MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Department of Physics

February, 1998

DOCTORAL GENERAL EXAMINATION

PART I

FIVE HOURS

INSTRUCTIONS:

1. This examination is divided into four sections, each containing either three or four problems. Although the problems in each section tend to concentrate on one area of physics, this does not mean that the subject matter of any problem will fit entirely, or even mostly, into that one area.
2. Use a separate fold of paper for each problem, and write your name on each fold. Include the problem number with each solution. A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a quantitative response, and is sometimes required by the problem.
3. Calculators may be used.
4. **No Books or Reference Materials May Be Used.**
5. Information on the next pages may be useful.

Contents

List of Figures

— Formulae —

Maxwell's equations in cgs units:

$$\nabla \cdot \vec{E} = 4\pi\rho, \nabla \cdot \vec{B} = 0, \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}.$$

Energy and momentum densities of the electromagnetic field:

$$u = \frac{|\vec{E}|^2 + |\vec{B}|^2}{8\pi}, \quad \vec{p} = \frac{1}{4\pi c}\vec{E} \times \vec{B}$$

Conservation of probability:

$$\partial_t \mathcal{P}(x,t) + \partial_x j(x,t) = 0,$$

where

$$\mathcal{P}(x,t) \equiv |\Psi(x,t)|^2$$

and

$$j(x,t) \equiv \frac{\hbar}{2mi}\left(\Psi^*(x,t)\partial_x\Psi(x,t) - \Psi(x,t)\partial_x\Psi^*(x,t)\right) = \frac{\hbar}{m}\,\mathrm{Im}\left\{\Psi^*(x,t)\partial_x\Psi(x,t)\right\}.$$

Ehrenfest's theorem:

$$\frac{d}{dt} < \hat{O} > \;=\; < \frac{1}{i\hbar}[\hat{O},\hat{H}] > + < \frac{\partial \hat{O}}{\partial t} > .$$

Normalized eigenstates of the infinite square well of width L:

$$E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2, \quad \psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right) \quad \text{for } 0 < x < L.$$

Harmonic oscillator operator relations:

$$\hat{a} \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}, \quad \hat{a}^\dagger \equiv \sqrt{\frac{m\omega}{2\hbar}}\hat{x} - \frac{i}{\sqrt{2m\omega\hbar}}\hat{p}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}), \quad \hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}^\dagger - \hat{a})$$

$$\hat{a}|n> = \sqrt{n}|n-1>, \quad \hat{a}^\dagger|n> = \sqrt{n+1}|n+1>$$

Angular momentum operator relations:

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \text{ (\& cycl. perm.)}, \quad [|\vec{\hat{L}}|^2, \vec{\hat{L}}] = 0;$$

$$\hat{L}_\pm \equiv \hat{L}_x \pm i\hat{L}_y;$$

$$\hat{L}_\pm|\ell\, m> = \hbar\sqrt{[\ell(\ell+1) - m(m\pm 1)]}\,|\ell\, m\pm 1> .$$

1 CLASSICAL MECHANICS (40 points *total*)

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1.1 System of Masses and Pulleys (14 points)

Consider a system of two masses $m_1 = m, m_2 = 2m$ connected by a massless string through a system of pulleys as Figure 1 shows. The pulleys are all identical, with radii R and moments of inertia $I = \frac{1}{2}mR^2$. They all have frictionless bearings, and they each rotate without slipping as the string moves.

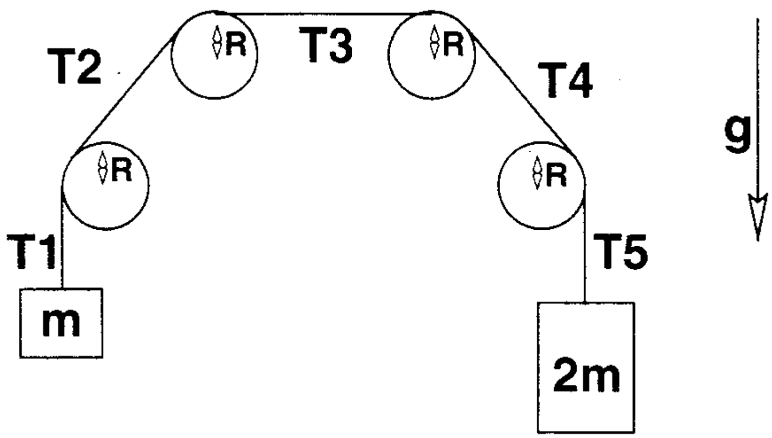


Figure 1: System of masses and pulleys

Lagrangian

Write down the Lagrangian for this system in a constant gravitational field of strength g in terms of the constants m, g, R, numerical factors and a minimal set of generalized coordinates.

Acceleration

If the masses are let free, at what rate will they accelerate?

Tensions

When the system accelerates freely under the action of gravity, what are the values for the tensions in each section of string in the figure, $T_1 \ldots T_5$?

1.2 Three Short Multiple-Choice Problems (12 points)

Instructions and Grading Policy: Ignoring such effects as friction, air resistance or evaporation, choose one of the three suggested answers for each scenario below. It is important that you write the corresponding letter in your answer booklet. No credit will be given for answers marked on the exam sheet. You will be *penalized* for incorrect answers by one-half of the four point value for each problem. Problems left blank will be assigned a score of zero. (The expected score with random guessing is also zero.)

Automotive power

If the power output of an automobile engine is directly proportional to the speed at which the car travels, the acceleration of the car will do which of the following?

a) Decrease with speed; b) Remain constant with speed (ignoring air resistance, etc.); c) Increase with speed.

Falling mass

This question concerns two point masses of equal mass m connected by a string and initially held stationary at equal heights and equal distances from the corner of a frictionless table. (See Figure 2.) If the mass m_2 is suddenly allowed to fall with the force of gravity, at which of the following times will m_2 hit the table?

a) Before mass m_1 reaches the edge; b) At the same time when m_1 reaches the edge (ignoring friction, etc.); c) After mass m_1 reaches the edge.

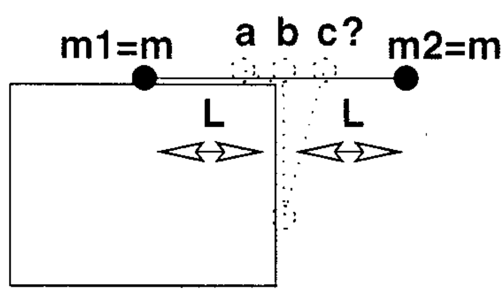


Figure 2: Masses on the edge of a table

Buoyancy

A boat containing a heavy brick initially floats in a (very small) lake. The brick is then thrown into the lake where it sinks to the bottom. (See Figure 3.) The final level of the *lake* will be which of the following?

a) Slightly higher; b) The same (ignoring evaporation, etc.); c) Slightly lower.

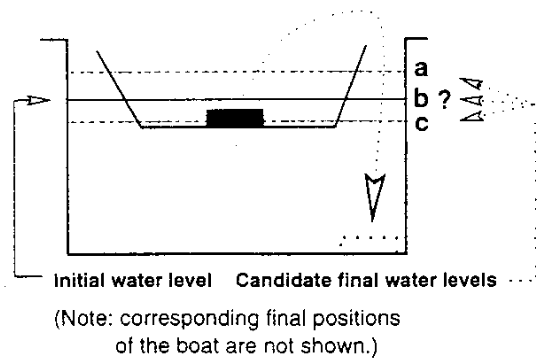
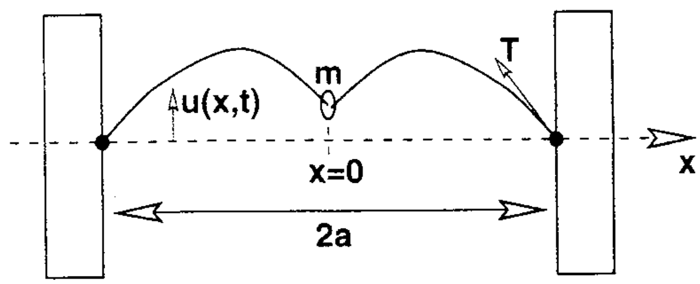


Figure 3: Boat on a lake

1.3 Waves on a String with a Point-Mass (14 points)

This problem concerns the transverse normal modes of a string of length $2a$, tension T and mass per unit length λ to which is attached a point mass of mass m at its center. Consider low amplitude vibrations only, and ignore the effects of gravity in this problem. (See Figure 4.)



(Mass per unit length:λ , g=0)

Figure 4: Vibrating string with point-mass

The case $m = 0$

For the case $m = 0$, give the equation of motion for the displacement of the string $u(x, t)$ in terms of λ and T.
What is the associated dispersion relation and phase velocity?

Equation of motion for the point-mass

Give the boundary conditions on the function $u(x, t)$ at both $x = a$ *and* $x = 0$ when $m > 0$.

Odd solutions

When $m > 0$, what are the frequencies for the modes of odd symmetry about $x = 0$?

Symmetric solutions

Derive a transcendental equation in terms of λ, m and a for the wave numbers k of the normal modes of even symmetry about $x = 0$ when $m > 0$.

2 STATISTICAL MECHANICS (40 points total)

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2.1 Ferromagnetic Spin-Waves (10 points)

A ferromagnet is system with localized spins where the spins tend to arrange so that each spin is aligned with its neighbors. The only fact about such a system which is relevant to this problem is that the low energy excitations of a ferromagnet are spin-waves with an unusual dispersion relationship. $\omega = \alpha k^2$, and a single polarization. When quantized, these excitations obey Bose statistics. Answer all questions below for a $d = 3$ dimensional system.

Density of states

Compute $g(\epsilon)$, the density of states per unit volume per unit interval in energy ϵ.

Energy density

Compute $u(T)$, the total spin-wave energy density (per unit volume) for these excitations at temperature T.
Hint: $\int_0^\infty dv\, v^{a-1}/(e^v - 1) = \Gamma(a)\zeta(a)$, where Γ and ζ are the gamma and Riemann zeta functions, respectively. (You may leave your result in terms of $\Gamma(a)$ and $\zeta(a)$ for the appropriate value of a.)

Entropy

Compute $s(T)$, the total spin-wave entropy density (per unit volume) for these excitations at temperature T.
Hint: You should be able to derive this from your result for $u(T)$ without any complicated integration.

2.2 Lattice Electrons in a Magnetic Field (10 points)

This problem involves a lattice of N sites among which N_{el} electrons are arranged. There is a strong interaction among the electrons so that no two electrons my occupy the same sight. The sites are sufficiently far apart, however, that the interaction among electrons is negligible when they are not on the same site. (See Figure 5 for an example with ten sites and seven electrons.) An external field B is applied to the system so that the energies of the electrons with up and down magnetic moments are $\epsilon_{\uparrow} = -\Delta$ and $\epsilon_{\downarrow} = +\Delta$, respectively.

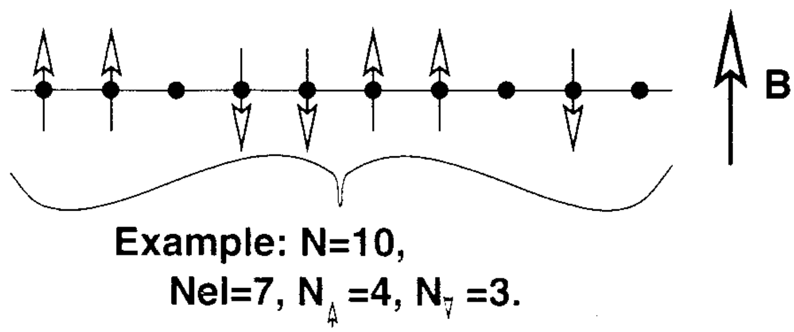

Figure 5: Mobile spins on a lattice

Free energy

Compute the Helmholtz free energy $F(N_{\uparrow}, T) \equiv E - TS$ of the system of N sites and N_{el} total electrons when there are $N_{\uparrow}$ electrons with up moments at temperature T.
To reduce writing, you may define constants to represent terms which depend only upon N and N_{el}.

Equilibrium

Show that in this case the equilibrium ratio of "up" to "down" moments at temperature T is given by the Boltzmann factor, $N_{\uparrow}/N_{\downarrow} = e^{2\Delta/kT}$.
Note: If two electrons were allowed to occupy the same site, this ratio would not be given by the Boltzmann factor.

2.3 Two-dimensional Bose Gas in a Magnetic Field (10 points)

This problem concerns the magnetic response of a gas of spin-zero Bosons of charge e and mass m constrained to move in two dimensions at a density of $n \equiv N/A$ particles per unit area at $T = 0$. The system is taken to be of unit height in the direction perpendicular to the area A so that density per unit volume is equivalent to density per unit area.

You will find the following relationships useful:

1. The energy spectrum for particles free to move in two dimensions in a constant magnetic field B consists of a discrete set of levels

$$E_n = (n + 1/2)\hbar\omega_c \quad \text{for } n = 0, 1, 2, \ldots,$$

where $\omega_c = e|B|/(mc)$ is the classical cyclotron frequency. The degeneracy of each of these levels is $g = A \cdot e|B|/(hc)$, where A is the area of the system.

2. The change in the internal energy density of a magnetic system is

$$du = T\,ds - m\,dB,$$

where s and m are the entropy density and magnetization density of the system, respectively.

3. When a system with magnetization density m is exposed to an externally applied magnetic field H, the net internal field B is given by

$$B = H + 4\pi m.$$

Energy at low temperature

In the limit $T \to 0$, compute the internal energy per unit area $u(B)$ of the two-dimensional Bose gas as a function of the field strength B.

Low temperature magnetization

In the same limit, what is the magnetization density $m(B)$ of the Bose gas as a function of the magnetic field strength?

Make a sketch of your result including *both* positive *and* negative values of B.

Response to an applied field

Use your result 2.3 to determine the behavior of the internal field B as a function of the externally applied field H. Sketch your result for B as a function of H as H varies from zero through $H \to \infty$.

For what range of applied fields H is the net field strength in the system equal to zero?

2.4 Confined, Directed Random Walk (10 points)

This question concerns the force which a constrained random chain exerts on it boundaries.
Consider a two-dimensional model in which a random chain progresses consistently along the positive $\hat{x}$ direction and. for each step along $-\hat{x}$. takes one step randomly along either $+\hat{y}$ or $-\hat{y}$. (See Figure 6.) The internal energy of the chain is completely independent of its configuration. Two rigid walls at separation w constrain the chain to always step back into the interior region. For the purposes of this problem, the unit of length is the length of one link of the chain.

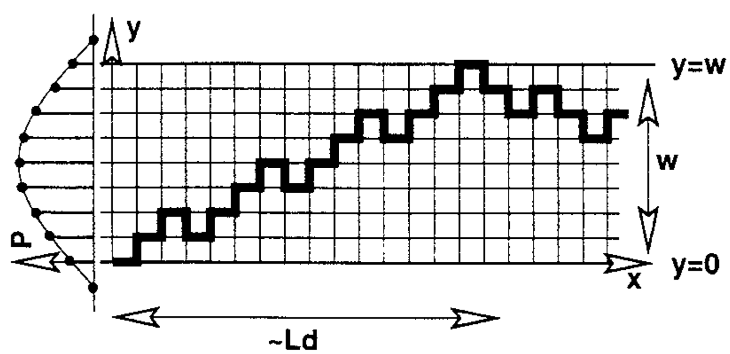

Figure 6: Random walker between two walls

Contributions to the entropy

a) What is the contribution ΔS to the total entropy of the chain for each step along x when the chain is located in the interior region $(0 < y < w)$?
b) What is the contribution ΔS to the entropy for each step where the chain is located on one of the boundaries ($y = 0$ or $y = w$)?

Entropy per unit length

Given that the probability distribution for the location of the chain along y is a half-sine wave as shown in the figure, <u>estimate</u> s the average entropy per unit length of the chain for large separations. $w >> 1$. Note that due to the shape of the probability distribution. the probability for being at one of the edges scales like $1/w^2$.
Your result need not be correct to numerical factors; you may ignore factors of π, etc.

Force per unit length

Within the approximation you made in 2.4, what force per unit length does the chain exert on the walls at temperature T?

3 ELECTRICITY AND MAGNETISM (40 points *total*)

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3.1 Currents in a Conducting Medium (10 points)

Two identical conducting spheres of radius a are embedded in an infinite medium with conductivity σ ($J = \sigma\vec{E}$) at a distance $A >> a$. (See Figure 7.) The medium has the same dielectric constant as free space. $\epsilon = 1$. There are no external sources of potential so that the electric field drops to zero at infinity.

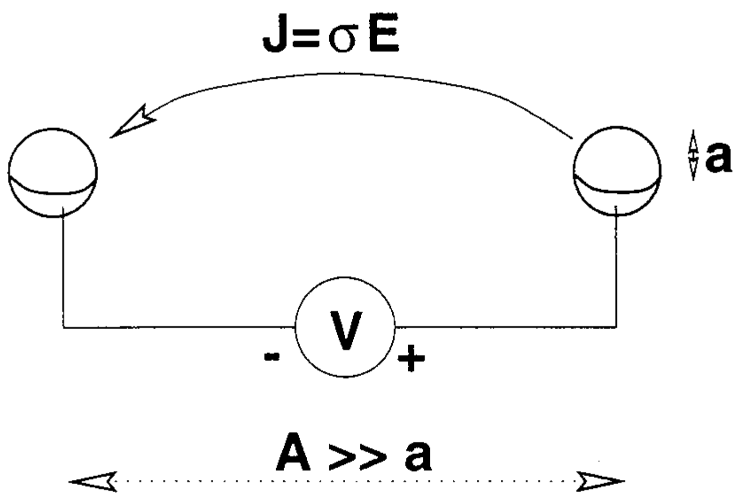

Figure 7: Two spheres embedded in a conducting medium

Ignoring corrections of order a/A, what is the resistance R between the spheres?

Hints: Note that the form of the answer will be very simple. You may wish to first determine the charge on the spheres for a given applied potential difference V and then evaluate the total current flowing out of the sphere held at higher potential.

3.2 Electromagnetic Energy and Momentum (10 points)

Two very large parallel plates each with mass $M/2$ and area A and with surface charge densities of $\pm\sigma$ travel at velocity $v << c$ at a fixed separation w in a direction parallel to the plates. Answer the questions below using a coordinate system aligned so that the plates are oriented perpendicular to the $\hat{z}$ direction and the velocity v is along the $\hat{x}$ direction. (See Figure 8.) You are to ignore edge effects.

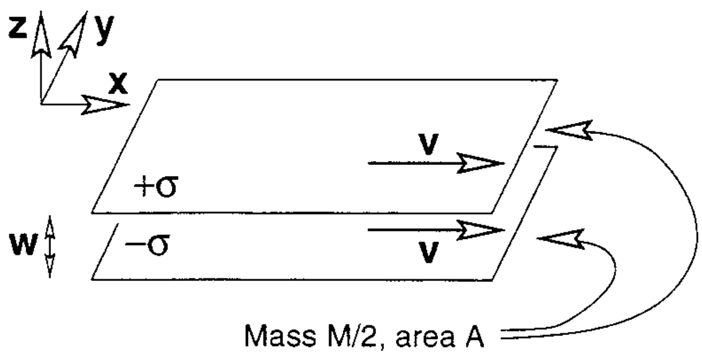

Figure 8: Two parallel. charged plates

Fields

What are the electric and magnetic fields ($\vec{E}$ and $\vec{B}$. respectively) in between and outside the plates? Be explicit with the direction and sign of the fields using the coordinate system in the figure.

Momentum and Energy

Show that. to lowest order in v/c. the energy stored in the electromagnetic field may be written in the form $U = const. + \frac{1}{2}\Delta M v^2$ and the momentum stored in the electromagnetic field may be written in the form $\vec{P} = \Delta M\, v\hat{x}$ for the *same* value of ΔM.

3.3 Electromagnetic Radiation in a Conducting Medium (10 points)

Consider an electromagnetic plane wave of frequency ω and wave vector $\vec{k}$ propagating in a conductive medium. The medium has conductivity σ, so that $\vec{J} = \sigma\vec{E}$. The dielectric constant is that of free space, $\epsilon = 1$.

It is a fact (which you need not show) that in this situation there are no oscillations in the charge density associated with the electromagnetic wave. You may find the following vector identity useful:

$$\nabla \times \left(\nabla \times \vec{F}(\vec{x})\right) = \nabla\left(\nabla \cdot \vec{F}(\vec{x})\right) - \nabla^2\vec{F}(\vec{x})$$

Orientation of the fields

Given that there is no oscillating component to the charge density, use Maxwell's equations to show that in this medium the oscillating electric and magnetic fields at each point in space are perpendicular both to each other and to the direction of propagation.

Skin-depth

Determine the length-scale λ over which the amplitudes of the electric and magnetic fields decay by a factor of e. You may leave your result in terms of elementary trigonometric functions without further simplification. (**Note:** If you cannot simplify your result to this extent, for partial credit you may leave your result in the form "$1/\lambda = \mathrm{Im}\{\ldots\}$".)

3.4 Work Required to Maintain an Imposed Magnetic Field (10 points)

Consider an apparatus consisting of two concentric coils with nearly identical radii R, both of length L and consisting of $N = nL$ counter-clockwise turns of wire. A constant, external magnetic field $\vec{H}$ is imposed on the inner coil by the outer coil which maintains a constant current I. The net internal magnetic field B within the coils may be varied by changing the current i' flowing in the inner coil. (See Figure 9.) You may neglect edge and displacement current effects.

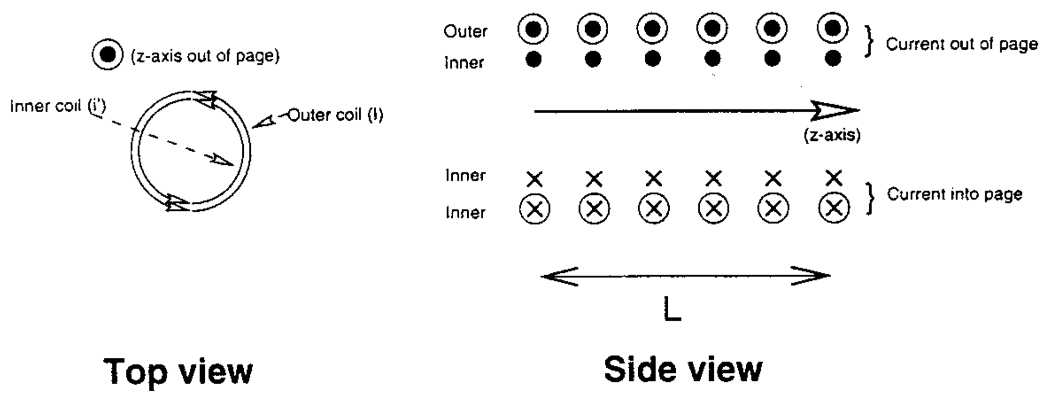

Figure 9: Two concentric coils

External field $\vec{H}$

Compute the external magnetic field $\vec{H}$ which the constant current I imposes on the interior of the coils in terms of I and other relevant parameters. Specify the direction of the field.

Electromotive force

Compute the voltage difference induced across the coil carrying the imposed current I when the inner coil causes the internal magnetic field to vary at the rate $d\vec{B}/dt = (dB/dt)\hat{z}$. Specify explicitly whether the potential increases or decreases with z when $dB/dt > 0$.

Net Work

In terms of H and ΔB, compute the work per unit volume of the interior of the coils required to maintain the constant external field H when the net internal magnetic field changes by the amount ΔB.

4 QUANTUM MECHANICS (40 points *total*)

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4.1 Scattering (10 points)

This question concerns a beam of particles incident from the left onto a potential step of height V_0 at the origin. The kinetic energy of the incoming particles is greater than the height of the step, $E > V_0$. (See Figure 10.)

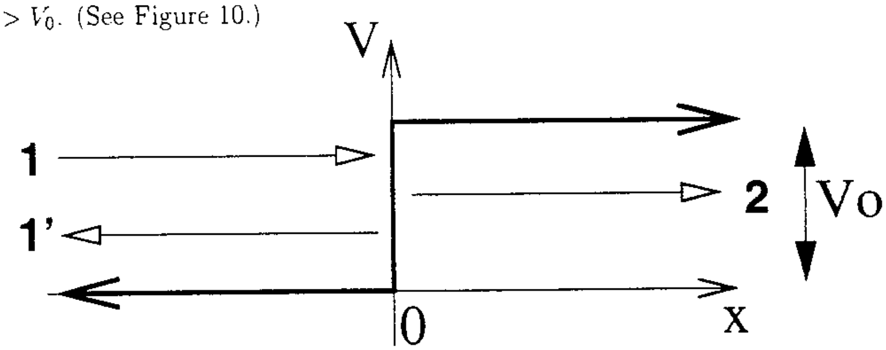

Figure 10: Scattering at a potential step

Wave function

Give the solution to the time independent Schrödinger equation for this potential which corresponds to a beam of incoming particles with energy $E \equiv \hbar^2 k_1^2/(2m)$ and momentum $p = +\hbar k_1$. You should leave your result in terms of k_1 and $k_2 \equiv \sqrt{2m(E-V)}/\hbar$ and normalize your result so that the incoming beam component is $e^{ik_1 x}$.

Particle flux

Maintaining the choice of normalization for the wave function you gave in 4.1, compute $|j_1|$, $|j_1'|$ and $|j_2|$, the average particle flows associated with the incoming, reflected, and transmitted beams, respectively. Express your results in terms of the incoming particle flux; i.e., $|j_1'| = \alpha \cdot j_1$, $|j_2| = \beta \cdot j_1$, with appropriate factors α and β.

Momentum flux

Compute the momentum flow carried by each of the beams, f_1, f_1' and f_2. Specify each flow with the appropriate sign for the momentum. You may leave your results in terms of the incoming particle flux j_1.

Momentum conservation

If the momentum of the particles is conserved, then $f_1 = f_1' + f_2$. For $V_0 > 0$, do you expect f_1 to be less than, equal to, or greater than $f_1' + f_2$. Give a brief physical explanation for your answer.

4.2 Bound States (10 points)

This problem concerns a particle of mass m in an infinite square well of width $2a$ ($V(x) \to \infty$ for $|x| > a$) containing a Dirac δ potential of strength $V(x) = -\kappa(\hbar^2/m)\delta(x)$ at its center, where $\kappa > 0$. (See Figure 11.)

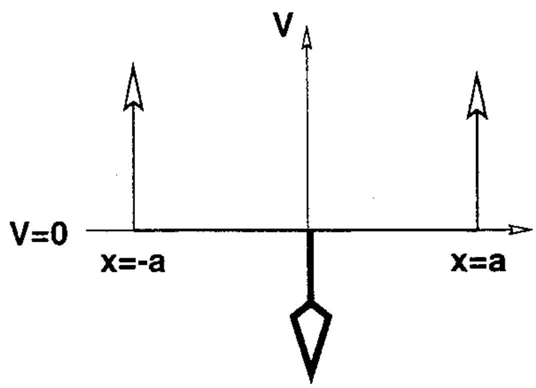

Figure 11: Dirac δ potential in center of infinite square well

For what width $W = 2a$ of the well is there a state with energy $E = 0$?

4.3 Spin-Orbit Coupling (10 points)

The outermost electron of a particular element is in a p orbital and experiences a spin-orbit coupling

$$\mathcal{H}_{SO} = \alpha \vec{L} \cdot \vec{S}.$$

where $\vec{L}$ and $\vec{S}$ are the orbital and spin angular momenta, respectively, of the electron. In this particular case, $\alpha > 0$.

Energy levels

Compute the shifts in the otherwise degenerate states of the outermost electron due to the spin-orbit coupling $\mathcal{H}_{SO}$. Give both the shifts and the degeneracies, if any, of the various levels you find.

Measurement of L_z

Suppose that you have a large collection of isolated atoms of this element at very low temperature so that the lowest energy spin-orbit states are occupied with equal probabilities and all higher energy states are completely unoccupied. If a measurement is made of L_z for the outermost electron, what fraction of the atoms will yield the result $L_z \neq 0$? Use the $\hat{L}_\pm$ operators given on the formula sheet to compute a <u>numerical</u> result.

4.4 Forced Harmonic Oscillator (10 points)

Initially, at $t = 0$, a particle of mass m rests in the ground state of a perfect simple harmonic oscillator. A small time dependent force $F(t) = F \sin \omega t$ is then applied to the particle at precisely the natural frequency of the oscillator, ω.

To lowest order in F, what is the probability of detecting the particle in the m^{th} excited state after a time $t >> 1/\omega$?

Useful formula: The equation of motion in the interaction picture is

$$i\hbar \frac{\partial}{\partial t} |\Phi(t)> = e^{i\hat{H}_o t/\hbar} \hat{V} e^{-i\hat{H}_o t/\hbar} |\Phi(t)>,$$

where the total Hamiltonian is $\hat{H} = \hat{H}_o + \hat{V}$ and the Schrödinger picture wave function is given by $|\Psi(t)> = \exp\left(-i\hat{H}_o t/\hbar\right) |\Phi(t)>$.