STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

**Graduate Placement Exam Part 2, August 26, 2009**

**General Instructions:** This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas.

Each solution should typically take on the order of 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

**Quantum Mechanics 1**

The Hamiltonian of the plane rotator is given by

$$H = -\frac{\hbar^2}{2I}\frac{\partial^2}{\partial \phi^2},$$

where $\phi$ is an angular coordinate and $I$ is the moment of inertia of the rotator. Initially, the rotator is in the state given by the wave function

$$\Psi(\phi, t = 0) = A \sin^2 \phi.$$

 a) Find the time dependence of the wave function $\Psi(\phi, t)$.

 b) In what time the rotator comes back to its initial state?

Solution

Eigenfunctions are $e^{in\phi}$ with eigenenergies $E_n = \frac{\hbar^2}{2I}n^2$. The initial state is $\Psi(\phi, t = 0) = \frac{A}{4}\left(2 - e^{2i\phi} - e^{-2i\phi}\right)$. Therefore,
a)

$$\Psi(\phi, t) = \frac{A}{2}\left[1 - e^{-2i\frac{\hbar t}{I}}\cos(2\phi)\right].$$

b) The rotator comes back to an initial state in time

$$T = \pi I/\hbar.$$

## Quantum Mechanics 2

A one-dimensional particle is moving inside an infinitely deep potential well of the width $a$ $(0 < x < a)$ in the presence of an additional potential $V(x) = V_0 \cos^2 \frac{\pi x}{a}$. Consider the additional potential as a perturbation.

a) Find the energy spectrum of the system in the first order of perturbation theory.

b) Find the energy spectrum of the system in the second order of perturbation theory.

c) What is the first order correction to the wave function of the ground state?

### Solution

a) The normalized eigenfunctions and eigenenergies are given by

$$\psi_n^{(0)}(x) = \sqrt{\frac{2}{a}} \sin \frac{n+1}{a} \pi x, \qquad E_n^{(0)} = \frac{\hbar^2}{2ma^2} \pi^2 (n+1)^2$$

with $n = 0, 1, 2, \ldots$. Matrix elements of the perturbation are easy to calculate

$$V_{nm} = \begin{cases} V_0/4, & n = m = 0 \\ V_0/2, & n = m \neq 0 \\ V_0/4, & n = m \pm 2 \\ 0 & \text{in other cases.} \end{cases}$$

In the first order of perturbation theory the energies shift by $E_n^{(1)} = V_{nn}$, i.e.,

$$
\begin{aligned}
E_0^{(1)} &= V_0/4, \\
E_n^{(1)} &= V_0/2, \qquad \text{for } n > 1.
\end{aligned}
$$

b) For the shifts of energy levels in the second order of perturbation theory we get

$$E_n^{(2)} = \sum_m{}' \frac{|V_{nm}|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{ma^2 V_0^2}{96\pi^2 \hbar^2} \cdot \begin{cases} -\frac{3}{2}, & n = 0, \\ -1, & n = 1, \\ \frac{6}{n(n+2)}, & n \geq 2. \end{cases} \qquad (1)$$

The perturbation theory is applicable if $|V_0| \ll \frac{\hbar^2 \pi^2}{ma^2}(n+1)$.

c) The first order correction to the wave function is given by

$$\psi_0^{(1)} = \sum_{n>0} \frac{V_{n0}}{E_0^{(0)} - E_n^{(0)}} \psi_n^{(0)} = \frac{V_0/4}{\frac{\hbar^2}{2ma^2}\pi^2(1 - 3^2)} \psi_2^{(0)} = -\frac{1}{16\pi^2} \frac{V_0}{\hbar^2/(ma^2)} \psi_2^{(0)}.$$

**Quantum Mechanics 3**

A particular system consists of three very weakly interacting subsystems. Classify all possible states of the system with respect to its full angular momentum $L$ knowing that the angular momenta of three subsystems are $l_1 = l_2 = 1$ and $l_3 = l \geq 2$ respectively. Namely, list all possible values of $L$ and corresponding degeneracies of the states.

<div align="center">Solution</div>

There are altogether $(2l_1 + 1)(2l_2 + 1)(2l_3 + 1) = 9(2l + 1)$ different states of the system. Adding angular momentum $l$ and 1 we obtain the direct sum of states with total momenta $l - 1$, $l$ and $l + 1$. Adding one more angular momentum 1 we get $(l - 2) \oplus 2 \cdot (l - 1) \oplus 3 \cdot l \oplus 2 \cdot (l + 1) \oplus (l + 2)$. Taking into account different $L_z$ we obtain that all states are divided into groups

$$
\begin{array}{ll}
(2l - 3) & \text{states with } L = l - 2, \\
2(2l - 1) & \text{states with } L = l - 1, \\
3(2l + 1) & \text{states with } L = l, \\
2(2l + 3) & \text{states with } L = l + 1, \\
(2l + 5) & \text{states with } L = l + 2.
\end{array}
$$

# Statistical Mechanics Placement.  Aug 26, 2009

**Problem 1**. Calculate the chemical potential of an ideal 2D gas of spin-0 Bose particles as a function of its density *n* (per unit area), and find out whether such gas can Bose-condense at low temperatures.

*Hint*: you may need the following integral:

$$\int_{a>0}^{\infty} \frac{dz}{\exp\{z\}-1} = \ln\frac{1}{1-\exp\{-a\}}.$$

*Solution*: The quantum density of 2D states $\nu \equiv \partial n/\partial E$ is independent of the particle energy $\varepsilon$:

$$\nu = \frac{gm}{2\pi\hbar^2},$$

where $g = 2s + 1$ is the spin degeneracy, in our current case equal to 1. Hence the gas density may be calculated as

$$n = \nu\int_0^{\infty} \frac{d\varepsilon}{\exp\left\{\dfrac{\varepsilon-\mu}{T}\right\}-1}.$$

Using the integral cited in the *Hint*, we get

$$n = \nu T \ln\frac{1}{1-\exp\{-\mu/T\}},$$

so that the chemical potential

$$\mu = -T \ln\left(1-\exp\left\{-\frac{n}{\nu T}\right\}\right).$$

Since the exponent in this expression is between 0 and 1 for any gas density *n* and any temperature $T > 0$, the argument of the logarithm is below 1, and the chemical potential is positive. Hence the Bose-Einstein condensation in 2 dimensions (which requires the gas to have $\mu = 0$) is impossible.

This might be evident already from the very beginning, because at $\mu = 0$ the integral for *n* diverges at its lower limit. This does not happen in three dimensions, where the different dependence of the density of states on the particle energy provides an extra factor of $\varepsilon^{1/2}$ under the integral, preventing its divergence at $\mu = 0$, and making the 3D BEC possible.

**Problem 2**. For a two-site Ising system with energy

$$E = -Js_1 s_2 - h(s_1 + s_2),$$

in thermal equilibrium, find:

    (a) the low-field susceptibility $\chi \equiv \partial\langle s\rangle/\partial h\,|_{h\to 0}$ and

    (b) the variance of energy fluctuations at $h = 0$.

Explore the low-temperature and high-temperature limits of the results, and give physical interpretations of the asymptotic behaviors.

*Solution*: The system has $2^2 = 4$ possible states corresponding to all combinations of its two "spins" $s_j = \pm 1$, with energies

$$E_n = \begin{cases} -J - 2h, & \uparrow\uparrow \\ +J, & \uparrow\downarrow \\ +J, & \downarrow\uparrow \\ -J + 2h. & \downarrow\downarrow \end{cases}$$

(Note the double degeneracy of one of the energy levels, and a similar degeneracy of another couple of levels at $h = 0$.) The resulting statistical sum is

$$Z = \exp\{(J + 2h)/T\} + 2\exp\{-J/T\} + \exp\{(J - 2h)/T\},$$

so that in weak fields, $|h| \ll J$, with the accuracy of $O(h)$,

$$Z \approx 4\cosh(J/T).$$

This accuracy is sufficient to calculate $\langle s\rangle$ (per unit particle) in the first approximation in $h$:

$$\langle s\rangle = \frac{1}{Z}\sum_n \left(\frac{s_1 + s_2}{2}\right)_n \exp\left\{-\frac{E_n}{T}\right\} \approx \frac{1}{4\cosh(J/T)}\left[\exp\left\{\frac{J + 2h}{T}\right\} - \exp\left\{\frac{J - 2h}{T}\right\}\right]$$

$$= \frac{\exp\{J/T\}}{2\cosh(J/T)}\sinh(2h/T) \approx \frac{h}{T}\frac{\exp\{J/T\}}{\cosh(J/T)} = \frac{2h}{T}\frac{1}{1 + \exp\{-2J/T\}}.$$

As a result, the low-field susceptibility

$$\chi \equiv \frac{\partial\langle s\rangle}{\partial h} = \frac{2}{T}\frac{1}{1 + \exp\{-2J/T\}} \to \begin{cases} 2/T, & \text{for } T \ll J, \\ 1/T, & \text{for } T \gg J. \end{cases}$$

The latter limit corresponds to our class result for the paramagnetism of independent spins, because strong thermal fluctuations destroy their correlation. On the other hand, at low temperatures the spins are always aligned. Since the total paramagnetic moment of a system of $N$ aligned spins is proportional to $N^2$, the susceptibility per particle scales as $N$, giving in our case the extra factor of 2.

    The average energy of the system at $h = 0$ is

$$\langle E \rangle = \frac{1}{Z} \sum_n E_n \exp\left\{-\frac{E_n}{T}\right\} = \frac{1}{4\cosh(J/T)}\left[2J\exp\left\{-\frac{J}{T}\right\} - 2J\exp\left\{\frac{J}{T}\right\}\right] = -J\tanh\frac{J}{T},$$

while its variance

$$\langle E^2 \rangle = \frac{1}{Z} \sum_n E_n^2 \exp\left\{-\frac{E_n}{T}\right\} = \frac{1}{4\cosh(J/T)}\left[2J^2\exp\left\{-\frac{J}{T}\right\} + 2J^2\exp\left\{\frac{J}{T}\right\}\right] = J^2,$$

so that

$$\langle \tilde{E}^2 \rangle = \langle E^2 \rangle - \langle E \rangle^2 = J^2\left[1 - \tanh^2\frac{J}{T}\right] = \frac{J^2}{\cosh^2(J/T)} \to J^2 \times \begin{cases} 4\exp(-2J/T), & \text{for } T \ll J, \\ 1, & \text{for } T \gg J. \end{cases}$$

At low temperature the fluctuations are exponentially small, due to the gap ($\Delta = 2J$) in the energy spectrum of the system, while at high temperatures the variance reaches is maximum (equal to the half-gap squared).

**Problem 3.** Calculate the low-frequency spectral densities of fluctuations of

     (a) coordinate $x$,

     (b) momentum $p$, and

     (c) energy $E$

of a weakly damped, 1D harmonic oscillator at thermal equilibrium with temperature $T \ll \hbar\omega_0$. Compare their dependences on temperature, damping $\eta$, and observation frequency $\omega$.

*Hint*: At $T \ll \hbar\omega_0$, the master-equation analysis of the system may be limited to its two lowest energy levels.

*Solution*:

     (a) At low frequencies $\omega \ll \omega_0$ (where $\omega_0 = (k/m)^{1/2}$ is the eigenfrequency of the oscillator), the fluctuation-dissipation theorem gives

$$S_x(\omega) = \frac{S_f(\omega)}{k^2} \approx \frac{\hbar\eta\omega}{2\pi k^2}\coth\frac{\hbar\omega}{2T},$$

where $k$ and $\eta$ are the coefficients in the Langevin-Heisenberg-Lax equation of motion

$$m\ddot{x} + \eta\dot{x} + kx = f(t). \qquad\qquad (*)$$

Note that even if $T \ll \hbar\omega_0$, the temperature dependence of $S_x(\omega)$ can still be significant, and disappears only if temperature is much lower than $\hbar\omega$. (It is crucial here to

distinguish the *observation* frequency $\omega$ from the *resonant* frequency $\omega_0$ of the oscillator.)

(b) Since $p \equiv m\dot{x}$, Fourier amplitudes of these variables are related as $p_\omega = -im\omega x_\omega$, and their spectral densities as $S_p(\omega) = m^2\omega^2 S_x(\omega)$, so that from our above result for $S_x(\omega)$ we readily get

$$S_p(\omega) = \frac{m^2\omega^2 S_f(\omega)}{k^2} \approx \frac{\hbar\eta m^2\omega^3}{2\pi k^2}\coth\frac{\hbar\omega}{2T}.$$

(c) The spectral density of fluctuations of energy cannot be found from the fluctuation-dissipation theorem, but may be recovered from the master-equation analysis of a two-level system:

$$S_E(\omega)\big|_{\omega\ll\Gamma} \approx \frac{\Delta^2}{\pi}\frac{\Gamma'\Gamma}{(\Gamma+\Gamma')^3}. \qquad (**)$$

For an oscillator, the distance $\Delta$ between the levels is just $\hbar\omega_0$. Rate $\Gamma$ of transfer from the excited energy state to the ground state may be expressed via the damping coefficient $\eta$ by requiring that it equals to the energy relaxation rate described by Eq. (*) with $\eta \ll m/\omega_0$:

$$\Gamma = \frac{\eta}{m}.$$

According to the Gibbs argument discussed in class, the excitation rate equals

$$\Gamma' = \Gamma\exp\left\{-\frac{\Delta}{T}\right\} = \Gamma\exp\left\{-\frac{\hbar\omega_0}{T}\right\},$$

and at $T \ll \hbar\omega_0$ is exponentially low, so that its contribution to the sum $\Gamma + \Gamma'$ is negligible. As a result, Eq. (**) yields

$$S_E(\omega) \approx \frac{(\hbar\omega_0)^2}{\pi}\frac{\Gamma'}{\Gamma^2} \approx \frac{\hbar^2 m\omega_0^2}{\pi\eta}\exp\left\{-\frac{\hbar\omega_0}{T}\right\} = \frac{\hbar^2 k}{\pi\eta}\exp\left\{-\frac{\hbar\omega_0}{T}\right\}.$$

We see that, in a sharp contrast with $S_x(\omega)$ and $S_p(\omega)$, the spectral density of *energy* fluctuations is

- exponentially small,

- virtually independent of the observation frequency, and

- *inversely* proportional to the damping coefficient.