General Instructions: This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas.

Each solution should typically take less than 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

Electromagnetism 1

Estimte the correction to the static potential between two opposite charges q and $-q$ separated by a distance d induced by placing the charges in the center of a cavity of size $R >> d$; assume that the electric field does not penetrate the cavity walls.

SOLUTION:

In general

$$\delta V(d) = \frac{q^2}{4\pi} \frac{d^2}{R^3}$$

Solution:

$$V(d) = \frac{1}{4\pi} \int d^3 r \, \mathbf{E_1}(\mathbf{r}) \mathbf{E_1}(\mathbf{d} + \mathbf{r})$$

At large distances $r \gg d$ the electric field of the charges is the field of a dipole. The correction is induced by the modification of the dipole field at large distances; since the electric field does not penetrate the walls of the cavity, the following piece has to be subtracted from the potential:

$$\delta V(d) \sim q^2 d^2 \int_R^\infty \frac{d^3 r}{r^6} \sim q^2 \frac{d^2}{R^3}.$$

Electromagnetism 2

Find the temporal Green's function of a medium whose complex dielectric constant is the following function of frequency:

$$\epsilon(\omega) = \epsilon_0 + \frac{nq^2}{m} \frac{1}{(\omega_0^2 - \omega^2) - 2i\omega\delta}$$

with low damping $\delta < \omega$.

SOLUTION:

$$G(t) = \frac{nq^2}{m\omega_0} \exp^{(-\delta t)} \sin(\omega_0 t)$$

Electromagnetism 3

Consider a relativistic point particle with electric charge $e < 0$ and mass m, coupled to time- and space-dependent electric and magnetic fields.

a. (6 pts) Write down the Lagrangian. Construct the conjugate momenta, and the Hamiltonian as a function of p and q.

b. (8 pts) Evaluate dH/dt. Derive the Lorentz force.

c. (6 pts) According to the Noether theorem, if an action $\int L dt$ has a continuous symmetry with a constant parameter α, there is a conserved charge Q. What is the general form of this Noether charge Q if the variation of L is a total time derivative, $\delta(\alpha)L = dK/dt$? If the electric and magnetic fields do not depend on time, derive the Noether charge for time translations.

SOLUTION:

(a.) The Lagrangian is

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{e}{c}A_\mu(\vec{x}, t)\frac{dx^\mu}{dt} = \ldots + \frac{e}{c}\vec{A}\cdot\vec{v} - e\phi \tag{1}$$

With this we find the conjugate momentum

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}\vec{A} \tag{2}$$

and the Hamiltonian

$$H = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}\vec{A} \cdot \frac{d\vec{x}}{dt} - L \tag{3}$$

$$= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} + e\phi(\vec{x}, t) \tag{4}$$

$$= \sqrt{c^2(\vec{p} - \frac{e}{c}\vec{A})^2 + m^2c^4} + e\phi(\vec{x}, t) \tag{5}$$

(b.) One always has

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{6}$$

So

$$\frac{dH}{dt} = e\frac{\partial \phi}{\partial t} - \frac{e}{c}\vec{v} \cdot \frac{\partial \vec{A}}{\partial t} \tag{7}$$

The Euler-Lagrange equations yield

$$\frac{d}{dt}\left(\frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} + \frac{e}{c}\vec{A}\right) - \frac{e}{c}\left(\vec{\nabla}A_\mu\right) \cdot \frac{dx^\mu}{dt} = 0 \tag{8}$$

which one can write in components as

$$\frac{d\Pi_i}{dt} = -\frac{d}{dt}\left(\frac{e}{c}A_i\right) + \frac{e}{c}\left(\partial_i\vec{A}\right) \cdot \frac{d\vec{x}}{dt} - e\partial_i\phi \tag{9}$$

$$= e\left(-\frac{1}{c}\frac{\partial A_i}{\partial t} - \partial_i\phi\right) + \frac{e}{c}\left(\partial_i A_j - \partial_j A_i\right)\frac{dx^j}{dt} \tag{10}$$

$$= eE_i + \frac{e}{c}(\vec{v} \times \vec{B})_i \tag{11}$$

where we have introduced

$$\vec{\Pi} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{12}$$

(c.) Start with the general formula for a Noether charge

$$\alpha Q = \frac{\partial L}{\partial \dot{q}}\delta q - K \tag{13}$$

For time translations $\delta q = \dot{q}\Delta t$. So then $\alpha = \Delta t$ and then

$$\delta L = \Delta t \frac{d}{dt}L \tag{14}$$

So $K = L\Delta t$ and $\alpha Q = p\dot{q}\Delta t - L\Delta t = H\Delta t$. Hence $Q = H$.

Classical Mechanics 1

We are going to study some canonical transformations generated by the generating functional $F_2(q, P, t)$ where q denotes the old coordinates, P the new conjugate momenta (and t the time).

a. (5 pts) Show that $F_2 = qP$ generates the identity transformation. Show that the Hamiltonian H generates the time evolution of the canonical coordinates.

b. (5 pts) Consider a free point particle in one dimension with mass m in an inertial frame, with Hamiltonian H, coordinate x, and also in a uniformly accelerated frame with Hamiltonian K and with coordinate $X = x - 1/2gt^2$, where g is the acceleration of the accelerated frame with respect to the inertial frame. Write down the Lagrangian in the accelerated frame, and derive from it the Hamiltonian K in the accelerated frame. Next construct $F_2(x, P, t)$ for the canonical transformation from the inertial frame to the accelerated frame, and express K in terms of the Hamiltonian H and F_2. Do both expressions for K agree? Is K conserved?

c. (5 pts) Now consider a free point particle in a two-dimensional Cartesian inertial frame with coordinates x, y and a rotating frame with Cartesian coordinates X, Y. The rotating frame has variable angular velocity $(\omega(t))$. Construct F_2 for the canonical transformation from the inertial frame to the rotating frame. Construct the Hamiltonian K in the rotating frame. Is K conserved? Is K equal the to the energy in the inertial frame?

d. (5 pts) Derive the Hamiltonian equations of motion in the rotating frame. Using this result, obtain the Lagrangian equation of motion in the rotating frame. What is the meaning of the various terms in the latter equation of motion?

SOLUTION:

(a.) Knowing that $F_2 = qP$, we have

$$p \equiv \frac{\partial F_2}{\partial q} = P \qquad Q \equiv \frac{\partial F_2}{\partial P} = q \tag{15}$$

Hence $F_2 = qP$ yields the identity transformation.
Now, using

$$F_2(q,\, P,\, \Delta t) = qP + H(q,\, P,\, t)\Delta t \tag{16}$$

yields

$$p = \frac{\partial F_2}{\partial q} = P + \frac{\partial H}{\partial q}\Delta t \qquad Q = \frac{\partial F_2}{\partial P} = q + \frac{\partial H}{\partial P}\Delta t \tag{17}$$

So then

$$P - p \equiv \delta p = -\frac{\partial H}{\partial q}\Delta t = \dot{p}\Delta t$$

$$Q - q \equiv \delta q = \frac{\partial H}{\partial p}\Delta t = \dot{q}\Delta t$$

(18)

and this yields the time evolution.

(b.) Denote the Lagrangian in the inertial frame by L and in the accelerated frame by $\bar{L}$. Then

$$L = \frac{1}{2}m\dot{x}^2 \qquad \bar{L} = \frac{1}{2}m\left[\frac{d}{dt}\left(X + \frac{1}{2}gt^2\right)\right]^2 = \frac{1}{2}m\left(\dot{X} + gt\right)^2 \qquad (19)$$

Then the Hamiltonian in the accelerated frame follows from

$$K = P\dot{X} - \bar{L} \quad ; \quad P \equiv \frac{\partial \bar{L}}{\partial \dot{X}} = m\left(\dot{X} + gt\right) = m\dot{x} \quad \Longrightarrow \quad P = p \qquad (20)$$

So

$$K = \frac{1}{2m}P^2 - Pgt \qquad (21)$$

For the canonical transformation, we have

$$\frac{\partial}{\partial P}\left[F_2(x,\,P,\,t)\right] \equiv X = x - \frac{1}{2}gt^2 \quad \Longrightarrow \quad F_2 = P\left(x - \frac{1}{2}gt^2\right) \qquad (22)$$

Then $K = H + \dfrac{\partial F_2}{\partial t}$ yields

$$K = \frac{1}{2m}p^2 + \frac{\partial F_2}{\partial t} = \frac{1}{2m}P^2 - Pgt \qquad (23)$$

This agrees with (21). Since

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} = -Pg \qquad (24)$$

K is not conserved.

(c.) The inertial coordinates $(x,\,y)$ and the rotating coordinates $(X,\,Y)$ are related via

$$X = x\cos\left(\omega(t)\right) + y\sin\left(\omega(t)\right) = \frac{\partial F_2}{\partial P_x}$$

$$Y = -x\sin\left(\omega(t)\right) + y\cos\left(\omega(t)\right) = \frac{\partial F_2}{\partial P_y}$$

(25)

Since this is a point transformation, the momenta transform as vectors

$$
\begin{aligned}
p_x &= P_x \cos\left(\omega(t)\right) - P_y \sin\left(\omega(t)\right) = \frac{\partial F_2}{\partial x} \\
p_y &= P_x \sin\left(\omega(t)\right) + P_y \cos\left(\omega(t)\right) = \frac{\partial F_2}{\partial y}
\end{aligned}
\tag{26}
$$

Hence

$$
F_2 = \left[x \cos\left(\omega(t)\right) + y \sin\left(\omega(t)\right)\right] P_x + \left[-x \sin\left(\omega(t)\right) + y \cos\left(\omega(t)\right)\right] P_y \tag{27}
$$

The Hamiltonian in the rotating frame is

$$
K = H + \frac{\partial F_2}{\partial t} \tag{28}
$$

Using

$$
p_x^2 + p_y^2 = P_x^2 + P_y^2 \tag{29}
$$

one gets

$$
\begin{aligned}
K &= \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \dot{\omega}\left[-x \sin\left(\omega(t)\right) + y \cos\left(\omega(t)\right)\right] P_x \\
&\quad - \dot{\omega}\left[x \cos\left(\omega(t)\right) + y \sin\left(\omega(t)\right)\right] P_y \\
&= \frac{1}{2m}\left(P_x^2 + P_y^2\right) + \dot{\omega}\left[Y P_x - X P_y\right]
\end{aligned}
\tag{30}
$$

K is conserved if (and only if) $\dot{\omega}$ is constant, but since $Y P_x - X P_y = y p_x - x p_y$, K differs from the total energy in the inertial frame by $-\dot{\omega} l_z$ where l_z is the angular momentum.

(d.) The Hamilton equations are

$$
\begin{aligned}
\frac{1}{m} P_x + \dot{\omega} Y &= \dot{X} \quad \text{and} \quad \dot{\omega} P_y = \dot{P}_x \\
\frac{1}{m} P_y - \dot{\omega} X &= \dot{Y} \quad \text{and} \quad \dot{\omega} P_x = -\dot{P}_y
\end{aligned}
\tag{31}
$$

Then

$$
\begin{aligned}
m\ddot{X} &= m\dot{\omega}^2 X + 2m\dot{\omega}\dot{Y} + m\ddot{\omega} Y \\
m\ddot{Y} &= m\dot{\omega}^2 Y - 2m\dot{\omega}\dot{X} - m\ddot{\omega} X
\end{aligned}
\tag{32}
$$

We recognize the centrifugal force, the Coriolis force, and a fictitious force due to angular acceleration.

Classical Mechanics 2

Consider an inverted pendulum. It consists of a massless rigid bar of length ℓ and a bob at the end of mass m. The equilibrium position at the top of the arc ($\theta = 0$) is unstable, but if the bottom of the pendulum is moved up and down along the z-axis by a harmonic forcing function with amplitude A and angular frequency ω, the motion of the inverted pendulum for small θ becomes stable provided ω is large enough.

a. (4 pts) Derive the Lagrangian for this forced system. Show that the equation of motion for θ can be written as

$$\theta'' + [a + q\cos z]\theta = 0$$

with $\theta' = d\theta/dz$, $z = \omega t$, $a = -g/(\ell\omega^2)$ and $q = A/\ell$.

b. (6 pts) If $\ddot{x} + \omega^2 x = 0$ and $\omega(t)$ is periodic, $\omega(t+T) = \omega(t)$, show that there exist solutions $x_1(t)$ and $x_2(t)$ satisfying $x_1(t+T) = \lambda_1 x_1(t)$; $x_2(t+T) = \lambda_2 x_2(t)$; $\lambda_1\lambda_2 = 1$. Show that if λ_1 is complex, the motion is bounded, but if λ_1 is real it is unbounded. Why does this imply that when the motion of the inverted pendulum becomes unstable, there exists a solution which is either periodic or antiperiodic with period T?

c. (6 pts) It is clear that if A gets small, the critical value of ω must become large. Hence if q gets small, the critical value of a must also become small. This suggests to expand a in a power series in q. Hence we set

$$\begin{aligned} a &= a(q) = q_0 + a_1 q + a_2 q^2 + ... \\ \theta(z) &= \theta(q,z) = \theta_0(z) + \theta_1(z)q + \theta_2(z)q^2 \end{aligned}$$

Write down the terms in the equation of motion which are of order q_0, q_1 and q^2. Assume $\theta(q,z)$ is periodic. Show that $\theta_0 = 1$ and $a_0 = 0$. Then show that periodicity of $\theta_1(z)$ requires that $a_1 = 0$, and that periodicity of $\theta_2(z)$ requires that $a_2 = -1/2$. Finally, find the critical frequency for which stability is reached when $A = 1$ cm and $\ell = 10$ cm.

d. (4 pts) When ω is increased from its critical value to even larger values, eventually a value is reached where instability sets in. How should this value of ω be determined?

SOLUTION:

(a.)
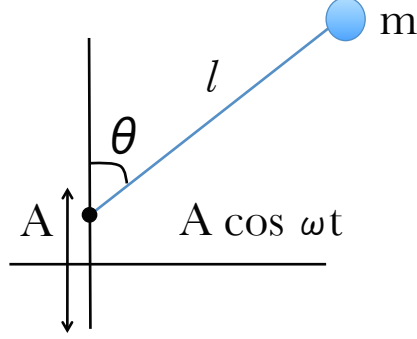
One introduces

$$x = l \sin(\theta) \qquad z = l \cos(\theta) + A \cos(\omega t) \qquad T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{z}^2 \qquad V = mgz \quad (33)$$

Substituting x and z in T and V gives

$$T = \frac{1}{2}ml^2\dot{\theta}^2 - mlA\omega\left(\frac{d\cos(\theta)}{dt}\right)\sin(\omega t) + \theta\text{-independent terms}$$

$$= \frac{1}{2}ml^2\dot{\theta}^2 + mlA\omega^2 \cos(\omega t)\cos(\theta) + \dots \tag{34}$$

$$V = mgl\cos(\theta)$$

(In the second step we integrated by parts; the total derivative does not contribute to the equation of motion.) From here one can get the equations of motion. Expanding around $\theta = 0$ yields the equation of motion in the desired form.

(b.) Write the system of equations

$$x_1(t+T) = ax_1(t) + bx_2(t) \tag{35}$$
$$x_2(t+T) = cx_1(t) + dx_2(t) \tag{36}$$

in matrix form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t+T) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) \tag{37}$$
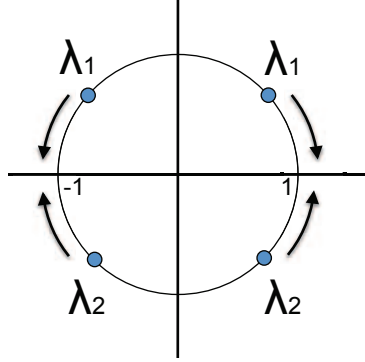
Diagonalizing gives $x_j(t+T) = \lambda_j x_j(t)$ with $j = 1, 2$. Using $\ddot{x}_j + \omega^2 x_j = 0$ we find that

$$x_1\ddot{x}_2 - x_2\ddot{x}_1 = 0 \quad x_1\dot{x}_2 - x_2\dot{x}_1 = c \tag{38}$$

But since we also have

$$(x_1\dot{x}_2 - x_2\dot{x}_1)(t+T) = \lambda_1\lambda_2(x_1\dot{x}_2 - x_2\dot{x}_1)(t) \tag{39}$$

8

we get $\lambda_1 \lambda_2 = 1$. If λ_1 is complex, $\lambda_2 = \lambda_1^*$, so λ_j are phases and we have bounded motion. If λ_1 is real but $\lambda_1 \neq \lambda_2$, then λ_1 or λ_2 is larger than one, so we have unbounded motion.

When λ_1 and λ_2 come together, the pendulum becomes unstable. Then $\lambda_1 = \lambda_2 = 1$ (periodic motion) or $\lambda_1 = \lambda_2 = -1$ (antiperiodic motion).

(c.) We have

$$\ddot{\theta}_0 + a_0 \theta_0 = 0 \tag{40}$$

$$\ddot{\theta}_1 + a_1 \theta_0 + a_0 \theta_1 + \cos(\tau)\theta_0 = 0 \tag{41}$$

$$\ddot{\theta}_2 + a_2 \theta_0 + a_1 \theta_1 + a_0 \theta_2 + \cos(\tau)\theta_1 = 0 \tag{42}$$

If there is a periodic or antiperiodic solution with period 2π, we have $\theta_0 = \cos\left(\frac{n}{2}\tau\right)$ or $\sin\left(\frac{n}{2}\tau\right)$. Begin with $n = 0$. Then from (40) it follows that $\theta_0 = 1$ and $a_0 = 0$. Equation (41) leads to

$$\ddot{\theta}_1 + a_1 + \cos(\tau) = 0 \quad \implies \quad \theta_1 = A + B\tau - \frac{1}{2}a_1\tau^2 + \cos(\tau) \tag{43}$$

But periodicity requires $\theta_1 = A + \cos(\tau)$. Then (42) yields

$$\ddot{\theta}_2 + a_2 + \cos(\tau)(A + \cos(\tau)) = 0 \tag{44}$$

For periodic solutions the constant term must vanish:

$$a_2 + \frac{1}{2} = 0 \quad \left(\text{use} \quad 2\cos^2(\tau) = 1 + \cos(2\tau)\right) \tag{45}$$

So then

$$a = -\frac{1}{2}q^2 \quad , \quad \text{or} \quad \omega_{\text{crit}}^2 = \frac{2gl}{A^2} \tag{46}$$

Then the critical frequency is

$$\nu_{\text{crit}} = \frac{1}{2\pi}\sqrt{\frac{2gl}{A^2}} = 22 \text{ Hz} \tag{47}$$

(d.) Start with $n = 1$, so $\theta_0 = \cos\left(\frac{1}{2}\tau\right)$ or $\sin\left(\frac{1}{2}\tau\right)$.

Classical Mechanics 3

We study some aspects of chaos theory and nonlinear dynamics.

a. (3 pts) Give a definition of chaos (not more than a few lines). Give an example of a system in classical mechanics that exhibits chaos.

b. (8 pts) Consider the tent map, defined by $x_{n+1} = f(x_n)$ where $f(x) = rx$ if $0 \leq x \leq 1/2$ and $f(x) = r(1 - x)$ if $1/2 \leq x \leq 1$. The parameter r lies between 0 and 2. Find the fixed points of this map. Are they stable or unstable? Show that this map has a period-2 orbit for certain values of r.

c. (9 pts) Consider a one-dimensional anharmonic oscillator with a cubic restoring force. The equation of motion is $\ddot{q} + q + cq^3 = 0$ where c is a constant and $\dot{q}$ denotes the time derivative. We want to find a periodic solution satisfying the initial condition $q(0) = A$ and $\dot{q}(0) = 0$. First show that nave perturbation theory (expanding $q(t) = q_0(t) + cq_1(t) + c^2 q_2(t) + ...$ and substituting this expansion into the equation of motion) does not give a periodic solution. (You should find that $q_1(t)$ is not periodic in t). The problem is the anharmonic term: show that if c gets large the period starts depending on the amplitude. To solve this problem we follow Poincare and Lindstedt and also expand the period as follows: $\omega = 1 + c\omega_1 + c^2\omega_2 + ...$ and $q(t) = q_0(s) + cq_1(s) + ...$ where $s = \omega t$. Inserting both expansions into the equation of motion find a periodic solution through terms linear in c.

SOLUTION:

(a.) A system is chaotic if it depends sensitively on the initial conditions, namely neighboring orbits separate **exponentially** fast. The double pendulum exhibits chaos for large angles.

(b.) The fixed points are $x^* = rx^*$ for $x^* \leq 1/2$, so $x_I^* = 0$ for all r, and further $x^* = r(1 - x^*)$ for $x^* \geq 1/2$, so $x_{II}^* = r/(1 + r)$ for $r \geq 1$. Fixed points are stable if $|f'(x^*)| < 1$. So x_I^* is stable if $r < 1$ but x_{II}^* is unstable. If for some value of r and some point $x_0 \leq 1/2$ the point $f(x_0)$ satisfies $f(x_0) \geq 1/2$, one can have a 2-cycle. Namely, $x^* = f(f(x^*)) = r(1 - rx^*)$ if

$$x^* = \frac{r}{1 + r^2} \quad \text{and} \quad f(x^*) \geq 1/2 \quad \text{if} \quad r > 1 \tag{48}$$

(c.) Naively we get $q_0 = A\cos(t)$ and q_1 must satisfy

$$\ddot{q}_1 + q_1 = -\frac{A^3}{4}(\cos(3t) + 3\cos(t)) \tag{49}$$

11

whose solution is

$$q_1(t) = A^3 \left(-\frac{3}{8} t \sin(t) - \frac{1}{32} \cos(t) + \frac{1}{32} \cos(3t) \right) \tag{50}$$

This solution is not periodic, the term $t \sin(t)$ blows up.

For large A the restoring force $q + cq^3$ becomes large, so the amplitude will start to depend on A. Expanding ω yields

$$q_0(s) = A \cos(s) \quad \text{where} \quad s = \omega t \tag{51}$$

and

$$\frac{d^2 q_1(t)}{dt^2} + q_1(t) = -\frac{A^3}{4} \left(\cos(3t) + 3 \cos(t) \right) + 2A\omega_1 \cos(t) \tag{52}$$

For $\omega_1 = \frac{3}{8} A^2$ the bad terms with $\cos(t)$ cancel, and one finds

$$q_1(s) = \frac{A^3}{32} \left(\cos(3s) - \cos(s) \right) \tag{53}$$

More explicitly

$$q(t) = A \cos\left[\left(1 + \frac{3}{8} A^2 + \dots \right) t \right] + \frac{cA^3}{32} \left(\cos(3t) - \cos(t) \right) + \mathcal{O}(c^2) \tag{54}$$