Well Come to the Course

MT-104 Linear Algebra

By

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For BS EE Students

Course Title	Linear Algebra	**Course Code**	MT- 104
Instructor	Dr. Syed Irfan Shah	**Semester**	Sp- 2021

Text Book(s)	**Title**	Linear Algebra and Its Applications (4th Edition).
	Author	David C. Lay
	Publisher	PEARSON
Ref. Book(s)	**Title**	Elementary linear algebra. Application version (11th Edition)
	Author	Howard Anton
	Publisher	Wiley

Course Learning Outcomes	**Description**	**Domain/ Taxonomy Level**	**PLO**
	1. Solve the systems of linear equations.	C3	01
	2. Compute properties of vector space and linear transformation.	C3	01
	3. Apply eigenvalue decomposition and SVD to a matrix.	C3	01

Linear Algebra

and its applications

FOURTH EDITION

David C. Lay

Chapter 1 Linear Equations in Linear Algebra 1

INTRODUCTORY EXAMPLE: Linear Models in Economics and Engineering

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Supplementary Exercises **88**

Google Classroom Code

Section A

EE A Sp 2021

Class code dp6wl7y

Section B

EE B

Class code fakxvls

For BS EE Students

Assessment items of Theory Part

Assessment Methods		**Weight %**
Assignment	4 - 5	08%
Quiz	5 - 7	12%
Midterm Exam	1 - 2	30%
Final Exam	1	50%
Grading criteria		Relative Grading

Chapter # 01

Linear Systems of Equations.

Linear Equation: A linear equation in n unknowns is an equation of the form:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

A sequence of numbers $s_1, s_2, \ldots, s_n$ that satisfies the equations is Known as the solution of the linear Equation .

Which of the following are linear equations in x_1, x_2, and x_3?

(a) $x_1 + 5x_2 - \sqrt{2}x_3 = 1$

(b) $x_1 + 3x_2 + x_1x_3 = 2$

(c) $x_1 = -7x_2 + 3x_3$

(d) $x_1^{-2} + x_2 + 8x_3 = 5$

(e) $x_1^{3/5} - 2x_2 + x_3 = 4$

(f) $\pi x_1 - \sqrt{2}x_2 + \frac{1}{3}x_3 = 7^{1/3}$

Linear Systems

An arbitrary system of m linear equations in n unknowns

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad\; \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{aligned}
$$

The system is called *linear* because each variable x_j appears in the first power only $a_{11}, \ldots, a_{mn}$ are given numbers, called the **coefficients** of the system. $b_1, \ldots, b_m$ on the right are also given numbers. If all the *bj* are zero, then linear system is **homogeneous**. If at least one *bj* is not zero, then it is called a **nonhomogeneous system**.

- A sequence of numbers $s_1, s_2, \ldots, s_n$ that satisfies the system of equations is called a solution of the system.
- A system that has **no** solution is said to be inconsistent ; if there is **at least** one solution of the system, it is called consistent.

Matrix Form of the Linear System

m equations may be written as a single vector equation i.e

$$\mathbf{Ax} = \mathbf{b}$$

where the **coefficient matrix A** = $[a_{jk}]$ is the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

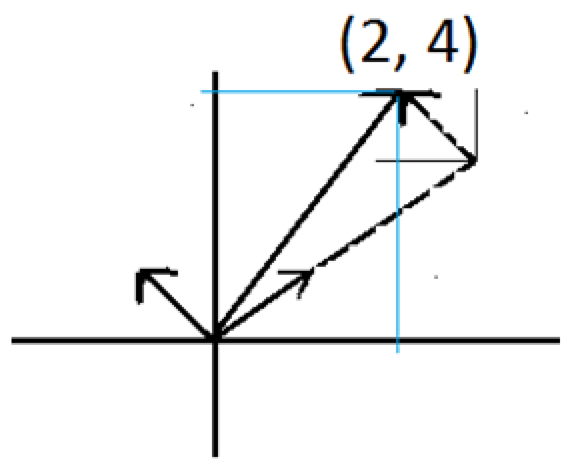
Vector Equation of the Linear System

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$x - y = 2$$
$$x + y = 4$$

$$x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$



Matrix Form of the Linear System . (continued)

The matrix

$$\mathbf{A}b = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

is called the **augmented matrix** of the system . The dashed vertical line could be omitted, as we shall do later. It is merely a reminder that the last column of $\tilde{\mathbf{A}}$ did not come from matrix **A** but came from vector **b**. Thus, we *augmented* the matrix **A**.

- *System:*

$$\begin{aligned} x - 4y + 3z &= 5 \\ -x + 3y - z &= -3 \\ 2x \qquad - 4z &= 6 \end{aligned}$$

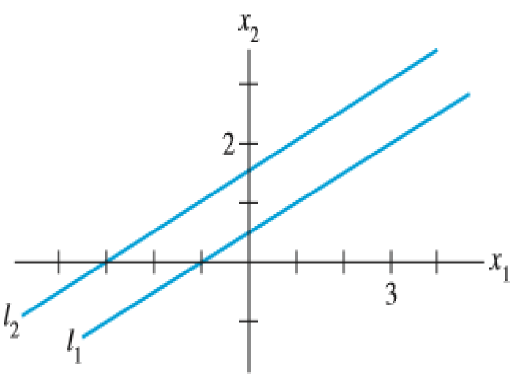
- *Augmented Matrix:* $\begin{bmatrix} 1 & -4 & 3 & \vdots & 5 \\ -1 & 3 & -1 & \vdots & -3 \\ 2 & 0 & -4 & \vdots & 6 \end{bmatrix}$

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

(a)

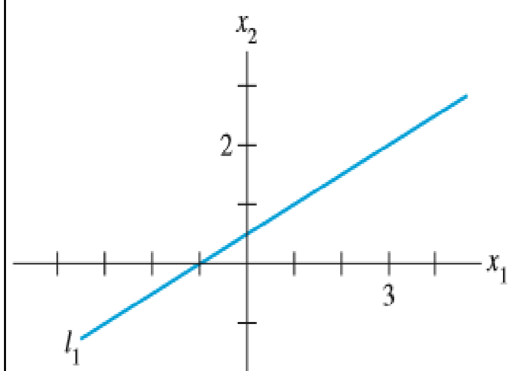
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 2x_2 & = & 3 \\ \hline 0 & + & 0 & = & 2 \end{array}$$



(a) No solution

(b)

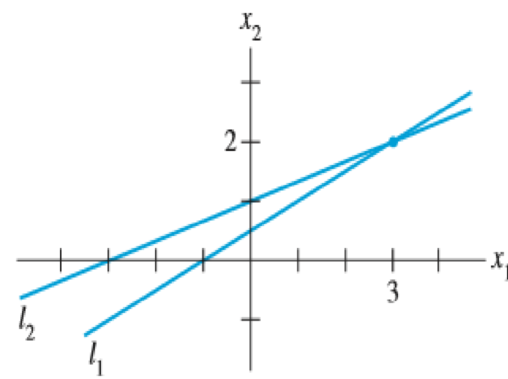
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 2x_2 & = & 1 \\ \hline 0 & + & 0 & = & 0 \end{array}$$



(b) Infinitely many solutions

(c)

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \\ \hline & & x_2 & = & 2 \end{array}$$



(c) One solution

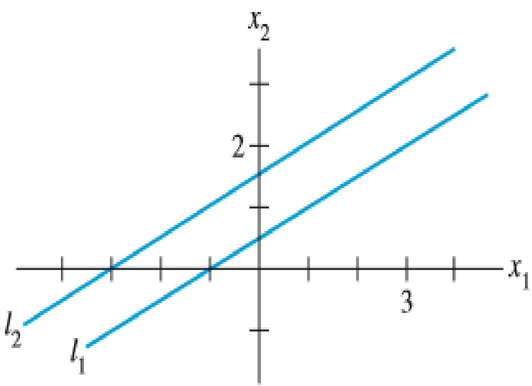
- *Every system of linear equations has either no solutions, infinitely many solutions or*
- *exactly one solution,*

Consistent System

\- Inconsistent System

(a)

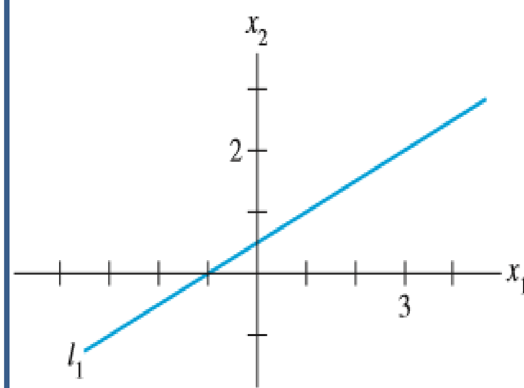
$$\begin{array}{r} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 3 \\ \hline 0 + 0 = 2 \end{array}$$



(*a*) No solution

(b)

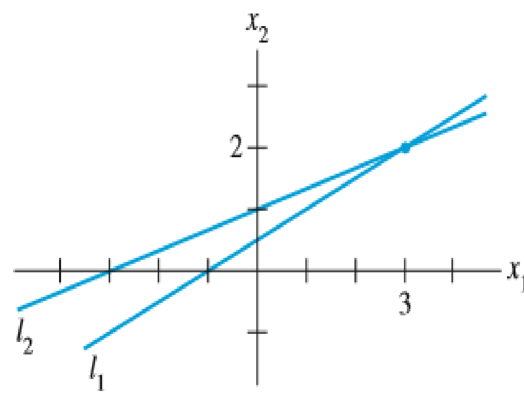
$$\begin{array}{r} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \\ \hline 0 + 0 = 0 \end{array}$$



(*b*) Infinitely many solutions

(*c*)

$$\begin{array}{r} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \\ \hline x_2 = 2. \end{array}$$



(*c*) One solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & \frac{3}{5} \\ 0 & 1 & 4 & -8 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Unique Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 12 & 70 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

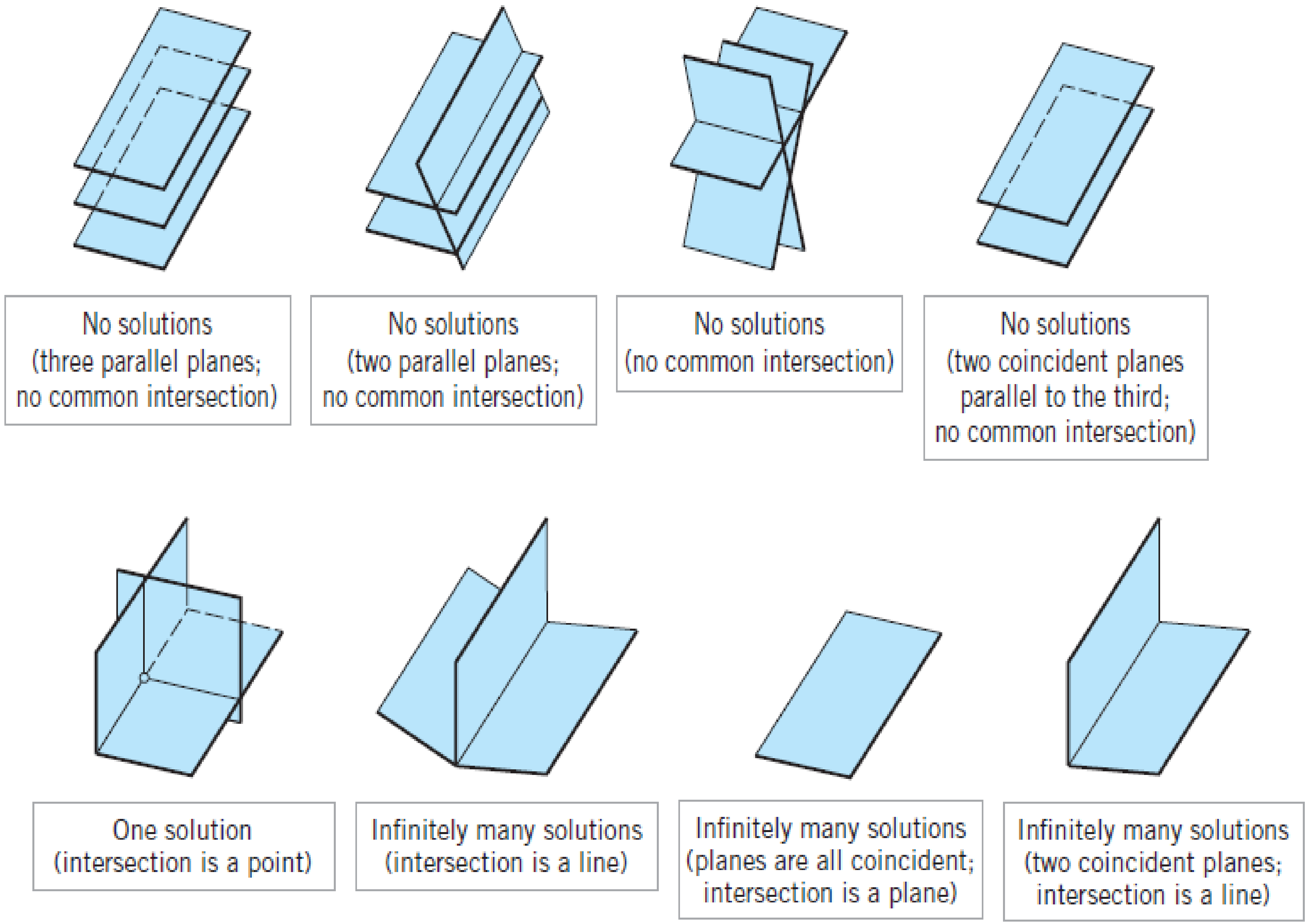
More than one solution

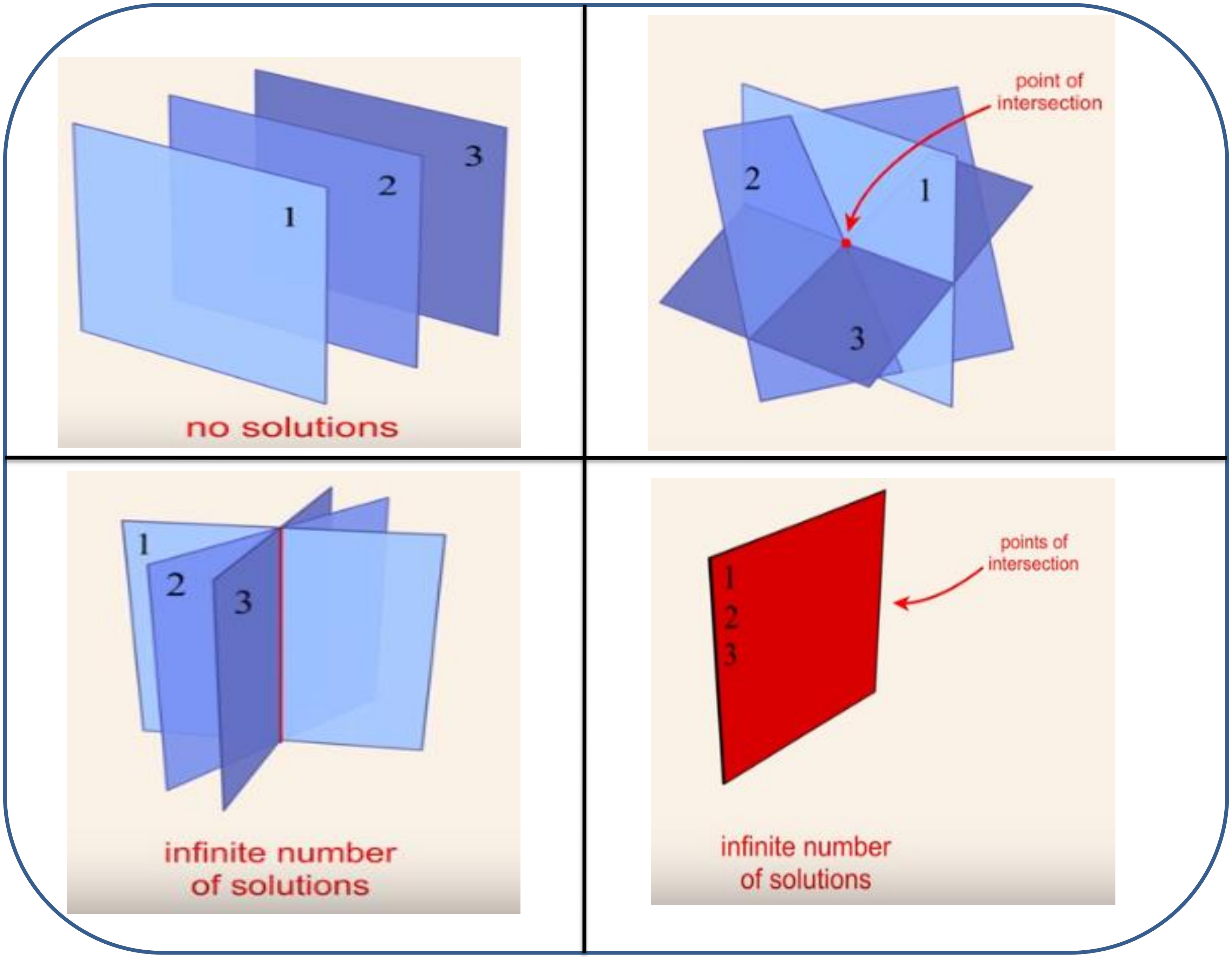
Consistent System

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

No Solution

Inconsistent System


No solutions
(three parallel planes;
no common intersection)
No solutions
(two parallel planes;
no common intersection)
No solutions
(no common intersection)
No solutions
(two coincident planes
parallel to the third;
no common intersection)
One solution
(intersection is a point)
Infinitely many solutions
(intersection is a line)
Infinitely many solutions
(planes are all coincident;
intersection is a plane)
Infinitely many solutions
(two coincident planes;
intersection is a line)



1
2
3
no solutions
point of intersection
2
1
3
1
2
3
infinite number of solutions
points of intersection
1
2
3
infinite number of solutions

$$\left.\begin{aligned} 3x+8y+4z&=18\\ 3x+2y-2z&=12 \end{aligned}\right\}$$

Solution

$$R_2-R_1\quad\begin{bmatrix} 3 & 8 & 4 & 18\\ 3 & 2 & -2 & 12 \end{bmatrix}$$

$$-\frac{1}{22}R_2\quad\begin{bmatrix} 3 & 8 & 4 & 18\\ 0 & -22 & -14 & -42 \end{bmatrix}$$

$$\begin{array}{cccc} x & y & z & \\ \end{array}$$
$$\begin{bmatrix} 3 & 8 & 4 & 18\\ 0 & 1 & \frac{7}{11} & \frac{21}{11} \end{bmatrix}$$

$$\boxed{z=s}\qquad -\infty<s<\infty$$

$$\boxed{y=\frac{21}{11}-\frac{7}{11}s}$$

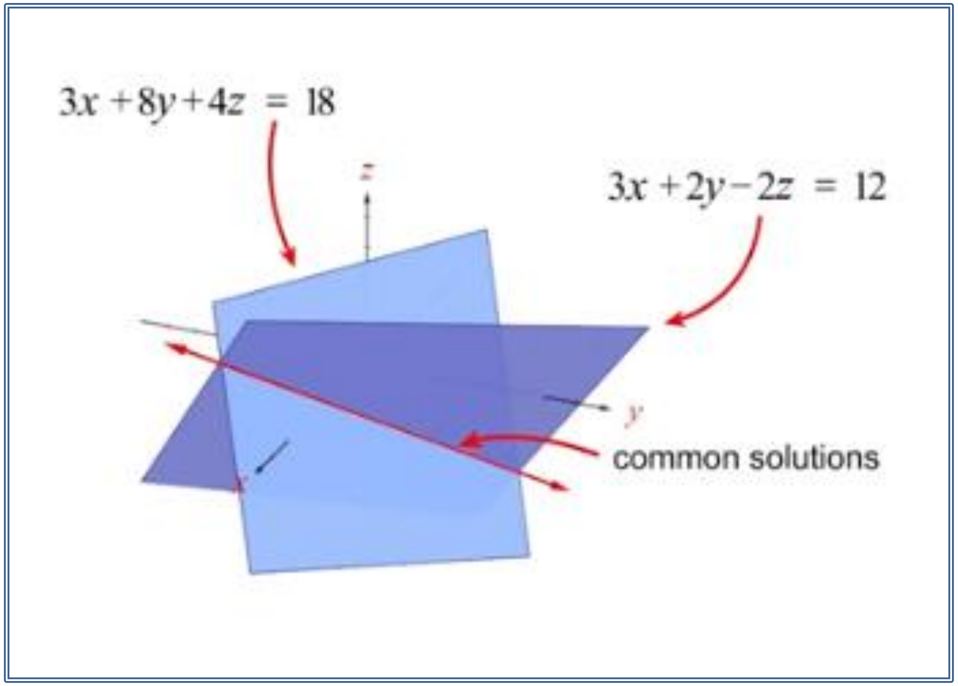
$$3x=18-8y-4z$$

$$3x=18-8\left(\frac{21}{11}-\frac{7}{11}s\right)-4(s)$$

$$3x=\frac{30}{11}-\frac{12}{11}s$$

$$\boxed{x=\frac{10}{11}-\frac{4}{11}s}$$

$$\begin{bmatrix} x\\ y\\ z \end{bmatrix}=\begin{bmatrix} \frac{10}{11}-\frac{4}{11}s\\ \frac{21}{11}-\frac{7}{11}s\\ s \end{bmatrix}$$

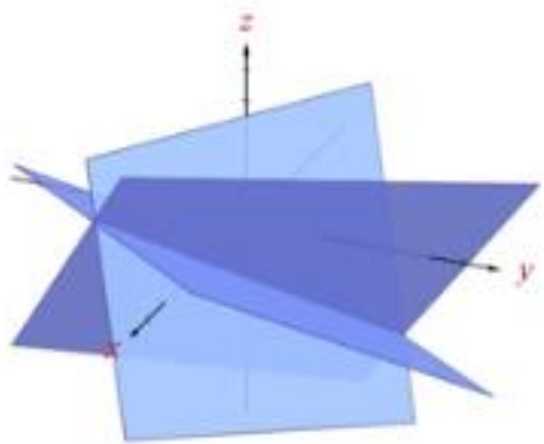


system of equations

$$3x + 8y + 4z = 18$$

$$3x - 2y - 6z = 8$$

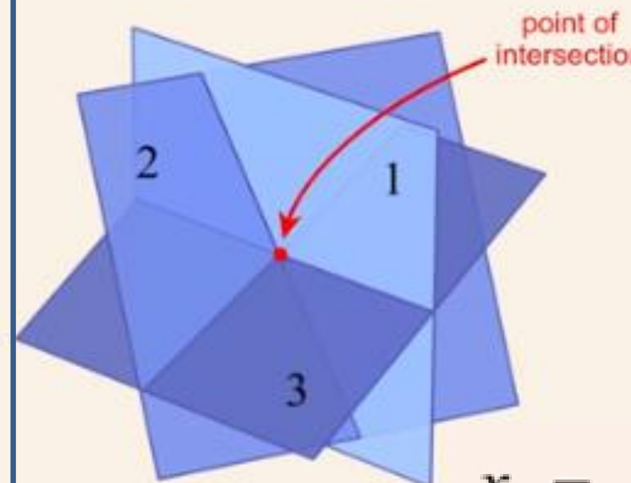
$$3x + 2y - 2z = 12$$



$$x + y - z = 1$$

$$-2x - y + 3z = 4$$

$$5x - 3y - 2z = 1$$



$$x = 3$$

$$y = 2$$

$$z = 4$$

Every system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Comments

- If a system has a solution, call it <u>consistent</u>
- If a system doesn't have a solution, call it <u>inconsistent</u>
- A homogeneous system always has the trivial solution $x_1 = x_2 = \cdots = x_n = 0$
- If two systems have the same solution, then they are called <u>equivalent</u>. The solution strategy for linear systems is to transform the system through a series of equivalent systems until the solution is obvious

Row Operations

Operations for equations	Row operations for matrices
Interchange of two equations	Interchange of two rows
Addition of a constant multiple of one equation to another equation	Addition of a constant multiple of one row to another row
Multiplication of an equation by a nonzero constant c	Multiplication of a row by a nonzero constant c

Theorem

Equivalent systems and equivalent matrices:

Two matrices are **row-equivalent** when one can be obtained from the other by a sequence of elementary row operations. If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\tfrac{1}{2}R_1 \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

The system can be solved by performing operations on the augmented matrix. Corresponding to the three operations used to obtain equivalent systems, the following row operations may be applied to the augmented matrix:

<u>Elementary Row Operations</u>

I. Interchange two rows.

II. Multiply a row by a nonzero real number.

III. Replace a row by its sum with a multiple of another row.

Elementary Row Operations—Notation

- We use the following notation to describe the elementary row operations:

Symbol	Description
kR_i	Multiply the i th row by k.
$R_i \leftrightarrow R_j$	Interchange the i th and j th rows.
$R_i + kR_j \rightarrow R_i$	Change the i th row by adding k times row j to it. Then, put the result back in row i.

Row- Echelon Form

- A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:
  1. All nonzero rows are above any rows of all zeros.
  2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
  3. All entries in a column below a leading entry are zeros.

$$\left[\begin{array}{ccc|c} 1 & -2 & 5 & \frac{3}{5} \\ 0 & 1 & 4 & -8 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

Unique Solution

$$\left[\begin{array}{ccc|c} 1 & 2 & 12 & 70 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

More than one solution

Consistent System

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right].$$

No Solution

Inconsistent System

- **Example 1:** Row reduce the matrix A below to echelon form, and locate the pivot columns of A.

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad R_{14}$$

- **Solution:** The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position.

Pivot Positions

- Now, interchange rows 1 and 4.

Pivot

$$\begin{array}{c} \\ R_2 + R_1 \\ R_3 + 2R_1 \\ \\ \end{array} \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

Pivot column

- Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and try to reach the matrix of the form:

$$\frac{1}{2}R_2 \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{matrix} R_3 - 5R_2 \\ R_4 + 3R_2 \end{matrix} \quad \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

Try to reach at the given form of matrix

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- There is no way a leading entry can be created in column 3. But, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

Pivot

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns

- The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of *A* are pivot columns.
- The pivots in the example are 1, 2 and -5.

Row Echelon Form and Information From It

The original system of m equations in n unknowns has augmented matrix $[\mathbf{A} \mid \mathbf{b}]$. This is to be row reduced to matrix $[\mathbf{R} \mid \mathbf{f}]$. The two systems $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Rx} = \mathbf{f}$ are equivalent: if either one has a solution, so does the other, and the solutions are identical.

From the echelon form, one can conclude three possible cases:

1. No solution
2. Precisely one solution
3. Infinitely many solutions

The following matrices are in row echelon form:

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

If a matrix in row echelon form is actually the augmented matrix of a linear system, the system is quite easy to solve by back substitution alone.

Solve the system

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - x_2 - 2x_3 &= -5 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

Row echelon form is

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The corresponding system is

$$\begin{aligned} x_1 - x_2 - 2x_3 &= -5 \\ x_2 + x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

back substitution gives

$$\begin{aligned} x_3 &= 2, \\ x_2 &= 3 - x_3 = 3 - 2 = 1, \\ x_1 &= -5 + x_2 + 2x_3 \\ &= -5 + 1 + 4 = 0. \end{aligned}$$

We write the solution vector as

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Gaussian Elimination

When row reduction is applied to the augmented matrix of a system of linear equations, we create an equivalent system that can be solved by back substitution. The entire process is known as ***Gaussian elimination.***

To solve a system of linear equations using Gaussian elimination method:

Step 1: Write the augmented matrix for the system.

Step 2: Use elementary row operations to convert the augmented matrix in row-echelon form.

Step 3: Use back substitution to solve the resulting system of equations.

Gaussian Elimination with Back-Substitution

$$\begin{aligned} x - 2y + 3z &= 9 \\ -x + 3y + z &= -2 \\ 2x - 5y + 5z &= 17 \end{aligned}$$

$$\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ -1 & 3 & 1 & -2 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

$$\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & -1 & -1 & -1 \end{array}\right]$$

$$\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

$$\left[\begin{array}{rrr:r} 1 & -2 & 3 & 9 \\ 0 & 1 & 4 & 7 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x = 1$, $y = -1$, and $z = 2$.

Reduce Row Echelon Form

- If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):
    4. The leading entry in each nonzero row is 1.
    5. Each leading 1 is the only nonzero entry in its column.
- Matrix in Reduce Echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Solve the system

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y &= -3 \end{aligned}$$

Solution The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

which can be row reduced as follows:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right] \xrightarrow{R_3 + 2R_2}$$

$$\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

In this, the leading variables are w and y, and the free variables are x and z.

If we assign parameters $x = s$ and $z = t$, the solution can be written in vector form as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

NOTE

- Any nonzero matrix may be **row reduced** (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique.

  Theorem : Uniqueness of the Reduced Echelon Form

  Each matrix is row equivalent to one and only one reduced echelon matrix.

Pivot Position in Reduce Echelon Form

A **pivot position** in a matrix *A* is a location in *A* that corresponds to a leading 1 in the reduced echelon form of *A*. A **pivot column** is a column of *A* that contains a pivot position.

The following matrix is in reduced row echelon form:

Pivot

Pivot column

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot

Pivot column

For 2×2 matrices, the possible reduced row echelon forms are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where * can be any number.

Theorem : Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—i.e., if and only if an echelon form of the augmented matrix has *no* row of the form

[0 … 0 *b*] with *b* nonzero.

- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least on free variable.

Row Reduction To Solve a Linear System

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. **Decide** whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the solution of the system in term of basic variable and free variables if exist

To solve a system of linear equations using Gauss-Jordan elimination method

Gauss-Jordan Elimination

In Gauss-Jordan elimination, we proceed as in Gaussian elimination but reduce the augmented matrix to *reduced* row echelon form.

Step 1: Write the augmented matrix of the system of linear equations.

Step 2: Use elementary row operations to reduce the augmented matrix to reduced row echelon form.

Step 3: If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

EXAMPLE Use Gauss–Jordan reduction to solve the system

$$\begin{aligned} -x_1 + x_2 - x_3 + 3x_4 &= 0 \\ 3x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 - x_2 - 2x_3 - x_4 &= 0 \end{aligned}$$

Solution

$$\left(\begin{array}{cccc|c} \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ \mathbf{0} & \mathbf{4} & \mathbf{-4} & \mathbf{8} & \mathbf{0} \\ 0 & 1 & -4 & 5 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{array}\right) \quad \text{row echelon form}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -1 & 0 & -2 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \quad \text{reduced row echelon form}$$

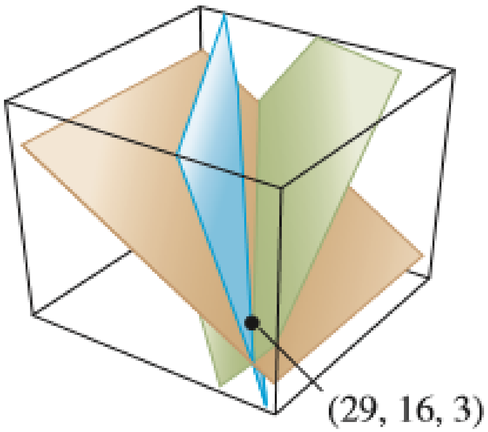
If we set x_4 equal to any real number α, then $x_1 = \alpha$, $x_2 = -\alpha$, and $x_3 = \alpha$. Thus, all ordered 4-tuples of the form $(\alpha, -\alpha, \alpha, \alpha)$ are solutions of the system.

Interpret the problem

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{cases} x_1 = 29 \\ x_2 = 16 \\ x_3 = 3 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$



Each of the original equations determines a plane in three-dimensional space. The point (29, 16, 3) lies in all three planes.

More on System of Equations

Overdetermined Systems

A linear system is said to be overdetermined if there are more equations than unknowns. Overdetermined systems are usually (but not always) inconsistent.

EXAMPLE 4 Solve each of the following overdetermined systems:

(a)
$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ -x_1 + 2x_2 &= -2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned}$$

System (a $\rightarrow$ $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right)$

System (b): $\rightarrow$ $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$

Underdetermined Systems

- A system of m linear equations in n unknowns is said to be underdetermined if there are fewer equations than unknowns **($m < n$).**
- Although it is possible for underdetermined systems to be inconsistent, they are usually consistent with infinitely many solutions
- **It is not possible for an underdetermined system to have a unique solution.**
- **A consistent underdetermined system will have infinitely many solutions.**

Solve the following underdetermined systems:

(a) $$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 2x_3 &= 3 \end{aligned}$$

(b) $$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 2 \end{aligned}$$

System (a): $$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$ inconsistent.

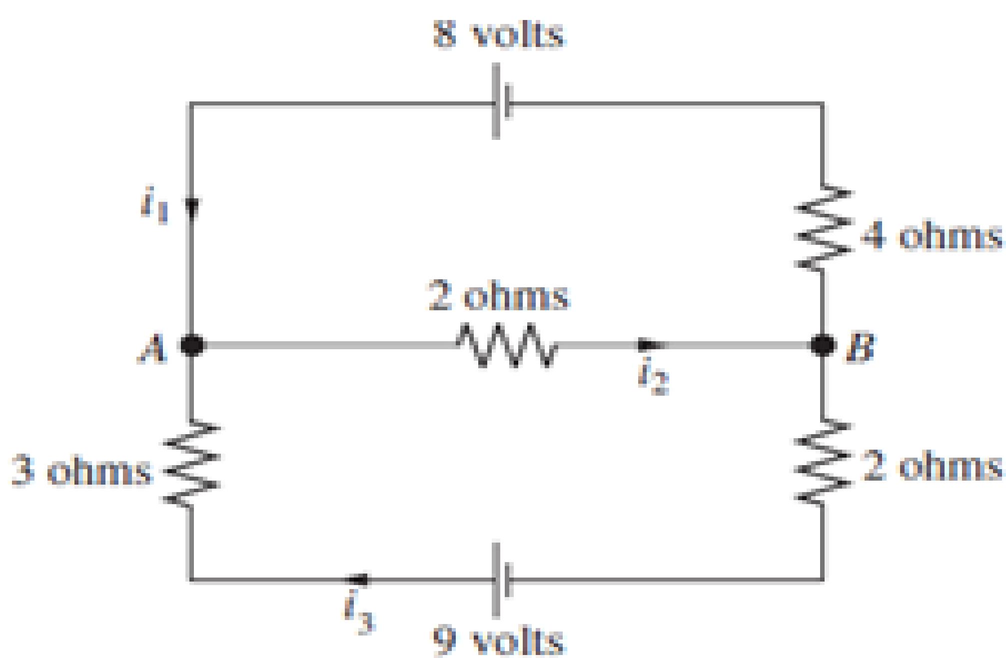
System (b): $$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$ consistent.

Modeling with Linear Systems

Electrical Networks

Kirchhoff's Laws

1. At every node the sum of the incoming currents equals the sum of the outgoing currents.
2. Around every closed loop, the algebraic sum of the voltage gains must equal the algebraic sum of the voltage drops. The voltage drops E for each resistor are given by Ohm's law: $V = IR$

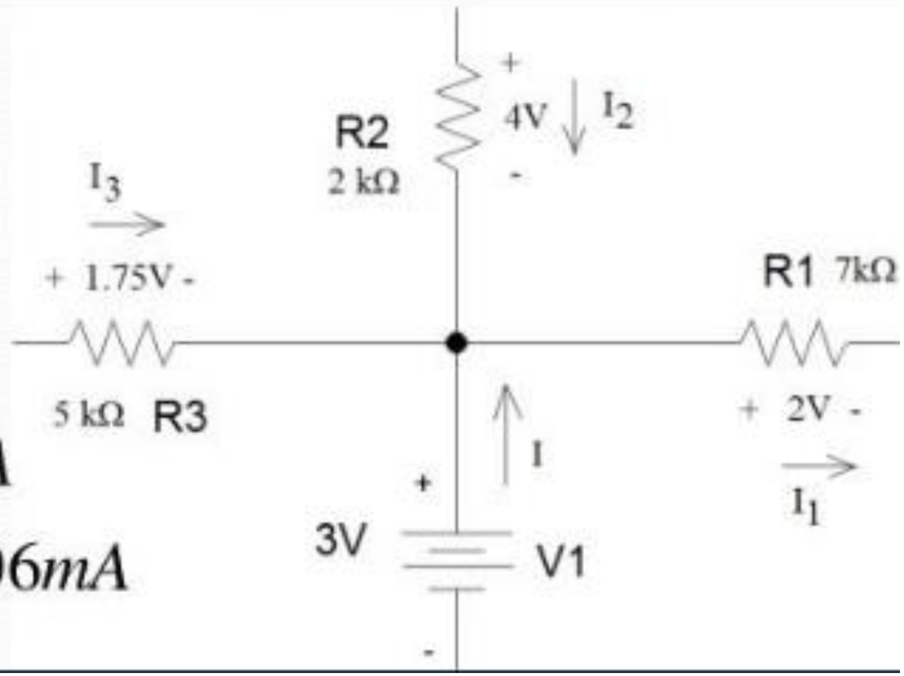


- I_1 is leaving the node.
- I_2 is entering the node.
- I_3 is entering the node.
- I is entering the node.

$$I_2 + I_3 + I = I_1$$

$$2mA + 0.35mA + I = 0.286mA$$

$$I = 0.286mA - 2.35mA = -2.06mA$$

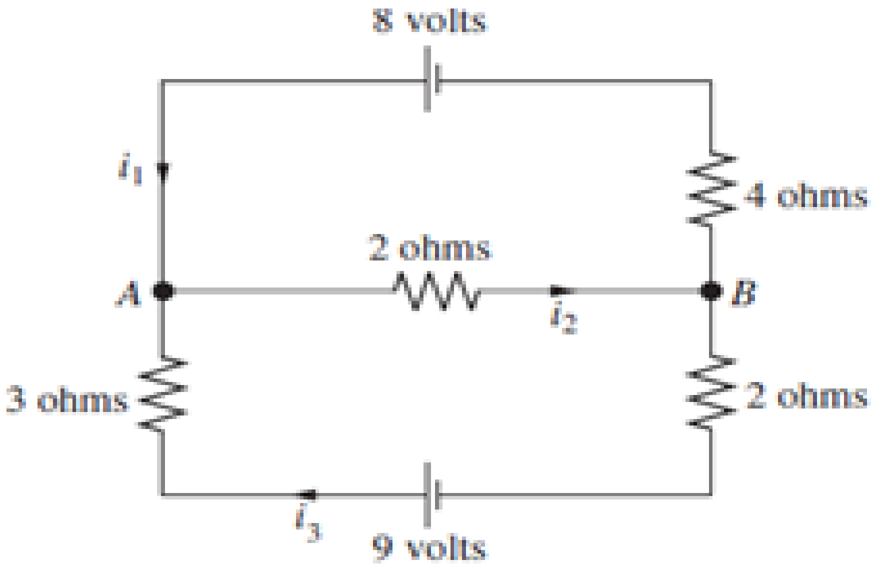


Application 1 Electrical Networks

In an electrical network, it is possible to determine the amount of current in each branch in terms of the resistances and the voltages. An example of a typical circuit is given in below Figure:

$$
\begin{aligned}
i_1 - i_2 + i_3 &= 0 \qquad \text{(node } A\text{)} \\
-i_1 + i_2 - i_3 &= 0 \qquad \text{(node } B\text{)}
\end{aligned}
$$

$$
\begin{aligned}
4i_1 + 2i_2 &= 8 \qquad \text{(top loop)} \\
2i_2 + 5i_3 &= 9 \qquad \text{(bottom loop)}
\end{aligned}
$$



Let us find the currents in the network

$$\begin{aligned} i_1 - i_2 + i_3 &= 0 \qquad \text{(node } A\text{)} \\ -i_1 + i_2 - i_3 &= 0 \qquad \text{(node } B\text{)} \end{aligned}$$

By the second law,

$$\begin{aligned} 4i_1 + 2i_2 &= 8 \qquad \text{(top loop)} \\ 2i_2 + 5i_3 &= 9 \qquad \text{(bottom loop)} \end{aligned}$$

The network can be represented by the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right]$$

This matrix is easily reduced to the row echelon form

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solving by back substitution, we see that $i_1 = 1$, $i_2 = 2$, and $i_3 = 1$.

Application 2 Nutritional Analysis

A nutritionist is performing an experiment on student volunteers.

He wishes to feed one of his subjects a daily diet that consists of a combination of three commercial diet foods:

MiniCal, LiquiFast & SlimQuick

For the experiment, it's important that, every day, the subject consume exactly :

500 mg of potassium, 75 g of protein and 1150 units of vitamin D

The amounts of these nutrients in one ounce of each food are given here.

	MiniCal	LiquiFast	SlimQuick
Potassium (mg)	50	75	10
Protein (g)	5	10	3
Vitamin D (units)	90	100	50

Assignment

How many ounces of each food should the subject eat every day to satisfy the nutrient requirements exactly?

Solution Let x, y, and z represent the number of ounces of MiniCal, LiquiFast, and SlimQuick, respectively, that the subject should eat every day.

Based on the requirements of the three nutrients, we get the system

$$\begin{cases} 50x + 75y + 10z = 500 & \text{Potassium} \\ 5x + 10y + 3z = 75 & \text{Protein} \\ 90x + 100y + 50z = 1150 & \text{Vitamin D} \end{cases}$$

From the reduced row-echelon form, we see that: $x = 5$, $y = 2$, and $z = 10$ is the solution to the problem. Which mean that every day, the subject should be fed by

05 oz of MiniCal

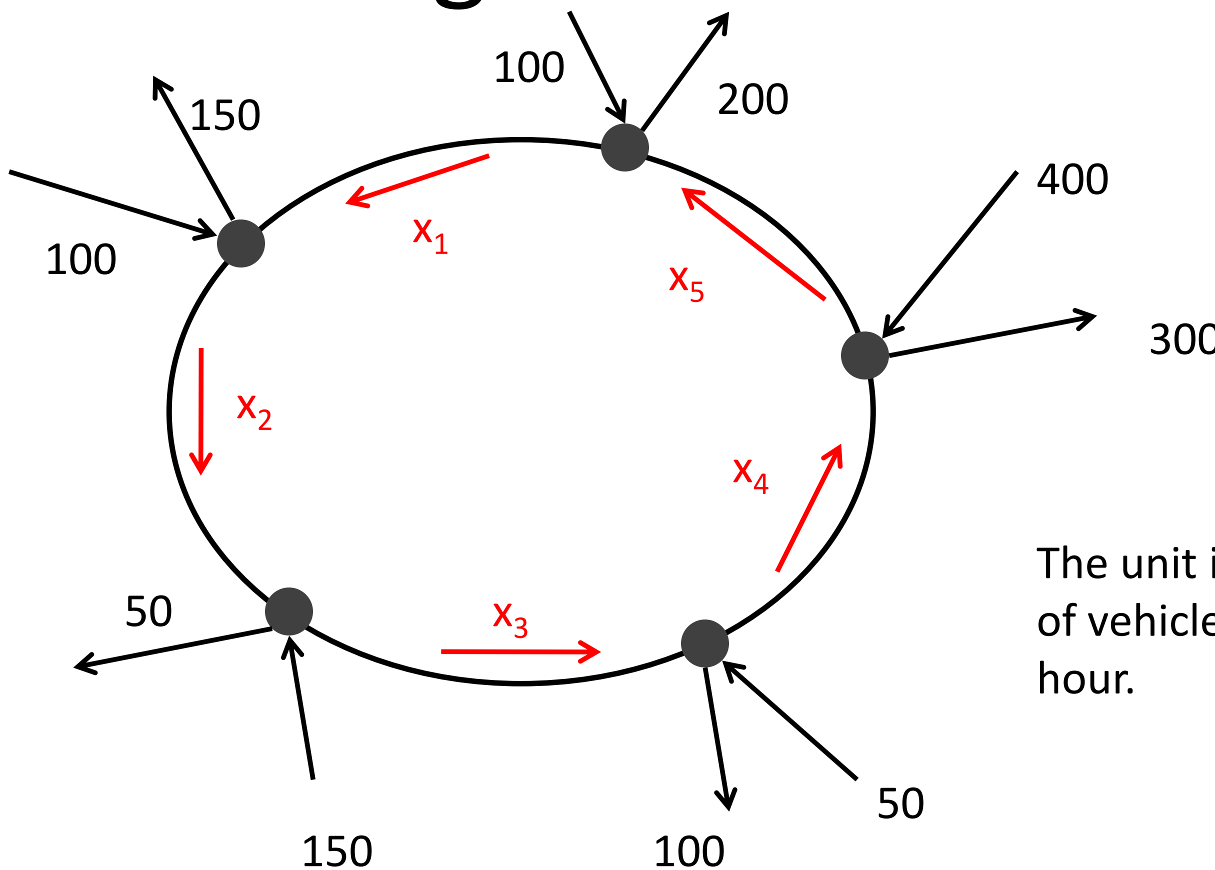
02 oz of LiquiFast &

10 oz of SlimQuick

Application 3 A Traffic flow problem

- A round-about connecting 5 roads
- The traffic within the circle is counter-clockwise.
- Question: model the traffic in each section of the round-about

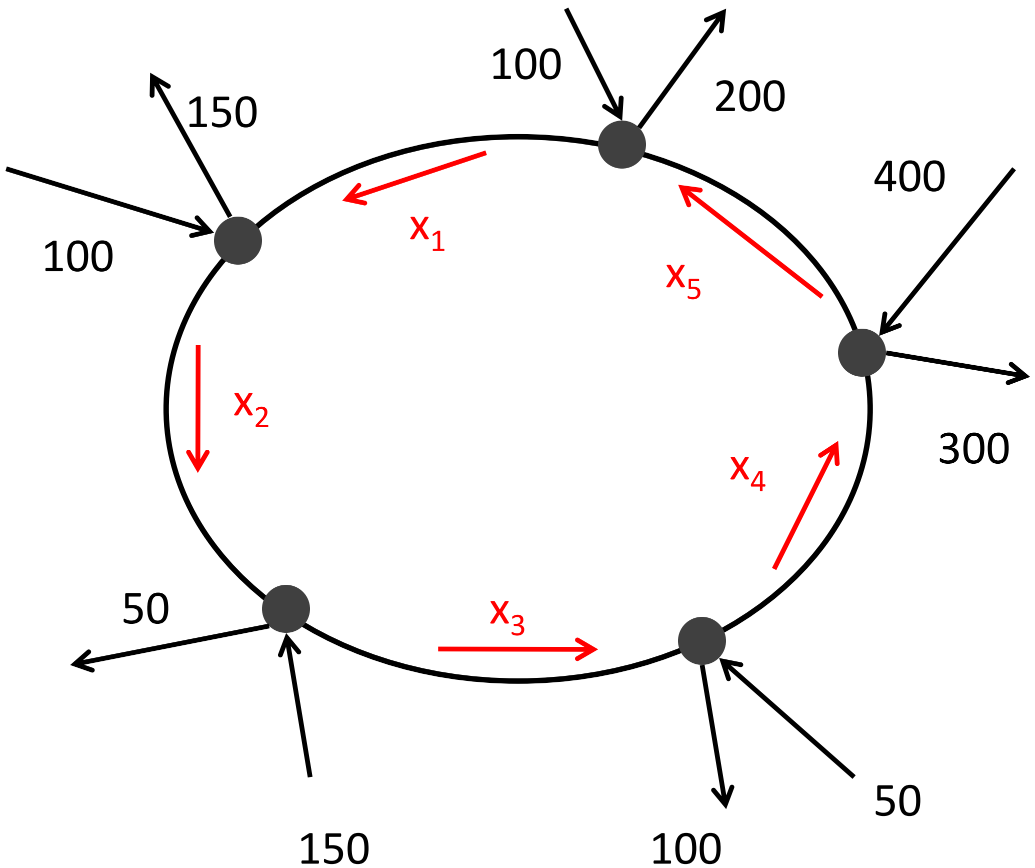
Modeling a round-about



The unit is number of vehicles per hour.

Can we find x_1, x_2, x_3, x_4 and x_5?

In-flow = out-flow



$$x_1 + 100 = x_2 + 150$$

$$x_2 + 150 = x_3 + 50$$

$$x_3 + 50 = x_4 + 100$$

$$x_4 + 400 = x_5 + 300$$

$$x_5 + 100 = x_1 + 200$$

We need to solve …

$$x_1 - x_2 = 150 - 100 = 50$$
$$x_2 - x_3 = 50 - 150 = -100$$
$$x_3 - x_4 = 100 - 50 = 50$$
$$x_4 - x_5 = 300 - 400 = -100$$
$$x_5 - x_1 = 200 - 100 = 100$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 & -1 & -100 \\ -1 & 0 & 0 & 0 & 1 & 100 \end{array}\right] \end{array}$$

Row operations

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 & -1 & -100 \\ -1 & 0 & 0 & 0 & 1 & 100 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 & -1 & -100 \\ 0 & -1 & 0 & 0 & 1 & 150 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & -1 & 0 & 1 & 50 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & 0 & -1 & 1 & 100 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 50 \\ 0 & 1 & -1 & 0 & 0 & -100 \\ 0 & 0 & 1 & -1 & 0 & 50 \\ 0 & 0 & 0 & 1 & -1 & -100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = f - 100$$
$$x_2 = f - 150$$
$$x_3 = f - 50$$
$$x_4 = f - 100$$
$$x_5 = f$$

where f is a real number larger than or equal to 150

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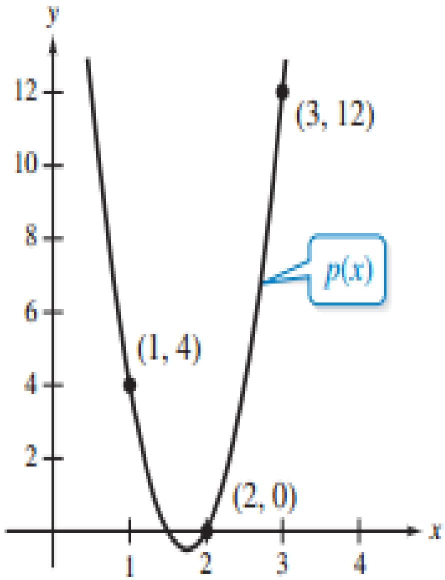

Figure 1.5

Simulation

Explore this concept further with an electronic simulation available at *www.cengagebrain.com*.

EXAMPLE 4 Polynomial Curve Fitting

Determine the polynomial $p(x) = a_0 + a_1x + a_2x^2$ whose graph passes through the points (1, 4), (2, 0), and (3, 12).

SOLUTION

Substituting $x = 1$, 2, and 3 into $p(x)$ and equating the results to the respective y-values produces the system of linear equations in the variables a_0, a_1, and a_2 shown below.

$$
\begin{aligned}
p(1) &= a_0 + a_1(1) + a_2(1)^2 = a_0 + a_1 + a_2 = 4\\
p(2) &= a_0 + a_1(2) + a_2(2)^2 = a_0 + 2a_1 + 4a_2 = 0\\
p(3) &= a_0 + a_1(3) + a_2(3)^2 = a_0 + 3a_1 + 9a_2 = 12
\end{aligned}
$$

The solution of this system is

$$a_0 = 24,\ a_1 = -28,\ \text{and } a_2 = 8$$

so the polynomial function is

$$p(x) = 24 - 28x + 8x^2.$$

Figure 1.5 shows the graph of p.

▶ EXAMPLE 5 Balancing Chemical Equations Using Linear Systems

Balance the chemical equation

$$\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + \mathrm{NaCl}$$

[hydrochloric acid] + [sodium phosphate] ⟶ [phosphoric acid] + [sodium chloride]

Solution Let x_1, x_2, x_3, and x_4 be positive integers that balance the equation

$$x_1\,(\mathrm{HCl}) + x_2\,(\mathrm{Na_3PO_4}) = x_3\,(\mathrm{H_3PO_4}) + x_4\,(\mathrm{NaCl})$$

Hydrogen (H)
Chlorine (Cl)
Sodium (Na)
Phosphorus (P)
Oxygen (O)

$$\Rightarrow \begin{bmatrix} \mathrm{H} \\ \mathrm{Cl} \\ \mathrm{Na} \\ \mathrm{P} \\ \mathrm{O} \end{bmatrix}$$

<u>The vector equation is</u>

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \\ 4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Equating the number of atoms of each type on the two sides yields

$$\begin{aligned} 1x_1 &= 3x_3 \\ 1x_1 &= 1x_4 \\ 3x_2 &= 1x_4 \\ 1x_2 &= 1x_3 \\ 4x_2 &= 4x_3 \end{aligned}$$

from which we obtain the homogeneous linear system

$$\begin{aligned} x_1 \quad - 3x_3 \quad &= 0 \\ x_1 \quad - x_4 \quad &= 0 \\ 3x_2 - x_4 \quad &= 0 \\ x_2 - \quad x_3 \quad &= 0 \\ 4x_2 - 4x_3 &= 0 \end{aligned}$$

We leave it for you to show that the reduced row echelon form of the augmented matrix for this system is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

from which we conclude that the general solution of the system is

$$x_1 = t, \quad x_2 = t/3, \quad x_3 = t/3, \quad x_4 = t$$

where t is arbitrary. To obtain the smallest positive integers that balance the equation, we let $t = 3$, in which case we obtain $x_1 = 3$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 3$. Substituting these values produces the balanced equation

$$3HCl + Na_3PO_4 \;=\; H_3PO_4 + 3NaCl \blacktriangleleft$$

Matrix Algebra with MATLAB

- Basic matrix definitions and operations were covered in Chapter 1. We will now consider how these operations are performed in MATLAB. All variables in MATLAB are considered as matrices. A simple scalar is considered as a 1 x 1 matrix. For MATLAB variables containing higher dimensions, certain special rules are required to deal with them.

Entering a Matrix in MATLAB

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix}$$

MATLAB Format

```
>> A = [2 -3 5; -1 4 5]

A =   2   -3   5
     -1    4   5
```

Entering a Row Vector in MATLAB

```
>> x = [1 4 7]

x =  1   4   7
```

Entering a Column Vector in MATLAB

$$\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

```
>> x = [1; 4; 7]

      1
x =   4
      7
```

Alternate Way to Enter a Column Vector

>> x = [1 4 7]'

$$x = \begin{matrix} 1 \\ 4 \\ 7 \end{matrix}$$

Matrix Addition and Subtraction

>> C = A + B

>> D = A - B

Matrix Multiplication

>> E = A*B

Determinant of a Matrix

>> a = det(A)

Inverse Matrix

```
>> B = inv(A)
```

Simultaneous Equation Solution

$$\boxed{\mathbf{Ax} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}}$$

MATLAB Format: `>> x = inv(A)*b`

Alternate MATLAB Format:

```
>> x = A\b
```

Example: Enter the matrices below in MATLAB. They will be used in the next several examples.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix}$$

```
>> A = [2  -3  5; -1  4  6];
>> B = [2  1; 7  -4; 3  1];
>> C = B'
```

$$C = \begin{matrix} 2 & 7 & 3 \\ 1 & -4 & 1 \end{matrix}$$

Example: Determine the sum of A and C and denote it as D.

```
>> D = A + C

D =

    4    4    8
    0    0    7
```

Example . Determine the product of A and B with A first.

```
>> A*B

ans =

    -2    19
    44   -11
```

Example: Determine the product of B and A with B first.

```
>> B*A

ans =

    3    -2    16
   18   -37    11
    5    -5    21
```

Example: Enter the matrix **A**. It will be used in several examples.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

```
>> A = [1  2  -1; -1  1  3; 3  2  1]
A =
    1     2    -1
   -1     1     3
    3     2     1
```

Example : Determine the determinant of A and denote it as a.

```
>> a = det(A)
a = 20
```

Example: Determine the inverse matrix of A and denote it as Ainv.

```
>> Ainv = inv(A)
Ainv =
   -0.2500   -0.2000    0.3500
    0.5000    0.2000   -0.1000
   -0.2500    0.2000    0.1500
```

Example: Use MATLAB to solve the simultaneous equations below.

$$x_1 + 2x_2 - x_3 = -8$$

$$-x_1 + x_2 + 3x_3 = 7$$

$$3x_1 + 2x_2 + x_3 = 4$$

Example :Enter the matrix **A**. It will be used in several examples.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

```
>> A = [1  2  -1; -1  1  3; 3  2  1]
A =
   1    2   -1
  -1    1    3
   3    2    1
```

Example. Continuation.

Assume that A is still in memory.

```
>> b = [-8; 7; 4]
b =
    -8
     7
     4
```

Example . Continuation.

>> x = inv(A)*b

x =

2.0000

-3.0000

4.0000

```
>> d=[B,c]

d =

     1     2    -1    -8
    -1     1     3     7
     3     2     1     4

>> rref(d)

ans =

     1     0     0     2
     0     1     0    -3
     0     0     1     4
```

Example 3-9. Continuation.

- Alternately,
- >> x = A\b
- x =
- 2.0000
- -3.0000
- 4.0000

Practice Question

Determine the values of a for which the system has no solutions, exactly one solution, or infinitely many solutions.

$$\begin{aligned} x + 2y - \quad 3z &= 4 \\ 3x - y + \quad 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

The row-echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & (a^2 - 16) & a - 4 \end{array}\right]$$

$$\begin{aligned} x \quad 2y - 3z &= 4 \\ y - 2z &= 10/7 \\ (a^2 - 16)z &= a - 4 \end{aligned}$$

$$(a^2 - 16)z = a - 4$$

$$(a + 4)(a - 4)z = a - 4$$

Case 1 Unique Solution

If $a \neq \pm 4$, then a will yield a unique solution,

Case II Infinitely Many Solutions

If $a = 4$, then the last equation becomes $0 = 0$, and hence there will be infinitely many solutions.

Case III No Solution

If $a = -4$, then the last equation becomes $0 = -8$, and so the system will have no solutions.