Physics PhD Qualifying Examination
Part I – Wednesday, August 22, 2007

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

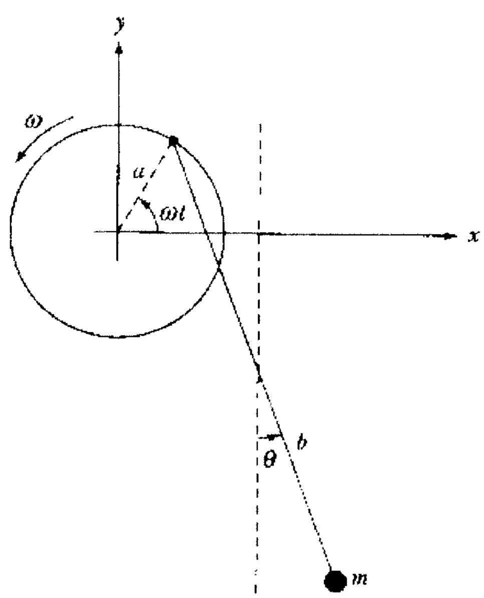
[I-1] **[10]**

A particle with mass m is projected vertically upward in a constant gravitational field g. The resisting force is proportional to the velocity v. The initial velocity of the particle is v_o.

(a) What is the maximum height (measured from the initial position) of the trajectory?
(b) What is the time required to reach the maximum height?

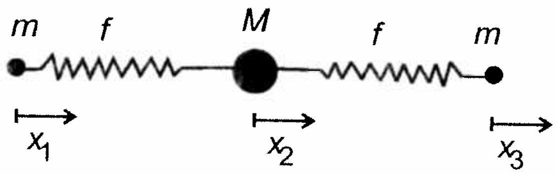
[I-2] **[10]**

A pendulum consists of a mass m attached to the end of a massless rod of length b. The point of support of the pendulum moves on a massless rim of radius a with constant angular velocity ω. Find the Lagrangian and equations of motion for mass m.



[I-3] **[10]**

Consider the stretching modes of a linear XY_2 molecule, such as CO_2. Three point masses representing the atoms can move along a fixed straight line. The three atoms are connected by two springs. Their equilibrium separation is d on both sides. Consider the two identical atoms of mass m to sit on the outsides, and the single atom of mass M at the center. The force constants f of the springs are identical (see sketch).



(a) Give the Lagrangian for the displacements x_1, x_2, and x_3 from their respective equilibriums positions along the axis of the molecule.
(b) Give the second order Lagrange equations of motion.
(c) Determine the Eigen frequencies.
(d) Now, assume identical masses ($m = M$) and determine the normal modes. Sketch the type of movements.

[I-4] **[10]**

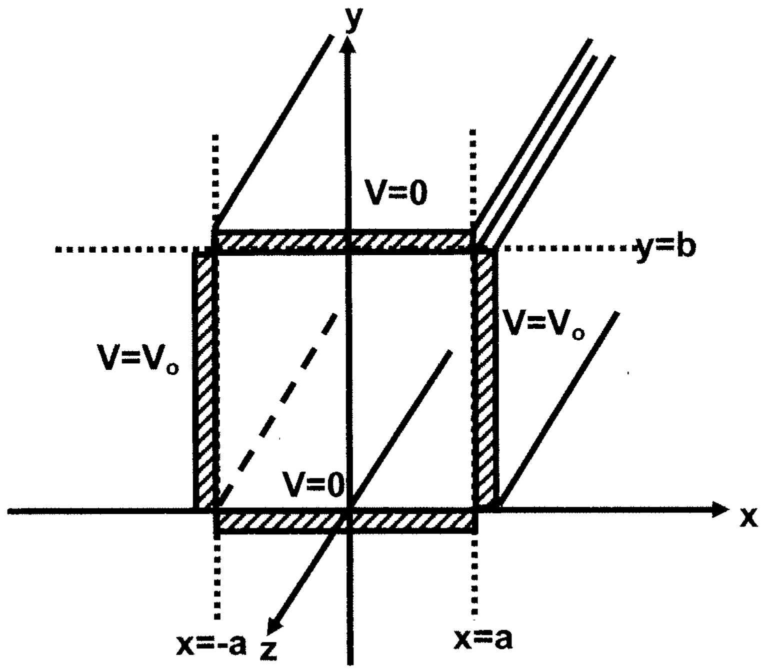
Consider a particle of mass m moving in a plane and subject to an inverse-square attractive force $F=-k/r^2$. Find the Lagrangian, the generalized momenta and equations of motion for mass m. Are the generalized momenta conserved?

[I-5] **[10]**

Consider two events that take place *at different points* (along the x-axis) *at the same instant* in an inertial reference frame $\mathcal{K}$. The two events are separated by a distance Δx in $\mathcal{K}$. Reference frame $\mathcal{K}'$ is moving with a constant velocity v along the x direction with respect to $\mathcal{K}$ (the orientation of the coordinate axes of the two frames coincide).
Are the two event simultaneous in $\mathcal{K}'$? If not, what is the time interval $\Delta t'$ separating the two events, as observed in $\mathcal{K}'$?

[I-6]   [10]

Two infinitely long grounded metal plates, at $y=0$ and $y=b$, are connected at $x=\pm a$ by metal strips maintained at potential V_o, as show in the figure below (a thin sliver of insulation at each corner prevent them from shorting out). Find the potential inside the resulting rectangular pipe.



[I-7] [10]

Check to see if the magnetic field $\vec{B} = \alpha\, xt\,\hat{z}$ or the electric field, each by itself, $\vec{E} = \frac{\beta}{r}\hat{r}$ are consistent with Maxwell's equations. If a special condition for ρ and **j** is needed discuss its physical reasonableness. $\hat{x}, \hat{y}, \hat{z}$ are unit vectors, α and β are arbitrary constants, ω is frequency, ε and μ are the electric permittivity and magnetic permeability.

You may find the following vector-analysis relationships useful:.

Cartesian $(x_1, x_2, x_3 = x, y, z)$

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial x_1} + \mathbf{e}_2 \frac{\partial\psi}{\partial x_2} + \mathbf{e}_3 \frac{\partial\psi}{\partial x_3}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right) + \mathbf{e}_3\left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_3^2}$$

Cylindrical (ρ, ϕ, z)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial\rho} + \mathbf{e}_2 \frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_3 \frac{\partial\psi}{\partial z}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_1) + \frac{1}{\rho}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$$

$$\nabla\times\mathbf{A} = \mathbf{e}_1\left(\frac{1}{\rho}\frac{\partial A_3}{\partial\phi} - \frac{\partial A_2}{\partial z}\right) + \mathbf{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial\rho}\right) + \mathbf{e}_3\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_2) - \frac{\partial A_1}{\partial\phi}\right)$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

Spherical (r, θ, ϕ)

$$\nabla\psi = \mathbf{e}_1 \frac{\partial\psi}{\partial r} + \mathbf{e}_2 \frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_3 \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_1) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_2) + \frac{1}{r\sin\theta}\frac{\partial A_3}{\partial\phi}$$

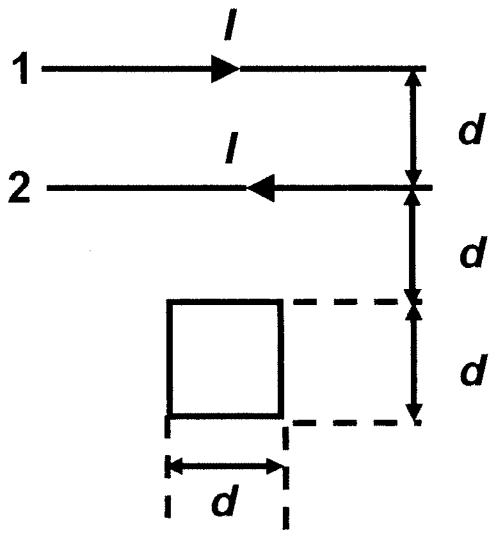
$$\nabla\times\mathbf{A} = \mathbf{e}_1\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_3) - \frac{\partial A_2}{\partial\phi}\right] + \mathbf{e}_2\left[\frac{1}{r\sin\theta}\frac{\partial A_1}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_3)\right] + \mathbf{e}_3\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_2) - \frac{\partial A_1}{\partial\theta}\right]$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\left[\text{Note that } \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi).\right]$$

[I-8] **[10]**

Two infinite parallel wires separated by a distance d carry equal currents I in opposite directions, with I increasing at the rate dI/dt. A square loop of wire of length d on a side lies in the plane of the wires at a distance d from one of the parallel wires, as illustrated in the figure bellow.



(a) Find the emf induced in the square loop.

(b) Is the induced current clockwise or counterclockwise? Justify your answer.

[I-9] **[5,5]**

Consider an electric dipole consisting of two tiny metal spheres separated by a distance d and connected by a fine wire. The charge is driven back and forth between the spheres with angular frequency ω such that the instantaneous charge on the upper and lower spheres is + and $-q(t)$, respectively, yielding a dipole moment of $\mathbf{p}(t) = q_0 d\cos(\omega t)\,\hat{z}$

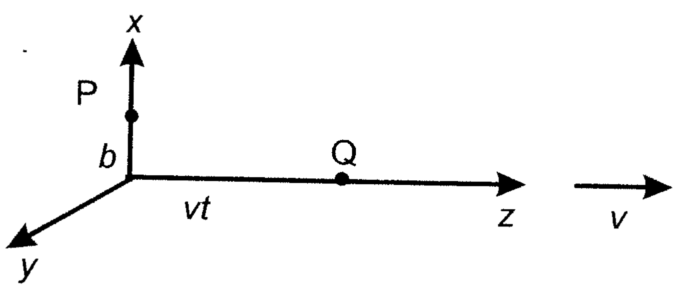
(a) Derive expressions for the scalar and vector potentials as a function of distance from the center of the dipole r which are lowest order in $\frac{1}{r}$, explicitly identifying each of the physical approximations which are needed.

(b) Derive expressions for the electric and magnetic fields at $\boldsymbol{r}$. Find the Poynting Vector and the total power radiated.

[I-10] **[10]**

A point charge e moves with a constant velocity in the z direction so that at time t it is at point Q with coordinates $x = 0$, $y = 0$, $z = vt$. Find at time t and at point P with coordinates $x = b, y = 0, z = 0$

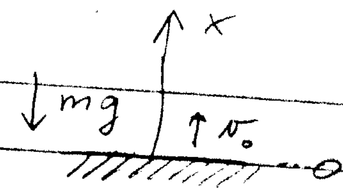
(a) the scalar potential ϕ;
(b) the vector potential A ;
(c) the electric field in the x direction E_x.



Solutions

I - 1

$$\boxed{m\frac{dv}{dt} = -mg - kv}$$



a) $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}\cdot v$

$x(0) = 0$
$v(0) = v_0$

$$mv\frac{dv}{dx} = -mg - kv$$

$$\frac{v}{g + \frac{k}{m}v}\,dv = -dx$$

$$\int \frac{v}{g+\frac{k}{m}v}\,dv = -x + C'$$

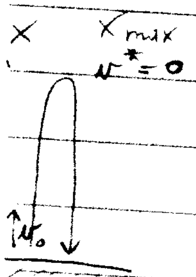
$$\int \frac{g + \frac{k}{m}v - g}{g + \frac{k}{m}v}\,dv = -\frac{k}{m}x + C$$

$$\int\left\{1 - \frac{g}{g+\frac{k}{m}v}\right\}dv = -\frac{k}{m}x + C$$

$$v - \frac{mg}{k}\ln\left(g + \frac{k}{m}v\right) = -\frac{k}{m}x + C$$

from initial condition: $v_0 - \frac{mg}{k}\ln\left(g+\frac{k}{m}v_0\right) = -\frac{k}{m}x_0 + C$

$$C = \frac{k}{m}x_0 + v_0 - \frac{mg}{k}\ln\left(g + \frac{k}{m}v_0\right)$$



Thus, $x(v) = \frac{m}{k}(v_0 - v) + \frac{m^2}{k^2}g\ln\left(\frac{g+\frac{k}{m}v}{g+\frac{k}{m}v_0}\right)$

at the highest point: $\boxed{v^* = 0}$ $\Rightarrow x(v^*) = \frac{m}{k}v_0 + \frac{m^2}{k^2}g\ln\left(\frac{g}{g+\frac{k}{m}v_0}\right) =$

$$= \frac{m}{k} v_0 - \frac{m^2 g}{k^2} \ln\left(\frac{g + \frac{k}{m} v_0}{g}\right) = \frac{m}{k} v_0 - \frac{m^2 g}{k^2} \ln\left(1 + \frac{k}{mg} v_0\right)$$

$$= \frac{m}{k} \left\{ v_0 - \frac{mg}{k} \ln\left(1 + \frac{k}{mg} v_0\right) \right\} = x(v^*) = x_{max}$$

b) $m \frac{dv}{dt} = -mg - kv$

$$\frac{dv}{g + \frac{k}{m} v} = -dt$$

$$\frac{m}{k} \ln\left(g + \frac{k}{m} v\right) = -t + C$$

initial condition: $v(0) = v_0$

$$\frac{m}{k} \ln\left(g + \frac{k}{m} v_0\right) = C$$

full solution: $t = \frac{m}{k} \ln\left(\frac{g + \frac{k}{m} v_0}{g + \frac{k}{m} v}\right)$ (*)

at the highest point: $v^* = 0$

$$t^* = \frac{m}{k} \ln\left(\frac{g + \frac{k}{m} v_0}{g}\right) = \frac{m}{k} \ln\left(1 + \frac{k}{mg} v_0\right)$$

Note: using equation (*) from (b) you can obtain the answer for (a) after integration

$$e^{\frac{k}{m}t} = \frac{g + \frac{k}{m} v_0}{g + \frac{k}{m} v}$$

$$g + \frac{k}{m} v = \left(g + \frac{k}{m} v_0\right) e^{-\frac{k}{m}t}$$

$$v(t) = \frac{m}{k}\left\{\left(g + \frac{k}{m} v_0\right) e^{-\frac{k}{m}t} - g\right\}$$

$$x(t) = \int v(t)\,dt = \frac{m}{k}\left\{\left(-\frac{m}{k}\right)\left(g + \frac{k}{m} v_0\right) e^{-\frac{k}{m}t} - g t\right\} + C$$

$x(0) = 0$:

$$0 = -\frac{m^2}{k^2}\left(g + \frac{k}{m} v_0\right) + C$$

$$C = \frac{m^2}{k^2}\left(g + \frac{k}{m} v_0\right)$$

full soln.: $$x(t) = \frac{m^2}{k^2}\left(g + \frac{k}{m} v_0\right)\left[1 - e^{-\frac{k}{m}t}\right] - \frac{mg}{k} t$$

using t^*:

$$x_{max} = x(t^*) = \frac{m}{k} v_0 - \frac{m^2 g}{k^2} \ln\left(1 + \frac{k}{mg} v_0\right)$$

Calculating the derivatives, we find

$$\ddot{r} - r\dot{\theta}^2 \sin^2\alpha + g \sin\alpha \cos\alpha = 0 \tag{7.31}$$

which is the equation of motion for the coordinate r.

We shall return to this example in Section 8.10 and examine the motion in more detail.

EXAMPLE 7.5

The point of support of a simple pendulum of length b moves on a massless rim of radius a rotating with constant angular velocity ω. Obtain the expression for the Cartesian components of the velocity and acceleration of the mass m. Obtain also the angular acceleration for the angle θ shown in Figure 7-3.

Solution. We choose the origin of our coordinate system to be at the center of the rotating rim. The Cartesian components of mass m become

$$\left.\begin{aligned} x &= a\cos\omega t + b\sin\theta \\ y &= a\sin\omega t - b\cos\theta \end{aligned}\right\} \tag{7.32}$$

The velocities are

$$\left.\begin{aligned} \dot{x} &= -a\omega\sin\omega t + b\dot{\theta}\cos\theta \\ \dot{y} &= a\omega\cos\omega t + b\dot{\theta}\sin\theta \end{aligned}\right\} \tag{7.33}$$

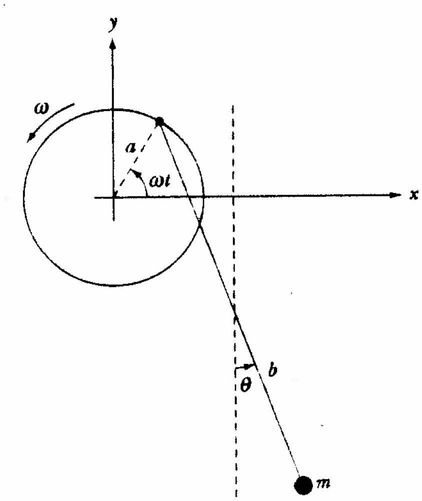


FIGURE 7-3 Example 7.5. A simple pendulum is attached to a rotating rim.

Taking the time derivative once again gives the acceleration:

$$\ddot{x} = -a\omega^2 \cos \omega t + b(\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta)$$

$$\ddot{y} = -a\omega^2 \sin \omega t + b(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$$

It should now be clear that the single generalized coordinate is θ. The kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

$$U = mgy$$

where $U = 0$ at $y = 0$. The Lagrangian is

$$L = T - U = \frac{m}{2}[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega \sin(\theta - \omega t)] - mg(a \sin \omega t - b \cos \theta) \tag{7.34}$$

The derivatives for the Lagrange equation of motion for θ are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega) \cos(\theta - \omega t)$$

$$\frac{\partial L}{\partial \theta} = mb\dot{\theta}a\omega \cos(\theta - \omega t) - mgb \sin \theta$$

which results in the equation of motion (after solving for $\ddot{\theta}$)

$$\ddot{\theta} = \frac{\omega^2 a}{b} \cos(\theta - \omega t) - \frac{g}{b} \sin \theta \tag{7.35}$$

Notice that this result reduces to the well-known equation of motion for a simple pendulum if $\omega = 0$.

EXAMPLE 7.6

Find the frequency of small oscillations of a simple pendulum placed in a railroad car that has a constant acceleration a in the x-direction.

Solution. A schematic diagram is shown in Figure 7-4a for the pendulum of length ℓ, mass m, and displacement angle θ. We choose a fixed cartesian coordinate system with $x = 0$ and $\dot{x} = v_0$ at $t = 0$. The position and velocity of m become

$$x = v_0 t + \frac{1}{2}at^2 + \ell \sin \theta$$

$$y = -\ell \cos \theta$$

$$\dot{x} = v_0 + at + \ell\dot{\theta} \cos \theta$$

$$\dot{y} = \ell\dot{\theta} \sin \theta$$

Part I – Mechanics
I-3 Normal Modes
SOLUTION

C.W.

a) Introduce $\mu = M/m$ and consider the Lagrangian

$$L = T - V = \frac{m}{2}\left(\dot{x}_1^2 + \dot{x}_3^2 + \mu \dot{x}_2^2\right) - \frac{f}{2}\left((x_2-x_1)^2 + (x_3-x_2)^2\right)$$

b) This leads to the second order Lagrange equations:

$$x_1: \quad m\ddot{x}_1 + f(x_2-x_1)(-1) = 0 \quad \Rightarrow m\ddot{x}_1 + f(x_1 - x_2) = 0$$

$$x_2: \quad \mu m \ddot{x}_2 + f((x_2-x_1) + (-x_2+x_3)) = 0 \Rightarrow m\mu\ddot{x}_2 + f(-x_1+2x_2-x_3) = 0$$

$$x_3: \quad m\ddot{x}_3 + f(x_3 - x_2) = 0$$

c) Consider a periodic excitation $x_i = a_i e^{j\omega t}$

which leads to

$$\begin{pmatrix} f - m\omega^2 & -f & 0 \\ -f & 2f - \mu m \omega^2 & -f \\ 0 & -f & f - m\omega^2 \end{pmatrix} \vec{a}\, e^{j\omega t} = \vec{0} \quad \forall t$$

and

$$\Rightarrow (f - m\omega^2)^2(2f - \mu m\omega^2) - f^2(f - m\omega^2)2 = 0$$

1.) $\omega^2 = \frac{f}{m}$ $\qquad \underline{\omega_{1,2} = \pm\sqrt{\frac{f}{m}}}$

2.) $2f^2 - f\mu m\omega^2 - 2fm\omega^2 + \mu m^2\omega^4 - 2f^2 = 0$

$$\omega^2 = 0 \rightarrow \underline{\omega_3 = 0}$$

$$-f\mu - f2 + \mu m\omega^2 = 0$$

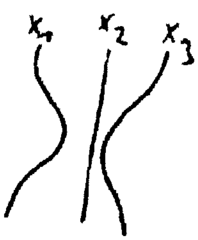
$$\omega^2 = \frac{f(\mu+2)}{\mu m} \qquad \underline{\omega_{4,5} = \pm\sqrt{\frac{f(\mu+2)}{\mu m}}}$$

d) Normal modes for $\mu = 1$

$\mu = 1;\quad \omega_1^2 = \frac{f}{m}$

$$\begin{array}{ccc|} 0 & -f & 0 \\ -f & f & -f \\ 0 & -f & 0 \\ \hline -f & 0 & -f \\ 0 & -f & 0 \\ 0 & 0 & 0 \\ \hline \end{array}$$

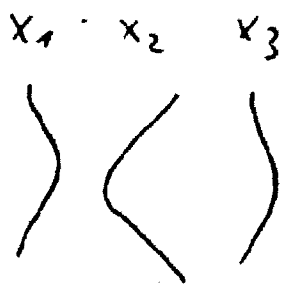
$\vec{e}_1 = t\begin{pmatrix}-1\\0\\1\end{pmatrix}$



$\omega_3^2 = 3\frac{f}{m}$

$$\begin{array}{ccc|} -2f & -f & 0 \\ -f & -f & -f \\ 0 & -f & -2f \\ \hline 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 2 \\ \hline 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ \hline \end{array}$$

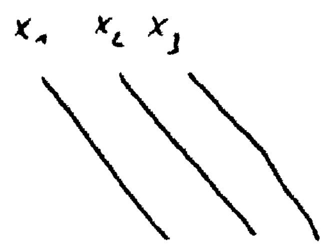
$\vec{e}_3 = t\begin{pmatrix}1\\-2\end{pmatrix}$



$\omega_2^2 = 0$

$$\begin{array}{ccc|} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \\ \hline 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ \hline \end{array}$$

$\vec{e}_2 = t\begin{pmatrix}1\\1\\1\end{pmatrix}$



L-4

$$L = T - U \qquad F = -\frac{\partial U}{\partial r}, \quad F = -\frac{k}{r^2}, \quad U = -\frac{k}{r}$$

$$x = r\cos\theta \qquad \dot{x} = \dot{r}\cos\theta + r\dot{\theta}(-\sin\theta)$$

$$y = r\sin\theta \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{k}{r}$$

$$\boxed{L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}}$$ Lagrangian

$\cancel{\dot{p}_\theta = \frac{d}{dt}\frac{\partial L}{}}$

$$\boxed{\begin{array}{lll} \dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 & p_\theta = \text{const.} & \text{conserved} \\ \dot{p}_r = \frac{\partial L}{\partial r} \neq 0 & p_r \neq \text{const.} & \text{not conserved} \end{array}}$$

generalized momenta

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \,/\, p_\theta = \frac{\partial L}{\partial \dot{\theta}} =$$

$$\boxed{\begin{array}{l} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = mr\ddot{\theta} + m2r\dot{r}\dot{\theta} \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - \frac{k}{r^2} \end{array}}$$

eqs. of motion

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = p_\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$\boxed{\text{I - 5}}$

K: two events, separated by Δx at the same time

thus, $\Delta t = 0$

Lorentz tr.
$$\Delta x' = \gamma\,(\Delta x - v\Delta t)$$
$$\Delta t' = \gamma\,(\Delta t - \frac{v}{c^2}\Delta x)$$
$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$\Delta t = 0$

Thus, $\Delta x' = \gamma\,\Delta x$

and $\Delta t' = -\gamma \frac{v}{c^2}\Delta x \neq 0$ if $\Delta x \neq 0$

Events which are simultaneous ($\Delta t = 0$) in one frame, are not generally simultaneous in an other reference frame ($\Delta t' \neq 0$)

I.-6

— The problem is to solve Laplace's equation

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} = 0$$

subject to the boundary conditions: (B.C.)

(1) $V=0$ when $Y=0$
(2) $V=0$ " $Y=b$
(3) $V=V_0$ " $X=a$
(4) $V=V_0$ " $X=-a$

(5) $V(X,Y)=\left(\frac{A}{2}e^{kX}+\frac{B}{2}e^{-kX}\right)(C \sin kY + D \cos kY$

— From symmetry: $V(-X,Y)=V(X,Y)$

It follows: $A=B$

From B.C. (3) and (4): $D=0$

$\therefore$ We have: $V(X,Y)=C \cosh(kX)\sin(kY)\ldots$

$\therefore$ A general linear combination: $V(X,Y)=\sum_k C_k \cosh(k$
$\sin($

$\ldots\ldots($

— From B.C.-(2), we have $\sin(kb) = 0$

$$\therefore k_n = \frac{n\pi}{b}, \quad n = 1, 2, 3, \cdots$$

From B.C.-(3), we have $V(a, y) = V_0$

$$\therefore V(a, y) = \sum_{n=1}^{\infty} C_n \cosh(k_n a) \sin ky = V_0$$

From Fourier Analysis:

$$\therefore C_n \cosh(k_n a) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4V_0}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

#

1

Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}} \qquad \vec{D} = \varepsilon\vec{E}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \vec{j} + \dot{\vec{D}} \qquad \vec{B} = \mu\vec{H}$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{j} + \mu\dot{\vec{D}}$$
$$= \mu\vec{j} + \mu\varepsilon\dot{\vec{E}}$$
$$\vec{\nabla}\times\vec{B} = \mu\sigma\cancel{j}\vec{E} + \mu\varepsilon\dot{\vec{E}}$$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\dot{\vec{B}}$$
$$\boxed{\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \Delta\vec{E} = -(\mu\sigma\dot{\vec{E}} + \mu\varepsilon\ddot{\vec{E}})}$$
differential equation for $\vec{E}$

$$\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}(\vec{\nabla}\cdot\vec{B}) - \Delta\vec{B} = \mu\sigma\,\vec{\nabla}\times\vec{E} + \mu\varepsilon\,\vec{\nabla}\times\dot{\vec{E}}$$
$$-\Delta\vec{B} = \mu\sigma(-\dot{\vec{B}}) + \mu\varepsilon(-\ddot{\vec{B}})$$
$$\boxed{-\Delta\vec{B} = -(\mu\sigma\dot{\vec{B}} + \mu\varepsilon\ddot{\vec{B}})}$$
differential equation for $\vec{B}$

(i)

$$\Delta\vec{B} = \frac{\partial^2\vec{B}}{\partial z^2} = \frac{\partial^2}{\partial z^2}(xt) = 0$$
$$\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}(xt\,\hat{z}) = x\,\hat{z}$$
$$\frac{\partial^2\vec{B}}{\partial t^2} = 0$$

$$-\Delta\vec{B} = -(\mu\sigma\dot{\vec{B}} + \mu\varepsilon\ddot{\vec{B}})$$
$$0 = -\mu\sigma x\,\hat{z} + 0$$
requires $\sigma = 0$ or $\vec{j} = 0$

no free currents $\Rightarrow$ reasonable

[illegible]

$$\vec{E} = \frac{p}{r}\,\hat{r} \qquad \dot{\vec{E}} = 0 \;, \quad \ddot{\vec{E}} = 0$$

$$\Delta \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial E}{\partial r}\right) \qquad \frac{\partial}{\partial r}\frac{1}{r} = -r^{-2}$$

$$\Delta \vec{E} = 0 \qquad r^2 \frac{\partial}{\partial r}\frac{1}{r} = -1$$

$$\vec{\nabla}(\vec{\nabla}\,\vec{E}) = 0 \qquad \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{1}{r}\right) = 0$$

$\vec{E}$ [illegible]

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial E_r}{\partial r}\right) = 0$$

I-3

The B-field produced by an infinite straight wire is:

$$B = \frac{\mu_0 I}{2\pi r}$$

The flux due to wire-1 is:

$$\phi_1 = \int_{2d}^{3d} \frac{\mu_0 I}{2\pi r}(d)\,dr = \frac{\mu_0 I d}{2\pi} \ln\frac{3}{2}$$

The flux due to wire-2 is:

$$\phi_2 = \int_{d}^{2d} \frac{\mu_0 I}{2\pi r}(d)\cdot dr = \frac{\mu_0 I d}{2\pi} \ln 2$$

"ϕ_1" is directing into the page.
"ϕ_2" is pointing out of the page.

The total flux: $\phi = \phi_2 - \phi_1 = \dfrac{\mu_0 I d}{2\pi} \ln\dfrac{4}{3}$

The emf induced in the square loop is:

$$\mathcal{E} = -\frac{d\phi}{dt} = -\frac{\mu_0 d}{2\pi} \cdot \ln\left(\frac{4}{3}\right) \cdot \frac{dI}{dt}$$

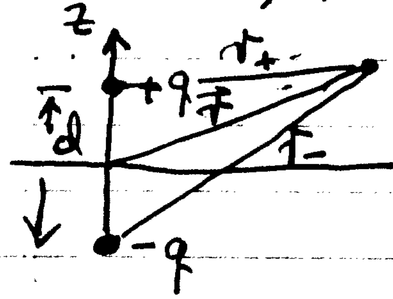
The induced field will "direct into the page" so as to oppose the change in flux.

$\therefore$ The induced current is "clockwise".

[I-9] Solution

(a) Retarded potential:

$$V(\vec{r},t)=\frac{1}{4\pi\epsilon_0}\left\{q_0\frac{\cos[\omega(t-r_+/c)]}{r_+}-q_0\frac{\cos[\omega(t-r}{r_-}\right.$$



where:

$$r_\pm=\sqrt{r^2\mp rd\cos\theta+\left(\frac{d}{2}\right)^2}$$

approx. 1: $d \ll r \Rightarrow r_\pm = r\left(1\mp\frac{d}{2r}\cos\theta\right)$

approx. 2: $d \ll \frac{c}{\omega} \Rightarrow$

$$\Rightarrow \cos[\omega(t-r_\pm/c)]\cong\cos[\omega(t-r/c)]$$
$$\pm\frac{\omega d}{2c}\cos\theta\sin[\omega(t-r$$

approx. 3: $r \gg \frac{c}{\omega}$ (radiation zone)

$$\Rightarrow V(r,\theta,t)=-\frac{p_0\omega}{4\pi\epsilon_0 c}\frac{\cos\theta}{r}\sin[\omega(t-r/c)]$$

$$\vec{A}(\vec{r},t)=\frac{\mu_0}{4\pi}\int_{-d/2}^{d/2}\frac{-q_0\omega\sin[\omega(t-r/c)]}{r}\hat{z}\,dz$$

$$\vec{A}(\vec{r},t)=\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\hat{z}$$

(b) $\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}$, $\vec{B}=\vec{\nabla}\times\vec{A}$

[I-9] continued.

$$\vec{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\theta}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{\mu_0}{4\pi c} p_0 \omega^2\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c})]$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{\mu_0}{c}\left\{\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-\frac{r}{c}\right.\right.$$

$$\langle P\rangle = \int\langle\vec{S}\rangle\cdot d\vec{a} = \frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\int\frac{\sin^2\theta}{r^2}\,r^2\sin\theta\, d\theta\, d\phi$$

$$\langle P\rangle = \frac{\mu_0 p_0^2\omega^4}{12\pi c}.$$

Solution

a) The Liénard-Wichert potentials at P due to the charge are given by

$$\phi = \frac{1}{4\pi\varepsilon_0\left(r - \frac{\vec{v}}{c}\cdot\vec{r}\right)}, \quad \vec{A} = \frac{e\vec{v}}{4\pi\varepsilon_0 c^2\left[r - \frac{\vec{v}}{c}\cdot\vec{r}\right]}.$$

Where $\vec{r}$ is the radius vector from the retarded position of the charge to the field point P, i.e.

$$\vec{r} = b e_x - v\left(t - \frac{r}{c}\right)e_z = b e_x - v t' e_z,$$

with

$$t' = t - \frac{r}{c}.$$

Thus

$$r^2 = \vec{r}\cdot\vec{r} = b^2 + v^2\left(t - \frac{r}{c}\right)^2 = b^2 + v^2\left(t^2 - \frac{2rt}{c} + \frac{r^2}{c^2}\right)$$

or

$$\left(1 - \frac{v^2}{c^2}\right)r^2 + 2\frac{v^2 t}{c} r - b^2 - v^2 t^2 = 0$$

This is the retardation condition, with the solutions

$$r = \frac{-\beta v t \pm \sqrt{(1-\beta^2)b^2 + v^2 t^2}}{1-\beta^2} \qquad \beta = \frac{v}{c}$$

Only the plus sign is to be taken due to r ≥ 0.

As $\vec{v} = v e_z$,

$$r - \frac{\vec{v}\cdot\vec{r}}{c} = r + \frac{v^2 t'}{c} = r + v\beta\left(t - \frac{r}{c}\right) = (1-\beta^2)r + v\beta t$$

$$= \sqrt{(1-\beta^2)b^2 + v^2 t^2}$$

The scalar potential is then

$$\phi = \frac{e}{4\pi\varepsilon_0 \gamma \sqrt{(1-\beta^2) + v^2t^2}}$$

b) The vector potential is then

$$\vec{A} = \frac{ev}{4\pi\varepsilon_0 c^2 \gamma \sqrt{(1-\beta^2)b^2 + v^2t^2}}\, e_z$$

c) The electric field at P is obtained by differentiating the Liénard-Wichert potentials:

$$\vec{E}(t) = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

For the spatial differentiation, b is to be first replaced by x. We then have

$$(\nabla\phi)_b = \left(\frac{\partial\phi}{\partial x}\right)_b e_x = \frac{e(1-\beta^2)b}{4\pi\varepsilon_0 [(1-\beta^2)b^2 + v^2t^2]^{3/2}}\, e_x$$

As $\vec{A}$ is in the z-direction, it does not contribute to E_x. Therefore

$$E_x = \frac{e(1-\beta^2)b}{4\pi\varepsilon_0 [(1-\beta^2)b^2 + v^2t^2]^{3/2}}$$

Physics PhD Qualifying Examination
Part II – Friday, August 24, 2007

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[10]**

Consider a particle in a one-dimensional box with its potential given by

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > a \end{cases} .$$

The wavefunction of this particle is given by

$$\psi(x) = \begin{cases} A\left(\dfrac{x}{a}\right) & 0 < x < a/2 \\ A\left(1 - \dfrac{x}{a}\right) & a/2 < x < a \end{cases},$$

where $A = \sqrt{12/a}$.

Calculate the probability that a measurement for the energy yields the eigenvalue E_n of this one-dimensional box.

[II-2] **[10]**

Let's explore the sensitivity of the electronic energy states on the actual size of the nucleus. Instead of a positive point charge, let us consider the proton as a thin uniform (hollow) shell of radius b and the same total charge. Use first-order perturbation theory to determine the relative energy change of the ground state of the hydrogen atom.

Hint: The ground state function is given by

$$\Phi_{00} = (\pi a_0^{\ 3})^{-1/2} \exp(-r / a_0)$$

[II-3] [10]

Show that the wavefunction $\psi_{lm}(\vec{r}) = x^2 + y^2 - 2z^2$ is an eigenstate of L^2 and L_z with eigenvalues $\hbar^2 l(l+1)$ and $\hbar m$. Find the values of l and m for Ψ_{lm}.
In addition, find a function $\Psi_{lm'}$ with the same eigenvalue $\hbar^2 l(l+1)$ for L^2 and the maximum possible eigenvalue m' for L_z.
You may find the following table for spherical harmonics useful.

Normalized spherical harmonics

$$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$$

$$Y_1^1 = -\frac{1}{2}\left(\frac{3}{2\pi}\right)^{1/2} \sin\vartheta\, e^{i\varphi}$$

$$Y_1^0 = \frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2} \cos\vartheta$$

$$Y_1^{-1} = \frac{1}{2}\left(\frac{3}{2\pi}\right)^{1/2} \sin\vartheta\, e^{-i\varphi}$$

$$Y_2^2 = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \sin^2\vartheta\, e^{2i\varphi}$$

$$Y_2^1 = -\frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2} \sin\vartheta\cos\vartheta\, e^{i\varphi}$$

$$Y_2^0 = \frac{1}{4}\left(\frac{5}{\pi}\right)^{1/2} (3\cos^2\vartheta - 1)$$

$$Y_2^{-1} = \frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2} \sin\vartheta\cos\vartheta\, e^{-i\varphi}$$

$$Y_2^{-2} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2} \sin^2\vartheta\, e^{-2i\varphi}$$

$$Y_3^3 = -\frac{1}{8}\left(\frac{35}{\pi}\right)^{1/2} \sin^3\vartheta\, e^{3i\varphi}$$

$$Y_3^2 = \frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2} \sin^2\vartheta\cos\vartheta\, e^{2i\varphi}$$

$$Y_3^1 = -\frac{1}{8}\left(\frac{21}{\pi}\right)^{1/2} \sin\vartheta(5\cos^2\vartheta - 1)e^{i\varphi}$$

$$Y_3^0 = \frac{1}{4}\left(\frac{7}{\pi}\right)^{1/2} (5\cos^3\vartheta - 3\cos\vartheta)$$

$$Y_3^{-1} = \frac{1}{8}\left(\frac{21}{\pi}\right)^{1/2} \sin\vartheta(15\cos^2\vartheta - 3)e^{-i\varphi}$$

$$Y_3^{-2} = \frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2} \sin^2\vartheta\cos\vartheta\, e^{-2i\varphi}$$

$$Y_3^{-3} = \frac{1}{8}\left(\frac{35}{\pi}\right)^{1/2} \sin^3\vartheta\, e^{-3i\varphi}$$

[II-4] **[10]**

A narrow beam of α-particles is incident normally on a thin gold foil, an the particles scattered at an angle θ to the initial direction are detected by a scintillator S that sees a small solid angle $d\Omega$ in point C, which is the center of the foil where the α-particles strike. Assume $d\sigma/d\Omega$ is the differential scattering cross section per nucleus for angle θ, N is the number of nuclei per unit volume, and t the thickness of the foil, which is thin enough so that the flux of particles stays constant. Then the fraction f of incident particles that get scattered into S is given by

$$f = \frac{d\sigma}{d\Omega} Nt\, d\Omega$$

In 1913 Geiger and Mardsen measured the following: Their ^{214}Po source emits α-particles with energy $E = 7.68$ MeV. The scintillator area was 1 mm^2 and it was 10 mm from point C. For $t = 2.1\text{x}10^{-7}$m and $\theta = 45°$ m, they found $f = 3.7\text{x}10^{-7}$. According to these measurements, what is the atomic number of the foil?
Assume a ratio of the foil's density ρ and the foil's atomic weight AW to be $\rho/AW = 98.0$ kg m^{-3}.

[II-5] **[10]**

Consider what is known as a Gaussian wave packet whose x-space wave function is given by

$$\psi(x) = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right] \exp\left[ikx - \frac{x^2}{2d^2}\right].$$

(a) Compute the expectation values of $\hat{x}$, $\hat{x}^2$, $\hat{p}$, and $\hat{p}^2$.
(b) Calculate $\langle(\Delta x)^2\rangle \equiv \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2$, $\langle(\Delta p)^2\rangle \equiv \langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2$, and check whether they satisfy Heisenberg's uncertainty relation.

[II-6] **[10]**

An electron travels in the y-direction through a homogeneous magnetic field $\boldsymbol{H}$ which is in the z-direction. Its spin is aligned in the *positive* z-direction. At a point which starts at $y=0$, which the electron passes at time $t=0$, an additional homogeneous magnetic field $\boldsymbol{H'}$ is switched on (and this field is aligned in the x-direction), and the electron leaves this additional field $\boldsymbol{H'}$ at time t, and at $y=l$. What is the probability that during this time interval, the spin of the electron is rotated into the *negative* z-direction?

In your calculations for the probabilities of the electron spin in the positive z-direction and negative z-direction should be an end result. Hence, ascertain from these results that the sum of these probabilities is unity.

[II-7] **[3,2,3,2]**

The Van der Waals equation of state for one mole of an imperfect gas reads

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT$$

[Note: part (d) of this problem can be done independently of part (a) to (c).]

(a) Sketch several isotherms of the Van der Waals gas in the P-V plane (V along the horizontal axis, P along the vertical axis). Identify the critical point.

(b) Evaluate the dimensionless ratio PV/RT at the critical point.

(c) In a portion of the P-V plane below the critical point the liquid and gas phases can coexist. In this region the isotherms given by the Van der Waals equation are unphysical and must be modified. The physically correct isotherms in this region are lines of constant pressure, $P_0\,(T)$. Maxwell proposed that $P_0\,(T)$ should be chosen so that the area under the modified isotherm should equal the area under the original Van der Waals isotherm. Draw a modified isotherm and explain the idea behind the Maxwell construction.

(d) Show that the heat capacity at constant volume of a Van der Waals gas is a function of temperature alone (i.e. independent of V).

[II-8] **[1,2,2,5]**

The Helmholtz free energy of a system is given by $F(T,V) = -\frac{a}{3}T^4V$.

(a) Obtain the equation of state.
(b) Obtain the internal energy of the system $U(T,V)$.
(c) Obtain the constant-volume heat capacity C_V.
(d) Consider that this gas undergoes "free expansion" from V_1 to V_2. (In this process, also referred to as the Joule experiment, the gas is thermally insulated from its environment and "suddenly" expands into vacuum.) The initial temperature of the gas is T_1. Obtain the final temperature T_2 of the gas. (You must express the final temperature T_2 in terms of T_1, V_1, and V_2.) Is the gas cooling down or warming up during this process?

[II-9] **[10]**

Consider N independent localized (hence, distinguishable) particles. The energy levels of a single particle are $\varepsilon_j = j\varepsilon_o$, $j = 1,2,\ldots,\infty$ .The degeneracy of the j^{th} level is $g_j = j$.

(a) Obtain the heat capacity $C(T)$ of the system (at constant ε_o and N). Further, from your result, obtain the *low-temperature* behavior ($\varepsilon_o / kT >> 1$) of $C(T)$.
(b) Obtain the entropy $S(T)$. Obtain the *low-temperature* behavior of $S(T)$.

[II-10] **[5,5]**

Consider a quantum-mechanical gas of non-interacting spin zero bosons, each of mass m which are free to move within a three-dimensional volume V.

(a) Find the energy and heat capacity in the very low temperature region. Discuss why it is appropriate at low temperatures to put the chemical potential equal to zero.
(b) Show how the calculation is modified for a photon (mass = 0) gas. Prove that the energy is proportional to T^4.

Note: Put all integrals in dimensionless form, but do not evaluate.

- The eigenstates of a 1D particle in a box:

$$\phi_n^{\circ}(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$$

- Demand: $\psi(x) = \sum_{n=1}^{\infty} A_n \phi_n^{\circ}(x)$

We have: $A_n = \int_0^a dx\, \psi(x) \cdot \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}$

$$\therefore A_n = \frac{\sqrt{24}}{a}\left[\int_0^{a/2} dx \left(\frac{x}{a}\right) \sin\frac{n\pi x}{a} + \int_{a/2}^{a} dx \left(1-\frac{x}{a}\right) \sin\frac{n\pi x}{a}\right.$$

- Let $u \equiv \frac{\pi x}{a}$ in the first integral.

And Let $\pi - u \equiv \frac{\pi x}{a}$ in the second integral.

$$\therefore A_n = \frac{\sqrt{24}}{a} \int_0^{\pi/2} du \cdot \frac{u}{\pi} \cdot \sin nu \left(1-(-1)^n\right)$$

$$\therefore \begin{cases} A_n = 0, & n = \text{even} \\ A_n = \frac{\sqrt{24}}{\pi} \cdot 2 \cdot \frac{1}{\pi n^2} (-1)^{n+1} & \end{cases}$$

Solution

For a point nucleus, the potential is given by

$$V^{(0)} = -\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{r}$$

For the thin spherical shell, the potential is given by

$$V = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r}, \ \forall\, r > b, \quad V = -\frac{e^2}{4\pi\varepsilon_0}\cdot\frac{1}{b} \quad \forall\, r < b.$$

The perturbation potential is therefore

$$V^{(1)} = V - V^{(0)} = \frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right) \qquad r < b$$

$$= 0 \qquad\qquad r > b$$

The first-order correction to the energy of the ground state then is

$$E^{(1)} = \int u_{1s}^* V^{(1)} u_{1s}\, d\tau$$, where $$u_{1s} = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0}$$

Therefore

$$E^{(1)} = \frac{4}{a_0^3}\frac{e^2}{4\pi\varepsilon_0}\int_0^b r^2\left(\frac{1}{r} - \frac{1}{b}\right)\exp(-2r/a_0)\, dr$$

Since $b/a \approx 10^{-5}$, the exponential term may be replaced by unity over the range of integration, and the integral becomes

$$\int_0^b \left(r - \frac{r^2}{b}\right) dr = \frac{1}{6}b^2$$

The ground-state energy for a point nucleus is

$$E^{(0)} = -\frac{e^2}{4\pi\varepsilon_0}\frac{1}{2a_0}$$

Combining the last three expressions we find

$$\frac{E^{(1)}}{E^{(0)}} = -\frac{4b^2}{3a_0^2}$$

The value of b is about 10^{-15} m; the Bohr radius $a_0 = 0.53 \times 10^{-10}$ m. These values give $E^{(1)}/E^{(0)} \approx 5\times10^{-10}$. So the fractional change in energy is rather small.

$L^2 Y_{\ell m} = \hbar^2 (\ell+1)\ell \, Y_{\ell m}$

$L_z Y_{\ell m} = \hbar m \, Y_{\ell m}$

$\psi_{\ell m}(\vec{r}) = x^2 + y^2 - 2z^2$

express $\psi_{\ell m}$ in spherical harmonics

~~$x = r\cos\theta\sin\varphi$~~

$x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$

$$x^2 + y^2 - 2z^2 = r^2\sin^2\theta - 2r^2\cos^2\theta$$

$$= r^2(1-\cos^2\theta) - 2r^2\cos^2\theta$$

$$= r^2(1-3\cos^2\theta)$$

$$\psi_{\ell m} = -r^2\sqrt{\frac{16\pi}{5}}\, Y_{20}(\theta,\varphi)$$

$\Rightarrow m = 0$, $\ell = 2$

for $\ell = 2$ maximum possible ~~eigenvalue~~ m is 2
$-\ell, \dots, 0, \dots, +\ell$

eigenfunction $\psi_{\ell m'} = C r^2 Y_{22}(\theta,\varphi)$

Solution

If the particles strike a small area A of the foil of thickness t, there are N A t nuclei in the incident beam. Each nucleus scatters the α-particles independently. So the total number scattered per second into dΩ is

$$n_{sc} = \frac{d\sigma}{d\Omega} F d\Omega\, NAt \qquad (1)$$

The number of incident particles striking the foil per second is $n_{inc} = FA$. Therefore, the fraction of particles scattered into dΩ is

$$f = \frac{n_{sc}}{n_{inc}} = \frac{d\sigma}{d\Omega} N\, t\, d\Omega \qquad (2)$$

Note that (1) assumes that the flux remains close to its initial value as the beam traverses the foil. This is only true if the foil thickness is small. Otherwise the flux decreases exponentially.

Insert the values E = 7.68 MeV = 7.68 x 1.602×10^{-13} J, and $\theta = 45^0$ in the Rutherford formula

$$\frac{d\sigma}{d\Omega} = \left\{ \frac{Ze^2}{8\pi\varepsilon_0 E \sin^2(\theta/2)} \right\}^2 \qquad (3)$$

gives

$$\frac{d\sigma}{d\Omega} = 4.10 \times 10^{-31} \times Z^2 m^2 \qquad (4)$$

The number of gold atoms per unit volume is

$$N = 10^3 \frac{N_A \rho}{AW}, \qquad (5)$$

Where ρ is the density of gold, AW is its atomic weight, and N_A is the Avogadro constant. This gives

$$N = 5.90 \times 10^{28} m^{-3} \qquad (6)$$

The solid angle covered by the scintillator at point C is

$$d\Omega = \text{foil area} / (\text{distance to C})^2 = 10^{-2} \text{ sr.} \qquad (7)$$

Inserting (4), (6), and (7) in (2) together with $f = 3.7 \times 10^{-7}$ and $t = 2.1 \times 10^{-7}$ m, we obtain

$$Z = 85. \qquad (8)$$

This compares rather well to the correct value of 79.

II.

- Given $\psi(x) = \left[\frac{1}{\pi^{1/4}\sqrt{d}}\right] \exp\left[ikx - \frac{x^2}{2d^2}\right]$

- $\langle x \rangle = \int_{-\infty}^{\infty} dx \cdot x \cdot |\psi(x)|^2 = 0$

$\langle x^2 \rangle = \int_{-\infty}^{\infty} dx \cdot x^2 \cdot |\psi(x)|^2 = \frac{1}{\sqrt{\pi} d} \int_{-\infty}^{\infty} dx \cdot x^2 \cdot \exp\left[\frac{-x^2}{d^2}\right]$ ...

$\langle x^2 \rangle = \frac{d^2}{2}$ ......

This leads to: $\langle \Delta x^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{d^2}{2}$ ...

- $\langle p \rangle = \hbar k$ (see next pages)...

$\langle p^2 \rangle = \frac{\hbar^2}{2d^2} + \hbar^2 k^2$ (see next pages)...

This leads to: $\langle \Delta p^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2 = \frac{\hbar^2}{2d^2}$ ...(

- In this case, the uncertainty product is:

$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle = \frac{d^2}{2} \cdot \frac{\hbar^2}{2d^2} = \frac{\hbar^2}{4}$ ......(7)

#

$$\langle p \rangle = \int_{-\infty}^{\infty} \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx' - \frac{x'^2}{2d^2}} \left(-i\hbar \frac{\partial}{\partial x'}\right) \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx' - \frac{x'^2}{2d^2}}$$

$$= \frac{-i\hbar}{\pi^{1/2} d} \int_{-\infty}^{\infty} e^{-ikx' - \frac{x'^2}{2d^2}} \left(ik - \frac{x'}{d^2}\right) e^{ikx' - \frac{x'^2}{2d^2}}$$

$$= \frac{-i\hbar}{\pi^{1/2} d} \left[ik \int_{-\infty}^{\infty} e^{-\frac{x'^2}{d^2}} dx' \;\; \cancel{- \frac{1}{d^2} \int_{-\infty}^{\infty} \underbrace{x'}_{\text{odd function}} e^{-\frac{x'^2}{d^2}} dx'} \to 0 \right]$$

$$= \frac{-i\hbar}{\pi^{1/2} d} \left[ik \sqrt{\frac{\pi}{\frac{1}{d^2}}} \right] = \hbar k \quad \#$$

$\langle p^2 \rangle$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi^{1/4}\sqrt{d}} e^{-ikx' - \frac{x'^2}{2d^2}} (-i\hbar)^2 \frac{\partial^2}{\partial x'^2} \frac{1}{\pi^{1/4}\sqrt{d}} e^{ikx' - \frac{x'^2}{2d^2}} dx'$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \int_{-\infty}^{\infty} e^{-ikx' - \frac{x'^2}{2d^2}} \frac{\partial^2}{\partial x'^2} e^{ikx' - \frac{x'^2}{2d^2}} dx'$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \int_{-\infty}^{\infty} e^{-ikx' - \frac{x'^2}{2d^2}} \frac{\partial}{\partial x'} \left[\left(ik - \frac{x'}{d^2}\right) e^{ikx' - \frac{x'^2}{2d^2}} \right] dx'$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \int_{-\infty}^{\infty} e^{-ikx' - \frac{x'^2}{2d^2}} \left[-\frac{1}{d^2} e^{ikx' - \frac{x'^2}{2d^2}} + \left(ik - \frac{x'}{d^2}\right)^2 e^{ikx' - \frac{x'^2}{2d^2}} \right] dx'$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \left\{ -\frac{1}{d^2} \int_{-\infty}^{\infty} e^{-\frac{x'^2}{d^2}} dx' + \int_{-\infty}^{\infty} e^{-\frac{x'^2}{d^2}} \left(-k^2 - 2ik\frac{x'}{d^2} + \frac{x'^2}{d^4}\right) dx' \right\}$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \left\{ -\frac{1}{d^2}\sqrt{\pi d^2} + (-k^2)\sqrt{\pi d^2} + 0 + \frac{1}{d^4} \underbrace{\int e^{-\frac{x'^2}{d^2}} x'^2 dx'}_{= \sqrt{\pi}\,\frac{1}{2} d^3} \right\}$$

$$= \frac{-\hbar^2}{\pi^{1/2} d} \left\{ -\frac{1}{2d}\sqrt{\pi} + (-k^2)\sqrt{\pi}\, d \right\}$$

$$= \frac{\hbar^2}{2d^2} + \hbar^2 k^2$$

#

[II-6] Time dependent perturbation problem (Non stationary problem)

Solution:

With the Schrödinger equation

$$-\frac{\hbar}{i}\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + \mu(\vec{\sigma},\vec{\mathcal{H}})\psi$$

where $\mu = \dfrac{e\hbar}{2mc}$, the Bohr magneton

and $\vec{\sigma}$ is the spin vector. As long as the field $\vec{H}$ exists $\psi = e^{i(ky-\omega t)}\alpha(s)$ and

upon substituting ψ into Schrödinger's equa. we have for the energy of the electron

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} + \mu H .$$

On switching on the field $\vec{H}'$ as the electron passes the point $y=0$, the state of the electro. begins to change, such that after a finite time interval a finite probability exist that the electron is in the state $\beta(s)$. We now assume:

$$\psi = e^{i(ky-\omega_0 t)}[a(t)\alpha(s) + b(t)\beta(s)]$$

Inserting the above into Schrödinger's equation using $\hbar\omega_0 = \hbar^2k^2/2m$ and separating into α- and β- parts:

$$-\frac{\hbar}{i}\dot{a} = \mu H a + \mu H' b \quad \text{and}$$

$$-\frac{\hbar}{i}\dot{b} = -\mu H b + \mu H' a .$$

[II-6] continued.

Solutions to these homogeneous systems of equations become:

$$a(t) = Ae^{-i\varepsilon t/\hbar}, \quad b(t) = Be^{-i\varepsilon t/\hbar}$$

hence, we obtain the algebraic system of equations,

$$(\mu H - \varepsilon)A + \mu H' B = 0$$

$$(\mu H' A) - (\mu H + \varepsilon) B = 0$$

the vanishing of the determinant leads to two solutions:

$$\begin{cases} \varepsilon_1 = +\mu\sqrt{H^2+H'^2} & B_1 = \dfrac{\sqrt{H^2+H'^2} - H}{H'} \\ \varepsilon_2 = -\mu\sqrt{H^2+H'^2} & B_2 = -\dfrac{\sqrt{H^2+H'^2} + H}{H'} \end{cases}$$

Out of the above the general solution becomes

$$\psi = e^{i(ky-\omega_0 t)}\Big\{ A_1\Big[e^{-\frac{i\mu}{\hbar}\sqrt{H^2+H'^2}\,t}\,\alpha(S) + *$$

$$* + \frac{\sqrt{H^2+H'^2} - H}{H'}\, e^{\frac{i\mu}{\hbar}\sqrt{H^2+H'^2}\,t}\,\beta(S)\Big] +$$

$$+ A_2\Big[e^{\frac{i\mu}{\hbar}\sqrt{H^2+H'^2}\,t}\,\alpha(S) - \frac{\sqrt{H^2+H'^2}+H}{H'}\, e^{-\frac{i\mu}{\hbar}\sqrt{H^2+H'^2}\,t}\,\beta(S)$$

now the integration constants A_1 and A_2 are determined for $t=0$ and the α-part

[II-6] continued.

of ψ, thus $A_1 + A_1 = 1$

and the β-part for this time dissapears, becomes:

$$A_1 \frac{\sqrt{H^2+H'^2}-H}{H'} - A_2 \frac{\sqrt{H^2+H'^2}+H}{H'} = 0$$

One obtains now

$$A_1 = \frac{\sqrt{H^2+H'^2}+H}{2\sqrt{H^2+H'^2}} \; ; \; A_2 = \frac{\sqrt{H^2+H'^2}-H}{2\sqrt{H^2+H'^2}}$$

Setting the above A_1 and A_2 into the general solution with

$$\tau = \frac{\mu}{\hbar}\sqrt{H^2+H'^2}\, t$$

the solution has the form:

$$\psi = e^{i(k_2 y - \omega_0 t)}\left\{\left(\cos\tau - i\frac{H}{\sqrt{H^2+H'^2}}\sin\tau\right)\alpha + i\frac{H'}{\sqrt{H^2+H'^2}}\sin\tau\cdot\beta\right\}.$$

Hence, the probability, that the electron at the time $t = l/v$ upon leaving the auxilia[ry] field with spin in the positive z-direction, will be:

[II-6] continued.

$$w_+ = \cos^2\tau + \frac{H^2}{H^2+H'^2}\sin^2\tau =$$

$$= 1 - \frac{H'^2}{H^2+H'^2}\sin^2\tau$$

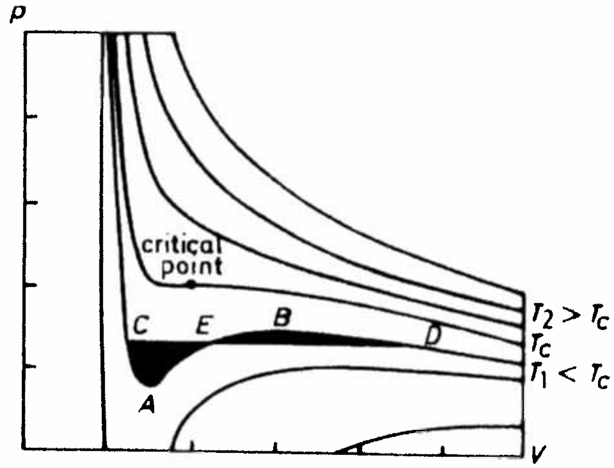
however the desired probability for the spin in the negative z-direction becom

$$w_- = \frac{H'^2}{H^2+H'^2}\sin^2\tau .$$

The sum of $w_+ + w_- = 1.$

[II-7] Solution:

(a) Figure:



As show in the above figure from

$$\left(\partial P/\partial V\right)_{T=T_c} = 0, \text{ and } \left(\partial^2 P/\partial V^2\right)_{T=T_c} = 0$$

we obtain $T_c = \frac{3a}{V_c^4}(V_c - b)^3/R$

$\therefore V_c = 3b$, $p_c = \frac{a}{27b^2}$, $T_c = \frac{8a}{27bR}$

(b) $p_c V_c / R T_c = 3/8$.

(c) In the above figure the horizontal line $\overline{CD}$ is the modified isotherm. The area of CAE is equal to the area of EB. The idea is that the common points, i.e., C and D of the Van der Waals isotherm and the physical isotherm have the same Gibbs free energy. Because of $G = G(T, p)$,

[II-7] continued
the equality of T's and p's, respectively will naturally cause the equality of G.
In this way,

$$\int_C^D dG = \int_C^D V dp = \int_C^A V dp + \int_A^E V dp + \int_E^B V dp + \int_B^D V dp = 0$$

That is,

$$\int_A^E V dp - \int_A^C V dp = \int_D^B V dp - \int_E^B V dp \quad \text{or}$$

$$\Delta S_{CAE} = \Delta S_{EBD} .$$

(d.) $$\left(\frac{\partial C_V}{\partial V}\right)_T = T \frac{\partial^2 S}{\partial V \partial T} = -T \frac{\partial^2}{\partial T^2}\left(\frac{\partial F}{\partial V}\right) = T\left(\frac{\partial^2 p}{\partial T^2}\right)$$

now for a Van der Waals gas, the equation of state gives

$$\left(\frac{\partial^2 p}{\partial T^2}\right)_V = 0$$

so that $$\left(\frac{\partial C_V}{\partial V}\right)_T = 0 .$$

$\boxed{\text{II}-8}$

Helmholtz free energy: $F = -\frac{1}{3} a T^4 V$ (given)

a) e.o.s.: $P = -\left(\frac{\partial F}{\partial V}\right)_T = \boxed{\frac{1}{3} a T^4}$ (you may realize that this is the photon gas)

b), c) internal energy: $F = U - TS$ where $S = -\left(\frac{\partial F}{\partial T}\right)$

$U = F + TS$ $\qquad S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4}{3} a T^3 V$

$U(T,V) = -\frac{1}{3} a T^4 V + \frac{4}{3} a T^4 V = \boxed{a T^4 V}$

c) $\Rightarrow$ heat capacity:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \boxed{4 a T^3 V}$$

or use $C_V = T\left(\frac{\partial S}{\partial T}\right)_V = 4a$

d) "free expansion" $V_1 \to V_2$ $\quad dU = \underbrace{dQ}_{\cancel{0}} - \underbrace{dW}_{\cancel{0}} = \cancel{0} \rightarrow U = \text{const.}$

$U(T,V) = a T^4 V = \text{const.}$ during free expansion (*)

$T_1^4 V_1 = T_2^4 V_2$

$T_2^4 = T_1^4 \frac{V_1}{V_2}$ $\qquad \boxed{T_2 = T_1 \left(\frac{V_1}{V_2}\right)^{1/4}}$

$V_2 > V_1$

$\Downarrow$

$T_2 < T_1$

"cools down"

"brute force" (if you desperately want to use equation provided
$C_V = 4aT^3V$

general relationship: (for "free expansion" $U = \text{const.}$)

$$\left(\frac{\partial T}{\partial V}\right)_U = -\frac{1}{C_V}\left(\frac{\partial U}{\partial V}\right)_T = -\frac{1}{4aT^3V}\,aT^4 = -\frac{1}{4}\frac{T}{V}$$

Thus,

$$\frac{dT}{dV} = -\frac{1}{4}\frac{T}{V} \qquad \text{for free expansion}$$

$$\frac{dT}{T} = -\frac{1}{4}\frac{dV}{V}$$

$$\ln(T) = -\frac{1}{4}\ln V + \text{const}$$

$$\boxed{TV^{1/4} = \text{const.}} \qquad \text{or} \qquad \underline{T^4V = \text{const.}}\ \text{, identical t}$$

$\boxed{\text{II}-9}$

localized (distinguishable) particles

$\varepsilon_j = \varepsilon_0 j \qquad j = 1, 2, \ldots$

$g_j = j$

$\beta \equiv 1/kT$

a) $$Z_1 = \sum_{j=1}^{\infty} g_j e^{-\beta\varepsilon_j} = \sum_{j=1}^{\infty} j e^{-\beta\varepsilon_0 j} = -\frac{\partial}{\partial(\beta\varepsilon_0)} \sum_{j=1}^{\infty} e^{-\beta\varepsilon_0 j}$$

$$= -\frac{\partial}{\partial(\beta\varepsilon_0)} \frac{e^{-\beta\varepsilon_0}}{1-e^{-\beta\varepsilon_0}} = -\frac{\partial}{\partial(\beta\varepsilon_0)} \frac{1}{e^{\beta\varepsilon_0}-1} = \boxed{\frac{e^{\beta\varepsilon_0}}{\left(e^{\beta\varepsilon_0}-1\right)^2}}$$

$$U = -N\frac{\partial}{\partial\beta}\ln Z_1 = -N\frac{\partial}{\partial\beta}\left\{\beta\varepsilon_0 - 2\ln\left(e^{\beta\varepsilon_0}-1\right)\right\}$$

$$= -N\varepsilon_0 + 2N\varepsilon_0 \frac{e^{\beta\varepsilon_0}}{e^{\beta\varepsilon_0}-1} = -N\varepsilon_0 + 2N\varepsilon_0 \frac{1}{1-e^{-\beta\varepsilon_0}}$$

$$\boxed{= -N\varepsilon_0 + 2N\varepsilon_0 \frac{1}{1-e^{-\varepsilon_0/kT}}}$$

$$C = \left(\frac{\partial U}{\partial T}\right)_{\varepsilon_0} = 2N\varepsilon_0 \frac{e^{-\varepsilon_0/kT}}{\left[1-e^{-\varepsilon_0/kT}\right]^2} \frac{\varepsilon_0}{kT^2}$$

$$\boxed{= 2Nk\left(\frac{\varepsilon_0}{kT}\right)^2 \frac{e^{-\varepsilon_0/kT}}{\left[1-e^{-\varepsilon_0/kT}\right]^2}}$$

low-T behavior $\boxed{\frac{\varepsilon_0}{kT} \gg 1}$

$$\boxed{C \simeq 2Nk\left(\frac{\varepsilon_0}{kT}\right)^2 e^{-\frac{\varepsilon_0}{kT}}}$$

$\left(\xrightarrow[T\to 0]{} 0\right)$

b)

$$F = -NkT \ln Z_1$$

$$F = U - TS \Rightarrow S = \frac{U-F}{T} = \frac{U}{T} - \frac{F}{T} =$$

$$\boxed{= \frac{U}{T} + Nk \ln Z_1}$$

$$S = -N\frac{\varepsilon_0}{T} + 2N\frac{\varepsilon_0}{T}\frac{1}{1-e^{-\varepsilon_0/kT}} + Nk\left\{\frac{\varepsilon_0}{kT} - 2\ln(e^{\varepsilon_0/kT} - 1\right.$$

$$= -Nk\left(\frac{\varepsilon_0}{kT}\right) + 2Nk\left(\frac{\varepsilon_0}{kT}\right)\frac{1}{1-e^{-\varepsilon_0/kT}} + Nk\left(\frac{\varepsilon_0}{kT}\right) - 2Nk\ln(e$$

$$\boxed{= 2Nk\left(\frac{\varepsilon_0}{kT}\right)\frac{1}{1-e^{-\varepsilon_0/kT}} - 2Nk\ln(e^{\varepsilon_0/kT} - 1)}$$

low-T behavior: $\varepsilon_0/kT \gg 1$ $(\rightarrow e^{-\varepsilon_0/kT} \ll 1)$

$$S \simeq 2Nk\left(\frac{\varepsilon_0}{kT}\right)(1 + e^{-\varepsilon_0/kT}) - 2Nk\ln[e^{\varepsilon_0/kT}(1 - e^{-\varepsilon_0/kT})]$$

$$= 2Nk\left(\frac{\varepsilon_0}{kT}\right)(1 + e^{-\varepsilon_0/kT}) - 2Nk\left(\frac{\varepsilon_0}{kT}\right) - 2Nk\ln(1 - e^{-}$$

$$= 2Nk\left(\frac{\varepsilon_0}{kT}\right)e^{-\varepsilon_0/kT} + 2Nk\,e^{-\varepsilon_0/kT} = 2Nk\,e^{-\varepsilon_0/kT}\left(\frac{\varepsilon_0}{kT}\right.$$

$$\approx 2Nk\left(\frac{\varepsilon_0}{kT}\right)e^{-\varepsilon_0/kT} \xrightarrow[T\to 0]{} \varnothing$$

[II-10] Solution

(a) The Bose distribution $\dfrac{1}{e^{(\epsilon-\mu)/kT}-1}$ requires that $\mu \leq 0$.

Generally

$$n = \int \frac{1}{e^{(\epsilon-\mu)/kT}-1} \cdot \frac{2\pi}{h^3}(2m)^{3/2}\sqrt{\epsilon}\, d\epsilon$$

When T decreases, the chemical potential, μ, ~~the~~ increases until $\mu = 0$, for which

$$n = \int \frac{1}{e^{\epsilon/kT}-1} \frac{2\pi}{h^3}(2m)^{3/2}\sqrt{\epsilon}\, d\epsilon.$$

Now Bose condensation occurs when the temperature continues to decrease with $\mu = 0$. Therefore in the limit of very low temperature, the Bose system can be regarded as having $\mu = 0$. The number of particles at the non-condensed state is not conserved. The energy density "u" and specific heat "c" are thus obtained as follows:

$$u = \int \frac{\epsilon}{e^{\epsilon/kT}-1} \cdot \frac{2\pi}{h^3}(2m)^{3/2}\sqrt{\epsilon}\, d\epsilon$$

$$u = \frac{2\pi}{h^3}(2m)^{3/2}(kT)^{3/2} \int_0^\infty \frac{x^{3/2}}{e^x-1}\, dx$$

[II-10] continued.

$$C = 5\pi\hbar\left(\frac{2mkT}{h^2}\right)^{3/2}\int_0^\infty \frac{x^{3/2}}{e^x-1}dx$$

(b) For a photon gas we have $\mu = 0$ at any temperature and $\epsilon = \hbar\omega$. The density of state is $\frac{\omega^2 d\omega}{\pi^2 c^3}$ and the energy density is

$$u = \frac{1}{\pi^2 c^3}\int \frac{\hbar\omega^3}{e^{\hbar\omega/kT}-1}d\omega =$$

$$= \frac{\hbar}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{x^3 dx}{e^x-1} .$$