

STONY BROOK UNIVERSITY
DEPARTMENT OF PHYSICS AND ASTRONOMY

Graduate Placement Exam, Part 2

Aug. 21, 2013 (6:00 pm - 10:30 pm)

General Instructions: This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas. Each problem is worth 20 points, and unless stated otherwise, all parts of each question have equal weight.

Each solution should typically take less than 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

Quantum Mechanics 1

A particle of mass m and energy $E = \hbar^2 k^2 / 2m$ scatters off a 3-dimensional radial potential:

$$V(r) = \begin{cases} -V_0 & a < r \\ 0 & r \geq a \end{cases} \quad (1)$$

- a. (4 pts.) Why does the $l = 0$ partial wave dominate the scattering near threshold (zero energy)?
- b. (8 pts.) Derive an expression for the S-wave phase shift $\delta_{l=0}$ by matching at $r = a$ the $l = 0$ radial waves.
- c. (8 pts.) What is the threshold cross section.

Note: It is useful to define $\hbar^2/2m(k_1^2 = k^2 + k_0^2)$ with $\hbar^2 k_0^2 / 2m = V_0$.

Solution

- a. At threshold with $k \approx 0$, the partial wave amplitude $f_l(k) \approx k^{2l}$, so the $l = 0$ dominates the cross section.
- b. Inside and outside the well the reduced radial wavefunctions are

$$\begin{aligned} u_{<}(r) &= rR_0 = A \sin(k_1 r) & \frac{u_{<}'}{u_{<}} &= k_1 \cotan(k_1 r) \\ u_{>}(r) &= B \sin(kr + \delta_0) & \frac{u_{>}'}{u_{>}} &= k \cotan(kr + \delta_0) \end{aligned} \quad (2)$$

with $\hbar^2/2m(k_1^2 = k^2 + k_0^2)$ and $\hbar^2 k_0^2 / 2m = V_0$. Matching the logarithmic derivatives yields

$$k_1 \cotan(k_1 a) = k \cotan(ka + \delta_0) \quad (3)$$

which can be rewritten as

$$\cotan(\delta_0) = \frac{k \sin(ka) + k_1 \cotan(k_1 a) \cos(ka)}{k \cos(ka) - k_1 \cotan(k_1 a) \sin(ka)} \quad (4)$$

- c. The total cross section involves the partial wave amplitude $f_l(k) = \sin(\delta_l(k))/k$.
Near threshold

$$\sigma(k) = 4\pi \sum_{l=0}^{\infty} (2l+1) \frac{\sin^2 \delta_l(k)}{k} \approx \frac{\delta_0^2(k)}{k^2} \quad (5)$$

From (4) it follows

$$\frac{\delta_0(k)}{k} \approx \frac{1}{k \cotan(\delta_0)} = a - \frac{\tan(k_0 a)}{k_0} \quad (6)$$

Thus

$$\sigma(k=0) = 4\pi \left(a - \frac{\tan(k_0 a)}{k_0} \right)^2 \quad (7)$$

Quantum Mechanics 2

Consider a particle on the x -axis with potential $U(x)$ such that $U(x)$ vanishes as $x \rightarrow \pm\infty$, and $U(x)$ is everywhere negative and nonsingular. Recall that the ground state for such a system is always a nondegenerate bound state $\phi_0(x)$.

- a. (4 pts.) Define $V(x) = U(x) - E_0$ where E_0 is the ground state energy. Write the Hamiltonian in factorized form as $H = A^\dagger A + E_0$ where $A = c \frac{d}{dx} + W(x)$ and c is a constant. Determine c and $W(x)$. (Hint: try the logarithmic derivative of $\phi_0(x)$ for W .)
- b. (6 pts.) Consider now two systems, one with Hamiltonian $H_1 = A^\dagger A + E_0$ and another with Hamiltonian $H_2 = AA^\dagger + E_0$. Let $A^\dagger A = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_1$ and $AA^\dagger = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_2$. Construct the $W_1(x)$ which gives $V_1 = a^2$, where a is a real constant, and construct the corresponding $V_2(x)$. (Hint: the solution of the Riccati equation $\frac{d}{dx}y + y^2 = 1$ is given by $y(x) = \tanh x$.)
- c. (4 pts.) Show that A annihilates $\phi_0(x)$. Show that $H_1 = A^\dagger A + E_0$ and $H_2 = AA^\dagger + E_0$ have the same non-vanishing eigenvalues. Draw a picture of the eigenvalues of H_1 and H_2 , both the discrete and the continuous ones. (Hint: act with A on H_1 .)
- d. (6 pts.) If $A^\dagger A$ has a constant potential $V_1 = a^2$, the solutions for H_1 are plane waves. Prove that then the potential $V_2(x)$ of H_2 is also reflectionless. (A potential is called reflectionless if every incoming plane wave of the continuous spectrum is transmitted without reflection. In other words, there is total transmission.)

Solution

- a. The ground state wave function satisfies $A^\dagger A \phi_0(x) = 0$. Hence $A \phi_0(x) = 0$. Assuming that $W = \alpha \phi'_0 / \phi_0$ one finds

$$A = \frac{\hbar}{\sqrt{2m}} \left(\frac{d}{dx} - \frac{\phi'_0}{\phi_0} \right) \quad (1)$$

- b.

$$-\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W_1 + W_1^2 = a^2 \quad \Rightarrow \quad W_1 = -a \tanh \left(\frac{\sqrt{2m}}{\hbar} ax \right)$$

$$V_2 = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} W_1 + W_1^2 = a^2 \left(-1 + 2 \tanh^2 \left(\frac{\sqrt{2m}}{\hbar} ax \right) \right) \quad (2)$$

c. From $A^\dagger A \phi_0(x) = 0$ we find after partial integration that

$$\int_{-\infty}^{\infty} |A \phi_0|^2 dx = 0 \quad (3)$$

Hence A annihilates ϕ_0 . Indeed

$$\left(\frac{d}{dx} - \frac{\phi'_0}{\phi_0} \right) \phi_0 = 0 \quad (4)$$

With

$$\begin{aligned} H_1 \phi_n^{(1)} &= (A^\dagger A + E_0) \phi_n^{(1)} = E_n^{(1)} \phi_n^{(1)} \\ H_2 \phi_n^{(2)} &= (A A^\dagger + E_0) \phi_n^{(2)} = E_n^{(2)} \phi_n^{(2)} \end{aligned} \quad (5)$$

we then find

$$\begin{aligned} A(A^\dagger A \phi_n^{(1)}) &= (E_n^{(1)} - E_0) A \phi_n^{(1)} \Rightarrow H_2 A \phi_n^{(1)} = E_n^{(1)} A \phi_n^{(1)} \\ A^\dagger(A A^\dagger \phi_n^{(2)}) &= (E_n^{(2)} - E_0) A^\dagger \phi_n^{(2)} \Rightarrow H_1 A^\dagger \phi_n^{(2)} = E_n^{(2)} A^\dagger \phi_n^{(2)} \end{aligned} \quad (6)$$

and therefore $A \phi_n^{(1)} = \phi_{n-1}^{(2)}$ and $E_n^{(1)} = E_{n-1}^{(2)}$. The spectrum has the following structure

$$\begin{array}{ccc} E_k^{(1)} & \text{-----} & E_{k-1}^{(2)} \\ & \text{-----} & \\ E_2^{(1)} & \text{-----} & E_1^{(2)} \\ E_1^{(1)} & \text{-----} & E_0^{(2)} \\ E_0^{(1)} & \text{-----} & \end{array}$$

d. H_1 is reflectionless, and the continuum eigenfunctions $\psi_k^{(2)}(x)$ of H_2 are proportional to $A \psi_k^{(1)}(x)$. Because $A = c \frac{d}{dx} + W_1$ tends to $c \frac{d}{dx} - a$ at infinity, if $\psi_k^{(1)}(x)$ is a plane wave at infinity, then so is $\psi_k^{(2)}(x)$.

Quantum Mechanics 3

Consider two spin-1/2 fermions of mass m , coordinates x_1, x_2 , confined to move on a circle of circumference L and interacting through a spin-dependent potential

$$V = -u\delta(x_1 - x_2)\vec{s}_1 \cdot \vec{s}_2, \quad u > 0,$$

where $\vec{s} = \{s_x, s_y, s_z\}$ is the operator of the spin 1/2 (in units of \hbar), so that the Hamiltonian H of the two-electron system (in the standard notations) is:

$$H = \frac{1}{2m}(p_1^2 + p_2^2) + V.$$

- a. (10 pts.) Derive the boundary conditions for the orbital part $\psi(x_1, x_2)$ of the wavefunction at $x_1 = x_2$ in different spin eigenstates of the spin part of the interaction V .
- b. (10 pts.) Find the energy E_0 and the wavefunction $|\psi_0\rangle$ of the ground state in the limit $u \rightarrow \infty$.

Solution

- a. As usual, making use of the operator $\vec{S} = \vec{s}_1 + \vec{s}_2$ of the total spin of the two fermions, one can write the spin part of the interaction V as

$$\vec{s}_1 \cdot \vec{s}_2 = [\vec{S}^2 - \vec{s}_1^2 - \vec{s}_2^2]/2 = \frac{1}{2}\vec{S}^2 - \frac{3}{4},$$

and see that the eigenstates of the operator $\vec{s}_1 \cdot \vec{s}_2$ are the triplet and the singlet states $|S = 1\rangle$ and $|S = 0\rangle$:

$$\vec{s}_1 \cdot \vec{s}_2 |S = 1\rangle = \frac{1}{4} |S = 1\rangle, \quad \vec{s}_1 \cdot \vec{s}_2 |S = 0\rangle = -\frac{3}{4} |S = 0\rangle.$$

Since these states are, respectively, symmetric and antisymmetric in the two spins:

$$|S = 1\rangle = \{|\uparrow\uparrow\rangle; |\downarrow\downarrow\rangle; \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle]\}, \quad |S = 0\rangle = \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle],$$

requirement that the total wavefunction of the two fermions is antisymmetric means that the orbital part $\psi(x_1, x_2)$ of the wavefunction is symmetric in the singlet and antisymmetric in the triplet states. This means that, in the states of the triplet, $\psi(x_1, x_2)\delta(x_1 - x_2) \equiv 0$ and the interaction V does not have any effect on the wavefunction. I.e., in this state, $\psi(x_1, x_2)$ is continuous together with first derivatives at $x_1 = x_2$, as at other regular points.

In the singlet state, the interaction V reduces to the form $V = (3u/4)\delta(x_1 - x_2)$, and its effect on $\psi(x_1, x_2)$ can be described in the same way as for the standard single-particle δ -function potential. Integrating the spatial Schrödinger equation over infinitesimal interval of x_1 around the point $x_1 = x_2$, one obtains that the wavefunction is continuous at $x_1 = x_2$ with discontinuous first derivative:

$$\left. \frac{\partial \psi}{\partial x_1} \right|_{x_1=x_2+0} - \left. \frac{\partial \psi}{\partial x_1} \right|_{x_1=x_2-0} = (3mu/2\hbar^2)\psi(x_1 = x_2).$$

Similarly for x_2 :

$$\left. \frac{\partial \psi}{\partial x_2} \right|_{x_2=x_1+0} - \left. \frac{\partial \psi}{\partial x_2} \right|_{x_2=x_1-0} = (3mu/2\hbar^2)\psi(x_1 = x_2).$$

The fact that the wavefunction $\psi(x_1, x_2)$ is symmetric in the singlet state, implies that

$$\left. \frac{\partial \psi}{\partial x_1} \right|_{x_1=x_2-0} = \left. \frac{\partial \psi}{\partial x_2} \right|_{x_2=x_1-0},$$

and we can combine the relation above to finally obtain the sought boundary condition:

$$\left(\frac{\partial \psi}{\partial x_1} - \frac{\partial \psi}{\partial x_2} \right) \Big|_{x_1=x_2+0} = (3mu/2\hbar^2)\psi(x_1 = x_2). \quad (1)$$

- b. To find the ground state of the system, we need to compare the lowest energies in the triplet and the singlet state. *In the triplet state*, as discussed above, the two fermions are effectively noninteracting, and the stationary states of the two fermions are constructed as usual out of the single-particle momentum states $\psi_k(x)$, with momenta quantized on the circle: $k = 2\pi\hbar n/L$, with integer n ,

$$\psi^{(t)}(x_1, x_2) = \frac{1}{\sqrt{2}}[\psi_{k_1}(x_1)\psi_{k_2}(x_2) - \psi_{k_1}(x_2)\psi_{k_2}(x_1)]. \quad (2)$$

Since $k_1 \neq k_2$, the triplet with the lowest energy $E_0^{(t)}$ corresponds to, e.g., $k_1 = 0$ and $k_2 = 2\pi\hbar/L$, so that

$$E_0^{(t)} = \frac{2}{m} \left(\frac{\pi\hbar}{L} \right)^2.$$

In the singlet state, the boundary condition (1) together with the fact that momenta are finite, in the limit $u \rightarrow \infty$, mean that $\psi(x_1 = x_2) = 0$. One can immediately see that such symmetric wavefunction $\psi^{(s)}(x_1, x_2)$ vanishing at coincident coordinates can be constructed out of the antisymmetric wavefunction (2) by the sign change:

$$\psi^{(s)}(x_1, x_2) = \psi^{(t)}(x_1, x_2) \operatorname{sgn}(x_1 - x_2). \quad (3)$$

Such a sign change implies however, that the single-particle wavefunctions on a circle should be antiperiodic, changing the quantization of momenta:

$$e^{ikL} = -1, \quad k = \pi\hbar(2n+1)/L.$$

In this case, the lowest-energy state corresponds to $k_{1,2} = \pm\pi\hbar/L$, giving the ground state energy (lower than the triplet energy $E_0^{(t)}$)

$$E_0 = \frac{1}{m} \left(\frac{\pi\hbar}{L} \right)^2.$$

The wavefunction that corresponds to this state is

$$|\psi_0\rangle = \frac{\sqrt{2}}{L} \sin \left[\frac{\pi}{L} |x_1 - x_2| \right] \otimes |S = 0\rangle.$$

Statistical Mechanics 1

A particle of charge e and mass m moves in an external magnetic field along the z -direction $\vec{H} = H\hat{z}$, in a volume $V_3 = L^3$ with $L \gg l_H = mc^2/eH$.

- (5 pts.) Determine the Landau levels of this charge and their corresponding degeneracy in terms of the free momentum p_z and the cyclotron frequency $\omega_0 = eH/mc$.
- (8 pts.) Use these levels to evaluate the partition function Ω_H at high temperature T in the limit where $x = \hbar\omega_0/2k_B T \ll 1$ (k_B is Boltzmann constant).
- (5 pts.) Calculate the magnetic susceptibility at high temperature. Show that it is diamagnetic and obeys Curie's law.
- (2 pts.) Explain why this law is expected on physical grounds.

Note: Recall that $\int_0^\infty e^{-x^2} = \sqrt{\pi}/2$.

Solution

- The energy levels of a charged particle in a magnetic field $H\hat{z}$ are given by Landau levels

$$\epsilon(p, n) = \frac{p_z^2}{2m} + \hbar\omega_0 \left(n + \frac{1}{2} \right) \quad (1)$$

with $n = 0, 1, \dots$. Each level is $g = (eH/hc)L^2$ degenerate.

- At high temperature the grand partition function follows from Boltzmann statistics

$$\Omega_H \approx \frac{2gL}{h} \sum_{n=0}^{\infty} \int_0^\infty dp_z z_T e^{-\epsilon(p,n)/T} \quad (2)$$

The sum over n is geometrical and the integral over p is Gaussian. The result is

$$\Omega_H \approx \frac{z_T g L}{\lambda_T} \frac{e^{-x}}{1 - e^{-2x}} \approx \frac{z_T g L}{\lambda_T} \frac{1}{2x} \left(1 - \frac{x^2}{6} \right) \quad (3)$$

with the Boltzmann fugacity $z_T = \lambda_T^3/V_3 = (2\pi\hbar^2/mk_B T)^{3/2}/V_3$ and $x = \hbar\omega_0/2k_B T \ll 1$.

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$$\chi_H = k_B T \frac{\partial^2 \ln \Omega_H}{\partial H^2} \frac{1}{V_3} \approx -\frac{z_T}{3k_B T \lambda_T^3} \left(\frac{e\hbar}{2mc} \right)^2 \quad (4)$$

Using $z_T = \lambda_T^3/V_3$ yields

$$\chi_H \approx -\frac{1}{3k_BTV_3} \left(\frac{e\hbar}{2mc} \right)^2 \quad (5)$$

which is Curie' s law. The negative sign accounts for diamagnetism.

d. The magnetization per particle at high temperature follows from Boltzmann

$$\frac{\mathbf{M}}{\mathbf{N}/V_3} \approx \frac{1}{3} \left(\frac{-\mu_0 H}{k_B T} \right) \mu_0 \quad (6)$$

with magnetic moment $\mu_0 = e\hbar/2mc$. This is Curie' s law.

Statistical Mechanics 2

Two classical particles are placed on a ring with $N > 2$ similar, available sites. The particles cannot occupy the same site, and if they occupy a pair of neighboring sites their repulsion increases the energy of the system by Δ .

- a. (12 pts.) Find the average energy $\langle E \rangle$ of the system at temperature T .
- b. (4 pts.) Find the energy variance $(\Delta E)^2 = \langle E^2 \rangle - \langle E \rangle^2$.
- c. (4 pts.) Is there any value of N at which the r.m.s. fluctuation ΔE is smaller than $\langle E \rangle$? If yes, find the corresponding temperature range.

Solution

Because of the rotation symmetry, fixation of one particle does not change anything (If we considered its N positions distinguishable, the state numbers given below would be multiplied by N ; the final answer would not change). Now, for the second particle, the system has $N - 1$ possible states: 2 states with an energy of Δ and $N - 3$ states with no interaction energy. From here, the partition function

$$Z = \sum_{i=1}^{N-1} \exp(-\Delta/T) = (N - 3) + 2 \exp(-\Delta/T).$$

According to the Gibbs distribution, the probability of each state, $p_i = Z^{-1} \exp(-E_i/T)$. From this,

$$\begin{aligned} \langle E \rangle &= \sum_{i=1}^{N-1} E_i p_i = \frac{2\Delta \exp(-\Delta/T)}{(N - 3) + 2 \exp(-\Delta/T)}, \\ \langle E^2 \rangle &= \sum_{i=1}^{N-1} E_i^2 p_i = \frac{2\Delta^2 \exp(-\Delta/T)}{(N - 3) + 2 \exp(-\Delta/T)}, \end{aligned}$$

Using these equations, the condition $\Delta E < \langle E \rangle$ may be readily reduced to

$$(N - 3) < 2 \exp(-\Delta/T)$$

This relation may be only satisfied for $N = 3$ (at any temperature) and for $N = 4$ (for $T < \Delta \ln 2 \approx 0.69\Delta$).

Statistical Mechanics 3

Consider the growth of an ice layer floating on a lake in winter. Take the temperature of the water to be at the freezing point, $T_{water} = 273$ K. For the ice thickness $x(t)$ to grow with time t , the air temperature must be lower, $T_{air} < 273$ K.

- a. (10 pts.) Derive an expression for the time dependence $x(t)$ of the growth in thickness of the ice. Assume the usual linear heat transfer mechanism, i.e., that the thermal energy flow rate is equal to $k \times (area) \times (temperature\ gradient)$, where k is a constant for the ice.
- b. (10 pts.) In the following use $k_{ice} = 1.6 \text{ J s}^{-1}\text{m}^{-1}\text{K}^{-1}$, $L_f = 334 \text{ kJ/kg}$ for the heat of freezing, and $\rho_{ice} = 920 \text{ kg/m}^3$ for the density of ice. Take the air above the ice to be at $T_{air} = 263$ K. Estimate how long it takes
 - i. for the ice to grow thick enough for walking, $x \approx 2.5 \text{ cm}$
 - ii. for the ice thickness to grow to equal the depth of the lake, which we take to be 10 m

Solution

- a. The main idea is that any heat which is transferred through the ice during time dt goes into freezing (increase x by dx). the amount of heat per area is

$$heat/area = \rho_{ice} \times L_f dx = k \delta T dt / x$$

and this is a differential equation because we have found dx/dt . Solving the equation with the obvious initial condition $x(0) = 0$ one gets

$$x^2(t)/2 = \frac{k \delta T}{\rho_{ice} L_f} t$$

- b. Putting numbers, one finds that for one inch it is about 2 hours (good for skating), for 10 meters about $10^9 \text{ s} = 30$ years (good for fishes: the lakes never freeze completely...unless during the ice ages).