MASSACHUSETTS INSTITUTE OF TECHNOLOGY

DEPARTMENT OF PHYSICS

FEBRUARY 6, 1996

DOCTORAL GENERAL EXAMINATION

PART I

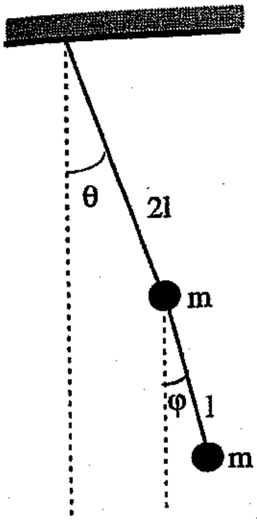
FIVE HOURS

INSTRUCTIONS

1. This examination is divided into four sections, each containing four problems worth 10 points each. Answer ALL of the problems. Although the problems in each section tend to concentrate on one area of physics, this does not mean that the subject matter of any problem will fit entirely, or even mostly, into that one area.
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution. A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a quantitative response.
3. Calculators may be used.
4. No Books or Reference Materials May Be Used.

SECTION I

Problem #1 (Mechanics)

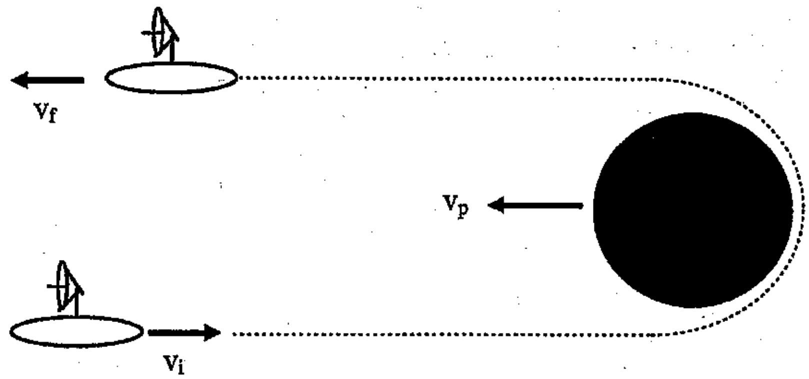


Consider the motion of a double pendulum consisting of two identical masses and two lengths of inextensible string. The top string has a length $2l$ and the bottom string has a length l. Assume that the motion of the masses stays in a plane.

a) For small oscillations, determine the normal mode frequencies.

b) Determine the relative amplitudes of the motions in θ and φ for each of the normal modes.

Problem #2 (Mechanics)

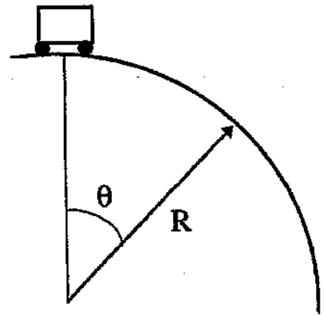


The figure above shows a space probe moving towards the planet Jupiter. The magnitudes of the initial velocities of the probe and the planet are v_i and v_p, respectively. These velocities are measured with respect to the sun. What is the magnitude of the final velocity of the probe after has gone around the planet as shown above? Ignore relativistic effects and neglect any effects from the sun.

SECTION I

Problem #3 (Mechanics)

A cart on a frictionless rail at height 2R is let go along a track which leads to a circular loop of radius R. At what angle θ from the top of the loop does the cart leave the trail?



SECTION I

Problem #4 (Mechanics)

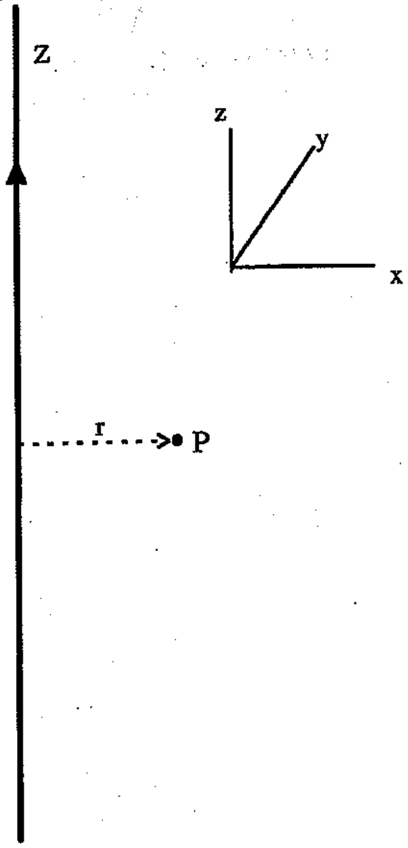
A linear molecule consists of two atoms of masses m_1 and m_2 which are interacting through the Lennard-Jones type potential:

$$U(r) = U_0\left[\left(\frac{r_0}{r}\right)^{a} - \left(\frac{r_0}{r}\right)^{2a}\right]$$

where $a>0$, and U_0 and r_0 are constants. Near the potential minimum, the molecule vibrates. Find the frequency of oscillation along the molecular axis when it can be approximated as a harmonic approximation.

SECTION II

Problem #1 (E&M)



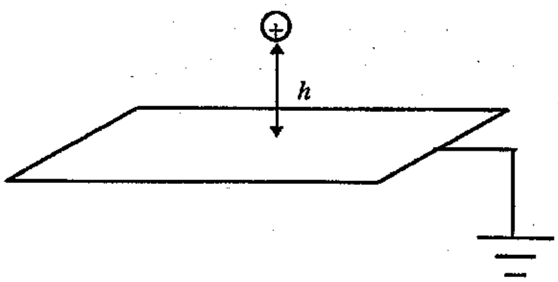
An infinite straight wire carries the current:

$$I(t) = \begin{cases} 0, & \text{for } t \le 0 \\ I_0, & \text{for } t > 0 \end{cases}$$

Write down complete expressions for the electric scalar potential and the magnetic vector potential at point P (in the x-z plane) a distance r away from the point z=0 along the wire for all times $t>0$. Do not solve any integrals.

Given these potentials, describe in words how you would find the resulting electric and magnetic fields at the point P. What are the directions of the electric and magnetic fields?

Problem #2 (E&M)



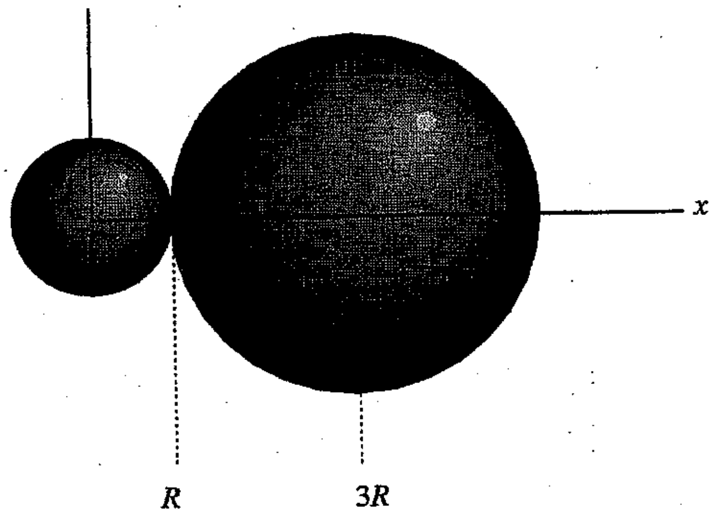
In the figure at left, a point charge q is fixed at a height h above an infinite grounded conducting plane. A field line emanates horizontally from the point charge. How far away (laterally) from the position of the point charge does the field line strike the plane?

SECTION II

Problem #3 (E&M)

Positive charge Q is distributed uniformly over the volumes of two spheres, one of radius R and the other of radius $2R$. The smaller sphere is centered at the origin and the other at x=$3R$. Find the magnitude and direction of the electric field at the following locations along the x axis:

a) x=0
b) x=R/2.

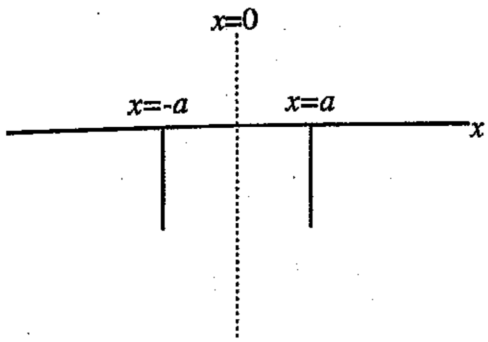


SECTION II

Problem #4 (E&M)

A particle with charge q is moving with a constant speed v_0 along the positive x direction. At time t=0, the particle is at the origin. Find the displacement current density at position $\vec{r}$. Assume v<<0, i.e., non-relativistic motion.

Problem #1 (Quantum)



A particle of mass m is in a potential $V(x) = \eta\delta(x-a) + \eta\delta(x+a)$, where $\eta<0$.

a) What is the wave function for the ground state? Do not normalize, but do determine relations between the coefficients of the wavefunctions for each position.
b) Derive an equation for the energy of the state. Do not solve the equation but show how you could graphically determine a solution.
c) Is this energy greater or less than what you would expect for a single delta function well?

SECTION III

Problem #2 (Quantum)

Consider the probability current density, $\vec{S}$, which obeys the continuity equation $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{S} = 0$, where ρ is the probability density. Determine the probability current density for all positions x and all times t in the following two cases.

a) An electron in the first excited state of an infinite square well of width d centered at x=d/2.

b) An electron in a mixed state of the same well. At $t=0$ this state is given by by $\psi(x) = 1/\sqrt{2}\,(\psi_0 + \psi_1)$, where $\psi_0 = \sqrt{2/d}\,(\sin(\frac{\pi}{d}x))$ and $\psi_1 = \sqrt{2/d}\,(\sin(\frac{2\pi}{d}x))$.

SECTION III

Problem #3 (Quantum)

A one-dimensional harmonic oscillator (mass m, classical frequency ω) in the first excited state is subject to the perturbation

$$H' = bxF(t), \qquad F(t) = \begin{cases} 0, & t < 0, t > T \\ 1, & 0 < t < T. \end{cases}$$

Here x is the position of the oscillator and b is a constant. Use first-order perturbation theory to find the probability that the oscillator will be in the ground state for $t>T$.

HINT: recall that $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)$

SECTION III

Problem #4 (Quantum Mechanics)

Consider a particle in a three dimensional square well potential of depth V_o and radius a. What is the necessary condition that at least one bound state exists when the particle has no angular momentum?

SECTION IV

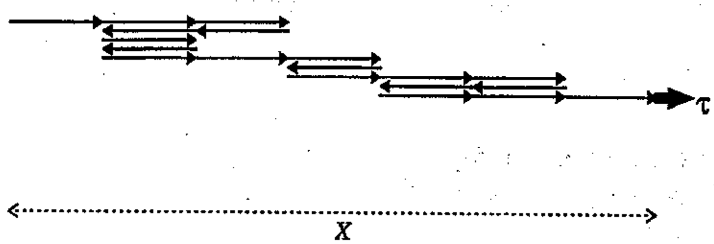
Problem #1 (Statistical Mechanics)

Consider a two-dimensional gas of N non-interacting electrons (spin ½) in a box of area A.

a) What is the density of single particle states in $\vec{k}$ space?
b) Calculate the density of single particle states in energy, $D(\varepsilon)$.
c) Find the Fermi wavevector k_F and the Fermi energy, $\mu(T=0)$.
d) Find the internal energy, E, at T=0 as a function of N and A.
e) Find the surface tension, γ, at T=0 as a function of N and A. (Note that $dE=TdS+\gamma dA+\mu dN$).
f) In three dimensions, $\mu(T)-\mu(0)$ is proportional to T^2. Will the temperature dependence be stronger, weaker, or the same in two dimensions. Explain.

SECTION IV

Problem #2 (Statistical Mechanics)



A one dimensional chain is made of N identical elements, each of length l. The angle between successive elements can be either 0° or 180°, but there is no difference in internal energy between these two possibilities. For the sake of counting, one can think of each element as either pointing to the right (+) or to the left (-). Then one has $N=n_{+}+n_{-}$ and $X=l(n_{+}-n_{-})=l(2n_{+}-N)$. The internal energy therefore *does not* depend on X. However, the tension τ in the chain is finite and arises from statistical effects alone. Use the microcanonical ensemble to find the following quantities.

HINT: Recall the approximation $\ln(N!) \approx N\ln(N)-N$

a) The entropy S as a function of N and n_{+}.
b) The Helmholtz free energy F as a function of N and n_{+}.
c) The tension τ as a function of T, N, and X.

Problem #3 (Statistical Mechanics)

Consider an isomerization process

$$A \leftrightarrow B$$

where A and B refer to two isomeric states of a molecule with degeneracy factors g_a and g_b, respectively. State A is lower in energy than state B by an amount $\Delta\varepsilon$.

a) Assuming the molecules to be in the gas phase, write the chemical potentials for the molecules in state A and the molecules in state B.

b) By setting the chemical potentials equal, find the ratio of the number of molecules in state A to the number in state B.

SECTION IV

Problem #4 (Statistical Mechanics)

Consider the random walk of a particle in one dimension, starting at x=0. The probablility of one step of length s is given by

$$w(s) \cdot ds = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(s-l)^2}{2\sigma^2} \right) \cdot ds$$

Find the probability $p(x)$ that the particle, after N steps (where N is very large), is in the interval between x and $x+dx$.