$\boxed{\text{SOLUTIONS}}$

(1) $T_1 = \frac{1}{2} m (2\ell\dot\theta)^2$

$$T_2 = \frac{1}{2} m \left[4\ell^2\dot\theta^2 + \ell^2\dot\varphi^2 + 4\ell^2\dot\theta\dot\varphi\cos(\theta-\varphi) \right]$$

$$V_1 = mg2\ell\,(1-\cos\theta) \qquad V_2 = mg2\ell(1-\cos\theta) + mg\,\ell\,(1-\cos\varphi)$$

For small angles

$$L = 4m\ell^2\dot\theta^2 + \frac{1}{2}m\ell^2\dot\varphi^2 + 2m\ell^2\dot\theta\dot\varphi - 2mg\ell\theta^2 - \frac{1}{2}mg\ell\varphi^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0 \quad\Rightarrow\quad 4\ddot\theta + \ddot\varphi + 2\omega_0^2\theta = 0$$

$$\omega_0^2 \equiv g/\ell$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\varphi} - \frac{\partial L}{\partial\varphi} = 0 \quad\Rightarrow\quad \ddot\varphi + 2\ddot\theta + \omega_0^2\varphi = 0$$

$$-4\omega^2\theta - \omega^2\varphi + 2\omega_0^2\theta = 0$$

$$-\omega^2\varphi - 2\omega^2\theta + \omega_0^2\varphi = 0$$

$$(2\omega_0^2 - 4\omega^2)(\omega_0^2-\omega^2) - 2\omega^4 = 0$$

$$\boxed{\omega^2 = \left(\frac{3}{2} \pm \frac{\sqrt5}{2}\right)\omega_0^2}$$

Amplitudes:

Using first equation

$$\frac{\theta}{\varphi} = \frac{\omega^2}{2\omega_0^2 - 4\omega^2} = \frac{3\pm\sqrt5}{-8\mp4\sqrt5}$$

equivalent

Second equation

$$\frac{\theta}{\varphi} = \frac{\omega_0^2 - \omega^2}{2\omega^2} = \frac{-1\mp\sqrt5}{6\pm2\sqrt5}$$

(2) $v_p \approx$ constant so in planet's frame

$$v_i' = v_i + v_p$$

to conserve K.E. in planet's frame

$$v_f' = v_i + v_p$$

In sun's frame

$$\boxed{v_f = v_i + 2v_p}$$

(3) $N - mg\cos\theta = -\frac{mv^2}{R}$ (Newton II)

$v^2 = 2gR(1-\cos\theta)$ (Energy conservation)

Look for $N=0$

$$mg\cos\theta = \frac{m}{R} 2gR(1-\cos\theta)$$

$$\cos\theta = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right)$$

(4) $U(r) = U_0\left[\left(\frac{r_0}{r}\right)^a - \left(\frac{r_0}{r}\right)^{2a}\right]$

Find equilibrium: $\frac{1}{U_0}\frac{dU}{dr} = -\frac{a}{r_0}\left(\frac{r_0}{r}\right)^{a+1} + \frac{2a}{r_0}\left(\frac{r_0}{r}\right)^{2a+1}$

$$= 0 \Rightarrow \quad \left(\frac{r_0}{r}\right)^{a+1} = 2\left(\frac{r_0}{r}\right)^{2a+1}$$

$$r_{eq} = 2^{1/a} r_0$$

$$\omega^2 = \frac{1}{\mu} \frac{d^2u}{dr^2}\Big]_{r=r_{eq}} \qquad \mu = \text{reduced mass}$$

$$\frac{1}{u_0}\frac{d^2u}{dr^2} = \frac{a(a+1)}{r_0^2}\left(\frac{r_0}{r}\right)^{a+2} - \frac{2a(a+1)}{r_0^2}\left(\frac{r_0}{r}\right)^{2a+2}$$

Evaluate at r_{eq} to get

$$\frac{1}{u_0}\frac{d^2u}{dr^2}\Big]_{r=r_{eq}} = \frac{a}{2r_0^2}\left[\frac{a+1}{2^{2/a}} - \frac{2a+1}{2^{2/a}}\right] = \frac{-a^2}{2\,2^{2/a}\,r_0^2}$$

(Note u_0 must be negative)

$$\omega^2 = \frac{-u_0\, a^2 (m_1+m_2)}{2\,2^{2/a}\, r_0^2\, m_1 m_2}$$

E&M # 1

retarded time

$$A(r,t) = \frac{\hat{z}}{c}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}\,dz$$

distance to points on wire

for $t > r/c$, integrate out to

$$z = \pm\sqrt{(ct)^2 - r^2}$$

$$A(r,t) = \frac{I_0\hat{z}}{c}\,2\int_0^{\sqrt{(ct)^2-r^2}}\frac{dz}{\sqrt{r^2+z^2}}$$

$V = 0$ – Nothing is ever charged

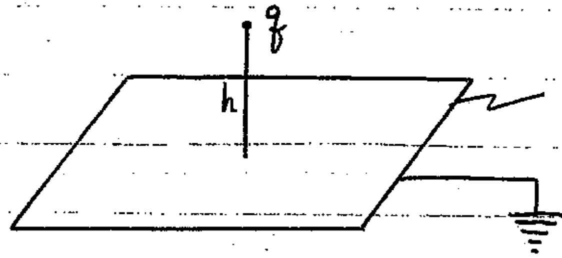
$$\vec{E}(r,t) = -\frac{\partial\vec{A}}{\partial t}$$

$|\vec{A}|$ is an increasing function of time, so $\vec{E}$ must point in the $-z$ direction

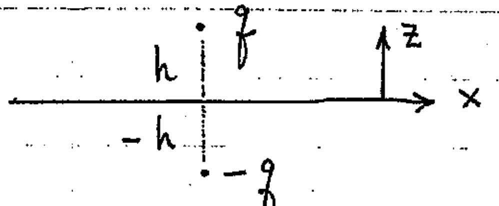
$|\vec{A}|$ clearly decreases with distance

B in $\hat{y}$ direction on right ↑ ↑ → this gives a curl pointing into
B in $-\hat{y}$ direction on left. the paper (out of paper on the left)

Electromagnetism #2



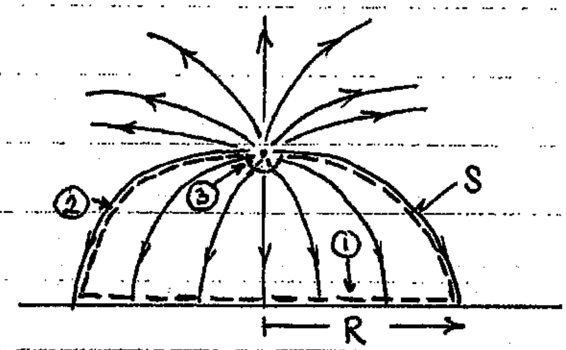
Replace the sheet by a negative image charge.



The potential is clearly zero everywhere on the plane $z=0$

For $z>0$ $\quad \vec{E} = \dfrac{q\left[(z-h)\hat{z}+x\hat{x}\right]}{\left[(z-h)^2+x^2\right]^{3/2}} - \dfrac{q\left[(z+h)\hat{z}+x\hat{x}\right]}{\left[(z+h)^2+x^2\right]^{3/2}}$ where y has been set $=0$ with no loss of generality

Consider the following Gaussian surface (dashed line) drawn on the field lines:



$$\int_S \vec{E}\cdot d\vec{A} = 0$$ since no charge is enclosed.

S is composed of ①, the region in the midplane bounded by the field lines which emanate horizontally from the charge q,

Electromagnetism #2 (cont.)

②, the surface formed by the field lines emanating horizontally from q, and ③, a hemisphere very near to the charge q.

$\int_{S_2} \vec{E}\cdot\vec{dA} = 0$ since the field lines are everywhere parallel to the surface

$\int_{S_3} \vec{E}\cdot\vec{dA} = -2\pi q$ since the field lines very near to the charge q are essentially radial and <u>un</u>affected by the charge $-q$ at $z=-h$.

$\Rightarrow \int_{S_1} \vec{E}\cdot\vec{dA} = 2\pi q$ to satisfy $\int_S \vec{E}\cdot\vec{dA} = 0$

$$\vec{E}\,(z=0) = \frac{q(-h\hat{z}+x\,\hat{x})}{[h^2+x^2]^{3/2}} - \frac{q(h\hat{z}+x\,\hat{x})}{[h^2+x^2]^{3/2}} = \frac{-2qh\,\hat{z}}{[h^2+x^2]^{3/2}}$$

$$\int_0^R \frac{-2qh\,\hat{z}}{[h^2+x^2]^{3/2}}\, 2\pi x\, dx\,(-\hat{z}) = 4\pi q h \int_0^R \frac{x\,dx}{[h^2+x^2]^{3/2}}$$

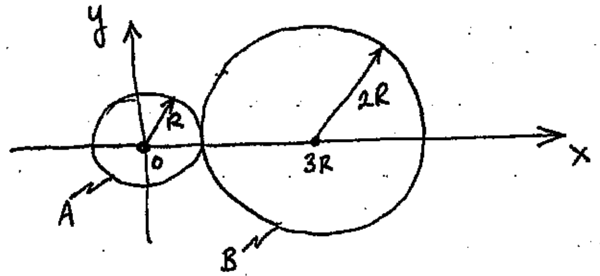
$$= 4\pi q h \left[\frac{1}{h} - \frac{1}{[h^2+R^2]^{1/2}}\right]$$

This should equal $2\pi q$, from above

So, $1 - \dfrac{h}{[h^2+R^2]^{1/2}} = 1/2$ or $1 = \dfrac{4h^2}{[h^2+R^2]}$

$$\boxed{R = \sqrt{3}\,h}$$

Electromagnetism #3



a. E field at the origin is centered is sphere A, outside of B

Thus, $E_x = \frac{-Q}{(3R)^2}$ from B $\qquad E_x = 0$ from A

$$\boxed{E_x = -\frac{Q}{9R^2}} \qquad E_y = E_z = 0 \text{ by symmetry}$$

b. At $x = R/2$

$$E_x = \frac{Q(<r)}{r^2} \text{ from A} \qquad E_x = -\frac{Q}{(5/2R)^2} \text{ from B}$$

$$= \frac{Q(r/R)^3}{r^2} = \frac{Q(1/2)^3}{(1/2)^2R^2}$$

$$\Rightarrow E_x = \frac{Q}{2R^2} - \frac{4Q}{25R^2} \quad (\text{at } r = R/2)$$

$$\boxed{E_x = \frac{17Q}{50R^2} \text{ at } r = R/2}$$

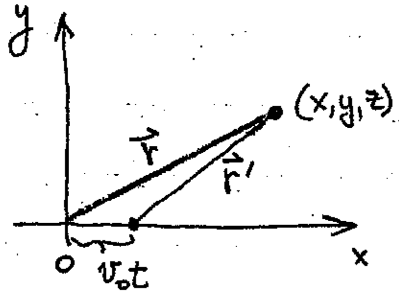
For the case where charge Q is distributed over the two spheres, weighted by volume, the corresponding answers are:

a. $E_x = -\frac{8Q}{81R^2}$

b. $E_x = \frac{1}{18}\frac{Q}{R^2} - \frac{32}{225}\frac{Q}{R^2} = -0.087\frac{Q}{R^2}$

note sign reversal

Electromagnetism #4



The electric field of a uniformly moving charge is

$$\vec{E} = \frac{q\gamma\vec{r}\,'}{[(\gamma x)^2+y^2+z^2]^{3/2}}$$

It always points radially away from the current position of the charge, but its strength depends on direction.

For low velocities $\gamma \to 1$

$$\vec{E} \cong \frac{q\vec{r}\,'}{r^3} \qquad \text{where } \vec{r}\,' = (x-v_0t)\hat{x} + y\hat{y} + z\hat{z}$$

or shorthand, $\vec{r}\,' = [(x-v_0t), y, z]$

The displacement current density is

$$\vec{J}_D = \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$$

$$= \frac{q}{4\pi}\left[\frac{-v_0\hat{x}}{r^3} + \frac{3\vec{r}\,'(x-v_0t)v_0}{r^5}\right]$$

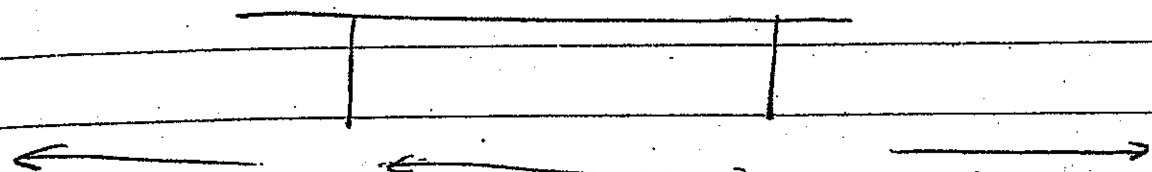
$$= \frac{qv_0}{4\pi r^5}\left[+3\vec{r}\,'(x-v_0t) - \hat{x}\,r'^2\right]$$

$$= \frac{qv_0}{4\pi r^5}\left[3(x-v_0t)^2\hat{x} + 3y(x-v_0t)\hat{y} + 3z(x-v_0t)\hat{z} - \hat{x}(x-v_0t)^2 - \hat{x}y^2 - \hat{x}z^2\right]$$

$$\boxed{\vec{J}_D = \frac{qv_0}{4\pi r'^5}\left[\left(2(x-v_0t)^2 - y^2 - z^2\right), 3y(x-v_0t), 3z(x-v_0t)\right]}$$

Quantum #1

a) Ground State is the symmetric state



$$A e^{K(x+a)} \qquad B\left(e^{-K(x+a)} + e^{K(x-a)}\right) \qquad A e^{-K(x-a)}$$

$$\boxed{A = B\left(1+e^{-2Ka}\right)}$$

b)

$$\psi'' + \frac{2m\eta}{\hbar^2}\,\delta(x)\,\psi = K^2\psi \quad \text{in each region}$$

integrate this around $x = a$ $\quad \displaystyle\int_{a-\varepsilon}^{a+\varepsilon}$

$$\psi'(a+\varepsilon) - \psi'(a-\varepsilon) + \frac{2m\eta}{\hbar^2}\,\psi(a) = 0$$

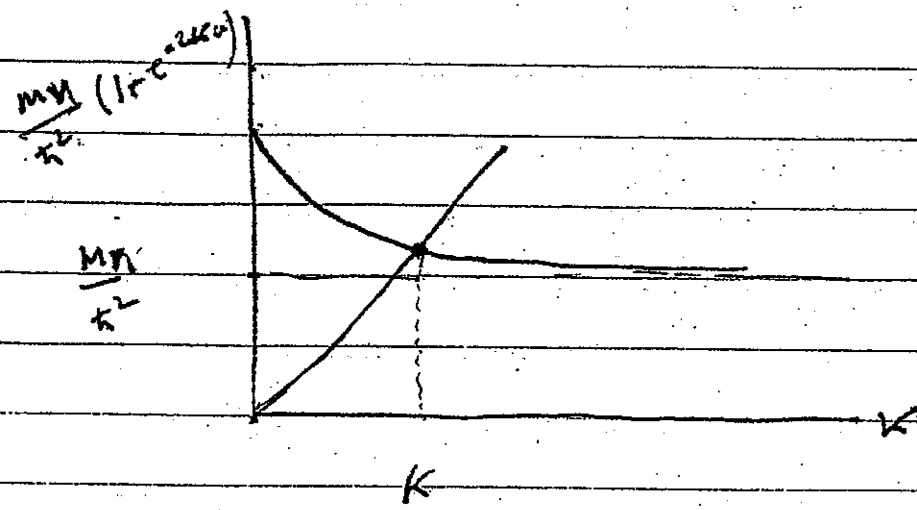
$$\psi'_{(x<a)}(x=a) = B\left(-e^{-2Ka} + 1\right)K$$

$$\psi'_{(x>a)}(x=a) = B\left(-e^{-2Ka} - 1\right)K$$

so

$$B\left(-e^{-2Ka} - 1\right)K - B\left(-e^{-2Ka} + 1\right)K + \frac{2m\eta}{\hbar^2}\,B\left(e^{-2Ka} + 1\right)$$

$$\therefore \quad \boxed{K = \frac{m\eta}{\hbar^2}\left(1 + e^{-2Ka}\right)} \qquad E = \frac{-\hbar^2 K^2}{2m}$$



c) If a is very large, $\kappa a >> 1$

$$\kappa = \frac{m\eta}{\hbar^2}\left(1 + \frac{1}{2\kappa a}\right)$$

If $a \to \infty$, this is the same as a single well in terms of energy. For $a < \infty$, κ is larger than it would be otherwise, therefore the Energy, $E = \frac{-\hbar^2 \kappa^2}{2m}$, is lower in the case of the double well when compared to the single well.

Quantum #2

a) Stationary State → ~~This~~ $S = 0$

b) This is probably easier to do if you remember that $S = -\frac{\hbar}{m} \text{Im}[\Psi(\nabla\Psi^*)]$ but you can do it using $\frac{\partial \rho}{\partial t} = \frac{-\partial S}{\partial x}$ as follows →

See next page

$$\Psi = \sqrt{\frac{1}{d}}\left[\sin\left(\frac{\pi}{d}x\right) + \sin\left(\frac{2\pi}{d}x\right)e^{-3i\omega_0 t}\right]e^{-i\omega_0 t}$$

$$\Psi^* = \sqrt{\frac{1}{d}}\left[\sin\left(\frac{\pi}{d}x\right) + \sin\left(\frac{2\pi}{d}x\right)e^{+3i\omega_0 t}\right]e^{i\omega_0 t}$$

$$\Psi^*\Psi = \frac{1}{d}\left[\sin^2\left(\frac{\pi}{d}\right) + \sin^2\left(\frac{2\pi}{d}\right) + \sin\left(\frac{\pi}{d}x\right)\sin\left(\frac{2\pi}{d}x\right)e^{-3i\omega_0 t}\right.$$

$$\left. + \sin\left(\frac{\pi}{d}x\right)\sin\left(\frac{2\pi}{d}x\right)e^{3i\omega_0 t}\right]$$

t dep part

$$\Psi^*\Psi = \frac{2}{d}\left[\sin\left(\frac{\pi}{d}x\right)\sin\left(\frac{2\pi}{d}x\right)\right]\cos 3\omega_0 t$$

$$\frac{\partial \rho}{\partial t} = -\frac{6\omega_0}{d}\sin\left(\frac{\pi}{d}x\right)\sin\left(\frac{2\pi}{d}x\right)\sin 3\omega_0 t = \frac{dJ}{dx}$$

$$J = \int_0^x \left(\frac{6\omega_0}{d}\sin 3\omega_0 t\right)\sin\left(\frac{\pi}{d}x\right)\sin\left(\frac{2\pi}{d}x\right)dx$$

$\sin 2u = 2\cos u \sin u$

$$\int = \frac{12 w_0}{d} \sin 3w_0 t \int_0^x \sin^2\left(\frac{\pi}{d}x\right)\cos\left(\frac{\pi}{d}x\right)dx$$

let $u = \frac{\pi}{d}x \quad du = \frac{\pi}{d}dx$

$$\int = \frac{12 w_0}{\pi} \sin 3w_0 t \int_0^u \sin^2 u \cos u \, du$$

let $\alpha = \sin u \quad d\alpha = \cos u \, du$

$$\int = \frac{12 w_0}{\pi} \sin 3w_0 t \int_0^{\alpha'} \alpha^2 \, d\alpha = \boxed{\frac{4 w_0}{\pi} \sin 3w_0 t \sin^3\left(\frac{\pi}{d}x\right)}$$

Quantum #3 Solution

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{\dagger})$$

$$P_{1\to 0} = \left| \frac{1}{\hbar} \int_0^T dt\, e^{i\omega_{10}t} \langle 0 | bx | 1 \rangle \right|^2$$

$$\omega_{10} = \frac{E_1 - E_0}{\hbar} = \omega$$

$$\langle 0 | x | 1 \rangle = \langle 0 | \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^{\dagger}) | 1 \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

$$\left| \int_0^T dt\, e^{i\omega t} \right|^2 = \left| \frac{e^{i\omega T} - 1}{i\omega} \right|^2 = \frac{4\sin^2\left(\frac{\omega T}{2}\right)}{\omega^2}$$

$$P_{1\to 0} = \frac{2b^2 \sin^2\left(\frac{\omega T}{2}\right)}{m\omega^3\hbar}$$

PART I - SPR 96
SECTION III

Problem #4 (Quantum Mechanics)

Consider a particle in a three dimensional square well potential of depth V_o and radius a. What is the necessary condition that at least one bound state exists when the particle has no angular momentum?

<Solution>

The Schrödinger equation for a bound state with no angular momentum is

$$-\frac{\hbar^2}{2m}\frac{d^2\chi(r)}{dr^2}=\begin{cases}(V_o-|E|)^2\chi & for \quad 0\le r<a\\ -|E|\chi & for \quad a<r\end{cases}$$

where

$$R(r)=\frac{\chi(r)}{r}$$

The solution is

$$\chi(r)=\begin{cases}A\sin\alpha r & for \quad 0\le r<a\\ Be^{-\beta r} & for \quad a<r\end{cases}$$

where

$$\alpha=\sqrt{\frac{2m(V_o-|E|)}{\hbar^2}} \quad \text{and} \quad \beta=\sqrt{\frac{2m|E|}{\hbar^2}}$$

The boundary condition at $r=a$ gives

$$\beta=-\alpha\cot\alpha a \quad and \quad \alpha^2+\beta^2=\frac{2mV_o}{\hbar^2}$$

Finding a condition that the two graphs have intersection is

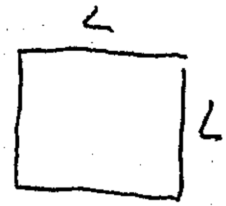
$$\frac{2mV_o}{\hbar^2}\le\left(\frac{\pi}{2a}\right)^2 \quad or \quad V_oa^2\le\frac{\pi^2\hbar^2}{8m}$$

<End>

Solutions, Statistical Mechanics Section Part I.IV
2/96

Problem 1 – 10 pts.

[1] (a) $D(\vec{k})\, dk_x\, dk_y = \underbrace{2}_{\text{2 spin states}} \cdot \overbrace{L^2}^{A} \cdot \frac{dk_x\, dk_y}{(2\pi)^2}$



box

$$D(\vec{k}) = A/2\pi^2$$

[2] (b) $\varepsilon_k = \hbar^2 k^2/2m$

$d\varepsilon = \hbar^2 k\, dk/m$

$dk_x\, dk_y = 2\pi k\, dk = \frac{2\pi m}{\hbar^2}\, d\varepsilon$

so $D(\varepsilon)\, d\varepsilon = \frac{A}{2\pi^2} \cdot \left(\frac{2\pi m}{\hbar^2}\, d\varepsilon\right)$

$$D(\varepsilon) = \frac{mA}{\pi\hbar^2} = \text{constant}$$

[2] (c) $N = \int_0^{\varepsilon_F} D(\varepsilon)\, d\varepsilon = \frac{mA}{\pi\hbar^2}\varepsilon_F \longrightarrow \varepsilon_F = \frac{\pi\hbar^2}{m}\frac{N}{A}$

$\hbar^2 k_F^2/2m = \frac{\pi\hbar^2}{m}\frac{N}{A} \longrightarrow k_F = \sqrt{2\pi N/A}$

[2] (d) $E = \int_0^{\varepsilon_F} \varepsilon D(\varepsilon)\, d\varepsilon = \frac{mA}{\pi\hbar^2}\frac{\varepsilon_F^2}{2} = \frac{\pi\hbar^2}{2m} \cdot \frac{N^2}{A}$

[2] (e) $dE = T\,dS + \gamma\, dA + \mu\, dT$, $\gamma = \left.\frac{\partial E}{\partial A}\right)_{S,T}$

also, $S = 0$ at $T = 0 \longrightarrow \gamma = -\frac{\pi\hbar^2}{2m}\left(\frac{N}{A}\right)^2$

[1] (f) Temperature dependence will be weaker in two dimensions because $D(\varepsilon)$ is constant.

Problem 2 – 10 points

$$N = n_+ + n_-$$
$$x = \ell(n_+ - n_-) = \ell(2n_+ - N) \longrightarrow n_\pm = \frac{N}{2} \pm \frac{x}{2\ell}$$

[1] (a) $S = k \ln \Gamma$

Γ = # ways of distributing N elements into n_+, n_-

$$\Gamma = \frac{N!}{n_+!\, n_-!}$$

$$S = k\underbrace{(\ln N! - \ln n_+! - \ln n_-!)}_{\xi}$$

$$\xi \cong (N \ln N - \cancel{N}) - (n_+ \ln n_+ - \cancel{n_+} + n_- \ln n_- - \cancel{n_-})$$

$$= N \ln N - n_+ \ln n_+ - \underbrace{n_- \ln n_-}_{(N-n_+)\ln(N-n_+)}$$

$$\xi = N \ln\left(\frac{N}{N-n_+}\right) - n_+ \ln\left(\frac{n_+}{N-n_+}\right), \quad S = k\xi$$

[2] (b) $F = U - TS = U - kT\xi$

U is independent of x and n_+, can shift to 0

[4] (c) $\tau = \frac{\partial F}{\partial x} = -kT \frac{\partial \xi}{\partial x}$

$$\frac{\partial \xi}{\partial x} = -\left[\frac{n_+}{n_+}\frac{dn_+}{dx} + \frac{dn_+}{dx}\ln n_+ + \frac{n_-}{n_-}\frac{dn_-}{dx} + \frac{dn_-}{dx}\ln n_-\right]$$

$$dn_\pm/dx = \pm 1/2\ell$$

$$\frac{d\xi}{dx} = -\frac{1}{2\ell}(\ln n_+ - \ln n_-) = -\frac{1}{2\ell}\ln\frac{n_+}{n_-}$$

$$\tau = \frac{kT}{2\ell}\ln\frac{n_+}{n_-} = \frac{kT}{2\ell}\ln\left(\frac{\frac{N}{2}+\frac{x}{2\ell}}{\frac{N}{2}-\frac{x}{2\ell}}\right) = \frac{kT}{2\ell}\ln\left(\frac{1+\frac{x}{NL}}{1-\frac{x}{NL}}\right)$$

Problem 3 – 10 points

canonical partition fn for a perfect gas: $Z = \dfrac{z^N}{N!}$

$$z = \sum_s e^{-\varepsilon(s)/kT} = \frac{V}{\lambda_T^3}\, g\, e^{-\varepsilon/kT}$$

s = microstate
λ_T = thermal wavelength
g = degenerag factor

$$F = -kT \ln Z, \quad \mu = (\partial F/\partial N)_{T,V}$$

$$\ln Z = N \ln z - \ln N! = N \ln z - N \ln N + N$$

$$\frac{\partial \ln Z}{\partial N} = \ln z - \ln N - \cancel{1} + \cancel{1} = \ln(z/N)$$

$$\mu = -kT \ln(z/N)$$

[7] (a) for the A (B)-state molecules,

$$\mu_A = -kT \ln\left(\frac{V}{\lambda_T^3}\,\frac{g_A}{N_A}\,e^{-\varepsilon_A/kT}\right)$$

$$\mu_B = -kT \ln\left(\frac{V}{\lambda_T^3}\,\frac{g_B}{N_B}\,e^{-\varepsilon_B/kT}\right)$$

[3] (b) $\mu_a = \mu_B$

$$\frac{g_A}{N_A} e^{-\varepsilon_A/kT} = \frac{g_B}{N_B} e^{-\varepsilon_B/kT}$$

$$\frac{N_A}{N_B} = \frac{g_A}{g_B} e^{\Delta\varepsilon/kT}, \quad \Delta\varepsilon = \varepsilon_B - \varepsilon_A$$

For a sequence of random variations, each with standard deviation σ (not necessarily distributed in a Gaussian manner), after a very large number of steps, N, the Central Limit Theorem applies — the distribution will be Gaussian with standard deviation $\sigma\sqrt{N}$. Also, since ~~the~~ in this problem the mean step length is ℓ, the distribution after N steps will be centered at $x = N\ell$:

$$P(x) = \frac{1}{\sqrt{2\pi N\sigma^2}} \exp - \frac{(x-N\ell)^2}{2N\sigma^2} .$$

<u>alternate solution</u> After N steps,

$$P(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\, dx_2 \cdots dx_N \; W(x_1)\, W(x_2-x_1) \cdots W(x-x_N) .$$

$P(x)$ may be obtained from the characteristic function,

$$\Phi(k) = \int_{-\infty}^{\infty} W(s)\, e^{iks}\, ds$$

$$P(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\Phi(k)]^N e^{-ikx}\, dk$$

For the given $W(s)$,

$$\Phi(k) = \exp(-k^2\sigma^2 + ik\ell)$$

and

$$P(x) = \frac{1}{\sqrt{2\pi N\sigma^2}} \exp - \frac{(x-N\ell)^2}{2N\sigma^2} .$$

Note that this latter solution holds even if N is not very large.