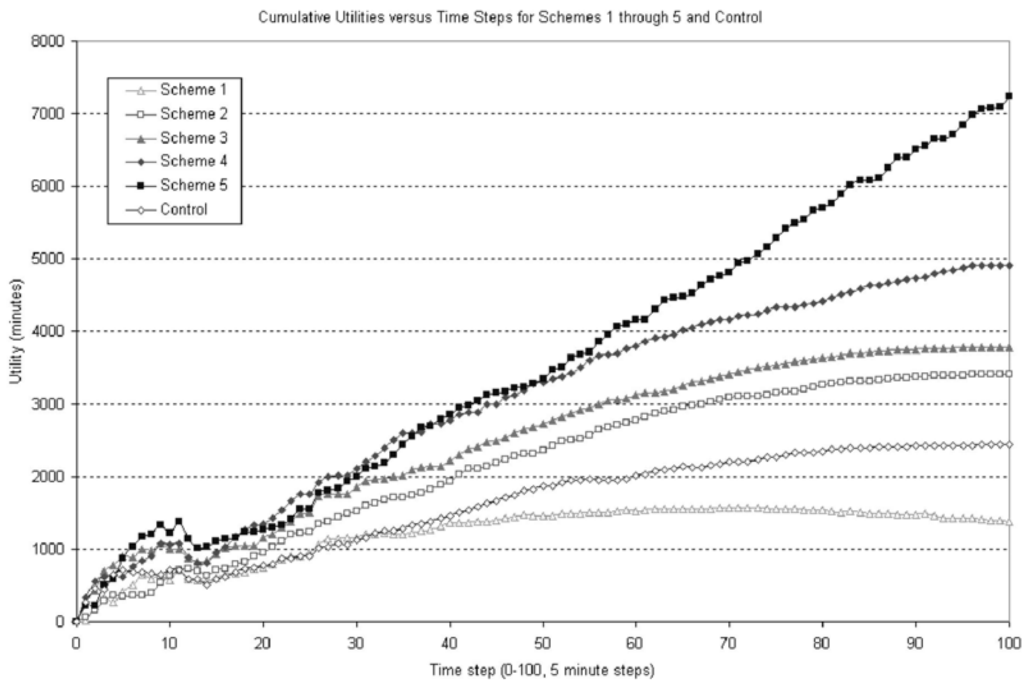



**Figure 4.** A graph demonstrating cumulative utilities and thus enjoyment levels associated with representative test schemes. Scheme 5 dominates after 50 time steps.

**Table 4.**

Overview of test scheme fairness as measured by QuickPass anomalies to assignments.

| Scheme # | 1 | 2 | 3 | 4 | 5 | Control |
|---|---|---|---|---|---|---|
| Ratio of QuickPass anomalies to QuickPass assignments | 0.00 | 0.00 | 0.36 | 0.03 | 0.04 | 0.00 |

Queue-loading systems do not allow anomalies in QP assignments and therefore are the most fair by that standard. In addition, to prevent unfair line-length distributions, we should use the constant-proportion service protocol. **Table 4** shows that all schemes except Scheme 3 result in about equal fairness levels.

On both criteria, Scheme 5 performs nearly as well as (if not better than) the alternatives. Ideally, then:

- the waiting time should be displayed, and
- people should be able to cancel a QP by activating another,
- the serving rate for the QP line should be proportional to the QP line length,
- QPs should be assigned via front-end loading (i.e., for the soonest time interval with available space).

## Further Study

Our results have a recurring theme: The more knowledge and greater number of choices an individual has, the higher the cumulative utility tends to be.

We propose that every ride should have an electronic display of the waiting times for normal and QP lines for *all* rides.

Our model does not allow QP selling and trading, but allowing it might improve cumulative utility and fairness, as discussed in Prado and Wurman [2002].

## Strengths of Model

The Aggregate-Decision model is a realistic and robust probabilistic framework that, for large population sizes, is statistically accurate at calculating aggregate behaviors. The utility functions are simple and realistic and reflect the processes of individual decision-making. The decisions, based on utility comparison, are reasonable and reflect the interest of the individual.

## Weaknesses of Model

The probabilistic nature of our model and related statistical flaws are the main source of weakness. Our model for the aggregate breaks down for small population sizes and small numbers of rides, where issues such as memory and changing preferences influence preference distributions. We assume that the distributions remain constant over time, but preferences change with experience (whether riding or waiting). Our model also assumes that preference distributions are independent of one another, while in reality we expect that more of the people who enjoy one roller coaster also enjoy a similar one.

Whereas most humans would plan out a series of rides rather than considering just one ride at a time, our model does not have such forethought capabilities.

Lastly, our assumptions on uniform distances neglect the actual geometric configuration of the park and the effect of that geometry on ride considerations.

# Conclusion

Our model takes into account the limited knowledge that influences the decisions of park-goers, based on economic assumptions. We apply our understanding of individual decision-making to develop a versatile model of aggregate decision-making for parks with and without QPs.

We tested different QP schemes systems using data from the Six Flags Magic Mountain. Factors in a QP scheme include:

- whether an individual has foreknowledge of the time interval for which a QP will be issued;
- whether people can cancel an active QP and obtain a new one;
- how people are fed onto the ride from the queues; and
- how QP times are allocated.

Our criteria for successful schemes were:

- Cumulative utility, summed over all people throughout the entire day, from taking rides and waiting in lines.
- Fairness, as measured by the ratio of the number of anomaly QP allocations to the total number QPs.

We compared QP schemes and found that the scheme with the greatest utility has the following properties:

- People have foreknowledge of when QPs are being issued for (perhaps by way of an electronic sign posting this information),
- people can cancel their QPs by switching to another QP,
- the QP line moves at a rate proportional to its length, and
- when people are allocated QP tickets, they receive tickets for the first available time interval.

This scheme provides a cumulative utility (measured in minutes) of 7,000, while the next highest cumulative utility was only 5,000 (the control had cumulative utility of 2,500 minutes). This scheme had an anomaly-to-allocation ratio of 0.04, while other schemes had values as high as 0.36.

# References

Acklam, Peter J. 2004. An algorithm for computing the inverse normal cumulative distribution function. `http://home.online.no/~pjacklam/notes/invnorm/` .

Ahmadi, Reza H. 1997. Managing capacity and flow at theme parks. *Operations Research* 45 (1): 1–13. `http://www.anderson.ucla.edu/x3241.xml` .

Browne, John M. Probabilistic Design Course Notes. 2001. `http://www.ses.swin.edu.au/homes/browne/probabilisticdesign/coursenotes/` .

Larson, Richard C. 1987. Perspectives on queues: Social justice and the psychology of queueing. *Operations Research* 35 (6) (November–December 1987): 895–905.

Levine, Arthur. 2004. Participating Six Flags Parks. http://themeparks.about.com/ .

Marden, Duane. 2004. Roller Coaster DataBase. http://www.rcdb.com/ .

Prado, Jorge E., and Peter R. Wurman. 2002. Non-cooperative planning in multi-agent, resource-constrained environments with markets for reservations. *AAAI 2002 Workshop on Planning with and for Multiagent Systems*, Edmonton, July–August, 2002, 60–66.

Press, William H., Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling. 1992. *Numerical Recipes in C: The Art of Scientific Computing*. New York: Cambridge University Press.

Six Flags Theme Parks. 2003. Six Flags Great Adventure. http://www.sixflags.com/parks/greatadventure/index.asp .

Weisstein, Eric. n.d. Central Limit Theorem. *Mathworld*. http://mathworld.wolfram.com/CentralLimitTheorem.html .

# Theme-Park Queueing Systems

Alexander V. Frolkin
Frederick D.W. van der Wyck
Stephen Burgess
Merton College, Oxford University
Oxford, England

Advisor: Ulrike Tillmann

## Summary

We determine an optimal system for allocating QuickPasses (QPs) to theme-park guests, subject to key criteria that we identify. We recommend a specific system—a way of deciding when a guest asks for a QP whether they should get one, and if so for what time. On the other hand, we warn against some plausible systems that would actually worsen the queueing situation. We also explain why some theme parks use an *unfair* way of allocating QPs, where late guests can fare better than early arrivals.

The keys to our approach are two very different simulations with the same parameters. The *Excel simulation* breaks the day into 10-min intervals, works with groups of people, and is nonrandom. It is fast and allows us to test quickly many different QP allocation systems. The *Perl simulation* breaks the day into 1-min intervals, models individual people, and includes randomness. It is more realistic and flexible. Thus, the simulations are useful in different contexts. The fact that their results are consistent provides a strong safeguard against incorrect results caused by coding errors, a risk in large simulations. In addition, we carry out extensive tests of the stability of our model and the robustness of our recommendation.

We conclude that it is best to allocate lots of QPs for slots early and late in the day, and fewer for the peak demand in the middle.

We also explore modifications to the QP concept, including charging.

---

*The UMAP Journal* 25 (3) (2004) 301–318. 

# Introduction

## The Problem

Theme parks have introduced two basic types of new queueing systems:

- In *virtual queue* systems, guests use a pager to register in a queue; it pages when "they" are near the head of the queue and should come to the ride. For example, Lo-Q Plc. has developed such a system, used in Six Flags theme parks in the U.S. [Six flags ... 2002].
- In *QuickPass* systems, rides can issue guests a QuickPass, allowing them to return to the ride at a specified time, when they can ride with minimal queueing. Examples of this type of system are Disney's FastPass® [Disney Tickets 2004] and Universal Studios Express Pass [Universal Studios n.d.].

Either type of system may or may not be free; Disney's FastPass® is free but guests pay for Lo-Q's pagers.

We focus on the QuickPass system and analyse how to implement it effectively. We conclude with a brief comparison with virtual queue systems.

## Criteria for a Good QuickPass system

We take the following as general guidelines in choosing a QuickPass system.

- At no time should more than 50% of a ride's capacity be QP users.
- No ride should have a queue longer than 45 min.
- The average waiting time should be as short as possible.
- Waiting times should be evenly distributed. (It is better that 100 people wait 20 min than that 50 people have no wait but another 50 wait 40 min.)
- The system should seem fair. People arriving later should not get QPs when previous guests have been refused them. Similarly, people arriving later should not be allocated earlier slots than previous guests.
- QPs should not be allocated for more than 4 h in the future. (We assume, based on personal experience, that people stay for only about 5 h.)

## Summary of Our Approach

- We collect data and perform calculations to obtain reasonable initial modelling assumptions.

- We construct two computer simulations and modify them with further assumptions. Eventually we find the behaviour agreeable with common sense and consistent between the two simulations.
- We use two very different simulations with the same parameters. The *Excel simulation* breaks the day into 10-min intervals, works with groups of people, and is nonrandom. The *Perl simulation* breaks the day into 1-min intervals, models individual people, and includes randomness. Each approach has its advantages. The fact that the two simulations give similar conclusions provides very powerful evidence for the validity of the conclusions.
- We list systems for allocating QPs that seem likely to work well and test them using the simulations.
- We analyse the results with graphical interpretations and summary statistics.
- We assess the stability of our model under variations in input parameters and the robustness of our recommendation under differing conditions.

# The Simulation Process

## Initial modelling assumptions

- We have in mind as an example a particular theme park, Thorpe Park, Surrey, UK, which we regard as a typical medium-sized park.
- We model a day running from 8 A.M.–6 P.M. The number of people in the park varies over the day and has a key impact on queue lengths.
- Thorpe Park has 2 million visitors per year, and the park is open for around 200 days per year [Thorpe Park Guide n.d.]. So we assume that 10,000 people visit the park on a typical day. Most arrive late morning or early afternoon and admissions stop well before closing time so that queues can subside.
- People who arrive early typically stay for about 5 h; later arrivals stay until a short time before closing.
- There is one overwhelmingly popular ride, the *Big Ride*, the only ride for which we issue QPs.
- We estimate from personal experience that popular rides take 40 people and leave every 5 min, so the Big Ride has a capacity of 8 people/min.
- We offer QPs for free. Since there is then no harm in taking a QP, whenever they are available, people take them.
- We do not give guests more than one QP at a time.

- For simplicity, we ignore the effect of people going around in groups; all guests behave independently of one another. The effect of grouping would be insignificant, because the size of each group is small compared to the total number of people.

## The Perl Simulation

The Perl simulation breaks the day into 1-min intervals, models individual people, and includes randomness. To implement it, we need only the following further very reasonable assumptions:

- We model arrivals as a Poisson process with rate $\lambda(t)$ per minute, where $\lambda(t)$ varies with the time of day:

| | | | |
|---|---|---|---|
| 8 A.M.–11 A.M.: | 13 | 11 A.M.–3 P.M.: | 27 |
| 3 P.M.–4.30 P.M.: | 13 | 4.30 P.M.–6 P.M.: | 0 |

  We chose these numbers roughly in accord with personal experience with the aim of making the number of arrivals per day about 10,000, consistent with the modeling assumption.

- We model departures as follows:
  - If someone arrives before 12.30 P.M., their length of stay (in hours) is a normally distributed $\mathrm{N}(5, 0.5^2)$ random variable.
  - If someone arrives after 12.30 P.M., their length of stay is a $\mathrm{N}(k, 0.25^2)$ random variable, where $k$ is the time between arrival and 5.30 P.M.. (So late arrivals typically leave at 5.30 P.M. and 97.5% of them leave by 6 P.M.).

## The Excel Simulation

The Excel simulation breaks the day into 10-min intervals, works with groups, and uses expected values and no randomness. So it requires more-significant further assumptions:

- We model guests as either on the Big Ride, queueing for the Big Ride, or elsewhere. So we do not worry about the effect of other rides.

- We adopt the following distribution of arrivals every 10 min to implement the arrival behaviour described above:

| | | | |
|---|---|---|---|
| 8 A.M.–11 A.M.: | 130 | 11 A.M.–3 P.M.: | 270 |
| 3 P.M.–4.30 P.M.: | 130 | 4.30 P.M.–6 P.M.: | 0 |

- We adopt the following departure distribution, in departures per 10 min, to implement the departure behaviour described above:

| | | | |
|---|---|---|---|
| 8 A.M.–12.30 P.M.: | 0 | 12.30 P.M.–3 P.M.: | 130 |
| 3 P.M.–4.30 P.M.: | 270 | 4.30 P.M.–6 P.M.: | 270–750 (incr. linearly) |

  The spreadsheet medium was not suited to modelling departures in the more realistic manner of the Perl simulation.

At the end of the day, queues shut at 6 P.M. and people then in the queues get a last ride after 6 P.M.

## Baulking

The initial simulations generated implausibly large queues, some more than a day long! We realised that we needed to introduce *baulking*—people being put off by long queues. Also, people with QPs for a ride should be much more easily discouraged from queueing for it than people without. (But if there is a small queue, QP holders *should* want to join it for an early extra ride.) We experimented with linear, exponential, and polynomial baulking models in both simulations, and found that the most realistic behaviour is given by using:

- for non-QP-holders: the inverse quartic model,

$$\mathrm{P}(\text{choosing to queue}) = \left(1 + \frac{qd}{40(1+p)c}\right)^{-4},$$

  where

  $q$ is the queue length (people),

  $d$ is the ride duration (minutes),

  $c$ is the ride capacity (people taken each time the ride runs), and

  $p$ is the relative popularity of the ride (the probability of choosing it).

  **Figure 1** shows a graph from this function family. Any guest will join an empty queue, but only one-sixteenth are prepared to wait 40 min. The adjustment factor $(1 + p)$ makes people more likely to persevere for more popular rides.

- for QP-holders: the linear baulking model,

$$\mathrm{P}(\text{choosing to queue}) = \max\left(0,\ 1 - \frac{qd}{15c}\right).$$

  An example graph is shown in **Figure 2**. Here, any guest will join an empty queue but none are prepared to wait for more than 15 min, irrespective of the popularity of the ride. (After all, they have a QP to come back later.)

Finally, we found a problem in the Perl simulation. All rides in the simulation ran for 5 min and were "in-sync." The result was an implausible 5-min-periodic behaviour in the Big Ride queue. Removing the synchronicity by staggering the ride departures (some rides at 8:01, 8:06, 8:11, . . . ; others at 8:02, 8:07, 8:12, . . . ) eliminated this problem.

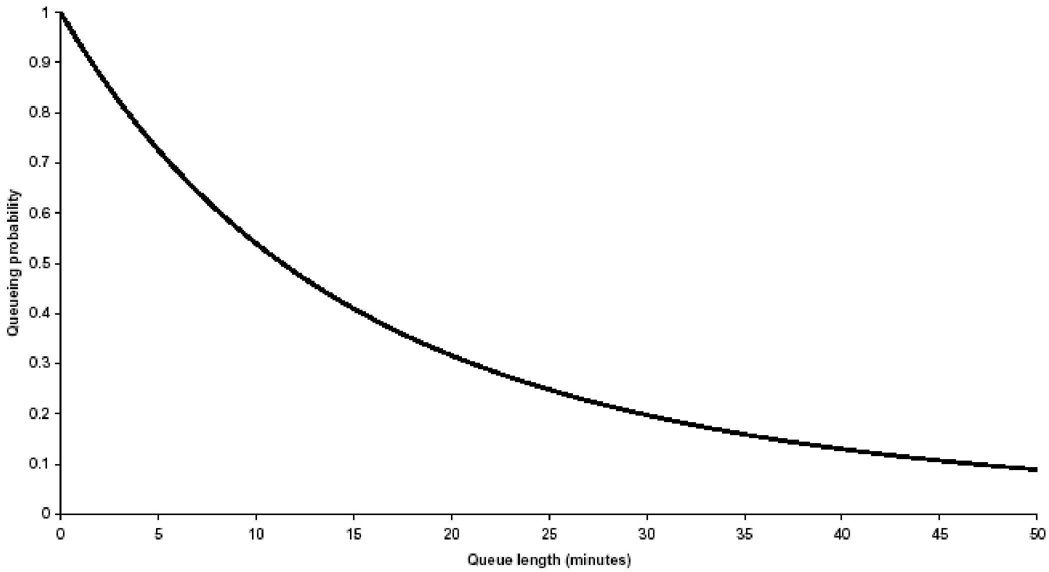



**Figure 1.** An inverse quartic baulking function.

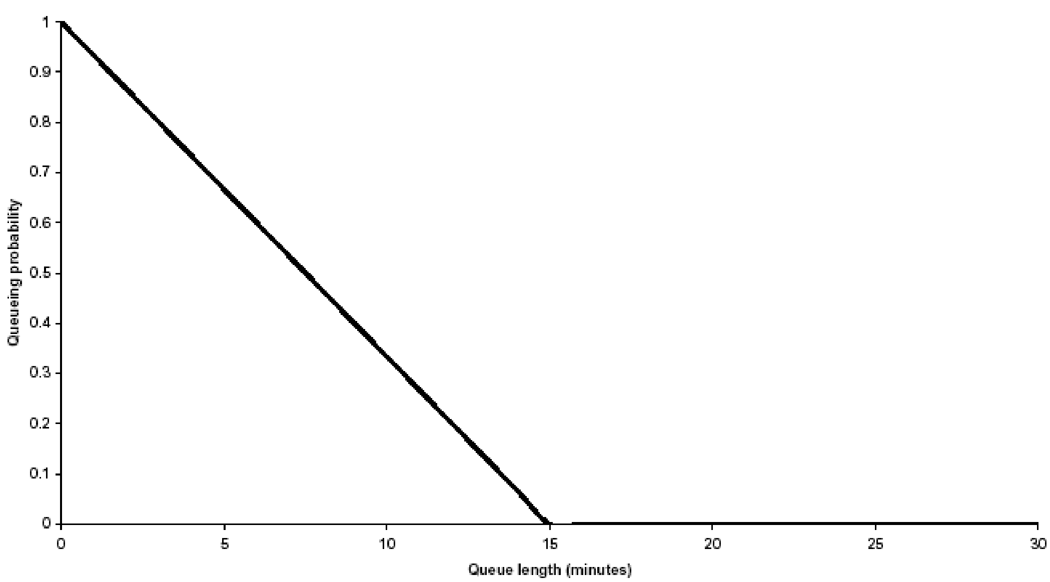



**Figure 2.** A linear baulking function.

## Allocation Systems

A QP system can be represented by an *allocation matrix*, a 10 $\times$ 10 matrix indicating the maximum number of QPs that can be issued by the end of a given hour for a particular future hour. (At the detail level of our simulations, a finer-grained system would make no difference.)

For example, the matrix in **Figure 3** indicates that by the end of the 08:00–09:00 hour, *at most* 100 QPs are issued for slots starting during the 11:00–12:00 hour and at most 50 for slots starting during the 12:00–13:00 hour. We always give out the earliest slots available at a given time, so that the system is seen to be fair.

| | 08:00 | 09:00 | 10:00 | 11:00 | 12:00 | 13:00 | 14:00 | 15:00 | 16:00 | 17:00 |
|---|---|---|---|---|---|---|---|---|---|---|
| 08:00 | | 0 | 0 | 100 | 50 | 0 | 0 | 0 | 0 | 0 |
| 09:00 | | | 0 | 100 | 50 | 0 | 0 | 0 | 0 | 0 |
| 10:00 | | | | 100 | 50 | 0 | 0 | 0 | 0 | 0 |
| 11:00 | | | | | 50 | 0 | 0 | 0 | 0 | 0 |
| 12:00 | | | | | | 0 | 0 | 0 | 0 | 0 |
| 13:00 | | | | | | | 0 | 0 | 0 | 0 |
| 14:00 | | | | | | | | 0 | 0 | 0 |
| 15:00 | | | | | | | | | 0 | 0 |
| 16:00 | | | | | | | | | | 0 |
| 17:00 | | | | | | | | | | |

**Figure 3.** Sample allocation matrix.

For a different example, the matrix in **Figure 4** indicates that by the end of the 08:00–09:00 hour, at most 100 QPs are issued for 13:00 slots, at most 150 for 14:00 slots, and so on. The values in a column are cumulative; so if 100 people take QPs between 08:00 and 09:00, only $150 - 100 = 50$ remain for people arriving between 09:00 and 10:00.

| | 08:00 | 09:00 | 10:00 | 11:00 | 12:00 | 13:00 | 14:00 | 15:00 | 16:00 | 17:00 |
|---|---|---|---|---|---|---|---|---|---|---|
| 08:00 | | 0 | 0 | 0 | 0 | 100 | 0 | 0 | 0 | 0 |
| 09:00 | | | 0 | 0 | 0 | 150 | 0 | 0 | 0 | 0 |
| 10:00 | | | | 0 | 0 | 200 | 0 | 0 | 0 | 0 |
| 11:00 | | | | | 0 | 250 | 0 | 0 | 0 | 0 |
| 12:00 | | | | | | 300 | 0 | 0 | 0 | 0 |
| 13:00 | | | | | | | 0 | 0 | 0 | 0 |
| 14:00 | | | | | | | | 0 | 0 | 0 |
| 15:00 | | | | | | | | | 0 | 0 |
| 16:00 | | | | | | | | | | 0 |
| 17:00 | | | | | | | | | | |

**Figure 4.** Another example of an allocation matrix.

Notice some features of the matrix:

- The lower triangular entries are irrelevant—we cannot issue QPs for the past, and we allocate them only for at least an hour in the future.
- The columns are nondecreasing because the entries are cumulative.

- If a column is strictly-increasing, the system is not *fair*. In **Figure 4**, if 120 people want QPs between 08:00 and 09:00, the last 20 will be refused them. But there will still be 50 available for people arriving between 09:00 and 10:00—later. Similarly, consider the matrix in **Figure 5**. Now if 120 people want QPs between 08:00 and 09:00, the last 20 get 14:00 slots, while the first 50 people arriving during 09:00–10:00— later–get 13:00 slots.

| | 08:00 | 09:00 | 10:00 | 11:00 | 12:00 | 13:00 | 14:00 | 15:00 | 16:00 | 17:00 |
|---|---|---|---|---|---|---|---|---|---|---|
| 08:00 | | 0 | 0 | 0 | 0 | 100 | 20 | 0 | 0 | 0 |
| 09:00 | | | 0 | 0 | 0 | 150 | 20 | 0 | 0 | 0 |
| 10:00 | | | | 0 | 0 | 200 | 20 | 0 | 0 | 0 |
| 11:00 | | | | | 0 | 250 | 20 | 0 | 0 | 0 |
| 12:00 | | | | | | 300 | 20 | 0 | 0 | 0 |
| 13:00 | | | | | | | 20 | 0 | 0 | 0 |
| 14:00 | | | | | | | | 0 | 0 | 0 |
| 15:00 | | | | | | | | | 0 | 0 |
| 16:00 | | | | | | | | | | 0 |
| 17:00 | | | | | | | | | | |

**Figure 5.** A third allocation matrix.

## How the Perl Simulation Works

The Perl simulation works in discrete time steps (usually 1-min long). Each person is assigned one of the *states*: Walking around, Queueing, Queueing with QuickPass, Riding, or Gone home.

At each time step:

- New arrivals are added to the people in the park; departures have their state set permanently to "Gone home."
- For each person in the park:
  - If the person is walking or has just entered, they check if they have overstayed their staying time, and if so, leave. If they have a QP for the current time, they move to the front of the queue for that ride; if not, they carry on walking with probability 0.8. (This is based on a geometric distribution with mean 5 min.) Otherwise, they choose a ride based on the rides' popularities and proceed as follows.
    - If the ride does not have QPs, they decide whether to queue or to carry on walking, based on the queue length, using the inverse quartic baulking model.
    - If the ride does have QPs, they try to get a QP if possible (i.e., if available and they do not already have a QP).
      - If they obtain a QP, they decide whether to queue for the ride or carry on walking, using the linear baulking model.

        - If they do not obtain a QP, they decide whether to queue or carry on walking, using the inverse quartic baulking model.
    - If the person is queueing for a ride, they check if they have a QP for the current time, in which case they will leave their current queue and move to the front of the queue for their QP ride.

- For each ride, carry out the following:
    - If it is time for the ride to take more people, all people on the ride are taken off (and put into the walking state), and the maximum number of people from the front of the queue (i.e., the ride's capacity, or the number of people queueing, if that is less) are put on the ride.

The Perl simulation allows the QP system to be easily modified, by simply changing the allocation matrix entries. In addition, all the following parameters can easily be adjusted:

- length of a time slot in the QP allocation matrix;
- length of a day;
- the function $\lambda(t)$ giving the mean number of people entering at time $t$;
- the function $\mu(t)$ giving the mean duration of a stay for a person entering at time $t$;
- the function $\sigma(t)$ giving the standard deviation of stay durations for people entering at time $t$;
- the probability that a walking person carries on walking at the next time step;
- the number of rides,
- for each ride, the capacity, the duration, the popularity, and the start-time delay (e.g., whether the ride runs at $0, 5, 10, \ldots$ or at $2, 7, 12, \ldots$ time units).

## How the Excel Simulation Works

The Excel simulation tabulates the number of people in different places at 10-min intervals. At each time, it knows:

- the number of people entering, the number of people leaving, and the total number of people in the park;
- the number of people in the queue for the Big Ride and hence its length;
- the number of people on the Big Ride;
- the number of people elsewhere without a QP for the Big Ride;

- the number of people elsewhere with a QP for the Big Ride; and
- the number of QPs issued for this slot and hence the remaining capacity on the Big Ride for this slot.

The number of people entering and leaving at each step is specified in advance. The main calculation at each step involves the following:

- Add people coming off the Big Ride from the previous step to "elsewhere."
- Check how many QPs can be issued now for slots later in the day and give them out to anybody elsewhere without a QP who is interested in the Big Ride (fixed interest level, no baulking); change their status to having a QP.
- Check how many people are in the queue for the Big Ride.
- Deduce via the appropriate baulking models what fraction of people elsewhere without a QP are willing to join the queue for the Big Ride, and what fraction with a QP are nonetheless willing to queue for an early go on the Big Ride. Add these people to the queue.
- Remove from the queue people taking the Big Ride without QPs.

The QP system can be easily modified (by changing the allocation matrix entries) and other parameters can be adjusted, but there is less flexibility than in the Perl simulation.

# Results and Interpretation

## Comparing the Two Simulations

We first ran the two simulations with no QP allocation (system QP1), to calibrate the simulations; in particular, this involved setting the Big Ride popularity parameter appropriately. The Perl simulation gave generally longer wait times, but the overall qualitative behaviour of the simulations was the same.

**Figure 6** shows the queue profile from one run of the Excel simulation, and **Figure 7** shows a waiting frequency barchart, averaged over 3 runs of the simulation. The Perl simulation generates similar results, but people tend to wait a little longer. In both simulations, the queue builds up over the day. It is long over several hours, so many people get a long wait.

After the initial calibration, we did *not* modify the parameters; nonetheless, the simulations continued to give similar results, so we felt justified in using only the Excel simulation for testing allocation strategies.

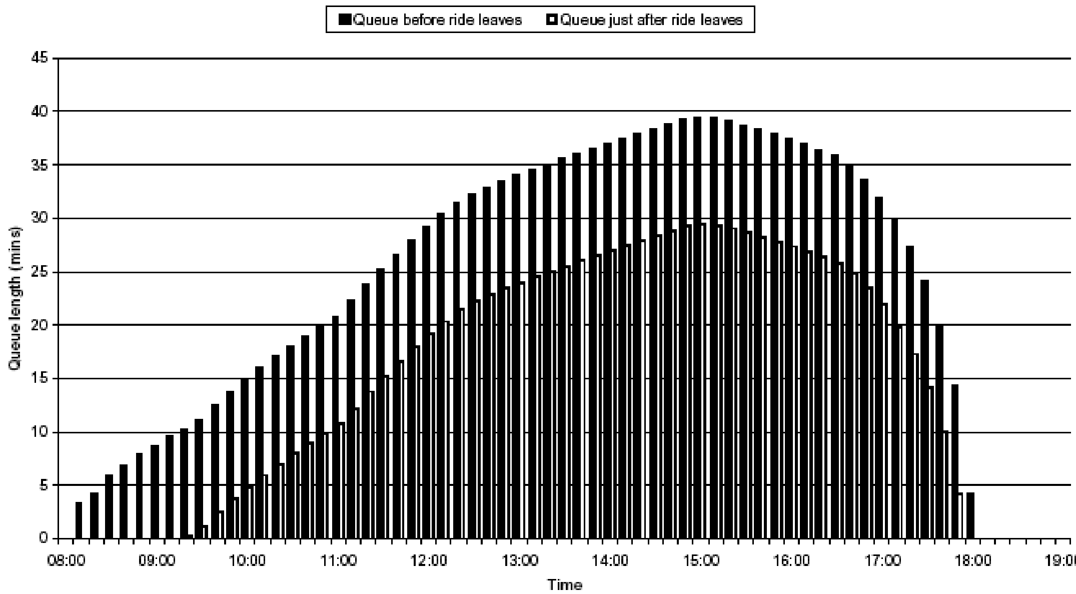



**Figure 6.** Results of run of Excel simulation for no QuickPasses: queue profile.

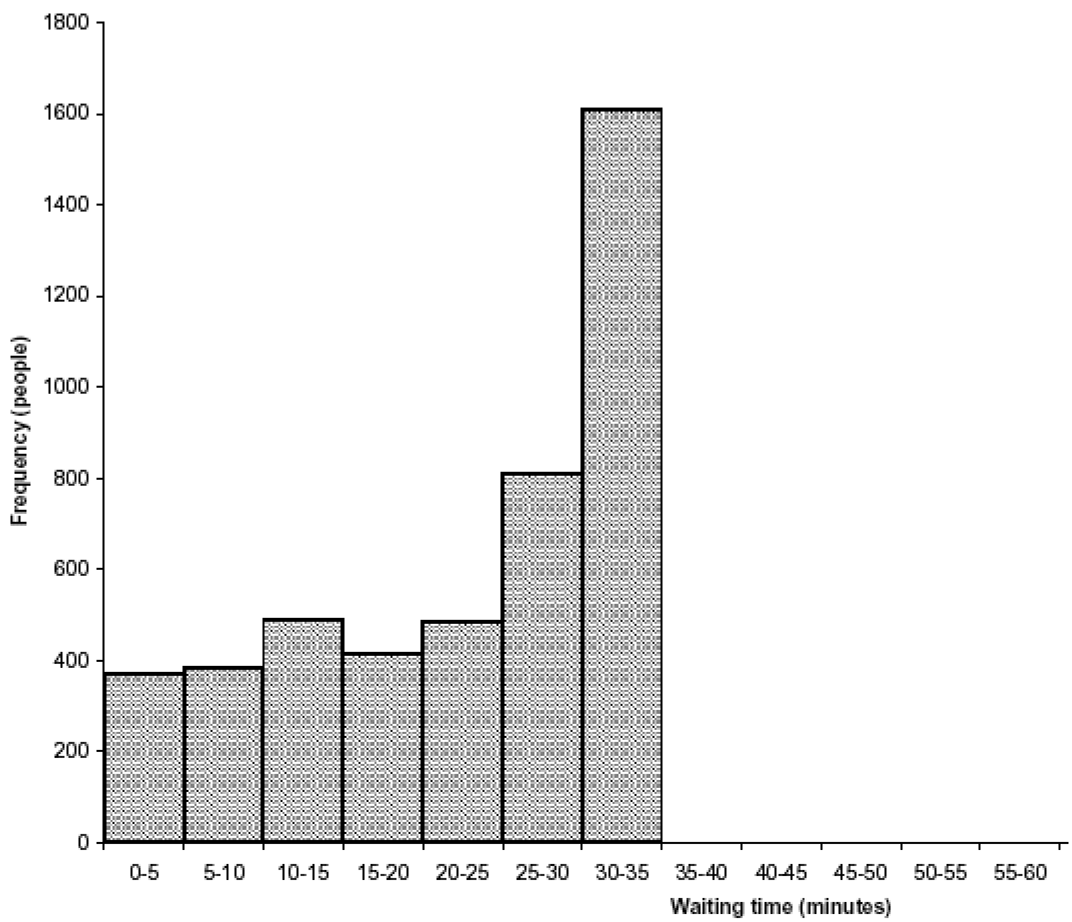



**Figure 7.** Results of run of Excel simulation for no QuickPasses: waiting frequency.

## Key Results

We tried several models:

- System QP2 allocates the full capacity of the ride to QPs, giving them out as soon as people want them. The queue for the Big Ride becomes huge, and many people have to wait a *very* long time, because no capacity remains for non QP-holders. This is a very bad way of allocating QPs.

- System QP3 allocates only half the capacity of the ride in QPs but still gives them out as soon as people want them. Now the queue stays short while QPs are issued (because people take a QP and tend to go away); but as soon as the QPs run out, the queue grows large, because the ride's capacity has been halved. Many people get a long wait. Things are worse than with no QPs, because the system shortens queues early (when they are short anyway), then reduces capacity at the busy time.

- System QP4 allocates QPs only up to 3 h in the future. This is better. A lot of people have a short wait and not many have an excessive one.

- System QP5 uses the allocation matrix in **Figure 8** to issue QPs. It gives out many QPs for slots early in the morning and late afternoon, in a way carefully adapted to the arrival distribution. This system gives the best results, shown in **Figures 9** and **10**. The queue is roughly constant for much of the day, many people have a short wait (the QP users!), and nobody waits more than 40 min.

| | 08:00 | 09:00 | 10:00 | 11:00 | 12:00 | 13:00 | 14:00 | 15:00 | 16:00 | 17:00 |
|---|---|---|---|---|---|---|---|---|---|---|
| 08:00 | | 240 | 120 | 60 | 30 | 0 | 0 | 0 | 0 | 0 |
| 09:00 | | | 240 | 120 | 60 | 30 | 0 | 0 | 0 | 0 |
| 10:00 | | | | 240 | 90 | 60 | 0 | 0 | 0 | 0 |
| 11:00 | | | | | 120 | 90 | 0 | 0 | 0 | 0 |
| 12:00 | | | | | | 120 | 0 | 0 | 0 | 0 |
| 13:00 | | | | | | | 0 | 0 | 30 | 0 |
| 14:00 | | | | | | | | 0 | 60 | 60 |
| 15:00 | | | | | | | | | 120 | 120 |
| 16:00 | | | | | | | | | | 240 |
| 17:00 | | | | | | | | | | |

**Figure 8.** Allocation matrix for scheme QP5.

## Recommendation

We recommend System QP5; it gives a lot of people a queue-free ride, while otherwise leaving the queue situation not much worse than without QPs. This is really the best we can hope for, because QPs cut normal capacity and so tend to worsen the normal queueing situation. Here is how QP5 performs on our criteria:

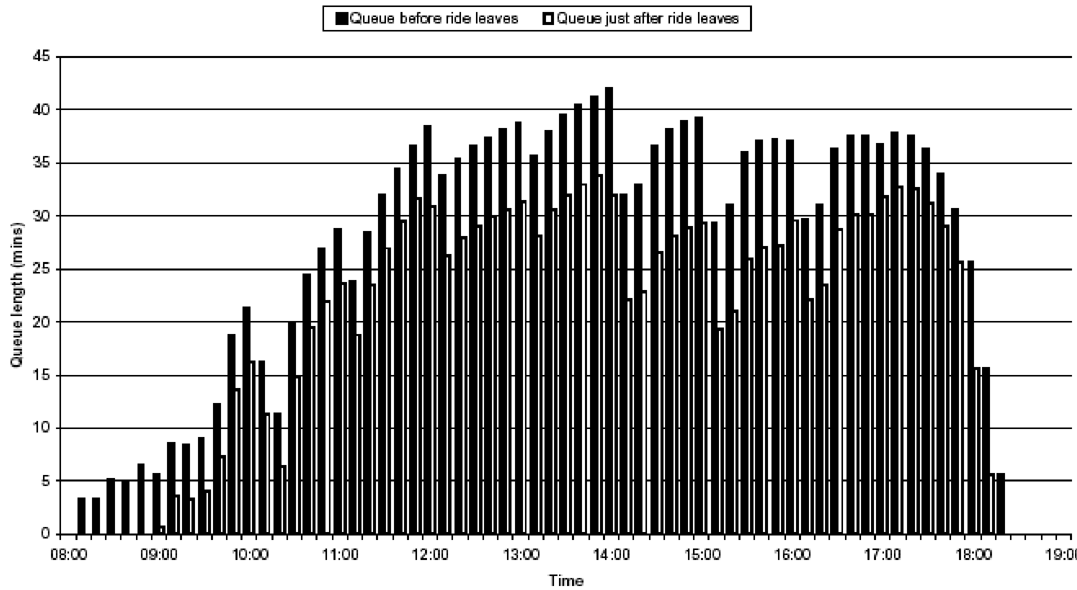



**Figure 9.** QP5: queue profile.

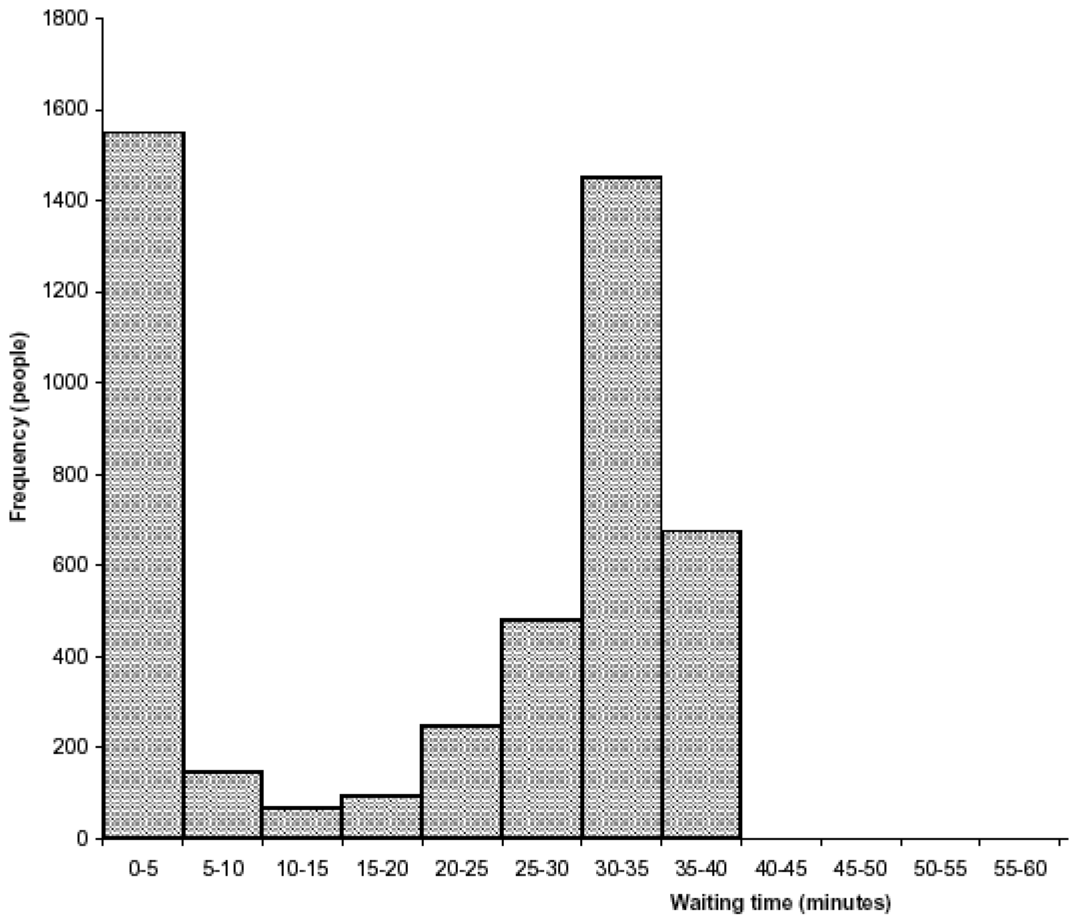



**Figure 10.** QP5: waiting frequency.

- No more than 50% of the ride is ever filled with QP users. (GOOD)
- The Big Ride never has a queue longer than 45 min. (GOOD)
- The average waiting time is similar to without QPs. (OK)
- Waiting times are not very evenly distributed, although better than in some systems. (POOR)
- The system is not fair. (POOR)
- QPs are not allocated for slots far in the future. (GOOD)

# Assessment

## Stability

### Overview

A good model should be stable: A small variation in the input parameters should cause a small change in the results. In addition, the direction of the change should usually be consistent with common sense.

We varied the following parameters to see how much they changed results and whether the change matched what we expected.

- relative popularity of the rides: We increased the popularity of the less popular rides (but kept them less popular than the Big Ride). This had the expected effect of decreasing the queue lengths.
- total number of rides: We removed 10 of the less popular rides (out of 20 rides). This had the expected effect of increasing the queue lengths by 50%.
- probability of continuing to walk around: We increased it from 0.8 to 0.95. This had the effect of decreasing the queue length, because people were now more likely to carry on walking, so fewer people queued for rides.
- baulking models: We changed the fourth power in the inverse quartic model to the sixth power and changed the 15-min cutoff in the linear model to 20-min. The maximum queue length decreased from 50 min to 35 min.
- arrival rate: We changed the arrival rate for the first hour from 13 people/min to 50 people/min. This made the queue length grow much more steeply at the start of the day, as expected.

### Conclusion

The results of the tests above are all favourable. The model is stable and responds in the expected way to changes.

As a further check, we calculated the standard deviation of the mean waiting times from 40 runs of the Perl simulation of systems QP1 and QP5. These were 0.46 min and 0.39 min, respectively, which is again favourable, as it indicates little change between the runs.

## Robustness

We checked the impact of big changes in the distribution of arrivals at the park (caused, for example, by weather conditions).

We modified the arrival distribution to reflect a typical weekend day, or a day with very good weather. Queues lengths increased significantly in both cases, but were only slightly worse with QPs, which is good.

Next, we modified the arrival distribution to reflect a day with bad weather in the morning—most arrivals after lunch. Again, the QP system coped with the change.

So our recommendation is quite robust.

# Improvements and Extensions

In this section we discuss possible improvements to our model and extensions to the QP system.

### Guests with Memory

Our model assumes that a guest's behaviour at a given time is independent of their previous behaviour. In fact, people don't go on the same ride time and again. We could modify the Perl simulation to account for this: the more times a person takes a ride, the less likely they are to take it again (except perhaps after their first go, which could actually encourage them). The Excel simulation's structure makes it impossible to implement this change.

### Finer-Grained QuickPass Allocation

A larger QP allocation matrix could be used; this would be easy to implement, but such a level of detail would be incompatible with the limited accuracy of the simulations.

### Charging for QuickPasses

Charging for QPs would increase revenue to the park and (by reducing QP uptake) leave more capacity on the Big Ride for the normal queue.

We model the extent to which cost deters guests. If we make the probability of a guest paying for a QP constant, the net effect is the same as issuing fewer QPs.

A more plausible model has long queues making people more willing to pay; **Figure 11** gives an example graph for a function from the family

$$\mathrm{P}(\text{pay for QuickPass}) = \min\left[1,\ \frac{1}{k}\log_{10}\left(1+\frac{qd}{2c}\right)\right],$$

where

$q$ is the queue length (people),

$d$ is the ride duration (minutes),

$c$ is the ride capacity (people taken each time the ride runs), and

$k$ is an arbitrary constant, initially set to 2.

In the morning, fewer QPs sell; later, when there are long queues, they all sell.

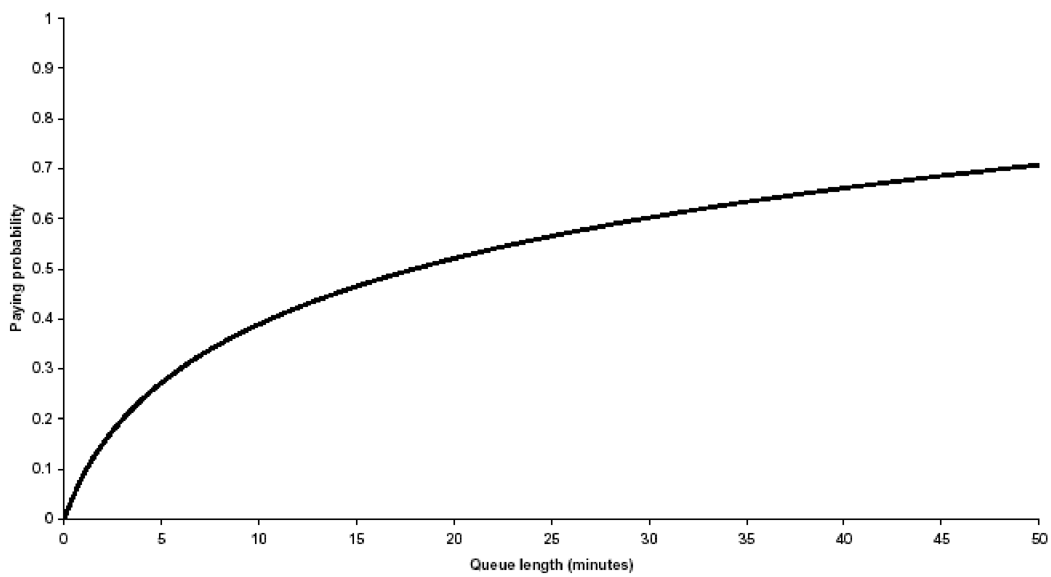



**Figure 11.** Logarithmic cost-reaction model: probability of paying for a QuickPass vs. queue length.

A bigger effect could be achieved by charging more, corresponding to increasing $k$. In principle, we could determine how $k$ varies with the cost of a QP; then running the simulation with different values of $k$ and recording the number of QPs sold, one could maximise revenue. **Figure 12** shows the response of sales to price, *assuming that a price of £$p$ per QP gives $k = p$.*

The graph's slope is sufficiently shallow that the highest price gives the highest revenue. However, the assumption that a price of £$p$ gives $k = p$ is unrealistic; a subtler model is needed.

At any rate, charging certainly improves the queueing situation as well as raising money. On the other hand, it reduces guest satisfaction, a major drawback.

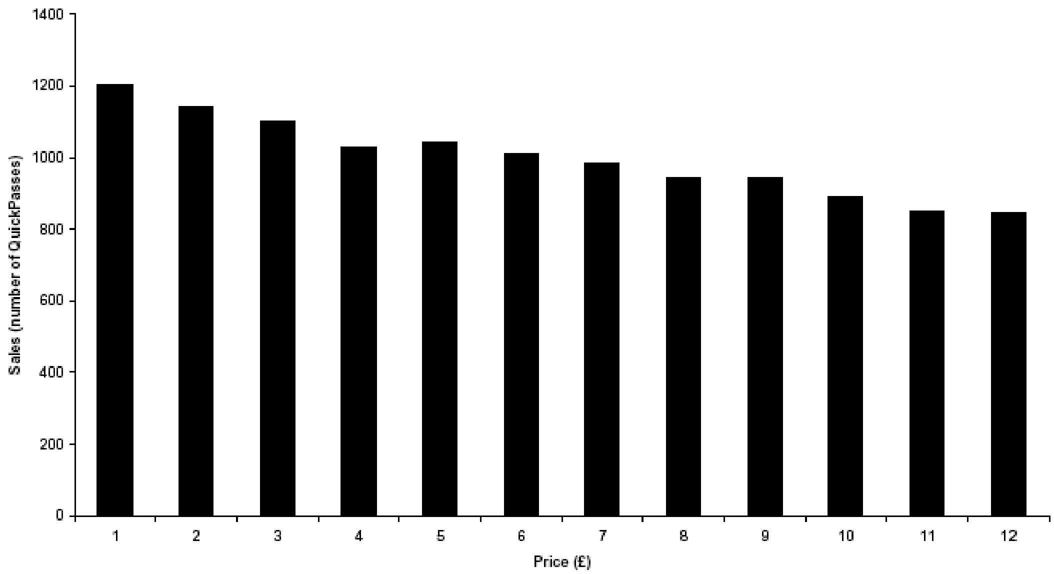



**Figure 12.** Sales of QuickPasses vs. price.

### More than One Ride Offers QuickPasses

We modified the Perl simulation to test what happens if QPs are issued for more than one ride (but never more than one at a time per guest); the behaviour is much the same as for QPs for only one ride. So we recommend the use of QPs for all popular rides.

## More than One QuickPass at a Time

Issuing people more than one QP at a time would require a sophisticated allocation system to avoid clashes, but the potential benefit is to move the system toward full scheduling and efficiency. We did not have time to test this idea.

# Conclusion

## Review of Our Approach

Our two different simulations provide a strong safeguard against incorrect results. Extensive testing of the stability of the model and the robustness of our recommendation provides further support for our conclusions.

On the other hand, our model has limitations. Treating guests as having a memory and using a finer-grained QP allocation matrix are important possible improvements.

## Key Conclusions

We recommend a QP allocation system that allows many people a queueing-free ride, without causing the normal queue to grow too much.

However, early guests may be given QPs for a late slot, or even refused a QP, while later arrivals fare better. Our investigations show that unfairness is crucial to a good QP system.

# References


Disney Tickets. 2004. http://www.disney-ticketshop.com/ .

Grimmett, G., and D. Stirzaker. 2001. *Probability and Random Processes*. Oxford, UK: Oxford University Press.

Grimmett, G., and D. Welsh. 2001. *Probability: An Introduction*. Oxford, UK: Oxford University Press.

Six Flags brings Lo-Q technology to nine parks for 2002. 2002. http://www.ultimaterollercoaster.com/news/stories/021402_03.shtml .

Thorpe Park Guide. http://www.thorpeparkguide.com/ .

Universal Studios. n.d. Universal Studios discount tickets. http://universalstudios-orlandoflorida.com/universalexpress.htm.

# Developing Improved Algorithms for QuickPass Systems

Moorea L. Brega
Alejandro L. Cantarero
Corry L. Lee
University of Colorado at Boulder
Boulder, CO

Advisor: Bengt Fornberg

## Summary

We model the arrivals at a "main attraction" of an amusement park by a Poisson process with constant rate; in a more advanced model, we vary the arrival rates throughout the day. The park is open 10 h/day, with a "peak arrival time" between 2.5 and 6 h of the park opening.

We model how a group arriving at the attraction decides whether to enter the normal queue or to obtain a QuickPass (QP)—a pass to return later for a shorter wait. Their decision is governed by their desire to ride, the length of the normal queue, and the return time for the QP.

We explore several models for assigning QPs. The basic model, which gives absolute priority to the QP line, is problematic, since it can bring the normal line to a halt. Our more advanced models avoid this problem by using a "boarding ratio"—either fixed or dynamically varying—for how many from each line to load onto the ride. We use polynomial regression to predict the behavior of the queues, and we determine a dynamic ratio that minimizes the wait times when the number of QPs that can be assigned per time interval is fixed. Finally, we combine these algorithms and determine dynamically both the boarding ratio and the number of QPs to issue throughout the day.

Our advanced models tend to be extrememly robust to small pertubations in starting parameters, as well as to moderate variations in the number of arrivals. We avoid the problems attributed to the current QP system, as long as the ride is not "slammed' with substantially more guests than its capacity. In addition, our system cannot print shorter return times than have previously been issued. Averaging over a two-month period, the total wait times, the

---

*The UMAP Journal* 25 (3) (2004) 319–336. 

number of people in each queue when the park closes, and the number of QPs issued are consistent across all models.

# Introduction

In amusement parks, guests spend a great deal of time waiting in line, especially for popular rides. To reduce the time in line and hence increase overall enjoyment, a "QuickPass" system has been implemented at various locations. Rather than waiting in the normal queue for a ride, a guest can choose to enter a virtual queue during a one-hour time window later in the day.

Our model takes into account various factors, including the length of the normal queue, the number of people with a QP for the ride, and the percentage of the ride capacity that the QPs can commandeer.

## Disney's "FastPass" System

We base some of our system design on the "FastPass" system implemented by Disney Theme parks for their most popular rides. A guest who approaches a ride sees the projected wait time in the normal queue, as well as the current FastPass time window; if the guest chooses a FastPass, the system prints a ticket for that time window, which tells the guest when they can enter the FastPass line. Wait times in the FastPass queue tend to range from 5 to 10 min [R.Y.I. Enterprises 2004]. FastPasses are set to commandeer 40–90% of the given ride's capacity [Jayne 2003]. A guest is allowed to get a FastPass every 45 min to 2 h, depending on how busy the park is. At popular attractions, FastPasses are often sold out before noon on busy days [Jayne 2003].

# Simplifying Assumptions

- At all times, we know the number of people in the amusement park (determined using turnstiles to count the entries and departures).
- At all times, we know the number of people in both the normal and the QP lines.
- Groups arriving together act together (e.g., all wait in the normal line or all obtain QPs).
- People who obtain QPs always return during their allotted time and enter the QP queue.

# Queue Flows and Wait Times

The flow rates (people / min) for the queues are determined as follows:

$$f_{\text{in}}^{\text{NL}} = \frac{\sum_{i=1}^{L} N_{\text{arrival}}^{\text{NL}}(t + i\Delta t)}{\Delta t_{\text{flow}}}, \qquad f_{out}^{\text{NL}} = \frac{\sum_{i=1}^{L} N_{\text{exit}}^{\text{NL}}(t + i\Delta t)}{\Delta t_{\text{flow}}},$$

$$f_{\text{in}}^{\text{QP}} = \frac{\sum_{i=1}^{L} N_{\text{arrival}}^{\text{QP}}(t + i\Delta t)}{\Delta t_{\text{flow}}}, \qquad f_{out}^{\text{QP}} = \frac{\sum_{i=1}^{L} N_{\text{exit}}^{\text{QP}}(t + i\Delta t)}{\Delta t_{\text{flow}}},$$

where

- $N_{\text{arrival}}^{\text{NL}}(t)$ is the number of people entering the normal queue at time $t$,
- $N_{\text{arrival}}^{\text{QP}}(t)$ is the number of people entering the QP queue at time $t$,
- $N_{\text{exit}}(t)$ is the number of people leaving each queue at time $t$,
- $L$ is a fixed constant defining the size of the interval over which we wish to compute the flow, and
- $\Delta t_{\text{flow}}$ is the total time over which the sum is computed, $L\Delta t - t$.

Note that $L$ can vary over the day and the spacing $\Delta t$ is not necessarily uniform.

We use the flow rates to estimate the waiting times in the two queues. Because the flow rates can change suddenly, we use linear regression on the previous two flow values and the current flow value to help smooth the data.

# Basic Model

We begin with a basic model describing the important aspects of the "primary attraction." We are interested in:

- the frequency of arrivals at the attraction,
- the number of people in each group that arrives at the attraction,
- the lengths of the normal and QP queues when the group arrives,
- how many groups obtain a QP, and
- the current state of the QP system (e.g., can it assign any more QPs today, or is it "sold out").

**Figure 1** is a flow-chart of the logic in the basic model. We first simulate the number of groups arriving at the attraction using a Poisson process with constant rate $\lambda$.

We use a Poisson random variable with a mean of 3.6 to simulate the size of a group; the sampled value is added to the minimum group size of 1 person,

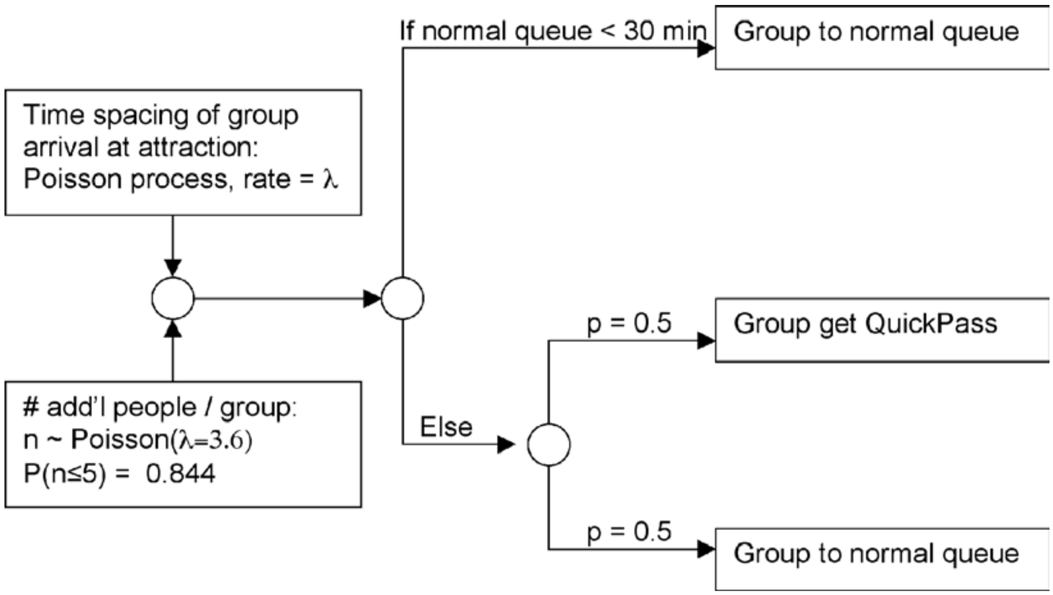



**Figure 1.** Flowchart for the primary processes in the basic model.

resulting in a mean group size of 4.6 people. This choice of rate gives 84% probability that groups have between 1 and 6 people.

If the wait-time of the normal queue is less than 30 min, the group enters the normal queue; if it is longer, they get a QP 50% of the time (unless the QP system is sold out, in which case the group enters the normal queue).

The return time for the QP is $\max\left(t_{\mathrm{NL}}, t_{\mathrm{syst}}\right)$, where $t_{\mathrm{NL}}$ is the predicted wait-time in the normal queue (based on the current number of people in both the queues and the ride capacity) and $t_{\mathrm{syst}}$ is the internal time of the QP system. The internal time is determined as follows:

1. When the QP system first turns on, the system time (and the start time for the first QP issued) is set to $t_C + t_{\mathrm{NL}}$, where $t_C$ is the current clock time (say, 1 h after the park opens).

2. From this point on, each time someone arrives at the attraction, we check:

   (a) If the QP system has reached the maximum number of QPs issued for a given start time, we increment the system time by a fixed value (e.g., 5 min).

   (b) If $t_C + t_{\mathrm{NL}} \leq t_{\mathrm{syst}}$, we issue a QP with $t_{\mathrm{start}} = t_{\mathrm{syst}}$; otherwise

   (c) if $t_{\mathrm{syst}} < t_C + t_{\mathrm{NL}}$ then we issue a QP with $t_{\mathrm{start}} = t_C + t_{\mathrm{NL}}$, and update the system time to $t_{\mathrm{syst}} = t_C + t_{\mathrm{NL}}$.

3. Once $t_{\mathrm{syst}} \geq T - 1.25$ h, where $T$ is the length of time that the park is open, we no longer issue QPs.

This system avoids the problem of the current system, where if the length of the normal line fluctuates drastically, a QP can be assigned for a time, say

4 h away, and a short while later for a time only 1 h away. By resetting the system time to $t_C + t_{\text{NL}}$ if this number is greater than the current system time, we guarantee that subsequent QPs always print a start time later than (or the same as) previous QPs.

The QP is issued to each guest with a time window from the specified $t_{\text{start}}$ to that time plus one hour. We assume that all guests who obtain a QP return during their allotted time; their return is simulated using a uniform distribution over the hour for which the ticket is valid.

Once the current group enters the normal queue or obtains a QP, we check to see if any of the attraction's cars have left since the last group arrived and update the number of people in each queue.

# Improvements on the Basic Model

## An Improved Decision Algorithm

We now include a decision algorithm that enables a group to make a choice based on three factors:

- their desire to ride the main attraction ($d \in [0, 1]$),
- the length of the normal queue ($L_{\text{NL}}$), and
- the return-time for the QP ($L_{\text{QP}}$).

We define

$$N \equiv L_{\text{NL}} - \mu_{\text{NL}}, \qquad Q \equiv 0.3\, L_{\text{QP}} - \mu_{\text{QP}}, \tag{1}$$

where

$$\mu_{\text{NL}} \equiv \min\bigl(f(d), L_{\text{NL}}\bigr), \qquad \mu_{\text{QP}} \equiv \min\bigl(f(1-d), L_{\text{QP}}\bigr).$$

The function $f(d)$ translates the group's desire to ride the attraction into the length of time that they are willing to wait in line. The function $f(1-d)$ determines how much later in the day the group would be willing to return for the QP queue. Using $0.3\, L_{\text{QP}}$ instead of $L_{\text{QP}}$ takes into account that people are more willing to wait in a "virtual" queue (where they can spend time riding other rides) than physically in line.

We define $f$ as a quadratic function passing through the points $(0, 0)$, $(1, T)$, and $(-1, T)$, where $T$ is the length of time that the park is open. Thus, a person with zero desire would be willing to wait for no time, and a desire level of 1 would indicate that the group would wait all day for this ride. As a more realistic example, a desire level $d = 0.6$ indicates that the group is willing to wait in the normal line for 3.6 h; they would prefer the QP option if the return time were less than 0.48 h.

The value for $d$, the group's level of desire to ride, is taken from a normal distribution $N(\mu = 0.5, \sigma = 0.1938)$, with 99.2% of its area contained in $[0, 1]$. We compute $N$ and $Q$ from **(1)**. The quantity with the minimum value determines whether the group enters the normal line or opts for the QP.

## Arrival Time at the Attraction

Amusement-park-goers know that the best times to ride the big attractions are early in the morning and late in the evening, when there are fewer people and the queues are shortest. In the basic model, we simulate the interarrival time between groups as an exponential random variable with constant rate $\lambda$. In a more realistic model, $\lambda$ should vary with the time of day—people arrive less frequently near the beginning and near the end of the day. We define this rate for the Poisson process as the piecewise continuous function:

$$\lambda = \begin{cases} \left( \dfrac{\dfrac{1}{M} - \dfrac{1}{m}}{a} \right) t + \dfrac{1}{m_B}, & 0 \le t < a; \\[2ex] \dfrac{1}{M}, & a \le t < b; \\[2ex] \left( \dfrac{\dfrac{1}{M} - \dfrac{1}{m_E}}{b - T} \right) t - \left( \dfrac{\dfrac{1}{M} - \dfrac{1}{m_E}}{b - T} \right) b + \dfrac{1}{M}, & b \le t < T - c; \\[2ex] \dfrac{1}{m_E}, & T - c \le t < T, \end{cases} \tag{2}$$

where

- $M$ is the expected number of groups per minute at the peak of the day,
- $m_B$ is the expected number of groups per minute at the start of the day,
- $m_E$ is the expected number at the end of the day,
- $a$ and $b$ are the beginning and end of the peak arrival time,
- $c$ is the time at which the arrival rate assumes a constant value of $1/m_E$, and
- $T$ is the number of hours that the park is open.

In our standard simulation, we take $M = 5$, $m_B = 1.5$, $m_E = 0.1$, $a = 2.5$ h, $b = 6$ h, $c = 1$ h, and $T = 10$ h.

## Dynamically-Calculated Number of QuickPasses

In the simple model, the wait time for the QP queue is kept low by placing all QP guests on the ride before anyone from the normal queue. As a result, the normal queue may build up to a wait time of several hours. A better system is to fix $\alpha$, the ratio of the number of people allowed to board from the normal queue to the capacity of the ride; $\alpha = 0$ reverts to the simple model of boarding all QP guests before anyone from the normal queue.

To ensure that the wait time of the normal line remains reasonable, we need an educated guess of how many QPs to give out for future time intervals. **Figure 2** shows the logic. The machine uses past and current information about the flow into and out of each queue to determine a projected wait time. The number of QPs for a time interval is determined by trying to keep the wait times of the queue below 20 min (QP) and 2.5 h (normal).

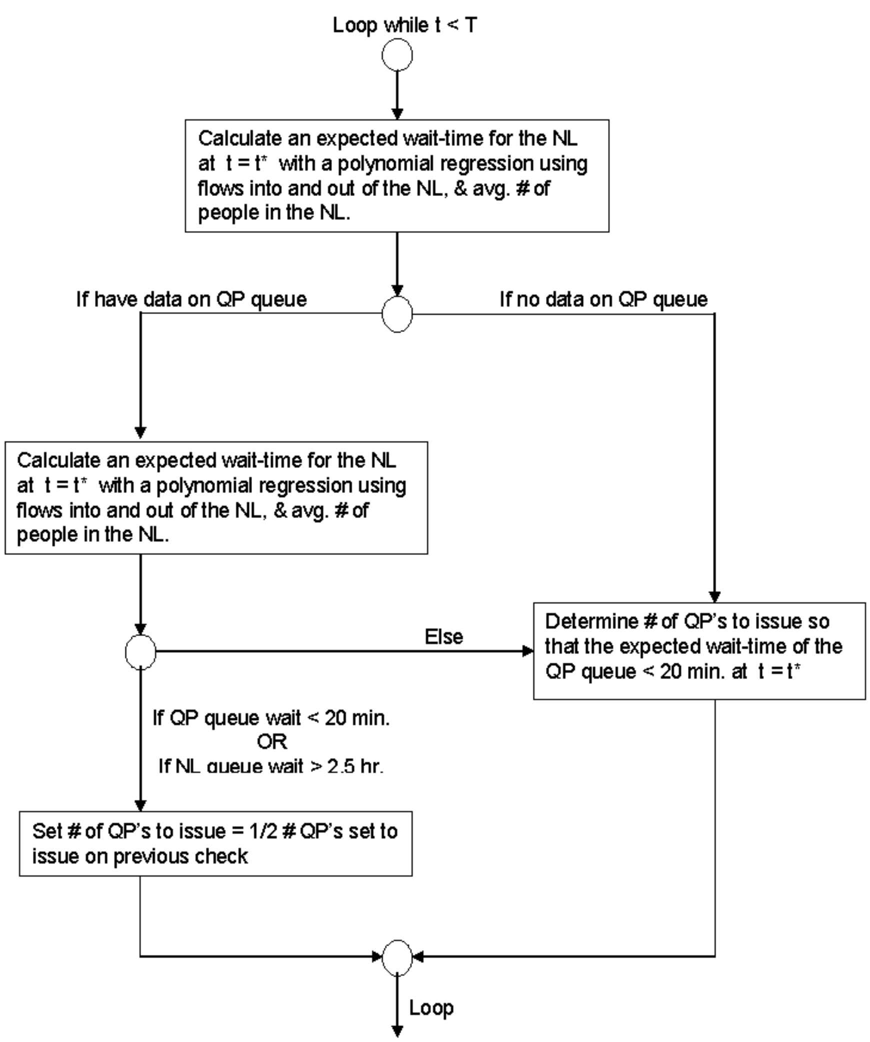



**Figure 2.** A flowchart of the QuickPass system that dynamically varies the number of QuickPasses offered for any one time interval.

Every 15 min, the QP machine calculates the average number of people entering and leaving each line during that time interval. Then, using a polynomial regression through the last three available flow rates, it calculates estimate of the flow rates at 15 min past the start of the QP return time window, $t_f = t_r + 15$ min. We approximate the flow rate from the current time $t_c$ to $t_f$ as the average of the current flow $f_c$ and the flow determined by polynomial regression: $\hat{f} = \frac{1}{2}(f_p + f_c)$. An approximation to the number of people in the queue at time $t_f$ is

$$N_f \approx \bar{N} - \hat{f}_{\text{out}} t_f + \hat{f}_{\text{in}} t_f = \bar{N} - \tfrac{1}{2}(f_{p,\text{out}} + f_{c,\text{out}})t_f + \tfrac{1}{2}(f_{p,\text{in}} + f_{c,\text{in}})t_f, \qquad \textbf{(3)}$$

where $\bar{N}$ is the estimated number of people currently in the queue. The number of people in the queue depends only on the current and projected flow rates for that queue and the estimated number of people currently in the queue. If **(3)** produces a negative number (indicating an unrealistic approximation of the flow rate), we use instead the current flow rate,

$$N_f = \bar{N}\,\frac{f_{c,\text{in}}}{f_{c,\text{out}}}.$$

The projected wait time $t_{\text{NL}}$ for the normal line is given by the projected number of people in the queue, $N_f^{\text{NL}}$, divided by the estimated future outflow rate:

$$t_{\text{NL}} = \frac{N_f^{\text{NL}}}{f_{p,\text{out}}^{\text{NL}}}.$$

The wait time for the QP queue is computed in a similar manner. If the wait times for the normal queue and the QP queue are below their maximum acceptable wait times and the flow for the QP line is zero, the number of QPs issued is determined using

$$n = 4(1-\alpha) f_{\text{out}_{\text{NL}}} \min\left(20 \text{ min}, \frac{t_{\text{NL}}}{3}\right).$$

If the flow for the QP queue is nonzero, we use

$$n = 4 f_{\text{out}_{\text{QP}}} \min\left(20 \text{ min}, \frac{t_{\text{NL}}}{3}\right).$$

This calculation is done every 15 min or whenever all the QPs for the time interval have been issued.

# Dynamic Ride-Loading Ratio

In the previous model, we fix the number of guests from each queue who can enter the ride and dynamically vary the number of QPs issued for each time interval. Here, we consider the opposite idea: The QP system issues a fixed number of tickets for every time interval, and the parameter $\alpha$ (the number of people from the normal queue divided by the total capacity of the ride) varies. **Figure 3** shows the logic chart for this system.

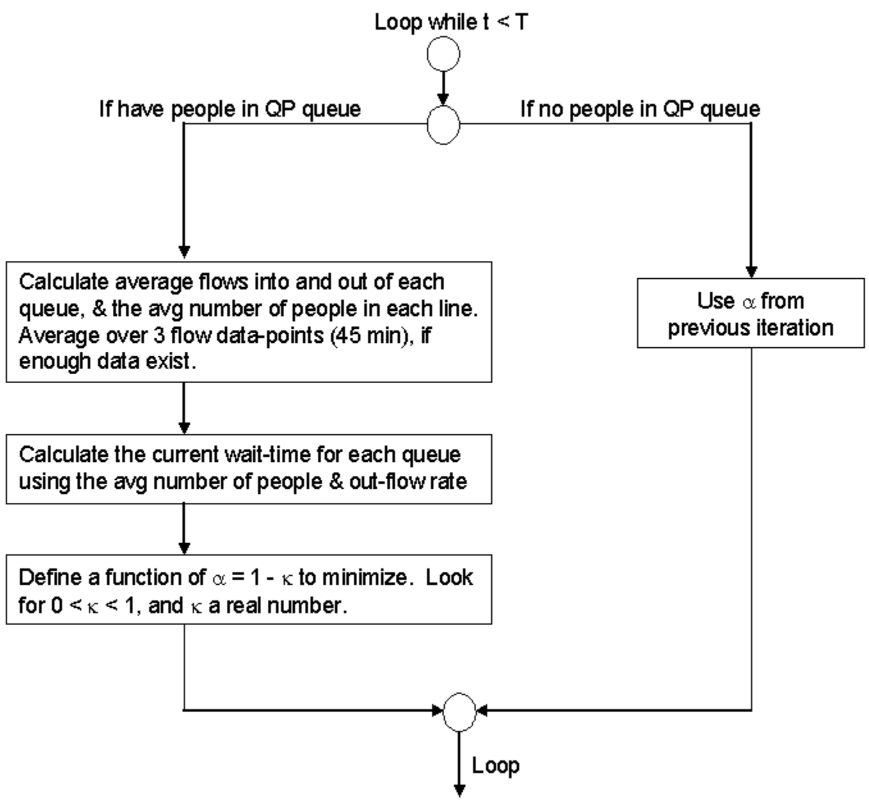



**Figure 3.** Regulating queue lengths by dynamically varying the ratio of people who board the ride from each queue.

We begin the day with an arbitrary $\alpha$ between 0 and 1. Once the QP queue forms, a new value is calculated by minimizing the dimensionless weighted sum of the wait times for each queue. First, the average wait time for each line is calculated by determining the average outflow rates for both lines and the average number of people in each queue during that time,

$$t_w = \frac{\bar{N}}{\bar{f}_{\text{out}}}.$$

Each waiting time is then normalized by the maximum acceptable waiting time, 20 min for the QP queue and 2.5 h for the normal queue, to create the weighting factors

$$\beta = \frac{t_w^{\text{QP}}}{1200}, \qquad \eta = \frac{t_w^{NP}}{9000},$$

with times in seconds. We then determine the value for $\alpha = 1-\kappa$ that minimizes the dimensionless wait time:

$$\text{dimensionless wait time} = \left(\beta\,\frac{f_{\text{in,QP}}}{144\kappa^2} + \eta\,\frac{f_{\text{in,NL}}}{144(1-\kappa)^2}\right)$$

by solving for the real root between 0 and 1 of the cubic polynomial

$$0 = (\mu+\gamma)\kappa^3 - 3\gamma\kappa^2 + 3\gamma\kappa - \gamma,$$

where

$$\gamma = \beta \bar{N}_{\text{QP}} f_{\text{in}}^{\text{QP}}, \qquad \mu = \eta \bar{N}_{\text{NL}} f_{\text{in}}^{\text{NL}}.$$

# Dynamic Ps and Boarding Ratio

In this model, we simultaneously vary both the number of QPs per time interval and the boarding ratio $\alpha$. Hence, we need only ensure that the initial parameter value for $\alpha$ is reasonable. If the initial $\alpha$ is too large, the algorithm will indicate that the system should issue no QPs at all; the algorithm to update $\alpha$ will then continue using the same $\alpha$ because no one has arrived in the QP queue. Hence, we expect this new model to be sensitive to the initial value of $\alpha$.

# Three-Tiered Queueing

The final improvement to the system adds a third queue, the Priority-OnePass (POP) queue, which is to the QP queue as the QP queue is to the regular line. This new queue will be shorter than the QP (now Priority II) queue, with a wait no longer than the arrival time between cars on the ride, but also with a return window of only 15 min that can be booked for as soon as 45 min from the current time. In essence, this new queue allows guests to "make an appointment" to ride the attraction. We assume that everyone who takes a POP or QP returns during their designated time window.

The POP queue has a maximum of 600 tickets for the day: 15 for every 15-min interval throughout the day. However, because the POP machine starts only after the wait time for the regular queue has exceeded half an hour, some POP tickets may not be issued.

The decision algorithm for the queues is relatively basic: If the return times for both queues are about the same, guests take the QP ticket 70% of the time because it allows more flexibility. The other cases are shown in detail in (**Figure 4**). This model can give first return times that are not in chronological order.

Once the group has decided to take a POP, their return time is chosen according to a $\chi^2(2)$ distribution, which favors return times closer to the current time while still allowing for times later in the day.

We did not fully implement this model; in comparing the models, we focus on the previously described systems.

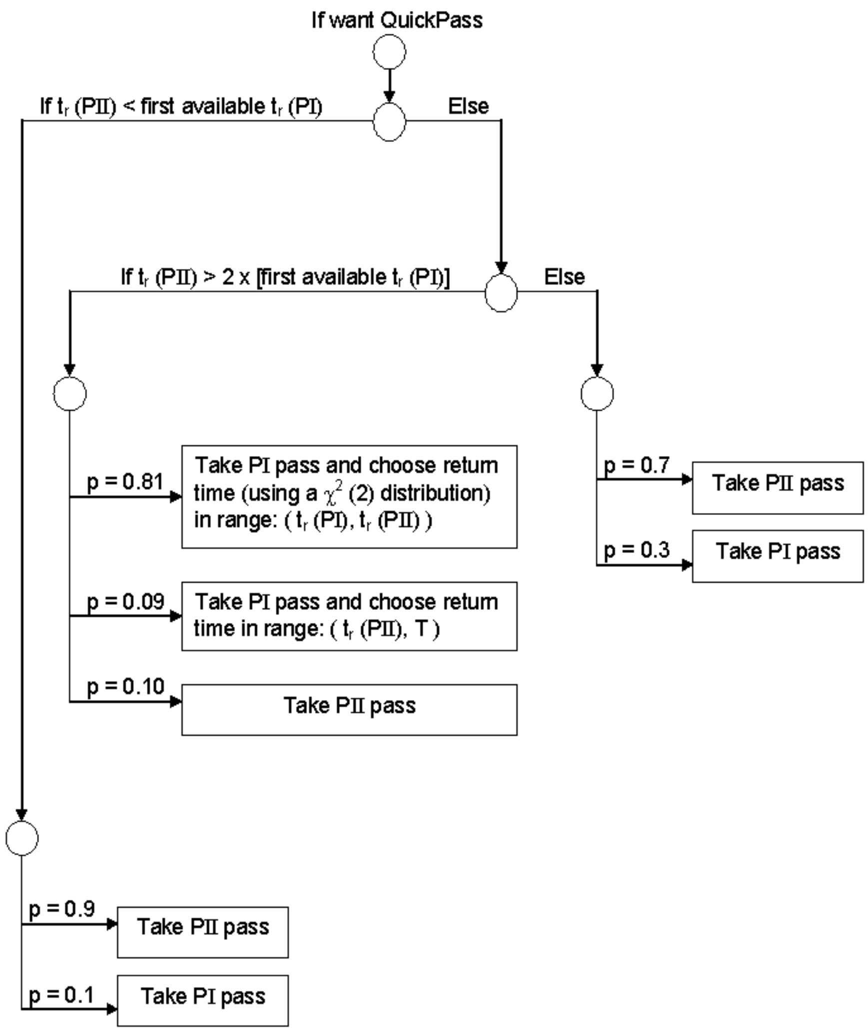



**Figure 4.** Flowchart of decision algorithm for choosing a QP) or a POP.

# Results

## Basic Model

The primary parameter in the basic model is $\lambda$, the rate for the Poisson process for the arrival of groups; changing $\lambda$ effectively changes the popularity of the ride.

In **Table 1** we present daily totals for the basic model with a constant inter-arrival rate. Results are shown for various values of $\lambda$, with a value of $\beta = 20$ QP tickets before incrementing the internal time by 5 min. The capacity of the ride is 7,200 people/day ($\lambda^{-1} = 23$).

As expected, when the total number of arrivals at the attraction is about 7,200 ($\lambda^{-1} = 23$), the QP system never activates, because we never have a wait-time in the normal queue of longer than 30 min. For both $\lambda^{-1} = 10$ and

**Table 1.**
Results for the basic model for various values of the parameter $\lambda$ (the rate of arrival of groups at the attraction).

| $\lambda^{-1}$ | Guests | QPs | Total riders | QP riders | Normal line riders |
|---|---|---|---|---|---|
| 10 | 16,657 | 2,335 | 7,200 | 2,335 | 4,865 |
| 17 | 10,047 | 2,362 | 7,200 | 2,362 | 4,838 |
| 21 | 7,863 | 441 | 7,157 | 441 | 6,716 |
| 23 | 7,371 | 0 | 7,025 | 0 | 7,025 |

$\lambda^{-1} = 17$, corresponding to numbers of guests that swamp the system, the basic model issues about the same number of QPs.

In addition to the day-end totals, it is interesting to look at the behavior of the queues (normal and QP), the flow-rates in and out of these queues, estimated waiting times in the queues, the expected wait-time for the ride (the total, at each time interval, of the predicted wait in the physical QP queue and that in the normal queue), start-times issued by the QP system, and the ratio of people choosing the QP queue over the normal queue, all as functions of time. [EDITOR'S NOTE: The authors' complete paper included numerous more graphs and analyses of these features; we cannot include them all here.]

## When Demand Is Near Capacity ...

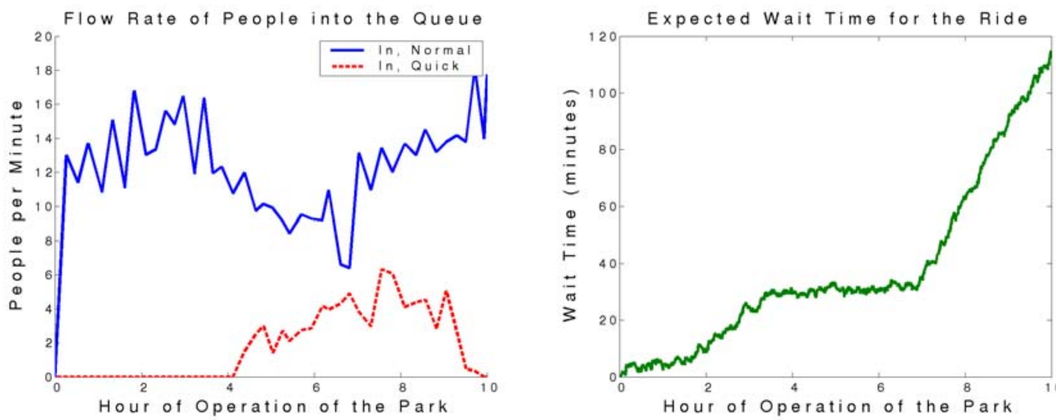
The only parameter of the QP model that can be adjusted is $\beta$, the maximum number of QPs that can be issued before the system's time clock ($t_{\text{syst}}$) advances. For example, with $\lambda^{-1} = 20$ (the mean interarrival time between groups is 20 sec), and with $\beta = 20$ QPs that can be issued before incrementing the system time by 5 min, we issue about 1,000 QPs. When we can distribute ten times as many—$\beta = 200$ QPs—before incrementing, we issue only about twice as many QPs; the limited number of people arriving during the 5-minute interval prevents a huge increase in the number of QPs issued (**Figure 5**).

## What Happens When Demand Swamps Capacity?

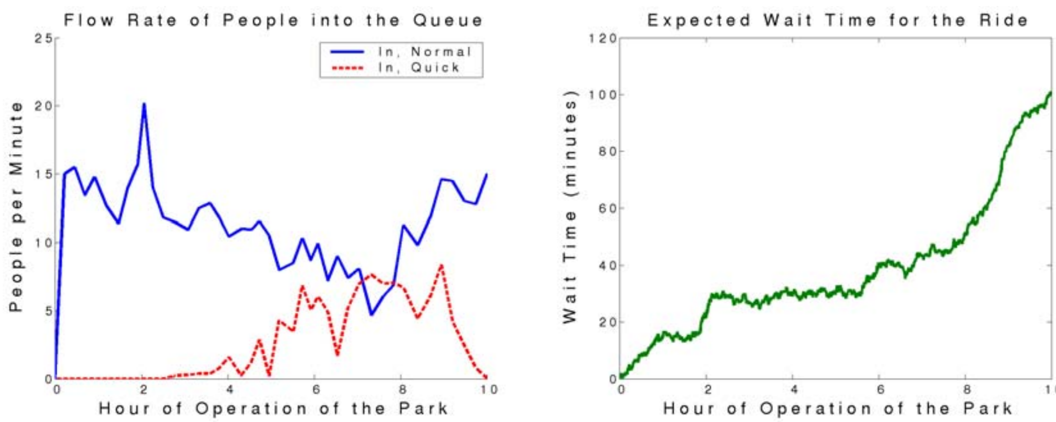
We compare the situation of $\lambda^{-1} = 20$ (the number of guests is approximately the capacity of the ride) with $\lambda^{-1} = 10$ (twice as many guests as the capacity), considering in each instance the cases $\beta = 20$ and $\beta = 200$, the maximum number of QPs issued before the QP clock is advanced by 5 min.

For the $\lambda^{-1} = 20$ cases, the overall wait-time increases linearly from when the park opens up until the QP system goes online, at which point it plateaus; the wait times increase again once the QP system is sold out for the day. The wait-time of the QP queue stays very short (< 4 min).

For $\lambda^{-1} = 10$, guests begin entering the QP queue only an hour after the park opens, and QPs sell out 3 to 5 h after the park opens (5 h is halfway through the day), depending upon how many QPs are allowed.




**a.** $\lambda^{-1} = 20$, $\beta = 20$.



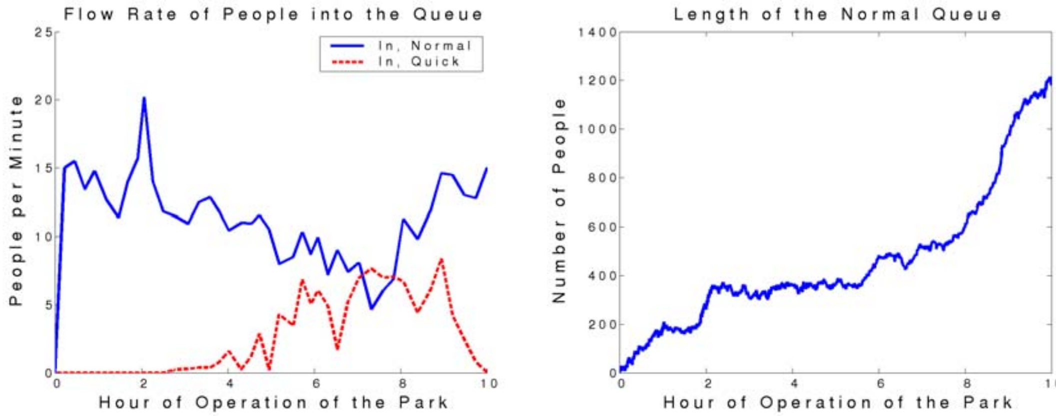

**b.** $\lambda^{-1} = 20$, $\beta = 200$.

**Figure 5.** Basic model: Inflow and expected wait time with $\lambda^{-1} = 20$ (8,000 guests), for two values of $\beta$, the number of QPs that can be issued before the QuickPass system increments the start time by 5 min: **a.** $\beta = 20$. **b.** $\beta = 200$.

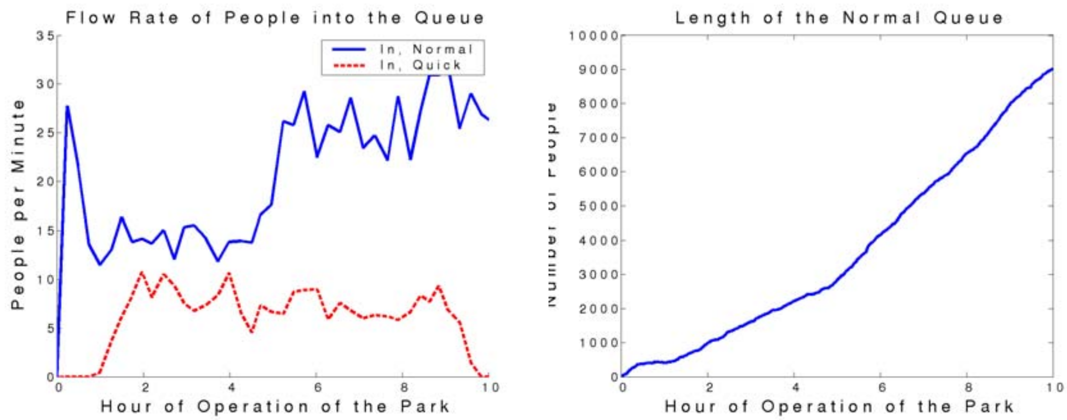
Because the number of total arrivals stays approximately constant throughout the day ($\sim$ 30 people/min), the high priority for QP riders substantially slows down the normal queue (even though when the QP system is active, the flow rate into the normal queue drops to $\sim 15$ people/minute), increasing the average wait-time for people in the normal line and leaving an enormous number of people in the normal line when the park closes (**Figure 6**). This is clearly not an ideal system.

## Improved Model

The basic model has groups take a QP 50% of the time when the normal queue had a predicted wait-time longer than 30 min. In the improved model, the decision of which queue to enter is based on the desire to ride the attraction, the length of the normal queue, and how much later in the day they would return if they took a QP.




**a.** $\lambda^{-1} = 20$ (8,000 guests), $\beta = 200$ max QPs before time increment.




**a.** $\lambda^{-1} = 20$ (8,000 guests), $\beta = 200$ max QPs before time increment.

**Figure 6.** Basic model: Inflow and length of normal queue with $\beta = 200$, for varying arrival intensity. **a.** $\lambda^{-1} = 20$ (8,000 guests). **b.** $\lambda^{-1} = 10$ (16,000 guests).

The behavior for the two decision algorithms is similar when the number of arrivals corresponds reasonably closely with the ride capacity ($\lambda = 1/20$). When arrivals overwhelm capacity, however, the new decision algorithm tends to "flood the queue." The plots in **Figure 7** give the predicted wait-time for the QP queue throughout the day with the new decision model, for two levels of ride demand. Because the number of arrivals is so much greater than the ride capacity when $\lambda^{-1} = 10$, the normal queue has no outflow during the middle of the day and its length grows so long that more and more people take a QP, until the QPs sell out. Even though many more people choose QPs in this situation, the QP queue still empties out nicely by the end of the day.

## Basic Model, Variable Interarrival Rate

We revert to the original "dummy" decision algorithm, where people choose the QP 50% of the time when the normal line's wait-time exceeds 30 min and the interarrival spacing is given in **(2)**. We choose a standard set of parameters

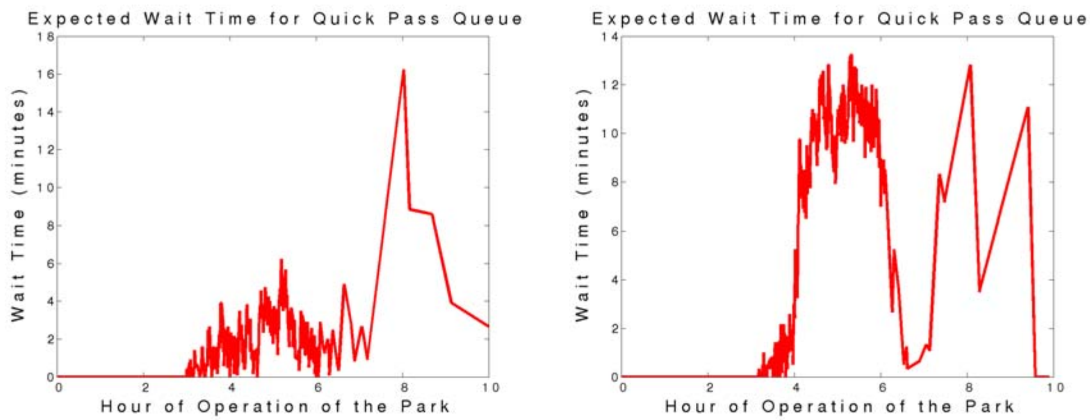



**Figure 7.** Basic model with improved decision algorithm: Expected wait time in the QuickPass with $\beta = 200$ QPs issued before the QP system increments $t_{\text{start}}$ by 5 minutes, for two levels of arrival intensity. **a.** $\lambda^{-1} = 20$. **b.** $\lambda^{-1} = 10$.

that result in a rush of arrivals during the peak hours but that does not cause a total demand greater than the ride's capacity. We define the peak arrival time to be between $2.5 < t < 6$ h after the park opens. The standard parameters that we use are: $M = 5$, $m_B = 1.5$, and $m_E = 0.1$ groups/min, where $M$ is the maximum number of groups/min during the peak-times of the day, $m_B$ is the minimum number as seen at the beginning of the day, and $m_E$ is the minimum as seen at the end of the day. With the interarrival spacing dependent on the time of day (and hence on the occupancy of the park), the normal queue nearly empties by closing time.

## Variable Rate and Improved Decision Algorithm

Using both of the improvements to the basic model, and approximately 7,400 people arriving at the attraction, we issue 1,500 QPs if we allow $\beta = 20$ QPs per 5 min and 3,000 if we allow $\beta = 200$ QPs per 5 min. The peak wait-times are respectively 3 h and 2 h.

# Dynamically Varying Parameters

[EDITOR'S NOTE: Although the authors presented results on separately varying the boarding fraction $\alpha$ and the number of QPs, space does not permit including those results here.]

## Varying Both $\alpha$ and the Number of QuickPasses

The final model allows both the borading ratio and the number of QPs issued per time interval to vary dynamically. **Figure 8** shows the results of this model with our standard set of parameters describing arrival times throughout the day. We begin the day with $\alpha = 0.5$. During the day, this value climbs as high as 0.6. The number of QuickPasses made available per 5 min period rises to 700 slightly more than 2 h after the ride opens, before declining to 0 about 2 h later. Although the expected wait time in the QP queue has a maximum at a little over 1 h, wait times in the normal queue reach 4 h, with the queue more than 1 h long at closing time.

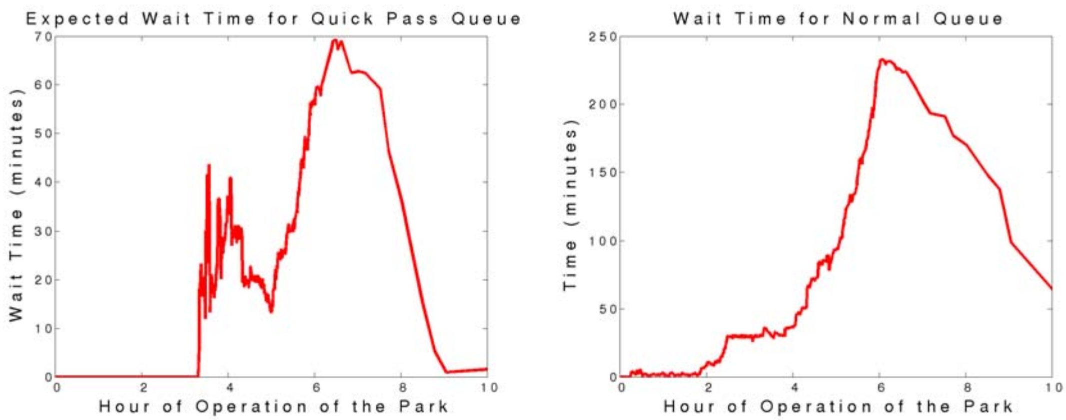



**Figure 8.** Expected wait times in the queues with dynamically varying $\alpha$ (proportion of people boarding from the normal queue) and $\beta$ (the number of QuickPasses issued before incrementing the system time), for an initial $\alpha = 0.5$.

# Statistical Analysis

We ran two-month trials for each QP system and summarized the overall performance of each QP system in terms of mean and standard deviation of various quantities. The average hourly wait-times are very similar for the four models, but with a larger variance (by as much as a factor of two) for the models that vary the number of QPs issued. Those tend to result in fewer people remaining in the queues when the ride shuts down for the day; they also tend to issue fewer QPs. The total and maximum queue wait-times are surprisingly consistent across the four models.

# Strengths and Weaknesses of the Models

## Strengths

- Our models are fairly robust to changes in parameters, including the two most important parameters, the boarding ratio and the number of QPs to issue.

- Our QP system cannot move "backward" in time. For example, it will not print out a QP for four hours in the future and then a half hour later print a ticket for one hour in the future.

## Weaknesses

- All the models rely on flow data, which can vary rapidly over short time intervals. Thuse, the wait times for the two queues can change rapidly, even when we use linear regression to estimate better the average flows. Because we use flows to determine wait time, the average wait time is only a rough approximation of the actual average; to obtain a better sense of the average, we would need to follow *individuals* through each queue and determine exactly how long each guest waits for the ride.

- The models assume that everyone who obtains a QP returns during their allotted window and that everyone in line stays in line until they reach the ride. In reality, some guests with QP tickets miss their window or decide not to return, and some in the normal queue get frustrated with the wait and leave.

- In addition, our models look at only a single ride. If several rides have a QP system, all ride systems must interact to determine how many QPs to give out for each ride in a single interval. A more complex model would have to take into consideration how people move between rides and how long they are willing to wait based on the lines of other rides in the park.

# Conclusion

We model the arrival of groups at an attraction and their decision process when faced with the option of obtaining a QP. We analyze the effect of different versions of a QP system, including dynamically adjusting the system. Our system avoids current problems, such as printing sooner return times than those previously issued.

For all our models, we obtain reasonable behavior when the number of people arriving at the attraction does not greatly exceed its capacity. Averaging behaviour over a two-month period, we find that the total waiting time, the number of people in each queue at the time the park closes, and the number of QPs issued per day are consistent across all our models within their statistical errors. Results of individual days show larger differences when the ride is "slammed."

Finally, we developed but did not implement a more sophisticated algorithm with an additional PriorityOnePass option.

# References


Burden, R.L., and J.D. Faires. 2001. *Numerical Analysis*. 7th ed. Belmont, CA: Brooks / Cole, Thomson Learning.

Jayne, A.W. 2003. How much time does Disney's Fast Pass save? `http://members.aol.com/ajaynejr/fastsave.htm` .

R.Y.I. Enterprises, LLC. 2004. Fastpass. `http://allearsnet.com/tp/fastpass.htm` .

Ross, S.M. 2002. *A First Course in Probability*. 6th ed. Upper Saddle River, NJ: Prentice Hall.

________ . 2003. *Introduction to Probability Models*. 8th ed. New York: Academic Press.

Strang, G. 1988. *Linear Algebra and its Applications*. 3rd ed. Philadelphia, PA: Harcourt College Publishers.

Tijms, H.C. 1994. *Stochastic Models: An Algorithmic Approach*. New York: John Wiley & Sons.

Wackerly, D.D., W. Mendenhall, and R.L. Scheaffer. 2002. *Mathematical Statistics with Applications*. 6th ed. Duxbury, CA: Duxbury, Thomson Learning.

Werner Technologies. n.d. Disney Fastpass information. `http://www.wdwinfo.com/wdwinfo/fastpass.htm` .

Yakowitz, S.J. 1977. *Computational Probability and Simulation*. Reading, MA: Addison-Wesley Publishing Company.

# KalmanQueue: An Adaptive Approach to Virtual Queueing

Tracy Clark Lovejoy
Aleksandr Yakovlevitch Aravkin
Casey Schneider-Mizell
University of Washington
Seattle, WA

Advisor: James Allen Morrow

## Summary

QuickPass (QP) is a virtual queueing system to allow some theme-park guests to cut their waiting time by scheduling a ride in advance. We propose innovative QP systems that maximize guest enjoyment. Only a small portion of guests can effectively use QP, and a good system maximizes this group subject to the constraints that regular users are not significantly affected and maximum waiting time for QP users is small.

We define and test a simple model for single-line formation and then develop two QP systems, GhostQueue and KalmanQueue. GhostQueue is intuitive and simple but would be far from optimal in practice. We then propose that the best model is one that adapts to its environment rather than trying to enforce rigid parameters. We implement KalmanQueue, a highly adaptive system that uses an algorithm inspired by the Kalman filter to adjust the number of QPs given today based on the maximum length of the QP line yesterday, while filtering out random noise. We simulate the KalmanQueue system with a C++ program and randomized input from our line-formation model. This system quickly converges to a nearly optimal solution. It is, however, sensitive to some parameters. We discuss the expected effectiveness of the system in a real environment and conclude that KalmanQueue is a good solution.

*The UMAP Journal* 25 (3) (2004) 337–352. 

# Introduction

The underlying idea of how to reduce waiting time for some theme-park guests is simple: Rather than wait in line, guests get tickets that tell them when to come back; when they return, they wait in a shorter line before going on the ride.

Many such systems have been implemented, including QuickPass, FastPass, Freeway, Q-lo, and Ticket-To-Ride; some have failed and others have thrived. The appeal to the parks is twofold: The systems increase guests' enjoyment, and guests spend more money in the park instead of waiting in line.

All systems that we researched assume that the number of QP users is small and manage the system either by restricting the total number of QPs or by charging a fee for them.

# Plan of Attack

We seek to maximize guest enjoyment.

- **Define Terms:** We state a definition of "guest enjoyment" and explain what affects it, why, and how we model it.
- **State Assumptions:** We restate the problem in a mathematical way.
- **Describe a Good Model:** An effective QP system has certain desirable characteristics. describing these steers our model in the right direction.

We then present our models:

- **Line-Formation Model:** The QP system is a line-manipulation system, and we cannot hope to design it without first understanding line formation.
- **GhostQueue: A Simple Model:** We describe a simple but limited approach to a QP system.
- **KalmanQueue: An Adaptive Algorithm:** We propose, model, and test an adaptive algorithm as a solution to the QP system. We provide a simple implementation using an adapted Kalman filter and test it using randomly generated input. We then discuss the strengths and weaknesses of our specific implementation and of the model in general.

# Increasing Enjoyment

- **Guests enjoy wandering freely more than they enjoy waiting in a line.** This is the basic assumption that makes virtual queueing potentially useful for increasing guest enjoyment. When not in line for a specific ride, guests can enjoy more of the park's attractions, including other rides, food courts, and shopping areas. We assume that enjoyment of the park increases as overall waiting time decreases.

- **All guests must perceive that they are treated fairly.** A QP system must operate logically and be comprehensible, at least in function, to the guests. A system that is perceived as random may cause discontent even if it minimizes waiting times. We see an example of this in the problem statement, where unexplained changes in scheduled times between adjacent tickets causes complaints.

- **QP must not significantly deter from the enjoyment of those not using it.** The population of QP users is small compared to the total park population. Regardless of QP implementation, rides must operate at capacity as long as there is demand, otherwise the general population is affected and upset. The QP system should be more enjoyable to those who use it and not significantly affect those who do not.

# Properties of a Good Model

- **Solves the problem.** Our model should maximize user enjoyment as we have defined it, subject to the constraints we defined. It need not be optimal, but it should be very good.

- **Ease of implementation.** We intend for this system to be used in an actual theme park. Thus, we aim for simplicity of implementation rather than mathematical complexity.

- **Ease of use.** We do not want a system that runs smoothly only when everybody shows up exactly on time but degenerates when this is not the case.

- **Not be sensitive to random events.** Park attendance varies, and how guests use rides can be modeled by various probability distributions. We want the effectiveness of our QP system not to decrease due to chance.

- **Adjustable and adaptive.** We do not want a model with a large number of parameters that must be re-set every day because of various conditions. We want a model that can easily be adjusted, or adjusts itself, based on its environment.

# Basic Queueing Theory: Is It Useful?

Queueing theory is a well-researched branch of mathematics, with applications ranging from grocery-store-line models to computer-processing event queues. We discuss its basic concepts and apply them to our problem.

The QP system operates during peak hours, when lines for major attractions are not increasing at a significant rate. Thus, we can assume that we are in a steady state. This assumption makes sense, because we expect guests to stop getting into lines if they grow too large.

In a steady state, we can make the following key assumptions:

1. Mean guests served per minute, $\mu$, is constant.
2. Mean guests arriving per minute, $\lambda$, is constant.
3. On average, more guests are served per minute than arrive. That is, $\mu > \lambda$.

Assumptions 1 and 2 mean that neither services nor arrivals depend on other factors, most importantly time and pre-existing line length. For both of these parameters, only the time-averaged input and output rates are considered, but the time between any two consecutive arrivals or departures need not be the same. This randomness leads to nonintuitive conclusions (below). Assumption 3 is valid, since if it were not the case, the line would continue to grow.

Given these assumptions, the results are [Ruiz-Pala et al. 1967]:

$$\begin{aligned} \text{mean number of guests in line} &= \frac{\lambda^2}{\mu(\mu-\lambda)}, \\ \text{mean waiting time for those who wait} &= \frac{1}{\mu-\lambda}, \\ \text{probability of having to wait} &= \rho = \frac{\lambda}{\mu}. \end{aligned} \tag{1}$$

Problems arise when $\lambda \approx \mu$; both the mean waiting time and the line grow arbitrarily large when $\rho$ is near 1. Consider the case of a ride with a wait time of 1 h, like those in **Table 1**:

$$60 \text{ min} = \frac{1}{\mu-\lambda} \quad \longrightarrow \quad \mu = \lambda + \frac{1}{60}.$$

To predict the waiting time accurately, even on the order of 1 h, one must know $\lambda$ and $\mu$ to at least two decimal places. This may be possible given accurate statistics over a period of time; however, these figures are not easily found, perhaps due to competition in the theme-park business.

In short, we need a new model for long lines that can predict the long wait times shown in **Table 1** and is not terribly sensitive to the parameters $\mu$ and $\lambda$.

We pursue this goal later; now we present a short example that suggests that queueing theory is useful when wait times and line lengths are small.

**Table 1.**
Statistics for 10 popular rides at Cedar Point Amusement Park (with somewhat tongue-in-cheek "Thrill rating") [Cedar Point Information 2003].

| Thrill rating (out of 5) | Average wait time (min) | Riders/hr | Ride |
|---|---|---|---|
| 3 | 15–30 | 1,400 | Blue Streak |
| 3 | 15-30 | 2,000 | Iron Dragon |
| 2 | 15 | 1,800 | Jr. Gemini |
| 4 | 30–45 | 2,000 | Magnum |
| 4.5 | 45 | 1,800 | Mantis |
| 5 | 60+ | 1,600 | Millennium Force |
| 4.5 | 45 | 1,800 | Raptor |
| 5 | 60–180 | 1,000 | Top Thrill Dragster |
| 4.5 | 45 | 1,000 | Wicked Twister |
| 3.5 | 30 | 1,800 | Wild Cat |

## Queueing Theory in Our Cafeteria

We collected data during the noon lunch rush in our university's cafeteria (**Figure 1**), from which we find $\lambda = 1$, $\mu_{\text{subs}} = 1.1$ and $\mu_{\text{pizza}} = 4$, and

$$\begin{aligned}\text{pizza mean wait time} &= \frac{1}{4-1} = 0.3 \text{ min},\\ \text{sub mean wait time} &= \frac{1}{1.1-1} = 10 \quad \text{min}.\end{aligned}$$

These figures agree with our experience. If our QP system has the same characteristics as the sub shop and the pizzeria, then we are in great shape.

# Long-Line Formation Model with Limited Sensitivity

We need to consider only one line, because we can treat every ride as independent. The number of visitors to a ride who have QPs for another ride is assumed to be small, so we neglect their impact.

The queueing-theory results assume that the average arrivals and the average rate of service are constant. Here we discard these assumptions in favor of a differential-equation approach to modeling a line. Then we selectively add assumptions as necessary to produce a realistic approach.

The rate of change of the length of a line should depend on the number of guests in the park, the probability that they want to join the line, and the constant service rate of the ride. For a line of length $L$, the rate of change is given the input rate $I$ minus the output rate $O$:

$$\frac{dL(t)}{dt} = I - O.$$

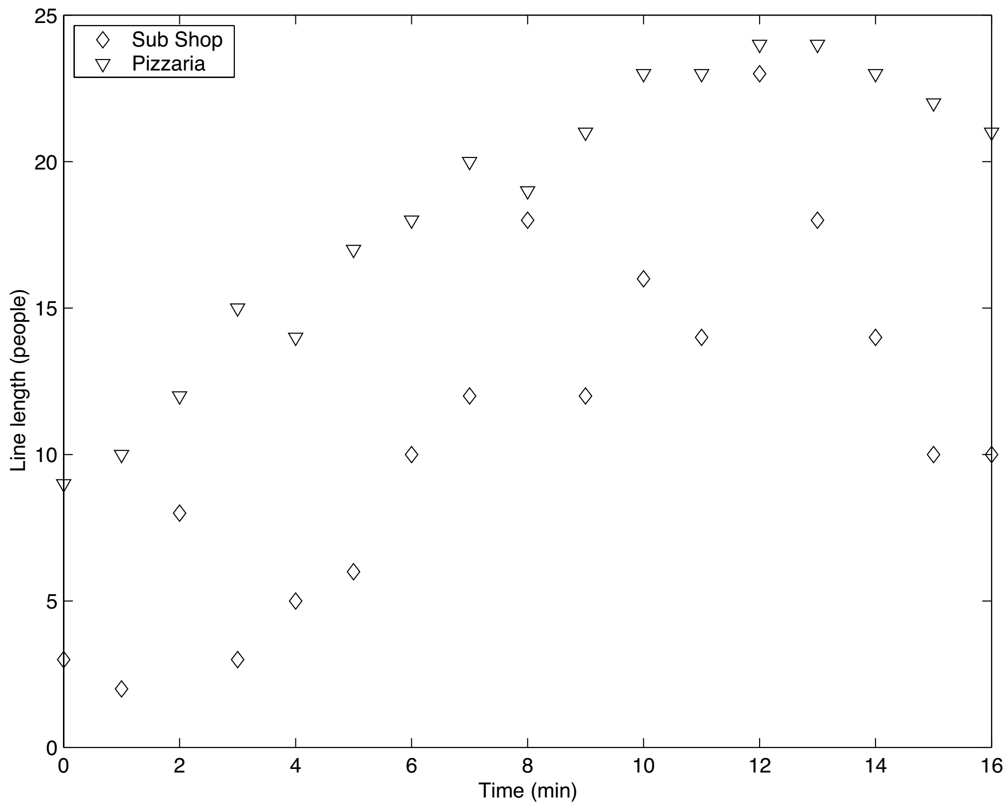



**Figure 1.** Observed line lengths as a function of time in the cafeteria during the lunch rush.

The input is the number of guests who join the line. This is given by the product of the population $P$ who could get on the ride with the probability $\alpha$ that they are interested in it during one time interval. Hence,

$$\frac{dL(t)}{dt} = \alpha P(t) - O. \tag{2}$$

Here

- $\alpha$, the probability that someone joins the line, is a function of the current length of the line and the perceived fun of that ride;
- $P(t)$, the number of guests in the park, is a function of time; and
- $O$ is constant, since rides run only as often as the machinery allows.

Let $\alpha$ be constant. Then, for the estimate of park attendance $P(t)$ shown in **Figure 2**, the solution to **(2)** is intuitive: The line has zero length until the park population reaches the $O/\alpha$ line, when it briefly has an increasing slope. Next, the slope is constant until park attendance begins to decrease. Only then does the line reach its maximum as park attendance falls below $O/\alpha$.

The longest lines occur around the peak. This is also the flattest part of the line-length curve, varying on the order of $\pm$ 10% during the time span about

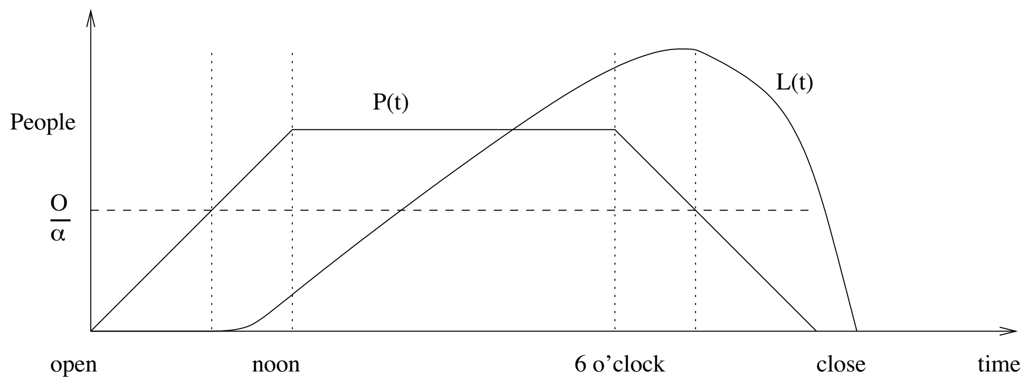



**Figure 2.** Line length $L(t)$ as predicted by **(2)** for constant $\alpha$ and the park population $P(t)$ shown.

the peak. Since this is exactly when our model should be effective, we assume that the line length does not change greatly over time.

The differential equations assume a lack of variation on the part of guests and ride operators; if both act with clockwork precision, the approach is sufficient. However, queueing theory and common sense tell us that the differential-equation approach is too deterministic; a good model must thus take statistical deviations into account. Even so, the line length predicted in **Figure 2** matches remarkably well with the data in **Figure 1**.

To incorporate statistical deviations, we introduce a computer simulation.

## Computer Simulation of Long Line Model

Our computer simulation dequeues (removes from the queue) at a fixed probability, enqueues (adds to the queue) at a probability dependent on time, and both are subject to noise from a random-number generator. We subdivide time into $N$ equal discrete time steps.

**Figure 3** shows an example of the output for $N = 2{,}000$, an average of 1 guest dequeued per time step, and 0 to 2 guests enqueued per time step. The shape of the figure closely resembles the model for line growth in **Figure 2** and our data in **Figure 1**.

# GhostQueue

The GhostQueue process behaves as if the guest has a "ghost" who stands in line instead, calling the guest back only when the ghost reaches the front of the line. We find that GhostQueue works well at very limited capacity; but as capacity grows, it suffers from the same problem as a normal line.

We assume that the wait time for the normal line is known to the system. Many virtual-queueing systems, such as Disneyland's FastPass, display this information at the QP kiosk [O'Brien 2001].

The GhostQueue system works in the following way:

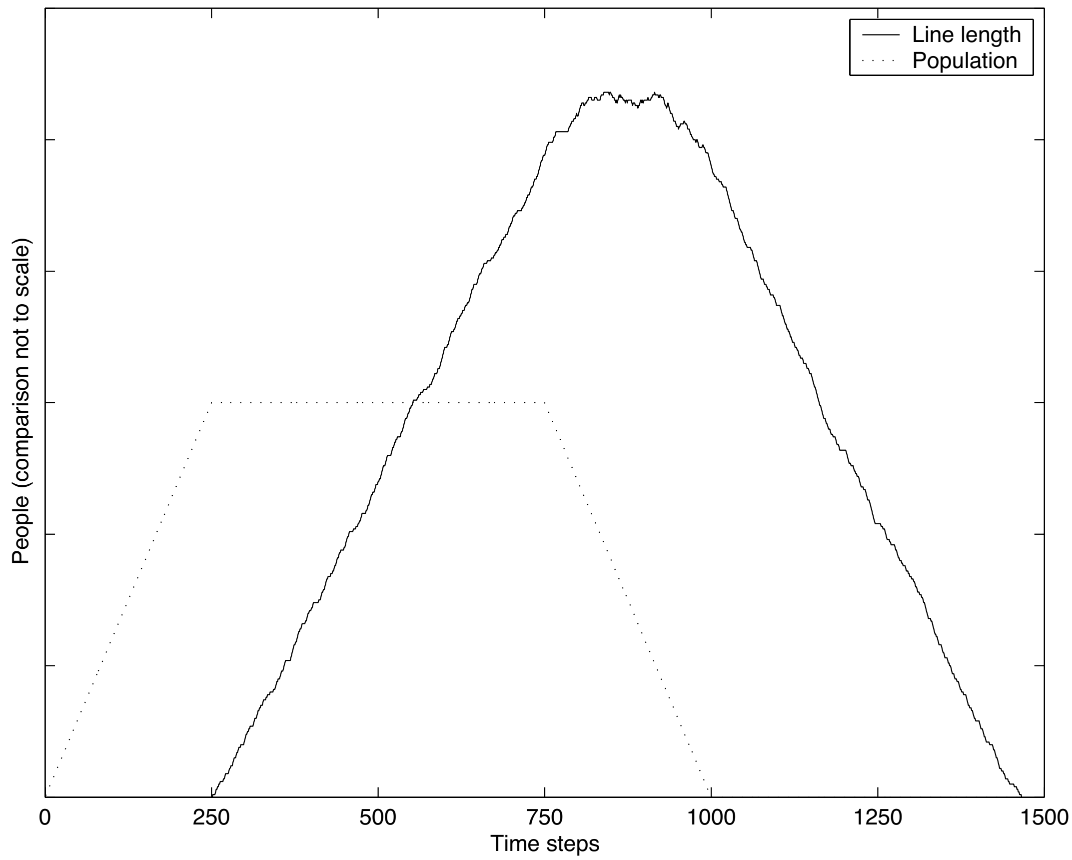



**Figure 3.** Simulated line length for the population input shown.

- The system checks the length of the normal line, computes the expected time when the guest would enter the ride, and gives the guest a ticket stamped with this time as the beginning of a short time window during which the guest can enter the GhostQueue line.
- The guest is free to roam about the park.
- When the guest's window is about to begin, the guest goes back to the ride.
- The ride takes guests first from the GhostQueue line, then from the normal line.

In theory, nothing changes for guests in the normal line; it acts exactly as if all guests were still present, even though only their ghosts wait. The average wait time $\bar{w}$ is given by

$$\bar{w} = \frac{\text{total time waited}}{\text{total guests}}.$$

With some guests not waiting, yet the total number of guests staying the same, $\bar{w}$ seems to go down with each new guest using GhostQueue. The optimal solution thus appears to be (but is not!) assigning everyone a ghost and watching the average wait time drop to zero.

Why is this not the best approach? Guests returning at a predetermined time is a probabilistic rather than a deterministic process, so queueing-theory results apply. To run at capacity and not create a line, guests must arrive at the same rate that the ride is boarding. In terms of the parameters of queueing theory, this means $\lambda = \mu$. However, **(2)** tells us that

$$\text{mean wait time for entry } = \frac{1}{\mu - \lambda} \xrightarrow{\lambda \to \mu} \infty.$$

Thus, even if there is a balance between arrivals and departures, not only is the expected wait time not zero, but if we run the fully ghosted system at capacity, *actual wait times get arbitrarily long*!

Reducing the number of ghost spots is equivalent to reducing $\lambda$. Since we wish to keep the wait time short for users of the ghost queue, we must both

- make $1/(\mu - \lambda)$ small, and
- keep the system stable to variations in $\lambda$.

The second goal is met when

$$\frac{d}{d\lambda}\frac{1}{\mu - \lambda} = \frac{1}{(\mu - \lambda)^2}$$

is small. The first goal implies that we should make $\mu - \lambda$ as large as possible. Both goals thus encourage the same end result. However, if the ride is not always filled by the ghost queue (which occurs sometimes because of the random distribution of arrivals), then the ride runs at less than full capacity. Therefore, there must be a normal line to keep the ride full.

From the perspective of guests, the length of the visible normal line must be related to its wait time, or else they will view the line as unfair. For example, if there are only 20 guests in the normal line but only 1 guest per minute is boarding from that line—the rest coming from the ghost queue—then this would not be an attractive line in which to stand. A balance needs to be created between perceived fairness and the average wait time.

Hence, the GhostQueue system is feasible only if the number of guests who use it is kept low relative to the number using the normal line.

In a similar system, Lo-Q at Six Flags amusement parks, the user limit comes from a fixed number of devices that must be rented to access the ghost-queueing feature. Based on the claim that 750,000 guests had used the system by October of 2001 [O'Brien 2001], and that the total 2001 attendance across the six parks was approximately 13 million visitors [O'Brien 2002], the utilization is 6%, in agreement with what we would predict.

# An Adaptive Algorithm

## The Idea of an Adaptive Algorithm

We introduce an adaptive model that does not try to maximize anything per se but instead adapts to make today's performance better than yesterday's.

A simple adaptive algorithm might count the number of guests who wait more than 10 min in the QP line today and assign that many fewer QPs tomorrow. Problems with this algorithm are the sensitivity to random variations in attendance and that lumping the whole day into one block of time is too coarse to capture many subtleties of park attendance. We propose an algorithm that breaks the day into more blocks and is not as sensitive to random fluctuations.

## Setup

The setup is described largely in **Figure 4**. We have a traffic control box that knows how many guests are in each line and determines how many QP tickets to make available in each hour. This information, as well as the wait times for the normal and QP lines, is fed into the kiosk, which displays waiting times and gives guests options for when to return. In the figure, the options noon and 1 P.M. are blacked out, indicating that those time slots are full.

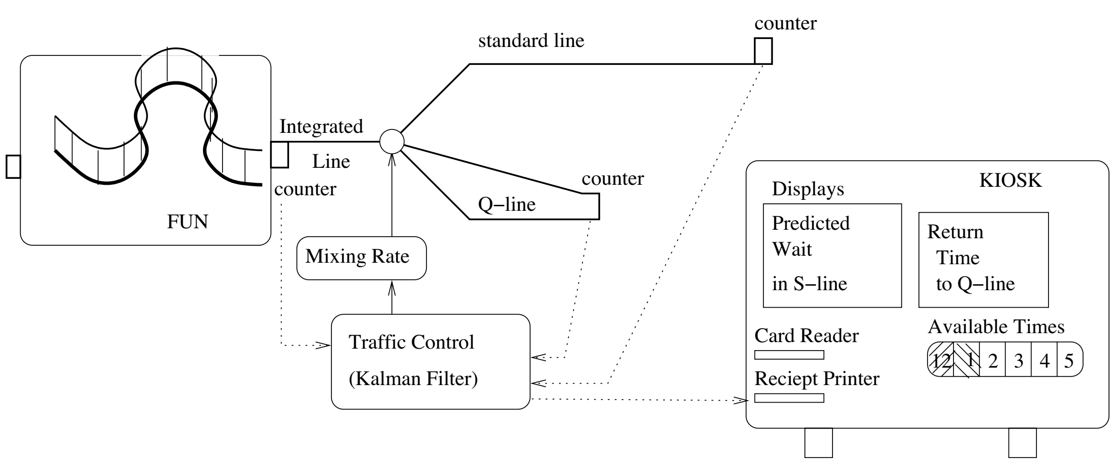



**Figure 4.** Schematic diagram of the park showing the the traffic control center, the kiosk, and, of course, the fun.

## Assumptions for the Adaptive Algorithm

- There is little day-to-day variation in the overall park attendance. The QP system is used during peak seasons and peak hours, when the pattern of line formation for a ride changes slowly.

- We set the number and times of QPs at the beginning of the day. Guests can get these at any point during the day, even nonpeak hours, on a first-come-first-serve basis. The system is thus fair and logical and will not behave strangely if there is variation in line formation.

- Line speeds at peak hours are nearly constant. As long as the ride does not break down, guests are using it at a constant rate.

- We allot a percentage of ride seats for QP users. The number of such seats per unit time is the same as the rate at which the normal and QP lines can be mixed. The maximum feasible mixing rate $M$ could be found from a pilot study or just taken to be reasonably small (5–10%).

- We declare a target maximum QP queue length.

## KalmanQueue

The Kalman filter is a set of recursive equations that provides a computationally efficient solution to the least-squares method [Welch and Bishop 2001]. Kalman filters have many applications, notably in autonomous navigation systems. They are appropriate here because our model is a discrete-time controlled process, and also because Kalman filters should satisfy our requirements for a good model. We briefly describe general Kalman filters and then adapt them for use in our model.

A Kalman filter estimates a state $X \in \Re^n$ at time $k+1$ from the state $X$ at time $k$ and an observation $Z$ at time $k$.

Kalman filters are adaptive, yet they filter out random noise to yield a stable system. A general Kalman filter with no control input is described by

$$\begin{aligned} X_{k+1} &= AX_k + w_k, \\ Z_k &= HX_k + v_k \end{aligned}$$

where

- $A$ relates the previous state to the next state,

- $w_k$ is Gaussian process noise,

- $Z_k$ is the observation at time $k$,

- $H$ relates the magnitude of the state to the magnitude of the observation, and $v_k$ is Gaussian measurement noise.

Now we outline our model:

- **State:** Number of QPs available for a given block of time.

- **Measurement $U_k$ = (Target QP length − Observed QP length):**

  If the difference is positive, we assign more QPs the next day, since the QP system is not running at capacity; if it is negative, the QP line is too long and we assign fewer QPs the next day.

- **Finding $H$:** The scalar $H$ relates the magnitudes of state and input. Since our state and our observation are on the same scale (guests in line), $H$ is simply 1.

We now give the recursive equations for our adaptive algorithm derived from the Kalman filter. We let $X_k$ be the number of QPs available on day $k$, while $P_k$ and $K_k$ determine how much we trust our observed data.

$$\begin{aligned} K_{k+1} &= \frac{R}{P_k + R}, \\ X_{k+1} &= X_k + K_k U_k, \\ P_{k+1} &= (P_k + V_k)\,\frac{P_k}{P_k + R}. \end{aligned}$$

Here $V_k$ is a measure of line fluctuation for day $k$, $R$ is the expected variance (from previous data), and $K_k$ is a scalar by which we weight the observation before we change state.

The variables $X$, $P$, and $K$ are related, since each adjusts per iteration depending on the others. The measurement $U_k$ is scaled by $K_k$, a measure of how much we trust the observation when computing the next state based on previous experience. Naturally, we expect $K_k$ to go to 1 when there are no fluctuations, and this indeed happens; it also should approach 0 when the observed variance is very large, and it does this too. Additionally, if the observed variance is about the same as expected, then we trust the observation with scale factor of approximately 1/2.

## Testing the Adaptive Algorithm

Our algorithm takes as input three parameters: $P_0$, $K_0$, and $R$. The first two are self-adjusting, so the filter is not sensitive to their initial values; but $R$ strongly affects convergence of the model. Amusement parks closely guard their attendance data, so we do not have data for these parameters and must guess reasonable values. However, even with rough guesses, our adaptive algorithm settles quickly to equilibrium.

We tested the filter by iterating our computer simulation. Given initial values, the first relevant output from the Kalman algorithm is the number of QPs assigned per hour block of time. We assume that guests arrive uniformly over their assigned block. We graph how many guests arrive as a function of time, exactly the input for our computer simulation.

This test does not capture the true power of our adaptive algorithm, because we do not know the value of the parameter $R$, and our model for line growth is a fairly rough and simple probability model.

**The Test:**

1. We assume that the peak time of day is subdivided into 6 equal blocks. We "couple" the blocks by having the final queue length of block $i$ as the initial queue length for block $i + 1$.
2. We use the same model for line formation as in our simulation. Additionally, we employ our algorithm on each block.
3. We guess the parameter $R$ and pick initial values for $K_0$, $P_0$, and $X_0$ (the number of QPs per time block).
4. Subject to noise, we input this $X_0$ into our line-formation model. Our algorithm measures the deviation of the actual line from the ideal line.
5. The Kalman filter outputs new values for $X$, $K$, and $P$. We now have a new value for the number of QPs for every block.
6. We iterate this process 1,000 times. The input to step 5 is the output of step 4.
7. We confirm visually that regardless of initial values, the filter converges to a steady optimal QP number per block that results in a stable and optimal QP queue length over time.

A pictographic version of the results of a trial is shown in **Figure 5**; a plot of the actual results is shown in **Figure 6**. The system output was programmed to vary around 100 guests/h, but we input an initial value of 120 QPs/h.

We can think of each time step as a previous similar day; for a Saturday, we look to last Saturday's data. For the first day, the line grows rapidly, as the ride cannot service the demand. By day 10, the line has visibly deformed. The distribution on day 30 is almost as good as on day 900, so the algorithm converges quickly, even with a bad initial guess, (in actual implementation we can start with very good initial values).

## Justification of Uniform Arrival Rate

Guests arrive with some probability distribution throughout their block. However, we should be able to overlap the blocks in such a way that the average arrival is constant. For example, **Figure 7** shows how we overlap the blocks when guests arrive with a normal distribution about the center of their block.

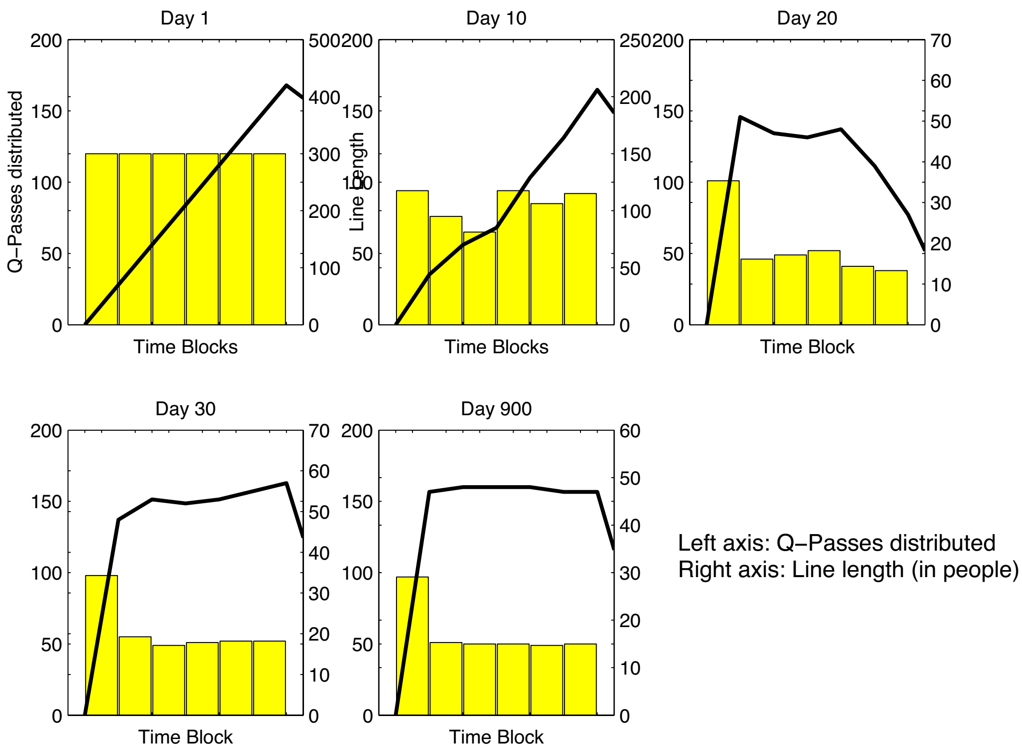



**Figure 5.** Number of QPs alloted per hour (histogram) and QP line length (black line) for days 1, 10, 20, 30, and 900.

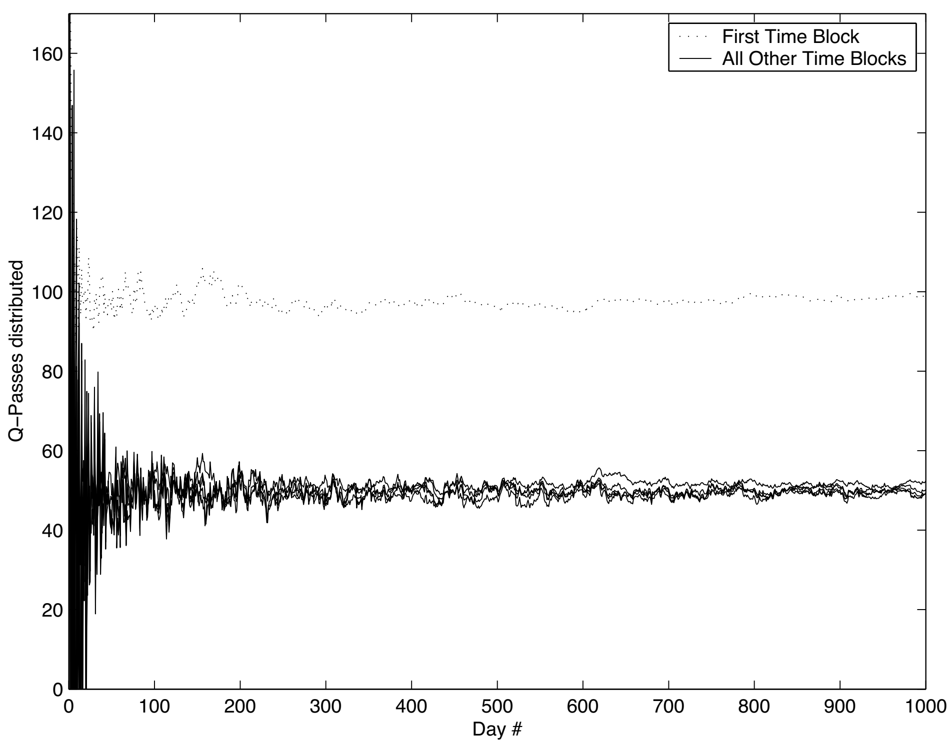



**Figure 6.** A plot of QPs alloted day by day, given wildly wrong initial values. Nevertheless, the algorithm stabilizes.

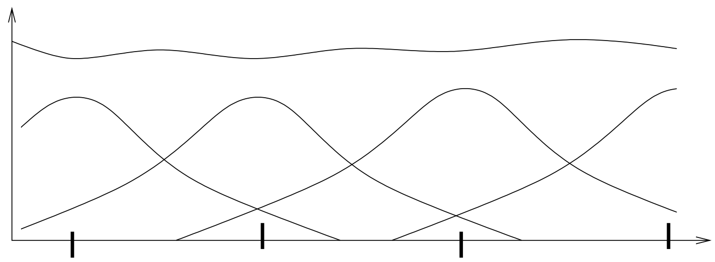


**Figure 7.** How to add normal distributions to achieve constant arrival.

# Conclusions

The GhostQueue decreases wait times and increases happiness, but in a way hard to optimize, and fairness is difficult to define and test. It is a good solution only if an external way (such as selling access) keeps utilization low.

The KalmanQueue process meets the criteria for a good model, despite some sensitivity in its parameters. The dynamic optimization approach is also easy to use, since the system adjusts itself to meet an ideal line condition. Hence, we recommend use of a KalmanQueue system for rides with long lines.

# Strengths & Weaknesses

## General Model

- **Strengths**
  - Our line-formation model agrees with our rough data, and our computer model agrees with both.
  - Our line model incorporates the natural randomness of human behavior.
- **Weaknesses**
  - Current line length is not taken into account by the line formation model. In real life, a guest is more likely to join a short line than a long one.
  - We ignore the effect of virtually-queued guests joining normal lines, thus increasing the normal line wait times.
  - Our model is valid only during peak hours.
  - Our model lacks a rigorous definition of optimality.

## GhostQueue

- **Strengths**
  - The return-time calculation is simple and comprehensible to the guest.

  - No guest waits longer than without the GhostQueue.

- **Weaknesses**
  - Utilization must be kept low for GhostQueue to be beneficial.
  - Fairness is a dominant factor in optimal utilization; but our assumptions do not quantify fairness, so optimizing is beyond the scope of the model.

### KalmanQueue

- **Strengths**
  - The primary input is the desired behavior of the QP line, and the model adjusts itself accordingly.
  - The KalmanQueue process satisfies all six properties of a good model.
  - The Kalman filter is highly adaptive to changes in the queueing process (e.g., time-varying output rates), so the core framework of our Kalman-based algorithm is valid for a wide variety of situations.
- **Weaknesses**
  - Our model assumes a constant mixing rate. An extension of the model (a two-dimensional Kalman filter) would allow for the determination of the mixing rate based on the relative lengths of the normal and QP lines.
  - $R$, the random variance of the number of guests in line, must be determined accurately for the model to be useful.
  - The Kalman filter tries to decrease the *maximum* QP line length. A future model should try to decrease the *average* QP line length.

## References

Cedar Point Information. 2003. The Point Online. `www.thepointol.com`.

Ruiz-Pala, Ernesto, Carlos Abila-Beloso, and William W. Hines. 1967. *Waiting-Line Models: An Introduction to Their Theory and Application.* New York: Reinhold Publishing Co.

Mouse Planet Inc. 2003. Disneyland Information Guide—What is Fastpass? `http://www.mouseplanet.com/al/docs/fast.htm`.

O'Brien, Tim. 2001. Six Flags debuts queue management. *Amusement Business* (5 March 2001). `http://www.lo-q.com/Press/Amusement%20Business%20March%202001.htm`.

________. 2002. North American parks down slightly from 2002. *Amusement Business* 115 (51) 9.

Welch, Greg, and Gary Bishop. 2002. An introduction to the Kalman filter. `http://www.cs.unc.edu/~welch/kalman/kalmanIntro.html`.

# Theme Park Simulation with a Nash-Equilibrium-Based Visitor Behavior Model

Andrew Spann
Daniel Gulotta
Daniel Kane
Massachusetts Institute of Technology
Cambridge, MA

Advisor: Martin Z. Bazant

## Summary

We build from the ground up a computer simulation consisting of a fictional theme park, MATHCOT. We populate MATHCOT with visitors and define an "enjoyment function" in which visitors gain points for going on rides and lose points as they stand in line.

We propose two QuickPass systems. In the Appointment System, QuickPasses represent an appointment to visit the ride later that day. In the Placeholder System, a QuickPass represents a virtual place in line. We then choose test cases to represent both systems and run the computer simulation.

With each set of parameters, we adjust the probability weights that govern visitor behavior to fit a Nash equilibrium. The Nash equilibrium adapts the behavior of park visitors to a greedy equilibrium that is not optimal for the group but represents individuals weighing to decisions based on immediate benefit.

Our results suggest that it is in the park's best interest to allocate a high percentage of the rides to QuickPass. Reserving too few seats for QuickPass users can result in lower average visitor enjoyment than without a QuickPass system. Both the Placeholder System and the Appointment System (with 75% of ride capacity allocated to QuickPass users) show strong increases in visitor enjoyment.

Varying the length of the time window for the QuickPass has little effect on visitor enjoyment.

*The UMAP Journal* 25 (3) (2004) 353. 

# Judges' Commentary: The Quick Pass Fusaro Award Paper

Peter Anspach
National Security Agency
Ft. Meade, MD
anspach@aol.com

Kathleen M. Shannon
Dept. of Mathematics and Computer Science
Salisbury University
Salisbury, MD 21801
kmshannon@salisbury.edu

The most distinctive feature of the Quick Pass Fusaro Award paper (summary on preceding p. 353) by the team from MIT was its creativity. The basic problem was a queueing problem, which the team members recognized and addressed. However, rather than choosing as their objective minimizing time spent in line, the team made a real effort to model human behavior and to maximize enjoyment. Although the judges questioned whether they had appropriately applied the Nash equilibrium, we were impressed by the idea of using game theory. The team referenced attempts by "real-world consultants" to simulate human behavior in virtual worlds.

Basically, the team simulated behavior by creating virtual visitors to their virtual theme park, giving them randomly generated preferences and tolerances. They then ran a simulation to find optimal parameters for the park itself, under various schemes for the QuickPass system.They treated visitors as individuals employing individual strategies but acknowledged that their assumption might not model reality fully, since people tend to come to theme parks in groups and group dynamics would definitely have an influence.

The team certainly developed the one of the most sophisticated and detailed models to address the problem, made well-thought-out and well-explained assumptions, went through all of the steps of the modeling process, and presented a well-written report. The purpose of the Fusaro Award is to recognize just such activities.

United States Postal Service

## Statement of Ownership, Management, and Circulation

| 1. Publication Title | 2. Publication Number | 3. Filing Date |
|---|---|---|
| The UMAP Journal | 0 1 9 7 - 3 6 2 2 | 11/8/2004 |
| **4. Issue Frequency** | **5. Number of Issues Published Annually** | **6. Annual Subscription Price** |
| Quarterly | 4 + annual collection | $99.00 |

7. Complete Mailing Address of Known Office of Publication *(Not printer) (Street, city, county, state, and ZIP+4)*

57 Bedford St., Suite 210, Lexington MA 02420

Contact Person: Kevin Darcy

Telephone: 788-862-7878

8. Complete Mailing Address of Headquarters or General Business Office of Publisher *(Not printer)*

SAME

9. Full Names and Complete Mailing Addresses of Publisher, Editor, and Managing Editor *(Do not leave blank)*

Publisher *(Name and complete mailing address)*

Solomon Garfunkel, 57 Bedford St., Suite 210, Lexington MA 02420

Editor *(Name and complete mailing address)*

Paul J. Campbell, Beloit College, 700 College St., Beloit WI 53511

Managing Editor *(Name and complete mailing address)*

Pauline Wright 57 Bedfors St., Suite 210, Lexington MA 02420

10. Owner *(Do not leave blank. If the publication is owned by a corporation, give the name and address of the corporation immediately followed by the names and addresses of all stockholders owning or holding 1 percent or more of the total amount of stock. If not owned by a corporation, give the names and addresses of the individual owners. If owned by a partnership or other unincorporated firm, give its name and address as well as those of each individual owner. If the publication is published by a nonprofit organization, give its name and address.)*

| Full Name | Complete Mailing Address |
|---|---|
| Consortium For Mathematics | 57 Bedford st., |
| And Its Applications, Inc. | Suite 210 |
| (COMAP, INC) | Lexington MA 02420 |
| | |
| | |

1. Known Bondholders, Mortgagees, and Other Security Holders Owning or Holding 1 Percent or More of Total Amount of Bonds, Mortgages, or Other Securities. If none, check box → ☐ None

| Full Name | Complete Mailing Address |
|---|---|
| | |
| | |
| | |
| | |
| | |

2. Tax Status *(For completion by nonprofit organizations authorized to mail at nonprofit rates) (Check one)*

The purpose, function, and nonprofit status of this organizati...

| 13. Publication Title | 14. Issue Date for Circulation Data Below | |
|---|---|---|
| The UMAP Journal | 11/30/2002 | |

| 15. Extent and Nature of Circulation | | Average No. Copies Each Issue During Preceding 12 Months | No. Copies of Single Issue Published Nearest to Filing Date |
|---|---|---|---|
| a. Total Number of Copies *(Net press run)* | | 900 | 950 |
| b. Paid and/or Requested Circulation | (1) Paid/Requested Outside-County Mail Subscriptions Stated on Form 3541. *(Include advertiser's proof and exchange copies)* | 750 | 817 |
| | (2) Paid In-County Subscriptions *(Include advertiser's proof and exchange copies)* | 0 | 0 |
| | (3) Sales Through Dealers and Carriers, Street Vendors, Counter Sales, and Other Non-USPS Paid Distribution | 70 | 70 |
| | (4) Other Classes Mailed Through the USPS | 0 | 0 |
| c. Total Paid and/or Requested Circulation *[Sum of 15b. (1), (2),(3),and (4)]* | | 820 | 8887 |
| d. Free Distribution by Mail *(Samples, compliment ary, and other free)* | (1) Outside-County as Stated on Form 3541 | | |
| | (2) In-County as Stated on Form 3541 | | |
| | (3) Other Classes Mailed Through the USPS | 30 | 20 |
| e. Free Distribution Outside the Mail *(Carriers or other means)* | | 0 | 0 |
| f. Total Free Distribution *(Sum of 15d. and 15e.)* | | 30 | 20 |
| g. Total Distribution *(Sum of 15c. and 15f)* | | 850 | 907 |
| h. Copies not Distributed | | 50 | 43 |
| i. Total *(Sum of 15g. and h.)* | | 900 | 950 |
| j. Percent Paid and/or Requested Circulation *(15c. divided by 15g. times 100)* | | 96 | 98 |

16. Publication of Statement of Ownership

☒ Publication required. Will be printed in the third issue of this publication. ☐ Publication not required.

| 17. Signature and Title of Editor, Publisher, Business Manager, or Owner | Date |
|---|---|
| Solomon Garfunkel | 11/08/04 |

I certify that all information furnished on this form is true and complete. I understand that anyone who furnishes false or misleading information on this form or who omits material or information requested on the form may be subject to criminal sanctions (including fines and imprisonment) and/or civil sanctions (including civil penalties).

## Instructions to Publishers

1. Complete and file one copy of this form with your postmaster annually on or before October 1. Keep a copy of the completed form for your records.

2. In cases where the stockholder or security holder is a trustee, include in items 10 and 11 the name of the person or corporation for whom the trustee is acting. Also include the names and addresses of individuals who are stockholders who own or hold 1 percent or more of the total amount of bonds, mortgages, or other securities of the publishing corporation. In item 11, if none, check the box. Use blank sheets if more space is required.

3. Be sure to furnish all circulation information called for in item 15. Free circulation must be shown in items 15d, e, and f.

4. Item 15h., Copies not Distributed, must include (1) newsstand copies originally stated on Form 3541, and returned to the publisher, (2) estimated returns from news agents, and (3), copies for office use, leftovers, spoiled, and all other copies not distributed.

5. If the publication had Periodicals authorization as a general or requester publication, this Statement of Ownership, Management, and Circulation must be published; it must be printed in any issue in October or, if the publication is not published during October, the first issue printed after October.

6. In item 16, indicate the date of the issue in which this Statement of Ownership will be published.

7. Item 17 must be signed.

*Failure to file or publish a statement of ownership may lead to suspension of Periodocals authorizat*