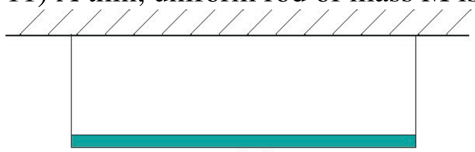
Fall 2007 Qualifier – Part II **12 minute questions**

11) A thin, uniform rod of mass M is supported by two vertical strings, as shown below.



Find the tension in the remaining string immediately after one of the strings is severed.

12) A small object moving on a circular orbit around a star suddenly explodes into two identical pieces. Right after the explosion, as observed from the star, the pieces move off with equal speeds, and the angle between their directions of motion is θ.
Are the trajectories of the pieces ellipses, parabolas, or hyperbolas

a) when $\theta = 60^\circ$;
b) when $\theta = 90^\circ$?

Give a quantitative explanation for your answers.

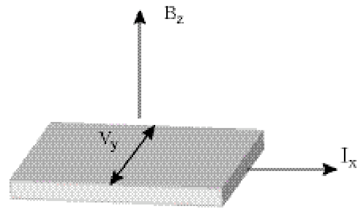
13) A classical gas in d dimensions has free energy $F = -cVT^{d+1}$ at high temperatures. Here, c is some positive numerical constant of order 1.

a) Show that this implies a relation between energy density $e = E/V$ and the entropy density $s = S/V$ of the form $e = c's^{\alpha}$, where c' is also a numerical constant, and determine α in terms of d.
b) The specific heat is defined by $C_V = T\left.\frac{dS}{dT}\right|_V$. Given the equation of state from part a), what is the range of values of α for which the specific heat is negative?
c) Is there a dimension d for which the specific heat of a classical gas at high temperatures becomes negative?

14) Consider two systems A and B with density of states $\Omega_A(E) = \alpha E^3$ and $\Omega_B(E) = \beta E^4$. These two systems are put into contact with each other so that they can exchange heat, and they come to thermal equilibrium. The total energy of the combined system is E_0.

a) How is the energy distributed between these systems in equilibrium (i.e. what is the energy of the system A, and what is the energy of system B, in terms of the total energy E_0?
b) What is the equilibrium temperature (in terms of the total energy E_0)?

15) A piece of p-doped silicon has a carrier density $n = 10^{15}\,\text{cm}^{-3}$ and the dimensions $\Delta x = 10$ mm, $\Delta y = 2$ mm, $\Delta z = 1$ mm. The magnetic field $B_z = 1\text{T}$ is applied in the z-direction, the current $I_x = 1\text{A}$ flows in the x-direction, and the voltage V_y is measured.



a) Write down the equilibrium force condition that determines V_y.
b) Express the current density j_x in terms of the carrier density n and the carrier velocity v_x.
c) Find V_y in volts.

16) Suppose a grounded conducting sphere of radius a is immersed in a uniform electric field $\vec{E} = E_0\hat{z}$. The resulting electric potential is $V(r,\theta) = -E_0 r\left[1-\left(\frac{a}{r}\right)^3\right]\cos\theta$ in spherical coordinates. Find the surface charge distribution on the sphere as a function of θ.

17) The radial momentum in spherical coordinates can be represented by the Hermitian operator $p_R = \frac{\hbar}{i}\frac{1}{r}\frac{\partial}{\partial r}r$.

a) Find a solution $\psi(r)$ of the eigenvalue problem $p_R\psi(r) = \hbar k\psi(r)$.
b) Give a physical interpretation of this wavefunction.

18) The Hamiltonian for a three level quantum system is

$$\hat{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$$

where a, b, and c are real numbers. You may assume that there are no degenerate states.

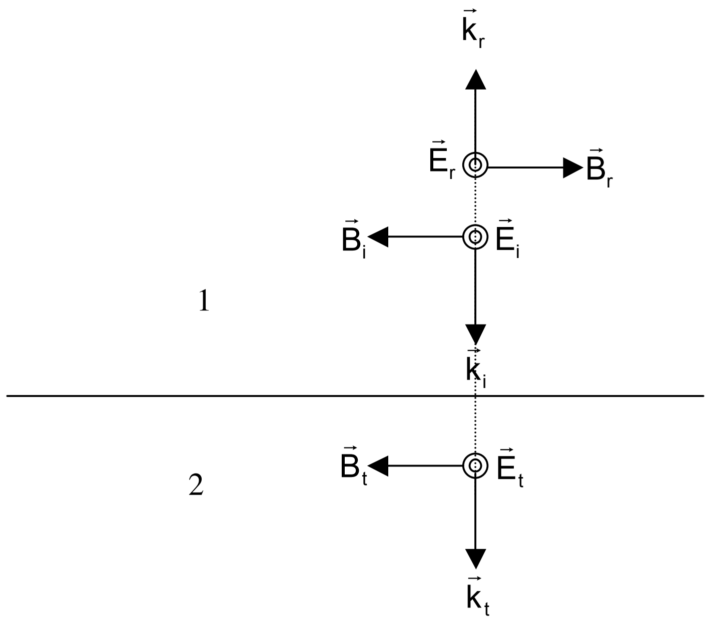
a. Find the eigenvalues and eigenvectors of $\hat{H}$.
b. The system is prepared in the state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Find the column vector $|\psi(t)\rangle$.

19) A pion of mass m_π decays into an electron and an antineutrino with masses m_e, and m_ν. Find the velocity of the antineutrino in the rest frame of the electron in terms of the masses of the three particles. Interpret your answer in the case where m_ν=0.

20) Consider an electromagnetic plane wave at normal incidence to an interface between two media. The polarizations of the incident (i), reflected (r), and transmitted (t) waves are shown below. The electric permittivity in the two regions is ε_1 and ε_2 , and the magnetic permeability is μ_1 and μ_2.



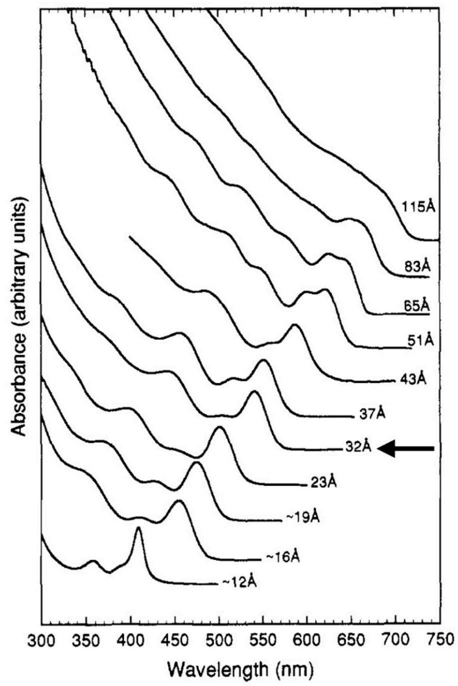
a. Write down an equation relating the amplitudes of the electric fields, E_{0i}, E_{0r} and E_{0t} at the boundary between the two media.
b. Write down an equation relating the amplitudes of the magnetic fields, B_{0i}, B_{0r} and B_{0t} at the boundary between the two media.
c. Write down equations relating the following pairs of quantities:
   i. B_{0i} and E_{0i}
   ii. B_{0r} and E_{0r}
   iii. B_{0t} and E_{0t}.

The relations in a, b, and c can be used to derive the amplitude reflection coefficient for a wave traveling from medium *1* into medium *2*:

$$\frac{E_{0r}}{E_{0i}} = \frac{n_1/\mu_1 - n_2/\mu_2}{n_1/\mu_1 + n_2/\mu_2}$$

d. Ferrites can be designed to have their relative electric permittivity equal to their relative magnetic permeability: $\varepsilon/\varepsilon_0=\mu/\mu_0$. Determine the value of the amplitude reflection coefficient for a wave traveling from vacuum into one of these materials.

21) The band gap E_g in semiconductor nanoparticles increases as the particle dimension d becomes smaller. The band gap can be measured using optical absorption spectroscopy with light of sufficient energy. The figure shows a series of optical absorption spectra for a series of CdSe nanoparticle samples with different particle diameters.



a) For the absorption spectrum indicated by the arrow, estimate the wavelength, λ_t, at the absorption threshold and from this the band gap in eV.

b) A simple model for this effect relates E_g to d, the bulk band gap, E_g^o, and the effective masses (m_e^* and m_h^*) of the electrons and holes:

$$E_g(\mathrm{eV}) = E_g^0(\mathrm{eV}) + \frac{2\hbar^2\pi^2}{ed^2}\left[\frac{1}{m_e^*} + \frac{1}{m_h^*}\right],$$ where S.I. units are used for all values except the band gap. For bulk CdSe, $E_g^o = 1.74\ \mathrm{eV}$, $m_e^* = 0.13\ m_e$ and $m_h^* = 0.3\ m_e$, where the free electron mass. Calculate the predicted size of the nanoparticle sample.

c) Assuming a 5% error on your estimate of λ_t, and no errors on the other parameters, what is the error on the calculated particle size? Is the model in agreement with experiment?

22) A non-magnetic medium has a dielectric constant that depends upon frequency ω and is given by $\varepsilon = \varepsilon_0\left(1 - A/\omega^2\right)$, where A is a constant and ε_0 is the permittivity of free space.

a) Write down the dispersion relation for an electromagnetic wave, which expresses ω as a function of wave number k.
b) Find the critical frequency below which a wave launched into the medium will not propagate through the medium, but rather will be evanescent or decaying.
c) Calculate the propagation distance over which the amplitude of the wave decays by a factor of *e*.

23) The total number of conduction electrons per unit length in a quantum wire is 5×10^9 /m. Assume that the electrons move freely in one dimension in a wire of length *L*. What is the Fermi energy of the wire in electron volts (eV)? (Hint: Recall that the wavelength must satisfy $n\lambda$ = L, where λ is the wavelength and *n* is an integer.)

24)

a) Give the definition of a semiconductor. How does a semiconductor differ from a metal?
b) What is the mechanism for the electrical conductivity of an intrinsic (undoped) semiconductor at low temperatures, $k_B T << \Delta$, where Δ is the energy gap between the valence band and the conduction band?
c) Give a physical argument to explain the approximate temperature dependence of the conductivity when $k_B T << \Delta$.

25) The Sun converts protons into He through a series of reactions amounting to $4p \rightarrow {}^4He + 2e^+ + 2v_e$. The solar constant describing the power of the solar radiation at Earth is P = 1400 W/m^2. The energy released per reaction corresponds to the binding energy of He (28.3 MeV)

a) What is the neutrino flux at Earth? In other words, how many neutrinos arrive at the Earth's surface (m^{-2} sec^{-1})?
b) Give a rough estimate of how the power of the neutrino flux compares to the solar optical radiation power. Assume that the average energy of a neutrino is 0.3 MeV.