Physics PhD Qualifying Examination
Part I – Thursday, 12 January 2006

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

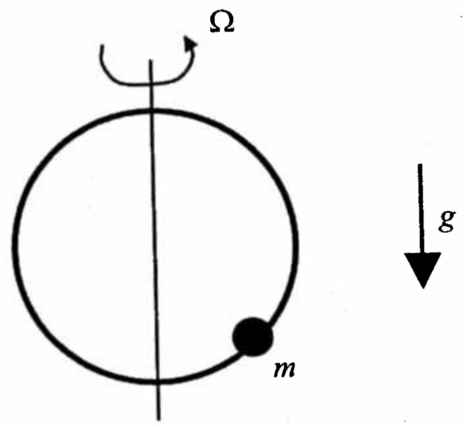
1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**

[I-1] **[10]**

A rocket in outer space in a negligible gravitational field starts from rest and accelerates uniformly at a until its final speed is v. The initial mass of the rocket is m_o, and the exhaust gas is ejected with velocity $-u$ with respect to the rocket. How much work does the rocket engine do?

[I-2] **[1,1,2,6]**

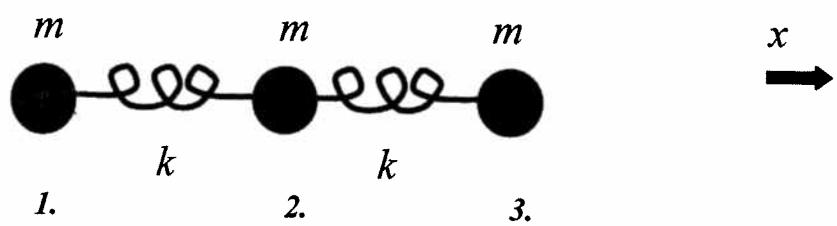
A particle of mass m is constrained to move (without friction) on a circular wire with radius a, which is rotating with a fixed angular velocity Ω about its vertical diameter in the presence of uniform gravity g.



(a) Find a convenient generalized coordinate and write down the Lagrangian of this system.
(b) Obtain the equation of motion.
(c) Find *all* possible equilibrium positions of the particle along the circular wire.
(d) Discuss the *stability* of the above equilibrium points depending on the value of the angular velocity Ω and obtain the *frequency of small oscillations* about the respective stable equilibrium positions on the wire.

[I-3] **[4,4,2]**

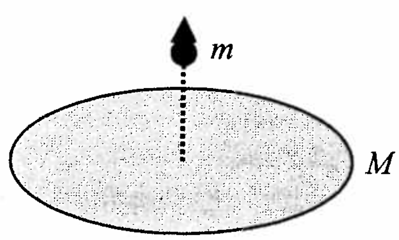
Three particles of equal mass move without friction in one dimension. Two of the particles are each connected to the third by a massless spring of spring constant k.



(a) Find the normal frequencies of the system.
(b) Determine the corresponding normal modes. If x_1, x_2, x_3 are the displacements of the respective masses from their equilibrium positions determine $x_1(t)$, $x_2(t)$, $x_3(t)$ for every normal mode frequency that exists for the system.
(c) Sketch the normal modes for the system.

[I-4] **[10]**

Consider a thin uniform disk of mass M and radius a. Find the force $\boldsymbol{F}$ on a mass m located along the axis through the center of the disk (perpendicular to the plane of the disc).



[I-5] **[10]**

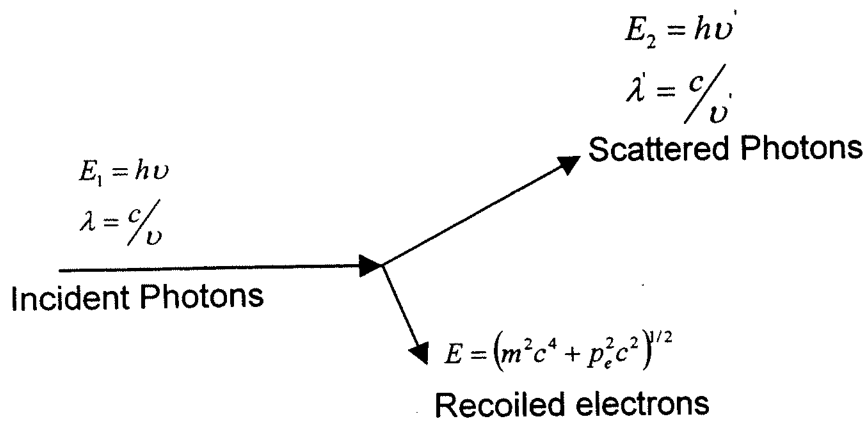
In the Compton effect, the incoming radiation is treated as a beam of "photons" with individual photons scattered elastically off individual electrons. The total energy of an electron is given by the relativistic relationship: $E = \left(m^2c^4 + p_e^2c^2\right)^{1/2}$. Here p_e is electron momentum and m is the rest mass of an electron.

Given the kinematics of the Compton effect shown below, derive the angular dependence of the scattering photons. More specifically, derive either of the following expressions.

(i) $2(h\upsilon)(h\upsilon')(1-\cos\theta) = 2mc^2(h\upsilon - h\upsilon')$ or

(ii) $\frac{h}{mc}(1-\cos\theta) = \lambda' - \lambda$.

Here "*h/mc*" is the Compton wavelength of the electron and is ~2.4 x 10^{-10} cm.
Also note that, for a photon, its rest mass is zero and the momentum and energy are related: $E(= h\upsilon) = pc$.



[I-6] [2,4,4]

Consider Laplace's equation in a source-free region.

(a) Consider two parallel plates if infinite extent in the x and y directions and separated by a distance d and a potential difference V_1 applied between them. Derive electric field strength in between the plates.

(b) Consider two, very long concentric, conducting cylinders along the z-direction. Their radius is ρ_1 and ρ_2, respectively. Given a potential difference V_2 between the two cylinders, find electric field strength in the region between the cylinders.

(c) Consider two concentric spheres, with radius r_1 and r_2, respectively. Given a potential difference V_3 between the two spheres, find electric field strength in the region between the cylinders.

[I-7] [4,3,3]

(a) Show that the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in a homogeneous medium without charges satisfy the telegraph equation (in Gaussian units):

$$\frac{\varepsilon\mu}{c^2}\frac{\partial^2 \boldsymbol{F}}{\partial t^2}+\frac{4\pi\mu\sigma}{c^2}\frac{\partial \boldsymbol{F}}{\partial t}=\nabla^2 \boldsymbol{F}$$

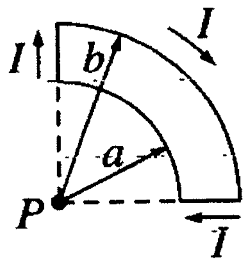
with $\boldsymbol{F}=\boldsymbol{E}$ and $\boldsymbol{B}$, and ∇^2 is the Laplacian operator. Herein ε is the dielectric constant, μ is the magnetic permeability, and σ is the electric conductivity. (Note that if you are more familiar, and intend to work with MKSA units, the above equation will have the form $\varepsilon\mu\frac{\partial^2 \boldsymbol{F}}{\partial t^2}+\mu\sigma\frac{\partial \boldsymbol{F}}{\partial t}=\nabla^2 \boldsymbol{F}$).

(b) Show furthermore that a planar monochromatic wave in such a medium is a transversal wave and that $\boldsymbol{E}$ is orthogonal to $\boldsymbol{B}$.

(c) What is the most general solution for monochromatic waves of $\boldsymbol{E}$ and $\boldsymbol{B}$ in an insulator ($\sigma=0$) propagating in z-direction?

[I-8] [10]

Calculate the magnetic induction $\boldsymbol{B}$ at point P for the steady-current configuration (with current magnitude I) shown below.



[I-9] [10]

An electron is released from rest and falls under the influence of uniform gravity g. After traveling an initial distance h, what fraction of the total potential energy lost is radiated away? (You should express your answer in terms of h, g, and constants related to the electron and to the relevant equations. In obtaining your result, you should consider the electro-magnetic fields far away from the electron.)

[I-10] [2,2,6]

The propagation of plane-wave in a dielectric medium along the z direction can be described as: $E(z,t) = E_0 \bullet e^{i(kz-\omega t)}$, where $k = \tilde{n}\dfrac{\omega}{c}$, and $\tilde{n}$ is the complex index of refraction. The complex refractive index of germanium at λ=400nm is given by $\tilde{n} = 4.141 + i2.215$. Calculate the following for germanium at λ=400nm:

(a) the phase velocity of light,
(b) the absorption coefficient, and
(c) the reflectivity.

QX: s 06

I-1) $\text{Work} = \int F\,dx = \int v\,dp$

WRT stationary reference system, during each dt,
$\Delta(\text{momentum}(\text{ROCKET} + \text{EXHAUST})) = 0$

$\Rightarrow \quad m\,dv\Big|_{\text{ROCKET}} = u\,dm\Big|_{\text{spent fuel}}$

$v = u \ln\left(\frac{M_0}{M}\right) \Rightarrow M\Big|_{\text{ROCKET}} = M_0 e^{-v/u}$

at all times, $a = \text{const} \Rightarrow v\Big|_{\text{ROCKET}} = at$

ROCKET: $dp = d(mv) = (ma)dt + (at)dm$

$= M_0 a e^{-at/u}\left(1 - \frac{at}{u}\right) dt$

$$W\Big|_{\text{ROCKET}} \equiv W_r = \frac{M_0 a}{u} \int_0^{t_f} at(u - at) e^{-at/u}\, dt$$

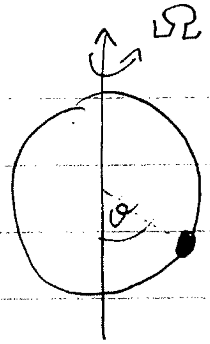
$$W\Big|_{\text{EXHAUST}} \equiv W_e = \frac{M_0 a}{u} \int_0^{t_f} (at - u)^2 e^{-at/u}\, dt$$

$t_f = v/a$

$$W_{total} = W_r + W_e = \frac{M_0 a}{u} \int_0^{v/a} (u^2 - uat) e^{-at/u}\, dt$$

$$= M_f u v_f \quad , \; M_f, v_f \Rightarrow \text{final values}$$

I-2



a) $L = \frac{m}{2}\left(a^2 \sin^2\vartheta\, \Omega^2 + a^2 \dot{\vartheta}^2\right) + mga\cos\vartheta$

ϑ is the <u>only</u> generalized coordinate

b) $\frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} = \frac{\partial L}{\partial \vartheta}$

$ma^2\ddot{\vartheta} = \Omega^2 m a^2 \sin\vartheta\cos\vartheta - mga\sin\vartheta$

$$\boxed{\ddot{\vartheta} = \Omega^2 \sin\vartheta\cos\vartheta - \frac{g}{a}\sin\vartheta}$$

c) stationary angle ϑ: $\vartheta = \text{const.}$ $\dot{\vartheta} = 0$ $\ddot{\vartheta} = 0$

$0 = \Omega^2 \sin\vartheta\cos\vartheta - \frac{g}{a}\sin\vartheta$

$0 = \left(\Omega^2\cos\vartheta - \frac{g}{a}\right)\sin\vartheta$ $\Omega_0^2 \equiv g/a$

$0 = \left(\Omega^2\cos\vartheta - \Omega_0^2\right)\sin\vartheta$

i) $\sin\vartheta = 0 \Rightarrow$ $\boxed{\vartheta_1 = 0}$ $\boxed{\vartheta_2 = \pi}$

ii) $\Omega^2\cos\vartheta - \Omega_0^2 = 0 \Rightarrow$ $\cos\vartheta_3 = \frac{\Omega_0^2}{\Omega^2}$ $(\Omega > \Omega_0)$

$$\boxed{\vartheta_3 = \cos^{-1}\left(\frac{\Omega_0^2}{\Omega^2}\right)}$$

d) expand equation of motion about respective equilibrium positions up to linear order in $(\vartheta - \vartheta^*)$

where $\vartheta^* = \vartheta_1, \vartheta_2,$ or ϑ_3

$$\ddot{\vartheta} = \Omega^2 \sin\vartheta \cos\vartheta - \Omega_0^2 \sin\vartheta \approx$$

$$\approx \underbrace{\Omega^2 \sin\vartheta^* \cos\vartheta^* - \Omega_0^2 \sin\vartheta^*}_{=0 \text{ since } \vartheta^* \text{ is eq.}} +$$

$$+ \left[\Omega^2 (\cos^2\vartheta^* - \sin^2\vartheta^*) - \Omega_0^2 \cos\vartheta^*\right](\vartheta - \vartheta^*) + \ldots$$

$$\boxed{\ddot{\vartheta} \approx \left[\Omega^2 (2\cos^2\vartheta^* - 1) - \Omega_0^2 \cos\vartheta^*\right](\vartheta - \vartheta^*)} \equiv -\omega^2(\vartheta - \vartheta^*)$$

stability: $\omega^2 > 0$

$\boxed{\text{for } \vartheta^* = \vartheta_1 = 0:}$ $\Omega^2 - \Omega_0^2 < 0$ if $\Omega^2 < \Omega_0^2$

i.e. $\vartheta_1 = 0$ stable if $\Omega < \sqrt{\frac{g}{a}}$ and $\omega = \sqrt{\Omega_0^2 - \Omega^2}$

$\boxed{\text{for } \vartheta^* = \vartheta_2 = \pi:}$ $\Omega^2 + \Omega_0^2$ is always > 0

<u>never stable</u>

$\boxed{\text{for } \vartheta^* = \vartheta_3 = \cos^{-1}\left(\frac{\Omega_0^2}{\Omega^2}\right):}$

$$\Omega^2\left(2\frac{\Omega_0^4}{\Omega^4} - 1\right) - \Omega_0^2 \frac{\Omega_0^2}{\Omega^2} = \Omega^2\left(\frac{\Omega_0^4}{\Omega^4} - 1\right) < 0 \text{ if } \Omega^2 > \Omega_0^2$$

i.e. ϑ_3 is stable if $\Omega > \sqrt{\frac{g}{a}}$ and $\omega = \Omega\sqrt{1 - \frac{\Omega_0^4}{\Omega^4}}$

Summary of "phase diagram": $\left(\Omega_0 \equiv \sqrt{\frac{g}{a}}\right)$

$\Omega < \sqrt{g/a}$: $\vartheta_1 = 0$ is stable and $\omega = \sqrt{\Omega_0^2 - \Omega^2}$

$\Omega > \sqrt{g/a}$: $\vartheta_3 = \cos^{-1}\left(\frac{\Omega_0^2}{\Omega^2}\right)$ is stable and $\omega = \Omega\sqrt{1 - \frac{\Omega_0^4}{\Omega^4}}$

[I-3] Solution

(a) Let x_1, x_2, x_3 be the displacement of the masses from their equilibrium position. Hence, the Lagrangian becomes

$$L = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - \frac{1}{2}k\left[(x_2-x_1)^2 + (x_3-x_2)^2\right]$$

and $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = 0$ gives.

$$m\ddot{x}_1 + k(x_1 - x_2) = 0$$

$$m\ddot{x}_2 + k(x_2 - x_1) + k(x_2 - x_3) = 0$$

$$m\ddot{x}_3 + k(x_3 - x_2) = 0$$

try solutions of the type $x_1 = Ae^{i\omega t}$, $x_2 = Be^{i\omega t}$, $x_3 = Ce^{i\omega t}$. Now write the above equations in matrix form

$$\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0$$

the secular equation becomes

$$\begin{vmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{vmatrix} = 0$$

[I-3] continued.

$$\left| \because \right| = m\omega^2(k - m\omega^2)(m\omega^2 - 3k) = 0$$

$\therefore$ the three non-negative roots are

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = \sqrt{\frac{3k}{m}}$$

these are the normal-mode angular frequencies of the system. The corresponding

(b) normal modes are as follows:

(i) $\omega_1 = 0$ from the matrix equation

$A = B = C \quad \therefore x_1 = x_2 = x_3$, then the first of the above three equations give

$\ddot{x}_1 = 0 \quad \Rightarrow x_1 = at + b$

(a and b) are constants, hence, in this mode the three masses undergo uniform translation together as a rigid body and no vibration occurs.

(ii) $\omega_2 = \sqrt{\frac{k}{m}}$ from matrix equation

$B = 0, \ A = -C$. In this mode the middle mass remains stationary while the other masses oscillate symmetrically with respect to it. The displacements are:

$$\left.\begin{aligned} x_1 &= A\cos(\omega_2 t + \varphi) \\ x_2 &= 0 \\ x_3 &= -A\cos(\omega_2 t + \varphi) \end{aligned}\right\} \text{with } \varphi \text{ being constant.}$$

(iii) $\omega_3 = \sqrt{\frac{3k}{m}}$, the matrix equation gives,

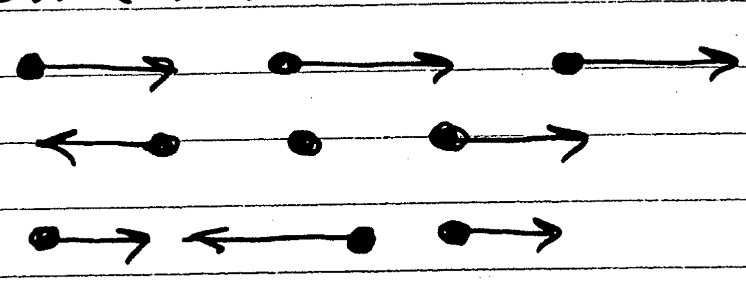
$B = -2A$, $C = A$, here the two outer masses oscillate with the same amplitude and phase, while the middle mass oscillate exactly out of phase with twice the amplitude with respect to the outer two masses. The displacements are:

$x_1 = A\cos(\omega_3 t + \varphi)$, $x_2 = -2A\cos(\omega_3 t + \varphi)$

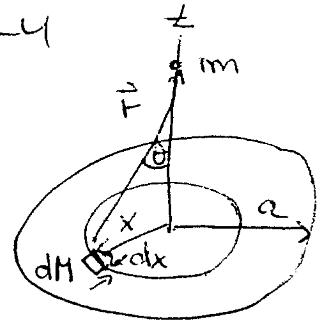
$x_3 = A\cos(\omega_3 t + \varphi)$

(c) The three normal modes are:

(i)
(ii)
(iii)



I-4



(1)

$$d\phi = -G\,\frac{dM}{r} \qquad \phi \text{ potential}$$

$$dM = \rho\, dA = \rho\, 2\pi x\, dx$$

$$d\phi = -G\rho 2\pi \frac{x\,dx}{r}$$

$$= -G\rho 2\pi \frac{x\,dx}{(x^2+z^2)^{1/2}}$$

$$\phi(z) = -\pi\rho G \int_0^a \frac{2x\,dx}{(x^2+z^2)^{1/2}} = -\pi\rho G 2\,(x^2+z^2)^{1/2}\Big|_0^a$$

$$= -2\pi\rho G\left[(a^2+z^2)^{1/2} - z\right]$$

$$\vec{F} = -m\vec{\nabla}\phi$$

$$F_z = -m\frac{\partial\phi(z)}{\partial z} = 2\pi m\rho G\left[\frac{z}{(a^2+z^2)^{1/2}} - 1\right]$$

(2)

$$dF_z = \cos\theta\,|d\vec{F}| = -mG\,\frac{\cos\theta\, dM}{r^2}$$

$$\cos\theta = \frac{z}{r}$$

$$dF_z = -mG\,\frac{z\,dM}{r^3}$$

$$= -mG\,\frac{z\,\rho\,2\pi x\,dx}{r^3}$$

$$F_z = -\pi m\rho G z\int_0^a \frac{2x\,dx}{(z^2+x^2)^{3/2}} = 2\pi m\rho G\left[\frac{z}{(a^2+z^2)^{1/2}} - 1\right]$$

$\boxed{\text{I-5}}$ (4)/2

From
i) Conservation of momentum: $\vec{p}_e = \vec{p}_1 - \vec{p}_2$ ...... ①

ii) Conservation of energy: $h\nu + mc^2 = h\nu' + (p_e^2c^2 + m^2c^4)^{1/2}$ ...... ②

from ①: $$|\vec{p}_e|^2 = |\vec{p}_1|^2 + |\vec{p}_2|^2 - 2|\vec{p}_1||\vec{p}_2|\cos\theta$$

$$|p_e|^2 = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\left(\frac{h\nu'}{c}\right)\left(\frac{h\nu}{c}\right)\cos\theta$$

$$\therefore p_e^2c^2 = (h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta \quad \text{...... ③}$$

(for $\theta = 0$, $p_e c = h\nu - h\nu'$; we have the forward-scattering.)

from ②: $$p_e^2c^2 + m^2c^4 = (h\nu - h\nu' + mc^2)^2$$

$$p_e^2c^2 = (h\nu - h\nu')^2 + 2mc^2(h\nu - h\nu') \quad \text{...... ④}$$

(for $\theta = 0$, $p_e c = h\nu - h\nu'$, $\xRightarrow{\text{from ④}}$ $h\nu = h\nu'$.) ~~We have elastic scattering~~

no energy or momentum transfer.

Now, combine equation ③ & ④:

$$(h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cos\theta =$$

$$(h\nu)^2 + (h\nu')^2 - 2(h\nu)(h\nu')\cdot 1 + 2mc^2(h\nu - h\nu')$$

$$\therefore \boxed{2(h\nu)(h\nu')(1-\cos\theta) = 2mc^2(h\nu - h\nu')}$$

$$\frac{h}{mc^2}(1-\cos\theta) = \frac{1}{\nu'} - \frac{1}{\nu'}$$

$$\therefore \boxed{\frac{h}{mc}(1-\cos\theta) = \lambda' - \lambda}$$

Here, $\frac{h}{mc}$ is called the Compton-Wavelength of the election.

$$\frac{h}{mc} = \frac{6.6\times10^{-34}}{9.1\times10^{-31}\times3\times10^{8}} \approx 2.4\times10^{-10}\ cm$$

#

I-6 ①/3

(i) In rectangular co-ordinates

Laplace's equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} = 0$$

Given $V = f(Z)$ only

$$\nabla^2 V = \frac{d^2 V}{d Z^2} = 0$$

$$\therefore \underline{\underline{V = k_1 Z + k_2}} \quad \text{-------------- (1)}$$

Let $V = 0$ at $Z = 0$

Let $V = V_1$ at $Z = d$

we have $k_2 = 0$, $k_1 = \frac{V_1}{d}$ .... (2)

$$\therefore V = \frac{V_1}{d} Z$$

$$\therefore \vec{E} = -\frac{dV}{dZ} = -\frac{V}{d}\hat{Z} \quad \# \quad \text{---------- (3)}$$

1-6 ②/3

(ii) In cylindrical coordinates

Laplace's equation is:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Given that $V = f(\rho)$ only

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$$

$\therefore$ and $V = k_1(\ln\rho) + k_2$ ....(4)

Let: $V = V_+$ at $\rho = \rho_1$
$V = V_-$ at $\rho = \rho_2$

We find: $k_1 = (V_+ - V_-)/(\ln\rho_1 - \ln\rho_2) = V_a/\ln(\rho_1/\rho_2)$ ----(5)

The electric field: $\vec{E} = -\vec{\nabla}V = -\frac{\partial V}{\partial\rho}\hat{\rho}$

$\therefore \vec{E} = -\frac{k_1}{\rho}\hat{\rho}$, $k_1 = \frac{V_a}{\ln(\rho_1/\rho_2)}$ ----(6) #

I-6 ③/3

(iii) In spherical coordinates with no variations in θ and ϕ, Laplace's equation is: $\nabla^2 V = \frac{1}{r^2}\left(\frac{\partial}{\partial r} r^2 \frac{\partial V}{\partial r}\right) = 0$

A solution is: $V = \frac{k_1}{r} + k_2$ ----- (7)

Given: $V = V_+$ at $r = r_1$

$V = V_-$ at $r = r_2$

We have: $V_+ = \frac{k_1}{r_1} + k_2$

$V_- = \frac{k_1}{r_2} + k_2$

$\therefore k_1 = \frac{V_3}{\left(\frac{r_2 - r_1}{r_1 r_2}\right)}$ -------- (8)

The electric field strength between the sphere is: $\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial r}\hat{r}$

$\therefore \vec{E} = \frac{k_1}{r_2}\hat{r}$ -------- (9)

#

Solution I-7:

I-7

Telegraph equation $\Delta \vec{F} = \frac{\varepsilon\mu}{c^2}\ddot{\vec{F}} + \frac{4\pi\mu\sigma}{c^2}\dot{\vec{F}}$ for $\vec{F} = \vec{E}, \vec{H}$

$\rho = 0$

$\varepsilon, \mu, \sigma \neq 0$

a) $$\Delta \vec{E} = \text{grad div}\, \vec{E} - \text{rot rot}\, \vec{E}$$
$$= \text{grad}\, 4\pi\rho - \text{rot}\left(-\frac{1}{c}\dot{\vec{B}}\right)$$
$$= \frac{\mu}{c}\,\text{rot}\, \dot{\vec{H}}$$
$$= \frac{\mu}{c}\left(\frac{4\pi}{c}\dot{\vec{j}} + \frac{1}{c}\ddot{\vec{D}}\right)$$
$$= \frac{\mu}{c}\left(\frac{4\pi}{c}\dot{\vec{j}} + \frac{1}{c}\varepsilon\ddot{\vec{E}}\right)$$
$$\Delta \vec{E} = \frac{4\pi\mu\sigma}{c^2}\dot{\vec{E}} + \frac{\mu\varepsilon}{c^2}\ddot{\vec{E}} \quad \square$$

$$\Delta \vec{H} = \text{grad div}\, \vec{H} - \text{rot rot}\, \vec{H}$$
$$= \text{grad}\, 0 - \text{rot}\left(\frac{4\pi}{c}\dot{\vec{j}} + \frac{\dot{\vec{D}}}{c}\right)$$
$$= -\frac{4\pi}{c}\,\text{rot}\,\sigma\dot{\vec{E}} - \frac{1}{c}\varepsilon\,\text{rot}\,\dot{\vec{E}}$$
$$\Delta \vec{H} = \frac{4\pi\mu\sigma}{c^2}\dot{\vec{H}} + \frac{\varepsilon\mu}{c^2}\ddot{\vec{H}}$$

b) assume $\vec{E} = \vec{E}_0 \, e^{i(\omega t - \vec{k}\vec{r})}$

then $\vec{\nabla}\vec{E} = -\vec{k}\vec{E}_0 \, e^{i(\omega t - \vec{k}\vec{r})} = -\vec{k}\vec{E} \Rightarrow \vec{\nabla} = -\vec{k}$

$\hookrightarrow$ $\text{rot}\,\vec{E} = \vec{\nabla} \times \vec{E} = -\vec{k} \times \vec{E} = -\frac{1}{c}\dot{\vec{B}}$

$\text{rot}\,\vec{B} = \vec{\nabla} \times \vec{B} = -\vec{k} \times \vec{B} = \frac{\mu 4\pi}{c}\vec{E} + \frac{\mu}{c}\dot{\vec{E}}$

$\text{div}\,\vec{E} = \vec{\nabla}\vec{E} = -\vec{k}\vec{E} = 0$

$\text{div}\,\vec{B} = \vec{\nabla}\vec{B} = -\vec{k}\vec{B} = 0$

$\hookrightarrow$ $\vec{k}, \vec{E}, \vec{B}$ are orthogonal to each other

c) $\sigma = 0 \Rightarrow \Delta \vec{F} = \frac{\varepsilon\mu}{c^2} \ddot{\vec{F}}$

one solution in z-direction

$$\vec{F}(\vec{z},t) = \vec{F}_0 \exp(i(\omega t - kz))$$

with $k^2 = \frac{\varepsilon\mu}{c^2}\omega^2$ and $k = \omega \frac{\sqrt{\varepsilon\mu}}{c} = \omega \frac{n}{c}$

$\hookrightarrow \vec{E}(\vec{z},t) = \vec{E}_0 \exp(i(\omega t - kz))$

$\vec{H}(\vec{z},t) = \vec{H}_0 \exp(i(\omega t - kz))$

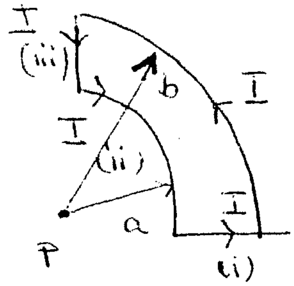
using $\operatorname{rot} \vec{H} = \frac{\varepsilon}{c} \dot{\vec{E}}$

$\hookrightarrow -ik H_0 = \frac{\varepsilon}{c} E_0 \cdot i\omega$

$$\frac{H_0}{E_0} = \frac{\varepsilon}{c} \frac{\omega}{k} = \frac{\varepsilon}{c} \frac{c}{\sqrt{\varepsilon\mu}} = \sqrt{\frac{\varepsilon}{\mu}}$$

I-8

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l} \times \vec{r}}{r^3}$$



(i), (iii) these segments of the current do not result in a contribution to $\vec{B}$ because $\vec{r}$ is parallel to $d\vec{l}$

(ii)

$$|\vec{B}_a| = \frac{\mu_0 I}{4\pi} \int_0^{\pi/2} \frac{a\,d\phi}{a^2} = \frac{\mu_0 I}{4\pi} \frac{1}{a} \frac{\pi}{2}$$

$$|\vec{B}_b| = \frac{\mu_0 I}{4\pi} \int_0^{\pi/2} \frac{b\,d\phi}{b^2} = \frac{\mu_0 I}{4\pi} \frac{1}{b} \frac{\pi}{2}$$

total B at P

$$B = \frac{\mu_0 I}{8} \left(\frac{1}{a} - \frac{1}{b} \right)$$

QX : s06

I-9) dipole moment $= \underline{p} = (-e) z \hat{z}$; $z = \frac{1}{2} g t^2 \Rightarrow \ddot{\underline{p}} = -ge\hat{z}$

$$\nabla V \doteq \frac{-1}{4\pi\epsilon_0 c^2} \frac{[\hat{r}\cdot\ddot{\underline{p}}(t_0)]}{r}\hat{r}, \quad t_0 = t - r/c$$

$$\frac{\partial A(t)}{\partial t} \doteq \frac{\mu_0}{4\pi} \frac{\ddot{\underline{p}}(t_0)}{r}$$

$$\underline{E}(\underline{r},t) = -\frac{\partial A}{\partial t} - \nabla V \Rightarrow \frac{\mu_0}{4\pi r}\left[\hat{r}\times(\hat{r}\times\ddot{\underline{p}}(t_0))\right]$$

$$\underline{B}(\underline{r},t) = \underline{\nabla}\times\underline{A} = \frac{-\mu_0}{4\pi r c}\left[\hat{r}\times\ddot{\underline{p}}\right]$$

$$\underline{S} = \frac{1}{\mu_0}(\underline{E}\times\underline{B}) = \frac{\mu_0}{16\pi^2 c}\left[\ddot{\underline{p}}(t_0)\right]^2\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}$$

$$\text{Power} \simeq \int \underline{S}\cdot d\underline{a} = \frac{\mu_0 \ddot{p}^2}{6\pi c} = \frac{\mu_0 (ge)^2}{6\pi c}$$

distance fallen $h = \frac{1}{2} g t_f^2 \Rightarrow t_f = \sqrt{\frac{2h}{g}}$

energy radiated $U_{RAD} = P t_f = \frac{\mu_0 (ge)^2}{6\pi c}\sqrt{\frac{2h}{g}}$

pot. energy lost $= mgh$

$$f_{rad} = \frac{U_{RAD}}{mgh} = \frac{\mu_0 e^2}{6\pi m c}\sqrt{\frac{2g}{h}} \rightarrow \left(2.76\times 10^{-22}\right)$$

I-10 ①/2

plane-wave propagation along z-direction.

$$\vec{E}(z,t) = \vec{E}_0 \, e^{i(kz-\omega t)}$$

where $k = \tilde{n} k_0$, k_0 - wave-vector in free-space.

(a) The phase velocity of light is the velocity of constant phase-front:

$$v_p = \frac{c}{n} = \frac{3\times10^8 \, m/s}{4.141} = \underline{7.2\times10^7 \, m/s}$$

$$\tilde{n} \equiv n + i\kappa$$

(b) The absoption constant (α) is measure of the attenuation of light Intensity

$$I \propto E^* E = |E_0|^2 \, e^{-2\cdot\kappa\cdot\frac{\omega}{c}\cdot z}$$

$$I \propto I_0 \, e^{-\alpha z}$$

$$\therefore \alpha = 2\cdot\kappa\cdot\frac{\omega}{c} = 4\pi\frac{\kappa}{\lambda}$$

Given $\kappa = 2.215$, $\lambda = 400nm$, $\underline{\alpha = 6.96\times10^7/m}$

(I-10) 2/2

(c) The reflectivity depends on both n and κ.

$$R = \left| \frac{\tilde{n}-1}{\tilde{n}+1} \right|^2 = \frac{(n-1)^2 + \kappa^2}{(n+1)^2 + \kappa^2}$$

Given: $n = 4.141$, $\kappa = 2.215$,

$R = 47\%$

Physics PhD Qualifying Examination
Part II – Monday, 16 January 2006

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box_of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**

[II-1] **[2,8]**

A particle is in a one-dimensional potential well given by $V(x) = -c\delta(x)$, where $\delta(x)$ is the Dirac delta-function and $c > 0$.

(a) Show that the derivative of the wavefunction has a finite jump at $x = 0$. Obtain an expression for this discontinuity, $\Delta \equiv \psi'(0^+) - \psi'(0^-)$. (Hint: Integrate Schrödinger's equation from $-\varepsilon$ to $+\varepsilon$ and take the $\varepsilon \to 0$ limit. Recall that $\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} \delta(x) f(x) dx = f(0)$ if $f(x)$ is continuous in $x = 0$.)

(b) Find the energy and the normalized wavefunction of the bound state(s).

[II-2] **[2,2,6]**

The Stark shift of the confined levels in a quantum well can be calculated by using second-order perturbation theory. Consider the interaction between the electrons in a quantum well of width d and a DC electric field of strength ε_z applied along the z-axis (the growth axis). Electrons have an effective mass of m^*.

(a) Explain why the perturbation to the energy of an electron is given by: $H' = e\varepsilon_z z$.

(b) Explain why the first order shift of the energy level, given by:

$$\Delta E^{(1)} = \int_{-\infty}^{\infty} dz\, \varphi(z)^* H' \varphi(z), \text{ is zero.}$$

(c) The second-order energy shift of the n=1 level is given by: $\Delta E^{(2)} = \sum_{n>1} \frac{|\langle 1 | H' | n \rangle|^2}{E_1 - E_n}$, where $\langle 1 | H' | n \rangle = \int_{-\infty}^{\infty} dz \varphi_1(z)^* H' \varphi_n(z)$. This can be evaluated exactly if we have infinite confining barriers. Within this approximation, show that the Stark shift is given approximately by: $\Delta E = -\alpha(\frac{e^2 \varepsilon_z^2 m^* d^4}{\hbar^2})$. α is a numerical constant.

[II-3] [10]

Consider a three-dimensional rigid rotator consisting of two particles, each of mass M, separated by a weightless rigid rod of length $2a$. The midpoint of the rotator is fixed in space.

(a) Find the Hamiltonian and write down the time-independent Schrödinger equation for this system.
(b) Find the energy eigenvalues of the system
(c) How many eigenfunctions exist for any eigenvalue?

[II-4] [10]

Calculate the total cross-section for scattering from a Yukawa potential, $V(r) = \beta \frac{e^{-\mu r}}{r}$, in the Born approximation. Express your answer as a function of particle energy E.

[II-5] [10]

Consider the expression

$$J = \frac{\hbar}{2mi}\left(\Psi^{*}\nabla\Psi - \Psi\nabla\Psi^{*}\right),$$

where J is the ***probability current density***, Ψ is the wave function, and ∇ is the gradient operator. This expression gives the probability that one particle per second will pass through a unit area normal to the direction of J. A beam of particles with uniform velocity v enters a region where some of them are absorbed. This absorption may be represented by the introduction of a constant complex potential $V_r - iV_i$ into the wave equation. Show that the cross section per atom for absorption is $\sigma = 2V_i/(\hbar N v)$, where N is the number of absorbing atoms per unit volume.

[II-6] [10]

A hydrogen atom in its ground state is placed between the parallel plates of a capacitor. For times $t < 0$, no voltage is applied. Starting at $t = 0$, a voltage, and hence, a uniform electric field $E(t) = E_0 \exp(-t/\tau)$ is applied, where τ is a constant. Derive the formula for the probability that the electron ends up in state "j" due to this perturbation. Evaluate the result for:

(a) when j is a 2s state;
(b) when j is a 2p state.

Hint: The first few normalized time-independent eigenfunctions are:

1S $$\varphi_{100} = \frac{2}{a_0^{3/2}} e^{-r/a_0} Y_0{}^0(\theta,\phi)$$

2S $$\varphi_{200} = \frac{2}{(2a_0)^{3/2}} (1 - r/2a_0) e^{-r/2a_0} Y_0{}^0(\theta,\phi)$$

2P $$\begin{pmatrix} \varphi_{211} \\ \varphi_{210} \\ \varphi_{21-1} \end{pmatrix} = \frac{1}{\sqrt{3}\,(2a_0)^{3/2}} \frac{r}{a_0} e^{-r/2a_0} \begin{pmatrix} Y_1{}^1(\theta,\phi) \\ Y_1{}^0(\theta,\phi) \\ Y_1{}^{-1}(\theta,\phi) \end{pmatrix}$$

with

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(x) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta$$

$$Y_1^0(x) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta$$

$$Y_1^1(x) = \frac{-1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta$$

[II-7] **[10]**

For a simple liquid-gas system the equation of state $P(T, \rho)$ is given by

$$P = -\frac{1}{2}ab\rho^2 - \frac{kT}{b}\ln(1 - b\rho),$$

where T is the absolute temperature, ρ is the density of particles (number of particles per unit volume), a and b are material-specific parameters, and k is the Boltzman constant.
Determine the critical point of the liquid-gas phase transition in this system, i.e., find T_c, ρ_c, and P_c. Sketch the $P(\rho)$ isotherms for $T > T_c$, $T = T_c$, and $T < T_c$. What is a physical interpretation of the parameter b based on the equation of state?

[II-8] **[10]**

Consider an elastic band with the equation of state $f = \gamma l T$, where f is the tension, l is the length of the band, T is the absolute temperature, and γ is a material-specific constant.

Obtain $\left(\frac{\partial E}{\partial l}\right)_T$.

Hint: First, write down the applicable form of the First Law and then use the appropriate Maxwell relation for the above system to obtain your answer.

[II-9] **[10]**

A container of volume V is initially divided in half: One half contains oxygen at pressure P, the other half contains nitrogen at the same pressure. Both gases may be considered ideal. The system is in an adiabatic enclosure at temperature T. Then we remove the wall separating the two halves so that the two gases are allowed to mix.

(a) Does the temperature change in the process? If so, by how much? Does the entropy change? If so, by how much?
(b) How would the result differ if both sides contained oxygen?
(c) Now consider one half of the enclosure filled with diatomic molecules of oxygen isotope ^{16}O and the other half with ^{18}O. Will the answer be different from parts (a) and (b)?

[II-10] **[7,3]**

(a) Derive the relation between ***pressure and volume*** for a free non-relativistic electron gas in three dimensions (obeying Fermi-Dirac statistics) at zero temperature.
(b) The formula obtained in (a) is approximately correct for sufficiently low temperatures (the so-called strongly degenerate gas). Discuss the applicability of this formula to ***common metals***. State the condition for strong degeneracy and express it in terms of an inequality relating temperature to Fermi energy. Are typical metals strongly degenerate even at room temperature?

II/1 [redacted] A particle is in a one-dimensional potential given by $V(x) = -c\,\delta(x)$, where $\delta(x)$ is the Dirac δ-function and $c > 0$.

a) Show that the derivative of the wave function, $\psi'(x)$ is not continuous at $x=0$ but has a finite jump. Give an expression for the value of this discontinuity, $\Delta = \psi'(0^+) - \psi'(0^-)$.

<u>Hint:</u> Integrate Schrödinger's equation between $-\varepsilon$ and $+\varepsilon$ and then let $\varepsilon \to 0$

Recall that $\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{+\varepsilon} \delta(x) f(x) = f(0)$ if $f(x)$ is cont func

b) Find the energy and the normalized wave function of the bound state(s).

<u>Solution</u>

a)

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

$$\psi''(x) = -[E - V(x)]\psi(x) \cdot \frac{2m}{\hbar^2}$$

$$\int_{-\varepsilon}^{+\varepsilon} \psi''(x)\,dx = -\int_{-\varepsilon}^{+\varepsilon} (E - V(x))\psi(x)\,dx \cdot \frac{2m}{\hbar^2}$$

$\psi(x)$ is continuous.

$\varepsilon \to 0$:

$$\psi'(0^+) - \psi'(0^-) = \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{+\varepsilon} -c\,\delta(x)\psi(x)\,dx \cdot \frac{2m}{\hbar^2} = -\frac{2mc}{\hbar^2}\psi(0)$$

b) $\quad x \neq 0 \qquad \psi'' = -\dfrac{2mE}{\hbar^2}\psi$

bound state(s) $\quad : E < 0$

$$\psi'' = \frac{2m|E|}{\hbar^2}\psi$$

$x < 0 \qquad \psi(x) = A e^{\varkappa x}$ $\qquad\qquad \varkappa \equiv \sqrt{\dfrac{2m|E|}{\hbar^2}}$

$x > 0 \qquad \psi(x) = B e^{-\varkappa x}$

i) $\psi(x)$ is continuous at $x=0$: $\qquad A = B$

ii) $\quad \psi'(0^+) - \psi'(0^-) = -\dfrac{2mc}{\hbar^2}\psi(0)$

$-\varkappa A - \varkappa A = -\dfrac{2mc}{\hbar^2} A \qquad , \; A \neq 0$

$\varkappa = \dfrac{mc}{\hbar^2} \qquad$ (one bound-state)

$\varkappa^2 = \dfrac{m^2c^2}{\hbar^4} = \dfrac{2m|E|}{\hbar^2} \Rightarrow |E| = \dfrac{mc^2}{2\hbar^2}$, or $E = -\dfrac{mc^2}{2\hbar^2}$

and

$\psi(x) = \sqrt{\varkappa}\, e^{-\varkappa|x|} \qquad$ from $\displaystyle\int_{-\infty}^{+\infty} |\psi|^2 dx = 1$

II-2 ①/3

(i) $H' = -p_z \cdot \mathcal{E}_z$, where $p_z = -eZ$

$\therefore H' = e\,\mathcal{E}_z Z$ #

(ii) Z is an odd function.

$\phi(z)\,\phi^*(z)$ is an even function.

$\therefore \int \phi^*(z) \cdot Z \cdot \phi(z)\, dZ = 0$

II-2 3/3

$$\Delta E^{(2)} \approx \frac{|\langle 1|H'|2\rangle|^2}{E_1 - E_2}$$

$$= e^2 \mathcal{E}_z^2 \frac{d^2}{\pi^4} \frac{2^8}{3^4}$$

$$E_1 - E_2 = -3 \frac{\hbar^2}{2m^*} \frac{\pi^2}{d^2}$$

$$\Delta E^{(2)} \approx e^2 \mathcal{E}^2 \frac{d^2}{\pi^4} \frac{2^8}{3^4} \frac{(-) 2m^* d^2}{3\hbar^2 \pi^2}$$

$$= -24 \left(\frac{2}{3\pi}\right)^6 \frac{e^2 \mathcal{E}^2 m^* d^4}{\hbar^2}.$$

$$E = E_0 + \Delta E^{(1)} + \Delta E^{(2)} + \cdots$$

$$\Delta E = \Delta E^{(1)} + \Delta E^{(2)} + \cdots$$

$$\approx \Delta E^{(2)}$$

II-2 ②/3

Q. Confined Stark Effect

(iii)

$$\Delta E^{(2)} = \sum_{n>1} \frac{|\langle 1|H'|n\rangle|^2}{E_1 - E_n}$$

$$= \frac{|\langle 1|H'|2\rangle|^2}{E_1 - E_2} + \frac{|\langle 1|H'|3\rangle|^2}{E_1 - E_3} + \frac{|\langle 1|H'|4\rangle|^2}{E_1 - E_4} + \cdots$$

$$\langle 1|H'|n\rangle = \int_{-\infty}^{\infty} \phi_1^* H' \phi_n \, dz \qquad \phi_n = \sqrt{\frac{2}{d}} \sin\left(\frac{n\pi z}{d}\right)$$

$$= \int_0^d \sqrt{\frac{2}{d}} \sin\left(\frac{\pi z}{d}\right) \sin\left(\frac{n\pi z}{d}\right) e\mathcal{E}_z z \, dz \qquad H' = e\mathcal{E}_z z$$

$$= \frac{2e\mathcal{E}_z}{d} \int_0^d \frac{1}{2} \left[\cos\frac{(n-1)\pi z}{d} - \cos\frac{(n+1)\pi z}{d}\right] z \, dz$$

$$= \frac{e\mathcal{E}_z}{d} \left[\int_0^d \cos\frac{(n-1)\pi z}{d} z \, dz - \int_0^d \cos\frac{(\pi+1)z}{d} z \, dz\right]$$

$$= \frac{e\mathcal{E}_z}{d} \left[\frac{-d^2}{(n-1)^2\pi^2}(1-\cos(n-1)\pi) + \frac{d^2}{(n+1)^2\pi^2}(1-\cos(n+1)\pi)\right]$$

$n:2$ $\quad \langle 1|H'|2\rangle = -e\mathcal{E}_z \dfrac{d}{\pi^2} \dfrac{16}{9}$

$n:3$ $\quad \langle 1|H'|3\rangle = 0$

$n:4$ $\quad \langle 1|H'|4\rangle = -e\mathcal{E}_4 \dfrac{d}{\pi^2} \dfrac{32}{9\times 25}$

$\vdots$

$$\frac{|\langle 1|H'|2\rangle|^2}{|\langle 1|H'|4|^2} = \left(\frac{\frac{16}{9}}{\frac{32}{9\times25}}\right)^2 = \frac{625}{4}$$

$$E_n = \frac{\hbar^2}{2m^*} \frac{n^2\pi^2}{d^2}$$

$$\frac{E_1 - E_2}{E_1 - E_4} = \frac{3}{15} = \frac{1}{5} \Rightarrow (\text{term } 1) \approx 625 \times (\text{term } 3)$$

II-3

moment of inertia about axis of rotation $I = 2Ma^2$

rotational energy $E = \frac{L^2}{2I}$

(a) Hamiltonoperator $\hat{H} = \frac{\hat{L}^2}{2I}$ with $\hat{L}^2$ angular momentum operator

time-independent Schroedinger equation

(b) $$\hat{H}\psi = \frac{\hat{L}^2}{2I}\psi = E\psi$$

$$\hat{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle$$

(c) $$E_l = \frac{\hbar^2 l(l+1)}{2I}$$

(d) $2l+1$ eigenfunctions for each E_l

QX : so6

II-4) $V = \beta \dfrac{e^{-\mu r}}{r}$

$$f(\theta) \cong \frac{-2m\beta}{\hbar^2 K} \int_0^\infty e^{-\mu r} \sin(Kr)\, dr \quad ; \quad K = 2k \sin\left(\frac{\theta}{2}\right)$$

$$= \frac{-2m\beta}{\hbar^2(\mu^2+K^2)}$$

$$\sigma = \int D(\theta) \sin\theta\, d\theta\, d\phi \quad ; \quad D(\theta) = |f(\theta)|^2$$

$$= 2\pi \left(\frac{2m\beta}{\hbar^2}\right)^2 \frac{1}{\mu_4} \left(\frac{\mu}{k}\right)^2 \int_{x_0}^{x_1} \frac{x\, dx}{(1+x^2)^2}$$

$x_0 = 0 \,,\; x_1 = \dfrac{2k}{\mu}$

$$\sigma = \pi \left(\frac{4m\beta}{\mu\hbar}\right)^2 \frac{1}{(\mu\hbar)^2 + 8mE} \quad ; \quad E = \frac{\hbar^2 k^2}{2m}$$

[II-5] Solution

$$\vec{J} = \frac{\hbar}{2mi}\left[\psi^* \vec{\nabla}\psi - \psi\vec{\nabla}\psi\right]$$

$$\vec{\nabla}\cdot\vec{J} = \frac{\hbar}{2mi}\left[\psi^*\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\psi^*\right] \quad \text{but:}$$

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi + (V_r - iV_i)\psi = i\frac{\partial\psi}{\partial t}$$

and $-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi^* + (V_r + iV_i)\psi^* = -i\frac{\partial\psi^*}{\partial t}$

now multiply the two equations above by ψ^* and ψ, respectively and subtract, then $\vec{\nabla}\cdot\vec{J} = \cdots$, becomes

$$\frac{\partial}{\partial t}(\psi^*\psi) + \vec{\nabla}\cdot\vec{J} = -\frac{2}{\hbar}V_i(\psi^*\psi)$$

Consequently, particles are being absorbed at the rate $\left(\frac{2}{\hbar}\right)V_i\,\psi^*\psi$ per unit volume.
By definition, the absorption cross section, thus must equal $N\sigma v\,\psi^*\psi$, where N is the density of absorbers.
Thus

$$\sigma = 2V_i/(\hbar N v).$$

Solution II-6

For time-dependent perturbations a general wave function is

$$\Psi(\vec{r},t) = \sum_j a_j(t)\, \psi_j(\vec{r})\, e^{-i\omega_j t} \tag{1}$$

Where the ψ_j satisfy

$$H_0 \psi_j = \hbar \omega_j \psi_j \tag{2}$$

For the time-dependent perturbation $V(t)$,

$$V(t) = -e|\vec{E}_0| z e^{-t/\tau} \tag{3}$$

From Schroedinger's equation we can derive an equation for the time development of the amplitudes $a_j(t)$:

$$i\hbar \frac{\partial}{\partial t}\Psi = [H_0 + V(t)]\Psi$$

$$i\hbar \frac{\partial}{\partial t} a_j(t) = \sum_\ell a_\ell \langle j|V(t)|\ell\rangle e^{it(\omega_j - \omega_\ell)} \tag{5}$$

If the system is initially in the ground state, we have $a_{1S}(0) = 1$ and then other values of $a_j(0)$ are zero. For small perturbations it is sufficient to solve the equation for $j \neq 1S$:

$$\frac{\partial}{\partial t} a_j(t) = \frac{ie|\vec{E}_0|}{\hbar} \langle j|z|1S\rangle e^{-t\left[\frac{1}{\tau} - i(\omega_j - \omega_{1S})\right]}$$

$$a_j(\infty) = \frac{ie|\vec{E}_0|\langle z\rangle}{\hbar} \int_0^\infty dt\, e^{-t\left[\frac{1}{\tau} - i(\omega_j - \omega_{1S})\right]}$$

$$= \frac{ie|\vec{E}_0|\langle z\rangle \tau}{\hbar[1 - i\tau(\omega_j - \omega_{1S})]}$$

The general probability P_j that a transition is made to state j is given by

$$P_j = |a_j(\infty)|^2 = \frac{(e|\vec{E}_0|\tau)^2 \langle j|z|1S\rangle^2}{\hbar^2[1 + \tau^2(\omega_j - \omega_{1S})^2]}$$

This probability is dimensionless. It should be less than unity for t this theory to be valid.

a) For the state $j = 2S$ the probability is zero. It vanishes because the matrix element of z is zero: $\langle 2S|z|1S\rangle = 0$ because of parity. Both S-states have even parity, and z has odd parity.

b) For the state $j = 2P$ the transition is allowed to the $L = 1$, $M = 0$ orbital state, which is called $2P_z$. The matrix element is similar to the problem of the Stark effect. The $2P$ eigenstate for $L = 1$, $S = 0$ is in Eqn. 5 and that for the $1S$ state is

$$\frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}$$

The integral is

$$\langle 2P_z|z|1S\rangle = \frac{2\pi}{\pi a_0^4\sqrt{32}}\int_0^\infty dr\, r^4 e^{-3r/2a_0}\int_0^\pi d\theta \sin\theta\cos^2\theta$$

$$= \frac{1}{3\sqrt{2}\, a_0^4}\int_0^\infty dr\, r^4 e^{-3r/2a_0} = a_0\left(\frac{2^{3/2}}{3}\right)^5$$

Where a_0 is the Bohr radius of the hydrogen atom.

$\boxed{\text{II-7}}$

$$P(T,\rho) = -\frac{1}{2}ab\rho^2 - \frac{kT}{b}\ln(1-b\rho)$$

at the critical point:

$$\frac{\partial P}{\partial \rho} = 0 \qquad \text{and} \qquad \frac{\partial^2 P}{\partial \rho^2} = 0$$

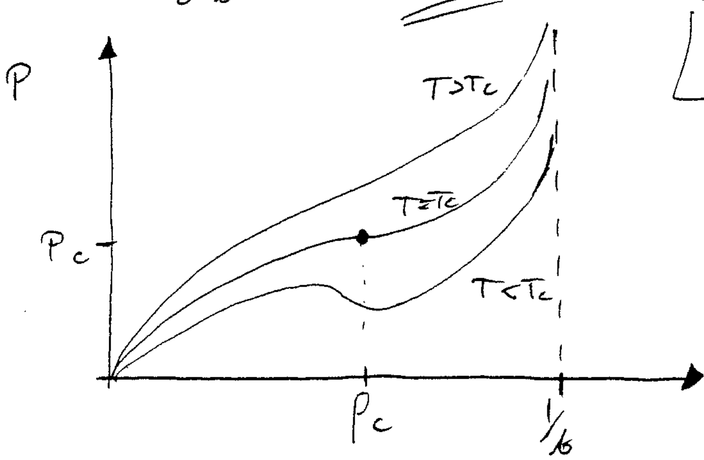
$$\begin{cases} \dfrac{\partial P}{\partial \rho} = -ab\rho + \dfrac{kT}{\cancel{b}}\dfrac{\cancel{b}}{1-b\rho} = 0 \\[2ex] \dfrac{\partial^2 P}{\partial \rho^2} = -ab + \dfrac{kTb}{(1-b\rho)^2} = 0 \end{cases}$$

$$\Downarrow$$

$$\rho_c = \frac{1-b\rho_c}{b} \quad \Longrightarrow \quad \underline{\underline{\rho_c = \frac{1}{2b}}}$$

$$kT_c = ab\rho_c(1-b\rho_c) = \frac{ab}{2b}\left(1-\frac{1}{2}\right) = \underline{\underline{\frac{a}{4}}}$$

$$P_c = -\frac{1}{2}ab\left(\frac{1}{2b}\right)^2 - \frac{a/4}{b}\ln\left(1-b\frac{1}{2b}\right) = -\frac{1}{8}\frac{a}{b} - \frac{a}{4b}\ln\left(1-\frac{1}{2}\right)$$
$$= \underline{\underline{\frac{1}{8}\frac{a}{b}\left[2\ln 2 - 1\right]}}$$



$\boxed{0 < \rho < 1/b}$

$\rho_{max} = \frac{1}{b} \sim \frac{N_{max}}{V}$ ("full packing")

$b N_{max} \sim V \Rightarrow b$ is the hard-core excluded volume of the particles

$\boxed{\text{II}-8}$ elastic band equation of state: $f = \gamma l T$

(1) First Law: $\underline{dE} = \delta Q + \delta W = \delta Q + f\,dl = \underline{T\,dS + f\,dl}$

consider $E(T,l)$: $\left(dE(T,l) = \left(\frac{\partial E}{\partial T}\right)_l dT + \left(\frac{\partial E}{\partial l}\right)_T dl \right)$

$F = E - TS$ (Helmholtz free energy)

$dF = -S\,dT + f\,dl$

$\Downarrow$ Maxwell relation

(2) $\underline{-\left(\frac{\partial S}{\partial l}\right)_T = \left(\frac{\partial f}{\partial T}\right)_l}$

Using (1): $\left(\frac{\partial E}{\partial l}\right)_T = T\left(\frac{\partial S}{\partial l}\right)_T + f$

using (2): $= -T\left(\frac{\partial f}{\partial T}\right)_l + f$

(using eq. of state:) $= -T\gamma l + \gamma l T \equiv \emptyset$

i.e., $\left(\frac{\partial E}{\partial l}\right)_T \equiv \emptyset$, no l-dependence of the internal energy, $E(l,T) = E(T)$.

Solution II-9

a) The energy of the mixture of ideal gases is simply the sum of energies of the two gases since we assume no interaction between them. Therefore, the temperature will not change upon mixing (**isothermal process**). The pressure also remains unchanged. The entropy of the mixture is simply the sum of the entropies of each gas in the total volume (as if there were no other gas).

For an ideal gas, the partition function can be separated into factors. The sum of N identical molecules must be divided by the number of interchanges $N!$ possible.

$$Z = \frac{1}{N!}\left(\sum_k e^{-E_k/\tau}\right)^N$$

Now, the Helmholtz free energy, F, is given by (we set $\tau = k_B T$)

$$F = -\tau \ln Z = -\tau \ln \frac{1}{N!}\left(\sum_k e^{-E_k/\tau}\right)^N$$

$$= -N\tau \ln\left(\sum_k e^{-E_k/\tau}\right) + \tau \ln N!$$

Using Sterling's formula,

$$\ln N! \approx N \ln (N/e)$$

we obtain

$$F = -N\tau \ln \sum_k e^{-E_k/\tau} + N\tau \ln \frac{N}{e} = -N\tau \ln\left[\frac{e}{N}\sum_k e^{-E_k/\tau}\right]$$

Using the explicit expression for the molecular energy E_k, we can rewrite F as

$$S_1 + S_2 = N_1 \ln \frac{eV_1}{N_1} + N_2 \ln \frac{eV_2}{N_2} - N_1 f_1'(\tau) - N_2 f_2'(\tau)$$

Therefore, the change in entropy ΔS, is given by

$$\Delta S = S - S_1 - S_2 = N_1 \ln \frac{V}{V_1} + N_2 \ln \frac{V}{V_2}$$

<u>Alternatively</u> we can start out with the well-known equation for isothermal expansion

$$dS = N\frac{dV}{V}\text{, which leads to } \Delta S = S_f - S_i = N\ln\frac{V_f}{V_i}.$$

This gives us the same formula.

In our case $V_1 = V_2 = V/2$, and

$$N_1 = N_2 = \frac{PV_1}{\tau} = \frac{PV}{2\tau}$$

So ΔS becomes

$$\Delta S = \frac{PV}{\tau} \ln 2$$

The entropy increased as it should because the process is clearly irreversible.

b) If the gases are the same, then the entropy after mixing is given by

$$S = (N_1 + N_2) \ln \frac{V_1 + V_2}{N_1 + N_2} - (N_1 + N_2) f'(\tau)$$

$$= 2N \ln \frac{V}{2N} - 2N f'(\tau)$$

and so $\Delta S = 0$. In the case of the identical gases, reversing the process only requires the reinsertion of the partition, whereas in the case where two dissimilar gases are mixed, some additional work has to be done to separate them again.

c) The same arguments as in a) apply for the mixture of two isotopes, ^{16}O and ^{18}O. The Gibbs free energy can be written in the form

$$G = N_1 \ln \frac{N_1}{N} + N_2 z \ln \frac{N_2}{N} + N_1 \mu_{01} + N_2 \mu_{02}$$

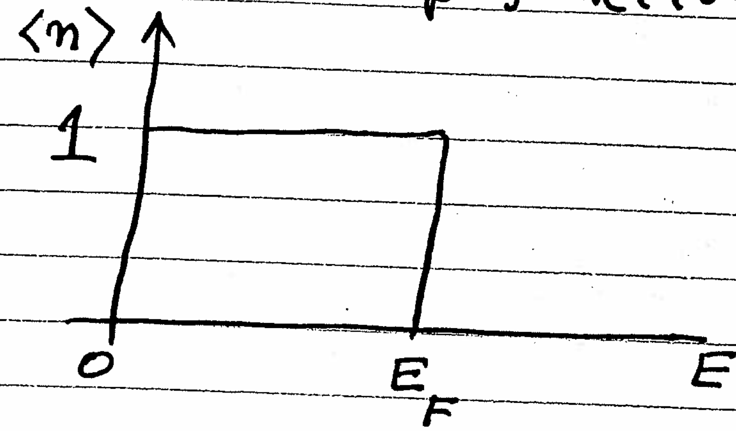
where μ_{01} and μ_{02} are the chemical potentials of pure isotopes. Therefore the potential G has the same form as in the mixture of two different gases, and there is no correction to the result of a). This is true as long as G can be written in this form, and it holds even after including quantum corrections to the order of h^2.

[II-10] <u>Solution</u> (Non-relativistic Electron gas)

(a) As $T \to 0$, the Fermi-Dirac distribution function

$$\langle n \rangle = \frac{1}{e^{(\epsilon-\mu)/kT}+1}$$

becomes a step function.



All states above a certain energy, $\epsilon > \mu$ are empty, and the states below, $\epsilon < \mu$, are filled (as shown above). This energy for an electron gas is called the Fermi energy. Physically this results from the simple fact that the total energy of the gas should be a minimum. However, we have to reconcile this with the Pauli principle, which prohibits more than one electron per quantum state (i.e. same momentum and spin). This means that the states are filled gradually from zero energy to the limiting energy, E_F. The number of

[II-10] continued.

states accessible to a free particle with absolute value of momentum between p and $p+dp$ is

$$d\Gamma = \frac{d^3p\, d^3q}{(2\pi\hbar)^3} = \frac{4\pi p^2 dp\, dV}{(2\pi\hbar)^3}$$

In each of these states, we can put two electrons with opposite spin, so if we consider the total number of electrons, N, contained in a box of volume V, then N is given by

$$N = 2V\int_0^{p_F} \frac{4\pi p^2 dp}{(2\pi\hbar)^3}$$

let $\frac{p^2}{2m} = \varepsilon$, we obtain

$$N = \frac{2\sqrt{2}\, m^{3/2} \varepsilon_F^{3/2} V}{3\pi^2 \hbar^3}$$ and therefore

$$\varepsilon_F = (3\pi^2)^{2/3} \frac{\hbar^2}{2m} \left(\frac{N}{V}\right)^{2/3}$$. To calculate

the total energy of the gas, we can write

$$E = 2\int^{\varepsilon_F} \varepsilon\, d\Gamma_p$$

with $d\Gamma_p = 4\pi p^2 dp\, dV/(2\pi\hbar)^3$.

[II-10] continued.

$$E = \frac{2\sqrt{2}}{5}\frac{Vm^{3/2}}{\pi^2\hbar^3}(2\pi^2)^{5/3}\left(\frac{\hbar^2}{2m}\right)^{5/3}\left(\frac{N}{V}\right)^{5/2}$$

$$E = \frac{3}{10}(3\pi^2)^{2/3}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{2/3}N \qquad \text{hence,}$$

$$P = -\left(\frac{\partial E}{\partial V}\right) = \frac{1}{5}(3\pi^2)^{2/3}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}$$

and we find: $\left[PV = \frac{2}{3}E\right]$

(b) The condition for strong degeneracy is that the temperature T should be much smaller than the Fermi energy:

$$T \ll \frac{\hbar^2}{m}\frac{1}{k_B}\left(\frac{N}{V}\right)^{2/3}.$$

For typical metals, if we assume that there is one free electron per atom and a typical interatomic distance $a \approx 2.5$ Å, we obtain an electron density $(N/V) \approx 5\cdot 10^{22}\ e/cm^3$, which indicates a Fermi energy of the order of 10^{-12} erg $\approx 10^4$ K. So, most of the metals are <u>strongly degenerate</u>, even at room temperature.