Problems with Solutions (QE-Part I, Aug 2003)

Physics PhD Qualifying Examination
Part I – Monday, August 18, 2003

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism), **and with at least one problem from problems 5 or 10 (Special Relativity).**
DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

I-1 [10]

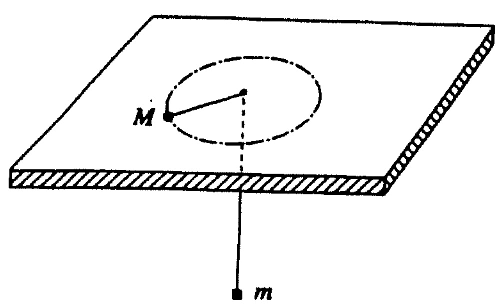
An automobile drag racer drives a car with acceleration a and instantaneous velocity v. The tires (of radius r_0) are not slipping.

(a) Derive an equation for the position of a point on the bearing surface of the tire as a function of time.

(b) Find which point on the tire has the greatest acceleration relative to the ground. What is this acceleration?

I-2 [2,2,6]

A particle of mass **M** is constrained to move on a horizontal plane; a second particle, of mass **m**, is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane, as shown in the figure below. The motion is frictionless.



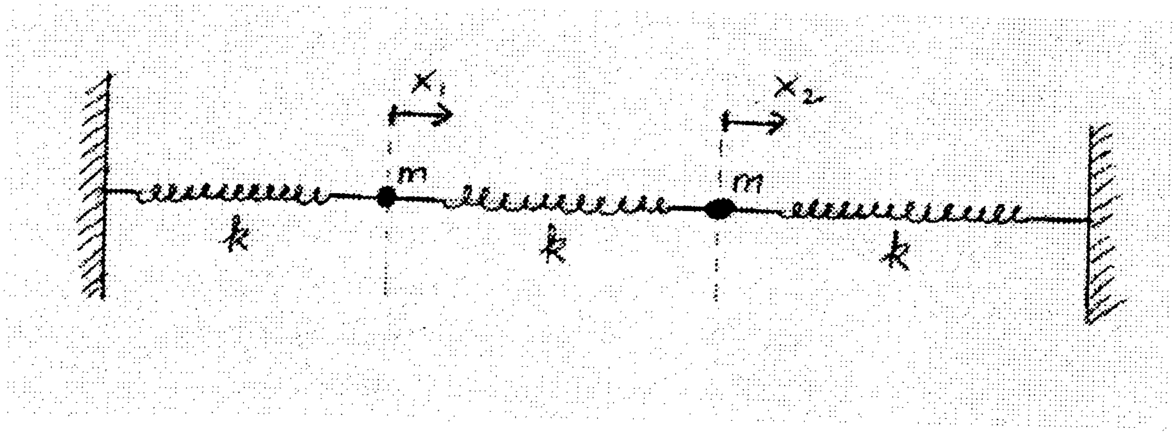
a) Find the Lagrangian of the system.
b) Derive the equations of motion for the system.
c) Show that the orbit is stable with respect to small changes in the radius and find the frequency of small oscillations.

I-3 [10]

Consider the coupled mass-spring system sketched below. The masses can only move horizontally. The springs are relaxed at equilibrium. Obtain the *complete* solution of the problem for the following set of initial conditions:

$x_1(0) = D$, $x_2(0) = 0$; $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$.

Note that simply finding the normal frequencies and normal modes is *not* sufficient for obtaining a passing score for this problem. You have to obtain $x_1(t)$ and $x_2(t)$ for the given initial conditions. (The displacements $x_1(t)$ and $x_2(t)$ are measured from the respective equilibrium positions of the masses.)

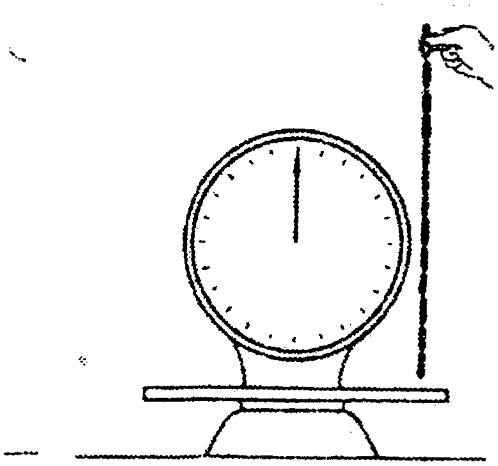


I-4 [10]

A butcher holding a long string of small link sausages upright just above the scale pan offers to charge the customer for just one-half of the maximum reading of the scale after she releases the string (see figure).

The customer, not knowing much physics, eagerly agrees. How much more did the customer pay than the proper charge?

Assume no overrun of the scale reading. Hint: At any moment the scale pan must support the weight of the sausage that has already arrived there, and must also absorb the momentum per second that is still arriving.



I-5 [10]

An anti-proton, $\bar{p}$, is produced in the reaction $p + p \rightarrow p + \bar{p} + p + p$ in which protons at rest are bombarded with protons of an energy E. Derive an expression for the minimal kinetic energy of the moving protons to allow the reaction.

I-6 [6,4]

A charged sphere of radius R has a surface charge density $\sigma(R, \theta) = \sigma(\theta)$, where θ is the polar coordinate.

a) Derive a general expression for the electrostatic potential for both $r < R$ and $r > R$.

b) Derive a general expression for the self energy.

I-7 [10]

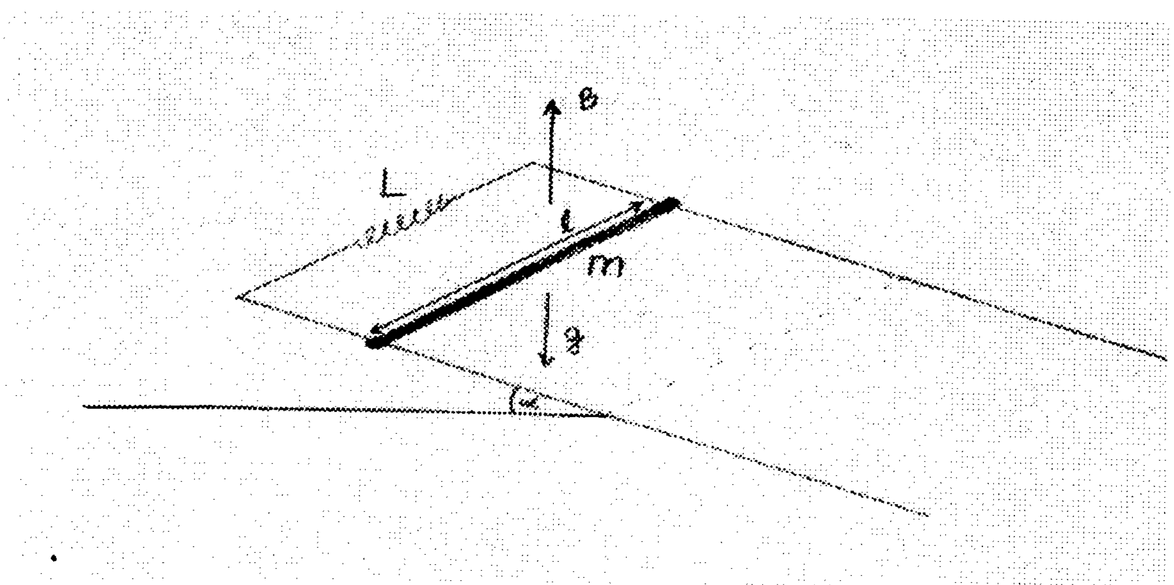
Show that in free space with $\rho = 0$, $\mathrm{J} = 0$, the Maxwell equations are correctly obtained from a single vector potential function $\bar{\mathbf{A}}$ satisfying

$$\nabla \cdot \bar{\mathbf{A}} = 0, \quad \nabla^2 \bar{\mathbf{A}} - 1/c^2 \, \partial^2 \bar{\mathbf{A}} / \partial t^2 = 0 \, , \; \Phi = 0 \, .$$

(Just writing out Maxwell's equation will give you zero points.)

I-8 [10]

Consider a conducting rod of mass m on a tilted pair of rails in the presence of gravity g and a constant and homogeneous vertical magnetic field B (see sketch below). The angle of inclination is α. The distance between the rails is l. The rails are connected with an inductance L. The resistance of the rod and the rails and the friction between the rod and the rails are negligible. Initially the rod is at rest and there is no current in the loop ($\mathrm{v}(0) = 0$, $I(0) = 0$). Describe the motion of the rod, i.e., obtain $\mathrm{v}(t)$.



I-9 [10]

Consider an electric dipole consisting of two tiny metal spheres separated by a distance d and connected by a fine wire. The charge is driven back and forth between the spheres with angular frequency ω such that the instantaneous charge on the upper and lower spheres is + and $-q(t)$, respectively, yielding a dipole moment of $\mathbf{p}(t) = q_0 d\cos(\omega t)\,\hat{z}$

.

(a) Derive expressions for the scalar and vector potentials as a function of distance from the center of the dipole r which are lowest order in $\frac{1}{r}$, explicitly identifying each of the physical approximations which are needed.

(b) Derive expressions for the electric and magnetic fields at $\mathbf{r}$. Find the Poynting Vector and the total power radiated.

I-10 [5,5]

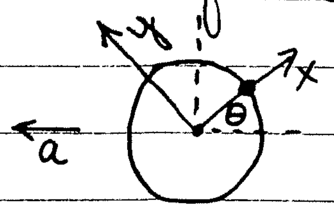
A plane electromagnetic wave with x-polarization propagates in free space along the +z axis. At the position $z=z_0$ the wave encounters a region of infinite extent in the x and y directions which is a low-density plasma of free electrons of number density n, mass m and charge e. For this plasma region it is found that the current density j and the electric field $\mathbf{E}=\mathbf{E}_0\exp(-i\omega t)$ are related by $\mathbf{j}=i(ne^2/(\omega m))\mathbf{E}$.

(a.) Using Maxwell's equation find the wave number k in the plasma region.

(b.) Describe the wave propagation in the plasma region for $\omega>\omega_p$ and $\omega<\omega_p$ where $\omega_p^2=ne^2/(\varepsilon_0 m)$ and specifically determine the depth δ of propagation in the plasma for the two cases.

I-1.)

a) fixed frame is ground; tire is fixed in rotating frame



$$\vec{a}_f = \vec{a}_r + \underbrace{\ddot{\vec{R}}_f + \dot{\vec{\omega}} \times \vec{r}}_{\text{translational + ang. acceleration}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_r$$

centrifugal

Coriolis

but $\ddot{\vec{R}}_f = -a \cos\theta\, \hat{x} + a \sin\theta\, \hat{y}$

$\vec{r} = r_0 \hat{x}$ $\qquad v_r = a_r = 0$

$\vec{\omega} = \dfrac{v}{r_0} \hat{z}$ $\qquad \dot{\vec{\omega}} = \dfrac{a}{r_0} \hat{z}$

$$\therefore \vec{a}_f = -a\cos\theta\, \hat{x} + a \sin\theta\, \hat{y} + a\hat{y} - \frac{v^2}{r_0}\hat{x}$$

$$= -\hat{x}\left(\frac{v^2}{r_0} + a\cos\theta\right) + \hat{y}\, a(\sin\theta + 1)$$

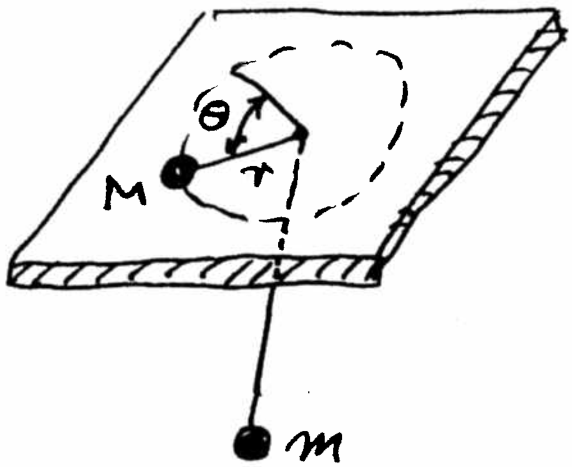
b) $$\frac{d|a_f|^2}{d\theta} = -\frac{2av^2}{r_0}\sin\theta + 2a^2\cos\theta$$

$$= 0 \text{ when } \tan\theta = \frac{a r_0}{v^2}$$ (identifies pt. of max. accel.)

substituting $\Rightarrow |a_f| = a + \sqrt{a^2 + v^4/r_0^2}$

I-2.) Solutions

a) We can write the Lagrangian in terms of the length r of the string on the table and angle θ as shown below:



$$\therefore \; L = \frac{1}{2} M(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} m\dot{r}^2 - mgr$$

b.) The equations of motion are:

$$(M+m)\ddot{r} - Mr\dot{\theta}^2 + mg = 0$$

$$\frac{d}{dt}(Mr^2\dot{\theta}) = 0$$

from the last equation we have angular momentum conservation: $Mr^2\dot{\theta} = \text{const} = l_0$

$\therefore \quad \dot{\theta} = l_0/Mr^2$ and

$$L = \frac{1}{2}(M+m)\dot{r}^2 + \frac{l_0^2}{2Mr^2} - mgr$$

c.) The equilibrium position is defined by taking the derivative of U_{eff} where

$$U_{eff} = mgr + l_0^2/2Mr^2$$

I-2) continued.

and $\left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_0} = 0$

$$r_0 = \left(\frac{l_0^2}{gMm}\right)^{1/3}$$

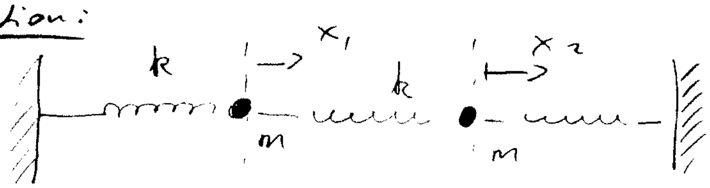
$\frac{\partial^2 U_{eff}}{\partial r^2} > 0$, so the orbit is stable with respect to small perturbations in the radius. The frequency of small ~~perturbatio~~ oscillations is given by

$$\omega^2 = \frac{1}{M_{eff}}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)\Bigg|_{r=r_0} = \frac{1}{M+m}\left(\frac{\partial^2 U_{eff}}{\partial r^2}\right)\Bigg|_{r=r_0}$$

$$\omega^2 = \frac{1}{M+m}\left(\frac{3l_0^2}{Mr_0^4}\right) = \frac{1}{1+\left(\frac{M}{m}\right)}\left(\frac{3g}{r_0}\right)$$

Solution:

I-3.)



$$x_1(0) = D \qquad x_2(0) = 0$$
$$\dot{x}_1(0) = 0 \qquad \dot{x}_2(0) = 0$$

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1) = kx_1 - 2kx_2$$

$$\bar{x} = \bar{a}e^{\pm i\omega t} \qquad -\omega^2\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -2k/m & +k/m \\ k/m & -2k/m \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} \omega^2 - 2\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \omega^2 - \frac{2k}{m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\omega^2 - \frac{2k}{m}\right)^2 - \left(\frac{k}{m}\right)^2 = 0$$

$$\omega^2 - \frac{2k}{m} = \pm\frac{k}{m} \quad \Rightarrow \quad \omega_1^2 = \frac{3k}{m}$$
$$\omega_2^2 = \frac{k}{m}$$

normal modes:

$$\omega_1: \quad \frac{k}{m}a_1 + \frac{k}{m}a_2 = 0 \quad \rightarrow \quad \bar{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \omega_1 = \frac{3k}{m}$$

$$\omega_2: \quad -\frac{k}{m}a_1 + \frac{k}{m}a_2 = 0 \quad \Rightarrow \quad \bar{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \omega_2 = \frac{k}{m}$$

I-3.) Continued.

Complete solution of the problem with given initial values:
(non-degenerate eigenvalues)

$$\overline{x}(t) = \left[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)\right]\overline{a}_1 + \left[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)\right]\overline{a}_2$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \left[A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)\right]\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \left[A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)\right]\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$(1) \quad \begin{cases} x_1(t) = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \\ x_2(t) = -A_1 \cos(\omega_1 t) - B_1 \sin(\omega_1 t) + A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t) \end{cases}$$

$$(2) \quad \begin{cases} \dot{x}_1(t) = -\omega_1 A_1 \sin(\omega_1 t) + \omega_1 B_1 \cos(\omega_1 t) - \omega_2 A_2 \sin(\omega_2 t) + \omega_2 B_2 \cos(\omega_2 t) \\ \dot{x}_2(t) = \omega_1 A_1 \sin(\omega_1 t) - \omega_1 B \cos(\omega_1 t) - \omega_2 A_2 \sin(\omega_2 t) + \omega_2 B_2 \cos(\omega_2 t) \end{cases}$$

$\underline{t=0}$, $x_1(0) = D$ $\quad x_2(0) = \emptyset$ $\qquad$ from (1):

$$\left.\begin{aligned} D &= A_1 + A_2 \\ 0 &= -A_1 + A_2 \end{aligned}\right\} \Rightarrow \quad A_1 = A_2 = \frac{D}{2}$$

$\dot{x}_1(0) = \dot{x}_2(0) = 0$ $\qquad$ from (2):

$$\left.\begin{aligned} \omega_1 B_1 + \omega_2 B_2 &= 0 \\ -\omega_1 B_1 + \omega_2 B_2 &= 0 \end{aligned}\right\} \Rightarrow \quad B_1 = B_2 = 0$$

Full solution:

$$x_1(t) = \frac{D}{2}\left(\cos(\omega_1 t) + \cos(\omega_2 t)\right) = D\cos\left(\frac{\omega_1+\omega_2}{2}t\right)\cdot\cos\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$x_2(t) = \frac{D}{2}\left(-\cos(\omega_1 t) + \cos(\omega_2 t)\right) = D\sin\left(\frac{\omega_1+\omega_2}{2}t\right)\sin\left(\frac{\omega_1-\omega_2}{2}t\right)$$

$$\omega_1 = \sqrt{\frac{3k}{m}} \quad , \quad \omega_2 = \sqrt{\frac{k}{m}}$$

$$\boxed{\begin{aligned} x_1(t) &= D \cos\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}+1}{2}\,t\right) \cos\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}-1}{2}\,t\right) \\ x_2(t) &= D \sin\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}+1}{2}\,t\right) \sin\left(\sqrt{\frac{k}{m}}\,\frac{\sqrt{3}-1}{2}\,t\right) \end{aligned}}$$

Solution

I-4.) Newtonian Mechanics

$$F = \frac{dp}{dt} = \frac{d}{dt}\left(m(t)\, v(t)\right)$$

$$= v(t)\frac{dm(t)}{dt} + m(t)\frac{dv(t)}{dt}^{*}$$

$$= v(t)\frac{dm(t)}{dt}^{**} + m(t)\cdot g \qquad (1)$$

$* \quad \frac{dv}{dt} = a = g$

$** \quad m(t) = \mu X(t)$ $\quad \mu$: mass per length of sausage

$v(t) = g\cdot t \; ; \; x = \frac{1}{2} g t^2$

continue from (1)

$$F = \frac{dp}{dt} = gt\cdot\mu\frac{dx(t)}{dt} + \mu x(t) g$$

$$= g\cdot t\mu \cdot gt + \mu\frac{1}{2} g t^2 \cdot g$$

$$= \mu g^2 t^2 + \frac{1}{2}\mu g^2 t^2$$

$$F = \frac{3}{2}\mu g^2 t^2$$

$$F_{max} = \frac{3}{2}\mu g^2 t^2_{max} \qquad t^2_{max} = \frac{2h}{g}$$

$$= \frac{3}{\cancel{2}}\mu g^{\cancel{2}}\frac{\cancel{2}h}{\cancel{g}}$$

$$= \frac{3}{\cancel{2}} mg \qquad m = \mu\cdot h$$

h total length of sausage

$F_{max} = 3\, mg$

one-half of maximum reading $\frac{3}{2}mg$

50% more than actual weight

I-5.)

We take advantage of the relation

$$\sum (E_i^2 - c^2 p_i^2) = (\sum m_i^2) c^4$$

Here m_i denotes the rest mass of particle i. We assume the masses of protons and antiprotons to b the same, E_i and p_i are the momenta respectively. In the initial state there is a proton with energy E and a proton at rest. In the final state four baryon. Hence

$$(2mc^2 + E)^2 - p^2c^2 = (4mc^2)^2$$

or

$$4m^2c^4 + 4mc^2E + E^2 - (2mc^2E + E^2)$$

$$= 16m^2c^4$$

$$2mc^2E = 14\, m^2c^4$$

$$E = 7mc^2$$

$$K = E - mc^2 = 6mc^2$$

I-6.)

E & M

1

a) $r < R$

$$V_1 = \sum A_n r^n P_n(\cos\theta)$$

$r > R$

$$V_2 = \sum \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

$$V_1(r=R) = V_2(r=R)$$

$$A_n R^n = \frac{B_n}{R^{n+1}}$$

$$B_n = A_n R^{2n+1}$$

$$\sigma(\theta) = -\left(\left(\frac{\partial V}{\partial r}\right)_{out} - \left(\frac{\partial V}{\partial r}\right)_{in}\right)\epsilon_0 \Big|_{r=R}$$

$$-\sum (m+1) \frac{B_m}{R^{m+2}} P_\ell(\cos\theta) - \sum m A_m R^{\ell-1} P_\ell(\cos\theta)$$

$$= -\frac{1}{\epsilon_0}\sigma(\theta)$$

$$\sum (2m+1) A_m R^{m-1} = \frac{1}{\epsilon_0}\sigma(\theta)$$

$$A_m = \frac{1}{2\epsilon_0} \int_0^\pi \sigma(\theta) P_m(\cos\theta) \sin\theta \, d\theta$$

b) $= \frac{1}{2}\int V(r=R)\,\sigma(\theta)\,d\theta$

$$= \frac{1}{2}\int \sum A_m R^m P_m(\cos\theta)\,\sigma(\theta)\sin\theta\,d\theta$$

$$\frac{1}{2}\sum R^m \frac{1}{2\epsilon_0}\frac{1}{R^{m-1}}\int_0^{\pi}\sigma(\theta)\,\cancel{P_m}(\cos\theta)\sin\theta\,d\theta$$

$$= \frac{1}{2}\frac{R}{\epsilon_0}\sum\int_0^{\pi}\sigma(\theta)\,P_m(\cos\theta)\sin\theta\,d\theta$$

I-7.)

Solution:

since $\phi = 0$, we have $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \vec{\nabla} \times \vec{A}$

$\vec{\nabla} \cdot \vec{B} = 0$ (automatically) and that

$\vec{\nabla} \cdot \vec{E} = 0$ follows from $\vec{\nabla} \cdot \vec{A} = 0$.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = -\frac{\partial \vec{B}}{\partial t}.$$ So now

only the curl $\vec{B}$, equation remains

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{\nabla} \times \vec{A} = -\vec{\nabla}^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$$

with $(\vec{\nabla} \cdot \vec{A}) = 0$, given,

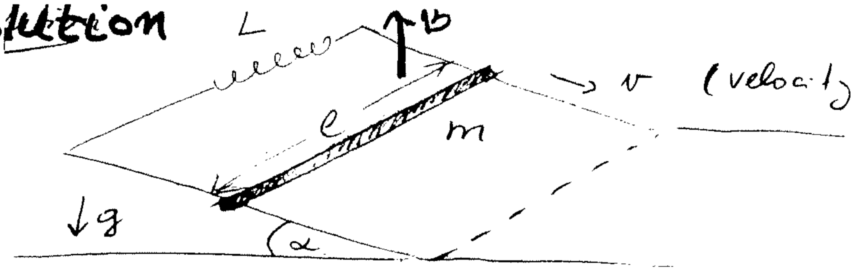
From the wave equation for $\vec{A}$,

$$\vec{\nabla}^2 \vec{A} = \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

So that, $\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \frac{\partial \vec{D}}{\partial t}$,

which is the remaining Maxwell's equation

I-8.) Solution



$$\boxed{\begin{aligned} v(0) &= 0 \\ I(0) &= 0 \end{aligned}}$$ initial co[nditions]

(1) $m\dot{v}(t) = mg\sin\alpha - I(t)\ell B\cos\alpha$ (Newton's II.)

(2) $v(t)\ell B\cos\alpha - L\dot{I}(t) = 0$ (voltages in closed loop)

$$m\ddot{v} = -\dot{I}\ell B\cos\alpha = -\frac{v\ell B\cos\alpha}{L}\cdot \ell B\cos\alpha$$

$$m\ddot{v} = -\frac{\ell^2 B^2\cos\alpha}{L}v(t)$$

$$\ddot{v} = -\frac{\ell^2 B^2\cos\alpha}{Lm}v(t) \quad \Rightarrow \quad \boxed{\omega = \frac{\ell B\cos\alpha}{\sqrt{mL}}}$$

(harmonic oscillations)

(3) $\begin{cases} v(t) = A\cos(\omega t) + B\sin(\omega t) \\ \dot{v}(t) = -\omega A\sin(\omega t) + \omega B\cos(\omega t) \end{cases}$

need $v(0)$ and $\dot{v}(0)$

$v(0) = 0$

and from $I(0)=0$ & (1): $m\dot{v}(0) = mg\sin\alpha \Rightarrow \dot{v}(0) = g\sin[\alpha]$

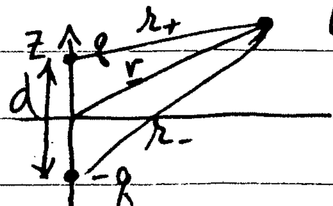
$t=0$: from (3): $0 = A$

$$g\sin\alpha = \omega B \quad \rightarrow \quad B = \frac{g\sin\alpha}{\omega} = \frac{\sqrt{mL}\cdot g\sin\alpha}{\ell B\cos\alpha}$$

$$\boxed{v(t) = \frac{g\sin\alpha\sqrt{mL}}{\ell B\cos\alpha}\cdot\sin\left(\frac{\ell B\cos\alpha}{\sqrt{mL}}t\right)}$$

I-9.) a) retarded potential:

$$V(\underline{r}, t) = \frac{1}{4\pi\epsilon_0}\left\{ \frac{q_0 \cos[\omega(t - r_+/c)]}{r_+} - \frac{q_0 \cos[\omega(t - r_-/c)]}{r_-} \right\}$$



where $r_\pm = \sqrt{r^2 \mp r d\cos\theta + \left(\frac{d}{2}\right)^2}$

approx. 1 : $d << r \Rightarrow r_\pm = r\left(1 \mp \frac{d}{2r}\cos\theta\right)$

approx. 2 : $d << \frac{c}{\omega}$

$$\Rightarrow \cos[\omega(t - r_\pm/c)] \cong \cos[\omega(t - r/c)] \mp \frac{\omega d}{2c}\cos\theta \sin[\omega(t - r/c)]$$

approx. 3 : $r >> \frac{c}{\omega}$ (radiation zone)

$$\Rightarrow V(r,\theta,t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\sin[\omega(t - r/c)]$$

$$\underline{A}(\underline{r}, t) = \frac{\mu_0}{4\pi}\int_{-d/2}^{d/2} \frac{-q_0\omega \sin[\omega(t - r/c)]\,\hat{z}}{r}\,dz$$

$$= -\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t - r/c)]\,\hat{z}$$

b) $\underline{E} = -\underline{\nabla} V - \frac{\partial \underline{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\theta}$

$\underline{B} = \underline{\nabla} \times \underline{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\,\hat{\phi}$

I-9. (cont'd)

$$\underline{S} = \frac{1}{\mu_0}(\underline{E}\times\underline{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\right\}^2 \hat{r}$$

$$\langle P\rangle = \int\langle\underline{S}\rangle\cdot d\underline{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{\mu_0 p_0^2\omega^4}{12\pi c}$$

I-10.)

Solution

Maxwell's equation

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\times\vec{B}=\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B}=0 \qquad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=-\Delta\vec{E}=-\vec{\nabla}\times\dot{\vec{B}}=-\frac{\partial}{\partial t}\left(\mu_0\vec{j}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\vec{j}=i\left(\frac{ne^2}{\omega m}\right)\vec{E}$$

$$-\Delta\vec{E}=-i\mu_0\frac{ne^2}{\omega m}\frac{\partial\vec{E}}{\partial t}-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Delta\vec{E}-\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}-i\frac{\mu_0 ne^2}{\omega m}\frac{\partial\vec{E}}{\partial t}=0 \qquad \frac{1}{c^2}=\mu_0\varepsilon_0$$

assume $\vec{E}=\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$

$$-k^2+\frac{\omega^2}{c^2}-\frac{\mu_0 ne^2}{m}=0$$

$$k^2=\frac{\omega^2}{c^2}-\frac{\mu_0 ne^2}{m}=\frac{\omega^2}{c^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)$$

$\omega>\omega_p$ $\quad k$ is real $\quad$ freely propagating wave

$$\vec{E}=\vec{E}_0 e^{-i(\vec{k}\cdot\vec{r}-\omega t)}$$

$\omega<\omega_p$ $\quad k$ is imaginary

$$\vec{E}=\vec{E}_0 e^{-\vec{\alpha}\cdot\vec{r}}\, e^{-i\omega t} \qquad \alpha=|k|$$

$$\delta=\frac{1}{\alpha}=\frac{1}{|k|}=\frac{c}{\sqrt{\omega_p^2-\omega^2}}$$

Problems with Solutions (QE-Part II, Aug 2003)

Physics PhD Qualifying Examination
Part II – Wednesday, August 20, 2003

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

II-1 [10]

Calculate the lowest order correction to the energy

of a one-dimensional harmonic oscillator perturbed by a potential $H_1 = \frac{1}{4}\alpha\, x^4$.

The Hamiltonian of the harmonic oscillator has the form: $H = \frac{p^2}{2m} + \frac{1}{2} k\, x^2$.

The ground state wave function of a 1-dimensional harmonic oscillator is given by :

$$\psi_0(x) = \left(\frac{m\,\omega}{\hbar\,\pi}\right)^{\frac{1}{4}} \exp\left(-\frac{m\omega\, x^2}{2\hbar}\right) .$$

<u>II-2 [10]</u>

The Yukawa potential (which is a crude model for the binding force in an atomic nucleus) has the form V(r) = $\beta \frac{e^{-\mu r}}{r}$, where β and μ are constants.

Calculate the amplitude $f(\theta)$ of a wave scattered from V as a function of angle from forward scattering θ in the Born approximation, and the total cross section in that same approximation.

II-3 [5,5]

Consider two identical particles of mass m and spin ½ . They interact only through a potential (spin-dependent r^{-1} potential):

$$V = (g/r)\,(\sigma_1 \cdot \sigma_2),$$

Where $g > 0$ and σ_j are Pauli spin matrices which operate on the spin of particle j.

a) Construct the spin eigenfunctions for the two particle states. What is the Expectation value of V for each of these states?

b) Give the eigenvalues of all of the bound states.

II-4 [4,3,3]

Use the variational method to estimate the ground state energy of the hydrogen atom using two possible trial functions u_1 and u_2 ,

$$u_1 \propto \exp(-r/a),$$

$$u_2 \propto \exp(-r^2/b^2).$$

a) Derive expressions for the variational parameters a and b.

b) Derive expressions for the corresponding ground state energies.

c) Explain briefly which wave function yields an energy closer to the exact energy of the ground state.

II-5 [10]

Consider the case of low-energy scattering from a spherical delta-function shell V(r) = $\alpha\delta$ (r – a), where α and a are constants. Assuming that $ka \ll 1$, so that only the $l = 0$ partial wave need be considered, calculate the scattering amplitude $f(\theta)$.

II-6 [10]

Consider a charged one-dimensional harmonic oscillator with mass m, frequency ω_o, and charge q. Initially the oscillator is in its unperturbed ground state when there is no electric field present. At $t = 0$ a weak uniform periodic electric field $E = E_o \sin(\omega t)$ is turned on (the field is parallel to the direction of motion of the oscillator). The frequency of the external field is chosen such that $\omega = \omega_o$. Using first-order *time-dependent perturbation theory*, find the probability that the oscillator is in its l^{th} excited state exactly one period after the field was turned on. Solve the problem in the number representation using the annihilation and creation operators

$$a = \sqrt{\frac{m\omega_o}{2\hbar}}\left(x + \frac{i}{m\omega_o}p\right), \quad a^+ = \sqrt{\frac{m\omega_o}{2\hbar}}\left(x - \frac{i}{m\omega_o}p\right)$$

and knowing their actions on the l^{th} eigenstate of the unperturbed Hamiltonian.

II-7 [6,4]

The Dietrici gas equation of state is given by: $p\,(V-b) = RT\exp\left(-\frac{a}{RTV}\right)$.

(a.) Calculate the critical volume V_k, critical temperature T_k and critical pressure p_k for the Dietrici gas and re-write the Dietrici equation of state with the variables

$$p' = \frac{p}{p_k},\quad T' = \frac{T}{T_k}\quad V' = \frac{V}{V_k}.$$

(b.) Next, calculate p'(T') for the condition $\left(\frac{\partial T'}{\partial p'}\right)_{H'} = -\frac{1}{c_p}\left(\frac{\partial H'}{\partial p'}\right)_{T'} = 0$. This is the inversion curve in the p'-T'-diagram.

II-8 [3,4,3]

a) Derive the relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

b) From electrodynamics it follows that the radiation pressure p and energy density u are related by

$$p = u(T)/3\,.$$

Show that ,

$$u = \tfrac{1}{3}\,T\,(du/dt) - \tfrac{1}{3}\,u\,.$$

c) Use the above result to obtain u(T), generally known as the Stefan-Boltzmann Law.

II-9 [2,2,3,3]

Consider two interacting distinguishable particles. Each particle can be in any of three possible states, labeled as $s_i = 1,2,3$; $i = 1,2$. The interaction energy between the two particles is $-J$ if they are in the same state, zero otherwise. We choose $J > 0$, so that particles prefer to be in the same state with one another.

(a) Obtain the average energy $E(T)$ of this two-particle system (T is the absolute temperature.)
(b) Obtain the specific heat for the system $c(T)$.
(c) Find the high-temperature ($J/kT << 1$) and the low-temperature ($J/kT >> 1$) behavior of the specific heat. Note, that you need to obtain the functional behavior of $c(T)$ for the above scenarios, not just the actual limiting values $c(\infty)$ and $c(0)$. Sketch $c(T)$.
(d) Find the entropy of the system $S(T)$. What is the value of the entropy *at* $T = 0$, and why is it non-zero?

II-10 [7,3]

(a.) Show that for an ideal gas of structureless fermions, the pressure is given by

$$\beta P = \frac{1}{\lambda^3} f_{5/2}(z) \quad \text{where} \quad z = e^{\beta\mu}, \text{ and } \lambda = (2\pi\beta\hbar^2/m)^{1/2} \text{ with m}$$

being the mass of the particle and $\beta = 1/kT$.

$$f_{5/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^{\infty} dx\, x^2 \ln\left(1 + ze^{-x^2}\right) \quad \text{and}$$

$$f_{5/2} = \sum_{l=1}^{\infty} (-1)^{l+1} z^l / l^{5/2}$$ with the chemical potential, μ, related to the average density, $\rho = <N>/V$ by

$$\rho\lambda^3 = f_{3/2}(z) = \sum_{l=1}^{\infty} (-1)^{l+1} z^l / l^{3/2} .$$

(b.) Similarly, show that the internal energy, $< E >$, obeys the relation

$$<E> = \frac{3}{2} pV.$$

Solution.

II-1.)

energy

$$E_1 = \langle \psi_0 | H_1 | \psi_0 \rangle = \int_{-\infty}^{+\infty} \psi_0^* H_1 \psi_0 \, dx$$

$$= \frac{1}{4} \alpha \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \int_{-\infty}^{+\infty} x^4 \exp\left(-\frac{m\omega x^2}{\hbar}\right)$$

$$= \frac{1}{4} \alpha \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \cdot 2 \int_0^{\infty} x^4 \exp\left(-\frac{m\omega x^2}{\hbar}\right) dx$$

$$= \frac{1}{4} \alpha \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \cdot 2 \; \frac{3\sqrt{\pi}}{8 \cdot \left(\frac{m\omega}{\hbar}\right)^{5/2}}$$

$$= \frac{1}{16} \alpha \cdot 3 \cdot \frac{\hbar^2}{m^2\omega^2}$$

$$= \frac{3}{16} \alpha \frac{\hbar^2}{m^2\omega^2}$$

$$\int_0^{\infty} x^n e^{-ax^2} dx = \frac{3\sqrt{\pi}}{2^3 \, a^{5/2}}$$

$n = 4 = 2k$

$k = 2$

II-2.)

a) $f(\theta,\varphi) = -\dfrac{m}{2\pi\hbar^2 A}\int e^{-i\underline{k}\cdot\underline{r}'}\,V(\underline{r}')\,\psi(\underline{r}')\,d^3r'$

iff $\psi_0 = Ae^{ikz}$

Born approx: $\psi \Rightarrow Ae^{ikz} \rightarrow Ae^{i\underline{k}'\cdot\underline{r}'}$ where $\underline{k}' \equiv k\hat{z}$

V sph. symmetric $\rightarrow$ do θ, φ integrals

$$\Rightarrow f(\theta) = \frac{-2m}{\hbar^2 K}\int_0^\infty r\,V(r)\sin(Kr)\,dr$$

$$\int_0^\infty e^{-\mu r}\sin(Kr)\,dr = \frac{K}{\mu^2+K^2}$$

$\underline{K} \equiv \underline{k}' - \underline{k}$

$|K| = 2k\sin\left(\frac{\theta}{2}\right)$

$$\therefore f(\theta) = -\frac{2m\beta}{\hbar^2(\mu^2+K^2)}$$

b) $\sigma = \int |f(\theta)|^2 \sin\theta\,d\theta\,d\varphi$ let $x = \dfrac{2k}{\mu}\sin\left(\dfrac{\theta}{2}\right)$

$$\therefore \sigma = 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2\frac{1}{\mu^4}\left(\frac{\mu}{k}\right)^2\int_0^{\frac{2k}{\mu}}\frac{x}{[1+x^2]^2}\,dx$$

$$\sigma = \pi\left(\frac{4m\beta}{\hbar^2}\right)^2\frac{1}{\mu^2}\,\frac{1}{\mu^2+4k^2}$$

$$= \pi\left(\frac{4m\beta}{\mu\hbar}\right)^2\frac{1}{(\mu k)^2+8mE}$$

where $E = \dfrac{\hbar^2k^2}{2m}$

II-3.)

Solution:

a.) The spin operator is $\vec{s} = \frac{\vec{\sigma}}{2}$. For spin $\frac{1}{2}$ the expression $\vec{s}\cdot\vec{s} = s(s+1)\tilde{I} = \frac{3}{4}\tilde{I}$ becomes, for Pauli Matrices, $\vec{\sigma}\cdot\vec{\sigma} = 3\tilde{I}$, where $\tilde{I}$ is the unit matrix. The total spin operator for the two particle system is

$$\vec{S} = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2)$$

$$\vec{S}\cdot\vec{S} = S(S+1)\tilde{I} = \frac{1}{2}[3\tilde{I} + \vec{\sigma}_1\cdot\vec{\sigma}_2]$$

$$\langle\vec{\sigma}_1\cdot\vec{\sigma}_2\rangle = 2S[S+1] - 3$$

For ~~singlet spin state~~ the spin singlet state $S = 0$, then $V_{S=0} = -\frac{3g}{r}$, while for the spin triplet state $(S = 1)$, then $V_{S=1} = g/r$.

b.) The potential is repulsive for the triplet state, and there are no bound states. There are bound states for the singlet state since the potential is attractive. For the Hydrogen atom the potential is $-e^2/r$, and the eigenvalues are

$$E_n = -\frac{e^4 m}{2\hbar^2 n^2} \quad \text{(hydrogen atom)}$$

Our two-particle bound state has $3g$ instead of e^2 and the reduced mass "$m/2$" instead of the mass "m", so we have eigenvalues

$$E_n = -\frac{9g^2 m}{4\hbar^2 n^2}.$$

$(\mathrm{II}-4.)$ QM

II-4.)

$$E(a) = \frac{\int u^* H u(r) r^2 dr}{\int u^*(r) u(r)}$$

$$E = \frac{\langle u|H|u\rangle}{\langle u|u\rangle}$$

$$\frac{dE}{da} = 0$$

$$\langle u|u\rangle = \frac{a^3}{4} \quad \langle u|T|u\rangle = \frac{\hbar^2 a}{8m}$$

$$\langle u|V|u\rangle = -\frac{ke^2}{a}$$

$$\frac{dE}{da} = 0$$

$$a_{\text{min energy}} \ \frac{\hbar^2}{mke^2}$$

$$E_{gr} = \frac{-ke^2}{2a_{min}}$$

In second case

$$E_{gr} = \frac{-8}{3\pi} \frac{ke^2}{2a_0}$$

II-5.)

radial eq: $-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$

$l=0$; wf cont. at $r=a$, $\frac{d}{dr}(wf) = ?$

integrate d.e. once from $r=a^-$ to $r=a^+$

$\Rightarrow -\frac{\hbar^2}{2m}\left(\frac{du}{dr}\Big|_{a^+} - \frac{du}{dr}\Big|_{a^-}\right) + \alpha u(r=a) = 0$

but $u = rR \Rightarrow \frac{d\psi}{dr}\Big|_{a^+} - \frac{d\psi}{dr}\Big|_{a^-} = \frac{2m\alpha}{\hbar^2}\psi(a)$

Let $\psi_- = b\, j_0(kr) = b\frac{\sin(kr)}{kr}$ $(r<a)$

$\psi_+ = A\left(j_0(kr)P_0(\cos\theta) + \frac{C_0}{\sqrt{4\pi}} h_0^{(1)}(kr)P_0(\cos\theta)\right)$

$= A\left\{\frac{\sin(kr)}{kr} - \frac{iC_0}{\sqrt{4\pi}}\frac{e^{ikr}}{kr}\right\}$ $(r>a)$

Applying bc at a:

$\Rightarrow \frac{C_0}{\sqrt{4\pi}} e^{ika}\left[1 + i\cot(ka) + i\frac{\phi}{ka}\right] = \frac{\phi}{ka}\sin(ka)$

where $\phi \equiv 2ma\alpha/\hbar^2$

but $ka \ll 1 \Rightarrow \sin(ka) \sim ka$ + $\cot(ka) \approx \frac{1}{ka}$

$\Rightarrow C_0 = -i\sqrt{4\pi}\, ka \frac{\phi}{1+\phi}$

$f(\theta) = \frac{1}{k}(-i)\frac{1}{\sqrt{4\pi}} C_0 = -\frac{a\phi}{1+\phi}$

II-6.) Solution:

$$H = H_0 + V(t)$$

$$H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2 \qquad V(t) = \begin{cases} -qE_0 X \cdot \sin(\omega t) & 0<t< \\ 0 & \text{otherwise} \end{cases}$$

$t=0$: s.h.o. is in ground state $|0\rangle$

recall:
$$\left.\begin{aligned} a &= \sqrt{\frac{m\omega_0}{2\hbar}}\left(x + \frac{i}{m\omega_0} p\right) \\ a^+ &= \sqrt{\frac{m\omega_0}{2\hbar}}\left(x - \frac{i}{m\omega_0} p\right) \end{aligned}\right\} \Rightarrow \quad x = \sqrt{\frac{\hbar}{2m\omega_0}}\,(a + a^+)$$

$$H_0 |n\rangle = \hbar\omega_0 (n + \tfrac{1}{2})\,|n\rangle \qquad n = 0,1,2,\dots$$

$$E_n = \hbar\omega_0 (n + \tfrac{1}{2})$$

$$a|n\rangle = \sqrt{n}\,|n-1\rangle$$
$$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad \omega_{\ell k} \equiv \frac{1}{\hbar}(E_\ell - E_k)$$

at $\frac{2\pi}{\omega}$: $m \neq 0$
$$W(0 \to \ell) = \frac{1}{\hbar^2}\left|\int_0^{\frac{2\pi}{\omega}} \langle \ell | V(t) | 0 \rangle e^{i\omega_{\ell 0} t}\, dt\right|^2$$

$$= \frac{1}{\hbar^2}\left|\int_0^{\frac{2\pi}{\omega}} (-qE_0)\langle \ell | x | 0 \rangle e^{i\ell\omega_0 t} \sin(\omega t)\, dt\right|^2$$

$$= \frac{q^2 E_0^2}{\hbar^2}\left|\langle \ell | x | 0 \rangle\right|^2 \left|\int_0^{\frac{2\pi}{\omega}} \sin(\omega t)\, e^{i\ell\omega_0 t}\, dt\right|^2$$

$$= \frac{q^2 E_0^2}{\hbar^2}\left|\langle \ell | 1 \rangle \sqrt{\frac{\hbar}{2m\omega_0}}\right|^2 \left|\int_0^{\frac{2\pi}{\omega}} \sin(\omega t)\, e^{i\ell\omega_0 t}\, dt\right|^2$$

$$= \delta_{\ell 1} \frac{q^2 E_0^2}{\hbar^2} \frac{\hbar}{2m\omega_0}\left|\int_0^{\frac{2\pi}{\omega}} \sin(\omega t)\, e^{i\omega_0 t}\, dt\right|^2$$

Kronecker-Delta

$$= \delta_{\ell 1} \frac{q^2 E_0^2}{2 m \hbar \omega_0} \left| \int_0^{\frac{2\pi}{\omega}} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \cdot e^{i\omega_0 t}\, dt \right|^2$$

$$= \delta_{\ell 1} \frac{q^2 E_0^2}{2 m \hbar \omega_0} \left| \int_0^{\frac{2\pi}{\omega}} \frac{e^{i(\omega+\omega_0)t}}{2i} dt - \int_0^{\frac{2\pi}{\omega}} \frac{e^{i(\omega_0-\omega)t}}{2i} dt \right|^2$$

$$= \delta_{\ell 1} \frac{q^2 E_0^2}{2 m \hbar \omega_0} \left| \frac{1}{-2(\omega+\omega_0)} \left(e^{i2\pi\frac{\omega_0}{\omega}} - 1\right) - \frac{1}{-2(\omega_0-\omega)} \left(e^{i2\pi\frac{\omega_0}{\omega}} - \right. \right.$$

clearly only $\underline{0 \to 1}$ transition is possible
(as a result of the nature of the x operator which is first order in a and a^+)

Now the $\omega \to \omega_0$ limit has to be taken carefully

$$W(0 \to 1) = \frac{q^2 E_0^2}{2 m \hbar \omega_0} \cdot \lim_{\omega \to \omega_0} \frac{\sin^2\left(\pi \frac{\omega_0}{\omega}\right)}{(\omega_0 - \omega)^2}$$

(first term in $| \; |$ trivially $\to 0$)

$$\frac{\sin\left(\pi\frac{\omega_0}{\omega}\right)}{\omega_0 - \omega} = \frac{1}{\omega} \cdot \frac{\sin\left(\pi\frac{\omega_0}{\omega}\right)}{\frac{\omega_0}{\omega} - 1} = \frac{1}{\omega} \frac{\sin(\pi x)}{x - 1} \quad \text{as } x \to 1 \; (\omega \to \omega_0)$$

$$\underset{x \to 1}{\simeq} \frac{1}{\omega_0} \frac{\pi \cos(\pi x)}{1} \underset{x \to 1}{\simeq} -\frac{\pi}{\omega_0}$$

$$W(0 \to 1) = \frac{q^2 E_0^2 \pi^2}{2 m \hbar \omega_0^3}$$

@ $t = \frac{2\pi}{\omega_0}$: $|\psi\rangle = C_0 |0\rangle + C_1 |1\rangle$ $\qquad |C_1|^2 = W(0 \to 1)$

$$|C_0|^2 = 1 - |C_1|^2 = 1 - \frac{q^2 E_0^2 \pi^2}{2 m \hbar \omega_0^3} \quad ; \; |C_\ell| = 0 \;\; \ell > 1$$

Solution

II-7.)

$$P(V-b) = RT\exp\left(-\frac{a}{RTV}\right)$$

$$p=\frac{RT}{(V-b)}\exp\left(-\frac{a}{RTV}\right)$$

determine V_k, T_k, P_k t

II-7

2 point $\frac{\partial p}{\partial V}=\frac{\partial^2 p}{\partial V^2}=0$

1 point eq. $p'(T', V')$

3 point P_K, T_K, V_K

$$\frac{\partial p}{\partial V} = -\frac{RT}{(V-b)^2}\exp\left(-\frac{a}{RTV}\right) \quad \frac{a}{TV^2}$$

$$= -\frac{RT}{(V-b)^2}\exp\left(-\frac{a}{RTV}\right)+\frac{a}{V^2(V-b)}\exp\left(-\frac{a}{RTV}\right)$$

$$\frac{\partial p}{\partial V}=0 = -\frac{RT}{(V-b)}+\frac{a}{V^2}$$

$$\frac{RT}{(V-b)} = \frac{a}{V^2}$$

$$T = \frac{a}{R}\ \frac{(V-b)}{V^2}$$

$$\frac{\partial^2 p}{\partial V^2} = +\frac{2RT}{(V-b)^3}\exp\left(-\frac{a}{RTV}\right) - \frac{RT}{(V-b)^2}\exp\left(-\frac{a}{RTV}\right)\left(\frac{a}{RT\right.$$

$$+\left[-\frac{2a}{V^3(V-b)}\exp\left(-\frac{a}{RTV}\right)-\frac{a}{V^2(V-b)^2}\exp\left(-\frac{a}{RTV}\right)\right.$$

$$\left.+\frac{a}{V^2(V-b)}\exp\left(-\frac{a}{RTV}\right)\left(\frac{a}{RTV^2}\right)\right]$$

$$\frac{\partial^2 p}{\partial V^2}=0 = \frac{2RT}{(V-b)^{3}} - \frac{\cancel{RT}}{(V-b)}\frac{a}{\cancel{RT}V^2} - \frac{2a}{V^3\cancel{(V-b)}} - \frac{a}{V^2(V-b)}$$

$$+\frac{a}{V^2\cancel{(V-b)}}\frac{a}{RTV^2}$$

$$0 = \frac{2RT}{(V-b)^2} - \frac{2a}{V^2(V-b)} - \frac{2a}{V^3} + \frac{a^2}{V^4 RT}$$

$$0 = \frac{2\cancel{R}}{(V-b)^{\cancel{2}}} \cdot \frac{a}{\cancel{R}} \frac{\cancel{(V-b)}}{V^2} - \frac{2a}{V^2(V-b)} - \frac{2a}{V^3} + \frac{a^{\cancel{2}}}{\cancel{V^4}_2\cancel{R}} \cdot \frac{\cancel{R}}{\cancel{a}} \frac{\cancel{(V^2)}}{V-b}$$

$$0 = \frac{2a}{V^2(V-b)} - \frac{2a}{V^2(V-b)} - \frac{2a}{V^3} + \frac{a}{V^2(V-b)}$$

$$0 = -\frac{2a}{V} + \frac{a}{(V-b)}$$

$$\frac{2a}{V} = \frac{a}{V-b}$$

$$2a(V-b) = aV$$

$$2aV - 2ab = aV$$

$$aV - 2ab = 0$$

$$V_k = 2b$$

$$T_k = \frac{a}{R} \quad \frac{2b-b}{4b^2} = \frac{a}{R} \quad \frac{1}{2b}$$

$$p_k = \frac{RT_k}{(V_k - b)} \exp\left(-\frac{a}{RT_k V_k}\right)$$

$$= \frac{\cancel{R} \cdot a}{2b\cancel{R} \cdot (2b-b)} \exp\left(-\frac{\cancel{a}\,\cancel{2b}\cancel{R}}{\cancel{R}\cancel{a}\,\cancel{2b}}\right)$$

$$p_k = \frac{a}{2b^2} e^{-1}$$

Joule-Thompson Coefficient:

$$\left(\frac{\partial T}{\partial p}\right)_H = -\frac{1}{c_p}\left(\frac{\partial H}{\partial p}\right)_T = 0$$

$$c_p - c_v = \left(\frac{\partial p}{\partial T}\right)_v \left(V - \left(\frac{\partial H}{\partial p}\right)_T\right)$$

$$c_p - c_v = -\frac{T\left[\left(\frac{\partial p}{\partial T}\right)_v\right]^2}{\left(\frac{\partial p}{\partial V}\right)_T}$$

$$-\frac{T\left[\left(\frac{\partial p}{\partial T}\right)_v\right]^2}{\left(\frac{\partial p}{\partial V}\right)_T} = \left(\frac{\partial p}{\partial T}\right)_v \left(V - \left(\frac{\partial H}{\partial p}\right)_T\right)$$

$$+T\frac{\left(\frac{\partial p}{\partial T}\right)_v}{\left(\frac{\partial p}{\partial V}\right)_T} + V = +\left(\frac{\partial H}{\partial p}\right)_T$$

$$\left(\frac{\partial H}{\partial p}\right)_T = V + T\frac{\left(\frac{\partial p}{\partial T}\right)_v}{\left(\frac{\partial p}{\partial V}\right)_T}$$

$$p = \frac{RT}{V-b} \exp\left(-\frac{a}{RTV}\right)$$

$$V' = \frac{V}{V_k} = \frac{V}{2b} \Rightarrow V = 2bV'$$

$$T' = \frac{T}{T_k} = \frac{T}{a} \cdot 2bR \Rightarrow T = T' \frac{a}{2bR}$$

$$p' = \frac{p}{p_k} = p \cdot \frac{2b^2}{a} e \qquad p = \frac{a}{2b^2 e} p'$$

$$\frac{\cancel{a}}{2b^{\cancel{2}} e} p' = \frac{\cancel{R} \cdot \cancel{a}}{\cancel{2bR}} T' \frac{1}{2bV'-b} \exp\left(-\frac{\cancel{a}}{\cancel{R} \cdot T' \frac{\cancel{a}}{\cancel{2bR}} \cdot \cancel{2b}V'}\right)$$

$$\frac{p'}{e} = \frac{T'}{\cancel{1b}(2V'-1)} \exp\left(-\frac{1}{T'V'}\right)$$

$$p' = e \frac{T'}{2V'-1} \exp\left(-\frac{1}{T'V'}\right)$$

$$\left(\frac{\partial p'}{\partial T'}\right)_{V'} = e\left[\frac{1}{2V'-1} \exp\left(-\frac{1}{T'V'}\right) + \frac{\cancel{T'}}{2V'-1}\left(\frac{1}{T'^{\cancel{2}}V'}\right) \exp\left(\frac{-1}{T'V'}\right)\right.$$

$$= e \frac{1}{2V'-1} \exp\left(-\frac{1}{T'V'}\right) \left[1 + \frac{1}{T'V'}\right]$$

$$\left(\frac{\partial p'}{\partial V'}\right)_{T'} = eT'\left[-\frac{2}{(2V'-1)^2} \exp\left(-\frac{1}{T'V'}\right) + \frac{1}{2V'-1} \exp\left(\frac{-1}{T'V'}\right) \frac{2}{T'V}\right.$$

$$= \frac{eT'}{(2V'-1)} \exp\left(-\frac{1}{T'V'}\right) \left[-\frac{2}{(2V'-1)} + \frac{2}{T'V'^2}\right]$$

$$V' + T' \frac{\left(\frac{\partial p'}{\partial T}\right)_V}{\left(\frac{\partial p}{\partial V}\right)_T} = V' + \frac{1 + \frac{1}{T'V'}}{2\left[\frac{1}{T'V'^2} - \frac{1}{(2V'-1)}\right]} = 0$$

$$V' + \frac{1 + \frac{1}{T'V'}}{2\left[\frac{1}{T'V'^2} - \frac{1}{(2V'-1)}\right]} = 0$$

$$2V'\left[\frac{1}{T'V'^2} - \frac{1}{(2V'-1)}\right] = -1 - \frac{1}{T'V'}$$

$$\frac{2}{T'V'} - \frac{2V'}{2V'-1} = -1 - \frac{1}{T'V'}$$

$$\frac{3}{T'V'} - \frac{2V'}{2V'-1} = -1$$

$$3(2V'-1) - 2T'V'^2 = -1\,T'V'(2V'-1)$$

$$6V' - 3 - \cancel{2T'V'^2} = -\cancel{2T'V'^2} + T'V'$$

$$V'(6-T') = 3$$

$$V' = \frac{3}{6-T'}$$

$$p' = e\,\frac{T'}{2\left(\frac{3}{6-T'} - 1\right)}\ \exp\left(-\frac{1}{T'\left(\frac{3}{6-T'}\right)}\right)$$

$$p' = \frac{e}{2}T' \cdot \frac{1}{\frac{3 - 6 + T'}{6-T'}}\ \exp\left(-\frac{6-T'}{3T'}\right)$$

$$p' = \frac{e}{2}\,\frac{T'(6-T')}{T'-3}\ \exp\left(-\frac{2}{T'} + \frac{1}{3}\right)$$

(II- 8.)

Stat Mech

II-8.)

a)

$$dF = -S\,dT - pV$$

$$\left(\frac{\partial F}{\partial T}\right)_V = -S \quad \left(\frac{\partial F}{\partial T}\right)_T = -p$$

Since $\frac{\partial^2 F}{\partial T \partial V} = \frac{\partial^2 F}{\partial T \partial V}$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$$

b) The total radiation energy is

$$U = V u(T)$$

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial S}{\partial V}\right)_T - p = T\left(\frac{\partial S}{\partial V}\right)_T - p$$

c) $u = \frac{1}{3} T \frac{du}{dT} - \frac{1}{3} u$

$$\frac{4u}{\cancel{3}} = \frac{1}{\cancel{3}} T \frac{du}{dT}$$

$$u = \sigma T^4$$

Solution

II-9.) distinguishable particle, each can be in 3 possible states: $s_i = 1, 2, 3$ $\quad i = 1, 2$

Interaction energy: $\mathcal{H}(s_1, s_2) = \begin{cases} -J & \text{is } s_1 = s_2 \\ 0 & \text{otherwise} \end{cases}$

(formally can write: $\mathcal{H}(s_1, s_2) = -J\,\delta_{s_1 s_2}$)

$\beta = \frac{1}{kT}$

$$Z(\beta) = \sum_{s_1, s_2} e^{-\beta \mathcal{H}(s_1, s_2)} = \sum_{s_1, s_2} e^{\beta J \delta_{s_1 s_2}} =$$

$$= 3e^{\beta J} + 6 = 3(2 + e^{\beta J})$$

(a)
$$E = \langle \mathcal{H}(s_1, s_2) \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta) = -J \frac{e^{\beta J}}{2 + e^{\beta J}}$$

$$E(T=0) = -J \qquad E(T=\infty) = -\frac{J}{3}$$

(b)
$$C = \frac{\partial E}{\partial T} = -\frac{1}{kT^2} \frac{\partial E}{\partial \beta} = +\frac{J^2}{kT^2} \frac{e^{\beta J}(2 + e^{\beta J}) - e^{\beta J} \cdot e^{\beta J}}{(2 + e^{\beta J})^2} =$$

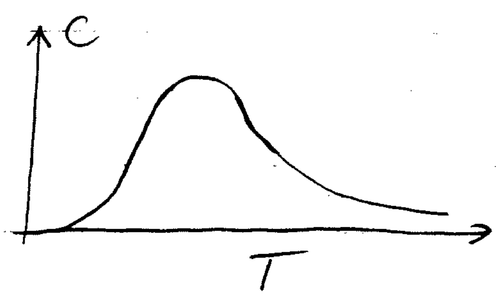
$$= k\left(\frac{J}{kT}\right)^2 \frac{2e^{\beta J}}{(2 + e^{\beta J})^2} = 2k\left(\frac{J}{kT}\right)^2 \frac{e^{J/kT}}{(2 + e^{J/kT})^2}$$

high-temperature behavior: $J/kT \ll 1$

$$C \simeq \frac{2}{9} k \left(\frac{J}{kT}\right)^2$$

low-temperature behavior: $J/kT \gg 1$

$$C \simeq 2k\left(\frac{J}{kT}\right)^2 e^{-J/kT}$$



(c) $F = -kT \ln Z = -kT \ln\left[3(2+e^{J/kT})\right]$

$$S = -\frac{\partial F}{\partial T} = k \log\left[3(2+e^{J/kT})\right] + kT \frac{e^{J/kT}}{2+e^{J/kT}}\left(-\frac{J}{kT^2}\right)$$

$$S = k \log 3 + k \log(2+e^{J/kT}) - \frac{J}{T} \cdot \frac{e^{J/kT}}{2+e^{J/kT}}$$

$$\lim_{T \to 0} S(T) = k \log(3) + k \log(e^{J/kT}) - \frac{J}{T} = k \log(3)$$

3 is the result of 3-fold degeneracy of the groundstate (00, 11, 22) $\rightarrow S_0 = k \log(3)$

II-10.)

<u>Solution</u>:

(a) For an ideal gas of structureless particles

$$\varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m}, \quad \vec{k} = \frac{\pi}{L}(n_x \hat{x} + n_y \hat{y} + n_z \hat{z})$$

Then, assuming a macroscopic volume,

$$\beta p V = \ln \mathcal{Z}_G = \sum_k \ln[1 + e^{\beta(\mu - \varepsilon_{\vec{k}})}] =$$

$$= \frac{V}{(2\pi)^3} \int_0^\infty 4\pi k^2 \ln[1 + e^{\beta(\mu - \hbar^2 k^2/2m)}]\, dk$$

$$= \frac{4V}{\pi^2}\left(\frac{m}{2\beta \hbar^2}\right)^{3/2} \int_0^\infty x^2 \ln(1 + z e^{-x^2})\, dx$$

with $z = e^{\beta\mu}$. So: $\beta p = \frac{4}{\sqrt{\pi}} \frac{1}{\lambda^3} \int_0^\infty x^2 \ln(1 + z e^{-x^2}) dx$

$\beta p = \frac{1}{\lambda^3} f_{5/2}(z)$; and $\lambda = \left(\frac{2\pi\beta\hbar^2}{m}\right)^{1/2}$

now substituting ~~for~~ the series for $\ln(1+x)$

$$f_{5/2}(z) = \frac{4}{\sqrt{\pi}} \sum_{k=1}^\infty (-1)^{k+1} \frac{z^k}{k} \int_0^\infty x^2 e^{-kx^2} dx$$

$f_{5/2}(z) = \sum_{k=1}^\infty (-1)^{k+1} \frac{z^k}{k^{5/2}}$. Now to calculate the density,

$$\langle N \rangle = \sum_k \langle n_k \rangle = \sum_k \frac{1}{e^{\beta(\varepsilon_k - \mu)} + 1} =$$

$$\langle N \rangle = \frac{V}{(2\pi)^3} \int_0^\infty \frac{4\pi k^2 z e^{-\hbar^2 k^2/2m}}{(1 + z e^{-\hbar^2 k^2/2m})} dk$$

$$\langle N \rangle = \frac{V}{\lambda^3} \frac{4}{\sqrt{\pi}} \int_0^\infty x^2 \frac{z e^{-x^2}}{1 + z e^{-x^2}} dx$$

(II-10) Solution Continued.

$$\therefore \rho\lambda^3 = \frac{4}{\sqrt{\pi}} \sum_{k=1}^{\infty} (-1)^{k+1} z^k \int_0^{\infty} x^2 e^{-kx^2} dx$$

$$\rho\lambda^3 = \sum_{k=1}^{\infty} (-1)^{k+1} z^k / k^{3/2} = f_{3/2}(z)$$

(b.) $$\langle E \rangle = \sum_k \langle n_k \rangle \varepsilon_k =$$

$$= \frac{V}{(2\pi)^3} \int_0^{\infty} 4\pi k^2 \left(\frac{\hbar^2 k^2}{2m}\right) \cdot \frac{1}{1+e^{\beta\left(\frac{\hbar^2 k^2}{2m} - \mu\right)}}\, dk$$

$$= \frac{m}{\beta} \frac{\partial}{\partial m} \left(\frac{V}{(2\pi)^3}\right) \int_0^{\infty} 4\pi k^2 \ln\left(1 + e^{\beta\left(\mu - \frac{\hbar^2 k^2}{2m}\right)}\right) dk.$$

$$= \frac{m}{\beta} \frac{\partial}{\partial m} \frac{V}{(2\pi)^3} f_{5/2}(z) = \frac{m}{\beta} \frac{3}{2m} \frac{V}{\lambda^3} f_{5/2}(z)$$

$$\langle E \rangle = \frac{3}{2\beta} (\beta V p) = \frac{3}{2} p V$$