

STONY BROOK UNIVERSITY  
DEPARTMENT OF PHYSICS AND ASTRONOMY

Graduate Placement Exam Part 2, Aug. 25, 2010

**General Instructions:** This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas.

Each solution should typically take less than 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

---

**Quantum Mechanics 1**

Calculate the transmission coefficient of the following 1D scatterer:

$$U(x) = W_1\delta(x) + W_2\delta(x - a)$$

as a function of the particle's energy and find its maximum value.

**Solution:**

Writing the scattering wavefunction with the appropriate scattering conditions at  $x \rightarrow \pm\infty$ , and matching the derivatives at  $x = 0, a$  we get

$$T = \frac{1}{|\alpha_1\alpha_2 \exp^{-ika} + (1 - i\alpha_1)(1 - i\alpha_2) \exp^{ika}|^2}, \text{ where } \alpha_{1,2} = \frac{mW_{1,2}}{\hbar^2 k}$$

To find the maximum transmission, one can view the denominator of this fraction as the squared magnitude of the difference of two vectors with lengths

$$|\alpha_1\alpha_2| \text{ and } |(1 - i\alpha_1)(1 - i\alpha_2)| = \sqrt{(1 - \alpha_1\alpha_2)^2 + (\alpha_1 + \alpha_2)^2} \geq |\alpha_1\alpha_2|.$$

By the proper choice of the particle's energy, i.e. parameter  $ka$ , the angle  $\theta = 2ka + \text{const}$  between these two vectors may be made zero, minimizing the denominator and maximizing the transparency:

$$T = \frac{1}{(\sqrt{(1 - \alpha_1\alpha_2)^2 + (\alpha_1 + \alpha_2)^2} - |\alpha_1\alpha_2|)^2}$$

## Quantum Mechanics 2

Calculate the differential and full cross sections of a spherical scatterer:

$$U(\vec{r}) = \begin{cases} U_0, & \text{for } r < R \\ 0, & \text{otherwise} \end{cases}$$

in the Born approximation.

### Solution:

For the spherically symmetric scatterer, one can use

$$f(\vec{k}, \vec{k}') = \frac{2mU_0}{\hbar^2 q} \int_0^R r dr \sin qr = -\frac{2mU_0}{\hbar^2 q^3} (\sin qR - qR \cos qR).$$

Thus,

$$\frac{d\sigma}{d\Omega} = 4 \left( \frac{mU_0}{\hbar^2 q^3} \right)^2 (\sin qR - qR \cos qR)^2.$$

In contrast to the previous problem in this case the scattering intensity has the axial symmetry about the axis of the incident wave propagation, so that the total cross section may be calculated as

$$\sigma = 2\pi \int_0^\pi \frac{d\sigma}{d\Omega} \sin \theta d\theta, \quad q = 2k \sin \frac{\theta}{2}.$$

As recommended in Landau and Lifshitz, the easiest way to calculate this integral is to notice that since

$$dq = k \cos \frac{\theta}{2} d\theta, \quad \text{and} \quad q dq = 2k^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} d\theta = k^2 \sin \theta d\theta,$$

we can replace  $\sin \theta d\theta$  with  $q dq / k^2$ , and write

$$\sigma = \frac{8\pi}{k^2} \left( \frac{mU_0}{\hbar^2} \right)^2 \int_0^{2k} \frac{(\sin qR - qR \cos qR)^2}{q^6} q dq = \frac{8\pi}{k^2} \left( \frac{mU_0}{\hbar^2} \right)^2 \int_0^{2kR} \frac{(\sin \zeta - \zeta \cos \zeta)^2}{\zeta^5} d\zeta.$$

Working out the integral, one has finally

$$\sigma = \frac{2\pi}{k^2} \left( \frac{mU_0 R^2}{\hbar^2} \right)^2 \left[ 1 - \frac{8(kR)^2 + 1 - \cos 4kR - 4kR \sin 4kR}{32(kR)^4} \right].$$

## Quantum Mechanics 3

A 1D quantum harmonic oscillator of frequency  $\omega$  is initially in its ground state. At time  $t = 0$ , a constant force  $F$  is abruptly applied to the oscillator. Express the force-related term in the Hamiltonian through the creation/annihilation operators  $a^\dagger, a$ , and using these operators find

- a. the state  $|\psi(t)\rangle$  of the oscillator at  $t > 0$  in the basis  $\{|n\rangle\}$  of the new energy eigenstates, and
- b. the time dependence of average coordinate  $\langle x \rangle(t)$  and momentum  $\langle p \rangle(t)$  of the oscillator.

**Solution:**

(a) Using the standard expression of the oscillator coordinate  $x$  in terms of the creation/annihilation operators  $a^\dagger, a$ , one can write the  $F$ -related term in the Hamiltonian as:

$$V = -Fx = -F\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a). \quad (1)$$

Then the total Hamiltonian of the oscillator at  $t > 0$  is:

$$H = \hbar\omega a^\dagger a - F\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a) = \hbar\omega b^\dagger b + \text{const}, \quad (2)$$

where

$$b = a - f, \quad f \equiv \frac{F}{(2\hbar m\omega^3)^{1/2}}.$$

Since the force was switched on abruptly, the state  $|\psi(0)\rangle$  of the oscillator right after switching on of the force  $F$  coincides with the initial ground state:  $a|\psi(0)\rangle = 0$ , and satisfies the following condition in terms of the new creation/annihilation operators  $b^\dagger, b$ :

$$b|\psi(0)\rangle = -f|\psi(0)\rangle,$$

i.e. it represents a coherent state. It can be expanded

$$|\psi(0)\rangle = \sum_n c_n |n\rangle$$

in the basis  $\{|n\rangle\}$  of the new energy eigenstates of the oscillator:

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n |0\rangle,$$

where the new ground state is defined as usual by

$$b|0\rangle = 0.$$

The expansion coefficients are found directly:

$$c_n = \langle n|\psi(0)\rangle = \frac{1}{\sqrt{n!}}\langle 0|b^n|\psi(0)\rangle = \frac{(-f)^n}{\sqrt{n!}}\langle 0|\psi(0)\rangle,$$

and after proper normalization

$$c_n = \frac{(-f)^n}{\sqrt{n!}} e^{-f^2/2}.$$

Taking into account the standard time evolution of the energy eigenstates, we obtain the wavefunction of the oscillator in the basis of the new energy eigenstates:

$$|\psi(t)\rangle = \sum_n \frac{[-f e^{-i\omega t}]^n}{\sqrt{n!}} |n\rangle e^{-f^2/2}. \quad (3)$$

(b) Using the wavefunction  $|\psi(t)\rangle$  and the standard properties of the creation/annihilation operators one finds:

$$\langle\psi(t)|b|\psi(t)\rangle = -f e^{-i\omega t}, \quad \langle\psi(t)|b^\dagger|\psi(t)\rangle = -f e^{i\omega t},$$

and from this, the average coordinate  $\langle x\rangle(t)$  and momentum  $\langle p\rangle(t)$ :

$$\langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}} (2f - 2f \cos \omega t) = \frac{F}{m\omega^2} (1 - \cos \omega t), \quad (4)$$

$$\langle p\rangle(t) = -i\sqrt{\frac{\hbar m\omega}{2}} f (e^{i\omega t} - e^{-i\omega t}) = \frac{F}{\omega} \sin \omega t. \quad (5)$$

Note that these are exactly the classical equations, in agreement with Ehrenfest's theorem.

### Statistical Mechanics 1

Derive the prediction from the Landau-Ginzburg mean field formalism for the critical exponents as  $T - T_c \rightarrow 0^+$  and  $T - T_c \rightarrow 0^-$  for the (isothermal, zero-field) susceptibility (e.g. of a ferromagnet) in a second-order phase transition.

#### Solution:

The second-order phase transition occurs for  $T = T_c$ ,  $H = 0$ . Here the Landau-Ginzberg expansion for the Helmholtz free energy is ( $T$ =temp,  $M$ =magnetism)

$$A(T, M) = c_0 + c_2 T^2 + c_4 T^4$$

where  $c_4 > 0$ . The coefficient  $c_2$  changes sign at the critical temperature,  $T_c$ , for the second-order phase transition. So we write it as  $c_2 = c_{21}(T - T_c)$  with  $c_{21} > 0$  and higher order terms in  $(T - T_c)$ . Now the external field  $H$  satisfies

$$H = \frac{\partial A}{\partial M}|_T = 2c_2 M + 4c_4 M^3 = 2M(c_2 + 2c_4 M^2)$$

and the isothermal compressibility  $\chi_T$  is given by  $\chi_T = \frac{\partial M}{\partial H}|_T$ , so

$$\chi_T^{-1} = \frac{\partial H}{\partial M}|_T = \frac{\partial^2 A}{\partial M^2}|_T = 2c_2 + 12c_4 M^2$$

The zero-field susceptibility is given by  $\chi_T(H=0)$ . If  $T > T_c$ , then setting  $H = 0 \rightarrow M = 0$ , i.e. there is no magnetism. If  $T < T_c$ , then for  $T$  near  $T_c$ ,  $M^2 = -c_2/(2c_4)$ , i.e.  $M = \pm\sqrt{c_{21}(T-T_c)/(2c_4)}$ . This implies that the critical exponent for magnetization  $\beta$  is  $\beta = 1/2$ . For  $\chi$ , consider  $T - T_c \rightarrow 0^+$ . Then  $\chi_{T,H=0}^{-1} = 2c_{21}(T - T_c)$  so  $\chi(T, H=0) = 1/(2c_{21}(T - T_c))$ . The critical exponent for the susceptibility is defined as

$$\chi(T, H=0) = \begin{cases} (T - T_c)^{-\gamma}, & (T - T_c) \rightarrow 0^+ \\ (T - T_c)^{-\gamma'}, & (T - T_c) \rightarrow 0^- \end{cases}$$

So from the above calculation, we read off that  $\gamma = 1$  is the prediction of the Landau-Ginzberg formalism for  $T > T_c$ . For  $T < T_c$ , we substitute  $M^2 = -c_2/(2c_4)$  in the expression for  $\chi_T^{-1}$ , so  $\chi(T, H=0) \approx 1/(4c_{21}(T_c - T))$  as  $T - T_c \rightarrow 0^-$  and thus  $\gamma' = 1$ .

## Statistical Mechanics 2

The Hamiltonian for the 1D Ising model may be written as  $\mathcal{H} = -J \sum_{\langle i,j \rangle} \eta_i \eta_j$ , where the classical Ising spin variable  $\eta_i = \pm 1$  on each site  $i$ , and  $\langle ij \rangle$  denotes nearest-neighbor pairs of sites. Derive the internal energy per site and the entropy per site for this model on a one-dimensional lattice in the thermodynamic limit.

### Solution:

For the zero-field Ising model, the partition function is

$$E = \sum_{\sigma_i} \exp^{-\beta H}$$

with  $\beta = k_B T$  and  $H = -J \sum \eta_i \eta_j$  and this sum over nearest neighbors. The energy  $E$  can be calculated exactly by a transfer matrix method or by summing the small  $\beta$  series. The resulting Gibbs free energy  $G(T, H) = G(T, 0)$  (for the zero field case) is  $G = U - TS$  with the value per site  $G = -k_B T \ln(2 \cosh K)$  where  $K = \beta J$ .

The internal energy  $U_{int} = -\partial Z / \partial \beta$  with the value per site

$$U = U_{int}/N = -J \tanh K$$

for  $N \rightarrow \infty$ , so

$$S = \frac{U - G}{T} = -J \tanh K + k_B T \ln(2 \cosh K)$$

This can also be written as

$$S = k_B [-K \tanh K + \ln(2 \cosh K)].$$

N.B.: The transfer matrix  $J$  is  $J_{ij} < \sigma_i | \exp^{-\beta H} | \sigma_j >$

## Statistical Mechanics 3

The Debye temperature of silicon is  $T_D \simeq 645$  K and its melting temperature is  $T_m = 1.69 \times 10^3$  K. Derive the functional dependence on temperature of the molar

specific heat of silicon in the Debye model, for  $T \simeq 10$  K (for this part of the problem, all that is asked for is the  $T$  dependence, not the prefactor). Be sure to explain the various assumptions of the model and the steps in the calculation. Give a rough sketch of the molar specific heat as a function of  $T$  for  $0 \leq T \leq 800$  K, labelling the horizontal and vertical axes carefully. What is the approximate value of this specific heat at  $T \simeq 800$  K and why?

**Solution:**

This is described in detail in e.g. Reif, Fundamentals of Thermal and Statistical Physics. The model approximates lattice vibrations in the harmonic oscillator approximation. The internal energy

$$U \propto \int_0^{\lambda_D} d^3k \hbar \omega D(\omega) / (\exp^{\beta \hbar \omega} - 1)$$

where  $\beta = 1/k_B T$ . Using  $\omega = vk$

$$U \propto \int_0^{\omega_D} \omega^2 d\omega \hbar \omega D(\omega) / (\exp^{\beta \hbar \omega} - 1)$$

in which  $\omega_D$  is defined by  $\hbar \omega_D = k_B T_D$ . The specific heat is given by

$$C = \frac{\partial U}{\partial T} = -k_B \beta^2 \frac{\partial U}{\partial \beta} \propto \frac{\beta^2}{v^3} \int \omega^2 d\omega (\hbar \omega)^2 D(\omega) / (\exp^{\beta \hbar \omega} - 1)^2$$

Converting to dimensionless form with  $x \equiv \beta \hbar \omega$  the integral becomes

$$C \propto \frac{1}{\beta^3} \int_0^{x_D} dx x^4 \exp^{-x} / (e^x - 1)^2$$

The defining  $x_D = \beta \hbar \omega_D = T_D/T$  allows us to see that for  $T \ll T_D$  the upper limit of the integral can be replaced with  $\infty$ . Then for silicon with  $T_D = 645$  K for silicon and  $C$  at  $T = 10$  K, so  $T \ll T_D$  and the result becomes

$$C \propto \beta^{-3} \propto T^3.$$

For  $T \gg T_D$  quantum effects are small, so we use classical equipartition theorem for the oscillator

$$H = \frac{1}{2} m \vec{r}^2 + \frac{1}{2} k \vec{r}^2$$

(with 3 translation degrees of freedom and 3 potential energy degrees of freedom). Each contribution gets  $\frac{1}{2} k_B T$  so  $U = 3N k_B T$  and  $C = 3N k_B$ . The molar specific heat is then  $C_m = 3R$  with  $R$  the ideal gas constant.