Placement Exam, Day 1, August 28 2007

You should do two of the three problems in any subject from which you wish to be excused from the corresponding graduate course. **Only turn in two problems!** If you attempt three, you must decide which two you want to submit for grading.

Use one exam book for each problem, and label it carefully with the problem designation and your name.

Remember that your goal is to demonstrate mastery of the topic, and so a carefully constructed analysis is much better than several pages of equations which may or may not be relevant.

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Electrodynamics and Special Relativity 1.

Imagine a point dipole which flashes on and off instantaneously at time $t = 0$. This can be described by a charge distribution

$$\rho(x, y, z, t) = P_0 \delta(x)\delta(y)\delta'(z)\delta(t).$$

a) (5 points) What is the corresponding current $\vec{J}(x, y, z, t)$?

b) (15 points) Use the vector potential in Lorentz gauge,

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x'}, t_{ret})}{|\vec{x} - \vec{x'}|} d^3x'$$

to find the magnetic field in the radiation zone (leading term in $|\vec{x}|$).

You might like to know that

$$\nabla\times\mathbf{A} = \hat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right] + \hat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right] + \hat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right].$$

Electrodynamics and Special Relativity 2.

Two thin lines of charge, linear density λ_0 in their rest frame, are separated by a distance d and are moving together, in the direction of their length, at a speed v, not necessarily small compared to c.

a) (10 points) What is the force per unit length between them?

b) (10 points) In their rest frame, what is the force per unit length between them?

c) (0 points) If your answers to these two questions differ, explain why.

Electrodynamics and Special Relativity 3.

A cubic cavity of edge a has walls that are very good conductors. It is excited in the mode

$$E(x, y, z, t) = E_0 \sin(\pi x/a) \sin(\pi y/a) \exp(-i\omega t)\hat{z}$$

with $H_z = 0$.

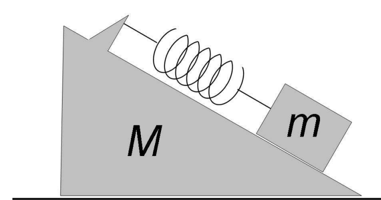
a) (6 points) Determine the other components of H.

b) (6 points) Give an expression for ω.

c) (8 points) Calculate the time-averaged force on each face.

Classical Mechanics 1.

A system consists of a wedge, a block, and a spring as sketched here. The wedge of mass M can slide along the horizontal plane while the block of mass m can slide along the wedge. Friction is negligible. The spring constant is k.



a) (5 points) Define (clearly!) convenient generalized coordinates and write the Lagrangian as a function of generalized coordinates and velocities.

b) (5 points) Write down the generalized momenta and the Lagrangian equations of motion.

c) (5 points) Write down the Hamiltonian of the system.

d) (5 points) Write down the conserved quantities (first integrals) of the system and identify the symmetries responsible for the conservation laws.

Classical Mechanics 2.

A uniform rope is rotating with angular frequency ω around the vertical axis z with one end attached to the axis of rotation. (Imagine that it is hung from the center of a ceiling fan.) The length of the rope is l. The problem is to find an equilibrium shape of the rope.

a) (5 points) Assume that an equilibrium shape is planar (the rope is confined to some vertical plane at any instant). Define the shape of the rope by function $r(z)$, where r is the distance from the z axis to the point of the rope at the height z. Write down the total energy (kinetic and potential) as a function of its shape.

b) (5 points) Write down the constraint (the total length of the rope is l) in terms of the shape of the rope.

c) (5 points) Write down the functional to be minimized by the equilibrium shape of the rope (total energy + Lagrange multiplier term corresponding to the constraint).

d) (5 points) Write down the Euler-Lagrange equations following from this variational problem. (Not necessary to provide a solution.)

Classical Mechanics 3.

A spool is made of two uniform disks, each of mass M and radius R, joined rigidly by a uniform hollow cylinder of mass m and radius $r < R$. The axis of the cylinder passes through the center of and perpendicular to the plane of each disk. A thread having negligible mass and thickness is wound around the central cylinder. The spool rests on a horizontal table and the free end of the thread is pulled by a force T acting perpendicular to the axis of the spool and at a fixed angle θ above the horizontal.

a) (5 points) Find the condition on T for the spool not to move vertically.

b) (5 points) Find the moment of inertia of the spool about its axis.

c) (10 points) Assuming that the spool rolls along the table without slipping, show that the thread may either wind onto, or unwind from, the spool, and find the conditions on θ for each to occur.

Placement Exam, Day 2, August 29 2007

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Quantum Mechanics 1.

Consider the quantum scattering by a repulsive sphere of radius a, $V(r) = \infty$ for $r < a$ and zero otherwise.

a) (10 points) Derive an expression for the S-wave phase shift.

b) (10 points) What is the total cross section σ_{TOT} in the low energy limit?

Quantum Mechanics 2.

The Hamiltonian in the tight binding approximation for a periodic chain of length Na is

$$H|n>= E_0|n> -\Delta|n-1> -\Delta|n+1> \tag{1}$$

with $|n>$ the normalized and periodic site states satisfying $|n+N>=|n>$.

a) (5 points) Identify the discrete symmetry of the chain and its properties.

b) (15 points) Use this symmetry to find the spectrum of H.

Quantum Mechanics 3.

A particle of mass m in two dimensions is confined by an isotropic harmonic oscillator of frequency ω, while subject to a weak and anisotropic perturbation of strength $\alpha \ll 1$. The total Hamiltonian describing the motion of this particle is

$$H = H_0 + V = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2) + \alpha\, m\omega^2\, xy \tag{2}$$

a) (5 points) What are the energies and degeneracies of the three lowest-lying unperturbed states?

b) (5 points) Use perturbation theory to correct the energies to first order in α.

c) (5 points) Find the exact spectrum of H.

d) (5 points) Check that the perturbative results in part b are recovered.

Statistical Physics and Thermodynamics 1.

Consider a two dimensional gas of nonrelativistic particles of spin S, density n (particles per unit area). Write the density of states $\partial n/\partial E$. (5 points)

Next, calculate the chemical potential of an ideal 2D gas of spin-0 Bose particles as a function of its density n, and find out whether such gas can Bose-condense at low temperatures. (15 points)

You may need the following integral:

$$\int_{a>0}^{\infty} \frac{dz}{\exp(z)-1} = \ln\frac{1}{1-\exp(-a)}$$

Statistical Physics and Thermodynamics 2.

A round cylinder of radius R, containing an ideal classical gas in thermal equilibrium at temperature T, is rotated about its symmetry axis with angular velocity ω (see sketch below). The gas pressure at the cylinder axis is P_a. Find the pressure P_r near the cylinder wall.

Statistical Physics and Thermodynamics 3.

Consider a one-dimensional, classical gas of N hard rots, each of length a. The system has length L and periodic boundary conditions (*i.e.*, confined to a ring of circumference L.) In this case, the "pressure" is actually a force F.

a) (5 points) Compute the second virial coefficient $B = -\frac{1}{2}\int[\exp(-V(r)/kT) - 1]dr$.

b) (10 points) Determine the exact equation of state.

c) (5 points) Compare the results of parts a and b by expanding the equation of state in density: *viz.* $FL = NkT(1 + B/L + C/L^2 + ...)$