Chapter 15

Oscillatory Motion

Oscillatory Motion

Periodic motion

Spring-mass system

Differential equation of motion

Simple Harmonic Motion (SHM)

Energy of SHM

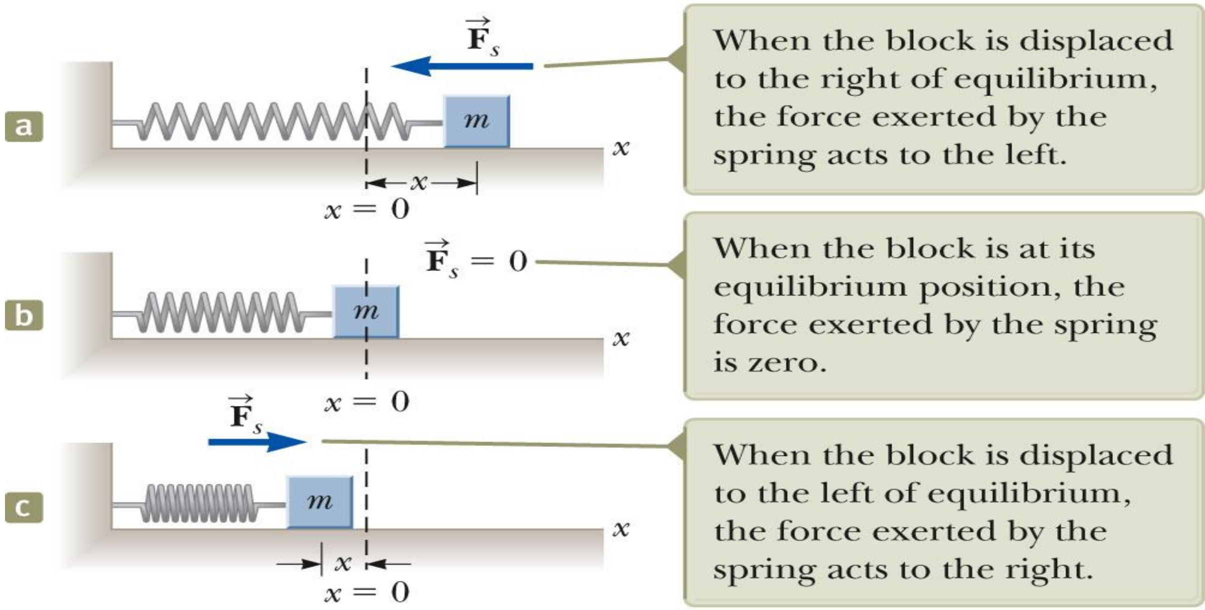
Pendulum

- Simple Pendulum
- Physical Pendulum
- Torsional Pendulum

Periodic Motion

- Periodic motion is a motion that regularly returns to a given position after a fixed time interval.
- A particular type of periodic motion is “simple harmonic motion,” which arises when the force acting on an object is proportional to the position of the object about some equilibrium position.
- The motion of an object connected to a spring is a good example.

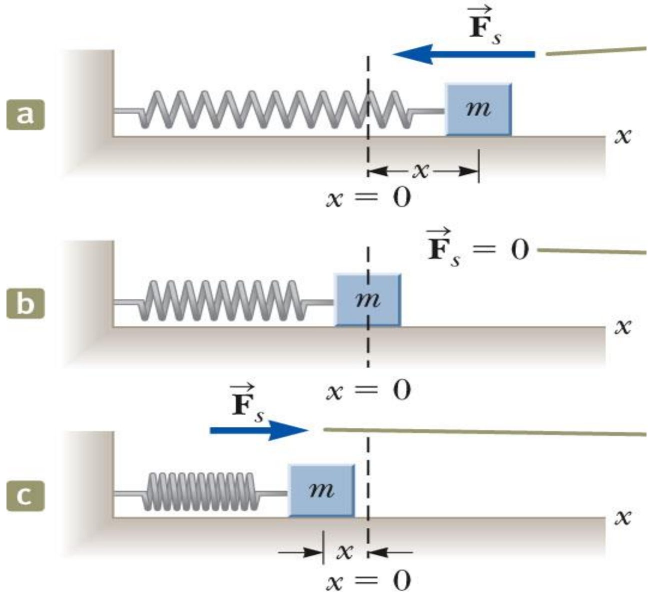
Examples: The beating of your heart, the ticking of a clock, and the movement of a child on a swing.
The Earth returns to the same position in its orbit around the Sun each year.
In alternating-current electrical circuits, voltage, current, and electric charge vary periodically with time



Recall Hooke's Law

Hooke's Law states $F_s = -kx$

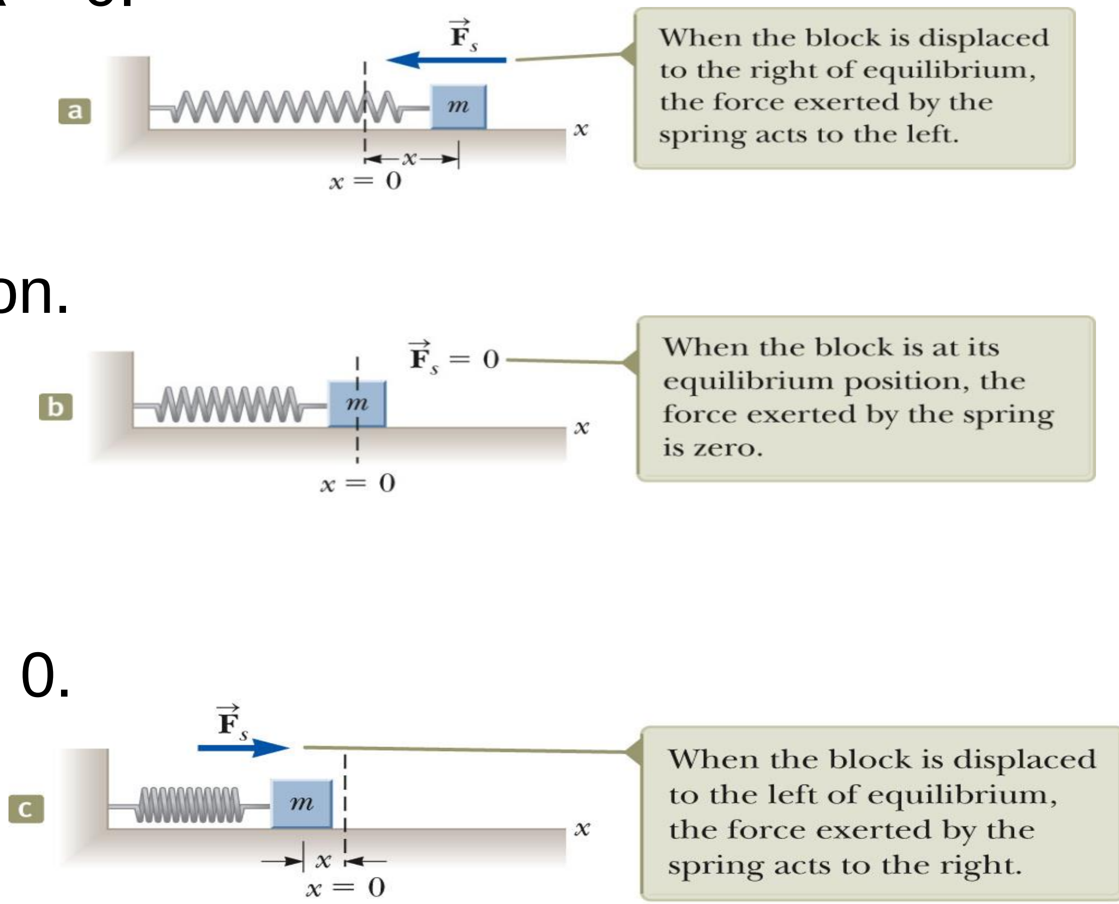
- F_s is the restoring force.
  - It is always directed toward the equilibrium position.
  - Therefore, it is always opposite the displacement from equilibrium.
- k is the force (spring) constant.
- x is the displacement.



CENGAGE Learning™

Restoring Force and the Spring Mass System

- In a, the block is displaced to the right of $x = 0$.
  - The position is positive.
  - The restoring force is directed to the left (negative).
- In b, the block is at the equilibrium position.
  - $x = 0$
  - The spring is neither stretched nor compressed.
  - The force is 0.
- In c, the block is displaced to the left of $x = 0$.
  - The position is negative.
  - The restoring force is directed to the right (positive).



CENGAGE Learning™

Differential Equation of Motion

Using $F = ma$ for the spring, we have $ma = -kx$

But recall that acceleration is the second derivative of the position:

$$a = \frac{d^2 x}{dt^2}$$

So this simple force equation is an example of a *differential equation*,

$$m\frac{d^2 x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2 x}{dt^2} = -\frac{k}{m}x$$

An object moves in simple harmonic motion whenever its acceleration is proportional to its position and has the opposite sign to the displacement from equilibrium.

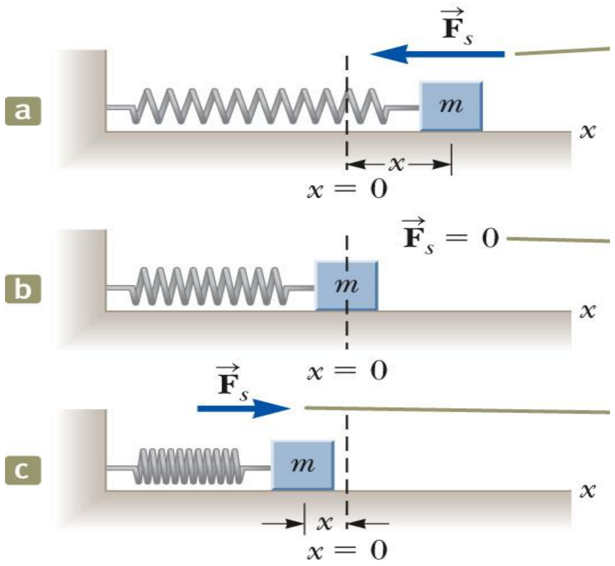
Acceleration

Note that the acceleration is NOT constant, unlike our earlier kinematic equations.

If the block is released from some position $x = A$, then the initial acceleration is $-kA/m$, but as it passes through 0 the acceleration falls to zero.

It only continues past its equilibrium point because it now has momentum (and kinetic energy) that carries it on past $x = 0$.

The block continues to $x = -A$, where its acceleration then becomes $+kA/m$.



Analysis Model, Simple Harmonic Motion

What are the units of k/m, in $a = \dfrac{d^2x}{dt^2} = -\dfrac{k}{m}x$?

They are $1/\mathrm{s}^2$, which we can regard as a frequency-squared, so let's write it as

$$\omega^2 = \frac{k}{m}$$

Then the equation becomes

$$a = -\omega^2 x$$

A typical way to solve such a differential equation is to simply search for a function that satisfies the requirement, in this case, that its second derivative yields the negative of itself! The sine and cosine functions meet these requirements.

Mathematical Representation of Spring Mass System

As we know

$$\Rightarrow a = \frac{dv}{dt} = \frac{dx^2}{dt^2}$$

Now Applying Hook's Law

$$\Rightarrow F = -kx$$

Applying Newton's second Law

$$\sum F_x = ma_x = -kx$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Let $\omega^2 = \frac{k}{m}$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x$$

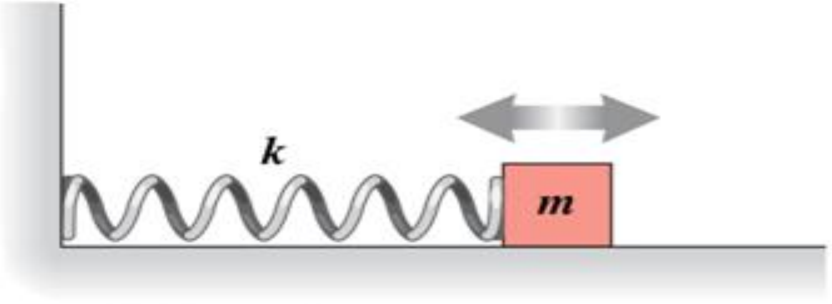
Solution of differential equation

$$\Rightarrow x(t) = A\cos(\omega t + \phi)$$

$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A\sin(\omega t + \phi)$$

$$\frac{d^2x}{dt^2} = -A\omega\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 \boxed{A\cos(\omega t + \phi)}$$

$$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x}$$

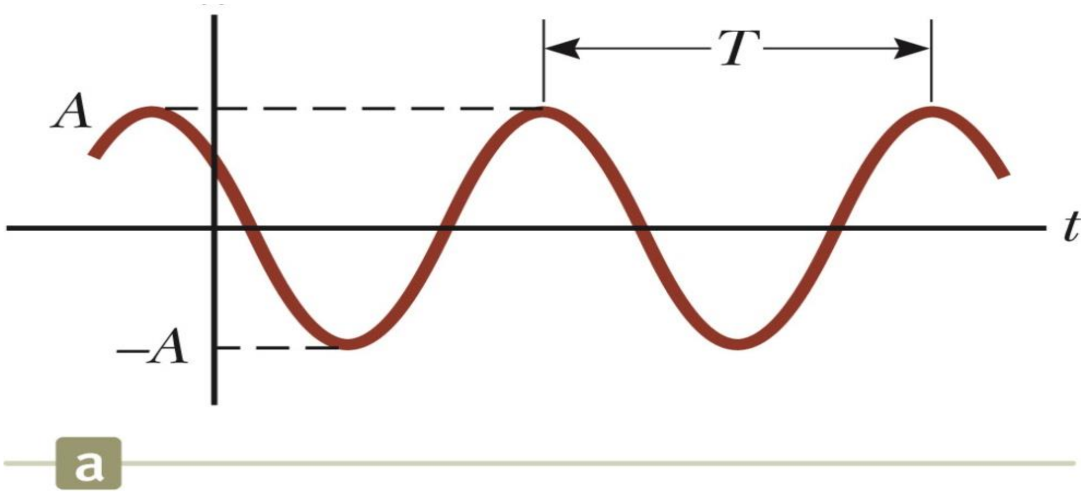


SHM Graphical Representation

A solution to the differential equation is

$$\boxed{x(t) = A\cos(\omega t + \phi)}$$

Position of a particle in simple harmonic motion



A, ω, ϕ are all constants:

A = amplitude (maximum position in either positive or negative x direction,

ω = angular frequency, $\sqrt{\frac{k}{m}}$

ϕ = phase constant, or initial phase angle.

A and ϕ are determined by initial conditions.

Remember, the period and frequency are:

$$T = \frac{2\pi}{\omega} \quad \left(f = \frac{1}{T} = \frac{\omega}{2\pi} \right)$$

Time Period and frequency of spring mass system

The ***period***, *T*, of the motion is the time interval required for the particle to go through one full cycle of its motion.

- The values of *x* and *v* for the particle at time *t* equal the values of *x* and *v* at *t* + *T*.

$$\Rightarrow [\omega(t+T)+\phi]-(\omega t+\phi)=2\pi$$

After simplification

$$\Rightarrow \omega T = 2\pi$$

$$\Rightarrow T = \frac{2\pi}{\omega}$$

$$\boxed{\Rightarrow T = 2\pi\sqrt{\frac{m}{k}}}$$

Time Period

$$\Rightarrow f = \frac{1}{T} = \frac{\omega}{2\pi}$$

The frequency represents the **number of oscillations the particle undergoes per unit time interval:**

$$\boxed{\Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Frequency

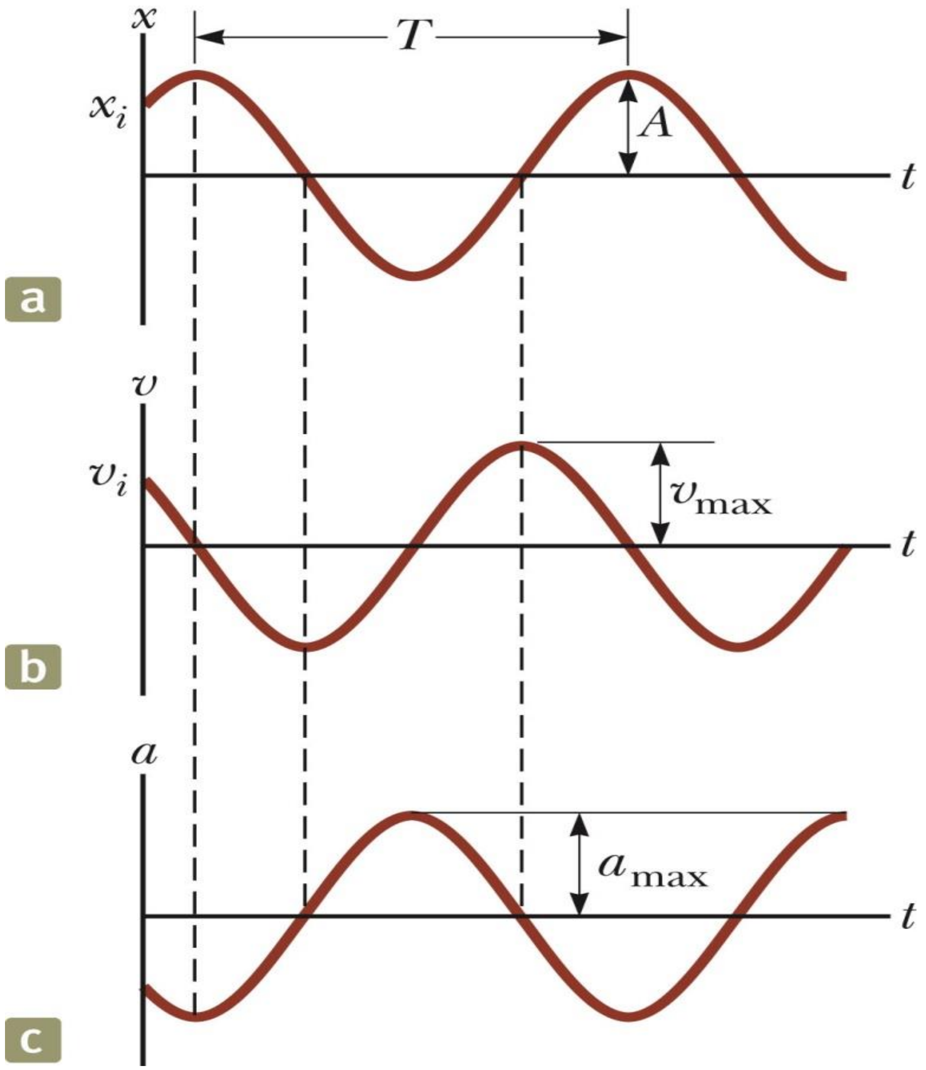
Motion Equations for SHM

$$x(t) = A\cos(\omega t + \phi)$$

$$v(t) = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$a(t) = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi)$$

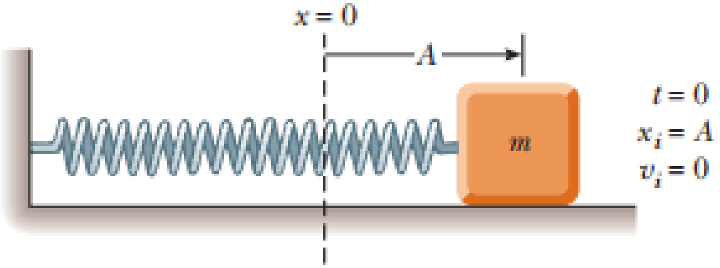
The velocity is 90º out of phase with the displacement and the acceleration is 180º out of phase with the displacement.



SHM Example 1

Initial conditions at $t = 0$ are

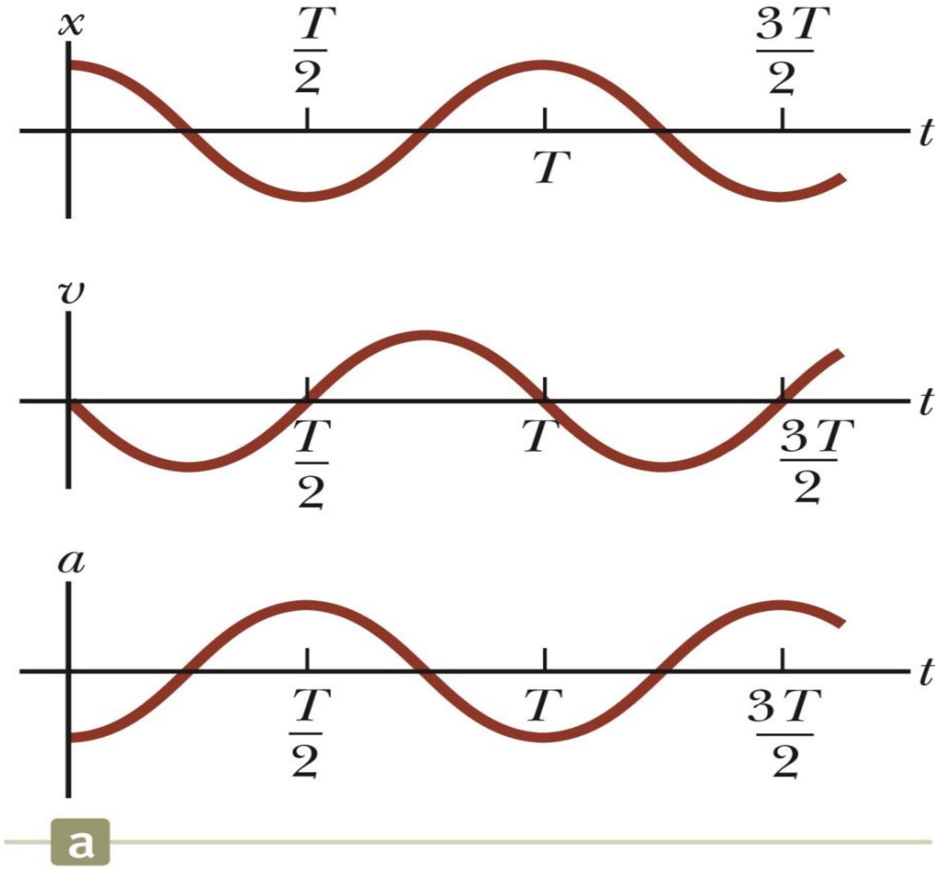
- $x(0) = A$
- $v(0) = 0$



This means $\phi = 0$

The acceleration reaches extremes of $\pm w^2 A$ at $\pm$A.

The velocity reaches extremes of $\pm wA$ at x = 0.



$$\Rightarrow x(t) = A\cos(\omega t + \phi)$$

$$\Rightarrow A = A\cos(\phi) \Rightarrow \frac{A}{A} = 1 = \cos(\phi) \Rightarrow \phi = 0$$

$$\Rightarrow v(t) = -\omega A\sin(\omega t + \phi)$$

SHM Example 2

Initial conditions at t = 0 are

- $x\,(0)=0$
- $v\,(0) = v_i$

This means $\phi = -\pi / 2$

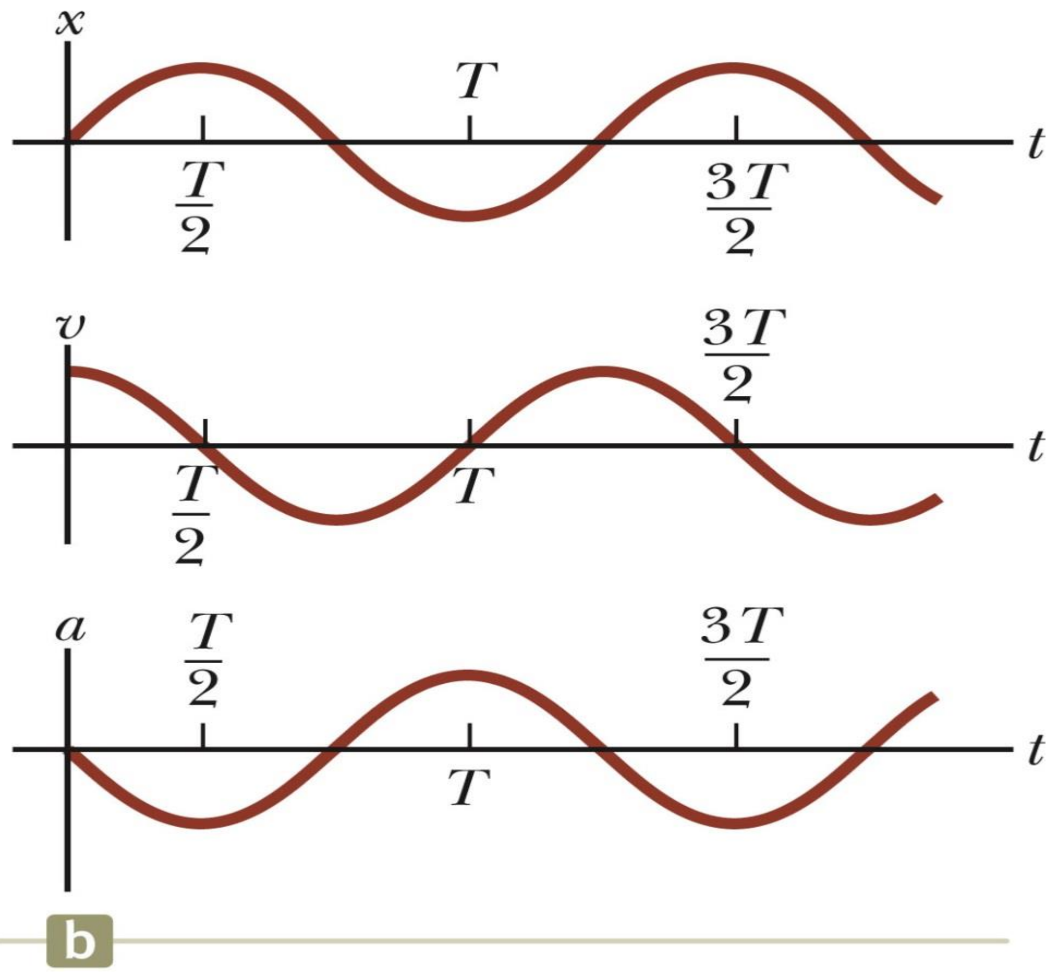
The graph is shifted one-quarter cycle to the right compared to the graph of $x\,(0) = A$.

$$\Rightarrow x(t) = A\cos(\omega t + \phi)$$

$$\Rightarrow 0 = A\cos(\phi)$$

$$\Rightarrow v(t) = -\omega A\sin(\omega t + \phi)$$

$$\Rightarrow v_i = -\omega A$$

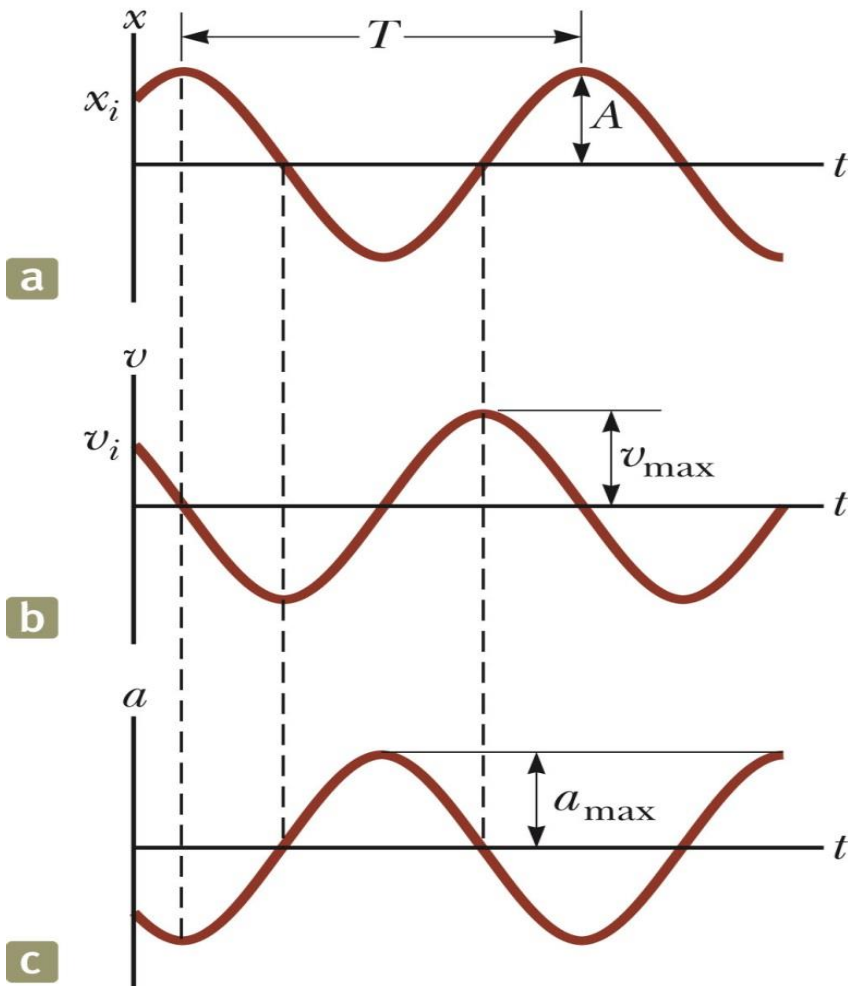


Maximum Values of v and a

Because the sine and cosine functions oscillate between ±1, we can easily find the maximum values of velocity and acceleration for an object in SHM.

$$v_{\max} = \omega A = \sqrt{\frac{k}{m}}\, A$$

$$a_{\max} = \omega^2 A = \frac{k}{m} A$$

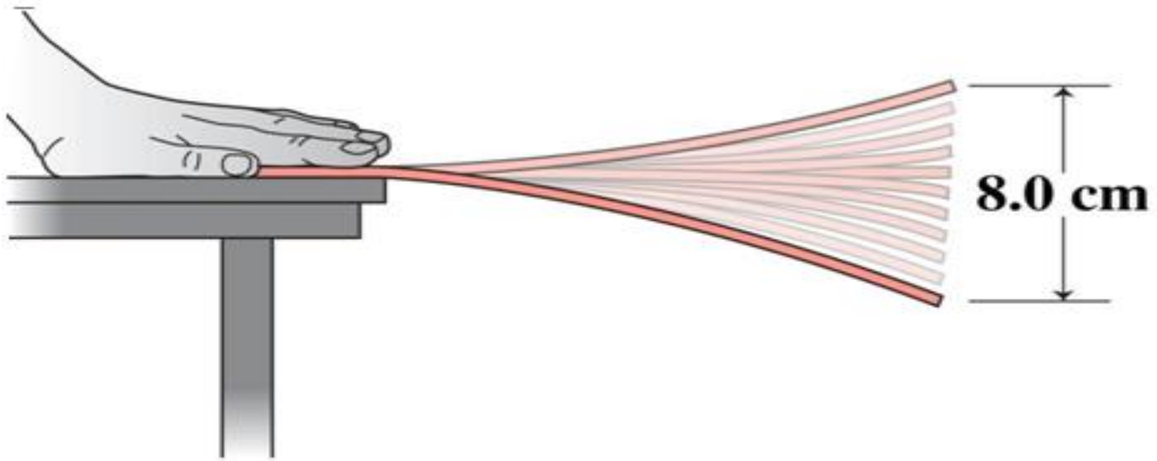


Quick Quiz

Quick Quiz 15.1 A block on the end of a spring is pulled to position $x = A$ and released. In one full cycle of its motion, through what total distance does it travel? (a) $A/2$ (b) A (c) $2A$ (d) $4A$

Problem. Oscillating Ruler

An oscillating ruler completes 28 cycles in 10 s & moves a total distance of 8.0 cm. What are the amplitude, period, & frequency of this oscillatory motion?



Amplitude = 8.0 cm / 2 = 4.0 cm.

$$T = \frac{10\ s}{28\ cycles} = 0.36\ s\,/\,cycle$$

$$f = \frac{1}{T} = \frac{28\ cycles}{10\ s} = 2.8\ Hz$$

CENGAGE Learning™

Frequency and Time Period in terms of system parameters

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

- Period in terms of system parameters

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

- Frequency in terms of system parameters

$$v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

- Velocity of a particle in simple harmonic motion

$$a = \frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t + \phi)$$

- Acceleration of a particle in simple harmonic motion

Problem 1:

An object oscillates with simple harmonic motion along the x axis. Its position varies with time according to the equation $x = (4.00\ m)\cos\left(\pi t + \frac{\pi}{4}\right)$, where t is in seconds and the angles in the parentheses are in radians.

(A) Determine the amplitude, frequency, and period of the motion.

(B) Calculate the velocity and acceleration of the object at any time t.

(C) Using the results of part (B), determine the position, velocity, and acceleration of the object at t = 1.00 s.

(D) Determine the maximum speed and maximum acceleration of the object.

(E) Find the displacement of the object between t = 0 and t = 1.00 s

Problem 1: Determine the amplitude, frequency, and period of the motion.

Solution: Given

$$\Rightarrow x = (4.00\ m)\cos\left(\pi t + \frac{\pi}{4}\right) \longrightarrow x(t) = A\cos(\omega t + \phi)$$

Amplitude: A = 4 m

And $\omega = \pi\ rad/s \rightarrow f = \frac{\omega}{2\pi} = \frac{\pi}{2\pi} = 0.500\ Hz$ **and** $T = \frac{1}{f} = 2.00\ s$

Using the results of part (B), determine the position, velocity, and acceleration of the object at t = 1.00 s.

Solution:

$$\Rightarrow x = (4.00\ m)\cos\left(\pi t + \frac{\pi}{4}\right) \Rightarrow v = \frac{dx}{dt} = -(4.00\ m)\sin\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$\Rightarrow v = \frac{dx}{dt} = -(4.00\pi\ m)\sin\left(\pi t + \frac{\pi}{4}\right)$$

$$\Rightarrow a = \frac{dv}{dt} = -(4.00\pi\ m)\cos\left(\pi t + \frac{\pi}{4}\right)\frac{d}{dt}(\pi t)$$

$$\Rightarrow a = \frac{dv}{dt} = -(4.00\pi^2\ m)\cos\left(\pi t + \frac{\pi}{4}\right)$$

Determine the maximum speed and maximum acceleration of the object.

Solution:

$$\Rightarrow x = (4.00\ m)\cos\left(\pi t + \frac{\pi}{4}\right)$$

At t= 1.00 s

$$\Rightarrow x = (4.00\ m)\cos\left(\pi + \frac{\pi}{4}\right)$$

$$\Rightarrow x = (4.00\ m)\cos\left(\frac{5\pi}{4}\right)$$

$$\Rightarrow x = (4.00\ m)(-0.707) = -2.83\ m$$

$$\Rightarrow v = \frac{dx}{dt} = -(4.00\pi\ m)\sin\left(\frac{5\pi}{4}\right)$$

$$\Rightarrow v = \frac{dx}{dt} = -(4.00\pi\ m)(-0.707) = 8.89\ m/s$$

$$\Rightarrow a = \frac{dv}{dt} = -(4.00\pi^2\ m)\cos\left(\frac{5\pi}{4}\right)$$

$$\Rightarrow a = \frac{dv}{dt} = -(4.00\pi^2\ m)(-0.707) = 27.9\ m/s^2$$

Find the displacement of the object between t = 0 and t = 1.00 s

Solution:

$$\Rightarrow v_{\max} = \omega A$$

$$v_{\max} = (4.00\pi) m/s = 12.6\, m/s$$

$$\Rightarrow a_{\max} = A\omega^2$$

$$a_{\max} = (4.00\pi^2) m/s^2 = 39.5\, m/s^2$$

Problem 1: Determine the amplitude, frequency, and period of the motion.

Solution:

$$\Rightarrow \boldsymbol{x} = (4.00\ \boldsymbol{m})\,\mathbf{cos}\left(\boldsymbol{\pi t} + \frac{\boldsymbol{\pi}}{4}\right)$$

The position at t = 0 s

$$\Rightarrow \boldsymbol{x} = (4.00\ \boldsymbol{m})\,\mathbf{cos}\left(0 + \frac{\boldsymbol{\pi}}{4}\right) = (4.00\ \boldsymbol{m})(0.707) = 2.83\ \boldsymbol{m}$$

In part C we found that the position at t = 1s, is -2.83 m

$$\Delta \boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{f}} - \boldsymbol{x}_{\boldsymbol{i}} = -2.83\boldsymbol{m} - 2.83\boldsymbol{m} = -5.66\boldsymbol{m}$$

Problem 2:

A car with a mass of 1 300 kg is constructed so that its frame is supported by four springs. Each spring has a force constant of 20 000 N/m. If two people riding in the car have a combined mass of 160 kg, find the frequency of vibration of the car after it is driven over a pothole in the road

Solution:

$$\Rightarrow \boldsymbol{F}_{total} = \sum(-\boldsymbol{k}\boldsymbol{x}) = -\left(\sum \boldsymbol{k}\right)\boldsymbol{x}$$

What if? Suppose the two people exit the car on the side of the road. One of them pushes downward on the car and release it so that it oscillates vertically. Is the frequency of the oscillation the same as the value we just calculated?

$$\Rightarrow \boldsymbol{k}_{eff} = \sum \boldsymbol{k} = 4 \times 20000 = 80000\ \boldsymbol{N}/\boldsymbol{m}$$

$$\boxed{\Rightarrow \boldsymbol{f} = \frac{1}{2\pi}\sqrt{\frac{80000}{1460}} = 1.18\ \boldsymbol{Hz}}$$

$$\boxed{\Rightarrow \boldsymbol{f} = \frac{1}{2\pi}\sqrt{\frac{80000}{1300}} = 1.25\ \boldsymbol{Hz}}$$

Problem: 3

A 200-g block connected to a light spring for which the force constant is 5.00 N/m is free to oscillate on a horizontal, frictionless surface. The block is displaced 5.00 cm from equilibrium and released from rest.

(A) Find the period of its motion.

(B) Determine the maximum speed of the block.

(C) What is the maximum acceleration of the block?

(D) Express the position, speed, and acceleration as functions of time.

What If? What if the block is released from the same initial position, x_i **= 5.00 cm**, but with an initial velocity of v_i **= 0.100 m/s?** Which parts of the solution change and what are the new answers for those that do change?

Solution:

(A) Find the period of it motion?

$$\Rightarrow \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5}{200\times10^{-3}}} = 5\, rad/s$$

$$\Rightarrow T = \frac{2\pi}{\omega} = \frac{2\pi}{5} = 1.26\, s$$

(b) Determine the maximum speed of the block?

$$\Rightarrow v_{max} = \omega A = (5)(5\times10^{-2}) = 0.250\, m/s$$

(c) What is the maximum acceleration of the block?

$$\Rightarrow a_{max} = \omega^2 A = (5)^2(5\times10^{-2}) = 1.25\, m/s^2$$

Solution:

(D) Express the position, speed and acceleration as function of time?

First we will find phase constant from the initial condition that *x* = *A* at *t* = 0:

$$\Rightarrow x(0) = A\cos\phi = A$$

$$\Rightarrow \phi = 0$$

$$\Rightarrow x = A\cos\omega t = (0.05\,m)\cos 5t$$

$$\Rightarrow v = -\omega A\sin\omega t = -(0.250m)\sin 5t$$

$$\Rightarrow a = -\omega^2 A\cos\omega t = -(1.25)\cos 5t$$

Problem: Tuned Mass Damper

The tuned mass damper in New York's Citicorp Tower consists of a 373-10^3 kg concrete block that completes one cycle of oscillation in 6.80 s. The oscillation amplitude in a high wind is 110 cm. Determine the spring constant & the maximum speed & acceleration of the block.

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \rightarrow \quad k = m\left(\frac{2\pi}{T}\right)^2 = \left(373\times10^3\ kg\right)\left(\frac{2\times3.1416}{6.80\ s}\right)^2 = 3.18\times10^5\ N/m$$

$$\omega = \frac{2\pi}{T} = \frac{2\times3.1416}{6.80\ s} = 0.924\ s^{-1}$$

$$v_{\max} = \omega A = \left(0.924\ s^{-1}\right)\left(1.10\ m\right) = 1.02\ m/s$$

$$a_{\max} = \omega^2 A = \left(0.924\ s^{-1}\right)^2\left(1.10\ m\right) = 0.939\ m/s^2$$

Energy of the SHM Oscillator

Mechanical energy is associated with a system in which a particle undergoes simple harmonic motion.

- For example, assume a spring-mass system is moving on a frictionless surface.

Because the surface is frictionless, the system is isolated.

- This tells us the total energy is constant.

The kinetic energy can be found by

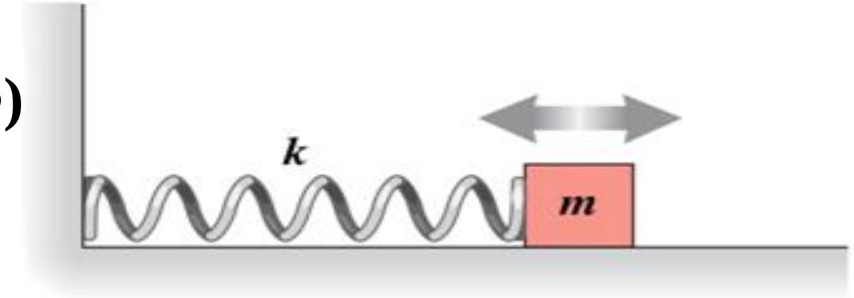
- $K = ½\, mv^2 = ½\, m\omega^2 A^2 \sin^2 (\omega t + \phi)$

$$\therefore v = \frac{dx}{dt} = -\omega A \sin(\omega t + \phi)$$

  - Assume a massless spring, so the mass is the mass of the block only.

The elastic potential energy can be found by

- $U = ½\, kx^2 = ½\, kA^2 \cos^2 (\omega t + \phi)$ $\quad \Rightarrow x(t) = A\cos(\omega t + \phi)$



Energy of the SHM Oscillator

The kinetic energy

$$k = \frac{1}{2} m\omega^2 A^2 \sin^2(wt + \phi) \qquad \Rightarrow k = \frac{1}{2} kA^2 \sin^2(wt + \phi) \qquad \therefore k = m\omega^2$$

And Potential energy

$$U = \frac{1}{2} kA^2 \cos^2(wt + \phi)$$

The total energy

$$E = k + U = \frac{1}{2} kA^2 \sin^2(wt + \phi) + \frac{1}{2} kA^2 \cos^2(wt + \phi) = \frac{1}{2} kA^2$$

Energy of the SHM Oscillator, cont.

The total mechanical energy is constant.

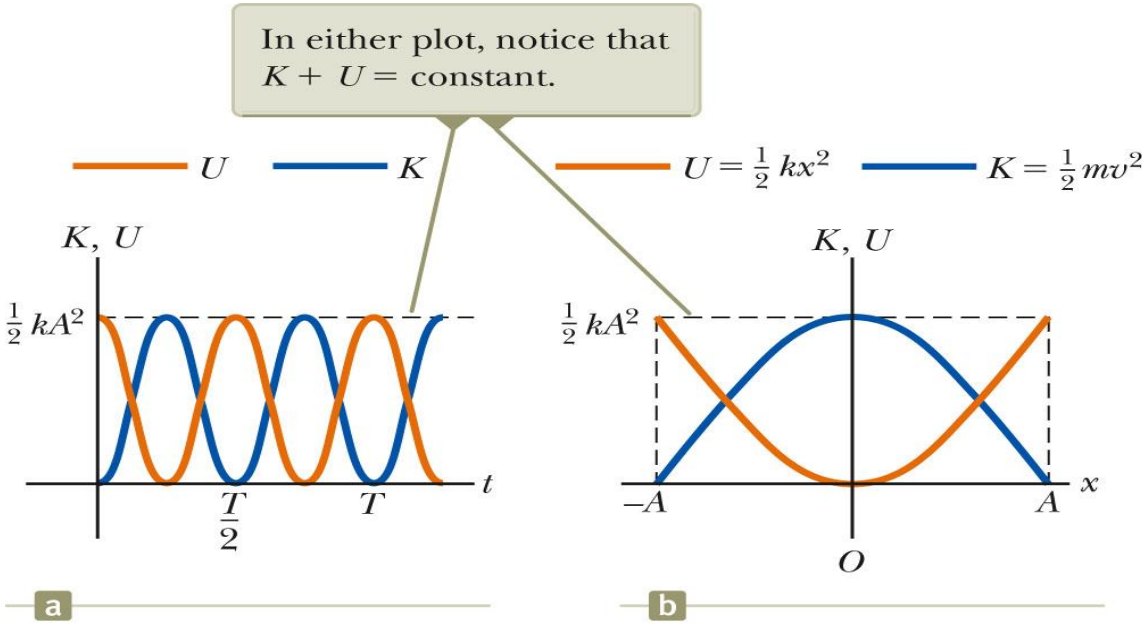
- At all times, the total energy is

½ k A^2

The total mechanical energy is proportional to the square of the amplitude.

Energy is continuously being transferred between potential energy stored in the spring and the kinetic energy of the block.

In the diagram, $\Phi = 0$



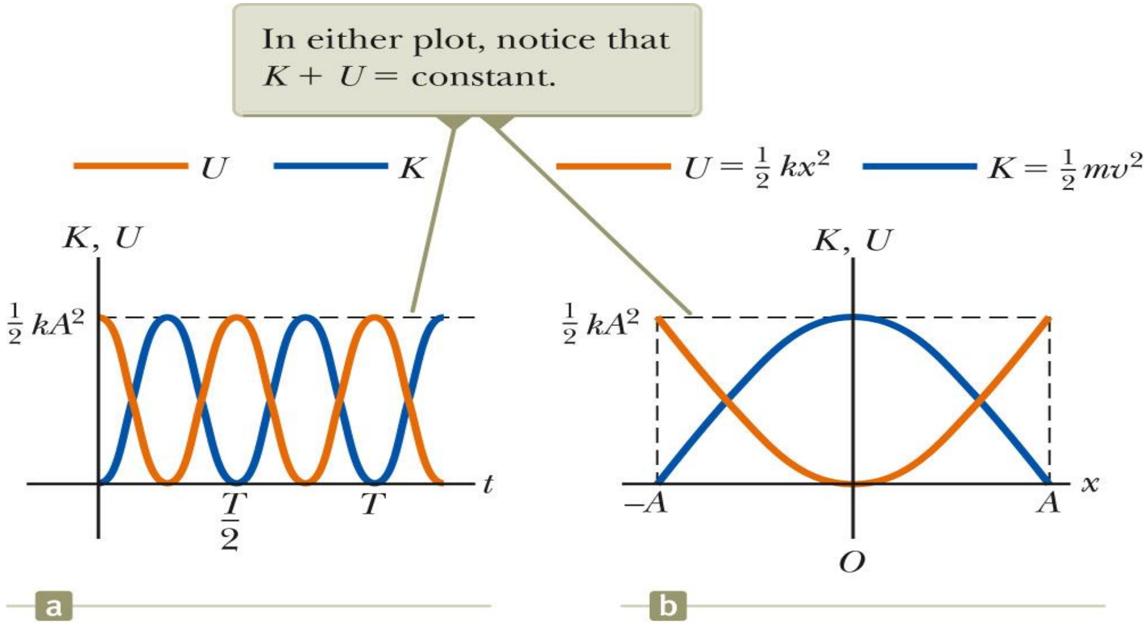
.

Energy of the SHM Oscillator, final

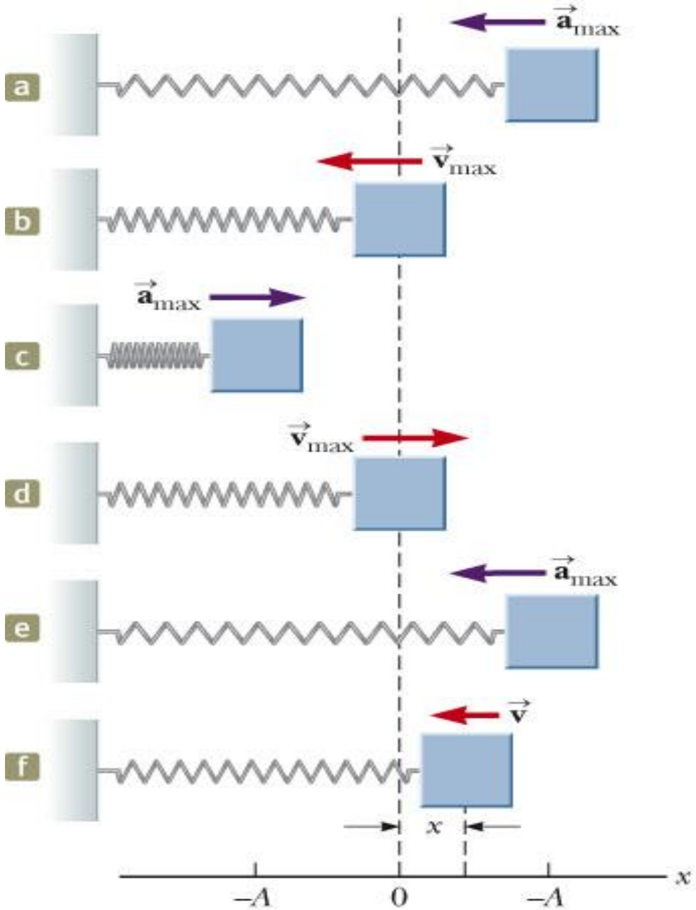
Variations of K and U can also be observed with respect to position.

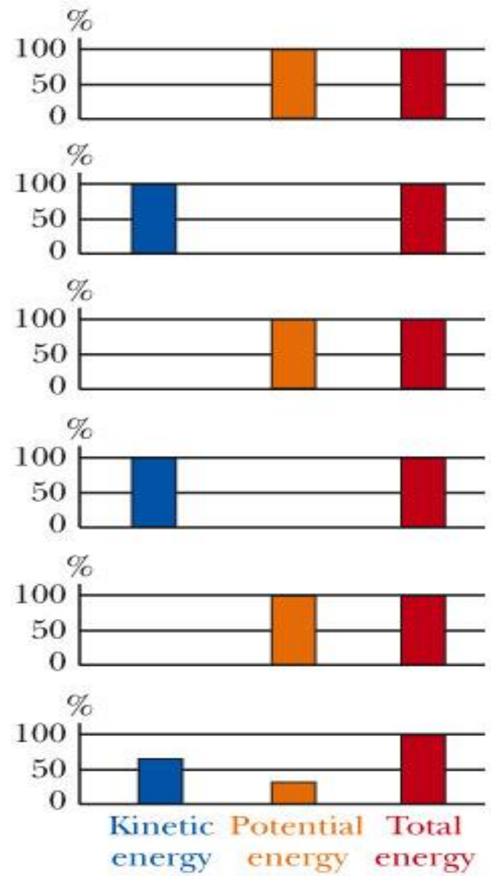
The energy is continually being transformed between potential energy stored in the spring and the kinetic energy of the block.

The total energy remains the same



Energy in SHM, summary





t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{T}{4}$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$\frac{T}{2}$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$\frac{3T}{4}$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
t	x	v	$-\omega^2 x$	$\frac{1}{2}mv^2$	$\frac{1}{2}kx^2$

Velocity at a Given Position

Energy can be used to find the velocity:

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \pm\sqrt{\frac{k}{m}\left(A^2 - x^2\right)}$$

$$= \pm\omega^2\sqrt{A^2 - x^2}$$

Problem 15.4:

A 0.500-kg cart connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless air track.
(A) Calculate the total energy of the system and the maximum speed of the cart if the amplitude of the motion is

(B) What is the velocity of the cart when the position is 2.00 cm?

(C) Compute the kinetic and potential energies of the system when the position is 2.00 cm.

Solution:

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(20.0\ \mathrm{N/m})(3.00 \times 10^{-2}\ \mathrm{m})^2$$

$$= \boxed{9.00 \times 10^{-3}\ \mathrm{J}}$$

Problem 15.4:

A 0.500-kg cart connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless air track.
(B) What is the velocity of the cart when the position is 2.00 cm?

(C) Compute the kinetic and potential energies of the system when the position is 2.00 cm.

Solution:

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)}$$

$$= \pm\sqrt{\frac{20.0\ \mathrm{N/m}}{0.500\ \mathrm{kg}}[(0.0300\ \mathrm{m})^2 - (0.0200\ \mathrm{m})^2]}$$

$$= \boxed{\pm 0.141\ \mathrm{m/s}}$$

Problem 15.4:

A 0.500-kg cart connected to a light spring for which the force constant is 20.0 N/m oscillates on a horizontal, frictionless air track.
(C) Compute the kinetic and potential energies of the system when the position is 2.00 cm.

Solution:

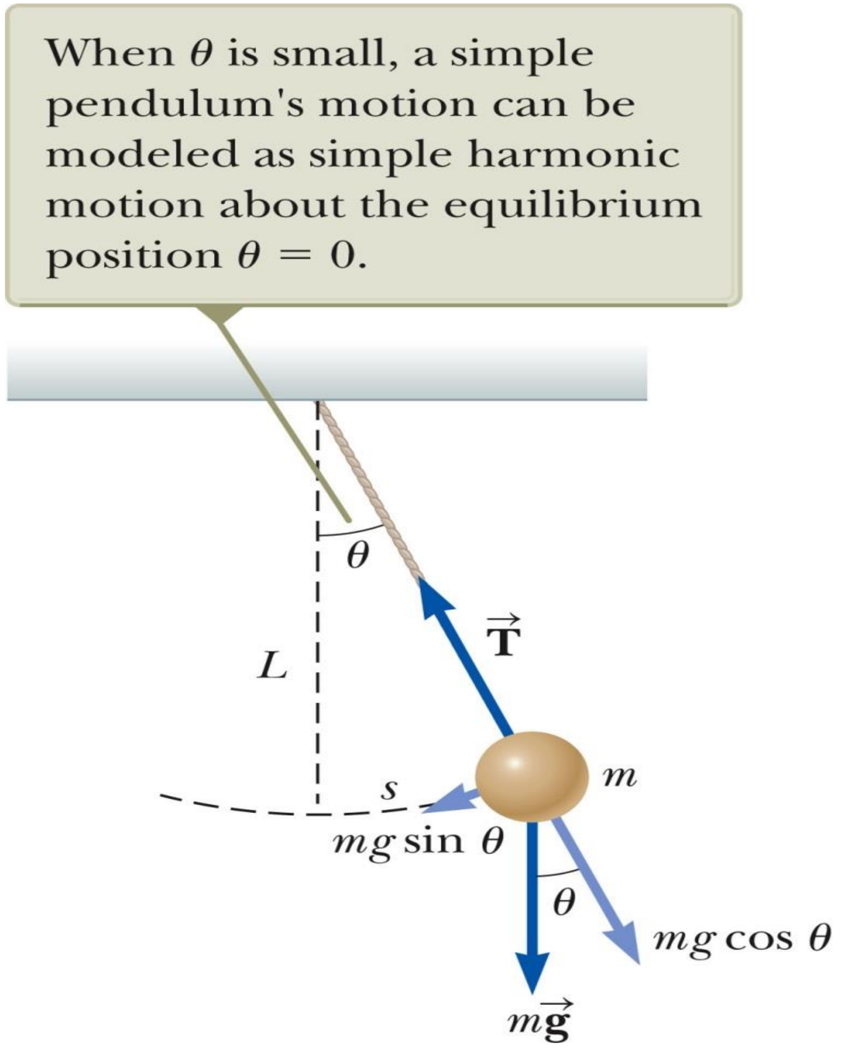
$$K = \frac{1}{2}mv^2 = \frac{1}{2}(0.500\text{ kg})(0.141\text{ m/s})^2 = \boxed{5.00 \times 10^{-3}\text{ J}}$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}(20.0\text{ N/m})(0.0200\text{ m})^2 = \boxed{4.00 \times 10^{-3}\text{ J}}$$

Simple Pendulum

The **simple pendulum** is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass *m* suspended by a light string of length *L*. The motion occurs in the vertical plane and is driven by gravitational force. The motion is very close to that of the SHM oscillator.

- If the angle is $<10^{o}$



CENGAGE Learning

Simple Pendulum

The forces acting on the bob are the tension and the weight.

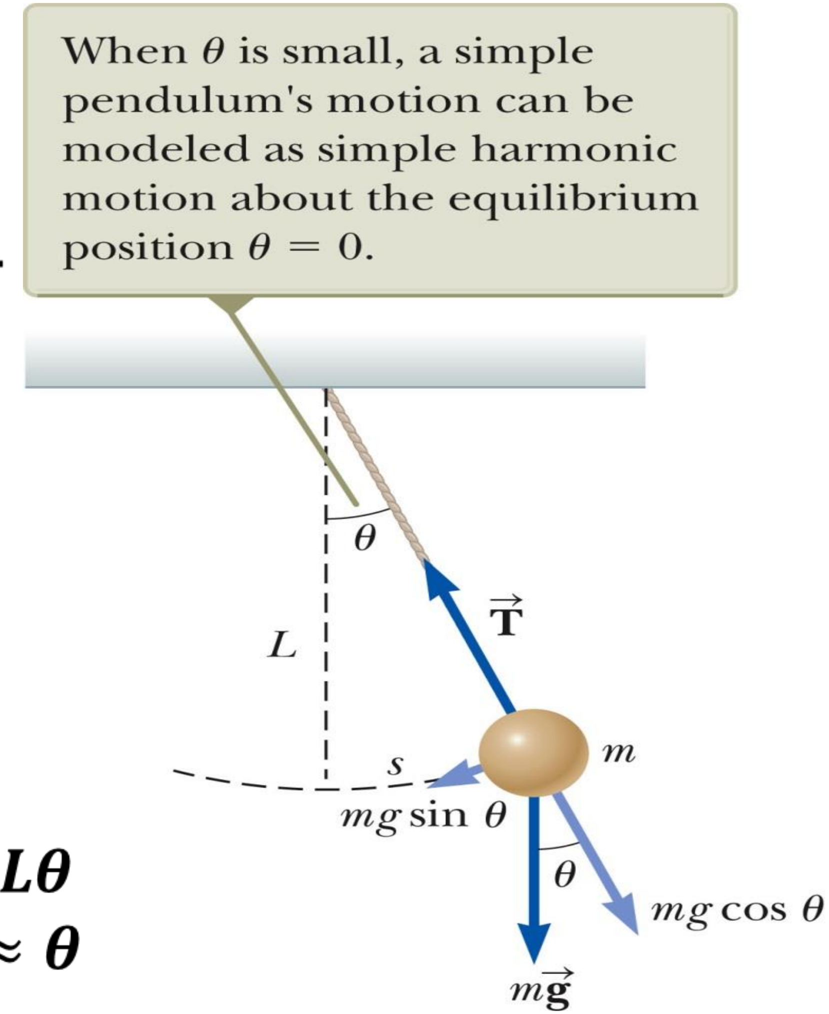
- $\vec{T}$ tension force and $m\vec{g}$ is the gravitational force.
- In the tangential direction,

$$\boxed{F_t = ma_t \rightarrow -mg\sin\theta = m\frac{d^2s}{dt^2}}$$

The length, *L*, of the pendulum is constant, and for small values of θ.

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta}$$

$$\therefore s = L\theta$$
$$\sin\theta \approx \theta$$



This confirms the mathematical form of the motion is the same as for SHM

CENGAGE Learning™

Simple Pendulum

The function θ can be written as $\theta = \theta_{max} \cos(\omega t + \phi)$.

The angular frequency is

$$\omega = \sqrt{\frac{g}{L}}$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\omega^2 x$$

$$\omega = \sqrt{\frac{k}{m}}$$

The period is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\frac{g}{L}\theta$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

Simple Pendulum Summary

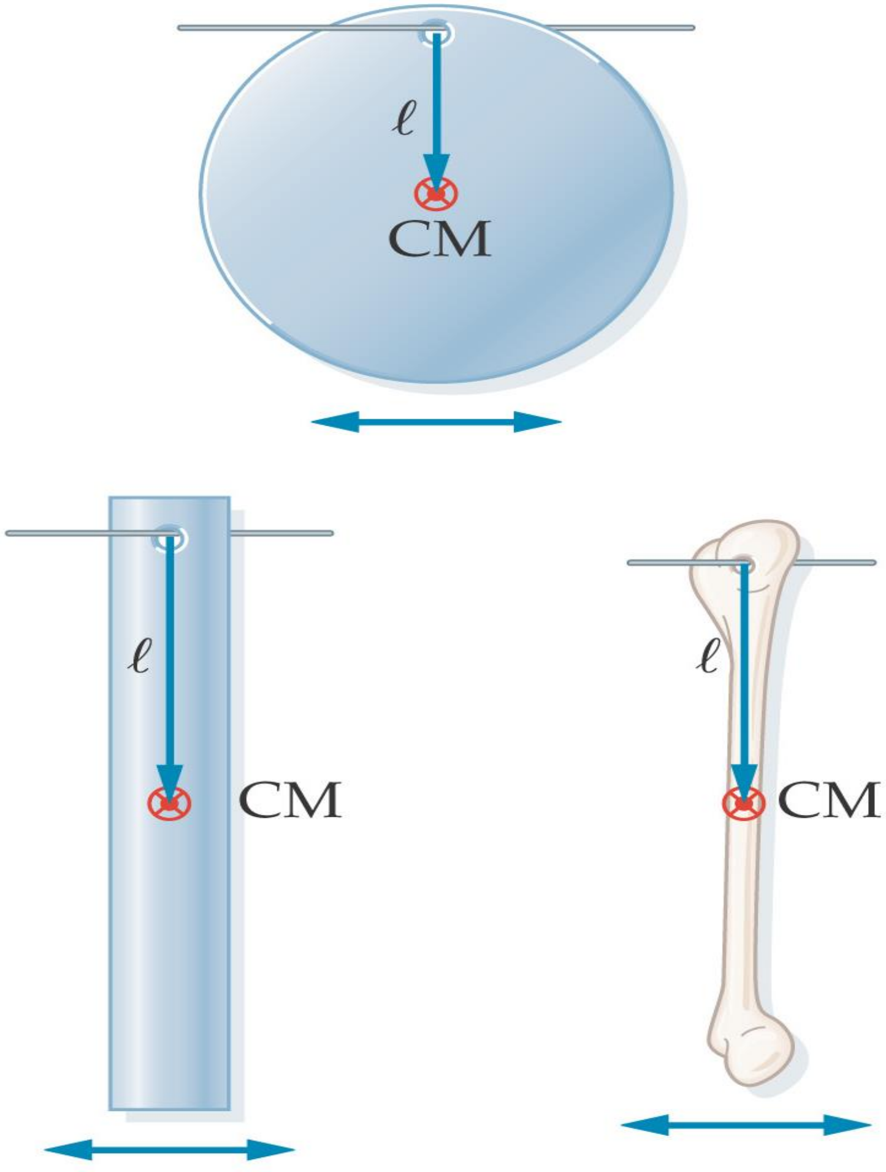
The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.

The period is independent of the mass.

All simple pendula that are of equal length and are at the same location oscillate with the same period.

The Physical Pendulum

A physical pendulum is a solid mass that oscillates around its center of mass, but cannot be modeled as a point mass suspended by a massless string. Examples:



Physical Pendulum

If a hanging object oscillates about a fixed axis that does not pass through the center of mass and the object cannot be approximated as a point mass, the system is called a ***physical pendulum.***

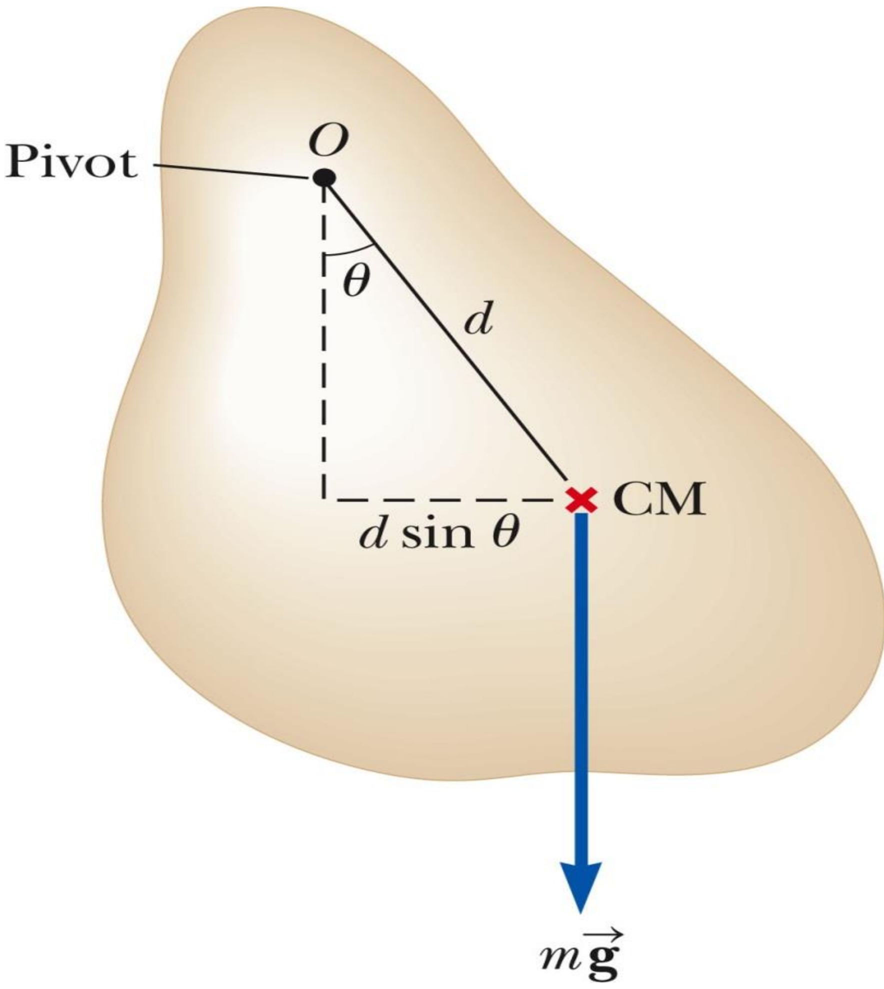
- It cannot be treated as a simple pendulum.

The gravitational force provides a torque about an axis through *O*.

The magnitude of the torque is

$$\tau = rF = (d\sin\theta)(mg) = mgd\sin\theta \qquad (1)$$

Also $\tau = I\alpha = I\frac{d^2\theta}{dt^2}$



Torque, Moment of Inertia and Newton's Law

Newton second law in linear form

$$\sum F = ma \qquad \therefore a = \frac{d^2x}{dt^2} \qquad \alpha = \frac{d^2\theta}{dt^2}$$

and Newton Second Law in angular form

$$\sum \tau = I\alpha$$

$$\tau = rF = (d \sin\theta)(mg) = mgd\sin\theta \qquad (1)$$

$$\tau = I\alpha = I\frac{d^2\theta}{dt^2} \qquad (2)$$

Moment of inertia is defined $I = mR^2$ **, the measure of resistance to a body in rotation**

Physical Pendulum

I is the moment of inertia about the axis through *O.*

From Newton's Second Law,

$$\boxed{-mgd\sin\theta = I\frac{d^2\theta}{dt^2}}$$

The gravitational force produces a restoring force.

Assuming θ is small, this becomes

$$\boxed{\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta}$$

Physical Pendulum

This equation is of the same mathematical form as an object in simple harmonic motion.

The solution is that of the simple harmonic oscillator.

$$\Rightarrow \frac{d^2 x}{dt^2} = -\omega^2 x$$

The angular frequency is

$$\omega = \sqrt{\frac{mgd}{I}}$$

$$\frac{d^2\theta}{dt^2} = -\left(\frac{mgd}{I}\right)\theta = -\omega^2\theta$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}}$$

Physical Pendulum

A physical pendulum can be used to measure the moment of inertia of a flat rigid object.

- If you know *d*, you can find *I* by measuring the period.

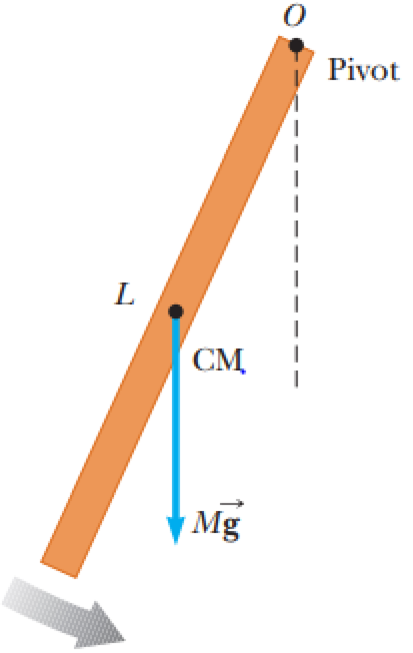
If $I = m d^2$ then the physical pendulum is the same as a simple pendulum.

- The mass is all concentrated at the center of mass.

Problem 15.6:

A uniform rod of mass *M* and length *L* is pivoted about one end and oscillates in a vertical plane (Fig. 15.18). Find the period of oscillation if the amplitude of the motion is small.

Solution: The moment of inertia of a uniform rod about an axis throug



$$I = \frac{1}{3}ML^2$$

The distance *d* from the pivot to the center of mass of the rod $\boldsymbol{\frac{L}{2}}$

$$T = 2\pi\sqrt{\frac{\frac{1}{3}ML^2}{Mg(L/2)}} = \boxed{2\pi\sqrt{\frac{2L}{3g}}}$$

Problem. Walking

When walking, the leg not in contact of the ground swings forward,

acting like a physical pendulum.

Approximating the leg as a uniform rod, find the period for a leg 90 cm long.

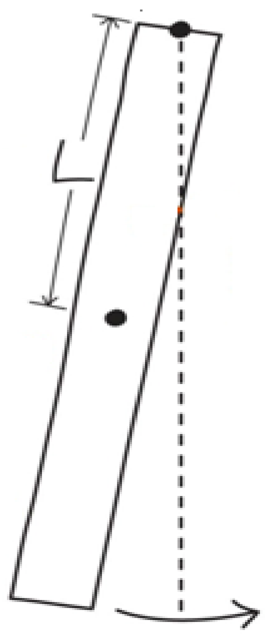


Table 10.2

$$\omega = \sqrt{\frac{m\,g\,L}{I}} \qquad I = \frac{1}{3}m(2L)^2$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4L}{3\,g}}$$

$$= 2\times 3.1416\sqrt{\frac{4\times(0.9\,m)}{3\times(9.8\,m/s^2)}}$$

$$= 1.6\,s$$

Forward stride = T/2 = 0.8 s

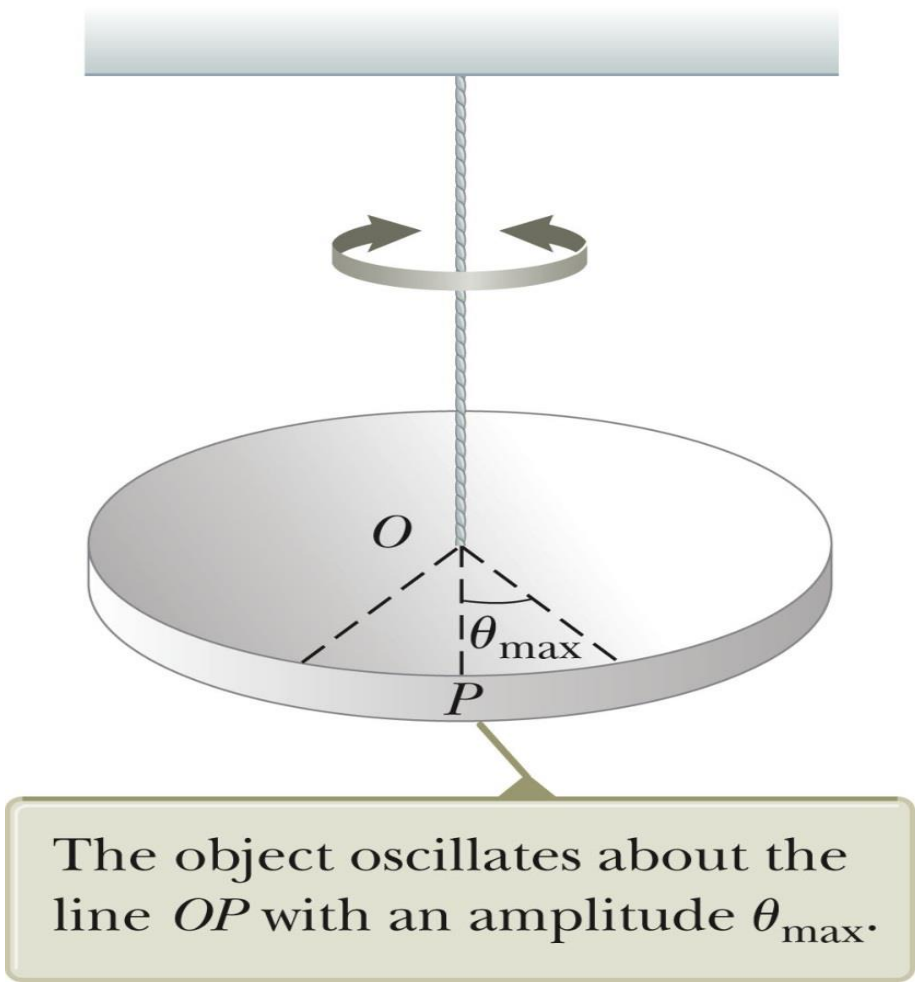
Torsional Pendulum

Assume a rigid object is suspended from a wire attached at its top to a fixed support.

The twisted wire exerts a restoring torque on the object that is proportional to its angular position.

The restoring torque is $\tau = -\kappa\,\theta$.

- κ is the *torsion constant* of the support wire.
- $\tau = -\kappa\theta$ is angular form of Hook's law



CENGAGE Learning™

Torsional Pendulum

Newton's Second Law gives

The torque equation produces a motion equation for simple harmonic motion.

$$\Rightarrow \tau = I\alpha \Rightarrow -k\theta = I\frac{d^2\theta}{dt^2}$$

The period is

$$\boxed{T = 2\pi\sqrt{\frac{k}{I}}}$$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{k}{I}\theta$$

No small-angle restriction is necessary.
Assumes the elastic limit of the wire is not exceeded

The angular frequency is

$$\boxed{\omega = \sqrt{\frac{k}{I}}}$$

$$\boxed{\therefore \frac{d^2x}{dt^2} = -\omega^2 x}$$

Damped Oscillations

In many real systems, non-conservative forces are present.

- This is no longer an ideal system (the type we have dealt with so far).
- Friction and air resistance are common non-conservative forces.

In this case, the mechanical energy of the system diminishes in time, the motion is said to be ***damped.***

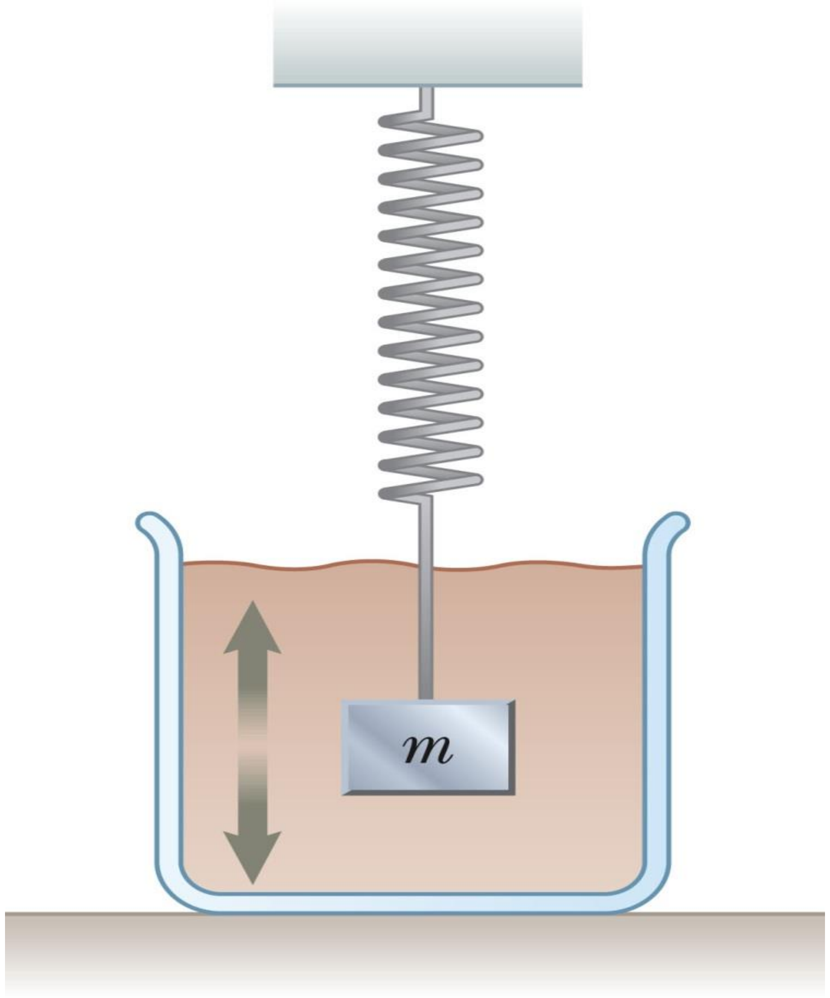
Damped Oscillation, Example

One example of damped motion occurs when an object is attached to a spring and submerged in a viscous liquid.

The retarding force can be expressed as

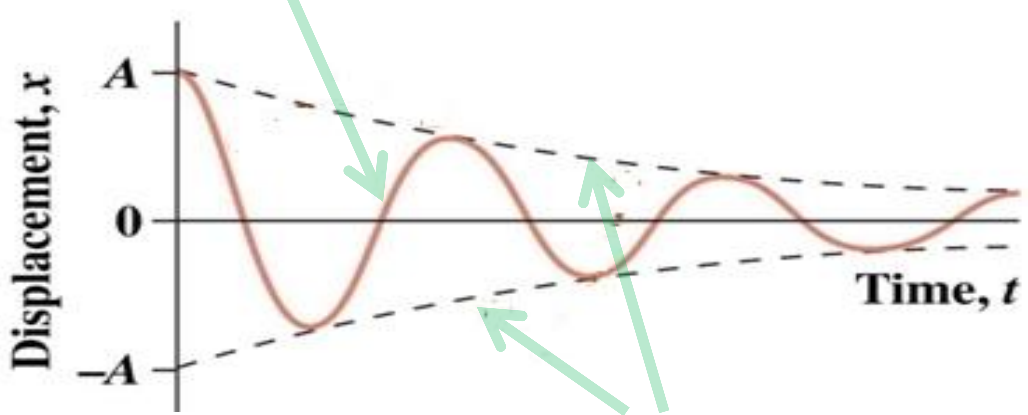
$$\vec{\mathbf{R}} = -b\vec{\mathbf{v}}$$

- *b* is a constant
- *b* is called the ***damping coefficient***



CENGAGE Learning™

Damped Harmonic Motion



Damping (frictional) force:

$$F_d = -b\,v \quad = -b\,\frac{dx}{dt}$$

Damped mass-spring:

$$m\,\frac{d^2x}{dt^2} = -k\,x - b\,\frac{dx}{dt}$$

Ansatz:

$$x(t) = A\,e^{-\alpha t}\cos(\omega t + \phi)$$

$$v(t) = A\,e^{-\alpha t}\left[-\alpha\cos(\omega t + \phi) - \omega\sin(\omega t + \phi)\right]$$

$$a(t) = A\,e^{-\alpha t}\left[\left(\alpha^2 - \omega^2\right)\cos(\omega t + \phi) + 2\alpha\,\omega\sin(\omega t + \phi)\right]$$

→

$$m\left(\alpha^2 - \omega^2\right) = -k + b\,\alpha$$

$$2m\,\alpha\,\omega = b\,\omega$$

$$\therefore \qquad \alpha = \frac{b}{2m} \qquad \omega = \sqrt{\frac{k}{m} - \alpha^2} = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Damped Oscillations, Equations

From Newton's Second Law

$$\Sigma F_x = -k\,x - bv_x = ma_x$$

When the retarding force is small compared to the maximum restoring force we can determine the expression for *x.*

- This occurs when *b* is small.

The position can be described by

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

$$\boxed{x = Ae^{-\left(\frac{b}{2m}\right)t}\cos(\omega t + \phi)}$$

The angular frequency will be

$$\boxed{\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}}$$

Damped Oscillations, Natural Frequency

When the retarding force is small, the oscillatory character of the motion is preserved, but the amplitude decreases exponentially with time.

The motion ultimately ceases.

Another form for the angular frequency:

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

- where ω_0 is the angular frequency in the absence of the retarding force and is called the **natural frequency** of the system.
-

$$\omega_o = \sqrt{k/m}$$

Damped Oscillations, Graph

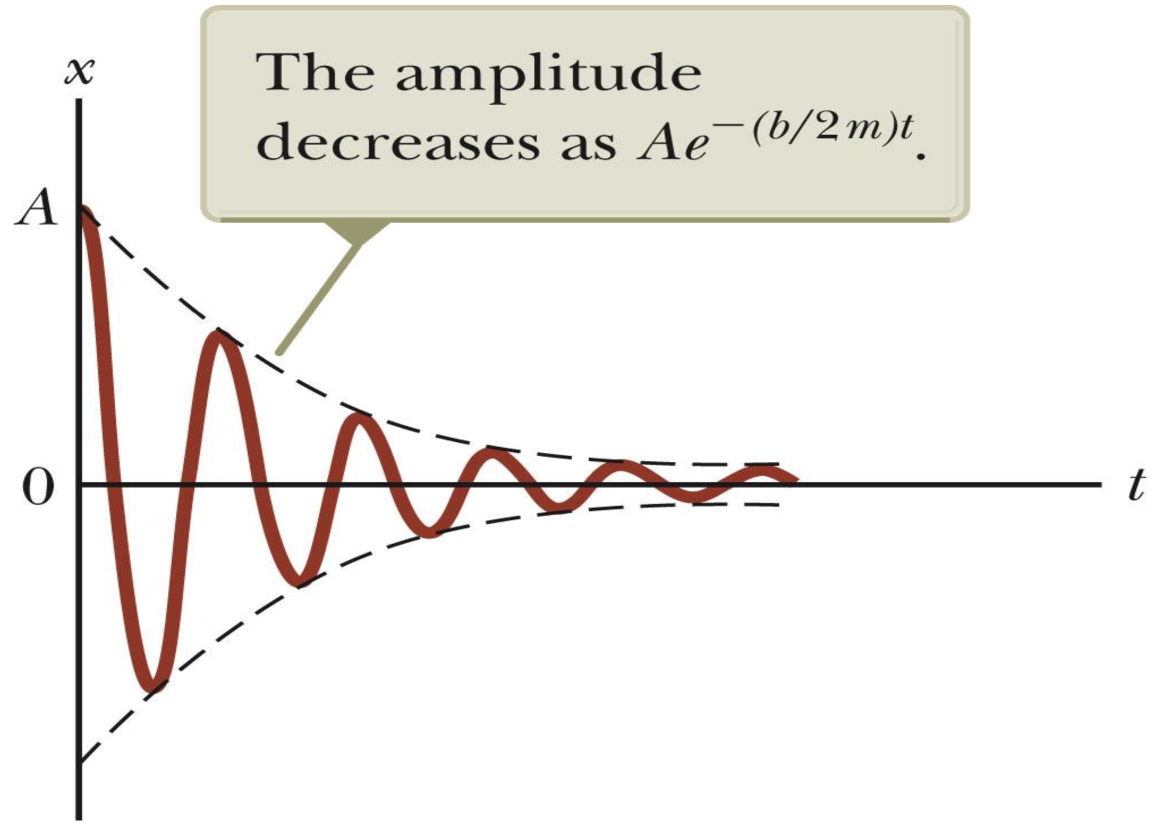
A graph for a damped oscillation.

The amplitude decreases with time.

The blue dashed lines represent the ***envelope*** of the motion.

Use the active figure to vary the mass and the damping constant and observe the effect on the damped motion.

The restoring force is – *kx.*



CENGAGE Learning™

Types of Damping

If the restoring force is such that $b/2m < \omega_o$, the system is said to be ***underdamped.***

When b reaches a critical value b_c such that $b_c / 2\,m = \omega_0$, the system will not oscillate.

- The system is said to be ***critically damped.***

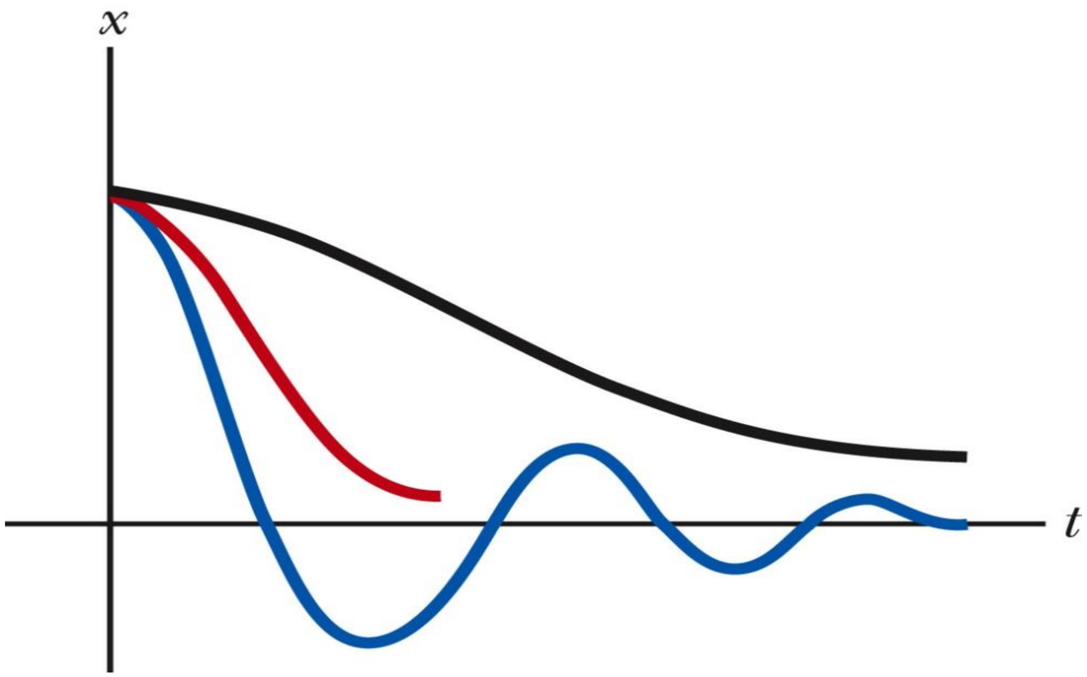
If the restoring force is such that $b/2m > \omega_o$, the system is said to be ***overdamped.***

Types of Damping, cont

Graphs of position versus time for

- An underdamped oscillator – blue
- A critically damped oscillator – red
- An overdamped oscillator – black

For critically damped and overdamped there is no angular frequency.



Forced Oscillations

It is possible to compensate for the loss of energy in a damped system by applying a periodic external force.

The amplitude of the motion remains constant if the energy input per cycle exactly equals the decrease in mechanical energy in each cycle that results from resistive forces.

After a driving force on an initially stationary object begins to act, the amplitude of the oscillation will increase.

After a sufficiently long period of time, $E_{\text{driving}} = E_{\text{lost to internal}}$

- Then a steady-state condition is reached.
- The oscillations will proceed with constant amplitude.

Forced Oscillations, cont.

The amplitude of a driven oscillation is

$$A = \frac{F_0 / m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\frac{b\omega}{m}\right)^2}}$$

- ω_0 is the natural frequency of the undamped oscillator.

Resonance

When the frequency of the driving force is near the natural frequency ($\omega \approx \omega_0$) an increase in amplitude occurs.

This dramatic increase in the amplitude is called ***resonance.***

The natural frequency ω_0 is also called the resonance frequency of the system.

At resonance, the applied force is in phase with the velocity and the power transferred to the oscillator is a maximum.

- The applied force and v are both proportional to sin ($\omega t + \phi$).
- The power delivered is $\vec{F} \cdot \vec{v}$
  - **This is a maximum when the force and velocity are in phase.**
  - **The power transferred to the oscillator is a maximum**.

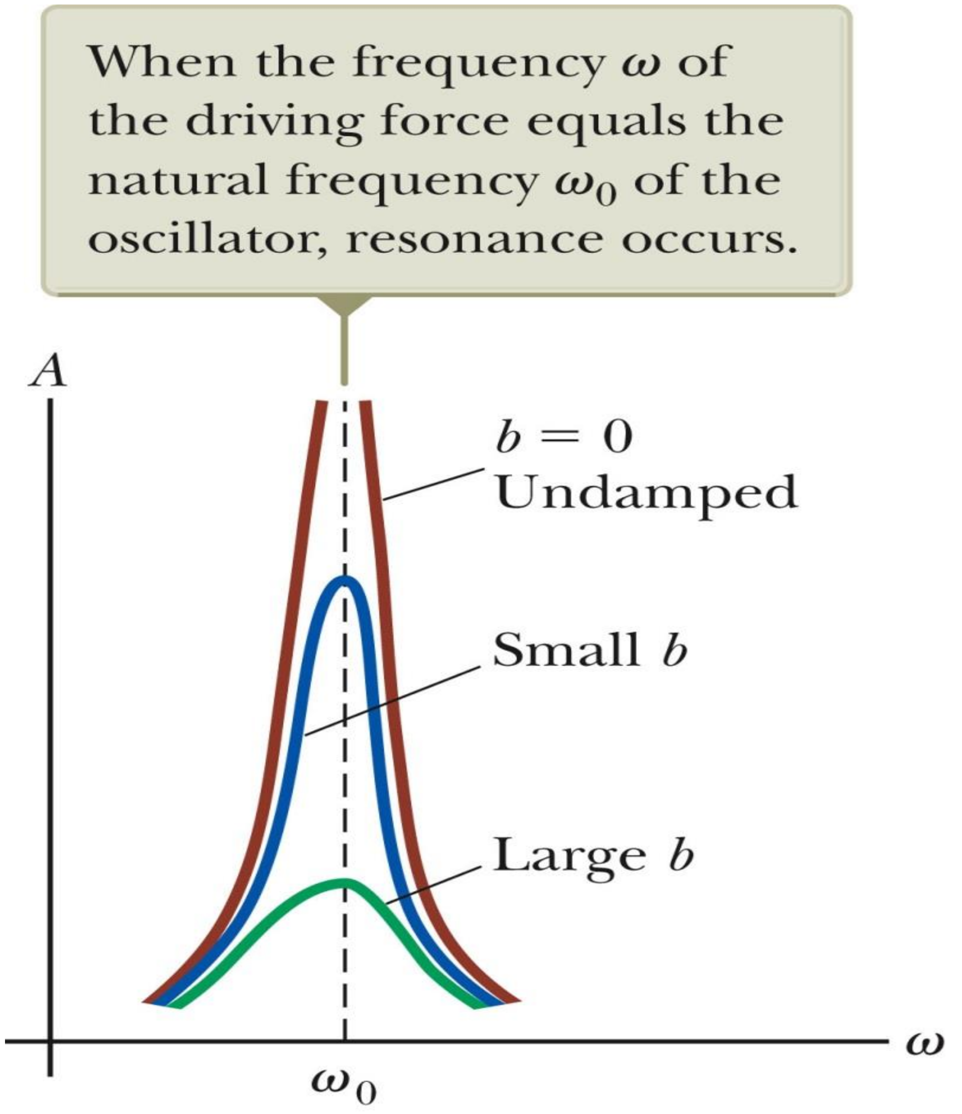
Resonance, cont.

Resonance (maximum peak) occurs when driving frequency equals the natural frequency.

The amplitude increases with decreased damping.

The curve broadens as the damping increases.

The shape of the resonance curve depends on *b.*



Practice Problem

A 50-g mass is attached to a spring and undergoes simple harmonic motion. Its maximum acceleration is 15 m/s^2 and its maximum speed is 3.5 m/s. Determine the (a) angular frequency, (b) spring constant, and (c) amplitude.

Practice Problem

A 2.00-kg object is attached to a spring and placed on a frictionless, horizontal surface. A horizontal force of 20.0 N is required to hold the object at rest when it is pulled to x = 0.200 m from its equilibrium position (at x = 0), and then released from rest and it subsequently undergoes simple harmonic oscillations.

(a) Find the force constant of the spring, k, and the frequency of the oscillations, f.

(b) Calculate the maximum speed of the object, v_{max}. Where does this maximum speed occur?

(c) Find the maximum acceleration of the object, $a_{max.}$ Where does the maximum acceleration occur?

(d) Calculate the total energy of the oscillating system.

(e) Determine the speed and the acceleration of the object when its position is equal to one-third the maximum value, $x = \frac{A}{3} \Rightarrow v = ?, \ a = ?$

(f) Express the position, velocity, and acceleration as functions of time, x(t), v(t), a(t)

CENGAGE Learning™

Practice Problem

A particle moving along the x axis in simple harmonic motion starts from its equilibrium position, the origin, at t = 0 sec and moves to the right. The amplitude of its motion is A = 2.00 cm, and the frequency is f = 1.50 Hz.

(a) Find an expression for the position of the particle as a function of time.

(b) Determine the maximum speed and maximum positive acceleration of the particle.

(d) Find the total distance traveled by the particle between t = 0 and t = 1.00 s.

Practice Problem

1. What is the period of 60.0 Hz electrical power?
2. If your heart rate is 150 beats per minute during strenuous exercise, what is the time per beat in units of seconds?
3. Find the frequency of a tuning fork that takes 2.50×10^{-3} s to complete one oscillation.
4. A stroboscope is set to flash every 8.00×10^{-5}s. What is the frequency of the flashes?
5. (a) What is the maximum velocity of an 85.0-kg person bouncing on a bathroom scale having a force constant of 1.50×10^{6} N/m, if the amplitude of the bounce is 0.200 cm? (b)What is the maximum energy stored in the spring?

Practice Problem

Consider a spring that is standing on end in the vertical position. You place 100 grams on the spring and it compresses a distance of 9.8 cm.

a) If an additional 200 grams are placed on top of the 100 gram mass, how much will the spring compress?
b) What is the spring constant?

Practice Problem

A spring is hanging freely from the ceiling. You attach an object to the end of the spring and let the object go. It falls down a distance 49 cm and comes back up to where it started. It continues to oscillate in simple harmonic motion going up and down a total distance of 49 cm from top to bottom. What is the period of the simple harmonic motion?

Practice Problem

You are on a boat, which is bobbing up and down. The boat's vertical displacement y is given by

$$y = (1.2m)\cos\left(\frac{1}{2s}t + \frac{\pi}{6}\right) \qquad y = A\cos(\omega t + \phi)$$

(a) Find the amplitude, angular frequency , phase constant, frequency, and period of the motion. (b) Where is the boat at t = 1 s? (c) Find the velocity and acceleration as functions of time t. (d) Find the initial values of the position, velocity, and acceleration of the boat.

Practice Problem

A 450-g mass on a spring is oscillating at 1.2 Hz, with total energy 0.51 J. What's the oscillation amplitude?

$$f = 2\pi\omega$$

$$\omega = \frac{f}{2\pi}$$

$$E = \frac{1}{2}kA^2$$

$$\frac{f}{2\pi} = \sqrt{\frac{k}{m}}$$