Sln I-1 Spin Measurements Part I Solutions

a) $\hat{x}\cdot\vec{S} = \frac{\hbar}{2}\sigma_x \qquad \sigma_x = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ ✓

$\hat{z}\cdot\vec{S} = \frac{\hbar}{2}\sigma_z \qquad \sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$S_x + S_z = \hat{x}\cdot\vec{S} + \hat{z}\cdot\vec{S} = \frac{\hbar}{2}(\sigma_x+\sigma_z) = \frac{\hbar}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

Eigenvalues are $\frac{\hbar}{2}\times$ eigenvalues of $\begin{pmatrix}1&1\\1&-1\end{pmatrix}$:

$0 = \det\left(\begin{pmatrix}1&1\\1&-1\end{pmatrix} - \lambda I\right) = (1-\lambda)(-1-\lambda) - 1$

$= -(1-\lambda^2) - 1 = \lambda^2 - 2 \Rightarrow \lambda = \pm\sqrt{2}$

✓

Thus, possible results are $\pm\sqrt{2}\,\frac{\hbar}{2}$

b) The eigenvectors $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ obey

$(1-\lambda)\alpha + \beta = 0$

$(1-m\sqrt{2})\alpha + \beta = 0$

if the eigenvalue is $m\frac{\sqrt{2}}{2}\hbar$.

So, unnormalized, the state after measurement $m\frac{\sqrt{2}}{2}\hbar$ is

$\begin{pmatrix}1\\1-m\sqrt{2}\end{pmatrix}$

So that

$P_{S_z=\frac{\hbar}{2}} = \frac{1}{1+(1-m\sqrt{2})^2} = \frac{1}{4-2\sqrt{2}m}$

$P_{S_z=-\frac{\hbar}{2}} = \frac{(1-m\sqrt{2})^2}{1+(1-m\sqrt{2})^2} = \frac{3-2\sqrt{2}m}{4-2\sqrt{2}m}$

✓

after measuring $S_x + S_z = m\frac{\sqrt{2}}{2}\hbar$, $m=\pm1$

Section I: Involving Quantum Mechanics

I -2 Electron Pressure

An electron is inside a sphere of radius R. What pressure P does it exert on the wall, if it is:

i) in the lowest S-state?

ii) in the lowest P-state? [Hint: $\psi_1 \sim d\psi_0/d(kr)$, solve approximately]

Sln I-2

Sln: Solve Schroed to get $E(R)$

- expand sphere by dR

Work done is: $dW = \mathbb{P}\, dV = 4\pi R^2 \, P\, dR = -dE(R) = -\frac{dE(R)}{dR} dR$

Hence : $\mathbb{P} = -\frac{dE}{dR} \frac{1}{4\pi R^2}$ (1)

i) lowest S-state: $\psi_0 \sim \frac{\sin kr}{kr}$ with $\psi(R) = 0 \rightarrow kR = \pi$

Then : $E = \frac{k^2\hbar^2}{2m} = \frac{\pi^2\hbar^2}{2mR^2}$

(1) $\longrightarrow$ $\boxed{\mathbb{P} = \frac{\pi\hbar^2}{4mR^5}}$

ii) Lowest P-state: $\psi_1 \sim \frac{\cos(kr)}{kr} - \frac{\sin(kr)}{(kr)^2} \approx \frac{d\psi}{d(kr)}$

$\psi(R) = 0 \rightarrow kR \cot(kR) = 1$ solve by iteration $kR \simeq 4.5$

$\longrightarrow \boxed{\mathbb{P} \approx (4.5)^2 \frac{\hbar^2}{4\pi m R^5}}$

I - 3 States and Observables of a System

Observable A has eigenstates $|+\rangle$ and $|-\rangle$, where $A|\pm\rangle = \pm 1|\pm\rangle$. The Hamiltonian for this system is defined by $\langle +|H|-\rangle = i\hbar\omega$, and $\langle +|H|+\rangle = \langle -|H|-\rangle = 0$.

a. Find the normalized eigenstates of H in terms of the states $|+\rangle$, and the corresponding eigenvalues.

b. What are the matrix representations of A and H in the H eigenstates basis?

c. Given that $|\Psi(t=0)\rangle = |+\rangle$, what is $|\Psi(t)\rangle$ for any t ?

d. Averaged over many identical experiments, what would one measure for the observables A and H when the system is in the state $|\Psi(t)\rangle$?

e. What is the probability of obtaining the result -1 in a measurement of A at time t ?

f. At time $t=10\,\omega^{-1}$, for example, the Hamiltonian suddenly changes to $H = \hbar\omega A$. What will $|\Psi(t)\rangle$ be at $t = 20\,\omega^{-1}$?

Sln I-3

a) In the $|\pm\rangle$ basis $(\pm)$:

$$H = \begin{pmatrix} 0 & \mp i\hbar\omega \\ \pm i\hbar\omega & 0 \end{pmatrix}$$

$$\left(A = \begin{array}{c|cc} & |+\rangle & |-\rangle \\ |+\rangle & 1 & 0 \\ |-\rangle & 0 & -1 \end{array} \text{ then sign inverted} \right)$$

Eigenstates from $\det(H - \epsilon I) = 0 \Rightarrow \epsilon^2 - \hbar^2\omega^2 = 0$

$\epsilon = \pm \hbar\omega$

If $\epsilon = +\hbar\omega$ $\begin{pmatrix} -\hbar\omega & -i\hbar\omega \\ i\hbar\omega & -\hbar\omega \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \alpha + i\beta = 0$

$e_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$ (Normalized) $\quad |\hbar\omega\rangle = \frac{|+\rangle \mp i|-\rangle}{\sqrt{2}}$

If $\epsilon = -\hbar\omega$ $\begin{pmatrix} \hbar\omega & -i\hbar\omega \\ i\hbar\omega & \hbar\omega \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \alpha - i\beta = 0$

$e_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ (Normalized) $\quad |-\hbar\omega\rangle = \frac{|+\rangle \pm i|-\rangle}{\sqrt{2}}$

b) $He_+ = \hbar\omega e_+$ $\Rightarrow H = \begin{pmatrix} \hbar\omega & 0 \\ 0 & -\hbar\omega \end{pmatrix}$ in $e_\pm$ basis

$He_- = -\hbar\omega e_-$ $\langle e_n | H | e_m \rangle = \hbar\omega \delta_{nm} \cdot n$; $n,m = \pm$

$Ae_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e_-$ $\Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $e_\pm$ basis

$Ae_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = e_+$ $\langle e_n | A | e_m \rangle = (1 - \delta_{nm})$ $n,m = \pm 1$

c) $|\psi(t=0)\rangle = |+\rangle = \frac{|\hbar\omega\rangle + |-\hbar\omega\rangle}{\sqrt{2}}$

$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|\hbar\omega\rangle e^{-i\omega t} + |-\hbar\omega\rangle e^{i\omega t}\right) = \frac{e^{-i\omega t} + e^{i\omega t}}{2}|+\rangle + \frac{ie^{-i\omega t} - ie^{i\omega t}}{2}|-\rangle$

$= \cos\omega t\,|+\rangle + \sin\omega t\,|-\rangle$

d) $\langle E \rangle = \overline{(\hbar\omega)\underbrace{\left|\frac{1}{\sqrt{2}}e^{-i\omega t}\right|^2}_{\frac{1}{2}} + (-\hbar\omega)\underbrace{\left|\frac{1}{\sqrt{2}}e^{i\omega t}\right|^2}_{\frac{1}{2}}} = 0$

$\langle A \rangle = (+1)\cos^2\omega t + (-1)\sin^2\omega t = \cos^2\omega t - \sin^2\omega t = \cos 2\omega t$

e) $P_{A=-1} = |\sin\omega t|^2 = \sin^2\omega t$

f) $|\psi(t=10)\rangle = \cancel{\cos}(10\omega)|+\rangle + \sin(10\omega)|-\rangle$

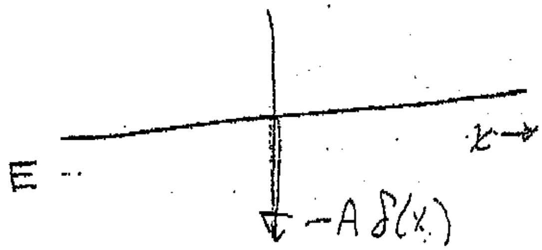
$|\psi(t>10)\rangle = \cos(10\omega)e^{-i\omega t}|+\rangle + \sin(10\omega)e^{i\omega t}|-\rangle$

$|\psi(t=20)\rangle = \cos(10\omega)e^{-i10\omega}|+\rangle + \sin(10\omega)e^{i10\omega}|-\rangle$

I - 4 Sudden Change in Potential

For $t<0$, an electron is in the ground state of the one dimensional potential $V(x)=-A\delta(x)$. At $t=0$, the potential suddenly changes to $V'(x)=-A'\delta(x)$. What is the probability that the electron ~~will be~~ in the ground state of the potential V' for times ~~$t>0$~~? $t \gg 0$?

For:



We have $-\frac{\hbar^2}{2m}\psi'' - A\delta(x)\psi = E\psi$

$$\psi'' + \frac{2mA}{\hbar^2}\delta(x)\psi = \frac{-2mE}{\hbar^2}\psi = \kappa^2\psi$$

For $x \neq 0$, this is $\psi'' = \kappa^2\psi \Rightarrow \psi = \alpha e^{\kappa x} + \beta e^{-\kappa x}$

Boundary conditions: To be normalizable no exponentially growing solution at $x = \pm\infty$

$$\Rightarrow \psi(x) = \begin{cases} \alpha e^{-\kappa x}, & x>0 \\ \beta e^{\kappa x} & x<0 \end{cases}$$

ψ is continuous at $x=0 \Rightarrow \alpha=\beta \Rightarrow \psi(x) = \alpha e^{-\kappa|x|}$

$$\int_{-\epsilon}^{\epsilon}\left(\psi'' + \frac{2mA}{\hbar^2}\delta(x)\psi\right)dx = \int_{-\epsilon}^{\epsilon}\kappa^2\psi\,dx$$

$$\psi'(\epsilon) - \psi'(-\epsilon) + \frac{2mA}{\hbar^2}\psi(0) = 0$$

$$(-\kappa\alpha) - (\kappa\alpha) + \frac{2mA}{\hbar^2}\alpha = 0$$

$$\Rightarrow \kappa = \frac{mA}{\hbar^2} \qquad \frac{mA}{\hbar^2}$$

Normalizing:

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} \alpha^2 e^{-2\kappa|x|} dx$$

$$= 2\alpha^2 \int_0^{\infty} e^{-2\kappa x} dx$$

$$= \frac{2\alpha^2}{2\kappa}$$

$$\Rightarrow \alpha = \sqrt{\kappa}$$

$$\Rightarrow \psi(x) = \sqrt{\frac{mA}{\hbar^2}}\, e^{-\frac{mA}{\hbar^2}|x|}$$

The probability is the square of the amplitude c:

$$P = |c|^2$$

$$c = \int dx\, \psi'^*(x)\, \psi(x)$$

$$= \int dx \sqrt{\frac{mA'}{\hbar^2}}\, e^{-\frac{mA'}{\hbar^2}|x|} \sqrt{\frac{mA}{\hbar^2}}\, e^{-\frac{mA}{\hbar^2}|x|}$$

$$= \frac{m}{\hbar^2}\sqrt{A'A} \int dx\, e^{-\frac{m}{\hbar^2}(A+A')|x|}$$

$$= \frac{m}{\hbar^2}\sqrt{A'A}\, 2 \cdot \frac{1}{\frac{m}{\hbar^2}(A+A')} = 2\sqrt{\frac{AA'}{(A+A')^2}}$$

$$\Rightarrow P = 4\frac{AA'}{(A+A')^2}$$

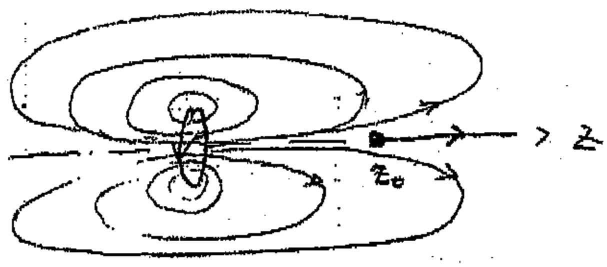
Sln II-1 Field of a Current Loop.

a) Field of the loop $\vec{B}$

Biot S. $\quad d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\, d\vec{l} \times \hat{r}}{r^3}$

$$\left[B = \frac{\mu_0}{4\pi} \frac{2\pi b}{(b^2+z_0^2)} \frac{b}{\sqrt{b^2+z_0^2}} = \boxed{\frac{\mu_0}{2} \frac{I b^2}{\sqrt{b^2+z_0^2}^{3}} \hat{z}} \right.$$

b)



c) Small Δy off axis

Use : $\quad \mathrm{div}\, B = 0 = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$

Note : from b) the cylindrical symmetry

$\rightarrow$ use $\nabla_{cyl} \cdot B = 0 = \frac{1}{r}\frac{\partial}{\partial r}(r B) + \underbrace{\frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi}}_{0} + \frac{\partial B_z}{\partial z}$

$r = \Delta y$ small : $\quad \frac{1}{\Delta y}\frac{\partial}{\partial y}(\Delta y\, B_y) = \frac{\partial B_y}{\partial y}$

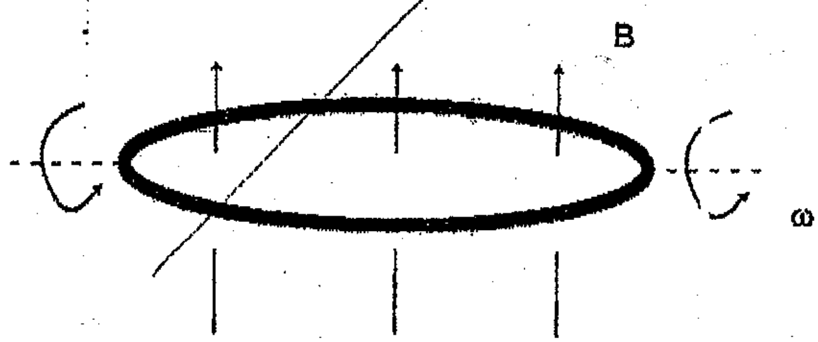
$$\boxed{B_y = \frac{\partial B_y}{\partial y}\Delta y = -\frac{\partial B_z}{\partial z}\Delta y = \frac{3}{4}\mu_0 \frac{I b^2 \Delta y}{\sqrt{b^2+z_0^2}^{5}} 2 z_0}$$

direction $+\hat{y}$

II - 2 Rotating Ring

A metal ring rotates in a weak homogeneous magnetic field B around an axis perpendicular to B, see sketch. Due to Joule heat dissipation it slows down to $\omega_0 e^{-1}$ in a time τ from its initial value ω_0.

Calculate τ for a ring made from a round wire of a metal with resistivity $\eta(\Omega \cdot m)$ and density ρ (g/cm^3). Neglect self induction and assume that only a small amount of energy is lost in one revolution.



Sln II-2

Flux $t \neq 0$: $\phi = B \cdot \pi a^2 = B \pi a^2 \cos \omega_0 t$

Faraday: $\mathcal{E} = -\frac{d\phi}{dt} = -B\pi a^2 \sin \omega_0 t = IR$

(a is radius of loop)

with $R = \eta \frac{2\pi a}{\pi b^2}$ (b is radius of wire)

Kin. E $E = \frac{1}{2}\theta\omega^2$ with mom of inertia θ

$\theta = \int r^2 dm = \int_0^{2\pi} (a \sin\theta)^2 d\ell \cdot \pi b^2 \rho$

$\theta = a^3 \pi b^2 \rho \int_0^{2\pi} \sin^2\theta \, d\theta = \pi^2 a^3 b^2 \rho$

$d\ell = a\,d\theta$

$a \sin\theta$

$dm = \pi b^2 \rho \, d\ell$

note: $\theta = \frac{ma^2}{2}$

E conservation: $\frac{d}{dt} \frac{ma^2}{4}\omega^2 = -\langle I^2 R\rangle = -\frac{B^2\pi^2 a^4 \omega^2}{2R} = \langle \frac{dE}{dt} \rangle$

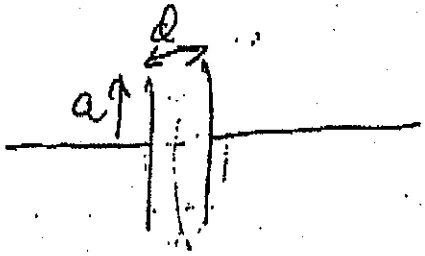
Diff eq : $\theta\omega\dot{\omega} = -\frac{B^2\pi^2a^4\omega^2}{2R}$

Sln : $\omega = \omega_0 \exp(-\underbrace{\frac{B^2\pi^2a^4}{2R\theta}}_{1/\tau} t)$

$\boxed{\tau = \frac{2R\theta}{\ } = \frac{2\eta\frac{2a}{b^2}\cdot\pi^2a^3b^2\rho}{\ } = \frac{4\eta\rho}{B^2}}$

Capacitance at Hi freq (3)

II-3



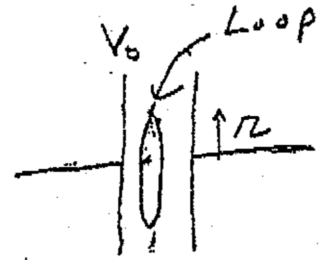
DC capacitance is C_0

$V = V_0 e^{i\omega t}$

(a) Electric field?

By definition, $E = \text{Volts}/m \doteq \dfrac{V_0 e^{i\omega t}}{d}$

(b) Magnetic field between plates?



$$\oint B \cdot dl = \mu_0 \iint_S \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}\right) \cdot dS$$

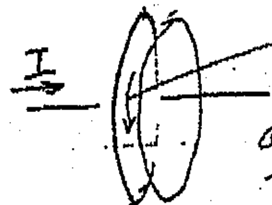
$$B \cdot 2\pi r = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot \pi r^2$$

$J = 0$

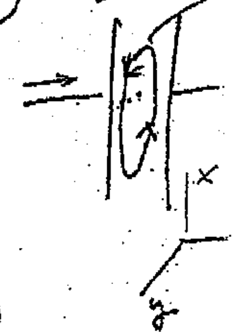
$\boxed{\dfrac{\partial \vec{E}}{\partial t} \neq 0}$

$$B = \frac{\mu_0 \epsilon_0 r}{2} \frac{\partial E}{\partial t}$$

$$\boxed{B = \frac{\mu_0 \epsilon_0 r}{2} \frac{V_0}{d} (i\omega) e^{i\omega t}}$$



(c) First order correction to electric field? $\Rightarrow$



Now $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ $\qquad \vec{E} = (0, 0, E_z)$

x cmp: $\dfrac{\partial E_z}{\partial y} - \underbrace{\dfrac{\partial E_y}{\partial z}}_{= 0} = -\dfrac{\partial B_x}{\partial t}$ $\qquad$ (appropriate on y axis) where $B = (B_x, 0, 0)$

$r \to y$

$\therefore \dfrac{\partial E_z}{\partial y} = -\dfrac{\partial B_x}{\partial t}$ $\qquad$ where $B_x = \dfrac{\mu_0 \epsilon_0 y}{2} \dfrac{V_0}{d} (i\omega) e^{i\omega t}$

$\therefore \dfrac{\partial E_z}{\partial y} = + \dfrac{\mu_0 \epsilon_0}{2} \dfrac{V_0}{d} \omega^2 y e^{i\omega t} \Rightarrow \boxed{E_z = \dfrac{\mu_0 \epsilon_0}{2} \dfrac{V_0}{d} \omega^2 \dfrac{y^2}{2} e^{i\omega t}}$

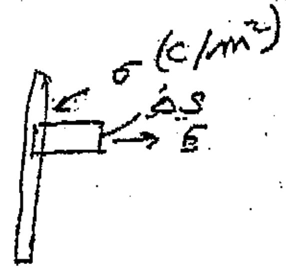
$y \to r$

(d) Charge density (net) on plates

$$E = E_0 + E_{con} = \frac{V_0}{d} e^{i\omega t}\left[1 + \frac{\mu_0 \epsilon_0 \omega^2 r^2}{4}\right]$$

From Gauss's law

$$\int E \cdot dS = \frac{q_{enc}}{\epsilon_0}$$



$$E \cdot \Delta S = \frac{\sigma \Delta S}{\epsilon_0}$$

$$E = \sigma / \epsilon_0$$

$$\therefore \sigma = E \epsilon_0$$

$$\boxed{\sigma = \frac{V_0}{d} \epsilon_0 e^{i\omega t}\left[1 + \frac{\mu_0 \epsilon_0 \omega^2 r^2}{4}\right]}$$

(e) Effective capacitance:

$$C = \frac{Q}{V} \quad \text{where } Q = \iint \sigma(r)\, dS = \int_0^a \sigma(r) \cdot 2\pi r\, dr$$

$$Q = \frac{V_0}{d} \epsilon_0 e^{i\omega t} \pi a^2$$

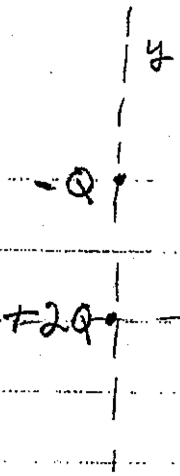
$$+ \frac{V_0}{d} \epsilon_0 e^{i\omega t} \frac{\mu_0 \epsilon_0 \omega^2}{4} \cdot 2\pi \underbrace{\int_0^a r^3\, dr}_{a^4/4}$$

$$C = \frac{Q}{V_0 e^{i\omega t}} = \frac{\epsilon_0}{d} \pi a^2 + \frac{\epsilon_0}{d}(\mu_0 \epsilon_0 \omega^2) \frac{\pi}{8} a^4$$

$$\boxed{C = \frac{\epsilon_0 \pi a^2}{d}\left[1 + \frac{\mu_0 \epsilon_0 \omega^2 a^2}{8}\right]}$$

PURCELL DOES

II-4



(a) on y axis, only zero is at $y > a$, by inspection.

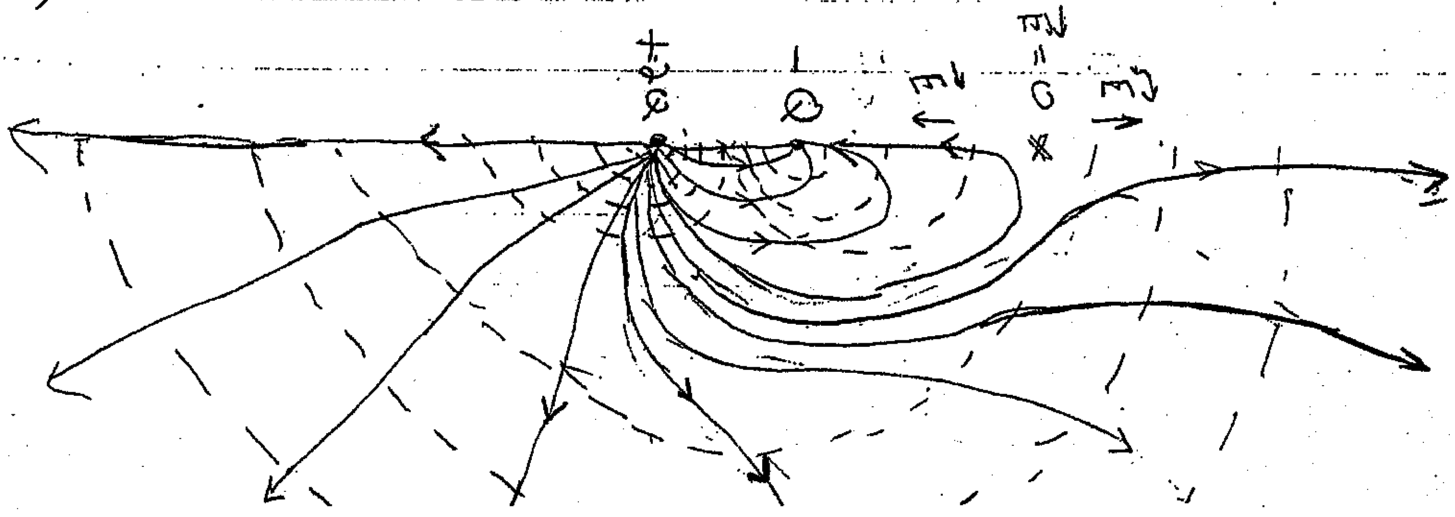
$$\frac{2Q}{y^2} - \frac{Q}{(y-a)^2} = 0 \quad \Rightarrow \quad y^2 - 4ay + 2a^2$$

$$y = (2 \pm \sqrt{2})\,a$$

$$y \approx \underbrace{0.6\,a}_{\text{excluded}},\ \underline{\underline{3.4\,a}}$$

(b) One half of field lines from 2Q must terminate on −Q. These will be the field lines starting in directions closest to that of the direction to −Q, i.e. those with $\theta < 90^\circ$. For $\theta > 90$, lines terminate at ∞.

(c)



Section III: Involving Statistical Mechanics and Thermodynamics

III - 1 Heat Capacity

~~Consider a solid piece of material containing N nuclei of spin 1, which do not interact. Each nucleus can be in m = 0 or ± 1 state. Due to the internal electric field in the solid the m = ± 1 states have the same energy ε > 0, while m = 0 has energy 0.~~

~~Deduce the entropy of the N nuclei as a function of temperature, and give the heat capacity for ε/kT << 1.~~

Sln III-1

sln: Part. fctn: $Z = \sum e^{-\frac{\varepsilon}{kT}} = \left(1+2e^{-\frac{\varepsilon}{kT}}\right)^N$

$$F = -kT \ln Z = -NkT \ln\left(1+2e^{-\frac{\varepsilon}{kT}}\right)$$

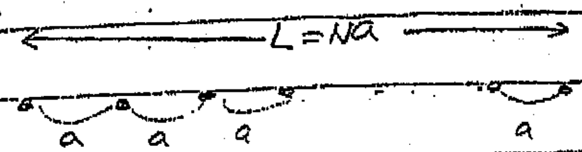
$S = -\left(\frac{\partial F}{\partial N}\right)_{V,N}$ $\quad S = -\frac{\partial F}{\partial T} = Nk \ln\left(1+2^{-\frac{\varepsilon}{kT}}\right) + \frac{2N\varepsilon}{T}\frac{e^{-\frac{\varepsilon}{kT}}}{\left(1+2e^{-\frac{\varepsilon}{kT}}\right)}$

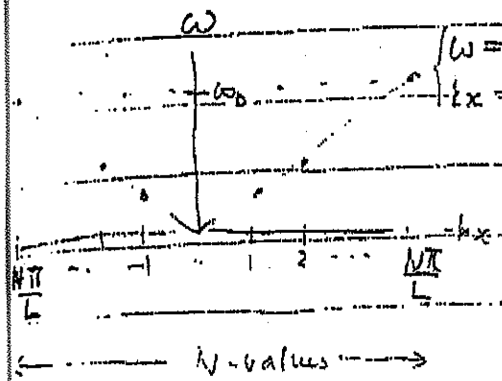
$E = U = TS$: $\quad U = \frac{2N\varepsilon\, e^{-\frac{\varepsilon}{kT}}}{\left(1+2e^{-\frac{\varepsilon}{kT}}\right)} \simeq \frac{2N\varepsilon}{3}\left(1 \cdot \frac{\varepsilon}{3kT}\right)$ for $\frac{\varepsilon}{kT} << 1$

$$C = \frac{\partial U}{\partial T} = \frac{2N\varepsilon^2}{3}\frac{1}{9kT^2} = \frac{2}{9}Nk\left(\frac{\varepsilon}{kT}\right)^2$$

Sln III-4

-4.





$\begin{cases} \omega = c k_x \\ k_x = \frac{\omega}{c} \end{cases}$

(a) $\frac{N}{2}$ - allowed values for $0 \le k_x \le \frac{N\pi}{L}$

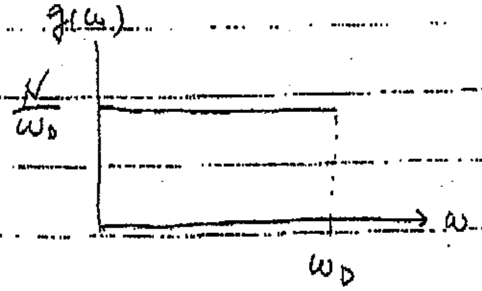
$\therefore\ g(k) = \left(\frac{N}{2}\right) \Big/ (N\pi/L) = \frac{L}{2\pi}$

for positive & negative

(b) $g(\omega)\,\Delta\omega = 2\,g(k)\,\Delta k_x$

$\therefore\ g(\omega) = 2\,g(k) \cdot \frac{\Delta k_x}{\Delta\omega} = 2\left(\frac{L}{2\pi}\right) \cdot \frac{1}{c} = \frac{L}{\pi c}$

(c) $\omega_D = c\,|k_x|_{max} = c\left(\frac{N\pi}{L}\right) \iff \frac{L}{c} = \frac{N\pi}{\omega_D}$



Insert into (b),

$g(\omega) = \frac{1}{\pi}\,\frac{N\pi}{\omega_D} = \frac{N}{\omega_D}$

$$\therefore\ U = \int_0^{\omega_D} \left(\frac{N\hbar\omega}{2} + N\hbar\omega \cdot \frac{1}{e^{\hbar\omega/kT}-1} \right) g(\omega)\, d\omega$$

$$\bar{\varepsilon} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT}-1}$$

$$\frac{\hbar\omega}{kT} \equiv x$$

(e) $$\doteq \int_0^{\omega_D} \frac{N\hbar\omega}{2}\, \frac{N}{\omega_D}\, d\omega$$

$$+ \underbrace{\int_0^{\infty} \frac{(NkT)\,x}{e^{x}-1} \cdot \frac{N}{\omega_D}\, dx\, \frac{kT}{\hbar}}_{\pi^2/6}$$

$$= \frac{N^2\hbar}{2\omega_D}\, \frac{\omega^2}{2}\Bigg|_0^{\omega_D} + \frac{\pi^2}{6} \cdot N(k_B T) \cdot \frac{N}{\omega_D}\, \frac{k_B T}{\hbar}$$

$$= \frac{1}{4} N^2 (\hbar\omega_D) + \frac{\pi^2}{6}\, \frac{N^2 \cdot (k_B T)^2}{\hbar\omega_D}$$

for $T \to 0$, the first term lasts; Zero Point Term //

(f) $$C_V = \frac{\partial U}{\partial T} = \frac{\pi^2}{6}\, \frac{N^2}{\hbar\omega_D} \cdot 2k_B^2 T$$

$$= \frac{\pi^2 N^2}{3(\hbar\omega_D)} k_B^2 T \quad //$$

III - 3 Cooling by relaxing B field.

At low temperatures one can cool further by reducing the magnetic field penetrating a paramagnetic substance. Assume M=a(T)*B, i.e. the magnetization is linear proportional to the B field applied. Find the temperature change ΔT for a magnetic field decrease ΔB in a thermally isolated sample of heat capacity c_B at constant B.

Sln III-3

sln: Suppose $M = a(T)\cdot B$

we have $dU = T\,dS + B\,dM$

thermal isolated: $\Delta S = 0$

Then: $\Delta T = \left.\frac{\partial T}{\partial H}\right|_S \Delta H$

$$d(U - BM) = T\,dS - M\,dB$$

$$\frac{\partial^2 (U - BM)}{\partial S\,\partial B} = \left.\frac{\partial T}{\partial B}\right|_S = -\left.\frac{\partial M}{\partial S}\right|_B \qquad \text{(Max Rel)}$$

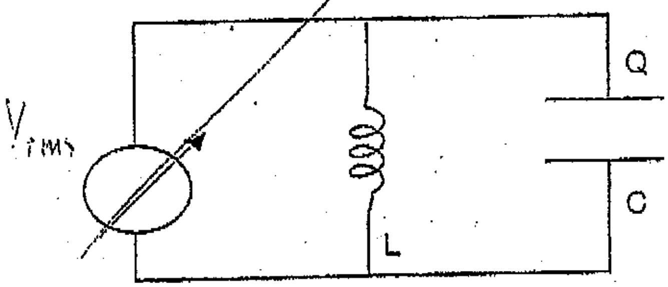
$$\longrightarrow \quad \Delta T = -\left.\frac{\partial M}{\partial S}\right|_B \Delta B = -\left.\frac{\partial M}{\partial T}\right|_B \Big/ \left.\frac{\partial S}{\partial T}\right|_B \Delta B$$

$$\boxed{\Delta T = \frac{-\frac{\partial a}{\partial T}\cdot B\,\Delta B}{C_B/T} = -\frac{T}{C_B}\frac{da}{dT} B\,\Delta B}$$

III - ~~7~~ 4 LC Thermometer

By measuring the noise (rms) voltage across a capacitor in parallel with an inductor one can determine the temperature, T. Find the relation between T and $V_{rms} = \langle \delta V^2 \rangle^{1/2}$.

Start with a Hamiltonian involving Q.



Sln III -4

Sln : $H = \frac{1}{2} L \left(\frac{dQ}{dt}\right)^2 + \frac{Q^2}{2C}$ $\rightarrow$ harm. osci $\omega = \frac{1}{\sqrt{LC}}$

$E_n = \hbar\omega (n + \frac{1}{2})$

- in circuit

$U = \langle E \rangle = \frac{\sum E_n e^{-E_n/kT}}{\sum e^{-E_n/kT}} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$

or

$U/2 = \langle \frac{CV^2}{2} \rangle = \langle \frac{LI^2}{2} \rangle$

$\langle V^2 \rangle = \frac{\hbar\omega}{C} \left\{ \frac{[illegible]}{e^{\cdots}} \right)$

$= \frac{\hbar\omega}{2C} \cosh\left(\frac{\hbar\omega}{2kT}\right)$

Class limit: $kT \gg \hbar\omega$; $\boxed{\langle V^2 \rangle = \frac{kT}{C}}$

Other ~ $kT \ll \hbar\omega$; $\langle V^2 \rangle = \frac{\hbar\omega}{2C}$

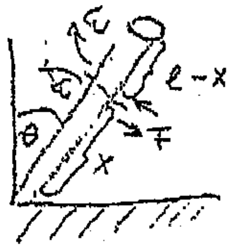
Section IV: Classical Physics of Mechanics

IV - 1 Chimney Breaking Point

A thin uniform brick chimney falls, pivoted by its low end. Consider the flexion stress at a section through the chimney and calculate the likely point of rupture.

Sln IV - I

Sln:



Ch. rotates: $\frac{M\ell^2}{3}\ddot{\theta} = Mg\frac{\ell}{2}\sin\theta$

$\Rightarrow \quad \ddot{\theta} = \frac{3g\sin\theta}{2\ell}$

Consider lower portion x: $\frac{Mx^2}{3}\frac{x}{\ell}\ddot{\theta} = Mg\frac{x^2}{2\ell} + \cancel{x}F - \tau$

shear force   internal flexion torque at x

Rotation of upper portion about its Center of mass: $\frac{M(L-x)^2}{12L}\ddot{\theta} = \frac{(L-x)}{2}\cancel{F} + \tau$

Solve 3 eq: $F = \frac{Mg\sin\theta(L-x)}{L^2(L+x)}\left\{\frac{(L-x)^2}{4} - x^2\right\}$

$\tau = \frac{Mx(L-x)^2 g\sin\theta}{4L^2}$

Since the chimney is thin width w : ~~τ >> F we dominate~~

Breaking at τ_{max} $\quad \frac{d\tau}{dx} = (L-x)^2 - 2x(L-x) = 0$

$L^2 - 2Lx + x^2 - 2Lx + 2x^2 =$

$L^2 - 4Lx + 3x^2 = 0$

$x^2 - \frac{4}{3}Lx + \frac{L^2}{3} = 0$

$x_{1/2} = \left\{\frac{2}{3} \pm \sqrt{\left(\frac{2}{3}\right)^2 - \frac{1}{3}}\right\}L$

$= \frac{1}{3}L$

Sln IV-2 Jet Engines

1. Jet Engines

a. The mass of area flowing into the intake per unit time is $\dot{m} = \frac{dn}{di} = v_i \rho A$.

b. The force applied to the engine equal to the momentum transferred to the air per unit time (Newton's Second Law), $F = \dot{m}(v_e - v_i) = \dot{m}\Delta v$. The power output of the engine is the energy output per unit time, $P = \dot{E} = \dot{m}\Delta v v_i$.

c. The kinetic energy increase per unit time of the air flowing in is $\dot{E}_{kin} = \frac{1}{2}\dot{m}\left(v_e^2 - v_i^2\right)$.

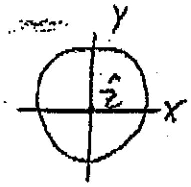
d. The propulsive efficiency η_p is the ratio of the power output of the engine P to the energy incrase per unit time of the gas $\dot{E}_{kin}$,

$$\eta_p = \frac{\dot{m}v_i\Delta v}{\frac{1}{2}\dot{m}\left(v_e^2 - v_i^2\right)} = \frac{2}{2 + \frac{\Delta v}{v_i}}$$

e. The mass flow is now $\dot{m}' = v_i \rho A'$. The force is the same as in part b, $(v_e - v_i) = \alpha(v'_e - v_i)$; in order to achieve the same power output, more air is moved at a lower velocity. The propulsive efficiency is then

$$\eta_p = \frac{\dot{m}v_i\Delta v'}{\frac{1}{2}\dot{m}\left(v_e'^2 - v_i^2\right)} = \frac{2}{2 + \frac{\Delta v'}{v_i}} = \frac{2}{2 + \frac{\Delta v}{\alpha v_i}}$$

For a loop:



$$\rho = \frac{M}{2\pi R}\,\delta(R\hat{s}-r)\,\delta(z)$$

$$I_{xx} = \int R\,d\theta\,\frac{M}{2\pi R}\left(R^2 - R^2\cos^2\theta\right)$$

$$= \frac{MR^2}{2\pi}\int_0^{2\pi}\sin^2\theta\,d\theta$$

a trig identity:

$$\sin^2\theta = \frac{-1}{4}\left(e^{2i\theta} - 2 + e^{-2i\theta}\right) = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$I_{xx} = \frac{MR^2}{2\pi}\left(\frac{1}{2}\right)\int_0^{2\pi}(1-\cos 2\theta)$$

$$= \frac{MR^2}{4\pi}\left(2\pi - \frac{1}{2}\cancel{\int_0^{2\pi}\cos 2\theta\,d2\theta}^{\,0}\right) = \frac{MR^2}{2}$$

$$I_{xx} = I_{yy}$$

$$I_{zz} = \int_0^{2\pi} R\,d\theta\,\frac{M}{2\pi R}\,R^2 = MR^2$$

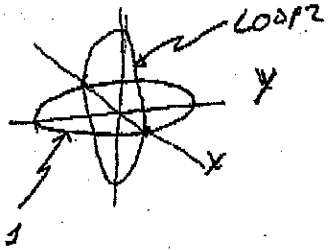
$$I_{xy} = \int_0^{2\pi} R\,d\theta\,\frac{M}{2\pi R}\,(-xy)$$

$$= -\frac{M}{2\pi}\int_0^{2\pi}\sin\theta\cos\theta\,d\theta = +\frac{M}{2\pi}\int_0 \cos\theta\,d(\cos\theta)$$

$$= \frac{M}{2\pi}\,\frac{1}{2}\cos^2\theta\Big|_0^{2\pi} = 0$$

$$I_{xz} = \int^{2\pi} R\,d\theta\,\frac{M}{2\pi R}\,(-x\overset{0}{\overset{\parallel}{z}}) = I_{yz} = 0$$

For loop 1:



$I_{xx,1} = I_{yy,1} = \frac{MR^2}{2}$

$I_{zz,1} = MR$

$I_{xy,1} = I_{xz,1} = I_{yz,1} = 0$

For loop 2:

$I_{xx,2} = I_{zz,2} = \frac{MR^2}{2}$

$I_{yy,2} = MR^2$

Then, $I_{xx} = MR^2$ since $I = I_1 + I_2$

$I_{yy} = \frac{3}{2} MR^2$

$I_{zz} = \frac{3}{2} MR^2$

all others are zero.

Principal axes: $\hat{x}, \hat{y}$ and $\hat{z}$

"By symmetry" not an acceptable answer.

c) $$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = \vec{N}$$

$$L_i = I_{ij}\,\omega_j$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} MR^2$$

$$\vec{L} = \begin{pmatrix} 1 & & \\ & 3/2 & \\ & & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \frac{MR^2\omega_0}{\sqrt{2}}$$

$$= \begin{pmatrix} 1 \\ 3/2 \\ 0 \end{pmatrix} \frac{MR^2\omega_0}{\sqrt{2}}$$

$\vec{N}$ is fixed such that $\frac{d\vec{L}}{dt} = 0$

$$\Rightarrow \vec{N} = \vec{\omega} \times \vec{L}$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 1 & 3/2 & 0 \end{vmatrix} \left(\frac{MR^2\omega_0}{\sqrt{2}}\right)\left(\frac{\omega_0}{\sqrt{2}}\right)$$

$$= \frac{MR^2\omega_0^2}{2} \begin{pmatrix} 0 \\ 0 \\ 3/2 - 1 \end{pmatrix} = \frac{MR^2\omega_0^2}{4}\,\hat{z}$$

d) For no net torque,

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = 0$$

$$L_x = \omega_x I_1 \quad \text{etc}$$

Then

$$\dot{\omega}_x I_x + \omega_y L_z - \omega_z L_y = 0$$

$$\dot{\omega}_x I_x + \omega_y \omega_z I_z - \omega_z \omega_y I_y = 0$$

$$\dot{\omega}_x I_x + \omega_y \omega_z (I_z - I_y) = 0$$

$$\dot{\omega}_y I_y + \omega_z \omega_x (I_x - I_z) = 0$$

$$\dot{\omega}_z I_z + \omega_x \omega_y (I_y - I_x) = 0$$

For $I_x = MR^2$

$I_y = \frac{3}{2} MR^2$

$I_z = \frac{3}{2} MR^2$

$$\dot{\omega}_x = 0 \qquad (I_z - I_y = 0)$$

$$\omega_y\left(\frac{3}{2}\right) + \omega_z \omega_x \left(-\frac{1}{2}\right) = 0$$

$$\omega_z\left(\frac{3}{2}\right) + \omega_x \omega_y \left(\frac{1}{2}\right) = 0$$

$\cancel{3\dot\omega_y - \omega_z \frac{\omega_0}{r}}$

at $t=0$, $\vec{L} = L_0 \hat{x} = I_x \omega_{x0}/\sqrt{2}$

$$\Rightarrow \omega_{x_0} = \frac{L_0}{MR^2/\sqrt{2}}$$

$\dot\omega_x = 0 \Rightarrow \omega_x$ remain constant

$$\left.\begin{aligned} 3\dot\omega_y - \omega_z \omega_{x0} &= 0 \\ 3\dot\omega_z + \omega_y \omega_{x_0} &= 0 \end{aligned}\right\} \quad 3\ddot\omega_z + \dot\omega_y \omega_{x_0} = 0$$

$\omega_z(0) = 0$

$\omega_y(0) = \frac{L_0}{\sqrt{2} MR^2}$

$$-\frac{3\ddot\omega_z}{\omega_{x_0}} = \dot\omega_y$$

$\Rightarrow \cancel{\omega_y(t) = \omega_y(0)\cos \Omega t}$

~~and~~

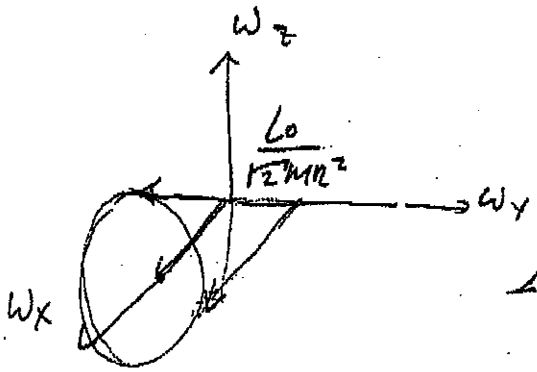
$$\Rightarrow 3\left(\frac{-3\ddot\omega_z}{\omega_{x_0}}\right) - \omega_z \omega_{x_0} = 0$$

$$\ddot\omega_z + \frac{\omega_{x_0}^2}{9}\omega_z = 0$$

$\Rightarrow \cancel{\omega_z(t) = \omega_0}$

$$\ddot\omega_y + \frac{\omega_{x_0}^2}{9}\omega_y = 0 \Rightarrow \omega_y(t) = \frac{L_0}{\sqrt{2} MR^2}\cos\left(\frac{\omega_{x_0}}{3} t\right)$$

$$\omega_z(t) = \frac{L_0}{\sqrt{2} MR^2}\sin\left(\frac{\omega_{x_0}}{3} t\right)$$



$\boxed{\vec{\omega} \text{ precesses around } \hat{x} \text{ with frequency } \frac{L_0}{\dots}\ ;\ \omega_x = \frac{L_0}{\sqrt{2} MR^2}}$ ω_{x_0}

Sln IV-4 Relativity

(a) $E^2 - p^2c^2 = m^2c^4$

$E_\pi = m_\pi c^2$

$E_\mu = c\sqrt{m_\mu^2 c^2 + p_\mu^2}$

$E_\nu = |p_\nu| c = |p_\mu| c$

$m_\pi c^2 = c\sqrt{m_\mu^2 c^2 + p_\mu^2} + |p_\mu| c$

$(m_\pi c - |p_\mu|)^2 = m_\mu^2 c^2 + p_\mu^2$

$$|p_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = m\gamma v$$

(c) $$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = m\gamma c^2$$

$p/E = v/c^2$ $\boxed{v = \frac{|p| c^2}{E}} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2} c$