Solutions

Physics PhD Qualifying Examination
Part I – Wednesday, August 22, 2012

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] [3,7]

A canal barge of mass m is traveling at speed v_i when it shuts off its engines. The drag force in the water acting on the barge is given by $-bv$.

(a) How long does it take until the speed of the barge is reduced to v_f?
(b) What distance does the barge travel during this time?

[I-2] [2,2,6]

Consider a binary star system.

(a) Write the Lagrangian for the system in terms of the Cartesian coordinates of the two stars $\vec{r}_1 and\ \vec{r}_2$ (Your expression should naturally include the two masses, m_1 and m_2.)

(b) Show that the potential energy is a homogeneous function of the coordinates of degree -1, i.e.

$$V(\alpha\vec{r}_1, \alpha\vec{r}_2) = \alpha^{-1} V(\vec{r}_1, \vec{r}_2)\text{, where } \alpha \text{ is a real scaling parameter.}$$

(c) Find and write down a coordinate transformation (based on the center-of-mass frame) which leaves the Lagrangian the same up to a multiplication constant (thereby leaving the physics unchanged) and thus, find Kepler's third law relating the period of revolution of the system to the size of the orbit. (When finding Kepler's law, for simplicity, assume circular orbits.)

[I-3] [10]

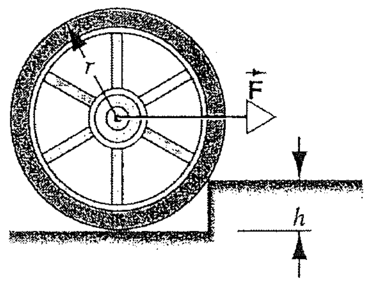
Three oscillators of equal mass m are coupled such that the potential energy of the system is given by

$$U = \frac{1}{2}\left[\kappa_1(x_1^2 + x_3^2) + \kappa_2 x_2^2 + \kappa_3(x_1x_2 + x_2x_3)\right]$$

where k_1, k_2 and k_3 are spring constants and $k_3=(2k_1k_2)^{1/2}$.
Find the eigenfrequencies of the coupled oscillators by solving the secular equation.
What is the physical interpretation of the zero-frequency mode?

[I-4] [10]

What minimum force F, applied horizontally at the axel of the wheel, is necessary to raise the wheel over a curb of height h? (See figure below.) The radius of the wheel is r and its total mass is M. The gravitational acceleration is g. You must express your answer in terms of h, r, M, and g.



[I-5] [10]

A train of rest length L_0 is travelling at a constant velocity v, relative to an observer in the laboratory reference frame. At the instant the middle of the train passes the observer, the observer sees flashes of light from both the front and the rear of the train.

<u>In the reference frame of the *observer:*</u>

(a) What was the delay between the flashes of light emitted from the front and the rear of the train? Which flash happened first?

(b) How far away was the observer from the front of the train when the first flash was emitted?

<u>In the reference frame of the *train:*</u>

(c) What was the delay between the flashes of light emitted from the front and the rear of the train? Which flash happened first?

(d) How far away was the observer from the front of the train when the first flash was emitted?

[I-6] **[10]**

A stationary current distribution is established in a medium that is isotropic but not necessarily homogeneous. Show that the medium will in general acquire a volume distribution of charge whose density is (in Gaussian units)

$$\rho = -\frac{1}{4\pi\sigma}(\sigma\vec{\nabla}\varepsilon - \varepsilon\vec{\nabla}\sigma)\cdot\vec{\nabla}\varphi \ ,$$

where σ and ε are the conductivity and the dielectric permittivity of the medium and φ is the potential.

[I-7] **[10]**

Consider a parallel-plate capacitor immersed in seawater and driven by an alternating voltage $V(t)=V_0\cos(2\pi ft)$.

Sea water at frequency $f=4\times10^8$ Hz has a permittivity $\varepsilon=81\varepsilon_0$, a permeability $\mu=\mu_0$, and a resistivity $\rho=0.23\ \Omega$m.

What is the ratio of amplitudes of conduction current to displacement current in sea water at $f=4\text{x}10^8$ Hz? Before obtaining the numerical answer you must express the solution in terms of the variables given.

$\varepsilon_0=8.85\times10^{-12}$ As/Vm, $\mu_0=4\pi\times10^{-7}$Vs/Am

[I-8] **[10]**

A square current loop lies in the xy plane. The sides of the square are of length a, and the center of the square is at the origin. Find the magnetic induction $\boldsymbol{B}$ at a height h above the xy plane along the z axis.

[I-9] **[10]**

A massive atom with an atomic polarizability α is subjected to an electromagnetic field (the atom being located at the origin), $\boldsymbol{E} = E_0 e^{i(kx-\omega t)}\hat{k}$. Find the asymptotic electric and magnetic field radiated by the atom and calculate the energy radiated per unit of solid angle.

[I-10] **[10]**

Show by direct substitution of the Lorentz transformation of space-time coordinates that the form of the wave equation for propagation with the speed of light is preserved. That is, show that if

$$\nabla^2\psi = \frac{\partial^2\psi}{c^2\partial t^2} \ ,$$

then

$$\nabla'^2\psi = \frac{\partial^2\psi}{c^2\partial t'^2} \ .$$

Part I Solutions

I-1 [3,7]

A canal barge of mass m is traveling at speed v_i when it shuts off its engines. The drag force in the water acting on the barge is given by $-bv$.

(a) How long does it take until the speed of the barge is reduced to v_f?
(b) What distance does the barge travel during this time?

a)

$$m\frac{dv}{dt} = -bv$$

$$\frac{dv}{v} = -\frac{b}{m}dt \Rightarrow \int_{v_i}^{v_f}\frac{dv}{v} = -\frac{b}{m}\Delta t$$

$$\ln(v_f) - \ln(v_i) = -\frac{b}{m}\Delta t$$

$$\boxed{\Delta t = \frac{m}{b}\ln\left(\frac{v_i}{v_f}\right)}$$

b)

$$v = \frac{dx}{dt} \Rightarrow dt = \frac{dx}{v}$$

$$m\frac{dv}{dx/v} = -bv$$

$$m\frac{dv}{dx} = -b$$

$$m\,dv = -b\,dx \Rightarrow m\int_{v_i}^{v_f}dv = -b\Delta x$$

$$\boxed{\Delta x = \frac{m}{b}(v_i - v_f)}$$

Alternative Solution to (b)

$$v(0) = v_i$$

$$\ln(v) = -\frac{b}{m} t + C$$

$$\ln(v_i) = C$$

$$\ln(v) = -\frac{b}{m} t + \ln(v_i)$$

$$\ln\left(\frac{v}{v_i}\right) = -\frac{b}{m} t$$

$$\frac{v}{v_i} = e^{-\frac{b}{m} t}$$

$$\boxed{v(t) = v_i e^{-b/m\, t}}$$

$$\boxed{(\Delta t =) t_f = \frac{m}{b} \ln\left(\frac{v_i}{v_f}\right)}$$

$$x(t) = \int v(t)\,dt$$

$$\Delta x = x_f - x_i = \int_0^{t_f} v(t)\,dt = v_i\left(-\frac{m}{b}\right) e^{-b/m\, t}\Big|_0^{t_f} =$$

$$= -\frac{m}{b} v_i \left\{ e^{-\frac{b}{m} t_f} - 1 \right\} = -\frac{m}{b} v_i \left\{ e^{-\frac{b}{m}\frac{m}{b}\ln(v_i/v_f)} - 1 \right\}$$

$$= -\frac{m}{b} v_i \left\{ e^{-\ln(v_i/v_f)} - 1 \right\} = -\frac{m}{b} v_i \left\{ \frac{v_f}{v_i} - 1 \right\}$$

$$\boxed{= \frac{m}{b}(v_i - v_f)}$$

QE-Ph.D. August 2012 (J. Schroeder) (28·VI·2012)

[I-2] Lagrangian Mechanics - Solutions

(a) Let $\vec{r}_1, \vec{r}_2$ be the radius vectors of the binary stars, masses m_1, m_2, respectively, from the origin of a fixed coordinate frame. Then

$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2, \quad V = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|},$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}\left(m_1 |\dot{\vec{r}}_1|^2 + m_2 |\dot{\vec{r}}_2|^2\right) + \frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

(b) $$V(\alpha\vec{r}_1, \alpha\vec{r}_2) = -\frac{G m_1 m_2}{|\alpha\vec{r}_1 - \alpha\vec{r}_2|} = -\frac{1}{\alpha}\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|} =$$

$$= \frac{1}{\alpha} V(\vec{r}_1, \vec{r}_2)$$ i.e. the potential energy is a homogeneous function of the coordinates of degree -1.

(c) Let $\vec{R}$ be the radius vector of the center of mass of the binary system from the origin of the fixed coordinate frame, and $\vec{r}_1{}', \vec{r}_2{}'$ be the radius vectors of m_1, m_2 from the center of mass respectively. By definition

$$(m_1 + m_2)\vec{R} = m_2 \vec{r}_2 + m_1 \vec{r}_1$$

$$\vec{r}_1 = \vec{R} + \vec{r}_1{}', \quad \vec{r}_2 = \vec{R} + \vec{r}_2{}'$$

or $$\vec{r}_1{}' = \frac{m_2 \vec{r}}{m_1 + m_2}, \quad \vec{r}_2{}' = -\frac{m_1 \vec{r}}{m_1 + m_2}$$

I-2] continued.

where $\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1' - \vec{r}_2'$, we may now write the Lagrangian as:

$$L = \frac{m_1+m_2}{2}|\dot{\vec{R}}|^2 + \frac{m_1 m_2}{2(m_1+m_2)}|\dot{\vec{r}}|^2 + \frac{G m_1 m_2}{|\vec{r}|}.$$

As L does not depend on $\vec{R} = (x, y, z)$ explicitly, $\frac{\partial L}{\partial \dot{x}}, \frac{\partial L}{\partial \dot{y}}, \frac{\partial L}{\partial \dot{z}}$ and hence $(m_1+m_2)\dot{\vec{R}}$ are constant. Therefore, the first term of L, which is the kinetic energy of the system as a whole, is constant. This term may be neglected when we are interested only in the internal motion of the system.

Thus, $L = \left(\frac{m_1 m_2}{m_1+m_2}\right)\left[\frac{1}{2}|\dot{\vec{r}}|^2 + \frac{G(m_1+m_2)}{|\vec{r}|}\right]$

$$L = \left(\frac{m_1}{m_1+m_2}\right)\left[\frac{1}{2}m_2|\dot{\vec{r}}|^2 + \frac{G m_2(m_1+m_2)}{|\vec{r}|}\right]$$

$$L = \left(\frac{m_2}{m_1+m_2}\right)\left[\frac{1}{2}m_1|\dot{\vec{r}}|^2 + \frac{G m_1(m_1+m_2)}{|\vec{r}|}\right],$$ which

may be considered as the Lagrangian, apart from a multiplicative constant, of the motion of one star in the gravitational field of a fixed star of mass (m_1+m_2). Let m_1 be this "moving" star and consider its centripetal force:

$$m_1 r \dot{\theta}^2 = \frac{G m_1(m_1+m_2)}{r^2}$$

[I-2] continued. (3.)

or $$\frac{T^2}{r^3} = \frac{4\pi^2}{G(m_1+m_2)}$$ with

$T = \frac{2\pi}{\dot{\theta}}$ being the period of m_1 about m_2, which is Kepler's third law.

<u>Note</u>: the same is true ~~about~~ for the motion of m_2 about m_1.

[redacted]

I-3

12-21. The tensors $\{\mathbf{A}\}$ and $\{\mathbf{m}\}$ are:

$$\{\mathbf{A}\} = \begin{bmatrix} \kappa_1 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 \end{bmatrix} \qquad A = \frac{\partial^2 U}{\partial x_i \partial x_j} \tag{1}$$

$$\{\mathbf{m}\} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{2}$$

thus, the secular determinant is

$$\begin{vmatrix} \kappa_1 - m\omega^2 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 - m\omega^2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 - m\omega^2 \end{vmatrix} = 0 \tag{3}$$

from which

$$\left(\kappa_1 - m\omega^2\right)^2 \left(\kappa_2 - m\omega^2\right) - \frac{1}{2}\kappa_3^2 \left(\kappa_1 - m\omega^2\right) = 0 \tag{4}$$

In order to find the roots of this equation, we first set $(1/2)\kappa_3^2 = \kappa_1\kappa_2$ and then factor:

$$\begin{gathered} \left(\kappa_1 - m\omega^2\right)\left[\left(\kappa_1 - m\omega^2\right)\left(\kappa_2 - m\omega^2\right) - \kappa_1\kappa_2\right] = 0 \\ \left(\kappa_1 - m\omega^2\right)\left[m^2\omega^4 - \left(\kappa_1 + \kappa_2\right)m\omega^2\right] = 0 \\ \left(\kappa_1 - m\omega^2\right)m\omega^2\left[m\omega^2 - \left(\kappa_1 + \kappa_2\right)\right] = 0 \end{gathered} \tag{5}$$

Therefore, the roots are

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{\kappa_1}{m}} \\ \omega_2 &= \sqrt{\frac{\kappa_1 + \kappa_2}{m}} \\ \omega_3 &= 0 \end{aligned}} \tag{6}$$

Consider the case $\omega_3 = 0$. The equation of motion is

$$\ddot{\eta}_3 + \omega_3^2 \eta_3 = 0 \tag{7}$$

[redacted]

so that

$$\ddot{\eta}_3 = 0 \tag{8}$$

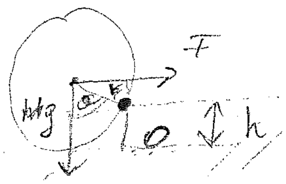
with the solution

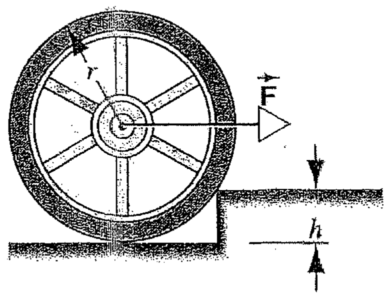
$$\eta_3(t) = at + b \tag{9}$$

That is, the zero-frequency mode corresponds to a *translation* of the system with oscillation.

I-4 [10]

What minimum force *F*, applied horizontally at the axel of the wheel, is necessary to raise the wheel over a curb of height *h*? (See figure below.) The radius of the wheel is *r* and its total mass is *M*. The gravitational acceleration is *g*. You must express your answer in terms of *h*, *r*, *M*, and *g*.





Torques from gravity and force F just balance / equal at the critical value

Torques about point O: $\vec{r} \times M\vec{g} + \vec{r} \times \vec{F} = 0$

$$Mg\cancel{r}\sin\theta = F\cancel{r}\cos\theta$$

$$F = Mg\frac{\sin\theta}{\cos\theta} = Mg\tan\theta$$ (Must express tanθ with r and h)

$$h + r\cos\theta = r$$

$$\cos\theta = \frac{r-h}{r} = 1 - \frac{h}{r}$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(1-\tfrac{h}{r}\right)^2} =$$

$$= \sqrt{2\frac{h}{r} - \left(\frac{h}{r}\right)^2}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{2\frac{h}{r} - \left(\frac{h}{r}\right)^2}}{1-\frac{h}{r}} = \frac{\sqrt{2hr-h^2}}{r-h} = \frac{\sqrt{h(2r-h)}}{r-h}$$

$$\boxed{F = Mg\frac{\sqrt{h(2r-h)}}{r-h}}$$

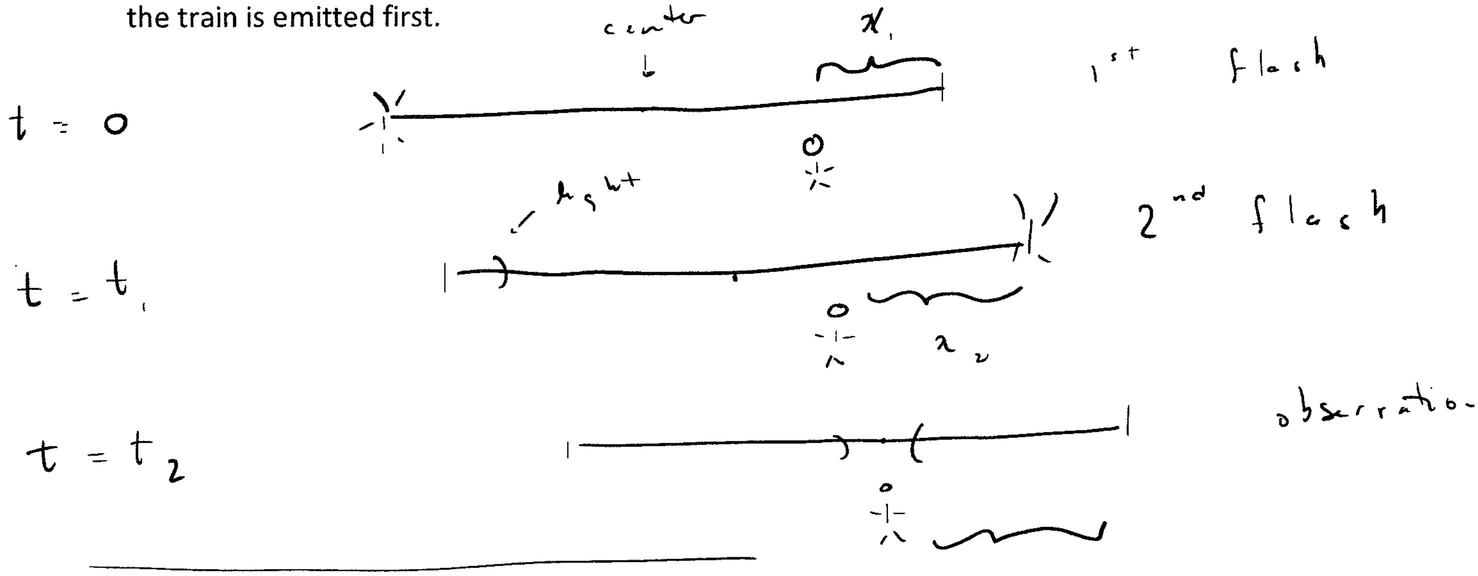
$(r > h)$

Solutions

I-5:

In the laboratory frame:

(a & b)The light flashes are not simultaneous in the laboratory frame, light from the rear of the train is emitted first.



$$\left.\begin{array}{ll} \dfrac{(x_2 - x_1)}{v} = t_1 & \dfrac{L - x_1}{c} = t_2 \\[2ex] \dfrac{(L/_2 - x_1)}{v} = t_2 & \dfrac{x_2}{c} = t_2 - t_1 \end{array}\right\} \text{solve for } x_1 \text{ \& } t_2.$$

where $L = L_0\sqrt{(1 - \left(\frac{v}{c}\right)^2}$.

In the reference frame of the train:

(c) An observer at the center of the train in the train reference frame would also observe the two flashes of light arriving simultaneously. Since the observer on the train is equidistant from the front and rear of the train, the flashes occurred simultaneously in the frame of reference of the train.

(d) The light from the two flashes arrive at the center of the train at the same time the observer does. It takes the light $\frac{L_0}{2c}$ to reach the center of the train after the flash. During that time the observer traveled, $\frac{L_0}{2c}v$. Therefore, the observer was, $\frac{L_0}{2} - \frac{L_0}{2c}v$ from the front of the train when the flash occurred.

Q.E.-Ph.D. August 2012. John Schroeder
(12·I·2012)

[I-6] Solution

From the Maxwell equation for $\vec{D} = \epsilon \vec{E}$

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho \quad \text{we now have}$$

$$4\pi\rho = \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\epsilon \vec{E}) = (\vec{\nabla}\epsilon) \cdot \vec{E} + \epsilon \vec{\nabla} \cdot \vec{E}$$

substituting $\vec{J}/\sigma$ for $\vec{E}$ above and $\vec{J}$ is the current density for an isotropic medium, we obtain

$$4\pi\rho = (\vec{\nabla}\epsilon) \cdot \vec{E} + \epsilon \vec{\nabla} \cdot \left(\frac{\vec{J}}{\sigma}\right) = (\nabla\epsilon) \cdot E - \frac{\epsilon}{\sigma^2}(\vec{\nabla}\sigma) \cdot \vec{J} + *$$

$$* \frac{\epsilon}{\sigma} \vec{\nabla} \cdot \vec{J}$$

however, the current is stationary $\therefore \vec{\nabla} \cdot \vec{J} = 0$ and substituting back for $\vec{J}$,

$$4\pi\rho = (\vec{\nabla}\epsilon) \cdot \vec{E} - \frac{\epsilon}{\sigma}(\vec{\nabla}\sigma) \cdot \vec{E} = -\frac{1}{\sigma}(\sigma \vec{\nabla}\epsilon - \epsilon \vec{\nabla}\sigma) \cdot \vec{\nabla}\phi$$

and so we have

$$\rho = -\frac{1}{4\pi\sigma}\left(\sigma \vec{\nabla}\epsilon - \epsilon \vec{\nabla}\sigma\right) \cdot \vec{\nabla}\phi$$

$\boxed{I-7}$

Solution

Conduction current $\quad j_c = \sigma E = \frac{1}{\rho}\frac{V}{d}$

displacement current $\quad j_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E)$

$$= \frac{\partial}{\partial t}\left(\varepsilon \frac{V(t)}{d}\right)$$

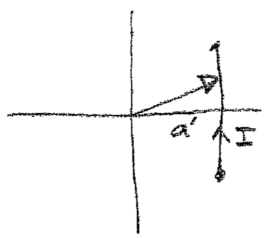
$$= \frac{\varepsilon}{d}\frac{\partial}{\partial t}\left(V_0 \cos(2\pi F t)\right)$$

$$= \frac{\varepsilon_0 V_0}{d}\left[-2\pi F \sin(2\pi F t)\right]$$

ratio of amplitudes

$$\frac{j_c}{j_d} = \frac{V_0}{\rho d}\,\frac{d}{2\pi F \varepsilon V_0} = \frac{1}{2\pi F \varepsilon \rho} \approx 2.41$$

Problem I-8 $\quad a' = a/2$



$$\vec{x}' = a'\hat{x} + y\hat{y}$$
$$\vec{x} = h\hat{z}$$
$$(\vec{x}-\vec{x}')^2 = a'^2 + y^2 + h^2$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{dy\,\hat{y} \times (h\hat{z} - a'\hat{x} - y\hat{y})}{(a'^2 + y^2 + h^2)^{3/2}}$$

By symmetry, all but $\hat{z}$ component vanishes when we sum the 4 sides.

$$dB_z = \frac{\mu_0 I a'}{4\pi} \frac{dy}{(a'^2 + y^2 + h^2)^{3/2}}$$

$$B_z = 4 \cdot \frac{\mu_0 I a'}{4\pi} \int_{-a'}^{a'} \frac{dy}{(a'^2 + y^2 + h^2)^{3/2}}$$

from 4 sides

$$= \frac{2\mu_0 I a'}{\pi} \int_0^{a'} \frac{dy}{(a'^2 + h^2 + y^2)^{3/2}}$$

$$= \frac{2\mu_0 I a'}{\pi} \frac{1}{a'^2 + h^2} \frac{y}{\sqrt{a'^2 + h^2 + y^2}} \Bigg|_0^{a'}$$

$$= \frac{2\mu_0 I a'}{\pi} \frac{1}{a'^2 + h^2} \frac{a'}{\sqrt{2a'^2 + h^2}}$$

$$\vec{B} = \hat{z} \frac{2\mu_0 I a'^2}{\pi (a'^2 + h^2)} \frac{1}{\sqrt{2a'^2 + h^2}}$$

$$\boxed{\vec{B} = \hat{z} \frac{\mu_0 I a^2}{2\pi\left(\frac{a^2}{4} + h^2\right)} \frac{1}{\sqrt{\frac{a^2}{2} + h^2}}}$$

I-9:

The atom acts as a Hertzian dipole at the origin with dipole moment, $\boldsymbol{p} = \alpha\boldsymbol{E} = \alpha E_0 e^{-i\omega t}\hat{k}$.

At large r, the asymptotic (radiation) electric and magnetic fields are given by

$$\boldsymbol{B}(\boldsymbol{r},t) = -\frac{\alpha E_0 \omega^2}{4\pi\epsilon_0 c^3 r} sin\theta e^{-i\omega t}\hat{\phi}$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\alpha E_0 \omega^2}{4\pi\epsilon_0 c^2 r} sin\theta e^{-i\omega t}\hat{\theta}$$

The average radiation intensity is given by the time average of the Poynting vector, $I = \langle N \rangle = \frac{1}{2} Re[\boldsymbol{E}^* \times \boldsymbol{H}] = \frac{1}{2\mu_0} Re[-c(\hat{r} \times \boldsymbol{B}^*) \times \boldsymbol{B}]$

$$< N > = \frac{c}{2\mu_0}|\boldsymbol{B}|^2$$

The energy radiated per unit solid angle is

$$\frac{dW}{d\Omega} = r^2 < N > = \frac{r^2 c}{2\mu_0}\boldsymbol{B}^* \cdot \boldsymbol{B} = \frac{c}{2\mu_0}\frac{\alpha^2 E_0^2 \omega^4}{16\pi^2\epsilon_0^2 c^6}\sin^2\theta = \frac{\alpha^2 E_0^2 \omega^4}{32\pi^2 \epsilon_0 c^3}\sin^2\theta$$

Problem I-10

Take the Lorentz transformation to be along the x-axis.

$$\frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial y'^2}, \quad \frac{\partial^2\psi}{\partial z^2} = \frac{\partial^2\psi}{\partial z'^2}$$

since $y = y'$, $z = z'$. Next

$$\frac{\partial\psi}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial\psi}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial\psi}{\partial t'}$$

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial x}\frac{\partial\psi}{\partial t'}$$

$$= \frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x}\frac{\partial^2\psi}{\partial x'^2} + \frac{\partial x'}{\partial x}\frac{\partial t'}{\partial x}\frac{\partial^2\psi}{\partial t'\partial x'}$$

$$+ \frac{\partial t'}{\partial x}\frac{\partial x'}{\partial x}\frac{\partial^2\psi}{\partial x'\partial t'} + \frac{\partial t'}{\partial x}\frac{\partial t'}{\partial x}\frac{\partial^2\psi}{\partial t'^2}$$

$$\frac{\partial^2\psi}{\partial t^2} = \left(\frac{\partial x'}{\partial t}\right)^2\frac{\partial^2\psi}{\partial x'^2} + 2\frac{\partial x'}{\partial t}\frac{\partial t'}{\partial t}\frac{\partial^2\psi}{\partial t'\partial x'}$$

$$+ \left(\frac{\partial t'}{\partial t}\right)^2\frac{\partial^2\psi}{\partial t'^2}$$

$$\frac{\partial x'}{\partial x} = \gamma \qquad \frac{\partial x'}{\partial t} = -\gamma v$$

$$\frac{\partial t'}{\partial t} = \gamma \qquad \frac{\partial t'}{\partial x} = -\gamma \frac{v}{c^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$= \gamma^2 \frac{\partial^2 \psi}{\partial x'^2} - 2\gamma^2 \frac{v}{c^2}\frac{\partial^2 \psi}{\partial t' \partial x'} + \gamma^2 \frac{v^2}{c^4}\frac{\partial^2 \psi}{\partial t'^2}$$

$$- \frac{1}{c^2}\gamma^2 v^2 \frac{\partial^2 \psi}{\partial x'^2} + 2\gamma^2 v \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t' \partial x'} - \frac{\gamma^2}{c^2}\frac{\partial^2 \psi}{\partial t'^2}$$

$$= \frac{\partial^2 \psi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t'^2}$$

using $\gamma^2 = \dfrac{1}{1 - \frac{v^2}{c^2}}$

$$\therefore \quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

$$= \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2} + \frac{\partial^2 \psi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t'^2}$$

//

Solutions

Physics PhD Qualifying Examination
Part II – Friday, August 24 2012

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1]   **[10]**

Suppose a lecture hall at the university is evacuated and (Schroedinger) cats are projected with speed v at the two doors leading into the lecture hall in a double-slit experiment.
The wavelengths of the interference fringes which are observed as the cats pile up against the wall of the lecture hall are larger than 1m. The mass of a cat is 1kg.

(a) Estimate the maximum speed for each cat.
(b) If the distance between the doors of the lecture hall to the wall is 33m, how long will it take to carry out the experiment?
(c) Compare this time with the age of the universe (10^{10} years).

Recall that $h = 6.6\times10^{-34}$Js.

[II-2]   [7,3]

The anharmonic oscillator has a Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x^4.$$

We will work in the limit of *small* λ.
(a) Calculate the leading correction to the ground state energy. Hint: it is proportional to λ.
(b) Write down a formula for the subleading correction to the ground state energy (proportional to λ^2) but do not evaluate the matrix elements.

[II-3] **[10]**

Consider the Pauli matrices σ_1, σ_2 and σ_3:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) Calculate the eigenvalues and eigenvectors of σ_1, σ_2 and σ_3.

(b) Calculate the commutator relations $[\sigma_i, \sigma_j]$ for $i{\neq}j$ and $i{=}j$.

[II-4] **[10]**

A nonrelativistic particle is scattered by a square-well potential

$$V(r) = \begin{cases} -V_0, & r < R, (V_0 > 0) \\ 0, & r > R \end{cases}.$$

(a) Assuming the bombarding energy is sufficiently high, calculate the scattering cross section in the first Born approximation (normalization is not essential).

(b) How can this result be used to measure R? (The smallest non-trivial solution of the transcendent equation $x = \tan(x)$ is approximately $x \approx 1.43\pi$.)

[II-5] **[10]**

A free particle of mass m moves in one dimension. At time $t = 0$ the normalized wave function of the particle is

$$\Psi(x,0) = (2\pi\sigma_x^2)^{-\frac{1}{4}} e^{-\frac{x^2}{4\sigma_x^2}}, \text{ where } \sigma_x^2 =< x^2 >.$$

(a) Compute the momentum spread $\sigma_p = \sqrt{< p^2 > - < p >^2}$ associated with this wave function. How can this be interpreted in terms of the uncertainty principle?

(b) Show that at time $t > 0$ the probability density of the particle has the form

$$|\Psi(x,t)|^2 = |\Psi(x,0)|^2 \text{ with } \sigma_x^2 \text{ replace by } \sigma_x^2 + \frac{\sigma_p^2 t^2}{m^2}.$$

[II-6] **[6,4]**

A hydrogen atom in its *ground state* is placed between the parallel plates of a capacitor. For times $t < 0$, no voltage is applied. Starting at $t = 0$, an electric field $\boldsymbol{E}(t) = E_o \hat{z} e^{-t/\tau}$ is applied, where τ is a constant.

(a) Derive the equation for the probability that the electron ends up in a state j due to this perturbation.

(b) Evaluate the result if state j is a:

(*i*) 2s state (parity argument may simplify the calculation);
(*ii*) 2p state.

The normalized eigenstates of the hydrogen atom:

$$\varphi_{100} = \frac{1}{\sqrt{\pi} a_0^{3/2}} e^{-r/a_0},$$

$$\varphi_{200} = \frac{1}{(2a_0)^{3/2}\sqrt{\pi}} \left(1 - \frac{r}{2a_0}\right) e^{-r/2a_0},$$

$$\varphi_{210} = \frac{1}{(2a_0)^{3/2}\sqrt{\pi}} \left(\frac{r}{2a_0}\right) e^{-r/2a_0} \cos\theta, \qquad \varphi_{21\pm1} = \frac{1}{8a_0^{3/2}} \left(\frac{r}{a_0}\right) e^{-r/2a_0} \sin\theta \, e^{\pm\phi},$$

where $a_0 = \frac{\hbar^2}{\mu e^2}$ (the Bohr radius).

[II-7] **[6,4]**

(a) Derive the Clausius-Clapeyron equation for the equilibrium of two phases of a substance. Consider a liquid or solid phase in equilibrium with its vapor.

(b) Using part (a) and the ideal gas law for the vapor phase, show that the vapor pressure follows the equation

$$\ln(P) = A - \frac{B}{kT} ,$$

where T is the temperature and k is the Boltzmann constant.

Make reasonable assumptions as required. What is the physical interpretation of B in this two-phase coexistence?

[II-8] **[10]**

A *monatomic ideal gas* consists of N atoms at initial temperature T. The gas is slowly compressed to one half of its initial volume through a *quasi-static reversible adiabatic* process.

What is the work done *on* the gas during this process? (Your answer must be expressed in terms of N, T, and the Boltzmann constant k.)

[II-9] **[10]**

A system consists of N independent localized (hence, distinguishable) particles. The single-particle energy spectrum has infinitely many energy levels, but precise information is only available on the lowest two levels. The energy of the single-particle ground state and the first excited state are ε and 3ε, with degeneracies $g_1 = 3$ and $g_2 = 6$, respectively.

Obtain the low-temperature ($\varepsilon / kT >> 1$) behavior of the heat capacity of the system, $C(N,T)$.

[II-10] **[6,2,2]**

Consider a cubical solid of dimensions $L\times L\times L$. Sound waves in this solid will behave much like photons (i.e., Planck distribution) in a cavity with conducting faces, except that there is a longitudinal polarization and a maximum mode number, determined by the fact that there can only be a total of $3N$ modes, if N is the number of atoms in the cubical solid. The corresponding quanta are called *phonons*. The relationship between the mode number and the angular frequency can be determined by recognizing that these are solutions to the (three dimensional) wave equation with wave speed v, with quantization of the wave vector $\boldsymbol{k}$ because of the boundary conditions on the faces of the cube. E.g., $k_x = \frac{\pi}{L} n_x$ where n_x is an integer.

(a) Determine the internal energy U of the phonons at temperature T, written in terms of an integral over over the mode number $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$ and the Debye temperature $= \left(\frac{\hbar v}{k_B}\right)(6\pi^2 N/V)^{1/3}$, where v is the velocity of sound, N is the number of atoms in the solid and $V = L^3$ is the volume.

(b) Find the heat capacity at constant volume in the low temperature limit $T \ll \theta$, by performing the integral in this approximation. You will want to know that $\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$.

(c) Find the heat capacity at constant volume in the high temperature limit $T \gg \theta$, by performing the integral in this approximation.

$\boxed{II-1}$

Part II Solution

<u>Solution</u>

$$\lambda = \frac{h}{p} = \frac{h}{mv} \Rightarrow v = \frac{h}{m\lambda}$$

$$v = \frac{6.6\times10^{-34}\ Js}{1\,kg \cdot 1\,m} = 6.6\times10^{-34}\ \frac{m}{s}$$

$$\Delta x = v\,\Delta t \Rightarrow \Delta t = \frac{\Delta x}{v} = \frac{33\ m}{6.6\ m/s}\ 10^{34}$$

$$\Delta t = 5\times10^{34}\ s$$

$$\frac{\Delta t}{\text{age of universe}} = \cancel{3\times10} \ \frac{5\times10^{34}\,s}{3\times10^{17}\,s} \approx 10^{17}\ !$$

$$1\ \text{year} = 365\times24\times60\times60\,s = 3.1536\times10^{7}\,s$$

$$10^{10}\ \text{years} = 3.1536\times10^{17}\ s$$

Problem II-2

(a)

$x = c\,(a + a^{\dagger})$ where

$a + a^{\dagger} = 2\sqrt{\frac{m\omega}{2\hbar}}\,x = \sqrt{\frac{2m\omega}{\hbar}}\,x$

so that $c = \sqrt{\frac{\hbar}{2m\omega}}$

$\langle 0|\,(a + a^{\dagger})^4\,|0\rangle =$

$\langle 0|\,(a^2 + a^{\dagger 2} + aa^{\dagger} + a^{\dagger}a)^2\,|0\rangle$

$= \langle 0|\,a^4 + a^{\dagger 4} + aa^{\dagger}aa^{\dagger} + a^{\dagger}aa^{\dagger}a + a^2a^{\dagger 2} + a^2aa^{\dagger}$
$+ a^2a^{\dagger}a + a^{\dagger 2}a^2 + a^{\dagger 2}aa^{\dagger} + a^{\dagger 3}a + aa^{\dagger}a^2$
$+ aa^{\dagger 3} + aa^{\dagger 2}a + a^{\dagger}a^3 + a^{\dagger}aa^{\dagger 2} + a^{\dagger}a^2a^{\dagger}\,|0\rangle$

$= \langle 0|\,aa^{\dagger}aa^{\dagger} + a^2a^{\dagger 2}\,|0\rangle$

$= \langle 0|\,[a, a^{\dagger}][a, a^{\dagger}] + [a^2, a^{\dagger 2}]\,|0\rangle$

$= 1 + \langle 0|\,a[a, a^{\dagger 2}] + [a, a^{\dagger 2}]a\,|0\rangle$

$= 1 + \langle 0|\,aa^{\dagger}[a, a^{\dagger}] + a[a, a^{\dagger}]a^{\dagger}\,|0\rangle$

$= 1 + 1 + 1 = 3$

$$\boxed{\Delta E_0^{(1)} = 3\lambda\left(\frac{\hbar}{2m\omega}\right)^2}$$

Alternatively,

$$\psi_0(x) = \frac{1}{\pi^{1/4}\sqrt{x_0}} \exp\left[-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right]$$

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

$$\lambda\langle 0|x^4|0\rangle = \lambda\int_{-\infty}^{\infty} dx\, \frac{x^4}{\sqrt{\pi}\, x_0} \exp\left[-\left(\frac{x}{x_0}\right)^2\right]$$

$$= \frac{\lambda x_0^4}{\sqrt{\pi}} \int d\left(\frac{x}{x_0}\right) \left(\frac{x}{x_0}\right)^4 e^{-\left(\frac{x}{x_0}\right)^2}$$

$$= \frac{\lambda x_0^4}{\sqrt{\pi}} \left[\frac{d^2}{d\alpha^2} \int dz\, e^{-\alpha z^2}\right]_{\alpha\to 1}$$

$$= \frac{\lambda x_0^4}{\sqrt{\pi}} \left[\frac{d^2}{d\alpha^2} \sqrt{\frac{\pi}{\alpha}}\right]_{\alpha\to 1}$$

$$= \lambda x_0^4 \, \frac{d}{d\alpha}\left(-\frac{1}{2}\alpha^{-3/2}\right)\Big|_{\alpha=1}$$

$$= \lambda x_0^4 \, \frac{3}{2^2} = 3\lambda\left(\frac{\hbar}{2m\omega}\right)^2$$

(b) $$\Delta E_0^{(2)} = \lambda^2 \sum_{n>0} \frac{\langle 0|x^4|n\rangle\langle n|x^4|0\rangle}{E_0^{(0)} - E_n^{(0)}}$$

$$= -\lambda^2 \sum_{n>0} \frac{\langle 0|x^4|n\rangle\langle n|x^4|0\rangle}{n\hbar\omega}$$

It would be fun to compute the matrix elements in this sum (only 3 of them are nonvanishing) and find a final answer.

II-3

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \qquad \lambda_{1,2} = \pm 1$$

eigenvectors

$\lambda = +1 \qquad -x + y = 0 \Rightarrow x = y \qquad x_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\qquad\qquad x - y = 0$

$\lambda = -1 \qquad x + y = 0 \Rightarrow = x - y \qquad x_1^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

for $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \lambda_{1,2} = \pm 1$

$\lambda = 1 \qquad \cancel{x_2^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \qquad x_2^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -1 \qquad \cancel{x_2^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \qquad x_2^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

for $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \lambda_{1,2} = \pm 1$

$\lambda = 1 \Rightarrow \qquad x_3^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1 \qquad x_3^4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

II-4

(a) Using the Born approximation we have

$$f(\theta) \propto -\frac{1}{q}\int_0^\infty rV(r)\sin(qr)\,dr$$

$$= \frac{V_0}{q}\int_0^R r\sin(qr)dr = \frac{V_0}{q^3}(\sin(qR) - qR\cos(qR)).$$

Hence, $\frac{d\sigma}{d\Omega} \propto \left(\frac{\sin(x)-x\cos(x)}{x^3}\right)^2$, with $x = qR = 2kR\sin(\frac{\theta}{2})$.

(b) The first zero of $\frac{d\sigma}{d\Omega}$ occurs at x for which x = *tan*(x), whose solution is x ~ 1.43π. This gives $R = \frac{1.43\pi}{2k\sin\left(\frac{\theta_1}{2}\right)}$. By measuring the minimum angle θ_1 for which $\frac{d\sigma}{d\Omega} = 0$, R can be determined.

II-5:

(a)

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar\frac{d}{dx}\right)\psi dx$$

$$= -i\hbar \int_{-\infty}^{\infty} (2\pi\sigma_x^2)^{-\frac{1}{2}}\left(-\frac{x}{2\sigma_x^2}\right) e^{-\frac{x^2}{2\sigma_x^2}}\, dx \;= 0$$

$$\langle p^2\rangle = \int_{-\infty}^{\infty} \psi^* \left(-\hbar^2\frac{d^2}{dx^2}\right)\psi dx = -\hbar^2 \int_{-\infty}^{\infty} (2\pi\sigma_x^2)^{-\frac{1}{2}}\left(-\frac{1}{2\sigma_x^2}+\frac{x^2}{4\sigma_x^4}\right)e^{-\frac{x^2}{2\sigma_x^2}}\, dx \;= \frac{\hbar^2}{4\sigma_x^2}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar}{2\sigma_x}$$

The uncertainties satisfy $\Delta x\Delta p = \sigma_x\sigma_p = \frac{\hbar}{2}$ (Gaussian wave-packet gives the minimum possible uncertainty product).

(b) By Fourier transform,

$$\psi(p,0) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}}\int e^{-\frac{ipx}{\hbar}}\psi(x,0)dx$$

$$= \frac{(2\pi\sigma_x^2)^{\frac{1}{4}}}{\sqrt{2}\pi\hbar}e^{-\sigma_x^2p^2/\hbar^2}$$

Hence, $\psi(p,t) = \psi(p,0)e^{-iEt/\hbar}$, with $E = \frac{p^2}{2m}$.

By inverse Fourier transform,

$$\psi(x,t) = \int e^{\frac{ipx}{\hbar}}\psi(p,t)dp = \frac{(2\pi\sigma_x^2)^{\frac{1}{4}}}{\pi\hbar\sqrt{2}}\int e^{\frac{ipx}{\hbar}}e^{-\sigma_x^2p^2/\hbar^2}e^{-ip^2t/2m\hbar^2}\, dp$$

Hence, $\psi(x,t) = \left(\frac{\sigma_x^2}{2\pi}\right)^{\frac{1}{4}}\left(\sigma_x^2+\frac{i\hbar t}{2m}\right)^{-\frac{1}{2}}e^{-\frac{x^2}{4\left(\sigma_x^2+\frac{i\hbar t}{2m}\right)}}$.

Therefore, $|\psi(x,t)|^2 = \left(\frac{\sigma_x^2}{2\pi}\right)^{1/2}\left(\sigma_x^2-\frac{i\hbar t}{2m}\right)^{-\frac{1}{2}}\left(\sigma_x^2+\frac{i\hbar t}{2m}\right)^{-\frac{1}{2}}e^{-\frac{x^2}{4(\sigma_x^2-\frac{i\hbar t}{2m})}}e^{-\frac{x^2}{4(\sigma_x^2+\frac{i\hbar t}{2m})}}$

$$= \left(\frac{1}{2\pi} \frac{\sigma_x^2}{\sigma_x^4 + \frac{\hbar^2 t^2}{4m^2}} \right)^{1/2} e^{-\frac{x^2}{2}\left(\frac{\sigma_x^2}{\sigma_x^4 + \frac{\hbar^2 t^2}{4m^2}} \right)}$$

To get the final answer, use the uncertainty principle, $\sigma_x \sigma_p = \frac{\hbar}{2}$.

Q.E.-Ph.D. August 2012 John Schroeder

(12·I·2012)

[II-6] Solution

For time-dependent perturbations a general wave function is

$$\Psi(\vec{r},t)=\sum_j a_j(t)\psi_j(\vec{r})e^{-i\omega_j t}$$

where the ψ_j satisfy

$$H_0\psi_j=\hbar\omega_j\psi_j$$

For the time dependent perturbation $V(t)$,

$$V(t)=-e|\vec{E}_0|z e^{-t/\tau}.$$

From Schrödinger's equation we can derive an equation for the time development of the amplitudes $a_j(t)$:

$$i\hbar\frac{\partial}{\partial t}\Psi=[H_0+V(t)]\Psi$$

$$i\hbar\frac{\partial}{\partial t}a_j(t)=\sum_\ell a_\ell\langle j|V(t)|\ell\rangle e^{i(\omega_j-\omega_\ell)t}$$

If the system is originally in the ground state, we have $a_{1s}(0)=1$ and the other values of $a_j(0)$ are zero. For small

[II-6] continued.

Perturbations it is sufficient to solve the equation for $j \neq 1s$:

$$\frac{\partial}{\partial t} a_j(t) = \frac{ie|\vec{E}_0|}{\hbar} \langle j|z|1s\rangle e^{-t\left\{\frac{1}{\tau} - i(\omega_j - \omega_{1s})\right\}}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle}{\hbar} \int_0^\infty dt\, e^{-t\left\{\frac{1}{\tau} - i(\omega_j - \omega_{1s})\right\}}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle \tau}{\hbar[1 - i\tau(\omega_f - \omega_{1s})]}$$

The general probability P_j that a transition is made to stat j is given by

$$P_j = |a_j(\infty)|^2 = \frac{(e|\vec{E}_0|\tau)^2 \langle j|z|1s\rangle^2}{\hbar^2[1 + \tau^2(\omega_j - \omega_{1s})^2]}$$

This probability is dimensionless. It needs to be less than unity for this theory to be valid.

(a.) For the state $j = 2s$ the probability is zero. It vanishes because the matrix element of z is zero: $\langle 2s|z|1s\rangle = 0$ due to parity.

[II-2] Continued.

Both S-states have even parity and z has odd parity.

(b.) For the state $j = 2P$ the transition is allowed to the $L=1$, $M=0$ orbital state, which is called $2P_z$. The matrix element is similar what is found for the Stark effect. The $2P$ eigenstate for $L=1$, $S=0$ is

$$|10\rangle = \frac{z}{\sqrt{32\pi a_0^5}} e^{-r/2a_0}$$ and that

for the 1S state is $(-r/a_0)/\sqrt{\pi a_0^3}$.

The integral becomes

$$\langle 2P_z | z | 1S \rangle = \frac{2\pi}{\pi a_0^4 \sqrt{32}} \int_0^\infty dr\, r^4 e^{-3r/2a_0} \int_0^\pi d\theta \sin\theta \cos^2\theta$$

$$\langle 2P_z | z | 1S \rangle = \frac{1}{3\sqrt{2}\, a_0^4} \int_0^\infty dr\, r^4 e^{-3r/2a_0} = a_0 \left(\frac{2^{3/2}}{3}\right)^5,$$

and a_0 is the Bohr radius of the hydrogen atom.

[II-7] Solution QE-Aug 2012 (J.Schroeder) 28.VI.2012.

(a.) At equilibrium, the chemical potential of two phases must be equal:

$$\mu_1[P(\tau),\tau] = \mu_2[P(\tau),\tau] \qquad (1)$$

Note: $P \equiv P(\tau)$ since the pressure depends on the temperature. Taking derivatives of μ with respect to temperature, we have

$$\left(\frac{\partial \mu_1}{\partial \tau}\right)_P + \left(\frac{\partial \mu_1}{\partial P}\right)_\tau \frac{dP}{d\tau} = \left(\frac{\partial \mu_2}{\partial \tau}\right)_P + \left(\frac{\partial \mu_2}{\partial P}\right)_\tau \frac{dP}{d\tau} \qquad (2)$$

Now using $\left(\frac{\partial \mu}{\partial \tau}\right)_P = -s$ and $\left(\frac{\partial \mu}{\partial P}\right)_\tau = v$,

where s and v are entropy and volume per particle, substituting the expressions for s and v into equn.(2) we have

(3.) $$\frac{dP}{d\tau} = \frac{s_1 - s_2}{v_1 - v_2},$$ with subscripts 1 and 2 refer to the two phases. Again $q = \tau(s_2 - s_1)$, with q being the Latent heat per particle, hence equation (3) becomes

(4.) $$\boxed{\frac{dP}{d\tau} = \frac{q}{\tau \Delta v}}$$ the Clausius-Clapeyron equation

[II-7] continued.

(b.) Consider the case of equilibrium between liquid and vapour. The liquid volume v_1 is usually much smaller than the vapour volume, v_2, hence neglect v_1, and equation (4) becomes

$$\frac{dP_v}{d\tau} = \frac{q}{\tau v_v}$$

and using the ideal gas law for the vapour $v_v = \tau/P_v$

We obtain $\frac{dP_v}{d\tau} = q\frac{P_v}{\tau^2}$ (5.) or

$\ln P_v = A - \frac{q}{\tau}$ (6.). We see that $B = q$ and rewriting eqn. (6) in normal units

$$\ln P_v = A - \frac{q}{k_B T} = A - \frac{q N_A}{k_B N_A T} = A - \frac{L}{RT}$$

with L being the Latent heat per mole, $N_A \to$ Avogadro's number, and R is the gas constant.

II-8 [10]
A *monatomic ideal gas* consists of N atoms at initial temperature T. The gas is slowly compressed to one half of its initial volume through a *quasi-static reversible adiabatic* process.

What is the work done *on* the gas during this process? (Your answer must be expressed in terms of N, T, and the Boltzmann constant k.)

monatomic ideal gas: $f = 3$

rev. adiabatic process: $\boxed{T V^{\gamma - 1} = \text{const.}}$

$(T, V) \rightarrow (T', V')$

$$\gamma = \frac{C_p}{C_v} = \frac{f+2}{f} = 1 + \frac{2}{f}$$

$$\gamma - 1 = \frac{2}{f} = \frac{2}{3}$$

$$T V^{2/3} = T' V'^{2/3} \qquad V' = V/2$$

$$T' = T \left(\frac{V}{V'}\right)^{2/3} = \boxed{T\, 2^{2/3}}$$

$$dU = \delta Q + \delta W \quad , \quad \delta Q = 0 \quad (\text{adiab.})$$

$$\delta W = dU \qquad U = \frac{f}{2} N k T = \frac{3}{2} N k T$$

(monatomic ideal gas)

$$W = \int_i^f dU = \Delta U = U_f - U_i = \frac{3}{2} N k (T' - T)$$

$$= \frac{3}{2} N k (2^{2/3} T - T) = \boxed{\frac{3}{2} N k T (2^{2/3} - 1)}$$

$$W = \frac{3}{2} N k T (2^{2/3} - 1) \simeq \frac{3}{2} N k T \cdot 0.5874$$

II-9 [10]

A system consists of N independent localized (hence, distinguishable) particles. The single-particle energy spectrum has infinitely many energy levels, but precise information is only available on the lowest two levels. The energy of the single-particle ground state and the first excited state are ε and 3ε, with degeneracies $g_1 = 3$ and $g_2 = 6$, respectively.

Obtain the low-temperature ($\varepsilon / kT >> 1$) behavior of the heat capacity of the system, $C(N,T)$.

$$Z = \sum_j g_j e^{-\varepsilon_j/kT} = g_1 e^{-\varepsilon_1/kT} + g_2 e^{-\varepsilon_2/kT} + \dots$$

$$\simeq g_1 e^{-\varepsilon_1/kT} + g_2 e^{-\varepsilon_2/kT} = 3e^{-\varepsilon/kT} + 6e^{-3\varepsilon/kT}$$

$$= 3e^{-\varepsilon/kT}\left\{1 + 2e^{-2\varepsilon/kT}\right\}$$

$$\ln Z = \ln(3) - \frac{\varepsilon}{kT} + \ln\left\{1 + 2e^{-2\varepsilon/kT}\right\}$$

$\varepsilon/kT >> 1$

$\Downarrow$

$e^{-2\varepsilon/kT} << 1$

$$\boxed{\ln Z \simeq \ln(3) - \frac{\varepsilon}{kT} + 2e^{-2\varepsilon/kT}}$$

$\ln(1+x) \simeq x$ $(x << 1)$

$$\ln Z \simeq \ln(3) - \beta\varepsilon + 2e^{-2\beta\varepsilon}$$

$\beta = 1/kT$

$$U = -N\frac{\partial}{\partial\beta}\ln Z = -N\left\{-\varepsilon - 4\varepsilon e^{-2\beta\varepsilon}\right\} = \boxed{N\varepsilon + 4N\varepsilon e^{-\frac{2\varepsilon}{kT}}}$$

$$C = \left(\frac{\partial U}{\partial T}\right)_N = 4N\varepsilon\left(\frac{2\varepsilon}{kT^2}\right)e^{-\frac{2\varepsilon}{kT}}$$

$$= \boxed{2Nk\left(\frac{2\varepsilon}{kT}\right)^2 e^{-\frac{2\varepsilon}{kT}}}$$

Problem II - 10

First determine the maximum mode number.

$$3N = \frac{3}{8}\, 4\pi \int_0^{n_D} n^2\, dn$$

because modes are labeled by

$\vec{n} = (n_x, n_y, n_z)$ with $n_i = 0, 1, 2, \ldots$

$\frac{1}{8}$ of sphere, and 3 polarizations.

$$3N = \frac{\cancel{3}\pi}{2}\,\frac{n_D^3}{\cancel{3}} = \frac{\pi n_D^3}{2}$$

$$n_D = (6N/\pi)^{1/3}$$

Next, these modes will have ~~the~~ Planck distribution

$$\langle s \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$$

where s is occupancy.

From the wave eq.

$$v^2 \vec{k}^2 = \omega_{\vec{k}}^2$$

But quantization due to boundary conditions gives

$$\vec{k} = \frac{\pi}{L} \vec{n}$$

so $$\omega_n^2 = \frac{\pi^2 v^2}{L^2} n^2$$

Thus we find:

$$U = \frac{3}{8} 4\pi \int_0^{n_D} n^2 dn \, \frac{\hbar \omega_n}{e^{\beta \hbar \omega_n} - 1}$$

$$\Theta = \frac{\hbar v}{k_B} \left(\pi^3 \frac{6N}{\pi} \frac{1}{L^3} \right)^{1/3}$$

$$= \frac{\hbar v \pi}{k_B L} (6N/\pi)^{1/3} = \frac{\hbar}{k_B} \frac{\pi v}{L} n_D$$

So we see

$\Theta = \frac{1}{k_B} \hbar \omega_{n_D}$, a reasonable definition.

$$\therefore \quad \boxed{U = \frac{3\pi}{2} \int_0^{(k_B L \Theta)/(\hbar \pi v)} n^2 dn \; \frac{\hbar \frac{\pi v}{L} n}{\exp(\beta \hbar \pi v n/L) - 1}}$$

For further evaluation, it is convenient to define

$$x = \beta \hbar \pi v n / L$$

$$dn = \frac{L}{\beta \hbar \pi v} dx$$

$$U = \frac{3\pi}{2} \left(\frac{L}{\beta \hbar \pi v} \right)^3 \frac{1}{\beta} \int_0^{x_D} \frac{x^3 dx}{e^x - 1}$$

$$x_D = \frac{\beta \hbar \pi v}{L} \frac{k_B L \Theta}{\hbar \pi v} = \beta k_B \Theta = \frac{\Theta}{T}$$

$$U = \frac{3 L^3}{2 \hbar^3 \pi^2 v^3} \frac{1}{\beta^4} \int_0^{\Theta/T} \frac{x^3 dx}{e^x - 1}$$

(b) $T \ll \theta$ then we can replace

$$\int_0^{\theta/T} \frac{x^3 dx}{e^x - 1} \rightarrow \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

$$= \frac{\pi^4}{15}$$

$$U = \frac{3 L^3 k_B^4}{2 \hbar^3 \pi^2 v^3} \frac{\pi^4}{15} T^4$$

$$C_V = \frac{dU}{dT} = \frac{12 L^3 k_B^4 \pi^2}{30 \hbar^3 v^3} T^3$$

$$\boxed{C_V = \frac{2 L^3 k_B^4 \pi^2}{5 \hbar^3 v^3} T^3}$$

This is traditionally rewritten as

$$C_V = \frac{12 \pi^4 N k_B}{5} \left(\frac{T}{\theta}\right)^3$$

(c) $T \gg \theta$

Then we can approximate the integrand ($x \ll 1$)

$$\frac{x^3}{e^x - 1} \approx \frac{x^3}{x} = x^2$$

$$\int_0^{\theta/T} dx\, x^2 = \frac{1}{3}\left(\frac{\theta}{T}\right)^3$$

$$U = \frac{3L^3 k_B^4 T^4}{2\hbar^3 \pi^2 v^3} \cdot \frac{1}{3} \frac{\theta^3}{T^3}$$

$$C_V = \frac{L^3 k_B^4 \theta^3}{2\hbar^3 \pi^2 v^3}$$

$$= \frac{\cancel{L^3} k_B^{\cancel{4}}}{\cancel{2}\cancel{\hbar^3}\cancel{\pi^2} v^3} \frac{\cancel{\hbar^3} v^3}{\cancel{k_B^3}} \cancel{\pi^2} \frac{\overset{3}{\cancel{6}} N}{\cancel{L^3}}$$

$$\boxed{C_V = 3 k_B N}$$

This agrees with classical equipartition.