Solutions

Physics PhD Qualifying Examination
Part I – Wednesday, 24 August 2005

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism), **and with at least one problem from problems 5 or 10 (Special Relativity).**
DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

[I-1] [10]

The rotational inertia is defined as: $I = \sum_i m_i r_i^2 = \int r^2 dm$ and the kinetic energy of a rotating body about a fixed axis is given by: $K = \frac{1}{2} I\omega^2$. (Here, ω is the angular frequency and m the mass.)

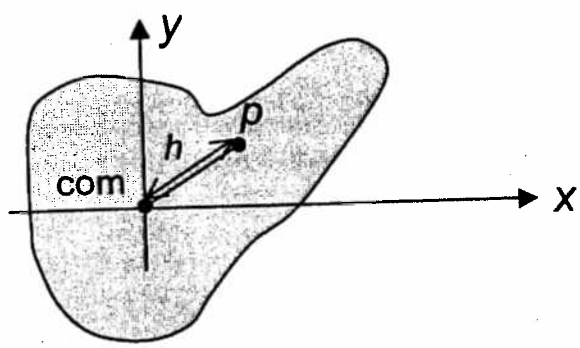
(a) Prove the parallel axis theorem, i.e. $I = I_{com} + M h^2$. Here,

M is the total mass of the rotating body;

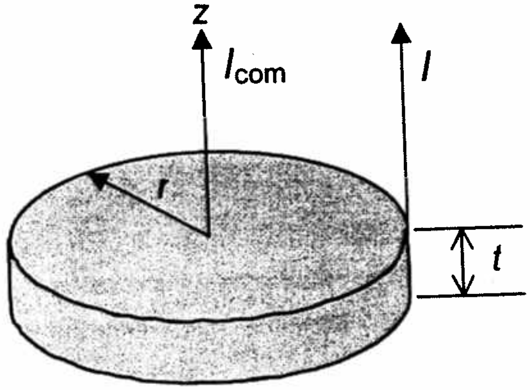
I is the rotational inertia about a given axis, along $\hat{z}$, through the point-p (see the figure below);

I_{com} is the rotational inertia about a parallel axis, $\hat{z}$, that extends through the body's center-of-mass (com); and

h is the perpendicular distance between the two axes.



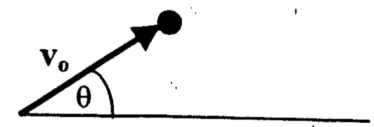
(b) Compute I and I_{com} for the solid disk shown below. The disk has a radius r, thickness t and a density ρ. The axes of rotation are indicated on the figure below.



[I-2] [10]

Consider the case of projectile motion under gravity in two dimensions. (Assume that the projectile is a point mass.) Let the mass of the projectile be m, the initial velocity of the projectile be $\mathbf{v_o}$ and the angle of elevation θ. At t=0 the projectile is at the position x_0 and y_0.

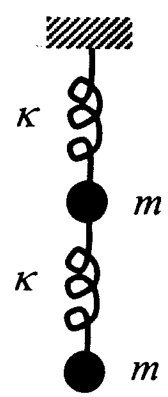
a) Find the Lagrange function of the projectile.
b) Derive the Lagrange equations of motion of the projectile.
c) By using the initial condition determine the appropriate equations of motion.



[I-3] [10]

A particle of mass m is attached to a rigid support by a spring with force constant κ. At equilibrium, the spring hangs vertically downward. To this mass-spring combination is attached an identical oscillator, the spring for the latter being connected to the mass of the former, as sketched below.

(a) Calculate the characteristic frequencies for (one-dimensional) vertical oscillations.
(b) Determine the normal modes of motion of the system.
(c) Calculate the frequencies when one or the other of the particles is held fixed while the other oscillates. How do they compare with the characteristic frequencies?



[I-4] [10]

A particle is dropped into a hole drilled through the center of the Earth. Find the equation of motion for the particle and obtain the period of the motion. In describing the motion, consider *only* the gravitational effects due to Earth (i.e., neglect the Earth's rotation, friction, air resistance, etc.). Assume that the Earth's mass is distributed homogeneously. The Earth's mass is M and its radius is R. The gravitational constant is G. You should express the period of the resulting motion in terms of these constants, M, R, and G.

[I-5] [4,6]

A rocket having initially a total mass M_0 ejects its fuel with constant velocity $-u$ ($u > 0$) relative to the rocket's instantaneous rest frame. According to Newtonian mechanics, its velocity V, relative to the inertial frame in which it was originally at rest, is related to its mass $M(V)$ by the formula

$$(M / M_0) = exp\,(-V / u)\,.$$

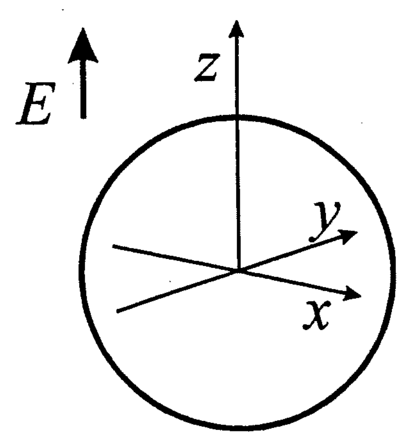
(a) *Derive* the above classical result.
(b) Now consider a *relativistic* rocket. Suppose the velocity of the ejected fuel is limited only by $0 \leq u \leq c$, where c is the speed of light; *derive the relativistic analogue of the above equation*. *Show* that it reduces to the Newtonian result in the appropriate limit.

[I-6] [6,4]

A perfectly conducting sphere is placed in a uniform electric field E pointing in the $\hat{z}$-direction.

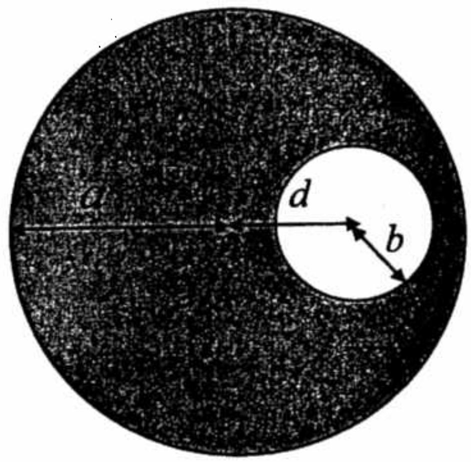
a) What is the surface charge density on the sphere?

b) What is the induced dipole moment of the sphere?



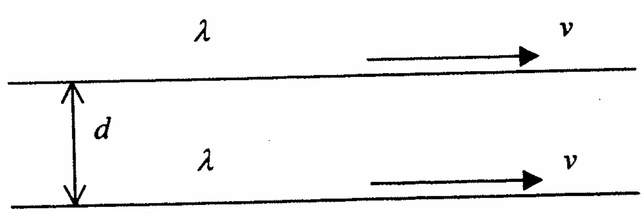
[I-7] [10]

A cylindrical conductor of radius a has a hole of radius b bored parallel to, and centered a distance d from, the cylindrical axis ($d + b < a$). The current density is uniform throughout the remaining metal of the cylinder and is parallel to the axis. Find the magnitude and the direction of the magnetic-flux density in the hole.



[I-8] [10]

Consider two parallel infinite straight line-charges λ, a distance d apart, moving along at a constant speed v. Both λ and v are measured in the Lab frame. What should be the value of v so that there is no net force on the lines? Is this a reasonable speed?



[I-9] **[10]**

The scalar potential V and vector potential A of a perfect electrical dipole in the far field $r>>c/\omega$ are given by Eqs. (1) and (2) below:

$$V(r,\theta,t) = \frac{p_0 \cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega}{c}\sin\left[\omega\left(t-r/c\right)\right]\right\} \quad (1)$$

$$\vec{A}(r,\theta,t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin\left[\omega\left(t-r/c\right)\right]\hat{z} \quad (2)$$

Calculate the electric field E, magnetic induction B and the total power radiated P in the far field.

[I-10] **[10]**

For a propagating EM-wave in the free-space, we have:

$\vec{E} = E_m \sin(kx - \omega t)\,\vec{y}$;

$\vec{B} = B_m \sin(kx - \omega t)\,\vec{z}$.

(a) Show that its phase velocity is given by: $v = \frac{\omega}{k}$.

(b) Further show, from the equation: $\oint \vec{E}\cdot d\vec{S} = -\frac{d\phi_B}{dt}$, that $\frac{\omega}{k} = \frac{E_m}{B_m}$.

(Here, ϕ_B is the magnetic flux.)

(c) Finally, from the equation: $\oint \vec{B}\cdot d\vec{S} = \mu_o \varepsilon_o \frac{d\phi_E}{dt}$, show that $(\frac{E_m}{B_m})^2 = \frac{1}{\mu_o \varepsilon_o}$ and therefore

$$c_o = \frac{1}{\sqrt{\mu_o \varepsilon_o}}.$$

(Here, c_o the speed of light in the free-space, ϕ_E is the electric flux.)

Qualifying Exam, August, 2005

$\boxed{\text{Part I}}$ Solutions

I-1 (a)

Solution:

- Let the two axis be along $\hat{z}$-direction.
- Let the center-of-mass (COM) be the origion: $(X, Y) = (0, 0)$
- Let the point-p be $(x, y) = (a, b)$

We have: $I \overset{def}{=} \int r^2 \, dm = \int [(X-a)^2 + (Y-b)^2] \, dm$

$$I = \int [X^2 + Y^2] \, dm + \int (a^2 + b^2) \, dm - 2a \int X \, dm - 2b \int Y \, dm$$

1st term $= \int [X^2 + Y^2] \, dm = I_{com}$

2nd term $= \int h^2 \, dm = h^2 M$

3rd term $= -2a \int X \, dm = 0$ (with-respect-to center-of-mass)

4th term $= -2b \int Y \, dm = 0$

$\therefore \; I = I_{COM} + M \cdot h^2$

#

I-1 (b)

To compute $I_{COM} = \int_{disk} r^2 \, dm$

Recognizing cylindrical symmetry:

$dm = 2\pi r (dr) \cdot (t)(\rho)$

$\therefore I_{COM} = \int_0^r 2\pi r^3 dr \, (t \cdot \rho) = \frac{2\pi r^4}{4} t \cdot \rho$

Noticing total mass $M = (\pi r^2 \cdot t \cdot \rho)$

We have: $I_{COM} = \frac{1}{2} M r^2$.

By the parallel axis theorem,

We have $I = I_{COM} + M r^2$

$\therefore I = \frac{3}{2} M r^2$

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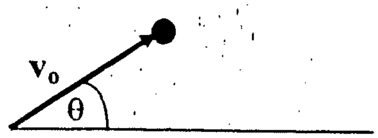
Part 1 1-2 Lagrangian mechanics

Consider the case of projectile motion under gravity in two dimensions. Let the mass of the projectile be m, the initial velocity of the projectile be v_0 and the angle of elevation θ. The initia position at $t=t_0$ is x_0 and the ~~same~~ y_0 pattern

- Find the Lagrange function of the projectile.
- Derive the Lagrange equations of motion of the projectile.
- By using the initial condition determine the appropriate equations of motion.



$$L = T - V$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2$$

$$U = mgy$$

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - mgy$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} = 0 \qquad \ddot{x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = m\ddot{y} + mg = 0 \qquad \ddot{y} = -g$$

$$\frac{d}{dt}\dot{x} = 0$$

$$\dot{x} = \text{const.} = c_1$$

$$dx = c_1 \, dt$$

$$x = c_1 t + c_2$$

$$x(t=0) = x_0 \qquad x(t=0) = c_2 = x_0$$

$$\dot{x}(t=0) = v_0 \cos\theta \qquad \dot{x}(t=0) = c_1 = v_0 \cos\theta$$

$$x(t) = v_0 \cos\theta \cdot t + x_0$$

$$\ddot{y} = -g$$

$$\frac{d}{dt}\dot{y} = -g$$

$$\dot{y} = -gt + c_3$$

$$\frac{d}{dt} y = -gt + c_3$$

$$y = -\frac{1}{2} g t^2 + c_3 t + c_4$$

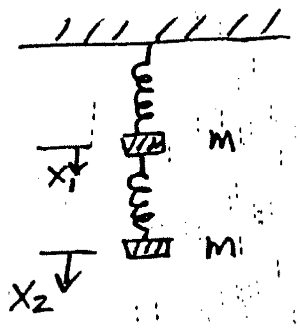
$$y(t=0) = y_0 \qquad c_4 = y_0$$

$$\dot{y}(t=0) = c_3 = v_0 \sin\theta$$

$$y(t) = -\frac{1}{2} g t^2 + v_0 \sin\theta \, t + y_0$$

6/21/05 : FALL 2005

I-3)



$$\begin{cases} m\ddot{x}_1 + 2Kx_1 - Kx_2 = 0 \\ m\ddot{x}_2 + Kx_2 - Kx_1 = 0 \end{cases}$$

$$\begin{cases} (2K - m\omega^2)B_1 - KB_2 = 0 \\ -KB_1 + (K - m\omega^2)B_2 = 0 \end{cases}$$

ev., es $\begin{cases} \omega_1^2 = \dfrac{3+\sqrt{5}}{2}\dfrac{K}{m} \;;\; \eta_1(t) = ma_{11}\left(x_{10} + \dfrac{1-\sqrt{5}}{2}x_{20}\right)\cos\omega_1 t \\ \omega_2^2 = \dfrac{3-\sqrt{5}}{2}\dfrac{K}{m} \;;\; \eta_2(t) = ma_{12}\left(x_{10} + \dfrac{1+\sqrt{5}}{2}x_{20}\right)\cos\omega_2 t \end{cases}$

MODE 1: antisymmetric; $x_{10} = -1.618\, x_{20}$; $\eta_2(t) = 0$

MODE 2: symmetric; $x_{10} = .618\, x_{20}$; $\eta_1(t) = 0$

Mass 2 fixed $\Rightarrow \omega_{10} = \sqrt{\dfrac{2K}{m}}$

Mass 1 fixed $\Rightarrow \omega_{20} = \sqrt{\dfrac{K}{m}}$

In contrast: $\omega_1 = \sqrt{\dfrac{3+\sqrt{5}}{4}}\,\omega_{10} > \omega_{10}$

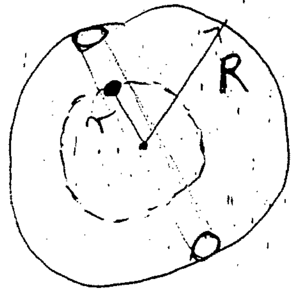
$\omega_2 = \sqrt{\dfrac{3-\sqrt{5}}{2}}\,\omega_{20} < \omega_{20}$

$\therefore \quad \omega_2 < \omega_{20} < \omega_{10} < \omega_1$

I-4

Use, e.g., Gauss's Law $\rho = \dfrac{M}{\frac{4\pi}{3}R^3}$

M: mass of the Earth
R: radius of the Earth
G: gravitational constant



$$\oiint \bar{g}\cdot d\bar{a} = -4\pi G m_{encl.}$$

$$g(r)\cancel{4\pi}r^2 = -\cancel{4\pi}G\rho\frac{4\pi}{3}r^3$$

$$g(r) = -G\frac{4\pi}{3}\rho r$$

("−" means points toward center)

So for the 1D motion along the line going through the center:

$$m\ddot{x} = m g(x)$$

$$\boxed{\ddot{x} = -G\frac{4\pi}{3}\rho x}$$

equation of motion for simple harmonic motion $\ddot{x} = -\omega_o^2 x$

$$\omega_o^2 = \frac{4\pi G\rho}{3} = \frac{\cancel{4\pi}G}{\cancel{3}}\frac{M}{\frac{\cancel{4\pi}}{\cancel{3}}R^3} = \frac{GM}{R^3}$$

$$T = \frac{2\pi}{\omega_o} = \frac{2\pi}{\sqrt{\frac{GM}{R^3}}} = 2\pi\sqrt{\frac{R^3}{GM}}$$

Q.E. Aug. 2005.

[I-5] Solution:

a) Let us consider the short interval "dt" in the center of mass frame moving with velocity "v;" the fuel is ejected with velocity "u" in this frame. At time $t' = t + dt$ the velocity of the rocket increases by dv. The mass $M(t)$ of the rocket decreases by dM ($dM < 0$) and the mass $|dM|$ of the ejected fuel will have a velocity $-u$ in this frame. Now momentum conservation gives

$$M(t+dt)dv - u|dM| = M(t+dt)dv + u\,dM = 0$$

where $M(t+dt)$ is the mass of the rocket at time $(t+dt)$. Expanding $M(t+dt)$ as

$$M(t+dt) = M(t) + M'(t)\,dt$$ and neglecting second-order terms in the differentials yields

$$M\,dv = -u\,dM$$

Transforming to the lab frame and using $dv = dV$, where "V" is the velocity of the rocket in the lab frame, we obtain a solution for the initial condition $v(0) = 0$:

$$\frac{M}{M_0} = e^{(-V/u)}$$

continued.

[I-5] (b) Write momentum conservation in the rocket's frame:

$$M\,dv - \gamma_u\,dm\,u = 0$$

where M is the mass of the rocket, dm is the mass of the fuel, and $\gamma_u = \dfrac{1}{\sqrt{1-u^2/c^2}}$.

Energy conservation in the frame of the rocket gives

$$M = \gamma_u\,dm + (M + dM)\ .$$

We ignored the relativistic corrections to the mass of the rocket in the above equation and terms such as $\left(\sqrt{1-(dv)^2/c^2}\right)^{-1}$ in the momentum conservation equation.

Substituting dm from the energy conservation into the momentum conservation, we have

$$M\,dv = -u\,dM$$

which is of course the same as obtained for the non-relativistic part in (a).

Now we must transform dv from the instantaneous rocket frame to the laboratory frame. Using the equation for the addition of the velocities, we obtain

[I-5] (cont'd)

$$V + dV = \frac{V + dv}{1 + V\,dv/c^2}$$

with $V + dV$ being the new velocity of the rocket in the lab frame. Rearranging gives

$$dV + \frac{V^2}{c^2}\,dv = dv, \quad \text{or} \quad dv = \gamma^2\,dV$$

where $\gamma = 1/\sqrt{1 - V^2/c^2}$. Upon substitution

$M\gamma^2\,dV = -u\,dM$ or

$$\frac{dV}{1 - V^2/c^2} = \frac{1}{2}\left(\frac{dV}{1 - V/c} + \frac{dV}{1 + V/c}\right) = -u\frac{dM}{M}$$

integrating the above expression, we obtain

$$\frac{c}{2}\left[-\ln(1-\beta) + \ln(1+\beta)\right] = u \ln\frac{M_o}{M}$$

where: $\beta = V/c$, and we find

$$\frac{M}{M_o} = \left(\frac{1-\beta}{1+\beta}\right)^{c/2u}$$ now, if $\beta \ll 1$

than the above reduces to:

$$\frac{M}{M_o} \cong e^{-V/2u} \cdot e^{-V/2u} = e^{-V/u}$$

again the same result as obtained in part (a)

I-6 Solution

a) The boundary conditions on the conductor surface are

$$\Phi = \text{constant} = \Phi_S$$

That is

$$\varepsilon_0 \frac{\partial \Phi}{\partial r} = -\sigma ,$$

Where Φ_S is the potential of the conducting sphere and σ is its surface charge density. Considering the symmetry of the problem, we use spherical coordinates with origin at the center of the sphere. The potential at a point (r,Θ,φ) outside the sphere then in general is given by

$$\Phi = \sum_{n=0} \left(C_n r^n + \frac{D_n}{r^{n+1}} \right) P_n(\cos\Theta) \tag{1}$$

We assume E_0 is the original uniform electric field intensity.

Then for $r \to \infty$ we expect

$$\Phi = -E_0 r \cos\Theta = -E_0 r\, P_1(\cos\Theta) . \tag{2}$$

By comparing the coefficients in Eqs. (1) and (2), we find

$C_0 = 0$, $\quad C_1 = -E_0$, $\quad D_1 = E_0\, a^3$, $\quad C_n = D_n = 0 \quad$ for $n > 1$.

And therefore

$$\Phi = -E_0 r \cos\Theta + \frac{E_0 a^3}{r^2} \cos\Theta , \tag{3}$$

where a is the radius of the sphere. The second boundary condition and Eq. (3) give

$$\sigma = 3\varepsilon_0 E_0 \cos\Theta . \tag{4}$$

b) Suppose that an electric dipole $\mathbf{P} = P e_z$ is placed at the origin, instead of the sphere. The potential at r produced by the dipole is

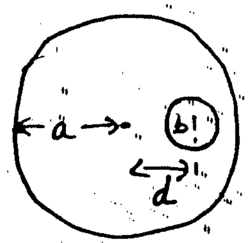
$$\Phi_p = -\frac{1}{4\pi\varepsilon_0}\mathbf{P}\cdot\nabla\left(\frac{1}{r}\right) = \frac{P\cos\Theta}{4\pi\varepsilon_0 r^2}. \tag{5}$$

Comparing this with the second term of Eq. (3) shows that the latter corresponds to the contribution of a dipole having a moment

$$\mathbf{P} = 4\pi\varepsilon_0 a^3\, \mathbf{E}_0\,, \tag{6}$$

which can be considered as the induced dipole moment of the sphere.

6/21/05 : FALL '05

I-7)  where $(d+b) < a$

Superpose $\underline{J}$ in complete cylinder with $-\underline{J}$ in hole.

For $r < a$, $B_\theta 2\pi r = \mu_0 J \pi r^2 \Rightarrow \underline{B}_{cylinder} = \frac{\mu_0 J}{2} r \hat{\theta}$

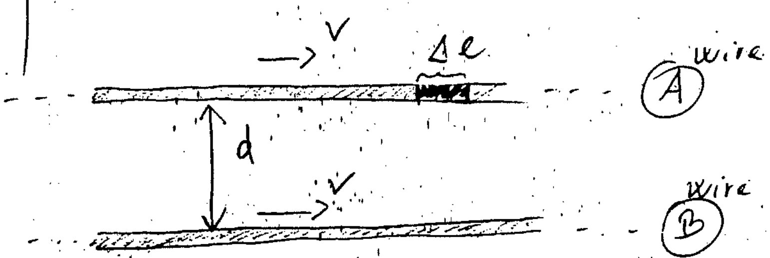
For $r' < b$, $B_{\theta'} 2\pi r' = \mu_0 (-J) \pi r'^2 \Rightarrow \underline{B}_{hole} = -\frac{\mu_0 J}{2} r' \hat{\theta}'$

In hole, $\underline{B}_{total} = \frac{\mu_0 J}{2} \left(r\hat{\theta} - r'\hat{\theta}' \right)$

where $\underline{r}' = \underline{r} - \underline{d}$

||

I-8



- find "current" as a result of moving: $\Delta Q = \lambda \Delta \ell$
$$J = \frac{\Delta Q}{\Delta t} = \lambda \frac{\Delta \ell}{\Delta t} = \lambda v$$
- find magnetic field, e.g., due to wire (A):
(Ampere's Law:) $B \cdot 2\pi d = \mu_0 J$

the magnetic field at a distance "d" from wire (A):
$$B = \frac{\mu_0 J}{2\pi d} = \frac{\mu_0}{2\pi d} \lambda v = \frac{\mu_0 \lambda v}{2\pi d}$$

- magnetic force per unit length on wire (B): $(\vec{J} \perp \vec{B})$

$$\Delta F_{BA} = \Delta Q\, v B = \Delta \ell\, \lambda v \cdot B \qquad (\text{toward } (A))$$
$$\boxed{\frac{\Delta F^{m}_{BA}}{\Delta \ell} = \lambda v B = \frac{\mu_0 \lambda^2 v^2}{2\pi d}}$$

- electrostatic field on wire (B) (note that λ is stationary!)
(Gauss's law:) $E\, 2\pi d\, \Delta \ell = \frac{\lambda \Delta \ell}{\varepsilon_0}$
$$E = \frac{\lambda}{2\pi \varepsilon_0} \frac{1}{d}$$

$$\Delta F^{el.}_{BA} = \Delta Q E = \Delta \ell \lambda E$$
$$\frac{\Delta F^{el.}_{BA}}{\Delta \ell} = \lambda E = \frac{\lambda^2}{2\pi \varepsilon_0} \frac{1}{d} \qquad (\text{pointing away from } (A$$

$$\frac{\Delta F_{BA}}{\Delta \ell} = \text{net force per unit length} = \frac{-\lambda^2}{2\pi \varepsilon_0 d} - \frac{\mu_0 \lambda^2 v^2}{2\pi d} = \frac{\lambda^2}{2\pi d}\left(\frac{1}{\varepsilon_0} - \mu_0 v^2\right)$$

$\boxed{\text{I}-8}$ (con't.)

$= \emptyset$ (for no net force on any wires)

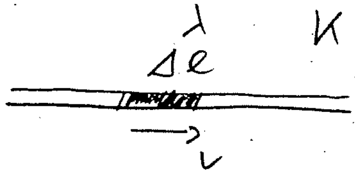
$$\frac{\Delta F_{BA}}{\Delta \ell} = -\frac{\Delta F_{AB}}{\Delta \ell}$$

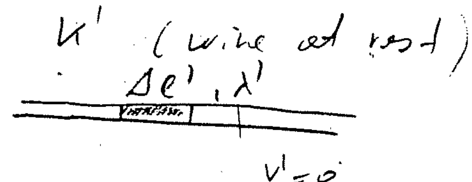
$$\Rightarrow v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c \leftarrow \text{speed of light}$$

Not realistic.

This problem can also be solved using the special Lorentz tr. for the line charge density

• find co-moving fr v, where $\lambda' = 0$





Lorentz contraction: $\Delta \ell = \sqrt{1-\frac{v^2}{c^2}}\,\Delta \ell'$, $\Delta Q = \Delta Q'$

$$\lambda' = \frac{\Delta Q'}{\Delta \ell'} = \frac{\Delta Q}{\Delta \ell / \sqrt{1-v^2/c^2}} = \sqrt{1-v^2/c^2}\,\frac{\Delta Q}{\Delta \ell} = \sqrt{1-v^2/c^2}\,\lambda$$

in this frame, there is no magnetic field since $v' = 0$
the only case when the electrostatic force is zero, if
$\lambda' = 0 \Rightarrow \boxed{v = c}$ (not realistic)

2)

Part I-9 Radiation dipole

The scalar potential V and vector potential **A** of a perfect electrical dipole in the far field r>>c/ω are given by eqs. (1) and (2):

$$V(r,\theta,t)=\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left\{-\frac{\omega}{c}\sin[\omega(t-r/c)]\right\} \quad (1)$$

$$\vec{A}(r,\theta,t)=-\frac{\mu_0 p_0\omega}{4\pi r}\sin[\omega(t-r/c)]\hat{z} \quad (2)$$

Calculate the electric field **E**, magnetic induction **B** and the total power radiated P in the far field.

griffith p. 447-448

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \quad (1)$$

$$\langle\vec{S}\rangle = \frac{1}{\tau}\int_0^\tau \vec{S}\,dt \quad (2) \quad \cancel{= \frac{\mu_0}{c}\left(\frac{p_0\omega^2}{4\pi}\frac{\sin\theta}{r}\cos[\omega(t-r/c)]\right)^2}$$

$$\langle P\rangle = \int \langle\vec{S}\rangle\cdot d\vec{a} \quad (3)$$

$$\vec{\nabla} V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$$

$$\cong \frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}$$

$$\frac{\partial\vec{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right](\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta})$$

$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$A_r = A_z\cos\theta$$

$$A_\theta = -A_z\sin\theta$$

for result of derivatives & integration refer to griffith

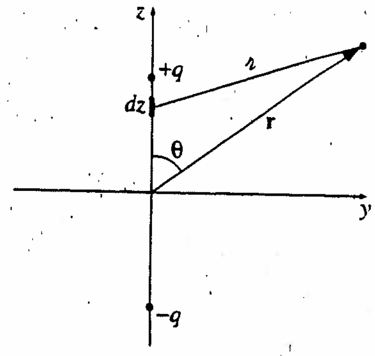


Figure 11.3

From the potentials, it is a straightforward matter to compute the fields.

$$\nabla V = \frac{\partial V}{\partial r}\,\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\,\hat{\boldsymbol{\theta}}$$

$$= -\frac{p_0\omega}{4\pi\epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}\right\}$$

$$\cong \frac{p_0\omega^2}{4\pi\epsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\mathbf{r}}.$$

(I dropped the first and last terms, in accordance with approximation 3.) Likewise,

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}),$$

and therefore

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\theta}}.} \tag{11.18}$$

Meanwhile

$$\nabla\times\mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]\hat{\boldsymbol{\phi}}$$

$$= -\frac{\mu_0 p_0 \omega}{4\pi r}\left\{\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)] + \frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right\}\hat{\boldsymbol{\phi}}.$$

The second term is again eliminated by approximation 3, so

$$\boxed{\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\mu_0 p_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t-r/c)]\,\hat{\boldsymbol{\phi}}.} \tag{11.19}$$

Equations 11.18 and 11.19 represent monochromatic waves of frequency ω traveling in the radial direction at the speed of light. $\mathbf{E}$ and $\mathbf{B}$ are in phase, mutually perpendicular, and transverse; the ratio of their amplitudes is $E_0/B_0 = c$. All of which is precisely what we expect for electromagnetic waves in free space. (These are actually *spherical* waves, not plane waves, and their amplitude decreases like $1/r$ as they progress. But for large r, they are approximately plane over small regions—just as the surface of the earth is reasonably flat, locally.)

The energy radiated by an oscillating electric dipole is determined by the Poynting vector:

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}. \tag{11.20}$$

The intensity is obtained by averaging (in time) over a complete cycle:

$$\langle \mathbf{S} \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\,\hat{\mathbf{r}}. \tag{11.21}$$

Notice that there is no radiation along the *axis* of the dipole (here $\sin\theta = 0$); the intensity profile[5] takes the form of a donut, with its maximum in the equatorial plane (Fig. 11.4). The total power radiated is found by integrating $\langle \mathbf{S} \rangle$ over a sphere of radius r:

$$\langle P \rangle = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}. \tag{11.22}$$

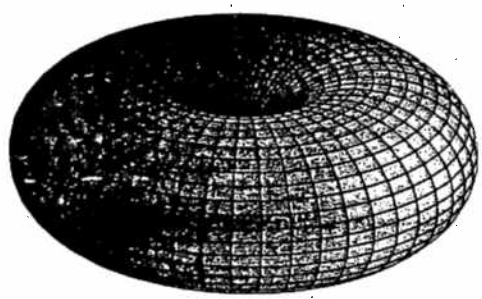


Figure 11.4

[5]The "radial" coordinate in Fig. 11.4 represents the magnitude of $\langle \mathbf{S} \rangle$ (at fixed r), as a function of θ and ϕ.

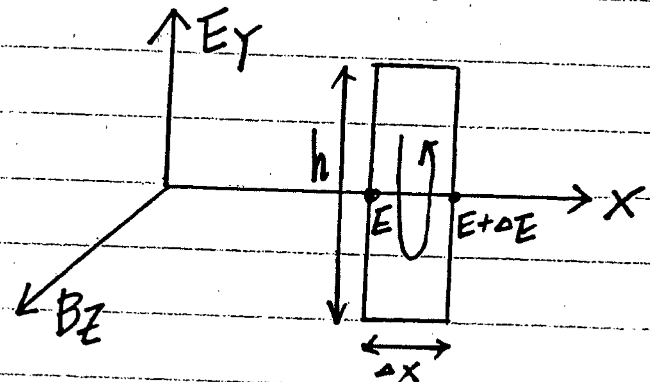
I-10

Solution

(1) The phase factor, $\theta = kx - wt)$
at ~~the~~ constant phase, $\Delta\theta = 0\ (= k\Delta x - w\Delta t$

$$\therefore V = \frac{\Delta x}{\Delta t} = \frac{w}{k} \quad \cdots\cdots (1)$$

(2) We construct a contour integral:



$\oint \vec{E} \cdot d\vec{s} = -\frac{d\phi_B}{dt}$ becomes:

$(E+dE)h - Eh = -(h \cdot dx)\frac{dB}{dt}$ and

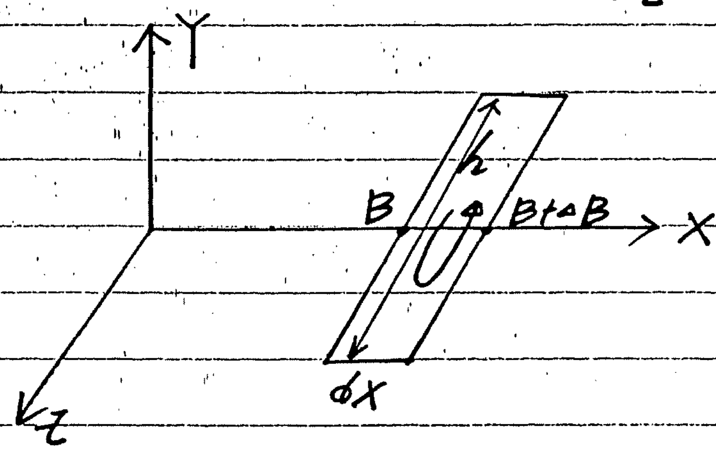
$$\frac{dE}{dx} = -\frac{dB}{dt} \quad \cdots\cdots\cdots (2)$$

$$E_m \cdot k \cdot \cos(kx - wt) = B_m \cdot w \cdot \cos(kx - wt)$$

Therefore, $\frac{E_m}{B_m} = \frac{w}{k}\ (= V) \cdots\cdots (3)$

I-10

③ From: $\oint \vec{B} \cdot d\vec{s} = \mu_0 \varepsilon_0 \dfrac{d\Phi_E}{dt}$,



we have: $-(B+dB)h + Bh = \varepsilon_0 \mu_0 \, h \, dx \dfrac{dE}{dt}$

and $-\dfrac{dB}{dx} = \varepsilon_0 \mu_0 \dfrac{dE}{dt}$ ...... ④

$\therefore \; -B_m \cdot k \cdot \cos(kx - wt) = -\varepsilon_0 \mu_0 \cdot w \cdot E_m \cdot \cos(kx - wt)$

Therefore: $\dfrac{E_m}{B_m} = \dfrac{1}{\varepsilon_0 \mu_0} \cdot \dfrac{1}{w/k}$ ..... ⑤

Combining ③ & ⑤, we have $\left(\dfrac{E_m}{B_m}\right)^2 = \dfrac{1}{\varepsilon_0 \mu_0}$

$c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$

#

Solutions

Physics PhD Qualifying Examination
Part II – Saturday, 27 August 2005

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

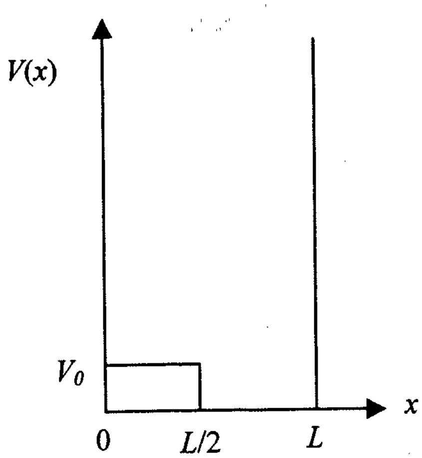
[II-1] **[10]**

Use $\frac{d}{dt}\langle\hat{Q}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle + \left\langle\frac{\partial\hat{Q}}{\partial t}\right\rangle$ (where $\hat{Q}$ is an arbitrary operator and $\hat{H}$ is the Hamiltonian), to prove the virial theorem (for stationary states) in three dimensions:
$2\langle T\rangle = \langle \mathbf{r}\cdot\nabla V\rangle$.
Here, T and V stand for the kinetic and potential-energy operators, respectively.
Apply the virial theorem to the case of hydrogen, and show that $\langle V\rangle = -2\langle T\rangle$.

[II-2] **[10]**

Consider a particle of mass m in a potential well as shown in the figure. The walls of the potential well are infinitely high. Suppose that the step (of height V_0 and width $L/2$) at the bottom of the well can be considered as a small perturbation.

a) Use first-order perturbation theory to calculate the eigenenergies E_n of the particle in the potential well.
b) What are the first-order corrected wavefunctions?
c) If the particle is an electron, how do the frequencies emitted by the perturbed system compare with those of the unperturbed system?
d) What smallness assumption is appropriate to V_0?



[II-3] **[10]**

A spin-1/2 particle interacts with a magnetic field $\boldsymbol{B} = B_0\hat{z}$ through the Pauli interaction $H = \mu\boldsymbol{\sigma}\cdot\boldsymbol{B}$, where μ is the magnetic moment and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices. At $t = 0$ a measurement determines that the spin is pointing along the positive x-axis. What is the probability that it will be pointing along the negative y-axis at a later time t ?

[II-4] **[6, 4]**

a) Particles are incident on a spherically symmetric potential

$$V(r) = \frac{\beta}{r}\exp(-\gamma r), \tag{1}$$

Where β and γ are constants. Show that, in the Born approximation, the differential scattering cross-section for the scattering vector $\boldsymbol{K}$ is given by

$$\frac{d\sigma}{d\Omega} = \left\{\frac{2m\beta}{\hbar^2(\kappa^2+\gamma^2)}\right\}^2. \tag{2}$$

b) Use this result to derive the Rutherford formula for the scattering of α-particles. This formula says that for α-particles of energy E which are incident on nuclei of atomic number Z, the differential scattering cross-section is given by

$$\frac{d\sigma}{d\Omega} = \left\{\frac{Ze^2}{8\pi\varepsilon_0 E\sin^2(\Theta/2)}\right\}^2. \tag{3}$$

Here Θ is the angle between scattered and incident particles.

Hint: Use the fact that for a spherical symmetric function $A(\mathbf{r}) = A(r)$ the following holds

$$\int_V A(\mathbf{r})\exp(i\,\boldsymbol{\kappa}\cdot\mathbf{r})\,dV = \frac{4\pi}{\kappa}\int_0^\infty A(r)\,r\sin(\kappa r)\,dr. \tag{4}$$

[II-5] [10]

For electrons incident on a narrow slit of width a, show (using the uncertainty principle) that the first diffraction minimum occurs at an angle θ, where

$$a\sin\theta \geq \frac{\lambda}{2\pi}:$$

Here, λ is the wavelength of the electrons.

[II-6] [10]

Consider a charged one-dimensional harmonic oscillator with mass m, frequency ω_o, and charge q. Initially the oscillator is in its unperturbed ground state when there is no electric field present. At $t = 0$ a weak spatially uniform electric field $E = E_o e^{-\gamma t}\cos(\omega t)$ is imposed (the field is parallel to the direction of motion of the oscillator) with $\gamma < \omega_o$. Using *time-dependent perturbation theory*, find the transition probabilities to all excited states for $t = \infty$. For fixed ω_o and γ, what value of ω maximizes these transition probabilities? You may find the number representation of the harmonic oscillator with the annihilation and creation operators useful

$$a = \sqrt{\frac{m\omega_o}{2\hbar}}\left(x + \frac{i}{m\omega_o}p\right),\ a^{+} = \sqrt{\frac{m\omega_o}{2\hbar}}\left(x - \frac{i}{m\omega_o}p\right).$$

[II-7] [5,5]

(a) Consider a region within a fluid described by the van der Waals equation

$$\beta p = \rho/(1-b\rho) - \beta a\rho^2 \;,$$

where $\beta = (k_b T)^{-1}$, and $\rho = \langle N \rangle / V$. The volume of the region is L^3. Due to the spontaneous fluctuations in the system, the instantaneous value of the density in that region can differ from its average value by an amount $\delta\rho$. Determine as a function of β, ρ, a, b, and L^3, the typical relative size of these fluctuations; that is, evaluate $\langle(\delta\rho)^2\rangle^{1/2}/\rho$; <u>hint</u>: use the following relation in the previous expression to determine the relative size of these fluctuations; $(\partial\rho/\partial\beta\mu)_{\beta,V} = \rho\,(\partial\rho/\partial\beta p)_{\beta,V}$; here μ is the chemical potential. Demonstrate that when one considers observations of a macroscopic system (i.e., the size of the region becomes macroscopic, $L^3 \rightarrow \infty$) the relative fluctuations become negligible.

(b) A fluid is at its "critical point" when

$$(\partial\beta p/\partial\rho)_\beta = (\partial^2\beta p/\partial\rho^2)_\beta = 0.$$

Determine the critical point density and temperature for the fluid obeying the van der Waals equation. That is, compute β_c and ρ_c as a function of a and b .

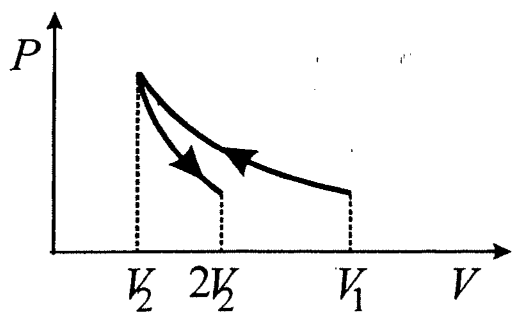
[II-8] **[4,3,3]**

An ideal gas is compressed at constant temperature T from volume V_1 to volume V_2.

a) Find the work done on the gas and the heat absorbed by the gas.

b) The gas now expands adiabatically to double its volume ($2V_2$).
What is the final temperature T_f ?
(Derive this result from the first and second law of thermodynamics)

c) Estimate T_f for $T_i = 300$ K for air.



[II-9] **[10]**

Interstellar space is sparsely filled with atomic hydrogen. The hydrogen gas is in thermodynamic equilibrium at temperature T. What is the number of hydrogen atoms which cross a given area A per second at temperature T in interstellar space? The mass of hydrogen is m.

[II-10] **[10]**

Consider a one-dimensional, non-interacting, spin=1/2 Fermi system with the energy-momentum dispersion relation $\varepsilon(p) = \alpha p^2 + m^2$, where α and m are phenomenological model parameters.
(a) Find the Fermi momentum p_F, as a function of the linear density of the particles N/L. (N is the number of electrons and L is the length of the interval to which the particles are confined.)
(b) Find the Fermi energy ε_F, as a function of the linear density of the particles N/L.
(c) Obtain the single-particle density of states $g(\varepsilon)$ and sketch this function.

Qualifying Exam, August, 2005

$\boxed{\text{Part II}}$ Solutions

6/21/05: Fall 2005

II-1) Let $\hat{Q} = \underline{r} \cdot \underline{p}$

$$\frac{d}{dt}\langle \underline{r}\cdot\underline{p}\rangle = \frac{i}{\hbar}\langle [\hat{H}, \underline{r}\cdot\underline{p}]\rangle + \underbrace{\left\langle \frac{\partial(\underline{r}\cdot\underline{p})}{\partial t}\right\rangle}_{0}$$

$$= \langle +2T - \underline{r}\cdot\underline{\nabla}V\rangle$$

But $\frac{d}{dt}\langle \underline{r}\cdot\underline{p}\rangle = 0$ for stationary state $\Rightarrow 2\langle T\rangle = \langle \underline{r}\cdot\underline{\nabla}V\rangle$

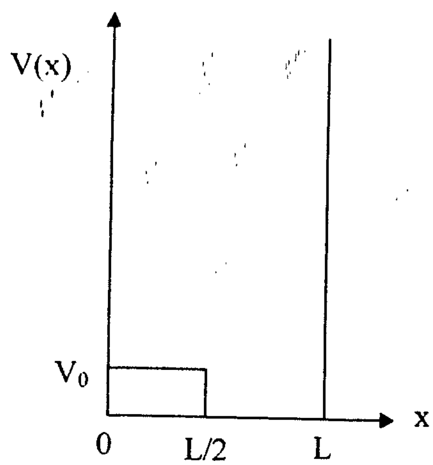
$V = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} \Rightarrow \underline{r}\cdot\underline{\nabla}V = -V \quad \therefore\ 2\langle T\rangle = -\langle V\rangle$

$\langle H\rangle = E_n = \langle T\rangle + \langle V\rangle \Rightarrow \langle T\rangle = -E_n$

Part II-2 Pertubation theory / time-independent

Consider a particle of mass m in an infinitely high potential well as shown in the figure. Suppose that the step at the bottom of the well can be considered as a small perturbation.

- Use first-order perturbation theory to calculate the eigenenergies E_n of the particle in the potential well.
- What are the first order corrected wavefunctions?
- If the particle is an electron, how do the frequencies emitted by the perturbed system compare with those of the unpertubed system?
- What smallness assumption is approbiate to V_0?



particle in a 1-dimensional potential well

$$E_n = n^2 E_1 \qquad E_1 = \frac{\hbar^2 \pi^2}{2mL^2} \qquad n = 1, 2, 3, \ldots$$

$$\varphi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

first order correction

$$E_n = E_n^{(0)} + H'_{nn} \qquad \varphi_n = \varphi_n^{(0)} + \sum_{i \neq n} \frac{H'_{in}}{E_n^{(0)} - E_i^{(0)}} \varphi_i^{(0)}$$

$$H'_{nn} = \langle \varphi_n | H' | \varphi_n \rangle \qquad H'_{in} = \langle \varphi_i^{0} | H' | \varphi_n^{0} \rangle$$

$$H' = V_0 \qquad 0 \leq x \leq \frac{L}{2}$$

$$H'_{nn} = \int_0^{L/2} \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) V_0 \, dx = \frac{2V_0}{L} \int_0^{L/2} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

frequencies $\Delta E = \hbar \omega$ $E^0 = n^2 E_1$ $E^1 = n^2 E_1 + \frac{V_0}{2}$

$\omega' = \frac{1}{\hbar} (E_n - E_{\ell}) = \frac{1}{\hbar} E_1 (n^2 - \ell^2)$ same as unperturbated

Smallness assumption $V_0 \ll E_1$

Q.E-Ph.D. Aug 2005

(II-3) Solution

Let us quantize the spin states along the z-axis so that the spin up and spin down are denoted b

$$\alpha \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H\alpha = \hbar\omega\alpha, \quad H\beta = -\hbar\omega\beta, \quad \hbar\omega = \mu B_0$$

The eigenstates of σ_x are ψ_x for pointing along the $+x$ axis, and $\psi_{\bar{x}}$ for the $-x$ axis

$$\psi_x = \frac{1}{\sqrt{2}}(\alpha + \beta) \qquad \sigma_x \psi_x = \psi_x$$

$$\psi_{\bar{x}} = -\frac{1}{\sqrt{2}}(\alpha - \beta) \qquad \sigma_x \psi_{\bar{x}} = -\psi_{\bar{x}}$$

$$\psi_y = \frac{1}{\sqrt{2}}(\alpha + i\beta) \qquad \sigma_y \psi_y = \psi_y$$

$$\psi_{\bar{y}} = \frac{1}{\sqrt{2}}(\alpha - i\beta) \qquad \sigma_y \psi_{\bar{y}} = \psi_{\bar{y}}$$

At time $t=0$, we start in the state ψ_x. Later this state becomes

$$\Psi_x(t) = \frac{1}{\sqrt{2}}(\alpha e^{-i\omega t} + \beta e^{i\omega t})$$

The amplitude for pointing in the negative y-directio is found by taking the matrix element with $\psi_{\bar{y}}$. The probability is the square of the absolute magnitude of this amplitude

$$\langle \psi_{\bar{y}} | \Psi_x(t) \rangle = \frac{1}{2}(e^{-i\omega t} + i e^{i\omega t})$$

$$P_{x\bar{y}}(t) = \cos^2(\omega t + \frac{\pi}{4}) .$$

II-4 Solution

a) The Born approximation gives the scattering amplitude

$$f(\Theta) = -\frac{m}{2\pi\hbar^2} I ; \tag{1}$$

where

$$I = \int_V V(\mathbf{r}) \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\, dV . \tag{2}$$

Using the identity given in the hint with $A(r) = V(r)$ and

$$V(r) = \frac{\beta}{r}\exp(-\gamma r)$$

we obtain

$$I = \frac{4\pi\beta}{\kappa}\int_0^\infty \sin(\kappa r)\exp(-\gamma r)\, dr . \tag{3}$$

The integral is readily evaluated by expressing the sine function as

$$\sin(\kappa r) = \frac{1}{2i}\{\exp(i\kappa r) - \exp(-i\kappa r)\}, \tag{4}$$

and noting that the integrated expression vanishes at $r \to \infty$ due to the factor $\exp(-\gamma r)$.

The result is

$$I = \frac{4\pi\beta}{\kappa^2 + \gamma^2}. \tag{5}$$

Thus from (1) and (5) we obtain

$$\frac{d\sigma}{d\Omega} = |f(\Theta)|^2 = \left\{\frac{2m\beta}{\hbar^2(\kappa^2+\gamma^2)}\right\}^2 . \tag{6}$$

b) α -particles are scattered by the electrostatic interaction between the charge $2e$ on the α -particle and the charge Ze on the nucleus. The potential is

$$V(r) = \frac{2Ze^2}{4\pi\varepsilon_0}\frac{1}{r}. \tag{7}$$

This is represented by the potential in part (a), with

$$\beta = \frac{2Ze^2}{4\pi\varepsilon_0}; \qquad \gamma = 0. \tag{8}$$

Inserting these values in (6) gives

$$\frac{d\sigma}{d\Omega} = \left\{\frac{2mZe^2}{\pi\varepsilon_0\hbar^2\kappa^2}\right\}^2. \tag{9}$$

The energy of the α - particle is

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}. \tag{10}$$

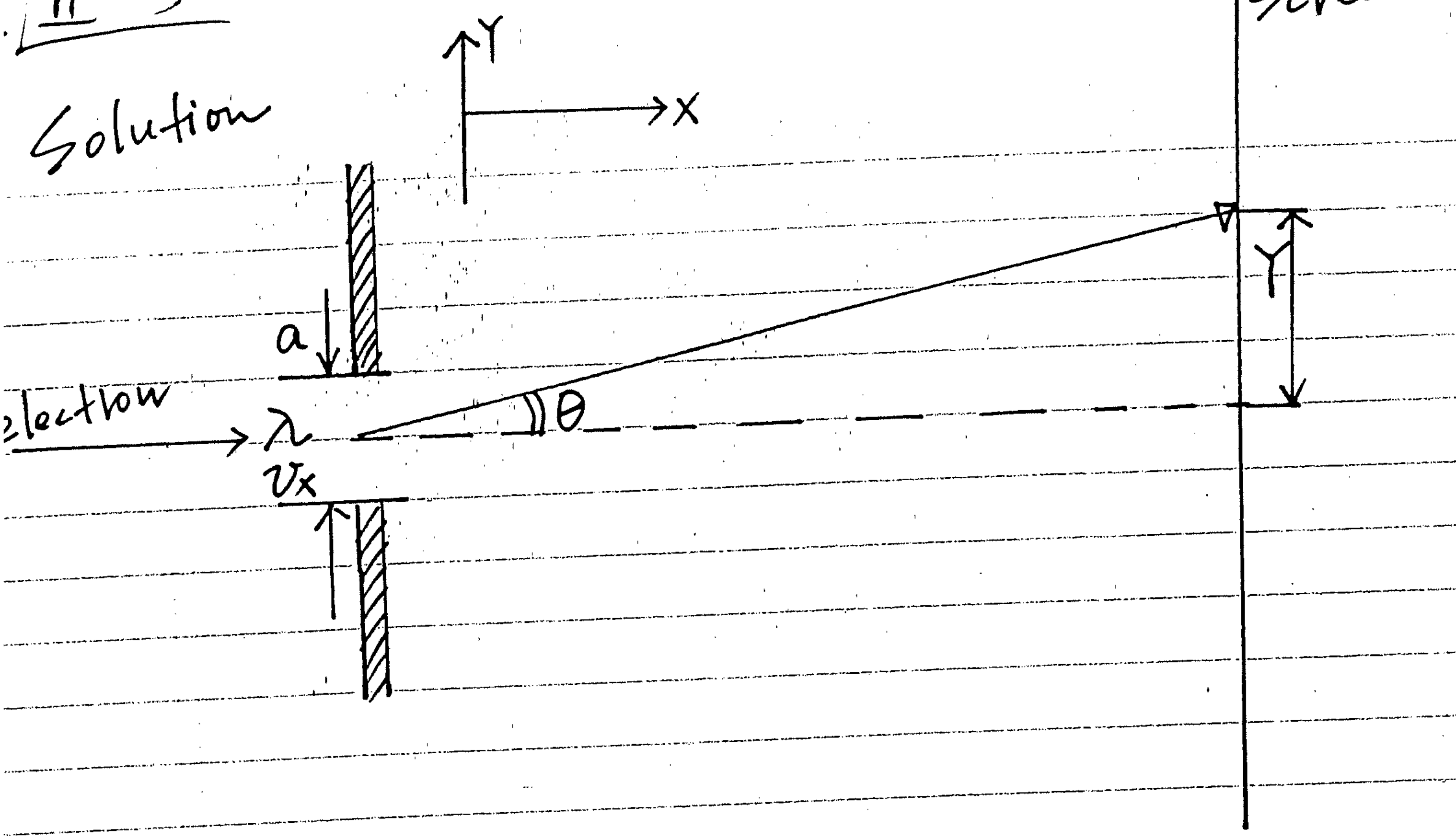
Where p is the momentum. Since $\kappa = 2k\sin(\Theta/2)$,

$$\hbar^2\ k^2 = 8mE\sin^2(\frac{\Theta}{2}). \tag{11}$$

Inserting (11) in (9) gives the required result.

II - 5

Solution



For a single slit, the uncertainty in Y-direction is $\Delta Y = a$. ------- ①

From the Uncertainty principle: $\Delta P_Y (= m_e V_Y) \geq \frac{\hbar}{a}$

Therefore $|\Delta V_Y| \geq \frac{\hbar}{m_e a}$. --------- ②

To determine the angle θ, we note from the drawing that: $\sin\theta \approx \frac{V_Y}{V} \geq \frac{\hbar}{m_e V a}$ ----(3)

By de-Broglie Postulate, an electron's wavelength is given by $\lambda = \frac{h}{p} \left(= \frac{h}{m_e V}\right)$ -------- ④

II – 5

By combining ③ & ④, we have

$$\sin\theta \geq \frac{\lambda}{2\pi a} \quad \text{or}$$

$$a\sin\theta \geq \frac{\lambda}{2\pi}.$$

$\boxed{\text{II-6}}$ Time-dependent Perturbation

$H = H_0 + V(t)$ $\qquad V(t) = -q E_0 x e^{-\gamma t} \cos(\omega t)$ $(t>0$

(charged) harmonic oscillator : $H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2$

$$\left.\begin{aligned} a &= \sqrt{\frac{m\omega_0}{2\hbar}}\left(x + \frac{i}{m\omega_0} p\right) \\ a^+ &= \sqrt{\frac{m\omega_0}{2\hbar}}\left(x - \frac{i}{m\omega_0} p\right) \end{aligned}\right\} \Rightarrow x = \sqrt{\frac{\hbar}{2m\omega_0}}\,(a + a^+)$$

$H_0 |n\rangle = \hbar\omega_0 (n + 1/2) |n\rangle \qquad n = 0, 1, 2, \ldots$

$E_n = \hbar\omega_0 (n + 1/2) \qquad \omega_{\ell k} \equiv \frac{1}{\hbar}(E_\ell - E_k) = \omega_0 (\ell - k$

$t = 0$: oscillator is in G.S. $|0\rangle$

Transition probabilities for $t = \infty$:

$$W(0 \to \ell) = \frac{1}{\hbar^2}\left|\int_0^\infty \langle \ell | V(t) | 0 \rangle e^{i\omega_{\ell 0} t}\, dt\right|^2$$

$$= \frac{q^2 E_0^2}{\hbar^2} |\langle \ell | x | 0 \rangle|^2 \left|\int_0^\infty e^{i\omega_{\ell 0} t} e^{-\gamma t} \cos(\omega t)\, dt\right|^2$$

$$= \frac{q^2 E_0^2}{\hbar^2} \frac{\hbar}{2m\omega_0} |\langle \ell | 1 \rangle|^2 \left|\int_0^\infty e^{i\omega_0 \ell \cdot t} e^{-\gamma t} \frac{e^{i\omega t} + e^{-i\omega t}}{2i}\, dt\right|^2$$

$$= \delta_{\ell 1} \frac{q^2 E_0^2}{4\hbar m\omega_0} \left|\int_0^\infty \left\{ e^{[i(\omega_0 + \omega) - \gamma] t} + e^{[i(\omega_0 - \omega) - \gamma] t} \right\} dt\right|^2 =$$

1) Kronecker-delta

$\boxed{\text{II}-6}$ (Cont'd)

$$= \delta_{e1}\frac{q^2E_0^2}{4\hbar m\omega_0}\left|\frac{1}{\gamma - i(\omega_0+\omega)} + \frac{1}{\gamma - i(\omega_0-\omega)}\right|^2$$

$$= \delta_{e1}\frac{q^2E_0^2}{4\hbar m\omega_0}\left|\frac{\gamma - i(\omega_0-\omega) + \gamma - i(\omega_0+\omega)}{[\gamma - i(\omega_0+\omega)][\gamma - i(\omega_0-\omega)]}\right|^2 =$$

$$= \delta_{e1}\frac{q^2E_0^2}{4\hbar m\omega_0}\left|\frac{2(\gamma - i\omega_0)}{\gamma^2 - i\gamma(\omega_0-\omega) - i\gamma(\omega_0+\omega) - (\omega_0-\omega)(\omega_0+\omega)}\right|^2$$

$$= \delta_{e1}\frac{q^2E_0^2}{4\hbar m\omega_0}\left|\frac{2(\gamma - i\omega_0)}{\gamma^2 - 2i\gamma\omega_0 + (\omega^2-\omega_0^2)}\right|^2$$

$$= \delta_{e1}\frac{q^2E_0^2}{4\hbar m\omega_0}\, 4\,\frac{\gamma^2+\omega_0^2}{[\gamma^2+\omega^2-\omega_0^2]^2 + 4\gamma^2\omega_0^2}$$

for $\gamma < \omega_0$, $W(0\to e)$ is maximum if $\omega = \sqrt{\omega_0^2-\gamma^2}$.

transition is only possible to the first excited state $|1\rangle$, and the tr. prob. is maximum if

$$\omega = \sqrt{\omega_0^2-\gamma^2}\,.$$

Q.E. Ph.D. Aug 2005

(II-7) Solution

$\beta p = \rho/(1-b\rho) - \beta a \rho^2$, $\rho = \frac{\langle n \rangle}{V}$, $V = L^3$

(a) For constant $V = L^3$

$$\langle (\delta\rho)^2 \rangle^{1/2} / \rho = \langle (\delta N)^2 \rangle^{1/2} / \langle N \rangle$$

$$\langle (\delta N)^2 \rangle = \left(\frac{\partial \langle N \rangle}{\partial (\beta\mu)} \right)_{\beta, V} = V \left(\frac{\partial \rho}{\partial \beta\mu} \right)_{\beta, V} = V\rho \left(\frac{\partial \rho}{\partial \beta p} \right)_{\beta, V}$$

$$= \langle N \rangle \left(\frac{\partial \beta p}{\partial \rho} \right)^{-1}_{\beta, V}$$

$$= \langle N \rangle \frac{(1-b\rho)^2}{1 - 2\beta a \rho (1-b\rho)^2} . \quad \text{or}$$

$$\frac{\langle (\delta\rho)^2 \rangle^{1/2}}{\rho} = \frac{1}{\sqrt{\rho V}} \cdot \frac{(1-b\rho)}{\sqrt{1 - 2\beta a \rho (1-b\rho)^2}}$$

which vanishes as the volume becomes infinit

(b) At the critical point

$$\left(\frac{\partial \beta p}{\partial \rho} \right)_{\beta} = \frac{1}{(1-b\rho)^2} - 2\beta a \rho = 0 \quad \text{and}$$

$$\left(\frac{\partial^2 \beta p}{\partial \rho^2} \right)_{\beta} = \frac{2b}{(1-b\rho)^3} - 2\beta a = 0 .$$

Now a lot of algebra must be done!

(II-7) Solution-continued

$$\therefore \boxed{\rho_c = \frac{1}{3b}} \quad \text{and}$$

$$\beta_c = \frac{1}{2a\rho_c(1-b\rho_c)^2} = \frac{3b}{2a\left(\frac{2}{3}\right)^2}$$

$$\boxed{\beta_c = \frac{27b}{8a}}$$

II-8 Solution

a) We can calculate the work as an integral, using the ideal gas law:

$$W = -\int_1^2 P\,dV = -\int_1^2 \frac{N\tau}{V}\,dV = N\tau \ln\frac{V_1}{V_2}\ , \tag{1}$$

where $\tau = k_B T$. Graphically, it is simply the area under the curve. Alternatively, we can say that the work done is equal to the change of free energy F of the system

$$\begin{aligned} W &= \Delta F = F_2 - F_1 \\ &= -N\tau\ln\frac{eV_2}{N} + N\,f(\tau) + N\tau\ln\frac{eV_1}{N} - N\,f(\tau) = N\tau\ln\frac{V_1}{V_2}. \end{aligned} \tag{2}$$

The total energy E of the ideal gas depends only on the temperature, which is constant, so the heat absorbed by the gas is

$$Q = \Delta E - W = -N\tau\ln\frac{V_1}{V_2}, \tag{3}$$

i.e., heat is rejected from the gas into the reservoir. Alternatively, since $\partial Q = \tau\,dS$

$$\begin{aligned} Q &= \int_1^2 \tau\,dS = \tau(S_2 - S_1) \\ &= \tau\left(-\frac{\partial F_2}{\partial\tau} + \frac{\partial F_1}{\partial\tau}\right) = \tau\left(N\ln\frac{eV_2}{\tau} - N\ln\frac{eV_1}{\tau}\right) \\ &= -N\tau\ln\frac{V_1}{V_2}. \end{aligned} \tag{4}$$

the same result as in (3)

b) For an adiabatic expansion the entropy is conserved, so

$$dE = -P\,dV \tag{5}$$

On the other hand

$$dE = C_v\,d\tau \tag{6}$$

Where Cv is the specific heat for an ideal gas at constant volume. From (5) and (6), and using the ideal gas law, we obtain

$$-\frac{dV}{V}=\frac{C_v}{N}\frac{d\tau}{\tau}=\frac{c_v d\tau}{\tau} \tag{7}$$

Where $C_v / N \equiv c_v$, the specific heat per one molecule. Integrating (7) yields

$$\tau_f=\left(\frac{V_i}{V_f}\right)^{1/c_v}\tau_i=\left(\frac{1}{2}\right)^{1/c_v}\tau_i.$$

c) For air we may take $c_v = 5/2$ (in regular units, $c_v = 5k_b/2$; it is almost diatomic). Therefore,

$$T_f=\left(\frac{1}{2}\right)^{2/5}T_i\approx 227 \text{ K}.$$

Part II-9 Classical Statistical Mechanics

Interstellar space is basically filled with atomic hydrogen. What is the number of hydrogen atoms which cross a given area A per second at temperature T. The mass of hydrogen is m.

$$\rho = \frac{N}{V}$$

$$n(\vec{v})\, d^3\vec{v} = \rho \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m\vec{v}^2}{2kT}}$$

\# of atoms per volume with velocity between $\vec{v}$ and $\vec{v} + d\vec{v}$

$$d^3\vec{v} = dv_x\, dv_y\, dv_z$$

$$g(v_z)\, dv_z = \int_{-\infty}^{+\infty} dv_x \int_{-\infty}^{+\infty} dv_y \; n(\vec{v})\, dv_z$$

\# of atoms per volume with velocity component v_z between v_z and $v_z + dz$ and arbitrary v_x, v_y

$$g(v_z)\, dv_z = \rho \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m v_z^2}{2kT}} dv_z \int_{-\infty}^{+\infty} e^{-\frac{m v_y^2}{2kT}} dv_y \int_{-\infty}^{+\infty} e^{-\frac{m v_x^2}{2kT}} dv_x$$

$$= \rho \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m v_z^2}{2kT}} dv_z \left(\frac{2\pi kT}{m}\right)$$

$$= \rho \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{m v_z^2}{2kT}} dv_z$$

atoms which are in the volume $A v_z \Delta t$ and have a velocity component v_z in $+z$ direction will cross the area A

$$\Delta n = \int_{-\infty}^{+\infty} A \Delta t\, N \left(\frac{m}{2\pi kT}\right)^{1/2} v_z\, e^{-\frac{m v_z^2}{2kT}} dv_z$$

$$= A \Delta t\, N \left(\frac{m}{2\pi kT}\right)^{1/2} 2 \underbrace{\int_{0}^{\infty} v_z\, e^{-\frac{m v_z^2}{2kT}} dv_z}_{\frac{1}{2\sqrt{\frac{m}{2kT}}}}$$

$$= \cancel{\frac{1}{2\sqrt{\frac{m}{2kT}}}\, A \Delta t N}$$

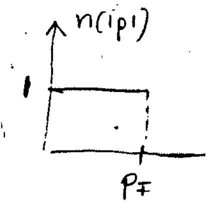
$$\frac{\Delta n}{\Delta t} = \rho\, \Delta A \left(\frac{2kT}{\pi m}\right)^{1/2}$$

$\boxed{\text{II-10}}$

$\varepsilon(p) = \alpha p^2 + m^2$ $\quad (m = \text{const.})$ $\quad$ fermions $\quad s = \frac{1}{2}$

1d

a) $N = \underbrace{2}_{\text{from spin}} \times \sum_p n_p = 2 \int_{-\infty}^{+\infty} n(p) \frac{L}{h} \, dp$ $\qquad T = 0:$



$N = 4 \int_0^{p_F} \frac{L}{h} \, dp = \frac{4L}{h} p_F$ $\qquad (p_F > 0$

$p_F = \frac{h}{4} \cdot \frac{N}{L}$ $\qquad$ particle number density: $\frac{N}{L}$

b)

$$\varepsilon_F = \varepsilon(p_F) = \alpha p_F^2 + m^2 = \alpha \frac{h^2}{16} \left(\frac{N}{L}\right)^2 + m^2$$

c) Note: $-\infty < p < \infty$ $\qquad m^2 \leq \varepsilon < 0$ $\qquad$ (gap!)

$$4 \int_0^{\infty} dp \frac{L}{h} (\ldots) \equiv \int_{m^2}^{\infty} d\varepsilon \, g(\varepsilon) (\ldots)$$

$$\frac{4L}{h} dp = d\varepsilon \, g(\varepsilon)$$

$$p = \sqrt{\frac{\varepsilon - m^2}{\alpha}}$$

$$\frac{dp}{d\varepsilon} = \frac{1}{2\sqrt{\alpha}} \frac{1}{\sqrt{\varepsilon - m^2}}$$

$$g(\varepsilon) = \frac{4L}{h} \frac{dp}{d\varepsilon} = \frac{4L}{h} \cdot \frac{1}{2\sqrt{\alpha}} \cdot \frac{1}{\sqrt{\varepsilon - m^2}}$$

$$= \frac{2}{\sqrt{\alpha}} \frac{L}{h} \frac{1}{\sqrt{\varepsilon - m^2}} \qquad \text{for } \varepsilon > m^2$$

$$g(\varepsilon) = \begin{cases} 0 & \varepsilon < m^2 \\ \frac{2L}{\sqrt{\alpha} h} \frac{1}{\sqrt{\varepsilon - m^2}} & \varepsilon \geq m^2 \end{cases}$$