Tong Problems

Set 1

1. Establish Stirlings formula. Start with $N! = \int_0^\infty e^{-x} x^N dx = \int_0^\infty e^{-F(x)} dx$
let the minimum of F be at x_0. Approximate $F(x)$ by $F(x_0) + F''(x_0)(x-x_0)^2/2$
and using one further approximation show that $N! \simeq \sqrt{2\pi N}\, N^N e^{-N}$
We will mostly be interested in $N \sim 10^{23}$. But what is the accuracy
of Stirling's formula for the paltry value of $N = 5$?

$$N! = \int_0^\infty e^{-x} x^N dx = \int_0^\infty e^{-x + N\ln(x)} dx \qquad F(x) = +x - N\ln(x)$$

$$x_0 \to \left.\frac{d}{dx}\left(+x - N\ln(x)\right)\right|_{x_0} = 0 \to -1 + \frac{N}{x_0} = 0 \quad \therefore x_0 = N$$

$$\frac{d^2}{dx^2}\left(x - N\ln(x)\right) = 0 - \frac{d}{dx}\left(\frac{1}{x}\right)\cdot N = \frac{N}{x^2}$$

$$\therefore F(x) \simeq F(x_0) + F''(x_0)(x-x_0)^2/2 \simeq N - N\ln(N) + \frac{N}{N^2}\cdot\frac{(x-N)^2}{2}$$

$$\therefore N! \simeq \int_0^\infty e^{-\left[N - N\ln(N) + \frac{(x-N)^2}{2N}\right]} dx$$

$$\simeq e^{-N} N^N \cdot \int_0^\infty e^{\frac{(x-N)^2}{2N}} dx$$

assuming $N \to$ large
$0 \simeq -\infty$
$x \to x - N$, $dx \to dx$

Gaussian integral

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \left(\frac{\pi}{\alpha}\right)^{1/2}$$

$$\therefore \boxed{N! \simeq \sqrt{2\pi N}\, e^{-N} N^N}$$

for $N = 5 \to N! = 5! = 120$ & $\sqrt{2\pi N}\, e^{-N} N^N \simeq \sqrt{10\pi}\, e^{-5} 5^5$
$\simeq 118.02$
pretty good

$$\ln(N!) = N\ln N - N + \frac{1}{2}\ln(2\pi N)$$

2i) Show that two coupled systems in the microcanonical ensemble maximize their entropy at equal temperature only if the heat capacity is positive.

—

The entropy of two coupled sysentems at equal temperature is

$$S(E_{total}) = k_B \ln(\Omega_{total}) \geq S(E_1) + S(E_2)$$
$$\approx S_1(E_*) + S_2(E_{total} - E_*)$$

$\rightsquigarrow$ See arguments on page 10 of notes

ii) in the canonical ensemble, show that the fluctuations in energy $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$ are proportional to the heat capacity.

—

$$Z = \sum_n e^{-\beta E_n} \quad , \quad \langle E \rangle = \sum_n E_n \, p(n) = \sum_n E_n \frac{e^{-\beta E_n}}{Z} = -\frac{\partial}{\partial \beta} \ln Z$$

$$p(n) = \frac{e^{-\beta E_n}}{Z}$$

$$\Delta E^2 = \langle (E - \langle E \rangle)^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2$$

$$C_V = \frac{\partial \langle E \rangle}{\partial T}\Big|_V \quad , \quad \Delta E^2 = \frac{\partial^2}{\partial \beta^2} \ln(Z) = -\frac{\partial \langle E \rangle}{\partial \beta} = k_B T^2 C_V$$

(see page 1 of my Tong notes) by chain rule on $\frac{\partial \langle E \rangle}{\partial T}$ definition

iii) Show that in the canonical ensemble the gibbs entropy can be written as

$$S = k_B \frac{\partial}{\partial T} (T \ln(Z))$$

—

$$S = -\frac{k_B}{Z} \sum_n e^{-\beta E_n} \ln\left(\frac{e^{-\beta E_n}}{Z}\right) = \frac{k_B \beta}{Z} \sum_n E_n e^{-\beta E_n} + k_B \ln Z$$

$$\parallel$$

$$-k_B \sum_n P_n \ln(P_n) \qquad = k_B \frac{\partial}{\partial T} (T \ln Z)$$

[illegible]

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3. Consider a system consisting of N spin-$\frac{1}{2}$ particles, each of which can be in one of two quantum states, up and down. In a magnetic field B, the energy of a spin in the up/down state is $\pm \mu B/2$ where

a) μ is the magnetic moment. Show that the partition function is

$$Z = 2^N \cosh^N\left(\frac{\beta \mu B}{2}\right)$$

b) Find the average energy E and entropy S. Check that your results for both quantities make sense in the $T \to 0$ + $T \to \infty$ limits.

c) Compute the magnetization of the system, defined by $M = N_\uparrow - N_\downarrow$ where $N_{\uparrow/\downarrow}$ are the number of up/down spins. The magnetic susceptibility is defined as $\chi = \partial M / \partial B$. Derive Curie's law which states that at high temperatures $\chi \sim 1/T$.

a) For $H = \pm \mu B/2$ we have $Z_1 = \sum_n e^{-\beta E_n} = e^{-\beta \mu B/2} + e^{+\beta \mu B/2}$

canonical ensemble

$= 2\cosh\left(\frac{\beta \mu B}{2}\right)$

& $Z = \prod_i^N Z_1$ for N independent systems

Partition function

i.e. $\boxed{Z = 2^N \cosh^N\left(\frac{\beta \mu B}{2}\right)}$

or $Z = \sum_{n,m} e^{-\beta(E_n + E_m)}$ for $N=2$ ... extrapolate

b) The average energy E is given by $\langle E \rangle = \frac{1}{Z} \cdot \left(\sum_n P(n) \cdot E_n\right)^N$

$$= \frac{\prod_1^N \sum_n E_n e^{-\beta E_n}}{\prod_1^N \sum_n e^{-\beta E_n}}$$

Energy

$\therefore \langle E \rangle = -\frac{N \mu B}{2} \tanh\left(\frac{\beta \mu B}{2}\right)$

Gibbs entropy

$*$ $\boxed{\langle S \rangle = -k_B \sum_n P(n) \ln(P(n))}$

$P(n) = \frac{e^{-\beta E_n}}{Z}$

$*$ $\boxed{\langle E \rangle = -\frac{\partial}{\partial \beta} \ln(Z)}$

$\boxed{\langle S \rangle = k_B \frac{\partial}{\partial T}\left(T \ln(Z)\right)}$

$= \left[2^N \cosh^N\left(\frac{\beta \mu B}{2}\right)\right]^{-1} \cdot 2^N$

$\cdot N \cosh^{N-1}\left(\frac{\beta \mu B}{2}\right)$

$\cdot \sinh\left(\frac{\beta \mu B}{2}\right) \cdot \frac{\mu B}{2}$

$\therefore \langle S \rangle = k_B \ln\left(2^N \cosh^N\left(\frac{\beta \mu B}{2}\right)\right) + k_B T \frac{2^N \cdot N \cosh^{N-1}\left(\frac{\beta \mu B}{2}\right) \cdot \sinh\left(\frac{\beta \mu B}{2}\right)}{\ldots}$

$$\text{So} \quad \langle S \rangle = N k_B \ln\left(2\cosh\left(\frac{\beta \mu B}{2}\right)\right) + \cdot N \frac{k_B T \tanh\left(\frac{\beta\mu B}{2}\right)}{-T^2 k_B} \cdot \frac{\mu B}{2}$$

$$= N k_B \ln\left(2\cosh\left(\frac{\beta\mu B}{2}\right)\right) + \frac{\langle E \rangle}{T}$$

for $T \to 0$ $\quad \langle E \rangle \Rightarrow -\frac{N\mu B}{2} \cdot -1 = \frac{N\mu B}{2}$ ✓

for $T \to \infty$ $\quad \langle E \rangle \Rightarrow 0$ ✓

for $T \to 0$ $\quad \langle S \rangle \Rightarrow \frac{\langle E \rangle}{T} = \frac{N\mu B}{2} \cdot \frac{1}{T}$

for $T \to \infty$ $\quad \langle S \rangle \Rightarrow N k_B \ln(2)$

Magnetization

c) $M = N_\uparrow - N_\downarrow$ comes from $N_\uparrow = \frac{\left(\langle E \rangle + \mu B/2 \cdot N\right)}{\mu B}$

$\quad$ & $N_\downarrow = N - N_\uparrow \quad \therefore M = 2N_\uparrow - N$

$$= N\left(\frac{2\langle E \rangle}{N\mu B} + \frac{2}{2} - 1\right)$$

$$M = \frac{2}{\mu B} \cdot -\frac{N\mu B}{2} \tanh\left(\frac{\beta\mu B}{2}\right) \qquad = \frac{2\langle E \rangle}{\mu B}$$

$$= -N \tanh\left(\frac{\beta\mu B}{2}\right) \quad \& \quad \boxed{m = -\tanh\left(\frac{\beta\mu B}{2}\right)}$$

$$\text{or} \quad m = \frac{1}{N}\frac{1}{\beta}\frac{\partial}{\partial B}\ln(Z)$$

by Grand canonical ensemble.

for $\beta \to 0$ we get $M \simeq -\frac{\beta \mu B}{2} \propto \frac{1}{T}$ Curie's law ✓

$$\chi = \frac{\partial M}{\partial B}$$

Susceptibility

4. Consider a system of N interacting spins. At low temperatures, the interactions ensure that all spins are either aligned or anti-aligned with the z-axis, even in the absence of an external field. At high temperatures, the interactions become less important and spins can point in either $\pm \hat{z}$ direction. If the heat capacity takes the form

$$C_V = C_{max}\left(\frac{2T}{T_0} - 1\right) \quad \text{for } \frac{T_0}{2} < T < T_0 \text{ and } C_V = 0 \text{ otherwise}$$

determine C_{max}.

Assuming in the large T limit this just becomes the 2-state system from before

$$C_V = \left.\frac{d\langle E\rangle}{dT}\right|_V = \frac{d}{dT}\left(-\frac{N\mu B}{2}\tanh\left(\frac{\beta\mu B}{2}\right)\right)$$

$$= -\frac{\mu N B}{2} \cdot \frac{\mu B}{2} \cdot \operatorname{sech}^2\left(\frac{\beta\mu B}{2}\right) \cdot \frac{d}{dT}\beta$$

$$\frac{d}{dT}\beta \rightarrow -\frac{1}{T^2 k_B}$$

$$\boxed{C_V(T_0) = +\frac{\mu^2 N B^2}{4} \frac{\operatorname{sech}^2\left(\frac{\mu B}{2T_0 k_B}\right)}{T_0^2 k_B}}$$

5. Compute the partition function of a quantum harmonic oscillator with frequency ω and energy levels

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad , \quad n \in \mathbb{Z}$$

a) Find the average energy E and entropy S as a function of temperature T.

b) Einstein constructed a simple model of a solid as N atoms, each of which vibrates with the same frequency ω. Treating those vibrations as a harmonic oscillator, show that at high temperatures, $k_B T \gg \hbar\omega$, the einstein model correctly predicts the Dulong-Petit law for the heat capacity of a solid,

$$C_V = 3 N k_B$$

c) at low temperatures, the heat capacity of many solids is experimentally observed to tend to zero as $C_V \sim T^3$. Was einstein right about this?

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a) In the canonical ensemble (fixed N in a solid) we have

for 3N modes of part B.

$$Z = \prod_1^{3N} \sum_n e^{-\beta E_n} = \prod_1^{3N} \sum_n^{\infty} e^{-\beta\hbar\omega\left(\frac{1}{2}+n\right)} = \prod_1^{3N} e^{-\frac{\beta\hbar\omega}{2}} \cdot \sum_n^{\infty} e^{-\beta\hbar\omega n}$$

$$= \left(\frac{e^{-\frac{\beta\hbar\omega}{2}}}{1-e^{-\beta\hbar\omega}}\right)^{3N} = \left(\frac{1}{2\sinh(\beta\hbar\omega}\right.$$

$$E = -\frac{\partial}{\partial\beta}\ln(Z) = -3N\,2\sinh(\beta\hbar\omega) \cdot -\frac{1}{2\sinh^2(\beta\hbar\omega)} \cdot \hbar\omega \cdot \cosh(\beta\hbar\omega)$$

$$\boxed{\langle E\rangle = 3N\hbar\omega\coth(\hbar\omega\beta)}$$

$d\beta/dT = -\frac{1}{T^2 k_B}$

$$S = k_B\frac{\partial}{\partial T}\left(T\ln(Z)\right) = 3Nk_B\ln\left(\frac{1}{2\sinh(\hbar\omega\beta)}\right) + k_B T \cdot \frac{\partial\beta}{\partial T}\frac{\partial}{\partial\beta}\ln(Z)$$

$$\boxed{S = 3Nk_B\ln\left(\frac{1}{2\sinh(\hbar\omega\beta)}\right) + \frac{\langle E\rangle}{T}}$$

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b) $C_v = \frac{d}{dT}\langle E\rangle\Big|_V = \overset{-\frac{1}{T^2 k_B}}{\frac{d\beta}{dT}} \frac{d}{d\beta}\left(3N\hbar\omega \coth(\hbar\omega\beta)\right)$

$= -\frac{1}{T^2 k_B} \cdot -\operatorname{csch}^2(\hbar\omega\beta)\cdot(\hbar\omega)^2\cdot 3N$

$\simeq k_B\beta^2\cdot(\hbar\omega)^2\cdot\frac{1}{(\hbar\omega\beta)^2}\cdot 3N \simeq \boxed{3N k_B = C_v}$

$\boxed{QED}$

c) $C_v \simeq 3N\hbar^2\omega^2 k_B\beta^2\cdot\left(\frac{2}{e^{\beta\hbar\omega} - \underset{0}{e^{-\beta\hbar\omega}}}\right)^2$ β large then

$\simeq 12 N\hbar^2\omega^2 k_B\beta^2\cdot\left(e^{-2\beta\hbar\omega}\right)$ $\times$ β^{-3} so einstein missed

?

6 i) A quantum violin string can vibrate at frequencies $\omega, 2\omega, \dots$. Each vibration mode can be treated as an independent harmonic oscillator. Ignore the zero point energy, so that the mode with frequency $p \cdot \omega$ has energy $n\hbar p\omega$, $n \in \mathbb{Z}$ + $p \in \mathbb{Z}$. Write an expression for the average energy of the string at temperature T. Show that at large temperatures the free energy is given by $F = -\frac{\pi^2}{6}\frac{k_B^2 T^2}{\hbar\omega}$ $\left(\zeta(2) = \pi^2/6\right)$

$$Z = \prod_p^\infty \sum_n e^{-\beta n\hbar\omega \cdot p} = \prod_p^\infty \left(\frac{1}{1-e^{-\beta\hbar\omega p}}\right)$$

Then $$E = -\frac{\partial}{\partial\beta}\ln(Z) = -\frac{\frac{\partial}{\partial\beta}\left(\prod_p^\infty \left(\frac{1}{1-e^{-\beta\hbar\omega p}}\right)\right)}{\prod_p^\infty\left(\frac{1}{1-e^{-\beta\hbar\omega p}}\right)} = \frac{-\prod_p p\frac{\hbar\omega}{(1-e^{-\beta\hbar\omega p})}}{\prod_p^\infty\left(\frac{1}{1-e^{-\beta\hbar\omega p}}\right)}$$

or better $$E = -\frac{\partial}{\partial\beta}\sum_p \ln\left(\frac{1}{1-e^{-\beta\hbar\omega p}}\right)$$

$$= -\sum_p p \cdot (1-e^{-p\beta\hbar\omega})^{1-2} \cdot -(-\hbar\omega)\, e^{-\beta\hbar\omega p}$$

$$\boxed{E = -\hbar\omega\sum_p^\infty p\frac{e^{-\beta\hbar\omega p}}{1-e^{-\beta\hbar\omega p}}}$$

$$-e^{-\beta\hbar\omega p}$$

free energy

$$F = -k_B T \ln(Z) = -k_B T \cdot \sum_p^\infty \ln\left(\frac{1}{1-e^{-\beta\hbar\omega p}}\right)$$

$$= -k_B T\sum_p (p\beta\hbar\omega)^{-1} = \sum_p -\frac{(k_B T)^2}{\hbar\omega p}$$

Tong Problems

Set 2

1. A particle moving in one dimension has Hamiltonian

$$H = \frac{p^2}{2m} + \lambda q^4$$

Show that the heat capacity for a gas of N such particles is $C_V = \frac{3Nk_B}{4}$.
Explain why the heat capacity is the same regardless of whether the particles are distinguishable or indistinguishable

$$Z = \prod_1^N \frac{1}{(2\pi\hbar)} \int dq\,dp\, e^{-\beta\left(\frac{p^2}{2m} + \lambda q^4\right)}$$

$$= \prod_1^N \frac{1}{(2\pi\hbar)} \left(\int dp\, e^{-\beta \frac{p^2}{2m}}\right)\left(\int dq\, e^{-\beta\lambda q^4}\right)$$

$$= \frac{1}{(2\pi\hbar)^N} \cdot \left(\sqrt{\frac{\pi}{\beta/2m}} \cdot \frac{2\Gamma(5/4)}{(\beta\lambda)^{1/4}}\right)^N$$

Then $\langle E \rangle = -\frac{\partial}{\partial\beta}\ln(Z) = -N \cdot \dfrac{\sqrt{\frac{\pi}{2m}}\,\frac{2\Gamma(5/4)}{\lambda^{1/4}} \cdot \beta^{-3/4-1} \cdot -3/4}{\sqrt{\frac{\pi}{2m}}\,\frac{2\Gamma(5/4)}{\lambda^{1/4}} \cdot \beta^{-3/4}}$

$$= +\frac{3}{4} k_B T \cdot N$$

$$\boxed{C_V = \left.\frac{d\langle E \rangle}{dT}\right|_V = \frac{3}{4} k_B N}$$

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$$

de Broglie thermal wavelength

and the $\frac{1}{N!}$ distinguishability factor on Z cancels out in the ln derivative

2. Derive the Sakur-Tetrode formula for the entropy of an ideal monatomic gas with $Z = Z_1^N / N!$. Show that the entropy is not extensive if we fail to include the $N!$ factor.

$$S = \frac{\partial}{\partial T}\left(k_B T \ln Z\right) = \frac{\partial}{\partial T}\left(k_B T \ln\left(\frac{Z_1^N}{N!}\right)\right) = \frac{\partial}{\partial T}\left(k_B T \ln\left(\frac{V^N}{N!} \cdot \left(\frac{m k_B T}{2\pi \hbar^2}\right)^{3N}\right)\right)$$

Where $Z_1 = \frac{V^N}{\lambda^{3N}}$, $\lambda = \sqrt{\frac{2\pi \hbar^2}{m k_B T}}$ for the ideal gas

$$S = k_B \cdot N \ln\left(V/\lambda^3\right) - k_B\left(\ln N!\right) + k_B \cancel{T} \cdot \frac{3N}{2} \cdot \frac{1}{\cancel{T}} \cdot \cancel{\frac{\sim}{\sim}} = \boxed{S = k_B N\left(\ln\left(\frac{V}{N\lambda^3}\right), + \frac{5}{2}\right)}$$

$+ \sim$

without including $\frac{1}{N!}$ part this $1/N$ wouldn't appear
(but it would still be extensive ....)

3. Consider an Ultra-Relativistic gas of N spinless particles obeying the energy-momentum relation $E = pc$, where c is the speed of light. (Here ultra-relativistic means that $pc \gg mc^2$ where m is the mass of the particle). Show that the canonical partition function is given by

$$Z(V,T) = \frac{1}{N!}\left[\frac{V}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3\right]^N$$

Hence show that an ultra-relativistic gas obeys the familiar ideal gas law $pV = Nk_BT$.

$$Z_1(V,T) = \frac{1}{(2\pi\hbar)^3}\int d^3q\, d^3p\, e^{-\beta pc} = \frac{V}{(2\pi\hbar)^3} \cdot 4\pi \cdot \int_0^\infty dp\, p^2 e^{-\beta pc}$$

$$\text{where } \int_0^\infty x^n e^{-x/a}\, dx = n!\, a^{n+1}$$

$$Z_1(V,T) = \frac{V}{(2\pi\hbar)^3} \cdot 4\pi \cdot 2!\left(\frac{1}{\beta c}\right)^3$$

$$\therefore Z = \frac{1}{N!} \cdot \left(\frac{V}{\pi^2}\left(\frac{1}{\hbar\beta c}\right)^3\right)^N$$

eqn of state

Then $p = -\frac{\partial F}{\partial V} = \frac{\partial}{\partial V}\left(k_B T \ln(Z)\right) = k_B T \cdot N \cdot \frac{1}{V} \cdot \cancel{\frac{\alpha}{\alpha}}$

$$\therefore \boxed{PV = Nk_BT} \quad \boxed{QED}$$

Since $Z \propto V^N$ then the ideal gas law holds (basically there is no interaction so duh!)

4. Consider a perfect classical gas of diatomic molecules for which each molecule has a magnetic moment m aligned along its axis. Let there be a magnetic field B, so that each molecule has a potential energy $-mB\cos\theta$ (θ being the angle between the axis of the molecule and the magnetic field). Show that the

a) rotational part of the partition function is $Z_{rot} = (z_{rot})^N$ where

$$z_{rot} = \left[\frac{2I}{\hbar^2 mB\beta^2}\right] \sinh(m\beta B)$$

b) Evaluate the total magnetization, $M = -\frac{\partial F}{\partial B}$ and sketch its dependence upon $m\beta B$. Show that, for large $m\beta B$, the average value of the potential energy is $NkT - NmB(1+2e^{-2m\beta B}+\ldots)$

a) $$H_{rot} = T_{rot} + V_{rot} = \frac{P_\theta^2}{2I} + \frac{P_\phi^2}{2I\sin^2\theta} - mB\cos\theta$$

$$\therefore z_{rot} = \frac{1}{(2\pi\hbar)^2}\int \underbrace{d\phi}_{2\pi}\, d\theta\, dp_\theta\, dp_\phi\, e^{-\beta\left(\frac{p_\theta^2}{2I} + \frac{p_\phi^2}{2I\sin^2\theta} - mB\cos\theta\right)}$$

gaussian $\rightarrow \sqrt{\frac{\pi}{\alpha}}$

$$= \frac{2\pi}{(2\pi\hbar)^2}\cdot\sqrt{\frac{2\pi I}{\beta}}\int_0^\pi d\theta\,\sqrt{\frac{2\pi I\sin^2\theta}{\beta}}\; e^{m\beta B\cos\theta}$$

$u = \cos\theta$, $du = -\sin\theta\, d\theta$

$0 \rightarrow 1$, $\pi \rightarrow -1$

$$= \frac{1}{\hbar^2}\frac{1}{\beta}\cdot I\int_{+1}^{-1} \cdot -du\, e^{m\beta B u}$$

$$= \frac{I}{\beta\hbar^2}\cdot\frac{1}{m\beta B}\, e^{m\beta B u}\Big|_{-1}^{1} = \boxed{2\cdot\frac{I}{mB\hbar^2\beta^2}\cdot\sinh(m\beta B) = z_{rot}}$$

$\boxed{QED}$

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b) **Magnetization**

$$M = -\frac{\partial F}{\partial B} = -\frac{\partial(-k_BT\ln(Z_{total}))}{\partial B}$$

$$\text{\& } Z_{tot} = (Z_{Trans}\cdot Z_{rot})^N = \left(V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\cdot 2\frac{I}{mB\hbar\beta^2}\sinh(m\beta B)\right)^N$$

$\beta^{-5/2}$

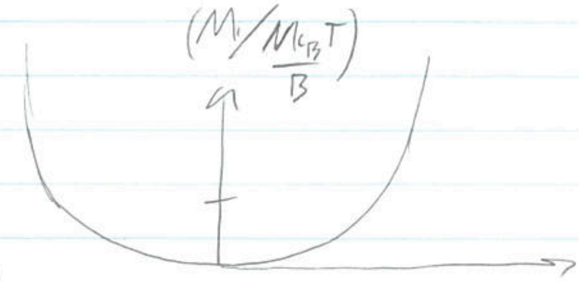
$$M = Nk_BT\cdot\frac{1}{V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\cdot\frac{2I}{m B\hbar\beta^2}\sinh(m\beta B)}\cdot V\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2}\cdot\frac{2I}{m\hbar\beta^2}\cdot\left[\frac{\sinh(m\beta B)}{-B^2}+\frac{1}{B}\cdot m\beta\cosh(m\beta B)\right]$$

B on top.

$$= Nk_BT\left(-\frac{1}{B}+m\beta\coth(m\beta B)\right)$$

$$= Nk_BT\cdot\frac{1}{B}\left(m\beta B\coth(m\beta B)-1\right)$$

$$\frac{M}{Nk_BT/B} =$$



Average Potential Energy

From looking at Z we see that $\langle u\rangle = +\frac{\partial}{\partial B}\ln(Z_{rot}^N)\times B$

$= -M\cdot B$

$= -Nk_BT\left(m\beta B\coth(m\beta B)-1\right)$

for large $m\beta B$

$\coth(x)\to$

$$= Nk_BT - NmB\left(\frac{e^{m\beta B}+e^{-m\beta B}}{e^{m\beta B}-e^{-m\beta B}}\right)$$

$$\simeq Nk_BT - NmB\left((1+e^{-2m\beta B})(1+e^{-2m\beta B})\right)$$

$$\boxed{\langle u\rangle \simeq Nk_BT - NmB(1+e^{-2m\beta B}+\dots)}$$

$\boxed{\text{QED}}$

5. A classical gas in three dimensions is constrained by a wall to move in the $x \geq 0$ region of space. A potential

$$V(x) = \frac{1}{2}\alpha x^2 \quad \text{for } x \geq 0$$

attracts the atoms to the wall. The atoms are free to move in an area A in the y & z directions. If the gas is at uniform temperature T, show that the number of particles varies as

$$N(x) = 2N\sqrt{\frac{\alpha\beta}{2\pi}}\, e^{-\alpha\beta x^2/2}$$

by considering a slab of gas between x and $x+\Delta x$, show that locally the gas continues to obey the ideal gas law. Hence determine the pressure that the gas exerts on the wall.

$$H = T + V = \frac{1}{2}\alpha x^2 + \frac{\vec{p}^{\,2}}{2m}$$

$$\therefore \quad Z_1 = \frac{1}{(2\pi\hbar)^3}\int d^3\vec{p}\, d^3\vec{r}\; e^{-\beta\left(\frac{1}{2}\alpha x^2 + \frac{\vec{p}^{\,2}}{2m}\right)}$$

$$= \frac{A}{(2\pi\hbar)^3}\cdot\left(\pi/(\beta/2m)\right)^{3/2}\int_0^\infty dx\, e^{-\frac{\beta\alpha x^2}{2}}$$

$\rightarrow \sqrt{\frac{\pi}{\alpha}}$ gaussian integral $\cdot \frac{1}{2}$

$$= \frac{(2m\pi)^{3/2}}{(2\pi\hbar)^3}\frac{A}{\beta^{3/2}}\cdot\frac{1}{2}\cdot\sqrt{\frac{\pi\cdot 2}{\beta\alpha}}$$

$$Z_{tot} = Z_1^N = \left(\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2}\cdot A\sqrt{\frac{\pi}{2\beta\alpha}}\right)^N = \left(\frac{1}{\beta^2}\frac{A}{4\pi}\left(\frac{m}{\hbar^2}\right)^{3/2}\frac{1}{\sqrt{\alpha}}\right)^N$$

$$\text{then } N(x) = \frac{\frac{dZ_{tot}}{dx}}{Z_{tot}} = \frac{N\tilde{A}^{N-1}\cdot\left(A\left(\frac{1}{2}\sqrt{\frac{2\pi}{\beta\alpha}}\right)^{N-1} e^{-\frac{\beta\alpha x^2}{2}}\right)}{\tilde{A}^N\cdot\left(\frac{1}{2}\sqrt{\frac{\pi\cdot 2}{\beta\alpha}}\right)^{N-1}} = \boxed{2N\sqrt{\frac{\beta\alpha}{2\pi}}\, e^{-\frac{\beta\alpha x^2}{2}} = N(x)}$$

basically $\frac{dZ}{Z} \propto$ probability distributions and the eqn of state is

$$P = -\frac{\partial F}{\partial V} = \frac{\partial}{\partial V}\left(k_B T \ln(Z_{tot})\right)$$

$\uparrow$ $dV = A\,dx$ for our case.

$$PA = k_B T \frac{d}{dx} \ln(Z_{tot}) = k_B T N(x)$$

$\uparrow$

$N(x) = \frac{dN}{dx}$ multiply through by $\int_0^\infty dx$

$$\therefore PA \int_0^\infty dx = k_B T \cdot 2N \sqrt{\frac{\beta\alpha}{2\pi}} \underbrace{\int_0^\infty dx\, e^{-\frac{\beta\alpha}{2}x^2}}_{\frac{1}{2}\cdot\sqrt{\frac{2\pi}{\beta\alpha}}} = N$$

$\therefore$ $PV = NkT$ as expected, since $V(x)$ doesn't depend on inter-particle interactions. $\rightarrow$ Total eqn of state (for all N)

and $P(x=0) = \frac{k_B T}{A} \cdot N(0) = \frac{k_B T}{A} \cdot 2N \sqrt{\frac{\beta\alpha}{2\pi}}$

$$\boxed{P(x=0) = \sqrt{\frac{2\alpha}{\pi}} \sqrt{k_B T} \cdot N}$$

6. Skip

7. Compute the equation of state, including the second virial coefficient, for a gas of non-interacting hard discs of radius $r_0/2$ in 2 dimensions.

van der waals

$$P = -\frac{\partial F}{\partial V} = \frac{Nk_BT}{V}\left(1 - \frac{N}{2V}\int d^3r\, f(r) + \ldots\right)$$

↑ 2nd virial coefficient.

$$f(r) = e^{-\beta U(r)} - 1 \quad \text{meyer f function}$$

$$+ \quad U(r) = \begin{cases} \infty & r < r_0/2 \\ 0 & r > r_0/2 \end{cases}$$

$$P = \frac{Nk_BT}{V}\left(1 - \frac{N}{2V}\int_0^{r_0/2}(\underbrace{0}_{e^{-\infty}=0} - 1)\, d^3r + \int_{r_0/2}^{\infty}\underbrace{(1-1)}_{0}\, d^3r\right)$$

$$= \frac{Nk_BT}{V}\left(1 + \frac{N}{2V}\cdot 4\pi \cdot \left.\frac{r^3}{3}\right|_0^{r_0/2}\right)$$

$$= \frac{Nk_BT}{V}\left(1 + \frac{4\pi}{2}\cdot\overbrace{\frac{r_0^3}{3\cdot 8}}^{V_0}\frac{N}{V}\right)$$

$$= \frac{Nk_BT}{V}\left(1 + \frac{V_0}{V}\frac{N}{16}\right) \Rightarrow \boxed{PV = Nk_BT + \frac{N^2k_BT}{16}\frac{V_0}{V}}$$

4

8. Determine the density of states for non-relativistic particles in $d=2$ and $d=1$ dimensions. (You should find that the density is constant for particles on a plane and decreases with energy for particles on a line.)

In general, $\sum_{\vec{n}} \simeq \int d^d n = \frac{V_d}{(2\pi)^d} \int d^d k = \frac{V_d \Omega_d}{(2\pi)^d} \int dk\, k^{d-1}$

$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ $\quad \Gamma(1/2) = \sqrt{\pi}, \ \Gamma(1) = 1, \ \Gamma(3/2) = \frac{\sqrt{\pi}}{2}$

3D $\quad \therefore \sum_{\vec{n}\,3D} = \frac{V \cdot 4\pi}{8\pi^3} \cdot \int dk\, k^2 = \frac{V}{2\pi^2}$ $\quad$ & $E = \frac{\hbar^2 k^2}{2m}$ $\quad dE = \frac{\hbar^2 k\, dk}{m}$

$\therefore \sum_{\vec{n}\,3D} = \int d^3 n = \frac{4\pi V}{(2\pi)^3} \int dk\, k^2 = \frac{V}{2\pi^2} \int dE \cdot \underbrace{\sqrt{\frac{2mE}{\hbar^2}} \frac{m}{\hbar^2}}$

$$\boxed{g(E) = \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E}$$

2D $\quad \therefore \sum_{\vec{n}\,2D} = \frac{V \cdot 2\pi/1}{4\pi^2} \int \underbrace{dk\, k}_{\frac{dE}{\hbar^2} \cdot m}$ $\quad$ & $E = \frac{\hbar^2 k^2}{2m}, \ dE = \frac{\hbar^2 k\, dk}{m}$

$$\int d^3 n = \frac{Vm}{2\pi\hbar^2} \int dE \rightarrow \boxed{g(E) = \frac{Vm}{2\pi\hbar^2}}$$

1D $\quad \therefore \sum_{n\,1D} = \frac{V}{2\pi} \cdot \frac{2 \cdot \pi^{1/2}}{\pi^{1/2}} \int dk = \frac{V}{\pi} \int \frac{m}{\hbar^2} \cdot \frac{dE}{k} = \frac{Vm}{\pi\hbar^2} \cdot \frac{\hbar}{\sqrt{2m}} \int \frac{dE}{E^{1/2}}$

$k \hookrightarrow \sqrt{\frac{2m}{\hbar^2} E}$

$$\therefore \boxed{g(E) = \frac{V}{\pi\hbar}\sqrt{\frac{m}{2}}\, E^{-1/2}}$$

$E_{relativistic} = ((\hbar k c)^2 + (mc^2)^2)^{1/2}$

9. In many experiments, particles are not trapped in a box, but instead in a quadratic potential. In d-dimensions, the potential energy felt by a single particle is

$$V(\vec{x}) = \frac{1}{2}\sum_{i=1}^{d} m\omega^2 x^2 \quad \text{find } g(E)$$

This has $E_n = \hbar\omega(|n| + \frac{3}{2})$ s.t. $\int d\vec{n}^3 = 4\pi\int n^2 dn = \frac{V}{(2\pi)^3}\int d^3k$

$\downarrow$ $\uparrow$ 3D or 2D ($3/2$) (1)

$dE = \hbar\omega\, dn$ $\qquad = 4\pi\int\left(\frac{E_n}{\hbar\omega} - \frac{3}{2}\right)^2 \cdot \frac{dE}{\hbar\omega}$

& $|n| = \frac{E_n}{\hbar\omega} - \frac{3}{2}$

$$\therefore\ g(E) = \frac{V}{(2\pi)^3}\frac{4\pi}{\hbar\omega}\left(\frac{E_n}{\hbar\omega} - \frac{3}{2}\right)^2$$

$$\boxed{g(E) \simeq \frac{V\,4\pi E^2}{(2\pi)^3(\hbar\omega)^3}} \quad 3D$$

& $\int d^2n = 2\pi\int n\,dn = \frac{A}{(2\pi)^2}2\pi\int\left(\frac{E_n}{\hbar\omega} - 1\right)\cdot\frac{dE}{\hbar\omega}$

$$\boxed{g(E) \simeq \frac{A\,2\pi}{(2\pi)^2(\hbar\omega)^2}E} \quad 2D \quad \checkmark$$

$$\left(\ \sum_n = \int d^3\vec{n} = \frac{V}{(2\pi)^3}\int d^3k = \frac{4\pi V}{(2\pi)^3}\int k^2 dk\ \right)$$

then take $E(n)$ or $E(k)$ to find Density of states. used as the measure in a $\sum_n \to \int dk$ integral

$\int dk \to \int dE\, g(E)$

11

10, Consider Blackbody radiation at temperature T. Show that the average number of photons grows as T^3. What is the mean photon energy? What is the most likely energy of a photon?

The density of states for relativistic photons is $g(E) = \underbrace{2}_{\text{polarizations}} \cdot \frac{VE}{2\pi^2\hbar^3 c^3}\sqrt{E^2 - \underbrace{m^2}_{\text{massless}} c^4}$

or $$g(E)dE = g(\omega)d\omega = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

Then $$Z = \prod_{\omega}^{\infty} \sum_{N}^{\infty} e^{-\beta \cdot N E_N(\omega)}$$ with $E_N(\omega) = \hbar\omega$ + N being the possible number of such E_N photons

product → sum
sum → integral

$$\therefore \ln Z = \int_0^\infty d\omega\, g(\omega) \ln\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)$$

Z_ω for a specific ω state (canonical ensemble) + our system is all ω's.

$$\therefore \boxed{\ln Z = -\frac{V}{\pi^2 c^3}\int_0^\infty d\omega\, \omega^2 \ln\left(1-e^{-\beta\hbar\omega}\right)}$$

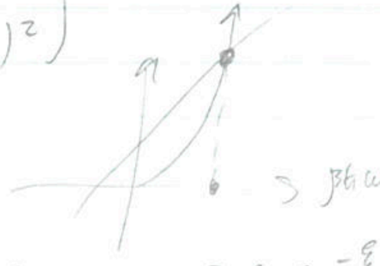
Then $$E = -\frac{\partial}{\partial\beta}\ln(Z) = \frac{V\hbar}{\pi^2 c^3}\int_0^\infty d\omega \frac{\omega^3}{e^{\beta\hbar\omega}-1} = \int_0^\infty E(\omega)\,d\omega$$

So, the most probable energy is where

$$\frac{dE(\omega)}{d\omega} = 0 \quad + \quad \frac{d^2E(\omega)}{d\omega^2} < 0$$ which happens at

$$0 = \frac{V\hbar}{\pi^2 c^3}\cdot\left(\frac{3\omega^2}{e^{\beta\hbar\omega}-1} + \omega^3\cdot\beta\hbar\cdot -1\cdot\frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega}-1)^2}\right)$$

$$-\frac{\omega\beta\hbar}{3} = e^{-\beta\hbar\omega} - 1$$



Occurs at $\omega_{max} = \xi \frac{k_B T}{\hbar}$ for $\xi = 2.822$ solving $3-\xi = 3e^{-\xi}$

So $\boxed{E_{\text{most probable}} = \hbar\omega_{max} = 2{,}822 \cdot k_B T \quad \text{Weins displacement law.}}$

And mean photon energy is then $\frac{E_{tot}}{N}$, but N is obtained by taking

$$N(\omega) = \frac{E(\omega)}{\hbar\omega} = \frac{V}{\pi^2 c^3} \frac{\omega^2}{e^{\beta\hbar\omega} - 1}$$

$$N = \int_0^\infty N(\omega)\, d\omega = \frac{V}{\pi^2 c^3} \frac{1}{(\beta\hbar)^2} \cdot \frac{1}{\beta\hbar} \cdot \underbrace{\int_0^\infty dx \frac{x^2}{e^x - 1}}_{\Gamma(3)\zeta(3)}$$

$\Gamma(3) \to 2$, $\zeta(3) \to 1{,}202$

$$\boxed{N \simeq \frac{2{,}4 \cdot V}{\pi^2 c^3} \frac{(k_B T)^3}{\hbar^3} \quad \text{Scales as } T^3}$$

+ E_{tot} is same but $\times \hbar\omega \to \Gamma(4)\zeta(4)$

$\hbar\omega \to$ leading to $\frac{1}{\beta}$ factor, $\Gamma(4) \to 6$, $\zeta(4) \to \pi^4/90$

$$\therefore \boxed{E/N = k_B T \left(\frac{\cdot 2.4}{\pi^4} \cdot 15\right)^{-1} \quad \text{Scales as } T.}$$

$$\boxed{\text{avg } E/N = 2{,}7\, k_B T \quad \text{is average energy.}}$$

Mean energy.

$$E/N = \frac{\sum_i^N E_i \cdot 1}{\sum_i \cdot 1}$$

Tong Problems

Sheet 3

1. A wigner crystal is a triangular lattice of electrons in a 2D plane. The longitudinal vibration modes of this crystal are bosons with dispersion relation $\omega = \alpha\sqrt{k}$. Show that, at low temperature, these modes provide a contribution to the heat capacity $\propto T^4$

Using the phonon energy $E = \hbar\omega(k) = \hbar\alpha\sqrt{k}$ we get the same partition function as for photons,

$$\ln Z_{phonon} = \int_0^{\omega_D} d\omega\, g(\omega) \ln\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right) \quad \text{with } \omega = \omega(k)$$

but the density of states in 2D changes from the 3D $\omega = k \cdot c_s$ case.

$$\omega = k\cdot c_s \text{ case} \Rightarrow \quad g(\omega) = \frac{3V}{2\pi^2 c_s^3}\omega^2 d\omega$$

Now we have $\displaystyle\sum_{\vec{n}} \simeq \int d^2\vec{n} = \frac{A}{(2\pi)^2}\int d^2k = \frac{2\pi A}{(2\pi)^2}\int dk \cdot k$

with $\omega = \alpha\sqrt{k}$ + $d\omega = \frac{1}{2}\alpha \cdot \frac{dk}{\sqrt{k}}$ we get

$$\sum_n = \frac{2\pi A}{(2\pi)^2}\cdot\int \frac{2\sqrt{k}}{\alpha}d\omega \cdot k = \frac{A}{\pi\alpha^3}\int d\omega\,\omega^3$$

s.t. $g(\omega) = \dfrac{A\omega^3}{\pi\alpha^3}$

Then $E = -\dfrac{d}{d\beta}\ln(Z) = \dfrac{A\cdot\hbar}{\pi\alpha^3}\displaystyle\int_0^{\omega_D}\frac{\omega^4}{e^{\beta\hbar\omega}-1}d\omega$ $\quad x = \beta\hbar\omega,\ dx = \beta\hbar d\omega$

$$E = \frac{A\hbar(k_BT)^5}{\pi\alpha^3}\cdot \text{Integral}$$

$$\therefore C_V = \frac{dE}{dT} = \frac{A\hbar\,\text{Integral}}{\pi\alpha^3}\cdot k_B^5\, T^4$$

$\boxed{QED}$

2. Use the fact that the density of states is constant in $d = 2$ dimensions (for a free, non-relativistic particle in a plane as seen in problem 2.8) to show that Bose-einstein condensation does not occur, no matter how low the temperature.

The number density of a bose gas is obtained by finding the Number of particles as z fugacity approaches 1 ($\beta \to \infty$, $T \to 0$ limit).

From $\quad \mathcal{Z} = \prod_r \frac{1}{1 - e^{-\beta(E_r - \mu)}}$

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln(\mathcal{Z}) = \sum_r \frac{1}{e^{\beta(E_r-\mu)} - 1}$$

$$\simeq \int dE\, g(E) \cdot \frac{1}{z^{-1}e^{\beta E} - 1}$$

Where $g(E)$ of 2D free gas is $\quad g(E) = \frac{A \cdot m}{2\pi\hbar^2} \quad$ independent of E.

$$x = \beta E, \quad dx = \beta\, dE$$

(N) $$N = \frac{Am}{2\pi\hbar^2} \cdot \frac{1}{\beta} \int_0^\infty \frac{dx}{z^{-1}e^x - 1} = \frac{1}{\lambda^2} \cdot g_1(z)$$

Where $g_n(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^n}$ & $g_n(1) = \zeta(n)$

but $\zeta(1) = \infty$

So then $T_{critical}$ occurs for $z = 1$ in (N)

$$T_c = \frac{N}{A}\frac{2\pi\hbar^2}{mk_B} \cdot \frac{1}{\zeta(1)} \to \frac{1}{\infty} = 0$$

So T_c is unobtainably low $\to 0°K$ $\boxed{\text{QED}}$

3. Consider N non-interacting, non-relativistic bosons, each of mass m, in a cubic box of side L. Show that the transition temperature scales as $T_c \sim N^{2/3}/mL^2$ and the 1-particle energy levels scale as $E_n \propto \frac{1}{mL^2}$. Show that when $T < T_c$, the mean occupancy of the first few excited 1-particle states is large, but not as large as $O(N)$.

$\Rightarrow$

The particle in a (3D) infinite square (box) well has energy

$$E_{\vec{n}} = \frac{(n_x \pi \hbar)^2}{2mL^2} + \frac{(n_y \pi \hbar)^2}{2mL^2} + \frac{(n_z \pi \hbar)^2}{2mL^2}$$

$$= \frac{\vec{n}^2 \pi^2 \hbar^2}{2mL^2}$$

Because we have such an expression in terms of n we could evaluate the Σn directly, or compute the density of states again.

$$\sum_{\vec{n}} \simeq \int d^3n = 4\pi \int n^2 dn \qquad \text{but } E_n \simeq \frac{n^2\pi^2\hbar^2}{2mL^2}$$

$$dE_n \simeq \frac{n\pi^2\hbar^2}{mL^2} dn$$

$$\therefore \sum_n \simeq \frac{4\pi \cdot mL}{\pi^2\hbar^2} \cdot \sqrt{\frac{2mL^2}{\pi^2\hbar^2}} \int dE\, E^{1/2}$$

$$\text{s.t. } g(E) = \frac{4\sqrt{2}\, m^{3/2} \overbrace{L^3}^{V}}{\pi^2\hbar^3} E^{1/2} = \frac{2}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} V E^{1/2}$$

(3.20) Then $N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln(Z) = \int dE \frac{g(E)}{z^{-1}e^{\beta E} - 1} = \frac{2}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} V \int \frac{E^{1/2} dE}{z^{-1}e^{\beta E} - 1}$

$x = \beta E$, $dx = \beta\, dE \quad \rightarrow \quad \frac{1}{\beta^{3/2}}$ factor

$N = \frac{2V}{\pi^2}\left(\frac{2m k_B T}{\hbar^2}\right)^{3/2} g_{3/2}(z) \cdot \Gamma(3/2) \qquad \Gamma(3/2) = \sqrt{\pi}/2$

$$g_{3/2}(1) = \zeta(3/2) \sim 2.612$$

then there is a critical temperature at $z = 1$ wherein the bottom states begin to fill up (and our $\propto E^{1/2}$ gives out, ignoring these states).

$$\text{s.t.} \quad N = \frac{2V}{\pi^2}\left(\frac{2mk_B T_c}{\hbar^2}\right)^{3/2} \cdot \zeta(3/2) \cdot \frac{\sqrt{\pi}}{\cancel{2}} \rightarrow \pi^{-3/2}$$

$$T_c = \left(\frac{N}{V}\right)^{2/3} \cdot \left(\pi^{3/2}\right)^{2/3} \cdot \left(\frac{1}{\zeta(3/2)}\right)^{2/3} \cdot \frac{\hbar^2}{2mk_B}$$

$$= \left(\frac{N}{V\,\zeta(3/2)}\right)^{2/3} \cdot \left(\frac{\hbar^2 \pi}{2mk_B}\right) \quad = \underline{\tfrac{1}{4}\, T_c \text{ of free particle}}$$

$\propto N^{2/3}$ as desired

Then, accounting for the $E = 0$ ground state in our counting we just include another state in our definition of N as

$$N_{tot} = N + N_{ground} \qquad \text{i.e.} \quad N_{ground} = N_{tot} - N \quad \rightarrow N = N(z)$$

$$= N_{tot} - \left(\frac{2mk_B T}{\pi\hbar^2}\right)^{3/2} V\zeta(3$$

Plug in T_c definition $\rightarrow$

$$= N_{tot} - \left(\frac{T}{T_c}\right)^{3/2} \cdot \frac{V}{V} \cdot \frac{N_{tot}\zeta}{\zeta}$$

$$= N_{tot}\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)$$

$$\therefore \frac{N_{ground}}{N_{tot}} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

nearly but not quite N_{tot} when $T < T_c$.

assuming $N(z) \simeq N_t$ right at critical poin before N_{ground} begins to fill.

4. Consider an ideal gas of bosons whose density of states is given by

$$g(E) = C E^{\alpha - 1}, \quad \alpha > 1$$

Derive an expression for the critical temperature T_c, below which the gas experiences Bose-einstein condensation.

$$N = \int dE \frac{C E^{\alpha-1}}{z^{-1}e^{\beta E}-1} = C \cdot \beta^{-\alpha+1-1} \Gamma(\alpha-1+1)\, g_{\alpha-1+1}(z)$$

$$N = \frac{C}{\beta^\alpha} \Gamma(\alpha)\, g_\alpha(z)$$

then $\boxed{T_c = \left(\frac{N}{C} \frac{1}{\Gamma(\alpha) \zeta(\alpha)}\right)^{1/\alpha} \cdot \frac{1}{k_B}}$ in general. $\therefore \zeta(1) = \infty$ has no BEC $\alpha > 1$ required

& $\boxed{N_0 = 1 - \left(\frac{T}{T_c}\right)^\alpha}$

4.ii) In BEC experiments, atoms are confined in magnetic traps which can be modelled by a quadratic potential of the type discussed in 2.9. Determine T_c for bosons in a 3D trap. Show that bosons in a 2D SHO trap will also condense at suitably low temperatures. In each case, calculate the number of particles in the condensate as a function of $T < T_c$

From 2.9 we see that $g(E) = \frac{4\pi E^2}{(\hbar\omega)^3}$ 3D & $g(E) = \frac{2\pi}{(\hbar\omega)^2} E$ 2D

$\therefore$ from above we get

$$T_c = \frac{1}{k_B}\left(\frac{N}{4\pi}(\hbar\omega)^3 \cdot \frac{1}{\Gamma(3) \cdot \zeta(3)}\right)^{1/3} \text{ in 3D}, \quad T_c = \frac{1}{k_B}\left(\frac{N}{2\pi}(\hbar\omega)^2 \frac{1}{\Gamma(2)\, \zeta(2)}\right)^{1/2}$$

$\alpha = 3$, $\Gamma(3) = 2!$, $\zeta(3) = 1.202$; $\alpha = 2$, $\Gamma(2) = 1!$, $\zeta(2) = \pi^2/6$

So $\exists\, T_c$ in 2 Dimensions, & 3D $N_0 = 1 - \left(\frac{T}{T_c}\right)^3$

2D $N_0 = 1 - \left(\frac{T}{T_c}\right)^2$ $\rightarrow N_0 = 1 - \left(\frac{T}{T_c}\right)^\alpha$

5. A system has two energy levels with energies 0 and ϵ. These can be occupied by (spinless) fermions from a particle and heat bath with temperature T and chemical potential μ. The fermions are non-interacting. Show that there are four possible microstates, and show that the grand partition function is

$$\mathcal{Z}(\mu, V, T) = 1 + z + z e^{-\beta\epsilon} + z^2 e^{-\beta\epsilon}$$

Where $z = e^{\beta\mu}$. Evaluate the average occupation number of the state of energy ϵ, and show that this is compatible with the result of the calculation of the average energy of the system using the fermi-dirac distribution. How could you take account of fermion interactions?

There are 4 possible microstates

$$P = e^{-\beta(E_r - N\mu)}$$

0 particles	$= 1$
1 particle in One state	$= e^{\beta\mu}$
1 particle in another state	$= e^{-\beta E + \beta\mu}$
1 particle in each state	$= e^{\beta\mu} \cdot e^{-\beta E + \beta\mu}$

Then $\mathcal{Z} = \sum_n P(n)$ or $\mathcal{Z} = \prod_r \mathcal{Z}_r = \prod_r \left(\sum_{n=0,1} e^{-\beta n(E_r - \mu)} \right)$

$$= \prod_r \left(1 + e^{-\beta(E_r - \mu)}\right)$$

$$= \left(1 + e^{\beta\mu}\right)\left(1 + e^{-\beta\epsilon + \beta\mu}\right)$$

$$= 1 + e^{\beta\mu} + e^{\beta\mu - \beta\epsilon} + e^{2\beta\mu - \beta\epsilon}$$

$$\therefore \boxed{\mathcal{Z} = 1 + z + z e^{-\beta\epsilon} + z^2 e^{-\beta\epsilon}}$$

$N(\epsilon) \Rightarrow N(\epsilon) = -\frac{1}{\beta} \frac{\partial}{\partial \epsilon} \ln(\mathcal{Z}) = -\frac{1}{\beta} \left(\frac{0 + 0 + z \cdot -\beta e^{-\beta\epsilon} + z^2 \cdot -\beta e^{-\beta\epsilon}}{\mathcal{Z}} \right)$

$$\therefore \quad N(\epsilon) = -\frac{1}{\beta} \cdot -\beta\left(Z_1 - (1+z)\right) \cdot \frac{1}{Z_1}$$

$$N(\epsilon) = 1 - \frac{1+z}{Z_1}$$ less than 1 occupant.

$$= \frac{1}{e^{\beta\epsilon - \beta\mu} + 1} \checkmark$$

Or $\rightarrow$ The fermi dirac distribution says $N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln(Z) = \sum_r \frac{1}{e^{\beta(E_r - \mu)}+1}$

$$n_\epsilon = \frac{1}{e^{\beta\epsilon - \beta\mu} + 1} \qquad = \sum_r n_r$$

and so avg number of particles must be

$$N = \frac{1}{e^{\beta\epsilon - \beta\mu} + 1} + \frac{1}{e^{-\beta\mu} + 1}$$

So we see that there are fewer particles in ϵ than intotal between ϵ & 0, so this makes sense.

or $$\langle E \rangle = \frac{\sum E_n P(n)}{Z_1} = \frac{0 + \epsilon N(\epsilon) \cdot Z_1}{Z_1} = \epsilon\left(e^{\beta\epsilon - \beta\mu} + 1\right)^{-1}$$

$$\text{So } \langle E \rangle = \langle N_\epsilon \rangle \cdot \epsilon \quad \text{or} \quad \langle N_\epsilon \rangle = \frac{\langle E \rangle}{\epsilon}$$

Which makes sense, the average number of particles in the state of energy ϵ would also be the average <u>total</u> energy!

I don't know how to include particle interactions.
Sommerfeld expansion?

Not average "average" energy!

ensemble average !!!

6. In an ideal fermi gas the average occupation numbers of the single particle state $|r\rangle$ is n_r. Show that the entropy

$$S = -\frac{\partial}{\partial T}\Phi = +\frac{\partial}{\partial T}\left(k_B T \ln(Z)\right)\Big|_{\mu, V} = -k_B \beta^2 \frac{\partial}{\partial \beta}\left(\frac{\ln(Z)}{\beta}\right)$$

Grand Potential

Can be written as

$$S = -k_B \sum_r \left[(1-n_r)\ln(1-n_r) + n_r \ln(n_r)\right]$$

and find the corresponding expression for an ideal bose gas.

Show that $(\Delta n_r)^2 = n_r(1-n_r)$ for the ideal fermi gas. Comment on this result, especially for very low T. What is the corresponding result for the ideal bose gas?

$$Z_1 = \prod_r \left(1+e^{-\beta(E_{r,\uparrow}-\mu)}\right)\left(1+e^{-\beta(E_{r\downarrow}-\mu)}\right)$$

single particle state

for $E_{s\uparrow} = E_{s\downarrow}$

$$Z_1 = \prod_r \left(1+e^{-\beta(E_r-\mu)}\right)^2 \rightarrow \ln Z = 2\cdot\sum_r \ln\left(1+e^{-\beta(E_r-\mu)}\right)$$

$$\frac{\partial}{\partial\beta}\ln(Z) = 2\sum_r \frac{-(E_r-\mu)\cdot e^{-\beta(E_r-\mu)}}{1+e^{-\beta(E_r-\mu)}}$$

$$= -2\sum_r (E_r-\mu)\, n_r$$

Eqns for Φ grand potential

$$\langle N\rangle = -\frac{\partial}{\partial\mu}\Phi$$

$$\langle U\rangle = \frac{\partial}{\partial\beta}(\beta\Phi) + \mu\langle N\rangle$$

$$\Phi = -\frac{1}{\beta}\ln Z_1$$

$$PV = -\Phi$$

$$S = -\frac{\partial}{\partial T}\Phi$$

~ eventually works out I guess

Skip rest of problems for now

→ distribution/continuum fermi/dirac gas

1

Tong problems

Sheet 4 (wait for help on tail of sheet 3 + all of 4)

1.