University of Illinois at Chicago
Department of Physics

Thermodynamics and Statistical Mechanics Qualifying Examination

January 6, 2006
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

1. Ideal Gas in a Gravitational Field: Consider a cubic container of edge L (volume $V = L^3$) containing an ideal gas of N particles, initially at temperature T.

(a) Near the surface of the earth, the gravitational field acting on all the particles of the gas is g (acting in the height direction). Find the density of the gas as a function of vertical position in the container, $\rho(z)$.

(b) Calculate the entropy of the gas as a function of N, the volume V and g, in the small-g limit $mgL/k_BT << 1$ (find the limiting behavior, don't just set $g = 0$!).

(c) Now, suppose the container is launched into deep space, so that now no gravitational field acts on the gas. Under the condition that the temperature of the container is held fixed during the flight, find the change in entropy relative to (b), again in the limit $mgL/k_BT << 1$. Explain your answer.

(d) Suppose you carry out the same experiment, except that now the container is heavily insulated so no heat can be transferred to or from the gas; it starts on earth at temperature T. Find the final temperature T_f of the gas when it reaches deep space where $g = 0$. Explain your answer.

2. Extensible molecule: Consider a long molecule which is composed of N chemical units ('monomers'), each of which can be in one of two states, of different lengths a and b, with $b > a$. The whole molecule therefore can be between Na and Nb in length. The energy of a monomer in the longer state is ϵ larger than a monomer in the shorter state. You may consider the thermodynamic limit $N >> 1$ to simplify the calculations.

(a) Calculate the equilibrium length of the entire molecule as a function of temperature T.

(b) Calculate the root-mean-square fluctuation in the length of the entire molecule as a function of temperature T.

(c) Now, suppose that the molecule is forced to be a *fixed* length $L(Na < L < Nb)$, so that $(L - Na)/(b - a)$ of its monomers are in the stretched (length b) state. Find the internal energy $E(N, L)$ and the entropy $S(N, T, L)$.

(d) From (c) calculate the Helmholtz free energy $F(N, T, L)$, and finally the force needed to extend the molecule to to length L at fixed temperature T.

3. Neutron Star: Consider a neutron star, a macroscopic body composed of neutrons, at a density of 10^{14} g/cm^3. The temperature of the star's interior is approximately 10^7 K. For this problem you should consider the star to be a noninteracting Fermi gas of neutrons.

(a) Determine whether the neutrons are relativistic or nonrelativistic, by estimating their kinetic energy.

(b) Determine whether or not the neutrons are reasonably well considered to be a zero-temperature Fermi gas.

(c) Estimate the pressure in the neutron star.

(d) Use (c) to estimate the *mass* of this neutron star.

4. Free (Joule) expansion of a gas: Consider a gas initially at temperature T_i, in a thermally isolated vessel of volume V_i. Suppose that a door is opened, allowing the gas to abruptly and freely expand into a final total volume V_f, without any work being done, and without any heat transfer to the gas. At the end of this process, the gas reaches a final temperature T_f.

(a) In general, what is the change in internal energy E as a result of this expansion?

(b) Suppose the gas is an ideal gas: what is the change in entropy of the gas as a result of the expansion? Explain your result in simple terms.

(c) Now consider a non-ideal (interacting) gas. Show that the temperature change can be expressed as

$$\Delta T = -\int_{V_i}^{V_f} dV \frac{T^2}{C_V} \left(\frac{\partial (p/T)}{\partial T} \right)_V$$

where the integration is done using the equilibrium equation of state of the gas. Make sure you justify the use of the equilibrium equation of state.

(d) Explain with justification whether in the general (non-ideal gas) case you expect the temperature change ΔT to be positive or negative.

5. Low-temperature paramagnet: Consider a paramagnetic crystal of spin-1/2 spins. You may neglect all interactions between the spins.

(a) At low temperature ($k_B T \ll \mu B$), find the entropy per spin as a function of external magnetic field B (the magnetic moment per spin is μ, so that each spin has energies $\pm \mu B$).

(b) Does the result of (a) violate any of the laws of thermodynamics? Explain your answer.

(c) Suppose the crystal is cooled in the presence of a magnetic field B_i to a temperature T_i, and then is thermally isolated. Now the magnetic field is slowly reduced; since the system is thermally isolated this is a reversible process. If the final value of the field is $B_f < B_i$, find the final temperature T_f of the spins.

Constants

At room temperature $T = 300$ K and $k_B T = 4.1 \times 10^{-21}$ J

Integrals

$$\int_{-\infty}^{\infty} dx \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2}$$

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ x^n \ \exp\left[-\frac{x^2}{2\sigma^2}\right] = 1 \cdot 3 \cdot 5 \cdots (n-1)\sigma^n$$

for $n = 2, 4, 6 \cdots$

Series

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}$$

$$1 + x + x^2 + \cdots x^{P-1} = \frac{1 - x^P}{1 - x} \qquad \text{(geometric series)}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = -\ln(1-x) \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{Zeta function} \quad \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \quad \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056 \cdots$$

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\frac{d}{dx} \sinh x = \cosh x \qquad \frac{d}{dx} \cosh x = \sinh x \qquad \cosh^2 x - \sinh^2 x = 1$$

Stirling's approximation for log of factorial

$$\ln n! = n \ln n - n + \mathcal{O}(1)$$

Combinations

$$C_n^N = \frac{N!}{n!(N-n)!}$$

Classical ideal gas partition function

$$Z(N,T,V) = \frac{1}{N!} \left(\frac{1}{h^3} \int_V d^3 r \int d^3 p \ \exp\left[-\frac{\mathbf{p}^2}{2mk_B T} \right] \right)^{3N}$$

$$= \frac{V^N}{N!} \left(\frac{2\pi m k_B T}{h^2} \right)^{3N/2}$$