# STONY BROOK UNIVERSITY

## DEPARTMENT OF PHYSICS AND ASTRONOMY

### Graduate Placement Exam Part 1, Aug. 21, 2012 (13:00 - 17:30)

**General Instructions:** This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas. Each problem is worth 20 points, and unless stated otherwise, all parts of each question have equal weight.

Each solution should typically take less than 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

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**Electromagnetism 1**

Calculate the energy of electrostatic interaction between a point charge $q$ placed in the center of a spherical cavity of radius $R$, which was cut inside a grounded conductor, and the conductor.

**SOLUTION:**
The energy of the interaction is given by the difference of the energies of the electric field in the presence of the cavity and in empty space. The Gauss theorem indicates that the electric field of the charge $q$ inside the cavity at $r < R$ is the same as in empty space, therefore the difference is only due to the absence of electric field at $r > R$ (the conductor is grounded):

$$U = U_{\text{cavity}} - U_{\text{empty}} = \frac{\epsilon_0}{2}\left(\int_{r<R} E^2 d^3r - \int E^2 d^3r\right) = -\frac{\epsilon_0}{2}\int_{r>R} E^2 d^3r;$$

Using $E = q/(4\pi\epsilon_0 r^2)$, we get

$$U = -\frac{\epsilon_0}{2}\int_R^\infty \left(\frac{q}{4\pi\epsilon_0 r^2}\right)^2 4\pi r^2 dr = -\frac{q^2}{8\pi\epsilon_0 R}.$$

Alternatively, one can use the image charge method and compute the force acting on the charge as a function of the displacement from the center of the cavity (the force vanishes for the charge located in the center by symmetry). The energy is then given by the work (integral of the force over the path) needed to move the charge from the cavity's center to infinity.

**Electromagnetism 2**

Find the angular distribution of the intensity of electromagnetic radiation of the electric dipole of size $L$ rotating in the plane with constant frequency $\omega$.

**SOLUTION**:
The angular distribution of intensity of the dipole radiation is given by

$$\frac{dI}{d\Omega} = \frac{1}{4\pi c^3}[\ddot{\vec{d}} \times \vec{n}]^2,$$

where $d$ is the dipole moment and $\vec{n}$ is the unit vector describing the direction of emitted radiation. Choosing the plane of rotation as $\{x, y\}$ we can write down the components of the dipole moment as

$$d_x = L\cos\omega t, \quad d_y = L\sin\omega t;$$

Substituting these expressions into the formula for the angular distribution, we find

$$\frac{dI}{d\Omega} = \frac{L^2\omega^4}{8\pi c^3}(1 + \cos^2\theta).$$

**Electromagnetism 3**

Consider a point-like particle with electric charge $e$ moving with constant velocity $v$ along the straight line.

a) Find the plane wave decomposition of the scalar and vector potentials of the induced electromagnetic field (7 points);

b) Find the plane wave decomposition of the electric and magnetic fields induced by the particle, $\vec{E}_k$ abd $\vec{B}_k$, where $k$ is the wave vector (momentum) of the field (8 points);

c) Find the quantiity $\vec{E} \cdot \vec{B}$ (so-called "Chern-Pontryagin density") for this field (5 points).

**SOLUTION**:
a) Let us write the charge density as

$$\rho = e\delta(\vec{r} - \vec{v}t).$$

The wave equation for the scalar potential is

$$\Box\varphi = -4\pi e\delta(\vec{r} - \vec{v}t).$$

Making a Fourier decomposition, we get

$$(\Box\varphi)_k = -4\pi e \exp(-i(\vec{v}\vec{k})t).$$

The l.h.s. can be written down as

$$(\Box\varphi)_k = -k^2\varphi_k - \frac{1}{c^2}\frac{\partial^2\varphi_k}{\partial t^2}.$$

Combining the expressions above, we find

$$\varphi_k = 4\pi e\ \frac{\exp(-i(\vec{v}\vec{k})t)}{k^2 - \frac{1}{c^2}(\vec{v}\vec{k})^2}.$$

Analogous derivation yields for the vector potential

$$\vec{A}_k = \frac{4\pi e}{c}\ \frac{\vec{v}\ \exp(-i(\vec{v}\vec{k})t)}{k^2 - \frac{1}{c^2}(\vec{v}\vec{k})^2}.$$

b) The electric and magnetic fields are

$$\vec{E}_k = -i\vec{k}\varphi_k + \frac{i}{c}\ (\vec{v}\vec{k})\ \vec{A}_k = i4\pi e\ \frac{-\vec{k} + \frac{1}{c^2}(\vec{v}\vec{k})\vec{v}}{\vec{v}k^2 - \frac{1}{c^2}(\vec{v}\vec{k})^2}\ \exp(-i(\vec{v}\vec{k})t),$$

and

$$\vec{B}_k = i\vec{k} \times \vec{A}_k = i\frac{4\pi e}{c}\ \frac{\vec{v} \times \vec{k}}{\vec{v}k^2 - \frac{1}{c^2}(\vec{v}\vec{k})^2}\ \exp(-i(\vec{v}\vec{k})t).$$

c) Since $\vec{v} \times \vec{k}$ is orthogonal to both $\vec{v}$ and $\vec{k}$, the scalar product $\vec{E} \cdot \vec{B} = 0$.

## Classical Mechanics 1

A particle moving with velocity $V$ (in the laboratory frame) decays into two massless particles (all particles have spin equal to zero).

a) Find the minimal angle between the decay products in the lab frame.

b) Find the distribution in the angle between the decay products in the lab frame.

**SOLUTION:**

In the center-of-mass (c.m.s.) frame, the decay products move back-to-back, so the angles between their momenta and $\vec{V}$ in that frame can be parameterized as $\theta_1 = \theta_0$ and $\theta_2 = \pi - \theta_0$. Using the rule for relativistic addition of velocities in the frame moving with velocity $V$ along the axis $x$:

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}; \quad v_y = \frac{v'_y \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{v'_x V}{c^2}}; \quad v_z = \frac{v'_z \sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{v'_x V}{c^2}};$$

we get the relations between the angles in the c.m.s. and lab frames:

$$\sin\theta = \frac{\sqrt{1 - \frac{V^2}{c^2}}}{1 + \frac{V}{c}} \sin\theta', \quad \cos\theta = \frac{\cos\theta' + \frac{V}{c}}{1 + \frac{V}{c}}.$$

From these formulae we find how the angle between the decay products in the lab frame $\theta = \theta_1 + \theta_2$ is related to the angle $\theta_0$:

$$\cos\theta_0 = \sqrt{1 - \frac{1 - V^2}{V^2} \cot^2(\theta/2)}$$

Because all particles have spin equal to zero, the distribution in $\theta_0$ has to be homogeneous in the solid angle $\Omega_0$,

$$dN = \frac{1}{4\pi} d\Omega_0 = \frac{1}{2} |d\cos\theta_0|.$$

Therefore in terms of the solid angle between the decay products in the lab frame $d\Omega = 2\pi \sin\theta d\theta$ we get

$$\frac{dN}{d\Omega} = \frac{1 - V^2}{16\pi V} \frac{1}{\sin^3(\theta/2)\sqrt{V^2 - \cos^2(\theta/2)}}.$$

The minimal value of the angle $\theta$ is $\theta_{min} = 2 \arccos V$.

**Classical Mechanics 2**

A particle of mass $m$ moves in the potential $U(x) = ax^4$. The strength of the potential $a$ is increased slowly to $3a$. Then the mass of the particle is increased slowly to $4m$. Find the change in the total energy of the particle.

**SOLUTION:**

The adiabatic invariant for this system is $I = \oint p dx = \oint \sqrt{2m(E - U(x))} dx \sim \sqrt{mE}(E/a)^{1/4}$. It is obtained up to the numerical prefactor using proper rescaling (change of variable $x = (E/a)^{1/4}s$). The adiabatic invariant does not change during the very slow change of parameters of the system. We have $m^{1/2}E^{3/4}a^{-1/4} = (4m)^{1/2}E'^{3/4}(3a)^{-1/4}$ and thus $E' = (3/16)^{1/3}E$.

**Classical Mechanics 3**

a) Write down the Lagrangian $L$ of a relativistic point particle with electric charge $q$ coupled to arbitrary time-dependent electromagnetic fields. Derive the Lorentz force from this Lagrangian. Construct the Hamiltonian $H$. Is $L = T - V$ , and is $H = T + V$? When is $L$ conserved, and when is $H$?

b) Show that $F_2(q, P) = q^\alpha P_\alpha$ is the generator of canonical transformation which generates the identity transformation. Show that the time evolution of a system in classical mechanics consists of a series of infinitesimal canonical transformations generated by the Hamiltonian.

**SOLUTION:**

(a) The Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c}\vec{A}(\vec{x}, t) \cdot \vec{v} - q\phi(\vec{x}, t).$$

Then

$$\frac{d}{dt}\Big(\frac{m\dot{x}^k}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{q}{c}A_k\Big) = \frac{q}{c}\partial_k A_l \dot{x}^l - q\partial_k\phi$$

which equals

$$\frac{d\vec{\pi}}{dt} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}, \qquad \vec{\pi} = \frac{m\dot{\vec{x}}}{\sqrt{1 - v^2/c^2}}$$

since $E_k = -\frac{1}{c}\dot{A}_k - \partial_k\phi$ and $B_k = \epsilon_{klm}\partial_l A_m$. Since $p_k = m\dot{x}_k/\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c}A_k$, we find

$$H = p_k\dot{x}^k - L = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + mc^2\sqrt{1 - \frac{v^2}{c^2}} + q\phi = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + q\phi = \sqrt{(p_k - \frac{q}{c}A_k)^2c^2 + m^2c^4} + q\phi.$$

So we find $H = T + V$, but $L \neq T - V$. Since

$$\frac{\partial H}{\partial t} = \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

$H$ is conserved if $\vec{A}$ and $\phi$ are time-independent, but $L$ is not conserved.

(b) $F_2(q, P) = q^\alpha P_\alpha$ generates

$$\frac{\partial F_2}{\partial q^\alpha} = P_\alpha = p_\alpha \;\text{ and }\; \frac{\partial F_2}{\partial P^\alpha} = Q_\alpha = q_\alpha$$

which is the identify transformation. Define

$$\delta q^\alpha \equiv \dot{q}^\alpha dt = \frac{\partial H}{\partial p_\alpha} dt \quad \text{and} \quad \delta p_\alpha \equiv \dot{p}_\alpha dt = -\frac{\partial H}{\partial q^\alpha} dt$$

The generator $F_2(q, P)$ of canonical transformations yields

$$\frac{\partial F_2(q, P)}{\partial q^\alpha} = p_\alpha \quad \text{and} \quad \frac{\partial F_2(q, P)}{\partial P_\alpha} = Q^\alpha.$$

Take $Q^\alpha(t) = q^\alpha(t + dt)$ and $p_\alpha(t + dt) = P_\alpha$ and $F_2 = q^\alpha P_\alpha + dtH$. Then

$$\frac{\partial F_2}{\partial q^\alpha} = P_\alpha + dt \frac{\partial H}{\partial q^\alpha} = p_\alpha$$

yields $P_\alpha = p_\alpha + dt\dot{p}_\alpha$ and idem for $q^\alpha$.