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DEPARTMENT OF PHYSICS

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DOCTORAL GENERAL EXAMINATION
PART 1
August 27, 2001

FIVE HOURS

1. This examination is divided into five sections, each consisting of four problems. Answer <u>all</u> the problems. Each problem is worth 5 points, thus the maximum score for the exam is 100.
2. Use a separate <u>fold</u> of paper for each problem. Write your name and the problem number (IV-3 for example) on each fold. A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a quantitative response.
3. Read the problem carefully and do not do more work than is necessary. For example "give" and "sketch" do not mean "derive".
4. Calculators may be used but are not necessary.
5. No books, notes or reference materials may be used.

Group I

1. Hamiltonian systems

A system is described by n coordinates q_i and their time derivatives $\dot{q}_i$. The Lagrangian for the system is $L(q_i, \dot{q}_i, t)$. There are no constraints.

Give a one (or at most two) sentence answer to each of the following questions.

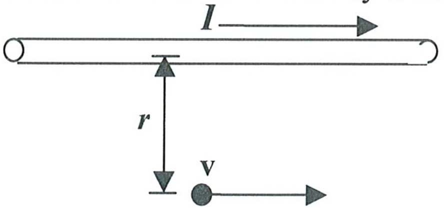
a. How many equations of motion are there? Give the equation for the ith coordinate.
b. Give the Hamiltonian describing the system and the momentum for each coordinate p_i.
c. Find dH/dt.
d. For each of the following transformations, give the conserved quantity if H is invariant under that transformation:
   I. $t' = t + \Delta$
   II. Rotation around the axis $(\hat{x} + \hat{y})/\sqrt{2}$.

2. Energy and potential from a wavefunction

The ground state (lowest energy state) eigenfunction of a particle of mass m is $\psi(x) = 1/\cosh(ax)$. Find $V(x)-E$ for this particle. What is the most logical assignment of $V(x)$ and E separately. Hint: It might help to sketch the potential.

3. Current carrying wire

An electron of mass m moves at velocity v parallel to a wire carrying current I. The electron is a distance r away from the wire. This is the frame F.



a. Find the force on the electron due to the current in the wire.
b. Find a frame F' in which there is no magnetic force on the electron. Find all forces on the electron in F'.

4. Fermi energy and chemical potential

A three dimensional electronic system has a Fermi energy ε_F.

a) How does the density of states $D(\varepsilon)$ depend on ε for a 3 dimensional, non-interacting gas?
b) Determine if the chemical potential rises above, falls below or stays fixed as the temperature is raised above a nonzero value. Explain your reasoning carefully.

Group II

1. Balloon in a train

You are sitting in a train car holding the string of a helium balloon. When the train car is at rest, the balloon points straight upward. Now the car begins to accelerate forward with uniform acceleration $a=g/10$, where g is the gravitational acceleration at the Earth's surface. What angle does the balloon make with the normal direction? Justify your answer carefully.

2. One dimensional relativistic gas

Find the entropy of a non-interacting one dimensional gas of ultra-relativistic $(E = pc)$ particles at temperature T confined to a one dimensional box of length L. If your result consistent with the third law of thermodynamics?

3. Expectation of x and p

If an operator O is time independent, then the derivative of it expectation value is given by

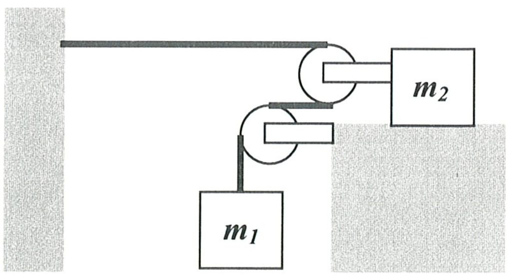
$$i\frac{d\langle O\rangle}{dt} = \frac{2\pi}{h}\langle[O,H]\rangle$$

where $<X>$ denotes the expectation value of X, H is the Hamiltonian of the system and [,] denotes commutation.

Consider a particle moving in a one-dimensional potential $U(x)$. Give expressions for the time derivatives of the expectation values of the position and momentum, $d<x>/dt$ and $d<p>/dt$.

4. Two masses, two pulleys and a rope

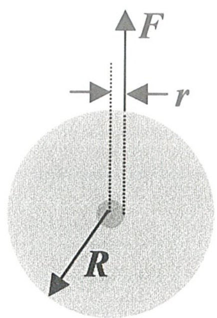
A block of mass m_1 is attached to a massless ideal rope. The rope goes around a massless pulley and then goes around a second massless pulley that is attached to a block of mass m_2 which is free to slide on a frictionless table. The other end of the rope is anchored to a wall. What is the acceleration of m_1 when the system is released?



Group III

1. Force on a yo-yo

A yo-yo consists of two disks with total mass m and radius R connected by a smaller massless cylinder of radius r. One end of a long string is fixed to the smaller cylinder and wound around the smaller cylinder several times. You pull with force F on the other end as you release the yo-yo from rest.



If sufficient force is applied to the yo-yo so that it accelerates neither up nor down, what will its angular acceleration be?

2. HD in equilibrium

A low density gas of HD (a molecule of hydrogen and deuterium) is contained at 300K in the presence of a catalyst which equilibrates the reaction

$$2HD \leftrightarrow H_2 + D_2$$

Given that the zero point energy of H_2 is 6000K, what will the final ratio of the concentration of HD to that of H_2?

3. Incoherent transmission through a dielectric slab

An electromagnetic wave of field strength E_o and wavelength λ is normally incident from vacuum on a dielectric slab of thickness l and dielectric constant n. Compute the field strength on the other side of the slab assuming the coherence length of the wave is much shorter than l.

4. Relativistic electron

The Lagrangian describing an electron (mass m_e) is $L = -\alpha\sqrt{1-v^2/c^2}$. Give the constant α.

Group IV

1. Relativistic collision

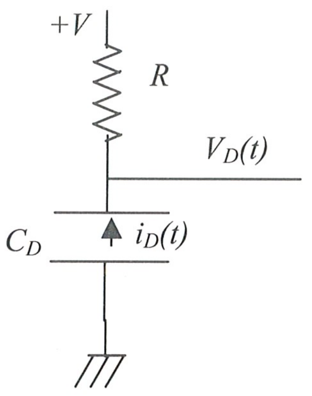
A proton of mass m_p and kinetic energy T collides via the Coulomb interaction with an electron (mass m_e) which is at rest. Use a transformation to the center of mass system. What is the maximum kinetic energy which the electron can have after the collision?

2. Detector bias

A charged particle passing through a detector makes a current pulse of the form

$$\begin{aligned} i_D(t) &= i_o \quad 0 < t < t_o \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Conceptually, the detector is just a capacitance C_D with current $i_D(t)$ flowing through t. In order to operate, the capacitor must have a bias voltage applied through a resistor R as shown. Find $V_D(t)$. Sketch assuming $RC_D << t_o$.



3. Bohr-Sommerfeld quantization

A particle of mass m is bouncing vertically on a hard surface in a uniform gravitational potential field. The surface is infinite in extent and lies on the $z=0$ plane. Use the Bohr-Sommerfeld quantization condition to find the energy eigenvalues. You may leave your answer in terms of a dimensionless integral.

4. A non-ideal gas

A theoretical model of a non-ideal gas gives the following expressions for the internal energy E and pressure P:

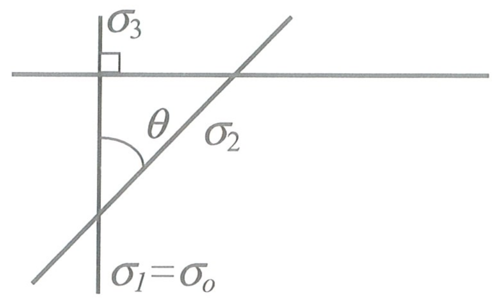
$$E(T,V) = aV^{-2/3} + bV^{2/3}T^2$$
$$P(T,V) = \frac{2}{3}aV^{-5/3} + \frac{2}{3}bV^{-1/3}T^2$$

Find an expression for the entropy of the gas, $S(T,V)$.

Group V

1. Charged sheets

Three charged sheets carrying surface charge σ_1, σ_2 and σ_3 are arranged as shown. Sheet 1 is known to have surface charge $\sigma_1 = \sigma_o$ and it is known there is no electric field in enclosed region shown on the figure. Find a solution for the surface charges on the other two sheets. Is your solution unique?



2. Phased array radar

A phased array radar makes a narrow beam by transmitting from a long row of N dipoles spaced a distance l apart. (a normal radar uses a single dipole in front of a parabolic antenna).

a) Find the beam width if all the dipoles transmit in phase. Take the array to be continuous of length Nl. The beam width is defined as half the angular distance between the first power minimum on each side of the central power maximum. You may assume θ is very small.
b) Find the angle of the second (diffraction) maximum. You do not need to solve the equation.
c) From your answer to b), estimate the beam power in the second maximum compared to the main beam.

3. Ground state energy

A particle moves in the one-dimensional potential $V(x) = ax^4$. $\psi(x) = \left(2\pi\sigma^2\right)^{-1/4} e^{-x^2/4\sigma^2}$ is a normalized wavefunction. It is not an energy eignefunction. Use $\psi(x)$ to find a bound on the ground state energy. Is it an upper or lower bound? Note: for this form of $\psi(x)$, $\langle x^{2n} \rangle = 1 \cdot 3 \cdot 5 \cdots (2n-1)\sigma^{2n}$.

4. Relativistic Astronauts

Astronaut A leaves the Earth at $t=0$, her watch synchronized with stay-at-home spouse E. A quickly accelerates to speed v$=3c/5$.

a) After one year has elapsed on Earth, E sends a radio transmission asking A to return to Earth to care for an ailing parent. In A frame, at what time was the transmission sent?
b) How far from Earth is A when she receives the transmission, measured in units of light-years in the Earth's rest frame?
c) As soon as she receives the transmission, A turns her ship around and heads back to Earth at $v=3c/5$. When she gets back, how much have A and E aged?