QM Random/preliminary list of important Comp topics/formula

9 Pauli Matrices (QM I and II midterm notes pg. 1)
3 Gaussian wave packets (2)
1 Schrodinger eqn (3)
2 + Heisenber eqn (4) + minimum uncertainty relation (Scratch notes pg. 2) + virial theorem (3)
4 Dirac delta (5)
7 Sudden approximation (QM comps pdf problem 3.b, QM I midterm problem 1)
5 Simple Harmonic oscillator (midterm notes pg. 6) $\langle x|n\rangle = \langle x|\frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$ since $a^{\dagger}|n\rangle = \sqrt{n+1}|n+$
6 Baker-Campbell-Hausdorf formula (6, 7)
10 ** Coherent States (7, + old comp Spring 2016 problem 2, QM I final review page 4 + 5,
8 Gaussian integrals (9)
15 Rotated Spinors (see midterm problem 3 from QM I + QM II midterm notes page 1 + 6)
11 ** Translation operator + Coherent States (" " problem 4) + *tight binding* (QM II midterm review pg 13)
4 12 * Bohm Aranov effect (QM I final review page 2)
1 missing 21 WKB method (3, QM Spring 2016 Comp problem 3)
13 I Landau levels (4)
16 II Spherical Tensors (QM II midterm review page 1, 12)
18 Clebsch-Gordon Coefficients (2, 9)
19 Operators review (3)
20 3D schrodinger equation (4, 17, QM Comp January 2016 Question 1)
22 Perturbation theory (5, final review pg 11)
17 Tensor tricks (1, 8, 12)
23 $[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$ proof (15)
24 Electric dipole potential (19)
25 Rotated $|lm\rangle$ states (20)
27 Time dependent perturbation theory (QM Final review pg. 1), applications (2, 10)
26 Interaction picture mechanics (Fall 2015 comp problem 1) + Rabi oscillations (QM Final review pg.
28 Variational principle (3)
29 Many particle Systems and Young Tableaux (3, 6)
30 ** Scattering theory + Born approximation (4, 8)
31 * Partial wave analysis (5, Fall 2014 problem 1 comp)
32 Trig identities (scratch notes page 1)
33 Operator identities (2)
34 Discrete symmetries + operators (3)
14 Spin Hamiltonians (QM II Final review pg 7)

1) Schrodinger Equation (stationary operators, time dependent states)

$U(t,0) = e^{-itH/\hbar}$, $i\hbar \frac{\partial}{\partial t} U(t) = H U(t)$, $|\alpha(t)\rangle = U(t)\,|\alpha(0)\rangle$

(TDSE) • $\therefore\ i\hbar \frac{\partial}{\partial t} |\alpha(t)\rangle = H|\alpha(t)\rangle$

For stationary states then we get $H|\alpha(t)\rangle = E|\alpha(t)\rangle$ as the eigenvalue equation

(TISE) • $\therefore \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)|\Psi(x)\rangle = E\,|\Psi(x)\rangle$

For non-stationary states we have their time evolution $|\alpha(t)\rangle = \sum_E e^{\frac{-itE}{\hbar}} C_E(0)\,|E\rangle$

2) Heisenberg Equation (time dependent operators, Stationary states)

$A_H(t) = U^\dagger(t)\, A_s\, U(t)$ and for non-time explicit dependent schrödinger operators we g

(HE) • $\frac{\partial}{\partial t} A_H = \frac{1}{i\hbar}[A_H, H]$ (from definition of $U(t)$)

Additionally $[x, F(p)] = i\hbar \frac{dF}{dp}$ & $[p, G(x)] = -i\hbar \frac{\partial G}{\partial x}$

Then, we have the minimum uncertainty relation

$\sigma_A^2 = (\Delta A)^2 = \langle\Psi|(A - \langle A\rangle)^2|\Psi\rangle$, $\langle A\rangle = \langle\Psi|A|\Psi\rangle$ applied to (HE)

• $\therefore\ \Delta A \Delta B = \sigma_A \sigma_B \geq \frac{1}{2}|\langle[A,B]\rangle|$

The virial theorem also follows from $\left\langle \frac{d(xp)}{dt}\right\rangle = \frac{1}{i\hbar}\langle[xp, H]\rangle = 0 = \langle x[p,H] + [x,H]p\rangle$

• for $H = \frac{p^2}{2m} + V(x)$ we get $\langle x \frac{\partial}{\partial x} V(x)\rangle = 2\left\langle \frac{p^2}{2m}\right\rangle \rightarrow \boxed{2\langle KE\rangle = \langle x\,\partial_x U\rangle}$

3) Gaussian Wavepackets (Saturate uncertainty principle) $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

$\alpha(x) = \frac{e^{\frac{-x^2}{2\sigma^2}} \cdot e^{\frac{ipx}{\hbar}}}{(2\sigma\sqrt{\pi})^{1/2}}$ $\bar{x} = \langle\alpha|\hat{x}|\alpha\rangle = 0$, $\bar{p} = \langle\alpha|\hat{p}|\alpha\rangle = p$

$\Delta x = \sigma/\sqrt{2}$, $\Delta p = \hbar/\sigma\sqrt{2}$ $\therefore \Delta x \Delta p = \hbar/2$ & Saturate.

4) Dirac Delta Function

for $H = \frac{p^2}{2m} + \beta\delta(x) = -\frac{\hbar^2}{2m}\nabla^2 + \beta\delta(x)$ we have $-\frac{\hbar^2}{2m}\phi''(x) + \beta\delta(x)\phi(x) = E\phi(x)$

Solve the (rest of the potential) eigen vectors the usual way ignoring the $\delta(x)$, but then integrate

$\int_{0+}^{0+} dx\left[-\frac{\hbar^2}{2m}\phi''(x) + \beta\delta(x)\phi(x) = E\phi(x)\right] \rightarrow -\frac{\hbar^2}{2m}\{\phi'(x)\}\Big|_{0+}^{0-} + \underbrace{\beta\int_{0-}^{0+}\delta(x)\phi(x)\,dx}_{=\beta\phi(x_0)} = \underbrace{E(\phi(0+)-\phi}_{=0}$

• leaving just $\frac{\hbar^2}{2m}\{\phi'(x)\}\Big|_{x_0 \text{ boundary left} \rightarrow \text{right}} = \beta\phi(x_0)$

Provides energy quantization for single bound state $\phi(x) \propto e^{-C|x|}$ & $E = -\frac{\beta^2 m}{2\hbar}$.

5) Simple Harmonic Oscillator

$H = \frac{P^2}{2m} + \frac{1}{2} m\omega^2 x^2$ let $a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega} P\right) = X + iP$ & $a^\dagger = (a)^\dagger$ obviously — $x = x_0 \cdot X$ & $x_0^2 = \frac{\hbar}{m\omega}$

$\therefore \quad x = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a)$, $P = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger - a)$

$[x, P] = i\hbar$, $[a, a^\dagger] = 1$ & $[a, f(a^\dagger)] = \frac{\partial f(a^\dagger)}{\partial a^\dagger}$ & $[a^\dagger, f(a)] = -\frac{\partial f(a)}{\partial a}$

Eigenvalue equations: $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$, $|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$

$H = \hbar\omega\left[a^\dagger a + \frac{1}{2}\right]$ with $a^\dagger a = N$ number operator, $[N, a^\dagger] = a^\dagger$

SHO Ground state is gaussian $\langle x|0\rangle = \left(\frac{1}{2\sqrt{\pi}}\right)^{1/2} \frac{1}{\sqrt{x_0}}\, e^{-\frac{x^2}{2x_0^2}}$ $\qquad \phi_n(x) = \frac{H_n(x/x_0)\, e^{-\frac{x^2}{2x_0^2}}}{\sqrt{2^n n!}\,\sqrt{x_0}\,\pi^{1/4}}$

6) BCH formula $e^A e^B = e^B e^A e^{[A,B]}$ ← $\mathbb{C}$ else series continues.

$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} = e^B e^A e^{+\frac{1}{2}[A,B]}$

7) Sudden Approximation

$$P_{i\to f} = |\langle f|i\rangle|^2 = \Big|\int_{\text{overlap}} d\alpha\, \langle f|\alpha\rangle\langle\alpha|i\rangle\Big|^2 = \Big|\int_{\text{overlap}} d\alpha\, \phi_i(\alpha)\,\phi_f^*(\alpha)\Big|^2$$

8) Gaussian Integrals

$\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$ & $\int_{-\infty}^{\infty} dx\, x^{2n} e^{-\alpha x^2} = \frac{\partial^n}{\partial\alpha^n}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\pi}\,(-1)^n \frac{d^n}{d\alpha^n}\left(\alpha^{-1/2}\right)$

$\int_0^{\infty} dx\, x^{2n} e^{-x^2/a^2} = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$ & $\int_0^{\infty} dx\, x^{2n+1} e^{-\frac{x^2}{a^2}} = \frac{n!}{2}\, a^{2n+2}$

$\int_0^{\infty} x^n e^{-x/a}\, dx = n!\, a^{n+1}$

9) Pauli Matrices

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_n = \vec{\sigma}\cdot\hat{n} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$ $\qquad \sigma_i^\dagger = \sigma_i$, $\sigma_i^2 = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_i\sigma_j = \delta_{ij} + i\,\varepsilon_{ijk}\,\sigma_k$, $\sigma_i\sigma^i = 3$

Spin matrices → $\vec{S} = s\hbar\,\vec{\sigma}$, $S_\pm = S_x \pm iS_y$, $S_x = \frac{1}{2}(S_+ + S_-)$, $S_y = \frac{1}{2i}(S_+ - S_-)$

↳ further → (18) Clebsch-Gordon $\quad S_\pm = \delta_{m', m\pm 1}\sqrt{s(s+1) - mm'}$

10) Coherent States

$$a|\lambda\rangle = \lambda|\lambda\rangle \;, \quad |\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} e^{\lambda a^\dagger}|0\rangle$$
$$= e^{-\frac{|\lambda|^2}{2}} \cdot \sum_{n=0}^{\infty} \frac{\lambda^n a^{\dagger n}}{n!}|0\rangle = e^{-\frac{|\lambda|^2}{2}} \cdot \sum_{n=0}^{\infty} \frac{\lambda^n}{\sqrt{n!}}|n\rangle$$

11) Translated SHO Ground State is a Coherent State

$$T_a|0\rangle = e^{-\frac{iaP}{\hbar}}|0\rangle = e^{-\frac{ia}{\hbar}\cdot i\sqrt{\frac{m\omega\hbar}{2}}(a^\dagger - a)}|0\rangle = e^{a\sqrt{\frac{m\omega}{2\hbar}}(a^\dagger - a)}|0\rangle = e^{La^\dagger - La}|0\rangle$$
$$= e^{La^\dagger}e^{-La}e^{-\frac{1}{2}[La^\dagger, -La]}|0\rangle = e^{La^\dagger}e^{-La}e^{-\frac{1}{2}|L|^2}|0\rangle = e^{-\frac{1}{2}L^2}e^{La^\dagger}|0\rangle$$
✓ $= |L\rangle$ Coherent st

12) Bohm-Aranov Effect

Path Integral $\mathbb{K}(x',t';x,t) = \int_{i\to f}^{\text{paths}} [dx]\, e^{iS/\hbar}\, P_{\Phi}[x]$ classical action Path dependent phase

$P_{\Phi_{i,j}}[x] = \exp[\frac{ie}{\hbar c}\int_{i,j} \vec{A}\cdot d\vec{x}$

The $\Psi_i \approx \Psi_0 \cdot P_{\Phi_i}$ and $\Psi_{F_{i,j}} = \Psi_i + \Psi_j$

$$|\Psi_{F_{ij}}/\Psi_0|^2 = |P_{\Phi_i} + P_{\Phi_j}|^2 = \left|1 + \exp\left(\frac{ie}{\hbar c}\left[\int_i \vec{A}\cdot d\vec{x} - \int_j \vec{A}\cdot d\vec{x}\right]\right)\right|^2$$
$$= 2 + 2\cos\left[\frac{e}{\hbar c}\oint \vec{A}\cdot d\vec{x}\right] = 4\cos^2\left[\frac{\gamma_{ij}}{2}\right]$$ if $i+j$ make a closed l

13) Landau Levels

A particle is in a magnetic field $B\hat{z}$, find its energy.

$H = \frac{\vec{\pi}^2}{2m}\ (\pm\vec{\mu}\cdot\vec{B})$ + we can use $\vec{\pi} = \vec{p} - \frac{e}{c}\vec{A}$ with a chosen gauge

let $\vec{B} = \vec{\nabla}\times\vec{A}$ with $\vec{A} = xB\hat{y}$ s.t. $\vec{B} = B\hat{z}$, then $\pi_y = p_y - \frac{e}{c}xB$

Separating $\Psi(x,y,z) = \phi(x)\cdot\frac{1}{(2\pi)^{1/2}}e^{ik_y y}\cdot\frac{1}{(2\pi)^{1/2}}e^{ik_z z}$ we get

$$H\phi(x) = E\phi(x) = \left(\frac{p_x^2}{2m} + \frac{1}{2m}\left(\cancel{\hbar\partial_y}\cdot\hbar k_y - \frac{e}{c}xB\right)^2 + \frac{\hbar^2 k_z^2}{2m} \pm \vec{\mu}\cdot\vec{B}\right)\phi(x)$$

letting $\frac{e^2B^2}{2mc^2} = \frac{1}{2}m\omega^2$ we see that $\frac{\hbar k_y c}{eB} = \frac{\hbar k_y}{m\omega} = \bar{X}$ s.t.

$$E\phi(x) = \Big(\underbrace{p_x^2/2m + \tfrac{1}{2}m\omega^2(x-\bar{x})^2}_{SHO} + \frac{\hbar^2 k_z^2}{2m} \pm \vec{\mu}\cdot\vec{B}\Big)\phi(x)$$

$E_x = \hbar\omega(n+1/2)$ + $E_z = \frac{\hbar^2 k_z^2}{2m}$ + $E_\mu = \pm\mu B$

shifted SHO in x axis.

then if this is fixed in a box of side length L

$0 < \bar{X} < L$, $0 < \frac{\hbar k_y}{m\omega} < L$, $\omega = eB/mc$ ∴ $0 < \frac{c\hbar k_y}{eB} < L$

but k_y is stuck to fit inside L too → $k_y = \frac{2\pi}{L}\cdot n_y$ Sinusoidal ∞ we

∴ $0 < n_y < eBL^2/2\pi\hbar c = \frac{e}{hc}BL^2 = \frac{\Phi}{\Phi_0}$ degeneracy determines node count

14) Spin Hamiltonians

$(\vec{J}_1 + \vec{J}_2)^2 = J_1^2 + J_2^2 + 2\vec{J}_1\cdot\vec{J}_2$ ∴ $\vec{J}_1\cdot\vec{J}_2 = \frac{1}{2}\left\{(\overbrace{J_1+J_2}^{J_{tot}})^2 - J_1^2 - J_2^2\right\}$ + $J \to L \to S$ all w

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15) Rotated Spinors

- $D_{\hat{n}}(\theta) = e^{-i\vec{\sigma}\cdot\theta\hat{n}} = \mathbb{1}\cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\sigma_{\hat{n}}$, $\sigma_{\hat{n}} = \vec{\sigma}\cdot\hat{n} = \begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}$
- $|+\rangle_{\hat{n}} = D_z(\gamma)D_y(\beta)|+\rangle_z = e^{-i\gamma/2}\cos(\beta/2)\begin{pmatrix}1\\0\end{pmatrix} + e^{i\gamma/2}\sin(\beta/2)\begin{pmatrix}0\\1\end{pmatrix}$ $\quad (|+\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$ etc.)
- $-i\sigma_z\cdot\hat{K}$ is the $\hat{n}$ parity operator for $P|+\rangle_{\hat{n}} = |-\rangle_{\hat{n}}$

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16) Spherical Tensors / Tensors

$$T_{ij} = \frac{1}{2}\left(T_{ij}+T_{ji}-\frac{2}{3}\delta_{ij}T\right) + \frac{1}{2}\left(T_{ij}-T_{ji}\right) + \frac{1}{3}\delta_{ij}T$$ symm traceless, antisym, trace

$$T_q^k = Y_m^l\cdot\text{Const} = \sum_{1,2}\langle k_1 q_1\, k_2 q_2 | k\, k_1 k_2\, q\rangle T_{q_1}^{k_1}T_{q_2}^{k_2}$$ for $-k \le q \le k$

$$k = k_1\otimes k_2 = |k_1-k_2|\oplus\cdots\oplus|k_1+k_2|$$ & $q = q_1+q_2$

17) Tensor Tricks

$\sum_i T_{ii}$ is a scalar? $\rightarrow \sum_i T_{ii} = \sum_{ijk} R_{ij}R_{ik}T_{jk} = \sum_{jk}\delta_{jk}T_{jk} = \sum_j T_{jj}$ $\boxed{QED}$

$\sum_{jk}\varepsilon_{ijk}T_{jk}$ is a vector $\rightarrow \sum_{jk}\varepsilon_{ijk}T_{jk} = A\sum_{jk}\varepsilon_{ijk}a_j b_k = A(\vec{a}\times\vec{b})_i$ $\boxed{QED}$

18) Clebsch-Gordon Coefficients

$\vec{J} = \vec{J}_1+\vec{J}_2 \rightarrow |j,m,j_1,j_2\rangle = \sum C_i |j_1,m_1\rangle\otimes|j_2,m_2\rangle$ where the available m & j values are subject to $m=m_1+m_2$, $|m_i| < j_i$

$\sum|C_i|^2 = 1$ normalization and $|j_1-j_2| \le j_{tot} \le |j_1+j_2|$

$$\vec{J}^2 = \vec{J}_1^2+\vec{J}_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

where $\vec{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$

$J_z|j,m\rangle = \hbar m|j,m\rangle$

$J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\,|j,m\pm1\rangle$

act these on the selection rule permitted combinations of states, and then use state orthonormality to determine C

19) Operator Review – Symmetry operator $U(\vec{a}) = e^{-\frac{i\vec{a}\cdot\vec{M}}{\hbar}}$ where $\vec{M}$ generates a cts. symmetry

Translation: $T(\vec{x}) = e^{-\frac{i\vec{x}\cdot\vec{P}}{\hbar}}$ $\quad \vec{P} = \frac{\hbar}{i}\frac{\partial}{\partial\vec{x}}$

Rotation: $D_{\hat{n}}(\theta) = e^{-\frac{i\theta\hat{n}\cdot\vec{J}}{\hbar}}$ $\quad \vec{J} = \hbar\vec{\sigma}_{3\times3}$ or $\vec{J} = \vec{S} = \hbar\cdot s\cdot\vec{\sigma}_{basis}$ for non $j=1$ spinors $\quad J_z = \frac{\hbar}{i}\frac{\partial}{\partial z}$

Time evolution: $U(t) = e^{-\frac{itH}{\hbar}}$ $\quad \hat{H}$ = Hamiltonian

Discrete operators

Parity $\quad \Pi = \Pi^{-1} = \Pi^\dagger$, $\Pi|\vec{x}\rangle = |-\vec{x}\rangle = \lambda|\vec{x}\rangle$ with $\lambda = -1$ for $\vec{x}$

$$\Pi|l\,m\rangle = (-1)^l|l\,m\rangle$$

Time reversal $\quad \Theta = -i\sigma_y\,\delta\,\vec{K}$ $\rightarrow$ spin 1/2, $\vec{K}$ is right acting antiunitary complex conjugator + δ is a phase

Translation $\quad T_{\vec{a}} = e^{-\frac{i\vec{a}\cdot\vec{K}}{\hbar}}$ + for bloch states $T_{\vec{a}}|\vec{x}\rangle = |\vec{x}+\vec{a}\rangle = \lambda_{\vec{a}}|\vec{x}\rangle$, $\lambda_{\vec{a}} = e^{-\frac{i\vec{a}}{\hbar}}$

S or J or L $\rightarrow (L_i)_{jk} = -i\hbar\varepsilon_{ijk}$, $\quad (L_iL_j)_{mn} = (L_i)_{ml}(L_j)_{ln}$ etc.

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20) 3D Schrödinger Equation $-$ $H\Psi(\vec{x}) = E\Psi(\vec{x})$ & $H = -\frac{\hbar^2 \vec{\nabla}^2}{2m} + V(r)$ $\rightarrow \Psi(\vec{x}) = Y_\ell^m(\theta,\phi)\cdot R_n$

$\vec{\nabla}^2 = \partial_r^2 + \frac{2}{r}\partial_r + \frac{L^2}{r^2}$ & $\partial_r^2 + \frac{2}{r}\partial_r = \frac{1}{r^2}\partial_r(r^2\partial_r)$

$L^2 Y_\ell^m(\theta,\phi) = \hbar^2 \ell(\ell+1) Y_\ell^m(\theta,\phi)$ & $L_z Y_\ell^m(\theta,\phi) = \hbar m Y_\ell^m(\theta,\phi)$

$\therefore \left(-\frac{\hbar^2}{2m}\left(\partial_r^2 + \frac{2}{r}\partial_r\right) + \frac{\ell(\ell+1)\hbar^2}{2mr^2}\right)R(r) = (E - V(r))R(r)$ let $R(r) = \frac{u(r)}{r}$

$-\frac{\hbar^2}{2m}u'' + V(r)u + \frac{\ell(\ell+1)\hbar^2}{2mr^2}u = Eu$

for $\ell = 0$ then we get $u'' + \frac{2m}{\hbar^2}(E - V(r))u = 0$ $\omega^2 = \frac{2m}{\hbar^2}(E - V(r))$

and $R(r)$ must be finite everywhere, so $u(r) \rightarrow 0$ at origin boundary condition

21) WKB Method double soft wall, $-1/4$ for each hard wall

$\oint p\,dx = h(n + \frac{1}{2})$ where $p = \sqrt{2m(E-V)}$ & turning points at $V(x_0) = E$

wave function $\propto e^{i\int p\,dx}/\sqrt{p}$

22) Perturbation Theory

$\overset{1}{E}_n = \langle \overset{\circ}{n}|V|\overset{\circ}{n}\rangle$, $|\overset{1}{n}\rangle = \frac{1}{\overset{\circ}{E}_n - H_0}\left(V - \overset{1}{E}_n\right)|\overset{\circ}{n}\rangle$, $\overset{2}{E}_n = \sum_{m \neq n} \frac{|\langle \overset{\circ}{n}|V|\overset{\circ}{m}\rangle|^2}{\overset{\circ}{E}_n - \overset{\circ}{E}_m}$

Degenerate perturbation theory

$\overset{1}{E}_n = \{\lambda_n\}$ where λ_n are the eigen values of $(V)_{nm} = \begin{array}{cc} & \\ \left(\begin{array}{cc} \langle 0|V|0\rangle & \langle 1|V|0\rangle \\ \langle 0|V|1\rangle & \langle 1|V|1\rangle \end{array}\right) & \begin{array}{c} |0\rangle \\ |1\rangle \end{array} \\ \begin{array}{cc} \langle 0| & \langle 1| \end{array} & \end{array}$

and use this eigenvalue equation to find the good $|\overset{\circ}{n}\rangle$ eigen states $\rightarrow$ or just guess a diagonalizing operator

23) $[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$ proof : $[L_i, L_j] = L_iL_j - L_jL_i \Rightarrow [L_i, L_j]_{mn} = (L_i)_{m\ell}(L_j)_{\ell n} - (L_j)_{m\ell}(L_i)_{\ell n}$

$= -\hbar^2[\delta_{in}\delta_{mj} - \cancel{\delta_{ij}\delta_{mn}} - (\delta_{jn}\delta_{mi} - \cancel{\delta_{ji}\delta_{mn}})] \leftarrow = -\hbar^2[\epsilon_{im\ell}\epsilon_{j\ell n} - \epsilon_{jm\ell}\epsilon_{i\ell n}]$

$= +\hbar^2[\delta_{jn}\delta_{im} - \delta_{in}\delta_{jm}] = \hbar^2\epsilon_{kji}\epsilon_{knm} = i\hbar\epsilon_{ijk}(-i\hbar\epsilon_{kmn})$

$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$ $\boxed{\text{QED}}$

24) Electric dipole potential

$H = \frac{p^2}{2m} + (-\vec{p}\cdot\vec{E})$ where $\vec{p} = d\vec{S}$ & $\vec{E} = -\vec{\nabla}V(r)$ so $H = \underset{\uparrow}{\frac{p^2}{2m}} + \underset{\uparrow}{d\vec{S}\cdot\vec{\nabla}V}$ scalar: rotation invariant

T even, P even   T odd, P odd

25) Rotated $|j\,m\rangle$ States

Rotate $|j,m\rangle$ by θ about $\hat{n}$ $\rightarrow$ $|j,m\rangle_\theta = D(R_{\hat{n}}(\theta))|j,m\rangle = e^{-i\frac{\theta\vec{J}\cdot\hat{n}}{\hbar}}|j,m\rangle$

and evaluate in small θ limit by expanding $e^{-x} = 1 - x + \frac{x^2}{2} + \ldots\ldots$

kets, see 27)

$$H = H_0 + V(t)$$

26) Interaction Picture Mechanics $\rightarrow$ $\bar{V}_I = e^{iH_0 t/\hbar} V(t) e^{-iH_0 t/\hbar}$ where $\bar{V}$ is the time-perturbati

$(TDSE)_I$ $\quad i\hbar \partial_t |\psi(t)\rangle_I = V_I |\psi(t)\rangle_I$ $\rightarrow$ hit with $\langle n|$ & $\sum_m |m\rangle\langle m|$ filter to get

$i\hbar \partial_t C_n(t) = \sum_m e^{i\omega_{nm}t} V_{nm} C_m(t)$ with $\omega_{nm} = \frac{E_n - E_m}{\hbar}$

$\hookrightarrow$ matrix form $\quad i\hbar \partial_t \begin{pmatrix} C_0(t) \\ C_1(t) \end{pmatrix} = \begin{pmatrix} V_{00} & V_{01} e^{i\omega_{01}t} \\ V_{10} e^{i\omega_{10}t} & V_{11} \end{pmatrix} \begin{pmatrix} C_0(t) \\ C_1(t) \end{pmatrix}$

$\rightarrow$ Solution may require C's $\rightarrow$ a+b's change of variables for ease

27) Time Dependent Perturbation Theory

$|\alpha, t\rangle_I = e^{iH_0 t/\hbar} |\alpha, t\rangle_S = \sum_n C_n(t) |n\rangle$ expressible in energy eigenkets

expand $C_n(t) = C_n^{(0)} + C_n^{(1)} + C_n^{(2)} + \cdots$ functions of t

- $C_n^{(0)} = \delta_{ni}$ where i is the energy state index of the initial condition {set}
- $C_n^{(1)} = \frac{-i}{\hbar} \int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')\, dt' = \frac{-i}{\hbar} \int_{t_0}^{t} \langle n | V_I(t') | i \rangle\, dt'$ $\qquad \omega_{ni} = \frac{E_n - E_i}{\hbar}$
- $C_n^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{nm}t'} V_{nm}(t') e^{i\omega_{mi}t''} V(t'')$

when taking expectation values be sure not to mix pictures.

Transition probability $\quad P_{i\rightarrow n}(t) \simeq |C_n^{(1)}(t)|^2 = |{}_I\langle n | i \rangle_I|^2$

Fermi's Golden Rule $\quad \omega_{i\rightarrow n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \rho(E_n \simeq E_i)$ $\quad$ density of states $\sim \frac{d\Omega}{dE}$ multiplicity

Rabi Oscillations $\rightarrow$ $H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \gamma \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$

$P_{1\rightarrow 2}(t) = \frac{\gamma^2}{\gamma^2 + \Delta^2} \sin^2\left(\frac{\sqrt{\gamma^2 + \Delta^2}}{2\hbar} t\right)$ & $\Delta = \hbar/2 (\omega - \omega_{21})$ & $\omega_{21} = \frac{E_2 - E_1}{\hbar}$

EM Harmonic Interaction

$\bar{\omega}_{i\rightarrow f} = \frac{2\pi}{\hbar} \left(\frac{e\omega A_0}{c}\right)^2 |\langle i | \hat{\varepsilon} \cdot \vec{x} | f \rangle|^2$ $\quad \rightarrow H = \frac{\Pi^2}{2m}$ & $\Pi = p - \frac{e}{c}\vec{A}$ & use linear in $\vec{A}$ term, not A^2 te

$\sigma_{i\rightarrow f} = 4\pi^2 \alpha \hbar\omega |x_{if}|^2 \delta(E_f - E_i \pm \hbar\omega)$

28) Variational Principle

$E(\alpha) \geq E$ reality

$E(\alpha) = \dfrac{\int d\vec{x}\, \phi^*(\alpha, \vec{x})\, H\, \phi(\alpha, \vec{x})}{\int d\vec{x}\, \phi^*(\alpha, \vec{x})\, \phi(\alpha, \vec{x})}$ then minimize for $\left.\frac{\partial E(\alpha)}{\partial \alpha}\right|_{\alpha_0} = 0$ fixes α_0 & minimu E lower limit on g

29) Many particle systems

Bosons fall into symmetric states, and fermions are overall antisymmetric (including spin states)

$\psi_{sym}(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\phi_\alpha(x_1) \phi_\beta(x_2) + \phi_\alpha(x_2) \phi_\beta(x_1) \right)$

$\psi_{anti}(x_1, x_2) = \frac{1}{\sqrt{2}} \left(\phi_\alpha(x_1) \phi_\beta(x_2) - \phi_\alpha(x_2) \phi_\beta(x_1) \right)$

for Bosons we use symmetric spin states trivially, and symmetric spatial wavefunction

for Fermions we use opposite spin-state-symmetry state to the spatial wavefunctions symmetry-sta

ex) Singlet is anti-symmetric and tripplet states are symmetric.

ex) if $\alpha = \beta$ then $\psi_{Fermion} = 0$ if it is in the tripplet state

$\hookrightarrow$ this is the Pauli-exclusion principle. $\quad \rightarrow$ (Young Tableaux ...

$\phi_k(\vec{x}) = \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}$ plane wave $\qquad G^+(\vec{x}-\vec{x}') = \frac{-1}{4\pi}\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}$

30) Scattering Theory and the Born approximation

Scattering Process : $\Psi^+(\vec{x}) = \phi_k(\vec{x}) + \frac{1}{E-H_0+i\varepsilon}V\,\Psi^+(\vec{x}) = \phi_k(\vec{x}) + \frac{2m}{\hbar^2}\cdot\frac{-1}{4\pi}\int d^3\vec{x}'\,\frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,V(\vec{x}')\,\Psi^+(\vec{x})$

local approximation : $\Psi^+(\vec{x}) = \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{1}{r}e^{ikr}f(\vec{k},\vec{k}')\right]$

$f(\vec{k},\vec{k}') = \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right)\int d^3\vec{x}'\,e^{-i\vec{k}'\cdot\vec{x}'}(2\pi)^{3/2}\,V(\vec{x}')\,\Psi^+(\vec{x}')$

differential cross-section : $\frac{d\sigma}{d\Omega} = |f(\vec{k},\vec{k}')|^2 \rightarrow \sigma_{tot} = \int\frac{d\sigma}{d\Omega}d\Omega \qquad d\Omega = -d\phi\,d(\cos\theta) \rightarrow 4\pi$

Born approximation : $\Psi^+(\vec{x})^{(0)} = \frac{1}{(2\pi)^{3/2}}e^{i\vec{k}\cdot\vec{x}}$ 0^{th} order in $f(\vec{k},\vec{k}')$

$f_B(\vec{k},\vec{k}') = \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right)\int d^3\vec{x}'\,e^{i\vec{q}\cdot\vec{x}'}\,V(\vec{x}') \rightarrow$ fourier transform of interaction with respect to $\vec{q} = \vec{k}-\vec{k}'$

Optical Theorem : $\sigma_{tot} = \frac{4\pi}{k}\,\mathrm{Im}\{f(\vec{k},\vec{k}'=\vec{k})\}$

31) Partial Wave Analysis

$f(k\hat{z},\vec{k}') = \sum_\ell (2\ell+1)\,f_\ell(E)\,P_\ell(\cos\theta)$

$f_\ell = -\frac{\pi}{k}T_\ell(E) \rightarrow$ transfer matrix elements $\quad f_\ell = \frac{e^{i\delta_\ell}\sin(\delta_\ell)}{k}$ phase shift from wavefunction matching boundary condition.

$f_\ell \propto k^{2\ell}$ for low energy scattering off very localized potentials.

32) Trig Identities

$\sin(A\pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

$\cos(A\pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$, $\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\sin^2(x) + \cos^2(x) = 1 \rightarrow \tan^2(x) + 1 = \sec^2(x) \rightarrow 1^2(x) + \cot^2(x) = \csc^2(x)$

$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$

$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$

$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

33) Operator Identities

$[x,p] = i\hbar$, $[x_i,x_j] = 0$, $[p_i,p_j] = 0$, $[L_i,L_j] = i\hbar\,\varepsilon_{ijk}L_k$

$[p,x^n] = n\,x^{n-1}[p,x]$, $[E,t] = i\hbar$, $[L^2,L_i] = 0$

$J_\pm = J_x \pm iJ_y$, $[J_z,J_\pm] = \pm\hbar J_\pm$, $[J_+,J_-] = 2\hbar J_z$

$[a,a^\dagger] = 1$, $N = a^\dagger a$, $[N,a^\dagger] = a^\dagger$, $[N,a] = -a$

0

Operator change of basis $A' = P A P^{-1}$ ($P^{-1} = P^{\dagger}$)

34)

	Operator	Parity	Time
	$\vec{x}$ position	$-$	$+$
	$\vec{v}$ velocity	$-$	$-$
	$\vec{a}$ acceleration	$-$	$+$
polar vectors	$\vec{p}$ momentum	$-$	$-$
	$\vec{F}$ force	$-$	$+$
	$\vec{j}$ current	$-$	$-$
	$\vec{E}$ field	$-$	$+$
	$\vec{D}$ field	$-$	$+$
	$\vec{P}$ polarization	$-$	$+$
	$\vec{A}$ vector potential	$-$	$-$
	$\vec{S}$ poynting vector	$-$	$-$
	t time	$+$	$-$
Scalars	m mass	$+$	$+$
	E energy	$+$	$+$
	P power	$+$	$-$
	ρ charge density	$+$	$+$
	V potential (electric)	$+$	$+$
	ρ energy density	$+$	$+$
Axial vectors	$\vec{L}$ angular momentum	$+$	$-$
	$\vec{B}$ field	$+$	$-$
	$\vec{H}$ field	$+$	$-$
	M magnetization	$+$	$-$
	$\overleftrightarrow{T}_{ij}$ stress tensor	$+$	$+$

Rotational invariance

- Scalars are rotationally invariant, but vectors are not.
- Also must not contain any non Y_0^0 spherical tensors

$$x^2 + y^2 + z^2 = r^2 \cdot \underbrace{Y_0^0}_{=1}$$

Real spherical harmonics

$$Y_{00} = s = Y_0^0 = \frac{1}{2}\sqrt{\frac{1}{\pi}} = \sqrt{\frac{1}{4\pi}}$$

$$Y_{1,-1} = p_y = i\sqrt{\frac{1}{2}}\left(Y_1^{-1} + Y_1^{1}\right) = \sqrt{\frac{3}{4\pi}}$$

$$Y_{1,0} = p_z = Y_1^0 = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$

$$Y_{1,1} = p_x = \sqrt{\frac{1}{2}}\left(Y_1^{-1} - Y_1^{1}\right) = \sqrt{\frac{3}{4\pi}}\,\frac{x}{r}$$

etc $\rightarrow$ $\ell \propto$ order of r^{-n} degree of trig funct

$m \rightarrow$ degree of azimuthal change enacted by op.

Scalar products are rotationally invariant

$+$ = even, = commutes

$-$ = odd, = anticommutes

$\Pi\, x\, \Pi^{\dagger} = -x$ $\quad \{\Pi, x\} = 0 \quad \Pi x = -x\Pi$

$\Pi\, L\, \Pi^{\dagger} = L$ $\quad [\Pi, L] = 0 \quad \Pi L = L\Pi$

Pseudoscalar	$\vec{L} \cdot \vec{x}$	$-$	$-$

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2) Virial theorem $\quad 2\langle KE \rangle = \langle x \frac{dU}{dx} \rangle \rightarrow \left\langle \frac{d[XP]}{dt} \right\rangle = \frac{-i}{\hbar}\langle [XP, H] \rangle$

$x\, \partial_x V - 2KE =$

then $[XP, H] = X[P, H] + [X, H]P$