Old QM Final 1

1. Consider 3 identical and non-interacting spin ½ particles,

a) What are all spin representations using Young Tableaux

$\square \times \square \times \square$ where the primitive $\square = \pm \frac{1}{2}$ or $0 + 1$

$$\square \times \square = \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array} + \begin{array}{|c|c|}\hline \; & \; \\ \hline \end{array} = \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 0 & 1 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$$

Then
$$\square \times \square \times \square = \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline \end{array} + \begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1} \end{array} + \begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline \end{array}$$

↳ invalid (no 2 value available)

answer →
$$= \left(\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1} \end{array} + \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1} \end{array} \right) \times 2 + \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline \end{array}$$
$$+ \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} = 8 \text{ permut}$$

Where $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array}$ are antisymmetric and $\begin{array}{|c|c|}\hline \; & \; \\ \hline \end{array}$ are totally symmetric.

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline \end{array} = |+,+,+\rangle$$

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline \end{array} = \frac{1}{\sqrt{3}} \left(|+,+,-\rangle + |+,-,+\rangle + |-,+,+\rangle \right)$$

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline \end{array} = \frac{1}{\sqrt{3}} \left(|+,-,-\rangle + |-,+,-\rangle + |-,-,+\rangle \right)$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} = |-,-,-\rangle$$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1} \end{array} = \frac{1}{\sqrt{4}} \left(\cancel{|+,+,-\rangle - |-,+,+\rangle + |+,+,-\rangle - |+,-,+\rangle} \right)$$

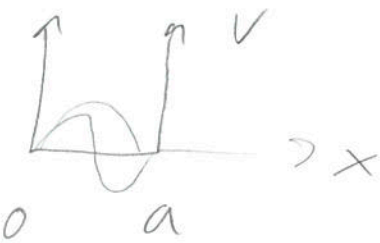
$$= \frac{1}{-} \left(\cancel{\sqrt{2}\,|+,+,-\rangle - |-,+,+\rangle} - |+,-,+\rangle \right)$$

Now we set all 3 particles in the same spin $|1/2, 1/2\rangle$ state and restrict them to move in an ∞-well in 1-d of width a.

b) Determine the energy and wave function of the ground state, when all particles have mass m.

The only spin state for 3 $|1/2, 1/2\rangle = |+,+,+\rangle$ and this is totally symmetric, Therefore we need a totally antisymmetric spatial wave function and it needs to have as low energy as possible

$\rightarrow$ The width is a $\rightarrow$

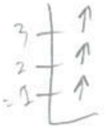


$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$$

$n = 1, 2, 3, \ldots$

$n = 1$ is the symmetric single particle ground state,

but we need antisymmetric (about $x = a/2$) for, so we choose $n = 2$ for one of the 3 particles. Which of the three? all 3.



Only possible state

$$\psi_{gs} = \frac{1}{\sqrt{6}}\begin{pmatrix} |123\rangle - |132\rangle - |213\rangle \\ +|231\rangle + |312\rangle - |321\rangle \end{pmatrix}$$

and $E_{gs} = E_1 + E_2 + E_3$ where $E_n = \dfrac{n^2\pi^2\hbar^2}{2ma^2}$

c) first excited state $\rightarrow$ Same but ~~with all 1's $\rightarrow$ 2's as we~~

use $|1,2,4\rangle$ as starting point now.

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(8)

$$\int_0^{\pi} \sin\theta\, d\theta = -\int_1^{-1} d(\cos\theta)\, d\theta$$

2. a) Determine the scattering cross-section for slow particles ($ka \ll 1$) in a spherical square well ($-V_0$, a)

$$\sigma_{TOT} = \frac{4\pi}{k} \operatorname{Im}\{f(\vec{k}, \vec{k}' = \vec{k})\} = \int d\Omega\, |f(\vec{k}, \vec{k}')|^2$$

first Order Born (+ local) approximation

$$f_B(\vec{k}, \vec{k}') = \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right) \int d^3\vec{x}'\, e^{i\vec{q}\cdot\vec{x}'}\, V(\vec{x}') \qquad \vec{q} = \vec{k} - \vec{k}'$$

$$= \frac{2m}{\hbar^2}\, \frac{-1}{4\pi} \cdot 2\pi \int_{-1}^{1}\int_0^a dx'\, d(\cos\theta)\, e^{iqx'(\cos\theta)}\, x'^2\, V_0$$

$$= -\frac{m}{\hbar^2} \int_0^a dx'\, x'^{\cancel{2}} \left(\frac{e^{+iqx'}}{iq\cancel{x'}} - \frac{e^{-iqx'}}{iq\cancel{x'}} \right) V_0$$

$$= -\frac{m}{\hbar^2} \int_0^a \frac{dx'\, x'}{q} \cdot 2\sin(qx')$$

x' $\quad \sin(qx')$

1 $\quad -\frac{1}{q}\cos(qx')$

0 $\quad -\frac{1}{q^2}\sin(qx')$

$$= +\frac{2m}{\hbar^2 q} \left(\frac{x}{q}\cos(qx) - \frac{\sin(qx)}{q^2} \right)\Bigg|_0^a$$

$$\boxed{= \frac{2m}{\hbar^2 q^2} \left(a\cos(aq) - \frac{1}{q}\sin(aq) \right)}$$

$$\approx \frac{2m}{\hbar^2 q^2}\left(\frac{a^3 q^2}{2}\right) \approx \frac{m a^3}{\hbar^2}$$

~~$\int dx\, x\, e^{iqx} = \frac{x}{iq} e^{iqx} + \frac{1}{q^2} e^{iqx}$~~

~~$= \frac{1}{iq} x e^{iqx} + \frac{1}{q^2} e^{iqx} \Big|_0$~~

~~$= \frac{a e^{iqa}}{iq} + \frac{e^{iqa} - 1}{q^2}$~~

~~$- \frac{a e^{-iqa}}{iq} - \frac{e^{-iqa} - 1}{q^2}$~~

~~$\frac{m}{2} \cdot \frac{2}{iq} \left(\frac{a}{q}\sin(qa) + \frac{i}{}\sin(qa) \right) V_0$~~

letting $ak \to$ small we continue.

$$\text{then } |f(\vec{k},\vec{k}')|^2 = \frac{4m^2}{\hbar^4 q^4}\left[a\cos(aq) \overset{-!}{+} \frac{1}{q}\sin(aq)\right]^{\textcircled{2}\,!}$$

$$q^2 = |\vec{k}-\vec{k}'|^2 = k^2+k^2-2k^2\cos\theta = 2k^2(1-\cos\theta)$$
$$= 4k^2\sin^2(\theta/2)$$
$$q = 2k\sin(\theta/2)$$

$$= \frac{4m^2 a}{\hbar^4 q^4}\left[\cos(aq) + \frac{1}{qa}\sin(qa)\right]$$

$$\simeq \frac{4m^2 a}{\hbar^4 q^4}\left[1 + \frac{qa}{qa}\right] \simeq \frac{8m^2 a}{\hbar^4 q^4}$$

$$\text{Then } \sigma_{tot} = \int_0^{2\pi}\int_{-1}^{1} d\cos\theta\, d\phi\, \frac{8m^2 a}{\hbar^4 \underbrace{(2k\sin(\theta/2))^4}_{[4k^2(1-\cos\theta)]^2}}$$

$$= \frac{2\pi\cdot 8m^2 a}{\hbar^4\cdot 16k^4}\int_{-1}^{1} du\,\frac{1}{(1-u)^2}$$

$du = -dv$

$v = 1-u \qquad v(-1) = 2$
$v(1) = 0$

$$= -\frac{\pi m^2 a}{\hbar^4 k^4}\int_2^0 \frac{dv}{v^2} = +\frac{\pi m^2 a}{\hbar^4 k^4}\,\frac{1}{v}\Big|_2^0 = \boxed{\infty - \frac{\pi a}{2\hbar^4}}$$

σ_{tot}

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3. At $t<0$ a particle is in the ground state of a 1D SHO

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$$

at $t=0$ an interaction is applied

$$V(t,x) = V_0 (a+a^\dagger)^2 e^{-t^2/\tau^2}$$

a) Use 1st order perturbation theory to find the $P_{0\to n}$ transition probability at $t \to \infty$.

$$P_{i\to n} \simeq |C_n^{(1)}(t)|^2 \simeq \left| \frac{-i}{\hbar} \int_{t_0}^{t} e^{i\omega_{ni} t'} V_{ni}(t') dt' \right|^2$$

$$V_{ni} = \langle n | V | i \rangle = V_0 e^{-t^2/\tau^2} \langle n | (a+a^\dagger)^2 | i \rangle$$

$$(a+a^\dagger)^2 = a^2 + aa^\dagger + a^\dagger a + a^{\dagger 2}$$

$$(a+a^\dagger)^2 |i\rangle = \sqrt{i}\sqrt{i-1}\,|i-2\rangle + \sqrt{i+1}\sqrt{i+1}\,|i\rangle + \sqrt{i}\sqrt{i}\,|i\rangle + \sqrt{i+1}\sqrt{i+2}\,|i+2\rangle$$

for $i=0$ $\quad V_{ni} = V_0 e^{-t^2/\tau^2} \sqrt{1}^2 \langle n|0\rangle + V_0 e^{-t^2/\tau^2}\sqrt{2}\langle n|2\rangle = V_0 e^{-t^2/\tau^2}(\delta_{n,0} + \sqrt{2}\,\delta_{n,2})$

$$P_{0\to n}(\infty) \simeq \frac{V_0^2}{\hbar^2} \left| \int_0^\infty e^{\underbrace{i\omega_{ni} t'}_{\text{insignificant}} - t'^2/\tau^2} \left(\delta_{n,0} + \sqrt{2}\,\delta_{n,2}\right) \right|^2$$

$$P_{0\to n} \simeq \frac{V_0^2}{\hbar^2}\left(\frac{\sqrt{\pi}}{2}\tau\right)^2 \left(\delta_{n,0} + 2\delta_{n,2}\right)$$

remaining in the ground state $P_{0\to 0}$ (c)

$P_{0\to 2}$ allowed transition (b)

5. Consider an ∞-well in $|x| \leq a$ with a repulsive δ function at the origin:

$$V(x) = \beta \, E_1 \, a \, \delta(x)$$

Find the correction to the energy levels to 2nd order in β. Explain your findings.

$$E_n^{(1)} = \int dx \, V(x) |\psi_n^{(0)}|^2 = \beta E_1 a |\psi_n(0)|^2$$

$n = $ odd $\rightarrow$ sines $\rightarrow$ no overlap
$\rightarrow$ no correction.

$n = $ even $\rightarrow$ $\psi_n(0) = \sqrt{\frac{2}{2a}} = \sqrt{\frac{1}{a}}$

$$\therefore \boxed{\begin{aligned} \beta E_1 &= E_{n \text{ even}}^{(1)} \\ 0 &= E_{n \text{ odd}}^{(1)} \end{aligned}}$$

To second order then

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle \overset{0}{n} | V | \overset{0}{m} \rangle|^2}{\overset{0}{E}_n - \overset{0}{E}_m}$$

$\rightarrow$ both n & m must be even.

$$= \sum_{m \neq n} \frac{|\beta E_1|^2}{n^2 - m^2} \cdot \frac{8 m a^2}{\pi^2 \hbar^2}$$

$$\boxed{\begin{aligned} E_{n \text{ even}}^{(2)} &= \sum_{\substack{m_{\text{even}} \\ \neq n}} \frac{\beta^2 E_1^2}{n^2 - m^2} \frac{8 m a^2}{\pi^2 \hbar^2} \\ E_{n \text{ odd}}^{(2)} &= 0 \end{aligned}}$$

non-degenerate

because odd solution (odd n) have no overlap with the origin