STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Graduate Placement Exam Part 1, Aug. 24 2010

General Instructions: This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas.

Each solution should typically take less than 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

Electromagnetism 1
Consider a capacitor at rest. The area of each plate is O, and the electric field between the plates is E. The plates are orthogonal to the z-axis. We assume that the mass of each plate is M^{pl} and there is a surface stress tensor in the plates due to the electric repulsion of the charges. The plates are kept a distance d apart by four thin columns. We assume that each of these columns have mass M^{col}, and there is a stress tensor in the columns due to the electric attraction of the plates.
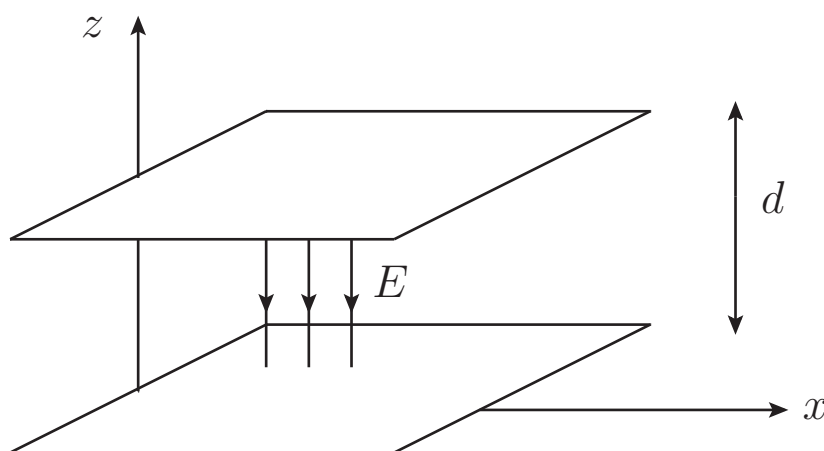
Do problems a,b,c, and **either d or e.**

a. (2 pts) Write down the expression for the energy-momentum tensor of the electromagnetic field in terms of the Maxwell field strength. What is the rest mass Mc^2 of the capacitor?

b. (4 pts) Prove that the volume integral of the space components of the sum of the stress tensor density for the electric field and the stress tensor density for the plates and columns vanishes, $\int T_{ij}^{total} d^3x = 0$. (Hint: consider $\frac{\partial}{\partial x^k}\left(x^i T_{kj}^{total}(x)\right)$) What are the nonvanishing components of the integrated stress tensor for the plates, and what are they for the columns? Express them in terms of the components of the integrated stress tensor of the electromagnetic field.

c. (6 pts) Consider now an observer who is moving in the positive z-direction with relativistic velocity v. According to special relativity the energy of the capacitor in the rest frame of this observer is γMc^2, where $\gamma = \left[1 - (v/c)^2\right]^{-1/2}$. Show that if one only takes into account the energy and stress of the electromagnetic field, and the mass of the plates and columns, but not the stress of the plates and columns, and one transforms T_{00}^{EM} according to the rules of special relativity, one does not get the correct answer.

d. (8 pts) Now repeat the calculations of c) but consider this time both the contributions to the total energy from the electromagnetic field and from the stress in the plates and the columns, and show that one obtains the correct result.

e. (8 pts) Repeat this calculation for an observer who is moving in the positive x direction. First calculate the electric and magnetic fields which the observer sees.

Solution:

a.

$$T_{\mu\nu}^{EM} \equiv \frac{1}{4\pi}\left(F_\mu{}^\alpha F_{\nu\alpha} - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$$

$$Mc^2 = \int T_{00}d^3x = 2M^{pl}c^2 + 4M^{col}c^2 + \frac{1}{8\pi}E^2Od \quad \text{in cgs units}$$

b. $\int \frac{\partial}{\partial x^k}\left(x^i T_{kj}^{total}(x)\right) d^3x = 0 = \int T_{ij}^{total}(x)d^3x$ because $\partial_k T_{kj} = \partial_0 T_{0j} = 0$ (the problem is independent of time). Only $\int T_{xx}^{pl}d^3x$, $\int T_{yy}^{pl}d^3x$ and $\int T_{zz}^{col}d^3x$ are nonvanishing. For the electromagnetic field $T_{i0}^{EM} = 0$ and

$$T_{ij}^{EM} = \frac{1}{4\pi}\left(F_i^0 F_{j0} + \frac{1}{2}\delta_{ij}F_{0k}F_0^k\right)$$

So $T_{xx}^{EM} = T_{yy}^{EM} = \frac{1}{8\pi}E^2$ and $T_{zz}^{EM} = -\frac{1}{8\pi}E^2$. Also $T_{i0} = 0$ for the plates and the columns. Then the total stress in the plates is $\int T_{xx}^{pl} = \int T_{yy}^{pl} = -\frac{1}{8\pi}E^2Od$ and the total stress in the four columns is $\int T_{zz}^{col} = +\frac{1}{8\pi}E^2Od$.

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c.
$$T'^{EM}_{00} = \frac{T^{EM}_{00} + (v/c)^2 \, T^{EM}_{zz}}{1 - (v/c)^2} = \frac{1}{8\pi} \frac{E^2 \left(1 - (v/c)^2\right)}{1 - (v/c)^2}$$

But there is Lorentz contraction, so the energy is $\frac{1}{8\pi}E^2 Od\sqrt{1 - (v/c)^2}$, which is not the correct result.

d.
$$\int T'^{EM}_{00} = \left(\frac{1}{8\pi}E^2\right)\left(Od\sqrt{1 - (v/c)^2}\right), \qquad \text{(from \textbf{c}) above)}$$

The energy of the plates transforms already correctly: $\int T'^{pl}_{00} = 2\gamma M^{pl}c^2$. The energy of the columns contains a part proportional to T^{col}_{00} which gives the correct result, plus a part due to T^{col}_{zz}

$$\int T'^{col}_{00} = 2\gamma M^{pl}c^2 + \frac{\int (v/c)^2 \, T^{col}_{zz}}{1 - (v/c)^2} = 2\gamma M^{pl}c^2 + \frac{\left(\frac{1}{8\pi}E^2 Od\right)}{1 - (v/c)^2}(v/c)^2\sqrt{1 - (v/c)^2}$$

Hence

$$\int T'^{EM}_{00} + \int T'^{pl}_{00} + \int T'^{col}_{00}$$

$$= \gamma\left(2M^{pl} + 4M^{col}\right) + \frac{1}{8\pi}E^2 Od\left(\sqrt{1 - (v/c)^2} + \frac{(v/c)^2}{\sqrt{1 - (v/c)^2}}\right) = \gamma Mc^2$$

e. Now a magnetic field $B'_y = \gamma(-v/c)E_z$ is present and $E'_z = \gamma E_z$

$$\int T'^{EM}_{00} = \int \frac{1}{8\pi}\left((E')^2 + (B')^2\right) = \frac{1}{8\pi}\frac{E^2 + (v/c)^2 E^2}{1 - (v/c)^2}Od\sqrt{1 - (v/c)^2}$$

Now the stress in the plates contributes

$$\int T'^{pl}_{00} = 4\gamma M^{col}c^2 + \frac{\int (v/c)^2 \, T^{pl}_{xx}}{1 - (v/c)^2} = 4\gamma M^{col}c^2 + \frac{-\frac{1}{8\pi}(v/c)^2 E^2 Od}{1 - (v/c)^2}\sqrt{1 - (v/c)^2}$$

Again these two contributions combine such that

$$\int T'^{EM}_{00} + \int T'^{pl}_{00} + \int T'^{col}_{00} = \gamma Mc^2$$

Electromagnetism 2
A circular capacitor of radius R and separation a, with $a \ll R$, is charged with a slow sinusoidal current, *i.e.* the charge on the plates is $Q(t) = \pm Q_o \sin(\omega t)$ as illustrated below. Neglect any fringing of the fields.

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$$-Q(t)$$
$$+Q(t)$$

R

a

3 D View   Side View
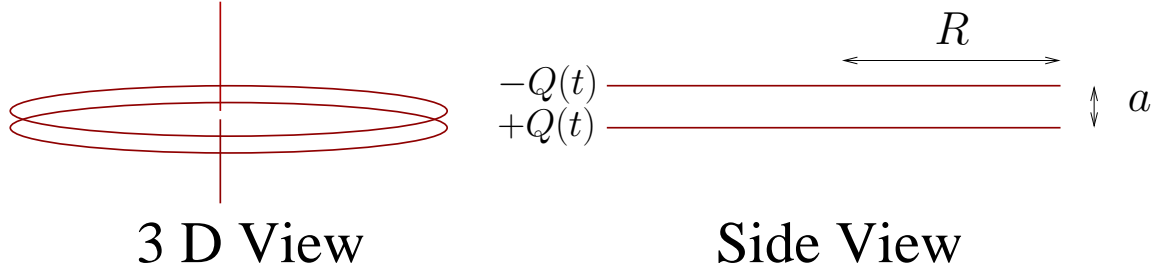
a. (6 pts) Determine the electric and magnetic fields in between the plates in the quasi-static approximation. Draw a picture to indicate the directions of the fields while the charge on the bottom plate is positive and increasing.

b. (2 pts) What is the size of typical corrections to the fields computed in part (a) due to the finite speed of light?

c. (6 pts) Write down the Maxwell equations for the gauge potentials ϕ and $\mathbf{A}$ in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$.

d. (6 pts) Determine the gauge potentials $(\phi, \mathbf{A})$ associated with the fields of part (1) and show that that they satisfy the Maxwell equations found in part (3) to the required order.

The curl of a vector field $\mathbf{F}$ in cylindrical coordinates is

$$\nabla \times \mathbf{F} = \left(\frac{1}{r} \frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z} \right) \hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r} \right) \hat{\boldsymbol{\theta}} + \left(\frac{1}{r} \frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \hat{\mathbf{z}} \quad (1)$$

The Laplacian is

$$\Delta^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{\partial^2 f}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} \quad (2)$$

Solution – Heaviside Lorentz Units

a. The electric field is

$$\nabla \cdot \mathbf{E} = \rho \qquad E^z = \frac{Q(t)}{A} \hat{\mathbf{z}} \quad (3)$$

The magnetic field is determined from Ampere's law without current

$$\frac{1}{c} \frac{\partial}{\partial t} \mathbf{E} - \nabla \times \mathbf{B} = 0 \quad (4)$$

So

$$B^\theta (2\pi r) = \frac{1}{c} \pi r^2 \partial_t E^z \quad (5)$$

Or

$$B^\theta = \frac{r\omega}{2c} \frac{Q}{A} \cos(\omega t) \quad (6)$$

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b. Corrections are of order

$$\left(\frac{R\omega}{c}\right)^2 \tag{7}$$

c. Then we have

$$\partial_\mu F^{\mu\nu} = -J^\nu/c \tag{8}$$

Or

$$\partial_i F^{i0} = -\rho \tag{9}$$

$$\partial_i(\partial^i A^0 - \partial^0 A^i) = -\rho \tag{10}$$

$$-\partial_i \partial^i A^0 = \rho \tag{11}$$

Similarly

$$\partial_0 F^{0i} + \partial_j F^{ji} = -j^i/c \tag{12}$$

$$\frac{1}{c}\partial_t(\partial^0 A^i - \partial^i A^0) + \partial_j(\partial^j A^i - \partial^i A^j) = 0 \tag{13}$$

$$-\frac{1}{c^2}\partial_t^2 A^i + \partial_j^2 A^i = \frac{1}{c}\partial_t \partial^i A^0 \tag{14}$$

d. Solving the Gauss Law equation for $A^0 = \phi$

$$-\partial_i \partial^i A^0 = 0 \qquad A^0 = -E^z(t)z \tag{15}$$

For **A** we have only a z component. And may drop ∂_t^2 in the quasi static approx

$$-\frac{1}{c^2}\partial_t^2 A^z + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A^z}{\partial r}\right) = \frac{1}{c}\partial_t \partial^z A^0 \tag{16}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A^z}{\partial r}\right) = \frac{-\omega}{c}\frac{Q}{A}\cos(\omega t) \tag{17}$$

Integrating this we find

$$A^z = -\frac{Q\omega}{4Ac}\cos(\omega t)r^2 \tag{18}$$

A straight forward sanity check gives $\mathbf{B} = \nabla \times A$

$$B^\theta = -\frac{\partial}{\partial r}A^z = \frac{Q}{A}\frac{r\omega_o}{2c}\cos\omega t \tag{19}$$

Electromagnetism 3
An electron of charge e and mass m_e moves in a circle of radius R_o with angular frequency ω_o. The electromagnetic field produced by the electron is observed at point O at a distance $R > R_o$ with frequency ω (see below). For different values of the

parameters $(\omega_o, \omega, R_o, R)$ relative to the constants $(e, m_e, \hbar$ and $c)$ the dynamics is vastly different, *i.e.* relativistic, non-relativistic, quantum or classical. In classical electrodynamics, the frequency of the observed radiation is of order $\omega \simeq \omega_o$ for non-relativistic motion, and $\omega \simeq \omega_o \gamma^3$ for relativistic motion, where $\gamma \equiv 1/\sqrt{1 - (v/c)^2}$.

$$\omega_O$$

[h!]
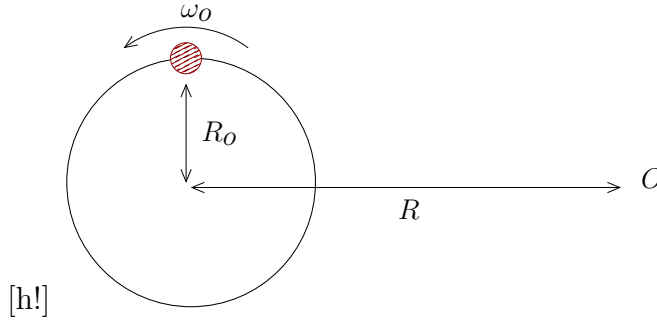
a. (7 pts) For each question give your answer as a condition on the parameters $(\omega_o, \omega, R, R_o)$ and constants $(e, m_e, \hbar$ and $c)$. For example, if the condition is that the motion is non-relativistic, write $v \ll c$ with $v = \omega_o R_o$.

   (a) For non-relativistic motion, what are the conditions for the electron's motion to be considered classical?

   (b) For non-relativistic motion, what are the conditions to treat the radiation as classical? Does this provide an additional constraint to part (a) ? Explain.

   (c) For relativistic motion, what are the conditions to treat the electron's motion as classical ?

   (d) For relativistic motion, what are the conditions to treat the radiation as classical? Does this provide an additional constraint to part (c) ? Explain.

b. (7 pts) This part discusses the classical electromagnetic field produced by classical non-relativistic motion:

   (a) Under what conditions of the parameters given above is the electromagnetic field observed at O near field or far field.

   (b) Compute (or remember) the power radiated per unit time. How does this power depend on the angular frequency.

   (c) A necessary condition for classical electrodynamics is that the energy lost to radiation does not significantly influence the motion of the particle over one revolution, *i.e.* that one can neglect the radiation reaction. Does this radiation reaction constraint provide additional constraints beyond those deduced in 1(a) and 1(b)? Explain.

c. (6 pts) Now consider an observer at O with $R = 8R_o$ in the ultra-relativistic (but still classical) regime. Is this near field or far field, explain. Describe qualitatively what the observer sees.

Solution:

a. Part 1:

(a) Non relativistically need

$$\frac{\hbar}{mv} \ll R_o \tag{20}$$

(b) The energy $\omega \simeq \omega_o$ needs to be less than E

$$\hbar\omega_o \ll \frac{1}{2}mv^2 \tag{21}$$

So with $\omega_o \sim v/R_o$ we see that this is the same conditions as part (a).

(c) Relativistically we need

$$\frac{\hbar}{p} \ll \frac{\hbar}{R_o} \tag{22}$$

but also require that the Compton wavelength is much less than the radius of orbit

$$\frac{\hbar}{mc} \ll R_o \tag{23}$$

which is in general more stringent than the de Broglie constraint.

(d) The radiation should be less than the energy

$$\hbar\omega \simeq \hbar\omega_o\gamma^3 \ll E \tag{24}$$

so this with $\omega_o \simeq c/R_o$ leads to the constraint

$$\gamma^2 \ll \frac{mcRo}{\hbar} \tag{25}$$

Thus γ can be large but not indefinitely large. This is a new constraint.

b. Part 2.

(a) Generally we will be in the far field region for $\lambda R \gg 1$. Another way to see this is to look at the fields from the Lienard potentials, which are composed of a coulomb piece and a radiative piece

$$\frac{e}{R^2} \qquad \frac{ea}{c^2R} \tag{26}$$

Putting $a = \omega_o^2 R$ leads to the requirement that $\omega_o^2 R^2/c^2 \gg 1$.

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(b) During a time R/c, the electron the electron make $\sim \omega_o R/c$ orbits. Thus the electron make many many orbits before the light reaches the observer.

(c) For the total power radiated we have the wonderfully simple formula, which can be remembered with dimensional reasoning. We see a characteristic ω_o^4 power

$$P = \frac{2}{3}\frac{e^2}{c^3}a^2 = \frac{2}{3}\frac{e^2}{c^3}\omega_o^4 R_o^2 \tag{27}$$

(d) Then we compare:

$$\frac{P}{\omega_o} \ll \frac{1}{2}mv^2$$

$$\frac{e^2}{\hbar c}\hbar\omega_o(\omega_o R_o)^2 \ll mc^2 v^2$$

$$\alpha(\hbar\omega_o) \ll mc^2$$

In the last line we used $\omega_o R_o \sim v$ and $\alpha = e^2/\hbar c \simeq 1/137$. Since we must have $\hbar\omega_o \ll mv^2$ this does not provide additional constraints. The radiation reaction is only important if observed for a long time.

c. Part 3. As the particle becomes relativistic the radiation becomes focused in a cone with $\theta \sim 1/\gamma$. And one sees blips of light as the strobe-light produced by the electron points in your direction.

Classical Mechanics 1

A particle is thrown up vertically with initial speed v_0, reaches a maximum height and falls back to ground. Show that the Coriolis deflection when it again reaches the ground is opposite in direction, and four times greater in magnitude, than the Coriolis deection when it is dropped at rest from the same maximum height.

Solution:

To make life easier just consider a vertical plane with coordinates z and x, such that the deflection will be in the x direction. First consider throwing it up from the ground. Then we know that $\frac{\partial^2 z}{\partial t^2} = -g$ and thus $z = -\frac{1}{2}gt^2 + v_0 t$. Now we use that $\frac{\partial^2 x}{\partial t^2}\hat{x} = -2\vec{\omega}\times\vec{v} = -2(\cos\theta\hat{z} - \sin\theta\hat{y})\times(v_0 - gt)\hat{z} = -2\omega\sin\theta(v_0 - gt)\hat{x}$ and then $x = -2\omega\sin\theta(-\frac{1}{6}gt^3 + \frac{1}{2}v_0 t^2)$

Thus $\vec{r} = (-\frac{1}{2}gt^2 + v_0 t)\hat{z} + (-2\omega\sin\theta(-\frac{1}{6}gt^3 + \frac{1}{2}v_0 t^2))\hat{x}$ This has time of flight $T = \frac{2v_0}{g}$ at the time

$$D_{corr,1} = -\frac{4}{3}\omega\sin\theta\frac{v_0^3}{g^2}$$

Looking at when $v_z = 0$ we can find the maximum height to be, $H = \frac{v_0^2}{2g}$.

Now considering it dropped from H. Then we find $z = -\frac{1}{2}gt^2 + \frac{v_0^2}{2g}$. Then $\frac{\partial^2 x}{\partial t^2}\hat{x} = -2\vec{\omega}\times\vec{v} = -2(\cos\theta\hat{z} - \sin\theta\hat{y})\times(-gt)\hat{z}$

Then $x(t) = \frac{1}{3}\omega\sin\theta gt^3$ and now the t_f when it hits the ground is $t_f = \frac{T}{2}$ and then

$$D_{corr,2} = \frac{1}{3}\omega\sin\theta\frac{v_0^3}{g^2}$$

and

$$\frac{D_{corr,1}}{D_{corr,2}} = -4.$$

Classical Mechanics 2

Calculate the differential scattering cross section of particles of energy E scattered by a central force potential $U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2}$ (α and β are positive constants).

Solution:

As usual $\theta = \pi - 2\int_{r_{min}}^{\infty}dr\frac{\frac{b}{r^2}}{\sqrt{1 - \frac{1}{r^2}(b^2 + \frac{\beta}{E}) + \frac{\alpha}{Er}}}$. This will give us $\cos\left(\frac{\sqrt{b^2 + \frac{\beta}{E}}}{2b}(\pi - \theta)\right) = \frac{\frac{\alpha}{2E}}{\sqrt{\frac{\alpha^2}{4E^2} + (b^2 + \frac{\beta}{E})}}$, which is a transcendental equation so let's approach this problem from another route. Let us make the assumption that the deflection angle is small. We then can use $\theta_1 = -\frac{b}{E}\int_b^{\infty}\frac{dU}{dr}\frac{dr}{\sqrt{r^2 - b^2}} = -\frac{p}{E}\int_b^{\infty}(\frac{\alpha}{r^2} - \frac{2\beta}{r^3})\frac{dr}{\sqrt{r^2 - b^2}}$ which gives us

$$\theta_1 = \frac{-\alpha}{Eb} + \frac{\beta\pi}{2Eb^2}$$

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which gives

$$b = \frac{-\alpha + \sqrt{\alpha^2 + 2\theta_1 E \beta \pi}}{2\theta_1 E}$$

and finally

$$d\sigma = 2\pi b\, db = \frac{2\pi E b^4}{\alpha b - \pi \beta}\, d\theta_1$$

Classical Mechanics 3

In 1915 A. Sommerfeld studied the relativistic corrections to Bohr's model of the hydrogen atom. He was motivated by the relativistic corrections to the orbits of planets which A. Einstein had obtained the same year. We retrace here Sommerfeld's steps.

Consider a relativistic point particle with mass m in three dimensions in an attractive potential $V = -\alpha/r$. We consider bound states and the particle moves in the equatorial plane $\theta = \pi/2$.

a) Write down the relativistic Lagrangian and Hamiltonian in Cartesian and polar coordinates. What are the relativistic canonical momenta in Cartesian and polar coordinates? Denote the polar coordinates by r and ϕ.

b) Write down the equation of motion for ϕ, and identify two conserved quantities. Solve the equation of motion for ϕ.

c) Obtain an equation for $\left(\frac{d\sigma}{d\phi}\right)^2$ where $\sigma = 1/r$. (Hint: Take the square of the relativistic kinectic energy $H - V$ and divide by the square of the relativistic momentum p_ϕ of ϕ.

d) Sommerfeld then went on to compute the relativistic binding energy by requiring that $\int_0^{2\pi} p_\phi d\phi = n_\phi \hbar$ and $\int p_r dr = n_r \hbar$. The expression for p_r contained a square root, and he used complex function theory to evaluate the contour integral around the corresponding cut in the complex p_r plane. Einstein wrote to him "A revelation", and Bohr wrote "I do not believe ever to have read anything with more joy than your beautiful work".

Solution:

(a) $L = -mc^2\sqrt{1 - \beta^2} + \alpha/r$ with $\beta^2 = (\vec{v}/c)^2$. With $\vec{p} = \frac{m\dot{x}}{\sqrt{1-\beta^2}}$ we

obtain $H = p\dot{q} - L = \frac{mc^2}{\sqrt{1-\beta^2}} - \alpha/r = mc^2\sqrt{1 + \vec{p}^2/(mc)^2} - \alpha/r$. In polar coordinates:

$$L = -mc^2\left[1 - \frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2}\right]^{\frac{1}{2}} + \frac{\alpha}{r}$$

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$$H = mc^2 \left[1 + \frac{p_r^2}{(mc)^2} + \frac{p_\phi^2}{(mrc)^2} \right]^{\frac{1}{2}} - \frac{\alpha}{r}$$

where $p_r = \gamma m \dot{r}$, $p_\phi = \gamma m r^2 \dot{\phi}$, $\gamma = 1/\sqrt{1 - \beta^2}$.

(b)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{mr^2\dot{\phi}}{\sqrt{1 - \beta^2}} \right) = 0 \rightarrow \frac{mr^2\dot{\phi}}{\sqrt{1 - \beta^2}} \equiv l \equiv constant$$

(c)

$$\left(E + \frac{\alpha}{r} \right)^2 = (cp_r)^2 + \frac{(cp_\phi)^2}{r^2} + (mc^2)^2$$

$$p_\phi = l, \quad \frac{p_r}{p_\phi} = \frac{m\dot{r}}{mr^2\dot{\phi}} = \frac{1}{r^2}\frac{dr}{d\phi} = -\frac{d\sigma}{d\phi}$$

$$\frac{(E + \alpha\sigma)^2}{l^2} = c^2\left(\frac{d\sigma}{d\phi} \right)^2 + (c\sigma)^2 + (mc^2/l)^2$$

(d)

$$\frac{d^2\sigma}{d\phi^2} + \left(1 - \frac{\alpha^2}{(lc)^2} \right)\sigma = \frac{E\alpha}{(lc)^2}$$

For $c \rightarrow \infty$ and E/c^2 fixed (note that E is the sum of mc^2 and the binding energy), the solution is $\sigma = \frac{1}{r} = \frac{a + \hat{c}\cos\phi}{b^2}$, where $a^2 = b^2 + \hat{c}^2$. In the relativistic case, the solution is $\sigma = \frac{a + \hat{c}\cos(\sqrt{A}\phi)}{b^2}$. The angular precession per revolution is $\left(\frac{1}{\sqrt{A}} - 1 \right)2\pi$.