DOCTORAL GENERAL EXAMINATION
PART 1
August 27, 2012

FIVE HOURS

1. This examination is divided into five sections, each consisting of four problems. Answer all the problems. Each problem is worth 5 points, thus the maximum score for the exam is 100.

2. For each section, use the separate labeled booklet you have been given. **DO NOT** put your name on it, as each booklet has an identification number that will allow the papers to be graded blindly.

3. A diagram or sketch is often useful to the grader and could improve your grade.

4. Read the problem carefully and do not do more work than is necessary. For example "give" and "sketch" do not mean "derive".

5. Calculators may **not** be used.

6. No books, notes or reference materials may be used.

Possibly Useful Fomulae

A few definite integrals:

$$\int_0^\infty e^{-px^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{p}}$$

$$\int_0^\infty \frac{x^{\nu-1}}{e^x + a}\,dx = \begin{cases} \Gamma(\nu)\zeta(\nu)\left(1 - \frac{1}{2^{\nu-1}}\right) & \text{for} \quad a = +1 \\[2mm] \Gamma(\nu) & \text{for} \quad a = 0 \\[2mm] \Gamma(\nu)\zeta(\nu) & \text{for} \quad a = -1 \end{cases}$$

where $\nu > 0$ and $\Gamma(\nu)$ and $\zeta(\nu)$ are the Gamma and Reimann zeta functions, respectively.

Some physical constants:

k_B	$=$	1.4×10^{-23}	$\mathrm{J\,K^{-1}}$	$=$	$10^{16}\,\mathrm{erg\,K^{-1}}$
$\hbar$	$=$	1×10^{-34}	$\mathrm{J.s}$	$=$	$10^{-27}\,\mathrm{erg\cdot s}$
c	$=$	3×10^8	$\mathrm{m\,s^{-1}}$	$=$	$3 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
e	$=$	1.6×10^{-19}	Coulomb	$=$	$4.8 \times 10^{-10}\,\mathrm{statcoulomb}$
m_e	$=$	9.1×10^{-31}	kg	$=$	$0.5\,\mathrm{MeV/c^2}$
$\hbar c$	$=$	3.2×10^{-26}	$\mathrm{J\cdot m}$	$\simeq$	$1.97 \times 10^{-13}\,\mathrm{MeV\cdot m}$
G	$=$	6.7×10^{-11}	$\mathrm{m^3\,kg^{-1}\,s^{-2}}$	$=$	$6.7 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}}$
ε_0	$=$	8.9×10^{-12}	$\mathrm{F\,m^{-1}}$	$=$	$1\,\mathrm{(cgs)}$
μ_0	$=$	$4\pi \times 10^{-7}$	$\mathrm{N\,A^{-2}}$	$=$	$1\,\mathrm{(cgs)}$

GROUP I

I-1 Coupled Oscillators
For the masses shown on the right, find the frequencies of all the normal modes and sketch the motions in these normal modes. Make sure to indicate the relative amplitudes of each mass for all the normal modes. Assume that the masses are constrained to move only along the springs and friction can be ignored. *Hint:* This problem can be done without lengthy calculations.
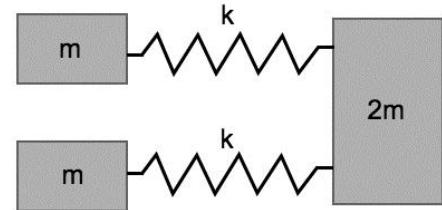
I-2 CMB versus CvB
Consider the relativistic particles present in the early universe. In particular, consider a sea of photons (spin 1 particles) at temperature T_γ and a sea of ultrarelativistic non-degenerate neutrinos (spin 1/2) at temperature T_ν. The two sets of particles do not interact with each other. What is the ratio of the photon-to-neutrino number density, expressed in terms of their respective temperatures?

I-3 Finding the potential
The wavefunction of a particle of mass m is:

$$\psi(x,t) = A\, e^{\left(-\frac{am}{\hbar}|x| - ibt\right)}$$

where A, a, and b are positive constants. What is the potential energy function $V(x)$ in the Hamiltonian for the particle? (Show this by explicit calculation; do not just state the result if you happen to know it.)

I-4 Charged Sphere
Consider a thin conducting spherical shell with radius R carrying a total charge Q. What is the force per unit area due to the electric field acting on the shell's surface? What is the direction of this force?

II-1 House of Cards

Consider a (minimalistic) house of cards shown on the right, built from two cards on a horizontal surface with coefficient of fraction μ. The length of the cards is L. What is the minimal angle between each card and the surface for which the house will be stable, neglecting the friction between the cards?
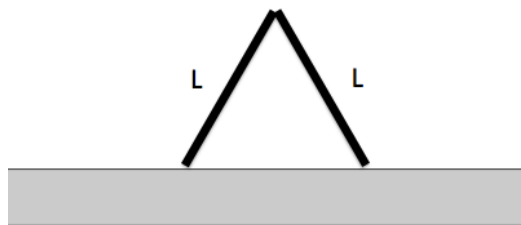
II-2 Harmonic oscillator in thermal equilibrium

A quantum mechanical one-dimensional harmonic oscillator has energy levels $E_n =(n + 1/2)\hbar\,\omega$. Suppose the oscillator is in a thermal contact with a heat reservoir at temperature T. Find the mean energy of the oscillator as a function of the temperature T, and show explicitly that it reduces to the appropriate limits for the cases $kT/\hbar\,\omega \ll 1$, and $kT/\hbar\,\omega \gg 1$.

II-3 Sudden Expansion

An otherwise free particle (mass m) is confined to a one-dimensional potential box that has infinite walls at $x = 0$ and $x = a$. If the particle is in its ground state when the wall at $x = a$ is moved suddenly to $3a$, calculate the probability that the particle will be in the *second* excited state (third energy level) of the expanded box.

### II-4 Interference and Diffraction from N slits

A monochromatic beam is incident on N slits, which results in an intensity pattern as a function of angle on a screen some distance away as shown in the figure. Each slit has a width D and the distance between the centers of the slits is d. The distance between the screen and the slits is large compared to both d and D. From the figure, deduce the following: (a) The number of slits N on which the beam is incident. (b) The ratio d/D. Explain your reasoning for both.
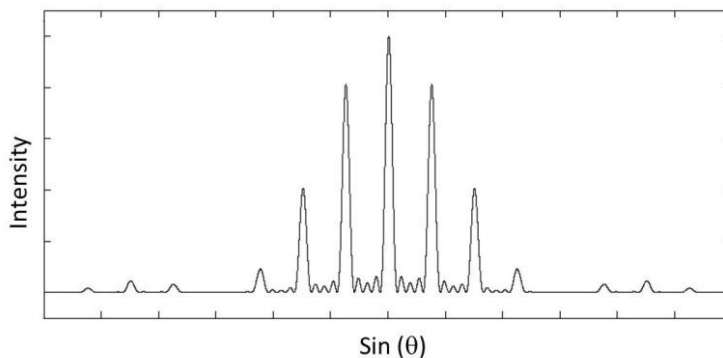
GROUP III

III-1 Two cylinders

Two solid cylinders of mass m_1 and m_2, respectively, and radius R_1 and R_2, respectively, are supported by frictionless bearings one above the other. At time $t=0$, cylinder with angular velocity $\omega_1=\Omega$, is brought into contact with cylinder 2 which has angular velocity $\omega_2=0$, initially. The normal contact force is the weight of cylinder 1 and the coefficient of friction between the cylinders is μ. Calculate the time $(t=t_r)$ at which the cylinders first roll without slipping.
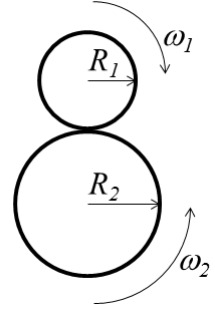
III-2 Isothermal compression

A certain gas has the equation of state $\left(P + \frac{a}{V^2}\right)(V - b) = Nk_BT$ where a and b are constants. Consider a quasistatic isothermal compression of this gas from an initial volume V_1 to a final volume $V_2 < V_1$ at temperature T. How much work has been done on the gas?

III-3 1D Harmonic Oscillator

Use the uncertainty principle to estimate the ground state energy of a one-dimensional harmonic oscillator.

III-4 Magnetic Monopole:

Imagine an electric charge moving in the field of a magnetic monopole (although none has yet been found). Set up the non-relativistic equation of motion for an electric charge q of mass m in the field of a magnetic monopole of strength Γ (a positive constant). Assume the particle at a particular moment in time is a distance r from the magnetic monopole and that the particle's velocity $\mathbf{v}$ is perpendicular to the line between the charged particle and the monopole. Give an expression for the force vector on the particle at this point. Are there low enough speeds for which you can see an approximation to the motion as a function of time? If so, what defines a "low speed"? What is the motion in this limit?

GROUP IV

IV-1 Gyroscope

A gyroscope wheel of mass M and moment of inertia I_0 is attached to one end of an axle of length l. The other end of the axle is suspended from a string of length L. The wheel, which has spin angular velocity ω_s, is set in motion so that the ω_s precesses uniformly in the horizontal plane. Find the angle β that the string makes with the vertical. You may assume that β is small, that $l \gg L \sin\beta$, and that the masses of the axle and string are negligible.
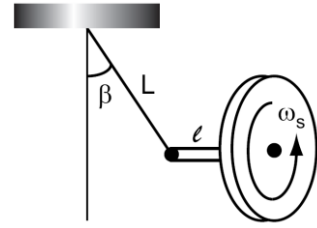
IV-2 Hot Atoms

A monoatomic gas of one-electron atoms is in thermal equilibrium at temperature T. Some of the atoms are in the s-state (l=0) and some are in a p-state (l=1). The ratio of p-state atoms to s-state atoms is R. Electronic transitions between the two states emit radiation of wavelength λ. Find an expression for the temperature of the gas.

IV-3 Harmonic Oscillator in three dimensions

A particle of mass m moves in a three-dimensional isotropic harmonic oscillator potential,

$$V(x, y, z) = \frac{1}{2}m\left(\omega^2 x^2 + \omega^2 y^2 + \omega^2 z^2\right)$$

where ω is a positive real constant. Derive the general formula for the degeneracy of the n-th energy level (that is, how many physically distinct states share the same energy)?

IV-4 Relativistic capacitor

A parallel plate capacitor charged to charge density σ is at rest in the lab frame, with its plates parallel to the x-y plane. Calculate the electric and magnetic fields measured by an observer moving in the x-direction with speed v relative to the lab frame.

GROUP V

V-1 The Sopwith Camel

The Sopwith Camel was a single-engine fighter plane flown by British pilots during WW I (and also by the character Snoopy in the Peanuts comic strip). It was powered by a radial engine, and the entire engine rotated with the propeller. The Camel had an unfortunate property: if the pilot turned to the right the plane tended to go into a dive, while a left turn caused the plane to climb steeply. These tendencies caused inexperienced pilots to crash or stall during takeoff. From the perspective of the pilot, who sat behind the engine, did the engine rotate clockwise (CW, ↻) or counter-clockwise (CCW, ↺) ? Explain *briefly* how you arrived at your answer.

V-2 C_P/C_V for Gases

Gas	$T\,°C$	γ
He	-180	1.66
Kr	19	1.68
Ar	15	1.67
H_2	100	1.40
O_2	100	1.40

The table at the left gives γ (the measured ratio of the specific heat at constant pressure to the specific heat at constant volume) for some gases at various temperatures.

(a) Quantitatively, are the numbers consistent with your expectations? Explain your answer.

(b) For the gases in the table, how do you expect γ to change as the temperature is increased to much higher values?

V-3 Uncertainty of S_x

Suppose that a spin ½ particle is in the state $\chi = A \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$ where as usual $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ denotes the eigenvector for S_z with eigenvalue $+\hbar/2$. What is the *uncertainty* of S_x? (Your answer should not contain any unknown parameters)

V-4 Scaling a circuit

An LC circuit consists of a capacitor and a coil with a large number of turns. Suppose all the linear dimensions of all elements of the circuit are changed by a factor γ while keeping the number of turns on the coil constant. How much does the resonant frequency of the circuit change? (Give simple arguments to justify your answer).