MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS

DOCTORAL GENERAL EXAMINATION
PART 1
August 26, 2002

FIVE HOURS

1. This examination is divided into five sections, each consisting of four problems. Answer <u>all</u> the problems. Each problem is worth 5 points, thus the maximum score for the exam is 100.

2. Use a separate <u>fold</u> of paper for each problem. Write your name and the problem number (IV-3 for example) on each fold. A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a quantitative response.

3. Read the problem carefully and do not do more work than is necessary. For example "give" and "sketch" do not mean "derive".

4. Calculators may be used but are not necessary.

5. No books, notes or reference materials may be used.

Group I

1. **One dimensional bound state**

   A particle of mass m is restricted to move within a one-dimensional infinite square well located at $-a/2 < x < a/2$. In addition, the particle feels a repulsive delta-function potential at the origin, $+W\delta(x)$. Treat the delta function potential as a perturbation and give the ground state energy to first order. Find the energy of first excited state exactly.

2. **Charged particle in a rectangular cavity**

   A rectangular conducting cavity has dimensions a, b, c along the x, y, z directions, respectively and $a < b < c$. Give the electric field vector and frequency for the lowest frequency mode. Inside the cavity, a neutron beta decays, leaving a proton (which is essentially at rest) and electron. [1] Just after emission, the proton and electron are very close together and you may assume their Coulomb fields cancel. Find the direction the electron must move in order to excite the lowest frequency mode of the cavity.

3. **Temperature scale**

   The following is a list of systems each of which has a characteristic temperature. Reorder the list in order of increasing temperature and give a very rough estimate of the temperature of each system.

   - The present day cosmic microwave background
   - The cosmic microwave background at the time of formation of atomic hydrogen
   - The center of the sun
   - Liquid He-4 at its boiling point
   - The highest temperature superconductors
   - The quark-gluon plasma in the early universe
   - Room temperature
   - Liquid nitrogen at its boiling point
   - Bose-Einstein condensates

4. **Relativistic Hamiltonian**

   For a relativistic free particle in one dimension, the Lagrangian is

   $$L(x, \dot{x}) = -mc^2\sqrt{1 - \dot{x}^2/c^2}.$$

   Find the conjugate momentum p and construct the relativistic Hamiltonian $H(x, p)$. Show that the non-relativistic Hamiltonian is recovered when you Taylor expand the relativistic Hamiltonian appropriately.

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   [1]There is also a neutrino, which plays no role in the problem.

Group II

1. **Cosmological red-shift**
   In a Friedman universe, the space-time interval ds is given by

   $$ds^2 = -dt^2 + a(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right]$$

   where (r, θ, ϕ) are the usual polar coordinates and t is time. $a(t)$ is a monotonically increasing function of time. Take $k = 0$. Consider two particles, one at the origin, the other at $r = r_o$, $\theta = 0$, $\phi = 0$, with r_o unchanging with time. An observer at rest at the origin measures the rate of change of the proper distnace between the two particles at time t, where "rate of change" is measured using a clock at rest at the origin. Give the rate of change measured by the observer at the origin.

2. **Carnot cycle between two bodies**
   Two identical bodies with temperature independent heat capacities C_o are initially at different temperatures, T_H and T_C. A Carnot cycle is run between them (with infinitesimal steps) until they have been reduced to a common temperature T_F. Find T_F.

3. **Force between comb and paper**
   With the exam, you received a plastic comb and several small pieces of paper. Put the paper on the desk and charge the comb by running it through your hair a few times. When you bring the comb close to the paper. you should observe evidence of a force between the paper and the comb. What is the nature of the force? How are electrical charges distributed on the comb and on the paper in order to produce the force you observe?

4. **Unequally spaced antenna array** Four identical antennas are driven with equal amplitudes and equal phases at a wavelength λ. The radiation pattern of a single antenna is isotropic. The antennas are spaced unequally along the y axis at positions $\pm$ 2d and $\pm$ 3d where $d = \lambda/2$.

   The radiation is detected a great distance away at an angle ψ with the normal to the array. Indicate in a drawing the complex vectors (amplitude and phase) of the four individual field at the detector and the resulting sum. Find an expression for the ratio of the observed intensity $< I >$ to that which would be observed if only one of the antennas were driven, $< I_o >$. Express your result in temrs of λ, d and ψ.

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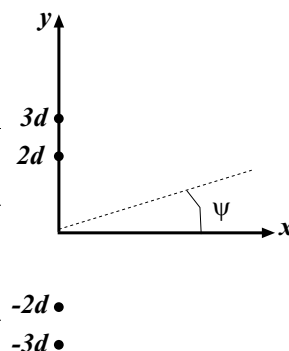
Group III

1. **Ferromagnet** A certain kind of ferromagnetic material has a magnetization M which is a function of applied field H, $M(H)$ which depends on temperature. Three representative curves for this substance are shown shown below.

a) Are T_1 and T_2 larger or smaller than T_c?
b) Sketch the curve of the susceptibility χ as a function of H for T_1, T_2 and T_c.
c) Give the units in the CGS or SI system for M, H and χ.
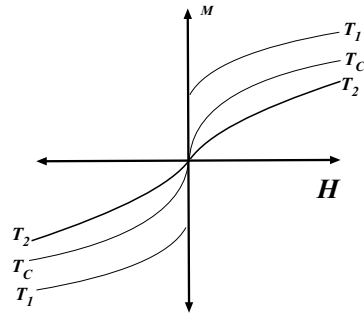
2. **Two state system** Consider a two state quantum mechanical system, whose basis states we denote $|+>$ and $|->$. The states $|+>$ and $|->$ are NOT eigenstates of the Hamiltonian. At time $t = 0$, the system is in the state $|+>$. The time evolution of the system is periodic such that at times $t = 0, T, 2T, 3T, ...$ the system is in the state $|+>$ while at times $T/2, 3T/2, ...$ the system is in the state $|->$. Write a two-by-two Hamiltonian matrix (using the states $|+>$ and $|->$ as your basis) which reproduces the above physics. Your Hamiltonian need only contain one parameter. Relate this parameter to T.

3. **Relativistic Rocket** A rocket moving with speed v passes a stationary observer. The observer waits a time T (according to his clock) after the rocket passes, and sends a pulse of light in the direction of the rocket. The rocket pilot notes that, according to her clocks, the time elapsed between the moment she passed the observer and the moment she receives the light pulse is T'. Express T' in terms of T, v and the speed of light c. According to the stationary observer, the light pulse has frequency ν. What frequency light does the rocket pilot detect when she receives the pulse?

4. **Young's Modulus** A beam of cross sectional area a and length $l >> \sqrt{a}$ is subject to a pressure P along l on each end. The Young's modulus E is defined as the ratio of the pressure P to the fractional compression $E = P/\Delta l/l$. Use Coulomb's law $U = Z_{eff}^2 e^2/r$ to estimate the order of magnitude of E for a typical material which has effective charge $Z = 5$ and atomic number density 10^{23} cm^{-3}).

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Group IV

1. **Particle in a conical well**
   A particle of mass m moves in a circle along the wall of a cone of opening angle α as shown below. The particle has angular momentum l and circles in a plane normal to the earth's gravitational field a distance h above the point of the cone. Find h as a function of l. What happens to the angular momentum if the angle α begins to change such that $\dot{\alpha}/\alpha << \omega$, where ω is the frequency at which the mass goes around the cone?
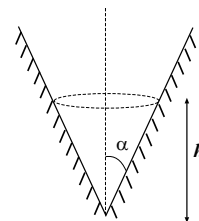
2. **Yield of the first atomic bomb** In 1946, the US Army released the first pictures of the denotation of the Trinity bomb, one of which, taken 0.94 ms after denotation, is shown below. The Soviet scientist LI Sedov reasoned the radius of the fireball at time t, $R(t)$ can only depend on the time t, the total energy released or yield, E, the density of the surrounding air, $\rho = 10^{-3}$ g/cm^3 and a dimensionless constant μ. Find an expression $R(t) = \mu f(E, \rho, t)$ where f has units of centimeters and contains only E, ρ and t (i.e. no numerical factors. How does R vary with t? If μ is close to one (a result of detailed analysis of the shock-wave), what is E? Express your result in tons of TNT, 1 ton TNT=4.2×10^9 J.

3. **Charged conductors over a grounded conducting plane** Two parallel infinite line charges are fixed a distance h above a grounded conducting plane. They are a distance $2h$ apart and have linear charge density $-\lambda$. A positive charge Q is placed a distance h above the plane midway between the two line charges and feels no net force. Find Q in terms of λ and h. Is Q stable for small perturbations along the vertical direction?

4. **Magnetic field from a 2D current distribution**
   A collection of current carrying wires in the z=0 plane generates a magnetic field $\vec{B}(x, y, z)$ throughout all space. The field at a specific point is known: $\vec{B}(x_0, y_0, z_0) = a\hat{x} + b\hat{y} + c\hat{z}$. What is the magnetic field at the mirror image point $\vec{B}(x_0, y_0, -z_0)$?

Group V

1. **Dipole antenna** A dipole antenna is shown below. Be sure to justify each of your answers below.
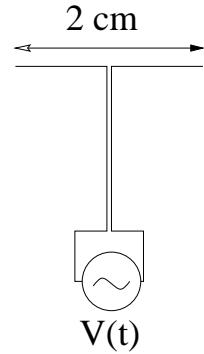
   a At what frequencies will the antenna transmit power most efficiently?

   b What is the polarization of the transmitted electromagnetic waves?

   c What is the angular distribution of the transmitted power?

2. **Driven harmonic oscillator** A damped harmonic oscillator has damping coefficient b and undamped resonant frequency ω. Find the solution $x(t)$ if the oscillator is initially at rest and is driven by a step function force $\vec{F}$ which has value F_o when $t > 0$ and is zero otherwise. That is find the solution to $m\ddot{x} + b\dot{x} + kx = F(t)$. Find $x(t \to \infty)$. Hint: You may wish to consider a function of the form $G(t, t') = (1/m\omega_1)e^{-\beta(t-t')} \sin \omega_1(t - t')$, $t > t'$ ($G(t, t') = 0$ otherwise) where $\beta = b/2m$ and $w_1 = \sqrt{\omega - \beta^2}$ gives $md^2G/dt^2 + bdG/dt + kG = \delta(t - t')$.

   Note: during the exam, a correction was given: $\omega_1 = \sqrt{\omega^2 - \beta^2}$

3. **Hydrogen atom in a magnetic field** In addition to the fine structure, the Hamiltonian for a hydrogen atom also has a perturbation from hyperfine structure and any applied magnetic field

$$H' = \frac{e}{2m_e}\vec{L} \cdot \vec{B} + \frac{e}{m_e}\vec{S} \cdot \vec{B} + \left(\frac{e^2}{8\pi\epsilon_o}\right)\frac{1}{m_e^2 c^2 r^2}\vec{S} \cdot \vec{L}$$

   where $\vec{B}$ is the applied magnetic field, $\vec{S}$ is the electron spin and $\vec{L}$ is the orbital angular momentum. Briefly describe each of the terms in the perturbing Hamiltonian. In the limit of $|\vec{B}|$ very large, which quantum numbers diagonalize the perturbing Hamiltonian? Which diagonalize H' in the limit $|\vec{B}| \to 0$?

4. **Fermions in a magnetic field** A time varying magnetic field is given by

$$\vec{B}(t) = B_o\left[\sin \alpha \cos(\omega t)\hat{x} + \sin \alpha \sin(\omega t)\hat{y} + \cos \alpha \hat{z}\right].$$

   At $t = 0$, a fermion of mass m and magnetic moment μ is at rest in the field and the fermion spin is oriented along the field. No other potentials are present. Assume that

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$|eB_o/\hbar m| >> \omega$. What are the energy eigenvalues and eigenfunctions as a function of time? Choose $\hat{z}$ as the quantization axis and describe the state of the system as a function of time.

Note: during the exam, the problem was amended: the fermion is meant to be spin 1/2 and $|\mu B_o/\hbar| >> \omega$

Replacement for problem II-3

A charge Q is located at the origin. A dipole $\vec{P} = p_o \hat{r}$ is located a distance l away. Find the net force on the dipole. Now consider a thin, spherical shell of dipoles surrounding the origin, all a distance l away. The surface density of dipoles is σ, the dipoles are all oriented along $\hat{r}$ and they are fixed in position so they cannot move. The charge Q remains at the origin. Find the electric field at $r >> l$.