3. $H = -J \sum_{\langle ij \rangle} \delta_{\eta_i \eta_j}$ $\quad J > 0.$

let $\beta J = K$

a) $Z_1 = \sum_i e^{+\beta J \sum_{\langle ij \rangle} \delta_{\eta_i \eta_j}} = \prod_{\langle i,j \rangle} \sum_i e^{\overbrace{\beta J}\delta_{\eta_i \eta_j}}$

b) $Z_1 = \mathrm{Tr}\begin{pmatrix} e^{K} & e^{0} \\ e^{0} & e^{K} \end{pmatrix}^{N \cdot q} = \mathrm{Tr}\begin{pmatrix} e^{k} & 1 \\ 1 & e^{k} \end{pmatrix}^{Nq}$

$\eta_i \neq \eta_j$ $\quad \eta_i = \eta_j = \pm 1$

Turn it into the Ising model chain problem with known solution!

c) $Z_1 = \mathrm{Tr}\begin{pmatrix} e^{k} & 1 \\ 1 & e^{k} \end{pmatrix}^{Nq} = \left(e^{k/2}\right)^{Nq} \cdot \mathrm{Tr}\begin{pmatrix} e^{k/2} & e^{-k/2} \\ e^{-k/2} & e^{k/2} \end{pmatrix}^{Nq}$

$$\boxed{Z_1 = e^{Nqk/2} \cdot \cosh^{Nq}(k) \cdot \left(1 + \tanh^{Nq}(k)\right)}$$

d) $\frac{F}{N} = -\frac{1}{N\beta} \ln(Z)$ as $N \to \infty$, $\therefore \tanh^{\infty}(k) = 0$ since $\tanh < 1$

$= -\frac{k_B T}{N} \cdot Nq \ln\left[e^{k/2} \cosh(k)\right]$

$= -k_B T q \left[J\beta/2 + \ln(\cosh(k))\right] = \boxed{F/N = -\frac{Jq}{2} - \frac{q}{\beta}\ln(\cosh(J\beta))}$

$\frac{E}{N} = \frac{-1}{N}\frac{\partial}{\partial\beta}\ln(Z) = \frac{-1}{N}\frac{\partial}{\partial\beta}\left[\frac{Nq}{2}J\beta + qN\ln\cosh(J\beta)\right]$

$= -\frac{qJ}{2} - qJ\frac{\sinh(J\beta)}{\cosh(J\beta)} = \boxed{E/N = \frac{-Jq}{2} - qJ\tanh(J\beta)}$

$$\frac{d}{dx}\tanh(x) = 1 - \tanh^2(x) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{+1}{\cosh^2 x}$$

e) $$C_V = \frac{dE}{dT} = \frac{d}{dT}\left(-\frac{Jq}{2} - Jq\tanh\left(\frac{J}{k_B T}\right)\right)$$

per site

$$= 0 - Jq \cdot \cancel{\frac{J/k_B \cdot -1/T^2}{\cosh^2(J/k_B T)}} = \boxed{\frac{J^2 q}{k_B T^2}\operatorname{sech}^2\left(\frac{J}{k_B T}\right) = C_V}$$

all per site → $$S = \frac{E-F}{T} = -\frac{Jq}{2T} - \frac{Jq}{T}\tanh(J\beta) - \left(-\frac{Jq}{2T} - \frac{q k_B T}{T}\ln(\cosh(J\beta))\right)$$

$$= \cancel{-\frac{Jq}{T}\tanh(J\beta) + q k_B \ln(\cosh(J\beta))}$$

f) Calculate E, C, + S in $T \to 0$ & ∞ limits. Physically + quantitatively explain results in both limits.

$$E(T\to 0) = -\frac{Jq}{2} - Jq\tanh(\infty) = -Jq(3/2) = \cancel{\boxed{-\frac{9J}{2} = E(T\to 0)}}$$

$q = 3$

$\tanh = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ ∴ $\tanh(\infty) = \frac{e^x}{e^x} = 1$

$$E(T\to\infty) = -\frac{Jq}{2} - 0$$

& $x \to 0$ ∴ $\tanh(0) = 0$

$$\therefore \cancel{\boxed{E(T\to\infty) = -\frac{3J}{2}}}$$

So at $T \to \infty$ the states are maximally misaligned, and at $T = 0$ they are maximally aligned together.

then $C_V(T\to 0) = \frac{J^2 q}{k_B T^2} \text{Sech}^2(J/k_B T) \Rightarrow \boxed{0 = C_V(T=0)}$

$$\left(\frac{1}{e^{\infty} + e^{-\infty}}\right)^2 = \left(\frac{1}{\infty}\right)^2 = 0$$

$C_V(T\to\infty) = \frac{J^2 q}{k_B T^2} \text{Sech}^2(J/k_B \infty) = \boxed{0 = C_V(T=\infty)}$

$$\left(\frac{1}{e^{x} + e^{-x}}\right)^2 \simeq \left(\frac{1}{1+x+\frac{x^2}{2}+1-x+\frac{x^2}{2}}\right)^2 = \frac{1}{4}$$

So the heat capacity flatlines at 0 for both $T = 0$ + ∞ limits, which we expect for $T\to 0$ due to the third law, and for $T\to\infty$ because the energy is already maximally disbersed.

Then $S(T\to 0) = \frac{E(T\to 0)}{(T\to 0)} - \frac{F(T\to 0)}{(T\to 0)} = -\frac{Jq}{T}\tanh(J/k_B T) + \frac{q}{T\beta}\ln(\text{Cosh}(J/k_B T))$

$\ln(\text{Cosh}(J/k_B T)) \to e^{J/k_B T}$ for small T

$$= -\frac{Jq}{T}\cdot 1 + \frac{q}{T}k_B T \cdot \frac{J}{k_B T}$$

$$= -\frac{Jq}{T} + \frac{Jq}{T} = \boxed{0 = S(T\to 0)}$$

Which is what we expect, as at $T\to 0$ the randomness decreases as everything falls into a single ground state.

$\frac{e^0 + e^{-0}}{2} = 1$

$\tanh(0) = 0$ $\qquad$ $\cosh(0) = 1$

and lastly $S(T \to \infty) = -\frac{Jq}{T} \tanh(J/k_B T) + \frac{q}{T} T k_B \ln(\cosh(J/k_B T))$

$\ln(1) = 0$

$\therefore S(T \to \infty) = 0$ also, meaning the ring settles into exactly the most ordered state possible, where all spins are anti-aligned.

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g) Symmetry breaking phase transition at finite T?

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No, because all the thermodynamic functions derived are continuous and differentiable. OK