Problem Solutions Part-I

Physics PhD Qualifying Examination
Part I – Wednesday, January 7, 2004

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
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	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism), **and with at least one problem from problems 5 or 10 (Special Relativity).**
   DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

I-1 [10]

Find the horizontal deflection from the plumb line caused by the Coriolis force acting on a particle falling freely in the Earth's gravitational field from a height h above the Earth's surface.

I-2 [2,2,6]

A simple pendulum of length l and bob with mass m is attached to a massless support moving *horizontally* with constant acceleration a in an inertial frame. The uniform gravity g is vertical. Working in the inertial frame:

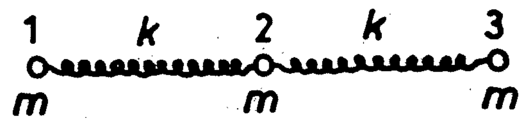
(a) Construct the Lagrangian and obtain the equation of motion.

(b) Determine the equilibrium angle ϑ_o (measured from the vertical direction) as a function of a and g.

(c) Determine the frequency of small oscillations about the equilibrium angle ϑ_o, in terms of a, g, and l.

I-3 [10]

Three particles of equal mass "m" move without friction in one dimension (see figure below). Two of the particles are each connected to the third by a massless spring of spring constant "k". Find normal modes of oscillation and their corresponding frequencies. Draw a schematic of the normal modes.



I-4. [10]

A simple pendulum of length b with bob of mass m is attached to a massless support moving horizontally with constant acceleration a. All motion is confined to the xy plane.

(a) Express the kinetic energy, potential energy, and Lagrangian using appropriate generalized coordinates. Give the corresponding equations of motion.

(b) What are the corresponding generalized momenta, Hamiltonian, and Hamilton's equations of motion?

(c) Does $H = E$? Is the total energy E conserved? Explain your answer.

I-5 [10]

A beam of light travels in the moving frame S' with velocity c in the y' direction. The S'- frame moves with a velocity V in the x-direction in the reference frame S:

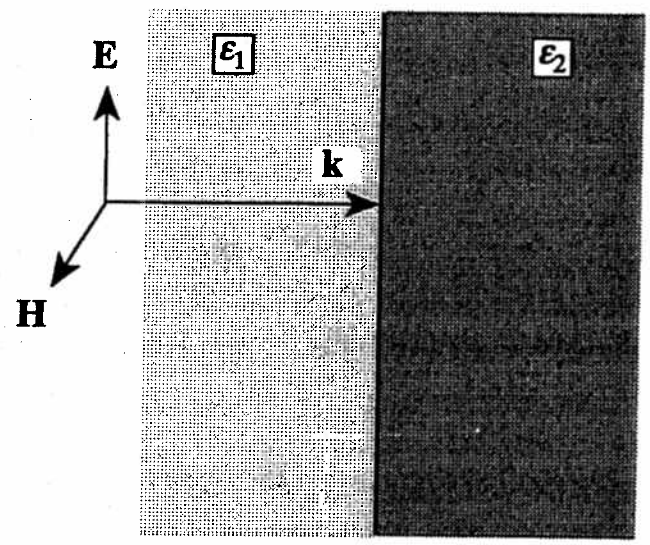
(a) Derive an expression for the velocity vector of light in the stationary frame.
(b) Determine the speed of light in the stationary frame.
(c) Determine the angle between direction of the pulse of light and the $\hat{y}$ axis in the stationary system S

I-6 [10]

Consider two very long thin-walled coaxial cylinders of radii r_A and r_B ($r_A < r_B$). The inner cylinder is kept at the potential $\Phi = V_A$, the outer cylinder is kept at $\Phi = V_B$.
Assume the space between the cylinders contains a space charge of density $\rho = kr$, where k is a constant and r is the distance from the axis. Find the potential Φ between the cylinders and find the surface charge on the outer surface of the inner cylinder and the inner surface of the outer cylinder.

I-7 [10]

A monochromatic plane wave of frequency ω propagates through a non-permeable (μ = 1) insulating medium with dielectric constant, ε_1. The wave is normally incident upon an interface with a similar medium of dielectric constant ε_2 (see Figure).

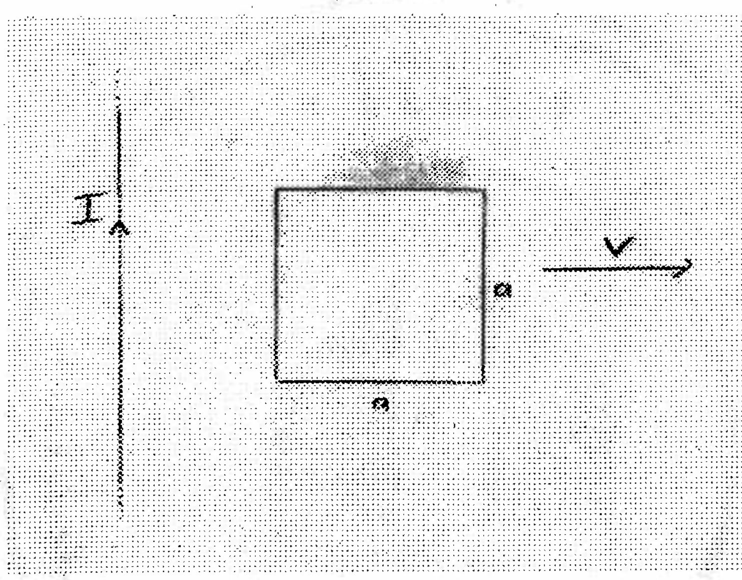


(a) State the boundary conditions and derive the expressions for the reflected and transmitted electric and magnetic fields at the interface.

(b) Find the fraction of the incident energy that is transmitted to the second medium, and the
fraction of the energy that is reflected back into the first medium from the interface.

I-8 [10]

A square-shaped loop of wire is moving perpendicularly away from an infinitely long wire carrying a current I, as shown in the figure. The loop and the infinite wire are in the same plane at all times. One side of the loop is parallel to the wire. For simplicity, assume that at $t = 0$ the (nearer) side of the loop and the infinite wire are at the same location. The side of the square-shaped loop is a. The velocity of the loop is v (constant).
Determine the induced emf (induced voltage) in the loop.



I-9. [10]

The fields at $(\mathbf{r}, t)$ due to a moving point charge are given by

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\xi|}{(\xi\cdot\mathbf{u})^3}[(c^2-|\dot{\mathbf{w}}|^2)\mathbf{u}+\xi\times(\mathbf{u}\times\ddot{\mathbf{w}})];\quad \mathbf{B}(\mathbf{r},t)=\frac{\hat{\xi}}{c}\times\mathbf{E}(\mathbf{r},t),$$

where $\xi \equiv \mathbf{r}-\mathbf{w}$, $\mathbf{u}\equiv c\hat{\xi}-\dot{\mathbf{w}}$, and the particle position $\mathbf{w}$ and its time derivatives are evaluated at a retarded time t_r satisfying $|\xi| = c(t-t_r)$.

(a) Derive the (Larmor) expression for the power emitted as radiation by an accelerating point charge.

(b) An electron is released from rest and falls under the influence of gravity. In the first centimeter, what fraction of the potential energy lost is radiated away? [q=1.60×10^{-19}C; m=9.11×10^{-31}kg; μ_0=$4\pi\times10^{-7}$N/A^2]

I-10 [10]

Consider a plane electromagnetic wave moving in the direction such that

$$\vec{E} = \vec{E}_0 \cos(kz - \omega t)\hat{x}$$

($\hat{x}$ is the unit vector in the x – direction)

(a) Obtain an expression for the magnetic field $\vec{B}$.
(b) Obtain an expression for the energy density u and the energy flux.

I-1. Solution

I-1 Rotational motion / Equation of Motion in Rotating Frame

Find the horizontal deflection from the plumb line caused by the Coriolis force acting on a particle falling freely in the Earth's gravitational field from a height h above the Earth's surface.

$$\vec{F} = \vec{S} + m\vec{g} - 2m\,\vec{\omega}\times\vec{v}_r$$

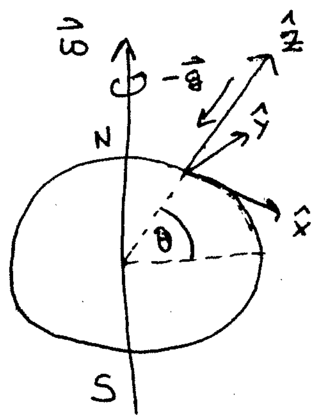
motion of object close to surface of earth

$\vec{S} = 0$ applied forces

$$\vec{F} = m\vec{a}_r$$

$$\vec{a}_r = \vec{g} - 2\vec{\omega}\times\vec{v}_r$$

acceleration of particle in the rotating coordinate system fixed on earth



$$F_G = \gamma \frac{M_E m}{(R_E + r)^2} = mg \qquad h \ll R_E$$

$$\omega_x = -\omega\cos\theta \qquad v_x = 0 \qquad g_x = 0$$

$$\omega_y = 0 \qquad v_y = 0 \qquad g_y = 0$$

$$\omega_z = \omega\sin\theta \qquad v_z = -gt \qquad g_z = -g$$

$$\vec{\omega}\times\vec{v}_r = -\omega g t\cos\theta\,\hat{y}$$

$$\vec{a}_r = \vec{g} - 2\vec{\omega}\times\vec{v}_r$$

$$a_{rx} = \ddot{x} = 0$$

$$a_{ry} = \ddot{y} = 2\omega g t\cos\theta \Rightarrow y(t) = \frac{1}{3}\omega g t^3\cos\theta$$

$$y(t=0) = 0 \qquad \dot{y}(t=0) = 0$$

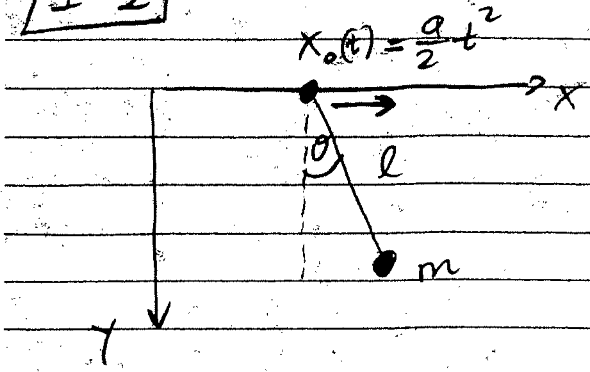
$$a_{rz} = \ddot{z} = -g$$

$$\to z(t) = h - \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2h}{g}} \qquad d = \frac{1}{3}\omega\cos\theta\sqrt{\frac{8h^3}{g}}$$

I-2. Solution

I-2



$$U = -mgy = -mgl\cos\theta$$

$$x(t) = x_0(t) + l\sin\theta \qquad \dot{x}(t) = at + l\cos\theta\,\dot{\theta}$$
$$y(t) = l\cos\theta \qquad \dot{y}(t) = -l\sin\theta\,\dot{\theta}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = a^2t^2 + l^2\cos^2\theta\,\dot{\theta}^2 + 2atl\cos\theta\,\dot{\theta} + l^2\sin^2\theta\,\dot{\theta}^2$$
$$= a^2t^2 + l^2\dot{\theta}^2 + 2atl\cos\theta\,\dot{\theta}$$

(a)

$$L = T - U = \frac{m}{2}\left(l^2\dot{\theta}^2 + 2atl\cos\theta\,\dot{\theta} + a^2t^2\right) + mgl\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left(ml^2\dot{\theta} + matl\cos\theta\right) = -matl\sin\theta\,\dot{\theta} - mgl\sin\theta$$

$$ml^2\ddot{\theta} + mal\cos\theta - \cancel{matl\sin\theta\,\dot{\theta}} = -\cancel{matl\sin\theta\,\dot{\theta}} - mgl\sin\theta$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{a}{l}\cos\theta$$

(b)

stationary angle: θ_0

$$0 = -\frac{g}{l}\sin\theta_0 - \frac{a}{l}\cos\theta_0 \Rightarrow \boxed{\tan\theta_0 = -a/g} \; (*)$$

$$\theta_0 = -\tan^{-1} a/g$$

(c)

small oscillations about stationary angle: (expand about θ_0)

$$\sin\theta \cong \sin\theta_0 + \cos\theta_0\,(\theta-\theta_0)$$
$$\cos\theta \simeq \cos\theta_0 - \sin\theta_0\,(\theta-\theta_0$$

$$\ddot{\theta} \cong -\frac{g}{\ell}\left(\sin\theta_0 + \cos\theta_0\,(\theta-\theta_0)\right) - \frac{a}{\ell}\left(\cos\theta_0 - \sin\theta_0\,(\theta-\theta_0)\right)$$

cancel because of θ_0 is the stationary angle solution of (*)

$$\ddot{\theta} = \left[-\frac{g}{\ell}\cos\theta_0 + \frac{a}{\ell}\sin\theta_0\right](\theta-\theta_0)$$

$$= -\left[\frac{g}{\ell}\cos\theta_0 - \frac{a}{\ell}\sin\theta_0\right](\theta-\theta_0) = \underbrace{-\frac{g}{\ell}\cos\theta_0\left[1-\frac{a}{g}\tan\theta_0\right]}_{\omega^2}(\theta-\theta_0)$$

$$\cos\theta_0 = \frac{1}{\sqrt{1+\tan^2\theta_0}} = \frac{1}{\sqrt{1+\frac{a^2}{g^2}}}$$

$$\ddot{\theta} = -\omega^2(\theta-\theta_0)$$
$$\varphi = \theta-\theta_0$$
$$\ddot{\varphi} = -\omega^2\varphi$$
small angle osc. about θ_0

Thus, $$\omega^2 = \frac{g}{\ell}\frac{1}{\sqrt{1+\frac{a^2}{g^2}}}\left[1-\frac{a}{g}\left(-\frac{a}{g}\right)\right] =$$

$$= \frac{g}{\ell}\frac{1}{\sqrt{1+\frac{a^2}{g^2}}}\left(1+\frac{a^2}{g^2}\right) = \frac{g}{\ell}\sqrt{1+\frac{a^2}{g^2}} = \frac{1}{\ell}\sqrt{g^2+a^2}$$

Finally, the frequency of small oscillations about θ_0:

$$\boxed{\omega = \sqrt{\frac{1}{\ell}\sqrt{g^2+a^2}} = \frac{1}{\sqrt{\ell}}\left(g^2+a^2\right)^{1/4}}$$

I-3. Solution

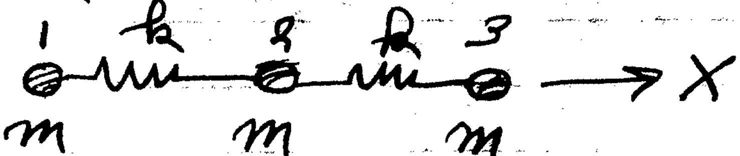
(John Schroeder)

Ph.D. Q.E. Jan 2004 3. Dec. 2003.

Mechanics (small oscillations)/normal modes.

[I-3] Problem: Three particles of equal mass "m" move without friction in one dimension. Two of the particles are each connected to the third by a massless spring of spring constant "k". Find normal modes of oscillation and their corresponding frequencies. Draw a schematic of the normal modes.

Solution:



let x_1, x_2, x_3 be the displacements of the respective masses from their equilibrium positions. The Lagrangian becomes:

$$L = T - V = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2} k [(x_2 - x_1)^2 + (x_3 - x_2)^2]$$

$$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \implies$$

$$m\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

Try the solutions: $x_1 = Ae^{i\omega t}$, $x_2 = Be^{i\omega t}$, $x_3 = Ce^{i\omega t}$. The matrix equation

$$\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

(I-3) continued

The secular equation becomes

$$\begin{vmatrix} k-m\omega^2 & -k & 0 \\ -k & 2k-m\omega^2 & -k \\ 0 & -k & k-m\omega^2 \end{vmatrix} =$$

$$= m\omega^2(k-m\omega^2)(m\omega^2-3k) = 0$$

We have three non-degenerate non-negative roots:
$\omega_1 = 0$, $\omega_2 = \sqrt{\frac{k}{m}}$, $\omega_3 = \sqrt{\frac{3k}{m}}$ and these are the normal mode angular frequencies of the system. The corresponding normal modes are as follows.

(1) $\omega_1 = 0$ the matrix equation gives
$A = B = C$ and thus $x_1 = x_2 = x_3$.
From the first equation of motion we obtain
$$\ddot{x}_1 = 0 \quad \text{or} \quad x_1 = at + b \quad (a, b\text{-const.})$$
Hence, in this mode the three masses undergo uniform translation together as a rigid body and no vibration occurs.

(2) $\omega_2 = \sqrt{k/m}$ now $A = -C$ and $B = 0$.
In this mode the middle mass remains stationary, while the outer masses oscillate

[I-3]. continued.

symmetrically with respect to it.
The displacements are

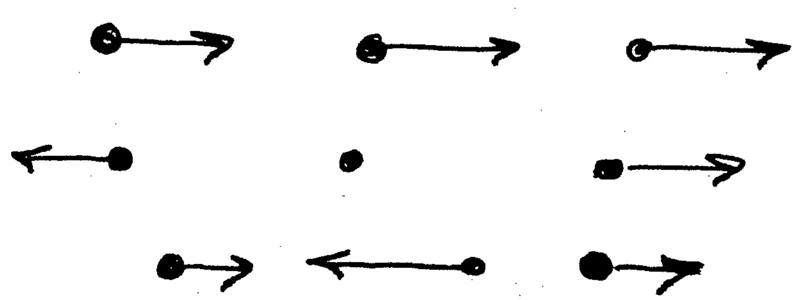
$$x_1 = A\cos(\omega_2 t + \varphi) \qquad \text{with } \varphi$$
$$x_2 = 0 \qquad \text{being a}$$
$$x_3 = -A\cos(\omega_2 t + \varphi) \qquad \text{constant.}$$

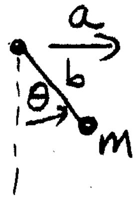
(3.) $\omega_3 = \sqrt{3k/m}$, $B = -2A$, $C = A$. In this mode the two outer masses oscillate with the same amplitude and phases while the middle mass oscillates exactly out of phase with twice the amplitude with respect to the other two masses. The displacements are:

$$x_1 = A\cos(\omega_3 t + \varphi)$$
$$x_2 = -2A\cos(\omega_3 t + \varphi)$$
$$x_3 = A\cos(\omega_3 t + \varphi)$$

The three normal modes are shown below

(i)
(ii)
(iii)



I-4) a) 

$x = \frac{1}{2}at^2 + b\sin\theta$; $\dot{x} = at + b\dot{\theta}\cos\theta$

$y = -b\cos\theta$; $\dot{y} = b\dot{\theta}\sin\theta$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(a^2t^2 + b^2\dot{\theta}^2 + 2atb\dot{\theta}\cos\theta\right)$$

$$U = mgy = -bmg\cos\theta$$

I-4. Solution

$$L = T - U$$

eq. of motion: $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$

$$\Rightarrow \ddot{\theta} + \frac{a}{b}\cos\theta + \frac{g}{b}\sin\theta = 0$$

b) $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = matb\cos\theta + mb^2\dot{\theta}$

$$H = p_\theta\dot{\theta} - L$$

$$= \frac{p_\theta^2}{2mb^2} - \frac{p_\theta at\cos\theta}{b} + \frac{1}{2}ma^2t^2\cos^2\theta - \frac{1}{2}ma^2t^2 - bmg\cos\theta$$

$$\dot{\theta} = \frac{1}{mb^2}\left(p_\theta - matb\cos\theta\right)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{at}{b}p_\theta\sin\theta - ma^2t^2\sin\theta\cos\theta + bmg\sin\theta$$

c) $E = T + U \neq H$; E not conserved because $H = H(t)$ and $\theta = f(x,t)$

I-5

I-5. Solution

Let $t' = 0$

$$t = \frac{t'}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$x = \frac{vt'}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

$$y = ct$$

$$z = 0$$

$$v_x = \frac{dx}{dt} = v$$

$$v_y = \frac{dy}{dt}\frac{dt'}{dt} = \sqrt{c^2 - v^2}$$

$$v_z = 0$$

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$\tan\theta = \frac{v_y}{v_x} = \frac{v}{v}$$

(I-6) Solution.

I-6. Solution

3) Separation of variables in 1-dimension

$$\Delta\phi(r,\varphi,z) = -\frac{\rho}{\varepsilon_0} = -\frac{k}{\varepsilon_0} r$$

$$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\phi = \phi(r)$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\frac{k}{\varepsilon_0} r$$

$$\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\frac{k}{\varepsilon_0} r^2$$

$$r\frac{d\phi}{dr} = -\frac{1}{3}\frac{k}{\varepsilon_0} r^3 + C_1$$

$$\frac{d\phi}{dr} = -\frac{1}{3}\frac{k}{\varepsilon_0} r^2 + \frac{C_1}{r}$$

$$\phi(r) = -\frac{1}{9}\frac{k}{\varepsilon_0} r^3 + C_1 \ln r + C_2$$

$$\phi(r_A) = -\frac{1}{9}\frac{k}{\varepsilon_0} r_A^3 + C_1 \ln r_A + C_2 = V_A \quad (\mathrm{I})$$

$$\phi(r_B) = -\frac{1}{9}\frac{k}{\varepsilon_0} r_B^3 + C_1 \ln r_B + C_2 = V_B \quad (\mathrm{II})$$

I−II $\quad V_A - V_B = C_1 \ln\frac{r_A}{r_B} - \frac{1}{9}\frac{k}{\varepsilon_0}\left(r_A^3 - r_B^3\right)$

$$C_1 = \frac{(V_A - V_B) - \frac{1}{9}\frac{k}{\varepsilon_0}\left(r_A^3 - r_B^3\right)}{\ln r_A/r_B}$$

I+II $\quad V_A + V_B = 2C_2 + C_1(\ln r_A + \ln r_B) - \frac{1}{9}\frac{k}{\varepsilon_0}\left(r_A^3 + r_B^3\right)$

②

surface charge $\quad \sigma = -\varepsilon_0 \frac{\partial \phi}{\partial n}$

$$\frac{\partial \phi}{\partial n} = \frac{\vec{r}}{r}\vec{\nabla}\phi(r) =$$

$$= \cancel{\frac{\vec{r}}{r}}\frac{\partial \phi}{\partial r}$$

$$\phi(r) = -\frac{1}{9}\frac{k}{\varepsilon_0}r^3 + c_1 \ln r + c_2$$

$$\frac{d\phi}{dr} = -\frac{1}{3}\frac{k}{\varepsilon_0}r^2 + \frac{c_1}{r}$$

$$\sigma_A = -\varepsilon_0 \frac{d\phi}{dr}\Big|_{r=r_A} = -\varepsilon_0\left(-\frac{1}{9}\frac{k}{\varepsilon_0}r_A^2 + \frac{c_1}{r_A}\right)$$

$$\sigma_B = \varepsilon_0 \frac{d\phi}{dr}\Big|_{r=r_B} = \varepsilon_0 \frac{1}{3}\frac{k}{\varepsilon_0}r_B^2 + \frac{c_1}{r_B}$$

b) $\cancel{C = \frac{Q}{U} \qquad U = \phi(R_A) - \phi(R_B) = -\int_{r_A}^{r_B} \vec{E}\cdot d\vec{l}}$

$$\cancel{C = \frac{2\pi\varepsilon_0 L}{\ln\frac{r_A}{r_B}}}$$

Gauß law

$$\int \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int \rho \, dV$$

$r_A < r < r_B$

$$E\cdot 2\pi r = \frac{1}{\varepsilon_0} k \int_0^{2\pi}\int_{r_A}^{r} r\, r\, dr\, d\varphi$$

$$E\cdot 2\pi r = \frac{1}{\varepsilon_0} k \cdot 2\pi \cdot \frac{1}{3}(r^3 - r_A^3)$$

$$E = \frac{k}{3\varepsilon_0}\left(r^2 - \frac{r_A^3}{r}\right)$$

$$\cancel{\phi = \int \vec{E}\cdot d\vec{l} = \frac{k}{3\varepsilon_0}\int r^2 dr}$$

$$\cancel{\phi = \frac{k}{9\varepsilon_0} r^3 + C_1}$$

$r > r_B$

$$\cancel{E\cdot 2\pi r = \frac{1}{\varepsilon_0}\int_0^{2\pi}\int_{r_A}^{r_B} k r\, r\, dr\, d\varphi}$$

$$\cancel{E\cdot 2\pi r = \frac{1}{\varepsilon_0} k \cdot 2\pi \; \frac{1}{3}(r_B^3 - r_A^3)}$$

$$\cancel{E = \frac{k}{3\varepsilon_0}\frac{1}{r}(r_B^3 - r_A^3)}$$

$r_A < r < r_B$

$$\phi(r) = -\int \vec{E}\cdot d\vec{l}$$

$$= \frac{k}{3\varepsilon_0}\left[\int r^2\cdot dr - \int \frac{r_A^3}{r} dr\right]$$

$$\phi(r) = \frac{k}{3\varepsilon_0}\left[\frac{1}{3} r^3 - r_A^3 \ln r + C_1 + C_2\right]$$

$$Q_{encl} = \int \rho \, dV = \cancel{\quad} \cdot k \cdot 2\pi \cdot \frac{1}{3}(r^3 - r_A^3)$$

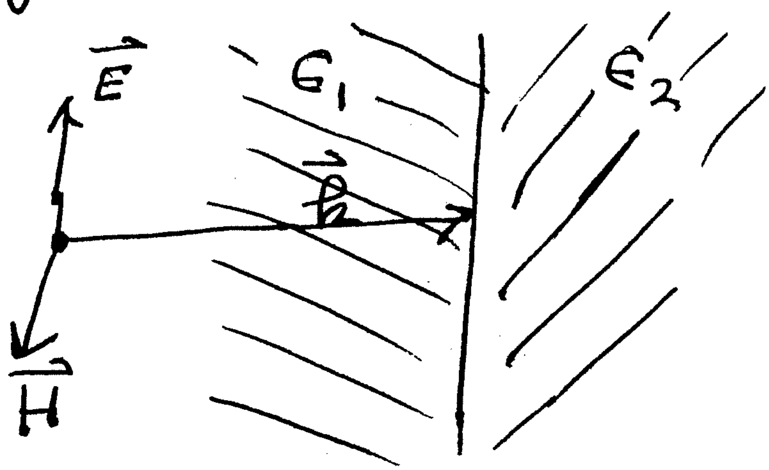
$$Q = 2\pi \int_{r_A}^{r} r\, r\, dr\, d\varphi$$

I-7. Solution

John Schroeder
(3·Dec·2003)

[I-7] Maxwell's Equations.
(Plane wave in dielectric medium.)

Problem: A monochromatic plane wave of frequency ω propagates through a non-permeable ($\mu = 1$) insulating medium with dielectric constant, ϵ_1. The wave is normally incident upon an interface with a similar medium of dielectric constant ϵ_2 (see Figure).



(a) Derive the boundary conditions for the electric and magnetic fields at the interface.

(b) Find the fraction of the incident energy that is transmitted to the second medium, and the fraction of the energy that is reflected back into the first medium from the interface. Show that $R + T = 1$.

[I-7] Solution:

(a) Assume that the dielectric constant is essentially real (no dissipations) For monochromatic waves travelling in the z direction with $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$, the ~~source~~less Maxwell equations ($\mu = 1$) become

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \frac{\epsilon}{c}\frac{\partial \vec{E}}{\partial t}$$

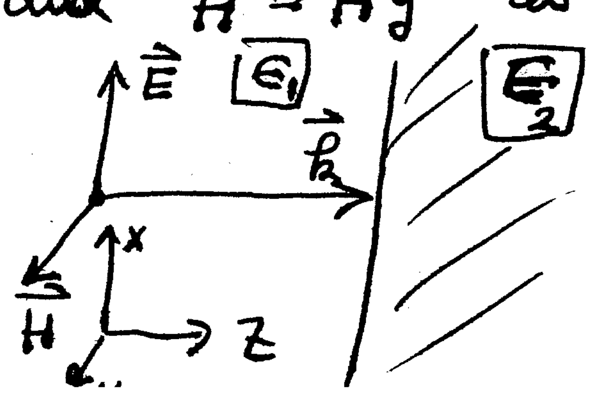
Now substituting the explicit form for $\vec{E}$ (and $\vec{H}$) produces the following exchange

$$\frac{\partial}{\partial t} \to -\omega, \quad \vec{\nabla}\times \to i\vec{k}\times$$

and Maxwell's Equations become:

$$i\omega \vec{H} = ic\,\vec{k} \times \vec{E}$$
$$i\omega \vec{E} = -ic\,\vec{k} \times \vec{H}$$

Now orient the axes so that $\vec{E} = E\hat{x}$ and $\vec{H} = H\hat{y}$ as shown below.



[I-7] continued.

The the boundary conditions, which require continuity for the tangential components of $\vec{E}$ and $\vec{H}$, become

$$E_x^{(1)} = E_x^{(2)}, \quad H_y^{(1)} = H_y^{(2)}$$

with (1) and (2) corresponding to ϵ_1, ϵ_2 media. From the Maxwell equations we obtain

$$H_y = \frac{ck}{\omega} E_x .$$

The field in medium 1 is the sum of the incident wave E_0 and the reflected wave E_1, whereas the field in medium 2 is due only to the transmitted wave E_2. Using the above given b.c.'s and the eqn. for H_y, we find

$$E_0 + E_1 = E_2 \quad \text{and} \quad k_1(E_0 - E_1) = k_2 E_2$$

Solving these equations for E_1 and E_2

$$E_1 = \frac{k_1 - k_2}{k_1 + k_2} E_0 = \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} E_0$$

$$E_2 = \frac{2k_1}{k_1 + k_2} E_0 = \frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} E_0$$

[I-7] part(b) continued.

(b) Energy flux of monochromatic wave is given by the magnitude of the Poynting vector. $\therefore$ $S = \frac{c}{8\pi}\sqrt{\frac{\epsilon}{\mu}}\, E E^*$ and the incident and transmitted fluxes S_0 and S_2, are

$S_0 = \frac{c}{8\pi}\sqrt{\epsilon_1}\,|E_0|^2$ and $S_2 = \frac{c}{8\pi}\sqrt{\epsilon_2}\,|E_2|^2 =$

$S_2 = \frac{c}{8\pi}\sqrt{\epsilon_2}\left(\frac{2\sqrt{\epsilon_1}}{\sqrt{\epsilon_1}+\sqrt{\epsilon_2}}\right)^2 |E_0|^2$ hence the fraction of the energy transmitted into the second medium is

$$T = \frac{S_2}{S_0} = \sqrt{\frac{\epsilon_2}{\epsilon_1}}\,\frac{4\epsilon_1}{(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2} = \frac{4\sqrt{\epsilon_1}\sqrt{\epsilon_2}}{(\sqrt{\epsilon_1}+\sqrt{\epsilon_2})^2} = \frac{4n_1 n_2}{(n_1+n_2)^2}$$

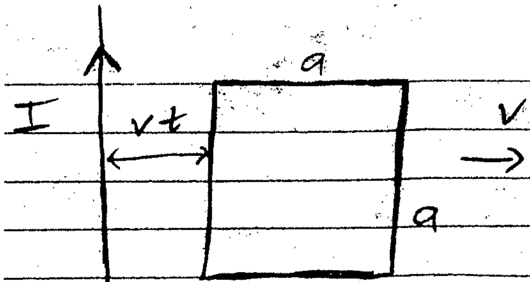
where $n_{1,2} = \sqrt{\epsilon_{1,2}}$ $\leftarrow$ the indices of refraction. Now the fraction of the energy that is reflected back into the first medium $R = \frac{S_1}{S_0} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$

with, S_1, the magnitude of the Poynting vector for the reflected wave.

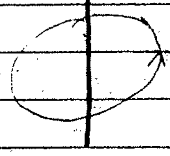
Also using the values for R and T we see that $T + R = 1$.

$\boxed{\text{I-8}}$

I-8. Solution



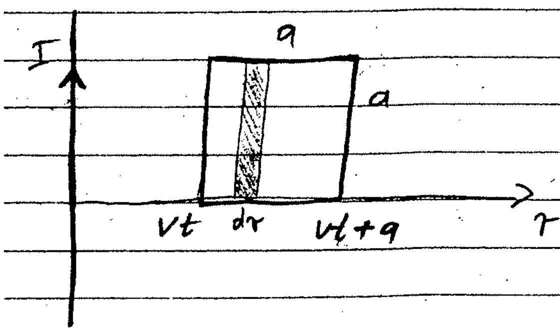
(at $t=0$ side of square wire and straight wire coincide)



B (perpendicular to square conducting wire)

$$\oint \bar{B}\cdot d\bar{s} = \mu_0 I \Rightarrow B(r)\,2\pi r = \mu_0 I \Rightarrow B(r) = \frac{\mu_0 I}{2\pi}\frac{1}{r}$$

$$\mathcal{E} = -\frac{d\phi}{dt}, \text{ where } \phi = \iint \bar{B}\cdot d\bar{A} \quad \text{magnetic flux}$$



$$\phi(t) = \int_{vt}^{vt+a} B(r)\,a\,dr = \int_{vt}^{vt+a} \frac{\mu_0 I}{2\pi}\frac{a\,dr}{r} =$$

$$= \frac{\mu_0 I a}{2\pi}\ln\left(\frac{vt+a}{vt}\right)$$

$$\frac{d\phi}{dt} = \frac{\mu_0 I a}{2\pi}\left(\frac{1}{vt+a} - \frac{1}{vt}\right)\cdot v = \frac{\mu_0 I a}{2\pi}\,\frac{\cancel{vt} - \cancel{vt} - a}{\cancel{v}t\,(vt+a)}\cdot\cancel{v}$$

$$= -\frac{\mu_0 I a^2}{2\pi}\,\frac{1}{t(a+vt)}$$

$$\boxed{\mathcal{E} = \frac{\mu_0 I a^2}{2\pi}\,\frac{1}{t(a+vt)}}$$

induced voltage in square-shape wire at time t.

I-9)

I-9. Solution

given
$$\underline{E}(\underline{r},t) = \frac{q}{4\pi\epsilon_0}\frac{|\xi|}{(\xi\cdot\underline{u})^3}\left[(c^2 - |\dot{\underline{w}}|^2)\underline{u} + \xi\times(\underline{u}\times\ddot{\underline{w}})\right]$$

$$\underline{B}(\underline{r},t) = \frac{\hat{\xi}}{c}\times\underline{E}(\underline{r},t)$$

$\xi \equiv \underline{r} - \underline{w}$, $\underline{u} = c\hat{\xi} - \dot{\underline{w}}$, $\underline{w}$ = position

position and $\dot{\underline{w}}$ are at retarded time $t_r \ni |\xi| = c(t - t_r)$

$$\underline{S} = \frac{1}{\mu_0}(\underline{E}\times\underline{B}) = \frac{1}{\mu_0 c}\left[E^2\hat{\xi} - (\hat{\xi}\cdot\underline{E})\underline{E}\right]$$

At large distance, $\underline{E}_{rad} = \frac{q}{4\pi\epsilon_0}\frac{|\xi|}{(\xi\cdot\underline{u})^3}\left[\xi\times(\underline{u}\times\underline{a})\right]$

$$\Rightarrow S_{RAD} = \frac{1}{\mu_0 c}E_{rad}^2\hat{\xi}$$

If treat $\dot{w}$ as zero, $E_{RAD} \to \frac{\mu_0 q}{4\pi\xi}\left[(\hat{\xi}\cdot\underline{a})\hat{\xi} - \underline{a}\right]$

$$S_{RAD} \to \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\xi^2}\right)\hat{\xi} \qquad \hat{\xi}\cdot\underline{a} = a\cos\theta$$

$$P = \oint d\underline{a}\cdot S_{RAD} = \frac{\mu_0 q^2 a^2}{6\pi c} \qquad \text{LARMOR FORMULA}$$

$$\text{distance} = \tfrac{1}{2}at^2 = \tfrac{1}{2}gt^2 = 10^{-2}\,m \Rightarrow t_0 = .045\,s$$

$$\int P\,dt = Pt_0$$

total PE lost = mg × distance

$$\frac{\text{RADIATION}}{\text{PE}} = 3\times10^{-22}$$

I-10. Solution

II . 10

$$\begin{cases} \vec{\nabla} \times \vec{B} + \frac{\partial E}{\partial t} = 0 \\ \vec{\nabla} \times H - \frac{\partial D}{\partial t} = 0 \end{cases}$$

$$\nabla^2 B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = 0$$

$$\nabla^2 D - \frac{1}{c^2} \frac{\partial^2 D}{\partial t}$$

$$E_0 = E_0 \cos(kz - \omega t)\,\hat{x}$$

$$B = B_0 \cos(kz - \omega t)\,\hat{y}$$

The average energy

$$= \frac{1}{2} \epsilon_0 E^2$$

The intensity is $c\langle u \rangle$

The momentum flux is $\frac{\langle u \rangle}{c}$

Problem Solutions Part-II

Physics PhD Qualifying Examination
Part II – Friday, January 9, 2004

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

II-1 [10]

A *plane* rotator with a moment of inertia I and an electric dipole moment $\mathbf{d}$ is placed in a uniform time-independent electric field $\boldsymbol{\varepsilon}$ (being in the plane of the rotation). Treating the interaction between the dipole and the electric field perturbatively, determine the first *non-vanishing* correction to the rotator's energy levels.

II-2. [10]

Calculate the total cross-section for scattering from a Yukawa potential, $V(r) = \beta\frac{e^{-\mu r}}{r}$, in the first Born approximation. Express your answer as a function of E.

II-3 [10]

The Hamiltonian H of a rigid rotor can be expressed as

$$H = \frac{1}{2}\left(\frac{L_x^2}{2I_{xx}} + \frac{L_y^2}{2I_{yy}} + \frac{L_z^2}{2I_{zz}}\right)$$

Where I_{xx}, I_{yy} and I_{zz} are the principal moment of inertia. Consider a rotor for which $I_{xx} = I_{yy} = I_1$ and $I_{zz} = I_3$.

(a) Show that the spherical harmonics $Y_{lm}(\theta, \varphi)$ are Eigenfunctions.
(b) Derive an expression for the energy E_{lm}.
(c) Derive an expression for the expectation value of L_z and L^2 in a state ψ, with

$$\psi = \sum_{lm} a_{lm} Y_{lm} \quad .$$

(d) Describe the situation when in addition $I_1 = 2$.

II-4 [10]

Calculate $<\vec{r}> = <x>\hat{x} + <y>\hat{y} + <z>\hat{z}$ in the Ψ_{nlm} state of hydrogen.

$$\psi_{nlm} = R_{nl} Y_l^m$$

$$R_{nl} = (2\kappa)^{3/2} A_{nl} \rho^l e^{-\rho/2} F_{nl}(\rho) \text{ with } \rho = 2\kappa r = \frac{2Z}{a_0 n} r \text{ and } A_{nl} = \sqrt{\frac{(n-l-1)!}{(2n[(n+l)!]^3}}$$

$$Y_l^m = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\frac{e^{im\varphi}}{\sqrt{2\pi}}$$

$$(2l+1)\cos\theta\, P_l^m(\cos\theta) = (l-m+1)\, P_{l+1}^m(\cos\theta) + (l+m)\, P_{l-1}^m(\cos\theta)$$

II-5 [10]

An electron moves in one dimension and is confined to the right half-space ($x > 0$) where it has potential energy

$$V(x) = -\frac{e^2}{4x} \quad \text{(one dimensional Coulomb potential)}$$

where e is the charge of an electron. This is the image potential of an electron outside of a perfect conductor.

(i) Find the ground state energy.

(ii) Find the expectation value $\langle x \rangle$ in the ground state and express in terms of the Bohr radius.

II-6. [10]

Consider a two-state system with Hamiltonian H_0 and states a and b, subjected to a time-dependent perturbation $H'(\mathbf{r}, t) = V(\mathbf{r})\cos(\omega t)$. If the system is in state a at time $t = 0$, calculate the probability that it will be found in state b at time t. Describe the behavior of the system as a function of t. At what time is the system most likely to be found in state b?

II-7 [10]

Derive from first principles an expression for

(a) the coefficient of volume expansion β and
(b) the compressibility κ ,

for a van der Waal's gas, starting from the definitions for β and κ .

(c) what happens to these equations in the limiting case of an ideal gas?

II-8 [10]

Calculate the difference between c_p and c_v for the van der Waal's gas. Here c_p and c_v are the specific heats at constant pressure and constant volume, respectively. Van der Waal's equation is given by:

$$\left(p+\frac{a}{V^2}\right)(V-b)=RT$$

with p, V and T being the usual thermodynamic variables; a and b are constants.

II-9 [10]

Consider a system of three dimensional rotators (with two degrees of freedom and no translational motion) in thermal equilibrium according to Boltzmann statistics; take account of the quantization of energy. Calculate

(i) the free energy
(ii) entropy
(iii) energy and
(iv) heat capacity (per rotator)

in the case of high temperature, making use of the Euler's approximation formula:

$$\sum_{j=0}^{\infty} f(j+\frac{1}{2}) = \int_0^{\infty} f(x)dx + \frac{1}{24}[f'(0) - f'(\infty)] + \cdots .$$

II-10 [2,2,6]

Consider the *two-dimensional relativistic* ideal (non-interacting) electron gas at zero temperature. The energy-momentum dispersion relation is $\varepsilon(p) = \sqrt{m^2c^4 + c^2p^2}$ (m is the rest mass of the electrons and c is the speed of light).

(a) Find the Fermi momentum p_F, as a function of the surface density of electrons N/A. (N is the number of electrons and A is the area to which the electrons are confined.)

(b) Find the Fermi energy ε_F, as a function of the surface density of electrons N/A.

(c) Obtain the single-electron density of states $g(\varepsilon)$ and sketch this function.

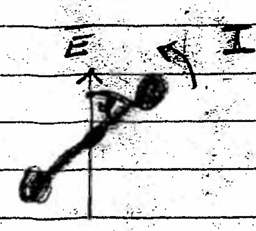
II-1. Solution

(II-1) (use $\bar{\mathcal{E}}$ for electric field to avoid confusion)

-1) Plane Rotation

$$H_0 \psi = E^{(0)} \psi$$

$$-\frac{\hbar^2}{2I}\frac{d^2\psi}{d\varphi^2} = E^{0}\psi$$



unperturbed Hamiltonian : $H_0 = -\frac{\hbar^2}{2I}\frac{d^2}{d\varphi^2}$

electric field ...

perturbation: $\mathbf{H}' = -E \cdot d \cos\varphi$ $\qquad E = |\bar{E}|$, $d = |\bar{d}|$

$$(H_0 + H')\psi = E\psi$$

unperturbed eigenfunctions & eigenvalues:

$$\psi_m^{(0)}(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi} \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$E_m^{(0)} = \frac{\hbar^2 m^2}{2I}$$

Despite the 2-fold degeneracy, non-degenerate perturbation theory can be applied since, $\langle \psi_{-m}^{(0)} | H' | \psi_m^{(0)} \rangle = 0$ for all m

i.e., the $-m$ state does not give any contribution to the corrections of the $+m$ state.

$$H'_{mm'} = \int_0^{2\pi} \psi_m^{(0)*} H' \psi_{m'}^{(0)} \, d\varphi = -\frac{Ed}{2\pi}\int_0^{2\pi} e^{i(m'-m)\varphi}\cos\varphi \, d\varphi$$

$$= -\frac{Ed}{2\pi}\frac{1}{2}\left\{ \int_0^{2\pi} e^{i(m'-m+1)\varphi} d\varphi + \int_0^{2\pi} e^{i(m'-m-1)\varphi} d\varphi \right\} =$$

$$= \frac{-Ed}{2}\left(\delta_{m',m-1} + \delta_{m',m+1}\right) \quad ; \text{ only } m' = m \pm 1 \text{ nonzero elements}$$

$$\left(H'_{m,-m} = 0 \quad \forall m \right)$$

$E_m^{(1)} = H'_{mm} = 0$, so one must go to 2nd order (formally non-degenerate) perturbation theory.

$$E_m^{(2)} = \sum_{j \neq m} \frac{|H'_{mj}|^2}{E_m^{(0)} - E_j^{(0)}} = \frac{|H'_{m\,m-1}|^2}{E_m^{(0)} - E_{m-1}^{(0)}} + \frac{|H'_{m,m+1}|^2}{E_m^{(0)} - E_{m+1}^{(0)}} =$$

$$= \frac{E^2 d^2}{4} \left\{ \frac{1}{\frac{\hbar^2 m^2}{2I} - \frac{\hbar^2 (m-1)^2}{2I}} + \frac{1}{\frac{\hbar^2 m^2}{2I} - \frac{\hbar^2 (m+1)^2}{2I}} \right\}$$

$$= \frac{I E^2 d^2}{2\hbar^2} \left\{ \frac{1}{m^2 - (m-1)^2} + \frac{1}{m^2 - (m+1)^2} \right\} = \frac{I E^2 d^2}{2\hbar^2} \left\{ \frac{1}{2m-1} + \frac{1}{-2m-1} \right\}$$

$$= \frac{I E^2 d^2}{2\hbar^2} \left\{ \frac{1}{2m-1} - \frac{1}{2m+1} \right\} = \frac{I E^2 d^2}{2\hbar^2} \cdot \frac{2m+1 - (2m-1)}{(2m-1)(2m+1)}$$

$$= \frac{I E^2 d^2}{\hbar^2} \frac{1}{4m^2 - 1}$$

Thus,

$$\boxed{E_m \simeq \frac{\hbar^2 m^2}{2I} + \frac{I E^2 d^2}{\hbar^2} \frac{1}{4m^2 - 1}} \qquad m = 0, \pm 1, \pm 2, \ldots$$

II-2) $V = \beta \dfrac{e^{-\mu r}}{r}$

II-2. Solution

$$f(\theta) \cong -\frac{2m}{\hbar^2 K}\int_0^\infty dr\, r\,V(r)\sin(Kr) \qquad 1^{st} \text{ BORN}$$

$$K = 2k\sin\left(\frac{\theta}{2}\right)$$

$$\simeq \frac{-2m\beta}{\hbar^2(\mu^2+K^2)} \qquad E = \frac{\hbar^2 k^2}{2m}$$

$$\sigma = \int d\phi\, d\theta \sin\theta\, |f(\theta)|^2$$

Let $x = \frac{2k}{\mu}\sin\left(\frac{\theta}{2}\right)$

$$\therefore \sigma = 2\pi\left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu^4}\left(\frac{\mu}{k}\right)^2 \int_0^{\frac{2k}{\mu}} dx\, \frac{x}{[1+x^2]^2}$$

$$\Rightarrow \sigma = \pi\left(\frac{4m\beta}{\mu\hbar}\right)^2 \frac{1}{(\mu\hbar)^2 + 8mE}$$

II-3. Solution

II-3

$$H = \frac{1}{2I_1^2}(L_x^2 + L_y^2) + \frac{L_z^2}{2I_3}$$

$$= \frac{L^2}{2I_1} + \frac{L_z^2}{2I_3}$$

$$\left(\frac{L^2}{2I_1} - \frac{L_z^2}{2I_3}\right)\psi_{lm} = E_{lm}\psi_l$$

$$E_{lm} = \frac{\hbar^2 \ell^2}{2I_1}[\ell(\ell+1) - m^2]$$

$$\langle E_{lm} \rangle = E_{lm}$$

$$\langle E_{lm} \rangle = \sum |a_{lm}|^2 E_{lm}$$

$$\langle L_z \rangle = \sum |a_{lm}| L_z$$

If $L_x \neq L_y \neq L_z$

Deagneracy removal

(II-4)

II-4. Solution

$$X = r \cos\varphi \sin\theta$$

$$y = r \sin\varphi \sin\theta$$

$$z = r \cancel{} \cos\theta$$

$$\langle \psi_{n\ell m} | x | \psi_{n\ell m} \rangle \propto \int_0^{2\pi} \cos\varphi \, d\varphi = 0$$

$$\langle \psi_{n\ell m} | y | \psi_{n\ell m} \rangle \propto \int_0^{2\pi} \sin\varphi \, d\varphi = 0$$

$$\langle \psi_{n\ell m} | z | \psi_{n\ell m} \rangle \propto \int_{-1}^{+1} P_\ell^m (\cos\theta) \cos\theta \, P_\ell^m (\cos\theta) \, d(\cos\theta)$$

$$\cos\theta \, P_\ell^m = (\ell - m + 1) P_{\ell+1}^m (\cos\theta) + (\ell + m) P_{\ell-1}^m (\cos\theta)$$

$$\langle \psi_{n\ell m} | z | \psi_{n\ell m} \rangle \propto \int_{-1}^{+1} P_\ell^m (\cos\theta) P_{\ell+1} (\cos\theta) \, d(\cos\theta) +$$

$$\int_{-1}^{1} P_\ell^m (\cos\theta) P_{\ell-1}^m (\cos\theta) \, d(\cos\theta) = 0$$

II-5. Solution

John Schroeder
11 · XII · 2003

Quantum Mechanics

[II-5.] An electron moves in one dimension and is confined to the right half-space ($x > 0$) where it has potential energy
$$V(x) = -\frac{e^2}{4x} \quad \text{(1-dim Coulomb potential).}$$
where e is the charge of ~~the~~ an electron. This is the image potential of an electron outside of a perfect conductor.

(i) Find the ground state energy

(ii) Find the expectation value $\langle x \rangle$ in the ground state.

Solution:

(i) Since the electron is confined to the right half-space, its wavefunction must vanish at the origin. Hence, an eigenfunction such as ($e^{-\alpha x}$) is unsuitable since it does not vanish at $x = 0$. Thus, the ground state wave-function must be of the form

$\psi = N x e^{-\alpha x}$, where α must be determined.

The operator p^2 acting on this form gives:
$$p^2(x e^{-\alpha x}) = \hbar^2 \alpha (2 - \alpha x) e^{-\alpha x}$$

[II-5] Solution: continued.

Now using this wave function in Schrödinger's equation yields

$$\frac{\hbar^2}{2m}\alpha(2-\alpha x)-\frac{e^2}{4}=Ex\ .$$

For this equation to be satisfied, the first and third terms on the left must be equal, and the second term on the left must equal the term on the right of the equal sign: $\frac{\hbar^2}{m}\alpha=\frac{e^2}{4}$

and $E=-\frac{\hbar^2\alpha^2}{2m}=-\frac{e^4 m}{32\hbar^2}=-\frac{E_R}{16}$

$\therefore \frac{1}{16}$ Rydberg: where E_R is the ground state energy of the hydrogen atom. $\alpha = 1/4(a_0)$ where a_0 is the Bohr radius.

(ii) Next find the expectation value $\langle x\rangle$. Let us find the Normalization coefficient:

$$1=N^2\int_0^\infty dx\, x^2 e^{-2\alpha x}=\frac{N^2}{4\alpha^3}$$

$$\langle x\rangle = N^2\int_0^\infty dx\, x^3 e^{-2\alpha x}=\frac{4\alpha^3 3!}{(2\alpha)^4}=\frac{3}{2\alpha}=6a_0\ .$$

The average value of x is 6-Bohr radii.

II-6) $H'(\underline{r},t) = V(\underline{r})\cos(\omega t)$

II-6. Solution

$$H'_{ab} = V_{ab}\cos(\omega t)\ ,\quad V_{ab} = \langle a|V|b\rangle$$

$$c_b(t) \simeq \frac{-i}{\hbar} V_{ba} \int_0^t dt'\, e^{i\omega_0 t} \cos(\omega t) \qquad \omega_0 = \omega_a - \omega_b$$

$$\simeq \frac{-i V_{ba}}{2\hbar} \int_0^t dt' \left[e^{i(\omega_0+\omega)t'} + e^{+i(\omega_0-\omega)t'} \right]$$

$$\simeq \frac{-i V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0-\omega} \right]$$

---

near ω_0, $\omega + \omega_0 \gg |\omega_0 - \omega| \Rightarrow$ drop "+" term

$$\rightarrow c_b(t) = -\frac{i V_{ba}}{\hbar} \frac{\sin[(\omega_0-\omega)t/2]}{\omega_0-\omega} e^{i(\omega_0-\omega)t/2}$$

$$P_{a\to b}(t) = |c_b(t)|^2 \simeq \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2[(\omega_0-\omega)t/2]}{(\omega_0-\omega)^2}$$

oscillates: Reaches <u>same</u> max. every $t = \frac{\pi}{\omega_0-\omega}$ sec., returns to old state in between.

(II-7)

II-7. Solution

a) $\beta = \left(\frac{1}{V}\frac{\partial V}{\partial T}\right)_p \quad - \frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$

$$\left(\frac{\partial v}{\partial T}\right)_p = -\frac{\left(\frac{\partial p}{\partial T}\right)_v}{\left(\frac{\partial p}{\partial v}\right)_T}$$

$$p = \frac{RT}{v-b} - \frac{a}{v^2}$$

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{R}{(v-b)^2} + \frac{2a}{v^3}$$

$$\beta = \frac{Rv^2(v-b)}{2Tv^3 - 2a(v-b)^2}$$

b) $k = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)$

$$k = -\frac{1}{v}\frac{v^2(v-b)^2}{RT^3 - 2(v-b)^2}$$

II-7

$PV = NRT$

$$\beta = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

In the limit $a = b = 0$

$$V = \frac{NRT}{P}$$

$$\left(\frac{\partial V}{\partial T}\right)_T = \frac{NR}{P}$$

$$\beta = \frac{NR}{P}\,\frac{1}{V} = \frac{NR}{NRT} = \frac{1}{T}$$

$$k = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$$

In the limit $a = b = 0$

$$V = \frac{NRT}{P}$$

$$k = \frac{1}{V}\,\frac{RT}{P^2} = \frac{1}{P}$$

II-8

II-8. Solution

$$\delta Q = dE + p\,dV$$

$$E = \frac{\nu}{2} N k T$$

$\nu \to$ degrees of freedom

$$\delta q = \frac{\nu}{2} R\,dT + p\,dV$$

per mol

$$C_V = \left(\frac{\delta Q}{\partial T}\right)_V = \frac{\nu}{2} R$$

$$C_p = \left(\frac{\delta Q}{\partial T}\right)_p = \frac{\nu}{2} R + p\left(\frac{dV}{\partial T}\right)_p$$

$$C_p - C_V = p\left(\frac{\partial v}{\partial T}\right)_p = -p\,\frac{\left(\frac{\partial p}{\partial T}\right)_v}{\left(\frac{\partial p}{\partial v}\right)_T}$$

van-der-Waals gas

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT$$

$$p = \frac{RT}{v-b} - \frac{a}{v^2}$$

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{R}{v-b}$$

$$\left(\frac{\partial p}{\partial v}\right)_T = -\frac{RT}{(v-b)^2} + \frac{2a}{v^3}$$

$$C_p - C_V = -\frac{\dfrac{pR}{(v-b)}}{\dfrac{2a}{v^3} - \dfrac{RT}{(v-b)^2}} = \frac{pR}{\dfrac{RT}{v-b} - \dfrac{2a(v-b)}{v^3}}$$

$$= \frac{pR}{\dfrac{RT}{v-b} - \dfrac{2a}{v^2} + \dfrac{2ab}{v^3}}$$

II-9. Solution

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11·XII·2003.

[II-9] Consider a system of three dimensional rotators (with two degrees of freedom and no translational motion) in thermal equilibrium according to Boltzmann statistics; take account of the quantization of energy. Calculate (i) the free energy, (ii) entropy, (iii) energy and (iv) heat capacity (per rotator) in the case of high temperature, making use of the Euler's approximation formula:

$$\sum_{j=0} f\left(j+\frac{1}{2}\right) = \int_0^{\infty} f(x)\,dx + \frac{1}{24}\left[f'(0) - f'(\infty)\right] + \cdots$$

[II-9] Solution.

The partition function is given by

$$Z = \sum_{j=0} (2j+1)e^{-A^2 j(j+1)/kT} =$$

$$Z = 2e^{A^2/4kT} \sum_{j=0} (j+\tfrac{1}{2})e^{-(A^2/kT)(j+\frac{1}{2})^2}$$

with $A^2 \equiv \hbar^2/2I$ and "I" is the rotational inertia (moment of inertia) of the rotator. With Euler's formula we have

$$Z = 2e^{A^2/4kT}\left\{\left[\int_0^\infty u e^{-A^2u^3/kT}du\right] + \frac{1}{24}\right\} =$$

$$= 2e^{A^2/4kT}\left(\frac{kT}{4A^2} + \frac{1}{24}\right).$$

Now the desired thermodynamic quantities may be obtained from

(i) $F = -kT \ln Z$

(ii) $S = \dfrac{(U-F)}{T} = -\dfrac{\partial F}{\partial T}$

(iii) $U = kT^2 \dfrac{\partial}{\partial T} \ln Z$

(iv) $C = \dfrac{\partial U}{\partial T}$

which is easily done once the partition functions is determined.

(II-9) Solution continued.

$$Z = 2e^{A^2/4kT}\left(\frac{kT}{4A^2} + \frac{1}{24}\right)$$

(i) $F = -kT\ln\left\{2e^{A^2/4kT}\left(\frac{kT}{4A^2} + \frac{1}{24}\right)\right\}$

(ii) $S = -\frac{\partial F}{\partial T} = +\frac{\partial}{\partial T}\left(kT\ln\left\{2e^{A^2/4kT}\left(\frac{kT}{4A^2} + \frac{1}{24}\right)\right\}\right)$

(iii) $U = kT^2\frac{\partial}{\partial T}\left[\ln\left(2e^{A^2/4kT}\left(\frac{kT}{4A^2} + \frac{1}{24}\right)\right)\right]$

(iv) $C = \frac{\partial U}{\partial T}$

(II-10)

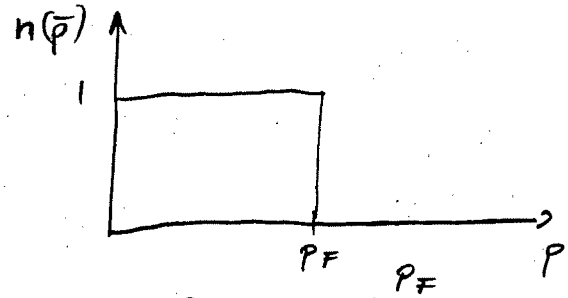
II-10. Solution

e^--gas : $s = 1/2$ $\quad L^2 = A$ (two dimension)

$$N = \underset{\text{spin multiplicity}}{2} \cdot \sum_{\bar{p}} n_{\bar{p}} = 2 \cdot \int n(\bar{p}) \frac{L^2}{h^2} d\bar{p} = 2 \frac{L^2}{h^2} \int_0^\infty n(\bar{p})\, 2\pi p\, dp$$

)

$T = 0$



$$N = 2 \cdot \frac{L^2}{h^2} \cdot 2\pi \int_0^{p_F} p\, dp = \frac{2\pi A}{h^2} p_F^2$$

$$p_F = \frac{h}{\sqrt{2\pi}} \left(\frac{N}{A}\right)^{1/2}$$

b)

$$\varepsilon(p) = \sqrt{m^2c^4 + c^2p^2} \Rightarrow \varepsilon_F = \varepsilon(p_F)$$

$$\varepsilon_F = \sqrt{m^2c^4 + c^2p_F^2} = \sqrt{m^2c^4 + \frac{h^2c^2}{2\pi}\left(\frac{N}{A}\right)}$$

c)

Note that since $0 < p < \infty \Rightarrow mc^2 < \varepsilon < \infty$ (rest energy)

this should be remembered when constructing $\rho(\varepsilon)$

$$2\int_0^\infty \frac{A}{h^2} d\bar{p}\, (\ldots) = \int_{mc^2}^\infty g(\varepsilon)\, d\varepsilon\, (\ldots)$$

$$\Rightarrow \frac{4\pi A}{h^2} p\, dp = g(\varepsilon)\, d\varepsilon$$

$$g(\varepsilon) = \frac{4\pi A}{h^2} p \frac{dp}{d\varepsilon}$$

$$\varepsilon(p) = \sqrt{m^2c^4 + c^2p^2} \Rightarrow p = \sqrt{\frac{\varepsilon^2}{c^2} - m^2c^2}$$

$\hookrightarrow$ (implies $mc^2 \leq \varepsilon < \infty$)

$$\frac{dp}{d\varepsilon} = \frac{\varepsilon}{c^2} \frac{1}{\sqrt{\frac{\varepsilon^2}{c^2} - m^2c^2}}$$

Thus, $g(\varepsilon) = \frac{4\pi A}{h^2} \cancel{\sqrt{\frac{\varepsilon^2}{c^2} - m^2c^2}} \cdot \frac{\varepsilon}{c^2} \frac{1}{\cancel{\sqrt{\frac{\varepsilon^2}{c^2} - m^2c^2}}}$

$= \frac{4\pi A}{h^2c^2} \varepsilon$ for $\varepsilon \geq mc^2$

and $g(\varepsilon) = 0$ for $\varepsilon < mc^2$

$$\boxed{g(\varepsilon) = \begin{cases} 0 & \text{for } \varepsilon < mc^2 \\ \frac{4\pi A}{h^2c^2}\varepsilon & \text{for } mc^2 \leq \varepsilon < \infty \end{cases}}$$