Qualifying Examination - Part I

Time: 9:30-11:30 a.m. **Saturday, September 17, 2005**

The examination is to be written on the single sheets that are provided; no more than one question is to be answered on a single sheet. Number each question. You are to answer all questions in Part I; however, if you do omit any questions, ***cross out those numbers on your title page***. When you are finished, collect the answer sheets in order and place them together (with the title page on top) back in the envelope.

Place your code letter (from your title page) on the back of each sheet of paper.

Part I counts one-third (1/3) of the final grade.
Part II is in this same room at 1:00 p.m.

PHYSICAL CONSTANTS

Planck's constant	$h = 6.63 \times 10^{-34}$ J•s $= 4.14 \times 10^{-15}$ eV•s
Vacuum speed of light	$c = 3.00 \times 10^{8}$ m/sec
	$\hbar c = 197$ MeV•fm $= 1.97 \times 10^{-5}$ eV•cm
Electron charge	$e = 1.60 \times 10^{-19}$ C $= 4.80 \times 10^{-10}$ esu
Boltzmann constant	$k = 1.38 \times 10^{-23}$ J/K $= 8.62 \times 10^{-5}$ eV/K
Gas constant	$R = 8.31$ J/(mol•K)
Gravitational constant	$G = 6.67 \times 10^{-11}$ N•m^2/kg^2
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H/m
Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}$ F/m
Electron mass	$m_e = 9.11 \times 10^{-31}$ kg $= 0.511$ MeV/c ^{2}
Proton mass	$m_p = 1.67 \times 10^{-27}$ kg $= 938$ MeV/c ^{2}
Bohr radius of hydrogen	$a_B = 5.3 \times 10^{-11}$ m
Ionization energy of hydrogen	13.6 eV
Avogadro's number	6.02×10^{23} / mole

Conversion Factors

1 eV $= 1.6 \times 10^{-19}$ J $= 1.6 \times 10^{-12}$ erg

1 m $= 10^{10}$ Å$= 10^{15}$ fm $= 6.25 \times 10^{-4}$ miles

1 atm $= 1.01 \times 10^{5}$ N/m^2 $= 760$ Torr

1 cal $= 4.186$ J

Divergence and curl in spherical coordinates

$$\nabla \cdot E = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

$$\nabla \times E = r\frac{1}{r sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\varphi) - \frac{\partial E_\theta}{\partial\varphi}\right] + \theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi} - \frac{\partial(rE_\varphi)}{\partial r}\right] + \varphi\frac{1}{r}\left[\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta}\right]$$

where r, θ, φ are the unit vectors associated with the spherical coordinates r, θ, φ.

Useful integrals

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty x e^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2 e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

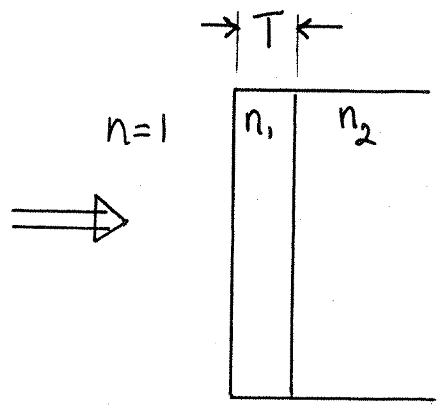
$$\int_0^x \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$\Sigma_{n=0}^{\infty} x^n = \frac{1}{(1-x)};\quad -\Sigma_{n=1}^{\infty}(-)^n\frac{x^n}{n} = \ell n\,(1+x)$$

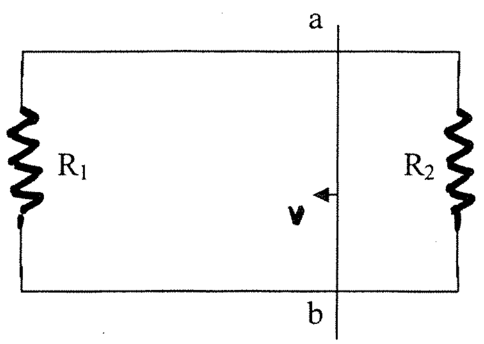
1. A block slides on a horizontal plane with known coefficient of friction μ. Find its initial speed if it stops after a distance d.

2. Light of vacuum wavelength λ=597 nm is normally incident on a flat glass plate which has on its front surface a coating with refractive index $n_1 = 1.30$. The refractive index of the glass is $n_2 = 1.40$.

   What is the thickness, T, of the thinnest coating layer that causes destructive interference in the reflected wave, minimizing reflection of the light?

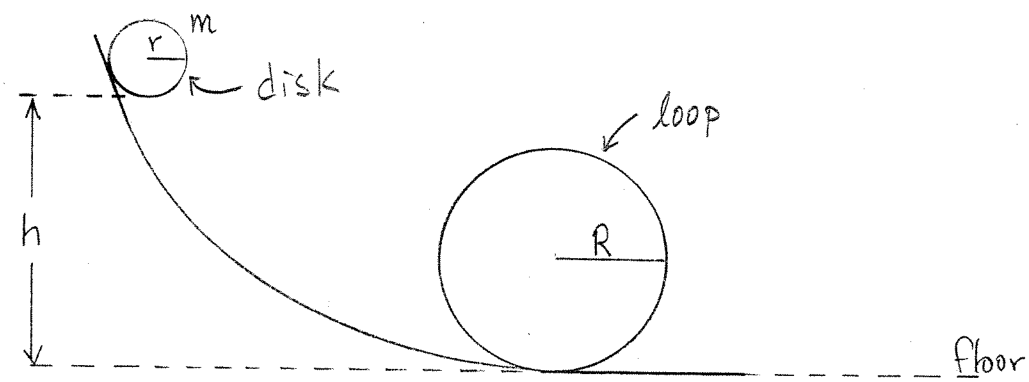


3. A conducting rod is free to slide along two parallel conducting rails 35.0 cm apart. Two resistors $R_1 = 2.00\ \Omega$ and $R_2 = 5.00\ \Omega$ are connected across the ends of the rails as shown in the sketch. A constant magnetic field of 1.50T perpendicular to the plane of the rails is directed into the page. An external force pulls the rod to the left at a constant speed of 6.5 m/s.



   a. Calculate the current through the rod.
   b. Calculate the total power dissipated by the resistors.
   c. Calculate the force needed to keep the rod moving with constant speed.

4. A disk of radius r (moment of inertia $I = \frac{1}{2}mr^2$) is released from rest at a point where its center of mass is a distance h higher than when the disk is on the floor. The disk then *rolls* without slipping down an inclined track. At floor level it rolls into a circular loop of radius R.



Find the minimum starting height h needed to ensure that the disk will remain in contact with the track at all times for the special case $r = \frac{1}{3}R$. Express the answer in terms of R.

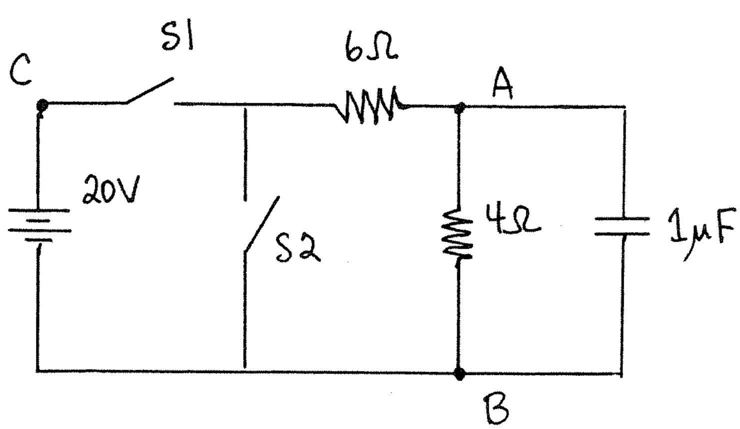
5. What is the ground-state energy of a system of ten noninteracting fermions, each with spin ½ and mass m, in a one-dimensional box of length L?

6. Initially, the capacitor in the circuit shown below is fully discharged.

a. At time t=0 the switch S1 is closed. What is the magnitude of the initial current at point C?

b. What is the final current at point C at long times (i.e., when the capacitor is fully charged)?

c. If S1 is now opened and S2 subsequently closed, how long will it take for the voltage between points A and B to reach 10 percent of its initial value?



7. A cylinder of volume 5.00 L contains oxygen at a temperature of 300.0 K and pressure of 2.00 atm. Assume that the oxygen behaves as an ideal gas with $C_p = 3.5$ R and $C_v = 2.5$ R.

The oxygen is taken through the cycle:

I. Heated at constant pressure from the initial state (state 1) to T = 500.0 K (state 2)
II. Cooled at constant volume to 250.0 K (state 3)
III. Cooled at constant pressure to 150.0 K (state 4)
IV. Heated at constant volume back to its initial state (state 1) at 300.0 K

a. Draw a clear sketch of this cycle on the pressure versus volume diagram.

b. Calculate the net work done by the oxygen.

c. What is the thermodynamic efficiency of this device as a heat engine?

8. A person with good vision can resolve two point sources if they are separated by a sufficient distance.

a. Suppose for now that our vision is limited by diffraction effects. If the two sources are the headlights of car, separated by 1.5 m, how close must the car be for us to resolve the two lights? [Hint: the Rayleigh criterion for circular apertures is $\theta_{min} = 1.22\ \lambda/D$.] Assume $\lambda = 500$ nm and a pupil diameter of 2 mm.

b. Given that the 20/20 letters on the Snellen eye chart subtend about 5 min of arc (1 cm at 6 m), would you expect to be visually limited by diffraction effects?

c. At what pupil diameter would you expect to be visually limited by diffraction effects?

9. A particle detector being designed at CERN will use a large solenoid, 3 m in radius and 12 m long. The solenoid will have 2000 turns and produce a 4 T magnetic field at its center. Assuming that the magnetic field is uniform within the entire volume,

a. find the total energy stored in the solenoid, and
b. the current in the windings.

10. In the H_2 molecule the two protons are separated by a distance d_0.

a. Write down the energy of the first excited rotational state in terms of d_0 and the proton mass.

b. In estimating the rotational specific heat of a dilute gas of hydrogen molecules, rotations around the internuclear axis are neglected. Explain why. (HINT: Estimate the excitation energy for such motions.)

Qualifying Examination - Part II

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11. Two particles, each of mass m, move in one dimension along the x axis, with positions x_1 and x_2. Both particles move in the same quadratic potential $\frac{1}{2}Kx^2$. In addition, the particles are attracted to each other by the potential $\frac{1}{2}L(x_1-x_2)^2$.

   a. Write the Lagrangian.
   b. Derive the equations of motion.
   c. Add and subtract these equations to get separated equations of motion.
   d. Describe the motion to be expected.
   e. Is energy conserved?
   f. Is momentum conserved?

12. An electron in the ground state (s-state) of hydrogen has wave function

$$\Psi = Ne^{-r/a}$$

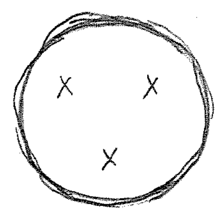
where N is a constant, r is the electron-proton separation, and a is the Bohr radius,

$$a = \frac{\hbar}{\alpha m_e c} \qquad \alpha = \frac{1}{137}.$$

   a. Find the root mean square velocity $\sqrt{\langle \vec{v} \cdot \vec{v} \rangle}$ of the electron, and compare to the speed of light.

   b. Find the probability that the electron is inside the proton, taking the proton radius to be $R = 10^{-15}$ m. Hint: Use a reasonable approximation in one of your integrals.

13. A circuit made of a metallic wire has 50 loops each of area 200 cm^2. The loop has a total resistance of $R = 5\Omega$. A magnetic field is applied perpendicular to its area and entering the page (see the plot). Its initial value is $B_0 = 5 \cdot 10^{-3}$ T, and it decreases linearly with time to 1/5 of its initial value in a time interval $t_0 = 0.2$ s.



   a. Calculate the induced emf.

   b. Find the induced current during the time interval t_0 and show its direction in the circuit.

   c. What total charge circulates past any given point in the wire during the time t_0 ?

   d. How much energy is dissipated by the induced current in the same time?

14. An LRC series circuit has an inductance L = 1.0 H, a resistance R = 1.2 kΩ, and a capacitance C = 71 nF.

a. Find the resonance frequency in Hz.

b. Find the Q of the circuit.

c. Find the sinusoidal, steady-state RMS voltage across the inductor with a sinusoidal input voltage of 1 Volt RMS on resonance.

15. A mass m moves under the influence of an attractive central potential

$$U(r) = -\frac{k}{r^{3/2}}.$$

a. Using polar coordinates r and φ, write down the Lagrangian, and show how to eliminate φ.

b. Write down the effective radial potential, and obtain a formula (in terms of the angular momentum) for the radius r_0 at which a circular orbit is possible.

c. Determine whether orbits that are close to satisfying the circular orbit condition will be stable (nearly circular) or unstable (some kind of spiral-like thing).

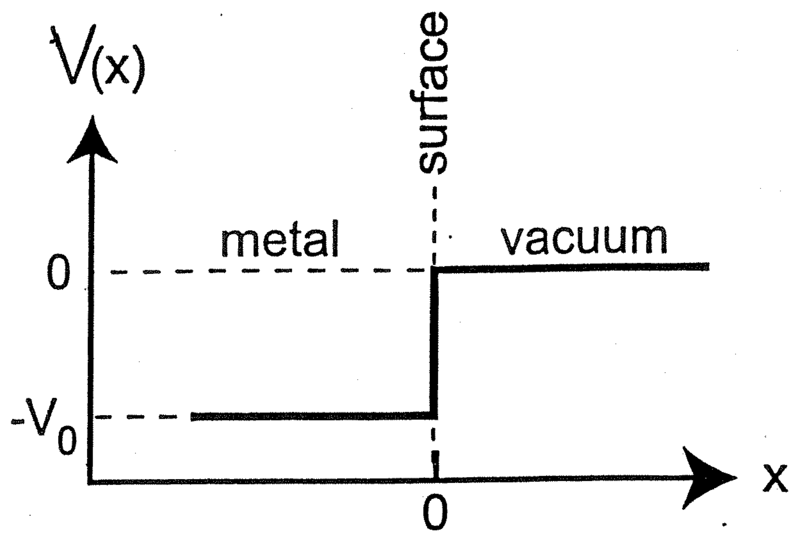
16. Quantum effects in ideal gases become important when the temperature is reducted to a value where the deBroglie wavelength of the particles becomes equal to the mean distance between them.

For a gas of particles of mass m and thermal energy $\frac{3}{2}k_B T$, find the temperature T_q for the deBroglie wavelength to be equal to the mean distance between particles, which is the cube root of the volume per particle, $(V/N)^{1/3}$.

Express your answer in terms of m, V/N, and fundamental constants.

17. Consider a two-electron atom in the (2p)(3d) electron configuration. Apply the LS-coupling to obtain all the levels (sometimes called "terms") within this configuration. For each level, list the values of all the quantum numbers that are appropriate for the LS-coupling. Express the values of these quantum numbers in the spectroscopic rotation.

18. Conduction electrons are confined in a metal by a potential barrier. Consider the one-dimensional model with potential given by $V(x) = -V_0$ for $x < 0$ and $V(x) = 0$ for $x \geq 0$ as shown in the figure.

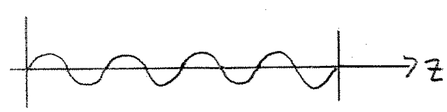


a. Calculate the quantum mechanical probability of transmission for a conduction electron approaching the surface at $x = 0$ from $x < 0$ with total energy $E > 0$.

b. For $E > 0$ write down the corresponding classical result for the transmission probability. In what limit do the quantum and classical results coincide?

c. Calculate the probability of transmission for $-V_0 < E < 0$.

19. An electromagnetic standing wave,



$$\vec{E}(\vec{r},t) = \hat{x}\ E_0 \ \sin kz \ \cos \omega t ,$$

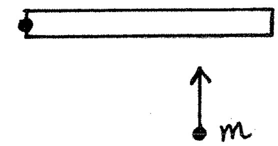
fills a region of empty space between two conducting plates. The region has volume V and the conductors are separated by an integral number of wavelengths.

a. Find a formula for the corresponding magnetic field.

b. Find the total electromagnetic energy, U, contained in the region between the plates, and show that U is independent of time.

20. Wave plates are made from birefrigent crystals cut to have orthogonal principal axes normal to the beam. For quartz, the refractive indices corresponding to the principal axes are $n_1 = 1.55$ and $n_2 = 1.54$ at a wavelength of $\lambda_0 = 600$ nm. Suppose you want to convert linear polarized light of $\lambda_0 = 600$ nm to circularly polarized light using a quartz wave plate. Determine the thickness of the wave plate, and indicate carefully how it should be oriented with respect to the direction of the linear polarization.

21. A uniform rod of mass M = 800 g and length L = 60 cm can rotate in a horizontal plane around a pivot O. The rod is initially at rest. A projectile of mass m = 50 g and negligible dimensions hits its edge at a distance x = 2/3 L with velocity v = 100 m/s perpendicular to the rod (see the figure). The projectile remains inside the rod.



a. Which of the following quantities are conserved in this collision:
1. the momentum;
2. the angular momentum;
3. the kinetic energy.

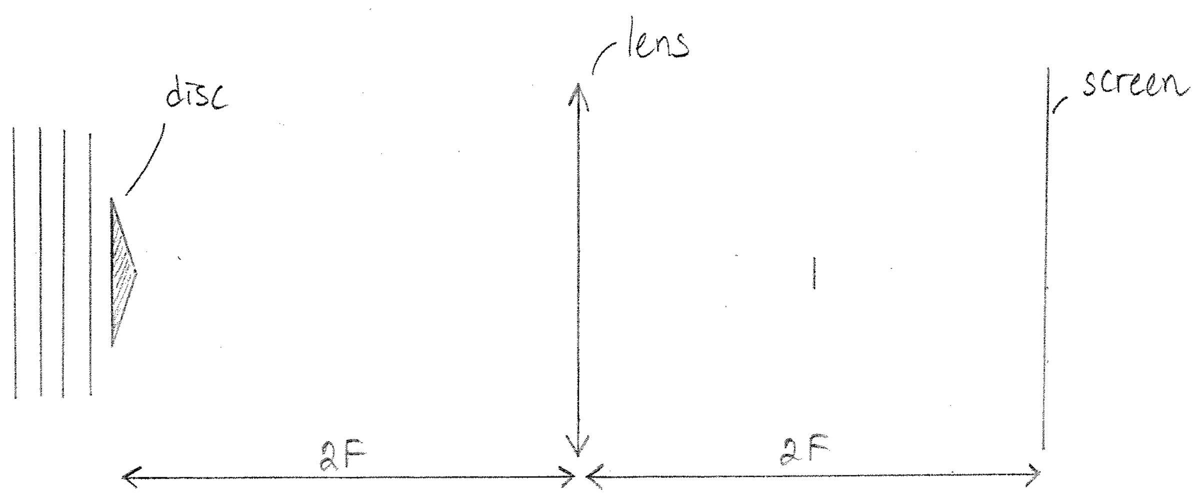
Calculate:

b. the momentum of inertia of the rod with respect to the rotation axis passing through O;

c. the momentum of inertia of the system (projectile + rod) after the collision with respect to the same rotation axis;

d. the angular velocity of the system (rod + projectile) after the collision.

22. A plane electromagnetic wave (intensity I_0, wavelength λ) illuminates a thin glass disc of diameter d and index of refraction n. The disc has a thickness t, which decreases linearly from t_0 at the center to zero at the edge, $t = t_0\,(1-2r/d)$. Light from the disc is imaged onto a screen (at unity magnification) by a lens of focal length F as shown in the drawing. A tiny opaque disk is placed in the focal plane of the regular lens in order to block out the plane wave. Plot the image intensity as a function of radius r, assuming that $nt_0 = 5\lambda$. You may neglect diffraction effects due to the lens.



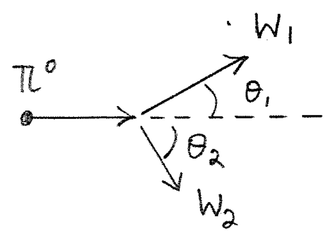
23. A one-dimensional monatomic ideal gas with N atoms at temperature T is confined to a line of length L. (An ideal gas has no collisions, so the atoms can move through each other.) Because the atoms are in motion they exert an effective (time averaged) force F on each end of the line.

a. Find F and show that F·L is related to the average energy of the atoms.

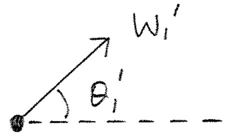
b. Use the equipartition theorem to find an expression for the heat capacity at constant length, C_L .

c. Find the equation of state (i.e., show how to write F·L in terms of the temperature T).

24. A $\pi°$ particle with <u>total</u> energy E = 400 MeV and rest mass m = 140 MeV/c^2 , moving along the z-axis, decays into two photons with energies W_1 and W_2 .



a. Complete the following diagram which shows the decay in the $\pi°$ rest frame. Show the direction θ_2' of photon #2, and give the photon energies of W_1' and W_2' .



b. Find the minimum and maximum photon energies in the lab frame.

c. For what values of θ_1' and θ_2' will the two photons have equal energies in the lab frame ($W_1 = W_2$)?

d. Find the angles θ_1 and θ_2 for this case.

25. Background: A one-dimensional solid can be represented as a very long chain of identical masses M connected by identical springs of constant K and unstretched lengths a.

The equation of motion for the nth mass is

$$M\frac{d^2x_n}{dt^2} = -K(x_n - x_{n-1}) - K(x_n - x_{n+1}) \ ,$$

where x_n is the deviation of mass n from its equilibrium position.

Phonons are wave-like excitations that travel along the chain, and one can show that the "wave"

$$x_n = A\cos(\omega t - kna) \ ,$$

solves the equations of motion provided that

$$\omega = 2\sqrt{\frac{K}{M}}\sin\frac{ka}{2} = \omega(k) \ .$$

a. Find expressions for the phase and group velocities of this wave.

b. Show that the waves become non-dispersive in the long-wavelength limit ($\lambda >> a$).