(old solutions written too)

QM II Midterm

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QM II midterm Memorization

Cyclotron ω | Spin precession

$\omega = \frac{eB}{mc}$, $\phi = \omega t$

Landé Formula

$$\boxed{\vec{\mu}_e = \frac{\overbrace{g_e}^{\simeq 2} q \vec{S}}{2mc}}$$

Pauli Spin Matrices $\quad Tr\{\sigma_i \sigma_j\} = 2\delta_{ij}$

① * $\vec{S} = \frac{1}{2}\hbar\vec{\sigma}$ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hydrogen G.S.

$\langle x|0\rangle = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$

In general $\quad S_+ = S_x + iS_y$, $S_- = S_x - iS_y$

$$S_x = \frac{1}{2}(S_+ + S_-), \quad S_y = \frac{1}{2i}(S_+ - S_-)$$

$$S_\pm = \delta_{m', m\pm 1}\sqrt{s(s+1) - m'm}$$

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② * $D_{\hat{n}}(\underset{\text{angle of rotation about } \hat{n} \text{ axis.}}{\theta}) = \mathbb{1}\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\sigma_{\hat{n}}$, $\quad \sigma_{\hat{n}} = \vec{\sigma}\cdot\hat{n}$

angle of rotation about $\hat{n}$ axis.

* $\vec{\sigma}\cdot\hat{n} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix}$ $\quad$ θ + ϕ polar angles

↓ yields

$$|+\rangle_{\hat{n}} = D_z(\gamma)D_y(\beta)|+\rangle_z = \begin{pmatrix} e^{-i\frac{\gamma}{2}}\cos\left(\frac{\beta}{2}\right) \\ e^{i\frac{\gamma}{2}}\sin\left(\frac{\beta}{2}\right) \end{pmatrix}$$

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$$T_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right) - \frac{1}{3}\delta_{ij}Tr\{T\} + \frac{1}{2}\left(T_{ij} - T_{ji}\right) + \frac{1}{3}\delta_{ij}Tr\{T\}$$

$$T_q^k = \sum_{1,2} \langle k_1 q_1 k_2 q_2 | k\, q\, k_1 k_2\rangle T_{q_1}^{k_1} T_{q_2}^{k_2} \qquad \begin{matrix} k \sim s \sim j \\ q \sim m \end{matrix}$$

where $-k_i \leq q_i \leq k_i$

$k \in |k_1 - k_2| \ldots |k_1 + k_2|$ $\sim j$'s.

In general any tensor $T_{ij} = Const \cdot A_i B_j$ and vice versa.

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$\nabla^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{L^2}{r^2}$, $\quad \int_0^\infty x^n e^{-\frac{x}{a}}dx = n!\, a^{n+1}$ ✓

$\int_0^\infty x^{2n} e^{-x^2/a^2}dx = \sqrt{\pi}\,\frac{(2n)!}{\ldots}\left(\frac{a}{\ldots}\right)^{2n+1}$ $\quad \int_0^\infty x^{2n+1} e^{-x^2/a^2}dx = \frac{n!}{2}a^{2n+2}$

$$\langle x | \hat{p}^2 | y \rangle = -\hbar^2 \delta^3(\vec{x}-\vec{y}) \vec{\nabla}^2$$

* Clebsch - Gorden coefficient computations

$$\vec{J} = \vec{J}_1 + \vec{J}_2 \quad \rightarrow |j_{tot}, m_{tot}, j_1, j_2\rangle \overset{!}{\propto} |j_1, m_1\rangle \otimes |j_2, m_2\rangle$$

Only proportional : There are many combinations of j's & m's that wo

$$m_{tot} = m_1 + m_2 \quad \& \quad |m_i| \leq j_i$$

$$|j_1 - j_2| \leq j_{tot} \leq |j_1 + j_2| \quad \& \quad \sum c_i^2 = 1 \text{ for } \underline{\text{coefficients}}$$

Then we write out all combinations of j_1, m_1, j_2, m_2 that satisfy the selection rules and then we act on the eqn. with

$$\vec{J}^2 = \vec{J}_1^{\,2} + \vec{J}_2^{\,2} + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

Wherein we utilize the eigenvalue relations (up to factors of $\hbar$.

$$\vec{J}^2 |j, m\rangle = \hbar^2 j(j+1) |j, m\rangle$$

$$J_z |j, m\rangle = \hbar m |j, m\rangle \qquad \text{iff } |m \pm 1| \leq j \,!$$

$$J_\pm |j, m\rangle = \hbar \sqrt{j(j+1) - m(m \pm 1)} \, |j, m \pm 1\rangle$$

Then exploit the orthonormality of both basis to elliminate all Bras and Kets and find a relationship for the coefficients which is then supplemented by either overall normalization or the use of ladder operators starting at unit normalized single option top or bottom rung possibilities.

3

Symmetry operators $U(\vec{w}) = e^{\frac{-i\vec{w}\cdot\hat{\vec{M}}}{\hbar}}$ where $\hat{\vec{M}}$ generates the sym

Translation $T(\vec{x}) = e^{\frac{-i\vec{x}\cdot\vec{p}}{\hbar}}$ where $\vec{p} = \frac{\hbar\vec{\nabla}}{i}$, $\vec{\nabla} = \frac{\partial}{\partial\vec{x}}$

$\textcircled{2} \overset{!}{=} \textcircled{2}$ Rotation $D_{\hat{n}}(\theta) = e^{\frac{-i\theta\hat{n}\cdot\vec{J}}{\hbar}}$ where $\vec{J} = \hbar\,\vec{\sigma}_{3\times3}$

Time reversal $\Theta = -i\sigma_y \underset{\text{phase}}{\delta}\ \underset{\text{right acting c.c.}}{\vec{K}}$ → on spin 1/2.

antiunitary

$$\Theta|Y_\ell^m\rangle = |\widetilde{\ell m}\rangle = (-1)^m|\ell,-m\rangle$$

Parity $\Pi = \Pi^+ = \Pi^{-1}$, $\Pi|\vec{x}\rangle = |-\vec{x}\rangle = \lambda_\pi|\vec{x}\rangle$

$\lambda_\pi = \pm 1$ ($\ominus$ for $\vec{x}$,

$\Pi|Y_\ell^m\rangle = \Pi|\ell m\rangle = (-1)^\ell|\ell m\rangle$ for spherical symmetric

Time $U(t) = e^{\frac{-i\hat{H}t}{\hbar}}$ where $\hat{H}$ is the Hamiltonian,

* Discrete $T_{\vec{a}} = e^{\frac{-i\vec{a}\cdot\vec{k}}{\hbar}}$: Bloch states $T_{\vec{a}}|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle = \lambda_{\vec{a}}|\vec{x}\rangle$

↳ $\lambda_{\vec{a}} = e^{\frac{-iak}{\hbar}}$ which can be useful

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$(S_i)_{jk} = -i\hbar\,\varepsilon_{ijk}$

$(S_iS_j)_{mn} = (S_i)_{m\ell}(S_j)_{\ell n}$ etc.

$$\varepsilon_{ijk}\varepsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

3D Schrodinger Eqn. $H\Psi(\vec{x}) = E\Psi(\vec{x})$ $\quad H = \frac{-\hbar^2\vec{\nabla}^2}{2m} + V(r)$

$$\vec{\nabla}^2 = \partial_r^2 + \frac{2}{r}\partial_r + \frac{\hat{L}^2}{r^2} \qquad \partial_r^2 + \frac{2}{r}\partial_r = \frac{1}{r^2}\partial_r(r^2\partial_r)$$

$$L^2 Y_\ell^m(\theta,\phi) = \hbar^2\ell(\ell+1)Y_\ell^m, \quad L_z Y_\ell^m = \hbar m Y_\ell^m$$

Spherical $V(r)$ symmetry then $\Psi(\vec{x}) = Y_\ell^m(\theta,\phi)R_n(r)$

let $R(r) = \frac{U(r)}{r}$ to simplify TISE to 1D

$\ell=0$ case only

$$\left[\frac{-\hbar^2}{2m}\frac{U''(r)}{\cancel{r}} + V(r)\frac{U(r)}{\cancel{r}} = \frac{EU(r)}{\cancel{r}}\right] \quad \text{DE}$$

Solve for $U(r)$ or $R(r)$ using B.C. $R(0) = 0$, $R(\infty)=0$ and derivative & R continuity.

from $$\left[\frac{-\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\ell(\ell+1)}{2mr^2}\hbar^2\right]R(r) = (E - V(r))R(r)$$

or $$U''(r) = \frac{-2m}{\hbar^2}(E - V(r))U(r) \qquad \rightarrow U'' + \omega^2 U = 0$$

or $$\frac{-\hbar^2}{2m}U''(r) = (E - V(r))U(r) \qquad U(r) = rR(r)$$

Perturbation theory (non-degenerate)

$$\overset{1}{E}_n = \langle \overset{0}{n} | V | \overset{0}{n} \rangle$$

$$| \overset{1}{n} \rangle = \frac{1}{\overset{0}{E}_n - H_0} \left(V - \overset{1}{E}_n \right) | \overset{0}{n} \rangle$$

$$\overset{2}{E}_n = \sum_{m \neq n} \frac{\left| \langle \overset{0}{n} | V | \overset{0}{m} \rangle \right|^2}{\overset{0}{E}_n - \overset{0}{E}_m}$$

Degenerate : $\overset{1}{E}_n = \lambda_n$ where λ_n are the eigenvalues of the interaction potential diagonalized in the basis of the unperturbed states.

$$\text{Take } V_{\substack{\text{matrix} \\ \text{elements}}} = \begin{array}{cc} \begin{pmatrix} \langle 0 | V | 0 \rangle & \langle 1 | V | 0 \rangle \\ \langle 0 | V | 1 \rangle & \langle 1 | V | 1 \rangle \end{pmatrix} & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \\ \begin{array}{cc} \langle 0 | \quad\quad & \langle 1 | \end{array} & \text{etc.} \end{array}$$

then diagonalize the matrix by $|V - I\lambda| = 0$ and solve for λ's. Then find the eigen vectors by

$$V \left(\Psi_n \right) = \lambda_n \left(\Psi_n \right)$$

- linearly independent eqns
- Orthonormalization of State

Or you can just guess a neutral operator which diagonalizes V for free, instead of using the original H_0 states.

Or to solve exactly reexpress hamiltonians in terms of L. Indipendent exactly solvable other H_0's.