MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS
CAMBRIDGE, MASSACHUSETTS 02139

DOCTORAL GENERAL EXAMINATION

PART I

January 30th, 2012

FIVE HOURS

1. This examination is divided into five sections, each consisting of four problems. Answer <u>all</u> the problems. Each problem is worth 5 points, thus the maximum score for the exam is 100.

2. For each section, please use the booklets that you have been given according to the section listed. Do not put your name on it, as each booklet has an identification number that will allow the papers to be graded blindly.

3. A diagram or sketch is often useful to the grader and could improve your grade.

4. Read each problem carefully and do not do more work than is necessary. For example "give" and "sketch" do not mean "derive".

5. Calculators may **not** be used.

6. No books, notes or reference materials may be used.

Possibly Useful Fomulae and Constants

A normalized Gaussian distribution with mean μ and standard deviation σ is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{-(x-\mu)^2/2\sigma^2}$$

A normalized Poisson distribution of mean $\bar{n}$ is

$$p(m) = \frac{\bar{n}^m}{m!} \, e^{-\bar{n}}$$

A normalized binomial distribution of mean $\mu = n\theta$ and standard deviation $\sqrt{n\theta(1-\theta)}$ is

$$p(x) = \frac{n!}{x!(n-x)!} \, \theta^x \, (1-\theta)^{n-x}$$

The magnetic field of a dipole with magnetic moment $\vec{m}$ at a distance r away is given by

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \left[\frac{3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}}{r^3} \right]$$

$$
\begin{array}{rclll}
k_B & = & 1.4 \times 10^{-23} & \mathrm{J\,K^{-1}} & = & 10^{16}\,\mathrm{erg\,K^{-1}} \\
\hbar & = & 1 \times 10^{-34} & \mathrm{J.s} & = & 10^{-27}\,\mathrm{erg\cdot s} \\
c & = & 3 \times 10^{8} & \mathrm{m\,s^{-1}} & = & 3 \times 10^{10}\,\mathrm{cm\,s^{-1}} \\
e & = & 1.6 \times 10^{-19} & \mathrm{Coulomb} & = & 4.8 \times 10^{-10}\,\mathrm{statcoulomb} \\
m_e & = & 9.1 \times 10^{-31} & \mathrm{kg} & = & 0.5\,\mathrm{MeV/c^2} \\
\hbar c & = & 3.2 \times 10^{-26} & \mathrm{J\cdot m} & \simeq & 1.97 \times 10^{-13}\,\mathrm{MeV\cdot m} \\
G & = & 6.7 \times 10^{-11} & \mathrm{m^3\,kg^{-1}\,s^{-2}} & = & 6.7 \times 10^{-8}\,\mathrm{cm^3\,g^{-1}\,s^{-2}} \\
\varepsilon_0 & = & 8.9 \times 10^{-12} & \mathrm{F\,m^{-1}} & = & 1\,(\mathrm{cgs}) \\
\mu_0 & = & 4\pi \times 10^{-7} & \mathrm{N\,A^{-2}} & = & 1\,(\mathrm{cgs})
\end{array}
$$

I-1 Pogo Stick

Two masses m_1 and m_2 are connected by a massless spring with spring constant k (see diagram). Initially m_1 is pushed down with force F. After some time, the force is removed. How big must F have been, if the subsequent motion causes m_2 to be lifted from the ground?
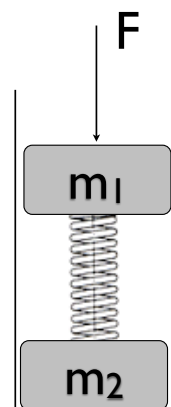
For m_2 to be lifted from the ground the tension in the spring has to be $T \geq m_2 g$. Therefore the spring is extended by $m_2 g/k$, with a potential energy of $\frac{1}{2}k(\frac{m_2 g}{k})^2$. Conservation of energy gives:

$$\frac{1}{2}k\left(\frac{m_2 g}{k}\right)^2 + m_1 g\frac{m_2 g}{k} = \frac{1}{2}k\left(\frac{m_1 g + F}{k}\right)^2 - m_1 g\frac{m_1 g + F}{k}$$

k drops out, and some algebra gives

$$\frac{1}{2}(m_2 g)^2 + m_1 g m_2 g = \frac{1}{2}F^2 - \frac{1}{2}(m_1 g)^2$$

and therefore

$$\underline{F = (m_1 + m_2)g.}$$

I-2 Electromagnetic Pendulum

Suppose a small sphere of charge $-q$ is suspended along the positive $\hat{z}$-axis by a massless string above a very large plate of charge Q and area A (where A is much larger than the distance from the charged surface to the sphere). Suppose the mass is moving horizontally with a velocity $\vec{v} = -v_0\hat{x}$ with respect to the plate. What is the electromagnetic force acting on the sphere in the reference frame of the sphere? You should not assume that $v_0 \ll c$, but you may ignore any radiative losses.

In the frame of the plate (call that S), the only contribution to the EM field is the electric field from the plate only, which yields

$$\vec{\mathbf{E}} = \frac{Q}{2A\epsilon_0}\hat{z}$$

When we boost to the frame of the sphere (S'), there is a Lorentz contraction that essentially increases the charge density. The corresponding electric field thus becomes...

$$\vec{\mathbf{E}}' = \frac{Q}{2A\epsilon_0\sqrt{1-v_0^2/c^2}}\hat{z}$$

In addition, there appears to be a sheet of charge moving in the $+\hat{x}$ direction, introducing a magnetic field component as well.

$$\vec{\mathbf{B}}' = -\frac{Qv_0\mu_0}{2A\sqrt{1-v_0^2/c^2}}\hat{y}$$

Using the Lorentz law allows one to calculate the force on the sphere in the S' frame

$$\vec{\mathbf{F}}' = -q(\vec{\mathbf{E}}' + \vec{\mathbf{v}} \times \vec{\mathbf{B}}')$$

$$\vec{\mathbf{F}}' = -\frac{qQ}{2A\epsilon_0}\frac{1-v^2\epsilon_0\mu_0}{\sqrt{1-v^2/c^2}}\hat{z}$$

$$\vec{\mathbf{F}}' = -\frac{qQ}{2A\epsilon_0}\sqrt{1-v^2/c^2}\hat{z}$$

$$\vec{\mathbf{F}}' = -\frac{qQ}{2A\epsilon_0\gamma}\hat{z}$$

4

Note that one can arrive at this final answer in a variety of ways, including transforming the E and B fields or the force itself. All approaches are correct, of course.

Note in cgs units, the expressions can be expressed as

$$\vec{\mathbf{E}'} = \frac{2\pi Q}{A\sqrt{1 - v_0^2/c^2}}\hat{z}$$

$$\vec{\mathbf{B}'} = -\frac{2\pi Q v_0}{Ac\sqrt{1 - v_0^2/c^2}}\hat{y}$$

$$\vec{\mathbf{F}'} = -\frac{2\pi q Q}{A\gamma}\hat{z}$$

I-3 1D Potential

A non-relativistic particle of mass m moves in a 1-dimensional potential

$$V(x) = A|x|,$$

where A is a positive constant. Estimate the minimum total energy of the particle by using the Heisenberg uncertainty principle and explicitly minimizing an expression for the energy.

We start with the energy expression as ...

$$E = \frac{p^2}{2m} + A|x|$$

Using the Heisenberg uncertainty principle, we have for the lowest energy configuration

$$\Delta x \Delta p \geq \hbar/2$$
$$x_{\min} p_{\min} \sim \hbar/2$$

Now write the energy with respect to one of them (for example, in terms of x)

$$E = \frac{\hbar^2}{8mx^2} + A|x|$$
$$\frac{\partial E}{\partial x} = 0 \rightarrow -\frac{\hbar^2}{4mx^3} + A = 0$$
$$x_{\min} = \pm \left(\frac{\hbar^2}{4mA} \right)^{\frac{1}{3}}$$

Substituting into the energy equation for this value, for either sign of x, one finds...

$$E_{\min} = \frac{3}{2} \left(\frac{A^2 \hbar^2}{4m} \right)^{\frac{1}{3}}$$

I-4 Relativistic Fermi Gas*

Consider an ideal Fermi gas of N uncharged particles with spin $1/2$ and rest mass m (each) confined to a small three dimensional volume V at temperature $T \ll \epsilon_F/k$. Find the Fermi energy ϵ_F of this gas in the extreme relativistic limit where $\epsilon_F \gg mc^2$.

As it's an ideal gas, the atoms have only kinetic energy and do not interact. The energy of an atom is $\epsilon = \sqrt{m^2c^4 + p^2c^2} - mc^2$. In the extreme relativistic limit this becomes $\epsilon = \sqrt{p^2c^2}$. At low temperatures the gas is degenerate: only energies less than ϵ_F are occupied. The density of states in momentum space is $D(\vec{k}) = V/(2\pi)^3$ and the energy is expressed as $\epsilon = \sqrt{p^2c^2} = \sqrt{\hbar^2k^2c^2} = \hbar ck$. Thus, the number of states that fulfill the above condition is given by
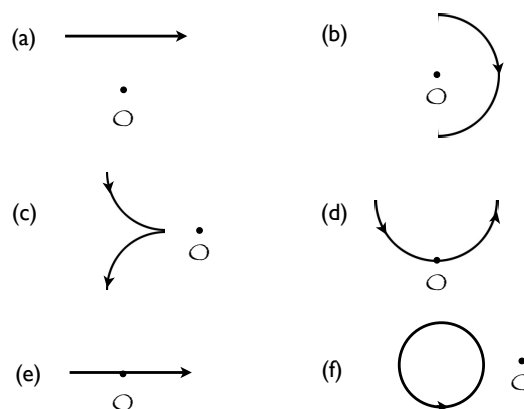
$$N(\epsilon) = \frac{V}{(2\pi)^3} \int_0^{\epsilon/\hbar c} 4\pi k^2 dk = \frac{V}{6\pi^2}\left(\frac{\epsilon}{\hbar c}\right)^3$$

Each state has two atoms, so the Fermi energy is given by

$$\epsilon_F = \left(\frac{3\pi^2 N}{V}\right)^{1/3} \hbar c$$

II-1 Angular Momenta Along Trajectories***

For each of the trajectories (a)-(f), state whether or not the angular momentum $\vec{\mathbf{L}}$ about the origin (denoted by $\mathcal{O}$) could be conserved. If it is not, explain why.

(a)

(b)

(c)

(d)

(e)

(f)

Items (c), (d), and (f) are not conserved. For (c), at the point at the start of the shown trajectory (top), $\vec{L} = \vec{r} \times \vec{p} \neq 0$ while at the cusp it is, since $\vec{r} \parallel \vec{p}$. Even if $\vec{r}$ is not parallel to $\vec{p}$, $\vec{p}$ and hence $\vec{L}$ changes sign and is not conserved.

For (d), $\vec{\mathbf{L}}$ is non-zero except at the origin, so it is not conserved.

For (f), the angular momentum changes sign (even in a perfect circle) as calculated from the point O, so it is not conserved.

II-2 Three Bosons***

Three non-identical spin-1 particles with spin operators $\vec{s}_1$, $\vec{s}_2$ and $\vec{s}_3$ interact via a Hamiltonian

$$H = \frac{E_0}{\hbar^2}(\vec{s}_1 \cdot \vec{s}_2 + \vec{s}_1 \cdot \vec{s}_3 + \vec{s}_2 \cdot \vec{s}_3).$$

How many independent states are there for the three particle system? What is the energy and the degeneracy of each of the lowest two energy levels?

Let $\vec{S} = \vec{s}_1 + \vec{s}_2 + \vec{s}_3$. Adding the first two, then a third spin of magnitude 1 shows that the states of total spin consist of one septet (S = 3), two quintets (S=2), three triplets (S=1), and one singlet (S=0). The total number of states is therefore 27. That can be deduced by counting the individual states of the three spins $(3 \times 3 \times 3 = 27)$ or by counting the number of total spin $(1 \times 7 + 2 \times 5 + 3 \times 3 + 1 \times 1 = 27)$.

$$\vec{S} \cdot \vec{S} = \vec{s}_1 \cdot \vec{s}_1 + \vec{s}_2 \cdot \vec{s}_2 + \vec{s}_3 \cdot \vec{s}_3 + 2(\vec{s}_1 \cdot \vec{s}_2 + \vec{s}_1 \cdot \vec{s}_3 + \vec{s}_2 \cdot \vec{s}_3)$$
$$(\vec{s}_1 \cdot \vec{s}_2 + \vec{s}_1 \cdot \vec{s}_3 + \vec{s}_2 \cdot \vec{s}_3) = \frac{1}{2}(\vec{S} \cdot \vec{S} - (\vec{s}_1 \cdot \vec{s}_1 + \vec{s}_2 \cdot \vec{s}_2 + \vec{s}_3 \cdot \vec{s}_3))$$
$$= \frac{1}{2}((S)(S+1) - 6)$$
$$E = \frac{E_0}{2\hbar^2}(S(S+1) - 6)$$

Therefore, the lowest state is the singlet (degeneracy = 1) with $E_{\text{gnd}} = -3\frac{E_0}{\hbar^2}$. The next goes to the triplet (degeneracy = $3 \times 3 \equiv 9$), with $E_1 = -2\frac{E_0}{\hbar^2}$

II-3 Magnetic Monopole***

Imagine an electric charge moving in the field of a magnetic monopole (although none has yet been found). Set up the non-relativistic equation of motion for an electric charge q of mass m in the field of a magnetic monopole of strength Γ (a positive constant).

Assume the particle at a particular moment in time is a distance r from the magnetic monopole and that the particle's velocity $\vec{\mathbf{v}}$ is perpendicular to the line between the charged particle and the monopole. Give an expression for the force vector on the particle at this point.

Since

$$\nabla \cdot \vec{\mathbf{B}} = 4\pi \rho_m$$

the field must be given by

$$\vec{\mathbf{B}} = \frac{\Gamma}{r^2} \hat{\mathbf{r}}$$

For a particle of mass m, charge q and velocity $\vec{v}$ the equation of motion must be

$$m\frac{d\vec{\mathbf{v}}}{dt} = q\frac{\vec{\mathbf{v}}}{c} \times \vec{\mathbf{B}} = \frac{q\Gamma}{c\,r^2}\vec{v} \times \hat{r}$$

The resulting force will be perpendicular to the plane of $\hat{r}$ and $\vec{v}$, and the distance to the monopole will increase.

Note this solution can be written in SI units as

$$m\frac{d\vec{\mathbf{v}}}{dt} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\Gamma}{r^2}\vec{v} \times \hat{r}$$

II-4 Decompression of Ideal Gas***

An ideal monoatomic gas at a temperature of 20 degrees Celsius expands adiabatically so that the final volume is 8 times the initial volume. What is the final temperature?

$pV^\gamma = \text{constant}$

$$
\begin{aligned}
\Rightarrow p_1 V_1^\gamma &= p_2 V_2^\gamma \Rightarrow \frac{p_1}{p_2} = \left(\frac{V_2}{V_1}\right)^\gamma = 8^{5/3} \\
\frac{p_1 V_1}{T_1} &= \frac{p_2 V_2}{T_2} \Rightarrow \frac{T_2}{T_1} = \frac{p_2 V_2}{p_1 V_1} = 8^{-5/3} \times 8 = 8^{-2/3} = \frac{1}{4}
\end{aligned}
$$

$$
T_2 = \frac{1}{4} T_1 = \frac{1}{4} \ 293 \ \text{K} = 73 \ \text{K} \quad (-199 \ C)
$$

III-1 The Sopwith Camel***

The Sopwith Camel was a single engine fighter plan flown by the British in WWI (and also by the character Snoopy in the Peanuts comic strip). It was powered by a radial engine, and the entire engine rotated with the propeller. The Camel had an unfortunate property: if the pilot turned to the right, the plane tended to go into a dive, while a left turn caused the plane to climb steeply. These tendencies caused inexperienced pilots to crash or stall during take-off. From the perspective of the pilot, who sat behind the engine, did the engine rotate clockwise or counter-clockwise? Explain *briefly* how you arrived at your answer.

Suppose the engine rotates clockwise (CW). Then the angular momentum vector $\vec{L}$ of the engine points straight ahead and a right turn will cause a change $\delta\vec{L}$ that points to the right. The airframe must then exert a torque τ on the engine that points to the right. By Newton's third law (since this is an isolated system), the engine exerts a torque on the airframe that points to the left. By the right-hand rule, this forces the plane into a dive.

III-2 Compton Scattering

In 1923, Compton performed a series of experiments in which he was scattering X-rays from a graphite scatterer. The wavelength of the X-rays emitted by his source was $\lambda = 0.7$ Angstroms. What was the wavelength of the longest wavelength scattered X-rays that he observed?

Using Compton scattering formula, the shift in wavelength is given by

$$\Delta\lambda = \frac{h}{mc}(1 - \cos\theta)$$
$$\lambda_1 = \lambda_0 + 0.0243\text{A}(1 - \cos\theta)$$

The angle factor (backscattering) can lengthen the wavelength at most by a factor of 2 times the Compton wavelength h/mc, so the longest possible wavelength is $\lambda_1 \simeq 0.75$ Å.

Students who get the Compton wavelength correct to within a factor of 2π should receive almost full credit.

III-3 Magnetic Flux***

A current I flows around the small loop of radius a shown in the figure. Calculate the total magnetic flux that falls to the right of the infinite line shown a distance $r \gg a$ from the loop. The current loop and the line are all in the same plane.
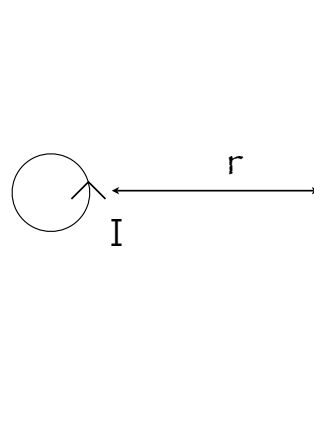
Consider the line to be part of an infinite loop, with the return part at infinity. By the reciprocal law ($M_{12} = M_{21}$ for mutual inductance), flux through the infinite loop due to current I in the small loop is the same as the flux through the small loop due to the current I in the infinite loop.

$$\text{Flux} = \pi a^2 \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 a^2 I}{2r}$$

$$\text{Flux} = \frac{2\pi a^2 I}{c\,r} \ (cgs)$$

III-4 Lattice Defects***

A cubic lattice of N atoms has M interstitial sites (site between lattice sites). An atom can be displaced from its lattice site to any one of the M interstitial sites at a cost of energy ϵ. Find the number of atoms located at interstitial sites as a function of temperature, $n(T)$. You may assume that $n \gg 1$ and that $M, N \gg n$ and for any large x you may use the approximation $x! \approx x \log x$.

The total number of states is the number of ways to have n defects in N sites times the number of ways to arrange n defects into M interstitial sites.

$$\Omega = \frac{N!}{n!(N-n)!} \cdot \frac{M!}{n!(M-n)!}$$

Since the entropy of the system is given by $S(n) = k \ln \Omega$

$$S(n) = \ln N! + \ln M! - \ln n! - \ln (N-n)! - \ln n! - - \ln (M-n)!$$
$$S(n) = N \ln N + M \ln M - 2n \ln n - (N-n) \ln (N-n) - (M-n) \ln (M-n)$$

We can relate the entropy at fixed N via

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_N = (\frac{\partial S}{\partial n} \frac{\partial n}{\partial E})_N$$
$$\left(\frac{\partial S}{\partial n}\right) = k(-2\ln n - 2 + \ln (N-n) + 1 + \ln (M-n) + 1)$$
$$\left(\frac{\partial S}{\partial n}\right) = k(-2\ln n + \ln (N-n) + \ln (M-n))$$
$$\left(\frac{\partial S}{\partial n}\right) = k \ln \frac{(N-n)(M-n)}{n^2}$$

$$E = n\epsilon \rightarrow dE = \epsilon d\epsilon$$
$$\frac{\partial n}{\partial E} = \frac{1}{\epsilon}$$

Since $M, N \gg n$, we have

$$\frac{1}{T} \simeq \frac{k}{\epsilon} \ln \frac{NM}{n^2}$$

$$\frac{\epsilon}{kT} = \ln \frac{NM}{n^2}$$

Finally yielding...

$$n(T) = \sqrt{NM} e^{-\frac{\epsilon}{2kT}}$$

IV-1 Radiative Transfer Between Plates***

Two parallel plates plates are maintained at temperatures T_L and T_R respectively and have emissivities ϵ_L and ϵ_R respectively. Given the Stephan-Boltzmann constant σ, express the net energy transfer rate per area from the left plate (L) to the right plate (R). *Hint:* this problem can be solved by using an infinite series, or by finding the energy transfer rate per area to the right and left, I_R and I_L, respectively.

One can express the net radiative transfer, I_{net}, in terms of the left and right portions of the system.

$$I_R = \epsilon_L \sigma T_L^4 + (1 - \epsilon_L)I_L$$

$$I_L = \epsilon_R \sigma T_R^4 + (1 - \epsilon_R)I_R$$

$$I_{\text{net}} = I_R - I_L$$

By looking at the quantity $\epsilon_R I_R - \epsilon_L I_L$ and isolating $I_R - I_L$, one finds...

$$\boxed{I_{\text{net}} = \frac{\epsilon_R \epsilon_L \sigma (T_L^4 - T_R^4)}{\epsilon_R + \epsilon_L - \epsilon_L \epsilon_R}}$$

For the infinite series approach, consider the contribution from each sequence. That is.

$$I_R' = \epsilon_R \epsilon_L \sigma T_L^4 (1 + (1 - \epsilon_R)(1 - \epsilon_L) + ((1 - \epsilon_R)(1 - \epsilon_L))^2 + ...)$$

and the same for I_L'. The infinite series term is just the geometric series with a value $(1 - (1 - \epsilon_R)(1 - \epsilon_L))^{-1}$. Therefore, the net effect is

$$I_{\text{net}} = \frac{\epsilon_R \epsilon_L \sigma (T_L^4 - T_R^4)}{1 - (1 - \epsilon_R)(1 - \epsilon_L)}$$

Which is the same as above.

IV-2 Particle on a Sphere

A particle initially sits on top of a large smooth sphere of radius R as shown in the figure. The particle begins to slide along the surface of the sphere. There is negligible friction between the particle and the surface. Let g denote the gravitational constant. Determine the angle θ_1 with respect to the vertical at which the particle will lose contact with the surface of the sphere.
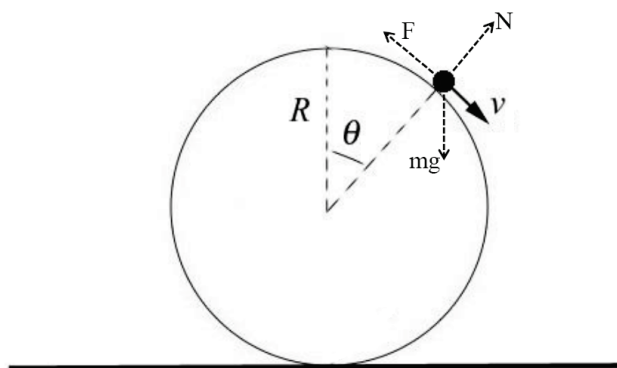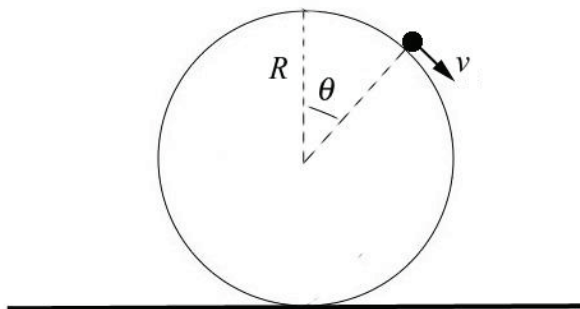
$$mg\cos\theta - N = m\frac{v^2}{R}$$

$$mgR(1 - \cos\theta) = \frac{mv^2}{2}$$

$$\Rightarrow \quad N \quad = \quad mg\cos\theta - 2(1 - \cos\theta)mg$$
$$= \quad 3mg\cos\theta - 2mg$$

$$N = 0 \Longleftrightarrow \cos\theta_1 = \frac{2}{3}$$

IV-3 Simple Harmonic Oscillator

A certain particle is free until $t = 0$, at which time a simple harmonic oscillator (SHO) potential is suddenly switched on. For convenience, assume that we have re-scaled the problem so that the time-independent Schrödinger equation for the SHO is

$$-\frac{d^2}{dx^2}\varphi_n(x) + x^2\varphi_n(x) = E_n\varphi_n(x),$$

from which the normalized ground state wavefunction is

$$\varphi_0(x) = (\pi)^{-1/4}e^{-\frac{1}{2}x^2},$$

with energy $E_0 = 1/2$.

If the free particle's wavepacket at time $t = 0$ is proportional to

$$e^{-\frac{1}{2}x^2}e^{ikx},$$

where k is real, what is the probability that a measurement of the energy at a later time will *not* yield $E = 1/2$?

Let $\psi(x) = Ae^{-\frac{1}{2}x^2+ikx}$

Normalization: $1 = \int dx|\psi|^2 = A^2 \int_{-\infty}^{\infty} dx e^{-x^2} = A^2\sqrt{\pi}$

$\Rightarrow \psi(x) = \pi^{-1/4}e^{-\frac{1}{2}x^2+ikx}$. At later time t, let $\psi(x,t) = \sum_n c_n\varphi_n(x)e^{-iE_nt/\hbar}$

$$
\begin{aligned}
P(E = 1/2) &= \left|\int_{-\infty}^{\infty} dx\psi(x,t)\varphi_o^*(x)\right|^2 = \left|c_oe^{-iE_ot/\hbar}\right|^2 = |c_o|^2 = \left|\int_{-\infty}^{\infty} dx\psi(x,o)\varphi_o^*(x)\right|^2 \\
&= \left|\int_{-\infty}^{\infty} dx\pi^{-\frac{1}{2}}e^{-x^2+ikx}\right|^2 \\
x^2 - ikx &= \left(x - \frac{ik}{2}\right)^2 + \frac{1}{4}k^2 \\
P(E = 1/2) &= \left|\pi^{-1/2}e^{-\frac{1}{4}k^2}\int_{-\infty}^{\infty} dxe^{-(x-\frac{i}{2}k)^2}\right|^2 \\
&= \left|e^{-\frac{1}{4}k^2}\right|^2 \\
&= e^{-k^2/2}
\end{aligned}
$$

$\Rightarrow \boxed{P(E \neq 1/2) = 1 - e^{-k^2/2}}$

IV-4 Pluto***

To a good approximation, the surface temperatures of planets and dwarf planets in our solar system can be estimated by assuming that they are thermodynamically ideal blackbodies that absorb all of the incident radiation from the Sun and re-radiate all of this energy as thermal radiation.

If the Earth's surface temperature is 300 K, estimate the temperature on the surface of Pluto. Possibly useful facts are that Pluto is 40 times more distant from the sun than the Earth is, and Pluto's radius is one-fifth that of the Earth's.

For each planet, let Φ be the incident energy flux from the sun, R be the distance from the sun, and r be the planet's radius. Then

$$\Phi r^2 \propto \frac{r^2}{R^2} = \text{constant} r^2 T^4$$

so

$$T_{\text{pluto}} = T_{\text{earth}} (\frac{R_e}{R_p})^{1/2} = 300K \sqrt{1/40} \simeq 48K$$

V-1 Needles

Consider two identical compass needles that are placed at a distance $\vec{r}$ relative to each other (with $|r| \gg$ the length of the needles). Determine the relative orientation of the needles for the following two cases (neglecting Earth's magnetic field):

(a)

1. The needles are mounted on a common axis running through their respective centers, such that each needle can rotate freely in a plane perpendicular to this axis.

2. The needles are displaced horizontally from each other, such that they both can rotate in the same horizontal plane

(b)

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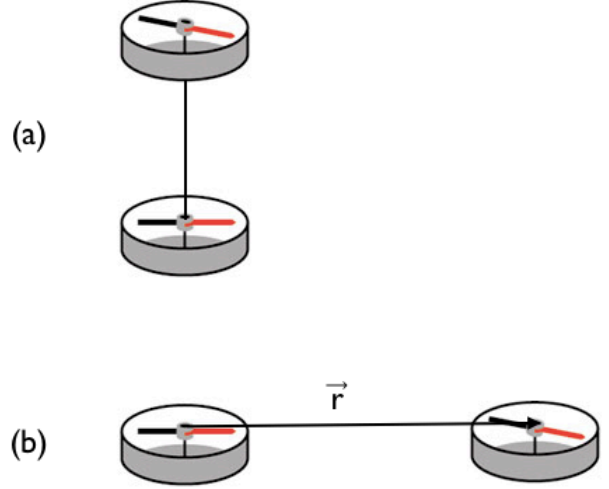
1. We have two dipoles $\vec{m}_1$ and $\vec{m}_2$. The magnetic field from dipole $\vec{m}_1$ at the position of dipole $\vec{m}_2$ is (formula sheet):

$$
\begin{aligned}
\vec{B}_{m_1 \to m_2} &= \frac{\mu_0}{4\pi} \left[\frac{3\vec{r}(\vec{m}_1 \cdot \vec{r})}{r^5} - \frac{\vec{m}_1}{r^3} \right] \\
&= \frac{\mu_0}{4\pi} \left[-\frac{\vec{m}_1}{r^3} \right], \quad \text{as } \vec{m}_1 \perp \vec{r}.
\end{aligned}
$$

The energy of dipole $\vec{m}_2$ in the field of dipole $\vec{m}_1$ is

$$
E = -\vec{m}_2 \cdot \vec{B}_{m_1 \to m_2} = -\frac{\mu_0}{4\pi} \left[-\frac{\vec{m}_2 \cdot \vec{m}_1}{r^3} \right]
$$

E is therefore minimal when $\vec{m}_1 \cdot \vec{m}_2$ is large and negative, i.e. when $\underline{\vec{m}_1 \text{ and } \vec{m}_2 \text{ are antiparallel}}$.

2. $\vec{r}$ is is the vector connecting the centers of dipole $\vec{m}_1$ and $\vec{m}_2$. $\vec{r}, \vec{m}_1$ and $\vec{m}_2$ are in the same plane. ψ_1 is the angle between $\vec{m}_1$ and $\vec{r}$, ψ_2 is the angle between $\vec{m}_2$ and $\vec{r}$.

In this case, the potential energy is

$$
\begin{aligned}
E &= -\frac{\mu_0}{4\pi}\left[\frac{3m^2r^2\cos\psi_2\cos\psi_1}{r^5} - \frac{m^2\cos(\psi_2 - \psi_1)}{r^3}\right] \\
&= -\frac{\mu_0}{4\pi}m^2\left[3\cos\psi_2\cos\psi_1 - \cos(\psi_2 - \psi_1)\right] \\
&= -\frac{\mu_0}{4\pi}m^2\left[\frac{1}{2}\cos(\psi_2 - \psi_1) - \frac{3}{2}\cos(\psi_2 + \psi_1)\right]
\end{aligned}
$$

E is minimal if $\cos(\psi_2 - \psi_1) = 1$ and $\cos(\psi_2 + \psi_1) = 1$, i.e. when both dipoles are aligned ($\psi_2 = \psi_1$) with each other along $\vec{r}$ ($\psi_1 = \psi_2 = 0$).

Students should receive full credit for simply writing down the correct answers.

V-2 Faulty Cup

Certain cups used for soups have the unfortunate design flaw that they tend to tip over when filled once tilted at a small angle, causing spills and burns. Imagine that you have a cup with a base radius R_1 and top radius R_2 (with $R_1 < R_2$) and filled to a depth h. At what angle with respect to the table surface do you need to tilt the cup for it to spill? You may treat the contents of the cup as one solid rigid object and the walls of the cup to be of negligible mass.

The first step is to find the center of mass of the cup. The COM is located along the central axis (by symmetry), so one needs only to find the height only.

The mass of the cup can be computed by integrating along the z-axis.

$$M = \int_0^h \rho \pi r^2(z) dz$$

where

$$r(z) = R_1 + \frac{(R_2 - R_1)z}{h}$$

The integration yields

$$M = \int_0^h \rho \pi (R_1^2 + 2R_1(R_2 - R_1)z/h + (R_2 - R_1)^2 z^2/h^2) dz$$
$$M = h\pi\rho(R_1^2 + R_1(R_2 - R_1) + (R_2 - R_1)^2/3)$$
$$M = \frac{h\pi\rho}{3}(R_1^2 + R_1 R_2 + R_2^2)$$

To find the COM, we calculate the 1st moment

$$Q = \frac{1}{M} \int_0^h \rho \pi r^2(z) z \, dz$$
$$Q = \frac{h}{4} \frac{(R_1^2 + 2R_1 R_2 + 3R_2^2)}{R_1^2 + R_1 R_2 + R_2^2}$$

23

The above expression can be re-expressed slightly to better answer the second part of the question:

$$Q = \frac{h}{2} + \frac{h}{4} \frac{R_2^2 - R_1^2}{R_1^2 + R_1 R_2 + R_2^2}$$

If the cup tilts an angle α, its center-of-mass will pass over the pivot point when

$$\tan \alpha = \frac{R_1}{Q}$$

V-3 Discharging a Plate

A very large metal plate carries a charge of $Q = -1$ C. The work function for the metal is $\phi = 3$ eV. The plate is illuminated by a 60 Watt light source with a wavelength λ of 330 nm. How long does it take to completely discharge the plate? How does the answer change for a light source with $\lambda = 660$ nm?

---

1. The energy per photon is

$$
\begin{aligned}
E_\gamma &= \frac{hc}{\lambda} \\
&= \frac{6.6 \cdot 10^{-34} \text{ m}^2 \text{ kg/s} \cdot 3 \cdot 10^8 \text{ m/s}}{3.3 \cdot 10^{-7} \text{ m}} \\
&= 6 \cdot 10^{-19} \text{ J.}
\end{aligned}
$$

The time to discharge the plate is given by total number of electrons divided by the rate of photons:

$$
\begin{aligned}
t &= Q/e \cdot \frac{E_\gamma}{P} \\
&= \frac{1.6 \cdot 10^{19} \cdot 6 \cdot 10^{-19} \text{ J}}{60 \text{ J/s}} \\
&= 0.016 \text{ s}
\end{aligned}
$$

2. For $\lambda = 660$ nm, $E_\gamma = 3 \cdot 10^{-19}$ J. $\phi = 3$ eV $= 4.8 \cdot 10^{-19}$ J. Therefore $E_\gamma < \phi$ and the plate will not be discharged.

V-4 Mixing Gases

Consider two containers with volume V_1 and V_2, respectively. V_1 contains oxygen at pressure p_1, while V_2 contains nitrogen at pressure p_2. At constant temperature T, a valve connecting the two containers is opened

1. What is the final pressure p in the two containers?

2. What is the change in entropy once the gases are mixed completely?

3. Would the change in entropy be different if both containers were filled with nitrogen initially? Give an argument, but no calculation is necessary.

---

1. We know that initially $n_1 = \frac{p_1 V_1}{RT}$ and $n_2 = \frac{p_2 V_2}{RT}$. After mixing the gases, we have:

$$
\begin{aligned}
p(V_1 + V_2) &= \left(\frac{p_1 V_1}{RT} + \frac{p_2 V_2}{RT} \right) RT \Rightarrow \\
p &= \frac{p_1 V_1 + p_2 V_2}{V_1 + V_2}
\end{aligned}
$$

2. For the change in entropy, ΔS, we find

$$
\begin{aligned}
\Delta S &= \Delta S_1 + \Delta S_2 \\
&= n_1 R \ln \frac{V_1 + V_2}{V_1} + n_2 R \ln \frac{V_1 + V_2}{V_2} \\
&= \frac{p_1 V_1}{T} \ln \frac{V_1 + V_2}{V_1} + \frac{p_2 V_2}{T} \ln \frac{V_1 + V_2}{V_2}
\end{aligned}
$$

3. As there would be no entropy increase from mixing different gases, ΔS would be smaller than in 2.

Students should get full credit if k_B is used instead of R (atoms versus moles).