STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination, August 25, 2014

Classical Mechanics

General Instructions:
Three problems are given. If you take this exam as a placement exam, you must do all three problems. If you have passed the Classical Mechanics core course, do two of the problems (if you do more than two problems, you must choose which two should be graded).
Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems may cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use a calculator, and with the proctor's approval, a foreign language dictionary. No help sheet or other materials may be used.

Some potentially useful formulas are listed on the next two pages.

Grad, Div, Curl, and Laplacian

CARTESIAN $\quad d\boldsymbol{\ell} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad d^3r = dx\,dy\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $\quad d\boldsymbol{\ell} = d\rho\hat{\boldsymbol{\rho}} + \rho d\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}} \qquad d^3r = \rho\,d\rho\,d\phi\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $\quad d\boldsymbol{\ell} = dr\hat{\mathbf{r}} + r d\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}} \qquad d^3r = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Figure 1: Grad, Div, Curl, Laplacian in cartesian, cylindrical, and spherical coordinates. Here ψ is a scalar function and $\mathbf{A}$ is a vector field.

Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\int_V d^3r\,\nabla\psi = \int_S dS\,\hat{\mathbf{n}}\psi$$

$$\int_V d^3r\,\nabla\times\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\times\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

Figure 2: Vector and integral identities. Here ψ is a scalar function and $\mathbf{A}, \mathbf{a}, \mathbf{b}, \mathbf{c}$ are vector fields.

Classical Mechanics 1

A relativistic electron in an EM field.

a) [5 pts.] Write down the Lagrangian for a nonrelativistic point particle with charge $e = -|e|$ and mass m coupled to external, possibly time-dependent, electromagnetic potentials $A^\mu = \left(\phi, \vec{A}\right)$. Derive the Lorentz force from this Lagrangian.

b) [5 pts.] Now consider the relativistic version of a). Write down the Lagrangian. Explicitly show that the corresponding action is relativistically invariant. Check signs in your result by taking the nonrelativistic limit.

c) [5 pts.] Construct the corresponding relativistic Hamiltonian.

d) [5 pts.] Derive the equations $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$ and $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$. When is the Hamiltonian for this system equal to the energy E, and when is the Hamiltonian conserved? Is the Lagrangian given by $T - V$, where T is the kinetic energy and V the potential energy? Is it conserved?

Solution

a) $L = \frac{1}{2}mv^2 - e\phi(x) + \frac{e}{c}\frac{dx^i}{dt}A_i(x)$. The Lorentz force is obtained as follows

$$\begin{aligned}
\tfrac{d}{dt}\left(m\vec{v}\right) + e\vec{\nabla}\phi + \tfrac{d}{dt}\left(\tfrac{e}{c}\vec{A}\right) - \tfrac{e}{c}v^i\vec{\nabla}A_i = 0 \\
F_j = e\left(-\nabla_j\phi - \tfrac{1}{c}\tfrac{\partial A_j}{\partial t}\right) + \tfrac{e}{c}v^i\left(\partial_j A_i - \partial_i A_j\right) \\
\vec{F} = e\vec{E} + \tfrac{e}{c}\vec{v}\times\vec{B}
\end{aligned}$$

b) $L = -mc^2\sqrt{1 - v^2/c^2} + \frac{e}{c}\frac{dx^\mu}{dt}A_\mu(x)$, where $v^i = \frac{dx^i}{dt}$. For small v we get $L = \frac{1}{2}mv^2 - e\phi + \cdots$, so $V = e\phi$ with $\phi = A^0$ the scalar potential, and thus $e = -|e|$ for the electron. The action reads $S = \int Ldt = -mc\int ds + \frac{e}{c}\int A_\mu dx^\mu$ where ds is the proper length, and is clearly relativistically invariant.

c) With $p_i = \frac{\partial}{\partial \dot{x}^i}L = \frac{m\dot{x}^i}{\sqrt{1-v^2/c^2}} + \frac{e}{c}A_i$, we find

$$H = p_i\frac{dx^i}{dt} - L = \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - v^2/c^2} + e\phi = \frac{mc^2}{\sqrt{1 - v^2/c^2}} + e\phi\,.$$

Expressing $\dot{x}^i$ in terms of p_i yields

$$v^2 = \frac{\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2}{m^2 + \frac{1}{c^2}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2} \text{ and then}$$

$$H = \left[\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 c^2 + m^2 c^4\right]^{1/2} + e\phi\,.$$

d) $\frac{dH}{dt} = \frac{d}{dt}\left(p\dot{q} - L(q,\dot{q},t)\right) = \ddot{q}\frac{\partial L}{\partial \dot{q}} + \dot{q}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\dot{q} - \frac{\partial L}{\partial \dot{q}}\ddot{q} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$.

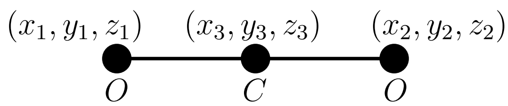
$\frac{\partial H}{\partial t} = \frac{\partial}{\partial t}\left(p\dot{q}(p,q,t) - L(q,\dot{q}(p,q,t),t)\right) = p\frac{\partial}{\partial t}\dot{q} - \frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial t} - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$.

The Hamiltonian is conserved if the EM fields are time-independent, and the Hamiltonian is equal to the total energy, $H = T_H + V$ with $T_H = \frac{mc^2}{\sqrt{1-v^2/c^2}}$ and $V = e\phi$, but the Lagrangian $L = T_L - V$ is not equal to $T_H - V$ because $T_L = -mc^2\sqrt{1 - v^2/c^2}$ is not the relativistic energy.

Classical Mechanics 2

Small and not so small oscillations.

Consider vibrations of the CO_2 molecule. The molecule consists of two oxygen atoms and one carbon atom which can be regarded as point particles. Electrons can be disregarded. Denote the masses of the oxygen atoms by m_O and the mass of the carbon atom by $m_C = \mu m_O$. (The value of μ is approximately $^3/_4$.) Denote the deviations of the oxygen atoms from their equilibrium position by (x_1, y_1, z_1) and (x_2, y_2, z_2), and those of carbon atom by (x_3, y_3, z_3). The distance between the oxygen atoms at rest is $2a$, and at rest the molecule lies along the z-axis.



a) [5 pts.] How many normal modes of oscillation with nonvanishing frequencies does this molecule have? Draw the motion of each such normal mode, but do not calculate the amplitudes of the atoms.

b) [5 pts.] Conservation of momentum and angular momentum can be used to choose a coordinate frame in which we can express all 9 coordinates in terms of a linearly independent set of coordinates. Choose such a set of coordinates, and write down the 9 relations expressing the 9 coordinates into this set. (There are many choices for this independent set, but you may choose any one.) Do not try to express the kinetic and potential energy in terms of this independent set of coordinates.

c) [5 pts.] Now consider anharmonic terms in the potential. What happens with the frequencies of the normal modes when anharmonic terms are included? As a model for a diatomic molecule we consider the following one-dimensional equation of motion (the Duffing oscillator)

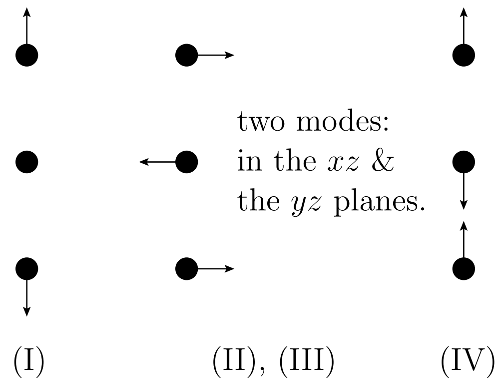
$$\ddot{x} + \omega_0^2 x = \lambda x^3 \, .$$

Show that naïve perturbation theory (by which we mean setting $x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \cdots$ and solving order-by-order in λ) leads to problems if we want to obtain a periodic solution.

d) [5 pts.] Indicate how these problems can be overcome. Briefly discuss how one can apply this "improved perturbation theory" to the pendulum.

Solution

a) There are 4 normal modes: $3N - 3 - 3 + 1$ (+1 because this is a linear molecule).



b) Conservation of momentum in COM frame:

$$x_1 + x_2 + \tfrac{3}{4}x_3 = 0$$
$$y_1 + y_2 + \tfrac{3}{4}y_3 = 0$$
$$z_1 + z_2 + \tfrac{3}{4}z_3 = 0\,.$$

Conservation of angular momentum imposes relations between the deviations due to infinitesimal rotations. To first order in deviations one obtains (X_i, Y_i, Z_i are the equilibrium positions)

$$\sum (Y_i\dot{z}_i - Z_i\dot{y}_i) = 0 \Rightarrow a\,(\dot{y}_1 - \dot{y}_2) = 0$$
$$\sum (Z_i\dot{x}_i - X_i\dot{z}_i) = 0 \Rightarrow a\,(\dot{x}_1 - \dot{x}_2) = 0$$
$$\sum (X_i\dot{y}_i - Y_i\dot{x}_i) = 0 \Rightarrow \text{no relation.}$$

Hence $y_1 = y_2$ and $x_1 = x_2$. We have then 5 linear equations for 9 unknowns, which yields 4 coordinates for the normal modes, for example x_1, y_1, z_1, z_2. The other 5 coordinates are expressed in terms of them as follows

$$x_2 = x_1\,; \quad x_3 = -\tfrac{8}{3}x_1\,; \quad y_2 = y_1\,;$$
$$y_3 = -\tfrac{8}{3}y_1\,; \quad z_3 = -\tfrac{4}{3}(z_1 + z_2)\,.$$

c) The frequencies of the normal modes are changed when one includes anharmonic terms, and also overtones (frequencies which are a multiple of the original frequency) can be excited. Solving in a naïve way $\ddot{x} + \omega_0^2 x = \lambda x^3$ gives

$$\ddot{x}_0 + \omega_0^2 x_0 = 0 \quad \Rightarrow \quad x_0 = A\cos\omega_0 t \quad \text{if at } t = 0 \text{ the velocity is zero.}$$

$$\ddot{x}_1 + \omega_0^2 x_1 = A^3 \left(\cos\omega_0 t\right)^3 = \tfrac{A^3}{4}\left(\cos 3\omega_0 t + 3\cos\omega_0 t\right)$$

whose solution is **not** periodic with period ω_0

$$x_1 = B\cos 3\omega_0 t + Ct\sin\omega_0 t$$
$$-9B\omega_0^2 + B\omega_0^2 = \tfrac{1}{4}A^3\,; \quad 2C\omega_0 = \tfrac{3}{4}A^3\,.$$

d) The way to obtain a periodic solution is to also expand the frequency

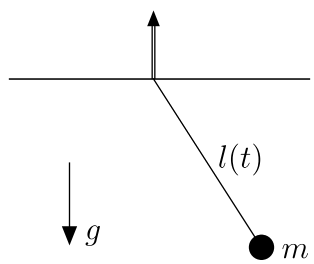
$$\omega_0 = \omega_0 + \omega_1\lambda + \cdots$$

This has been used (by Poincaré and Lindstedt) to describe planetary motion. It can also be used to find the perturbative corrections to the period of a nonlinear pendulum.

Classical Mechanics 3

A gravitational pendulum with variable length.

The mass m of a simple gravitational pendulum is attached to a wire of length l. At time $t = 0$, the wire is pulled up slowly through a hole in the ceiling, so that the length l of the pendulum is reduced. Neglect dissipation.



a) [7 pts.] Calculate the work W that must be done to pull up the wire by a small amount dl. Show that W is independent of the (long) time interval it takes to pull up the wire.

b) [7 pts.] Calculate the change dE in the oscillation energy E of the pendulum. (By oscillation energy we mean the difference between the total energy of the pendulum and the energy of the same pendulum if it is brought to rest.) Calculate the change $d\omega$ in the oscillation frequency $\omega = 2\pi\nu$. Show that $\frac{dE}{E}$ is equal to $\frac{d\omega}{\omega}$.

c) [6 pts.] Determine the l dependence of the amplitudes for the angular and linear deviations of the mass from equilibrium.

Solution

a) The force F pulling the wire down consists of the centrifugal force and the component of the gravitational force along the wire. The velocity of the mass is $v = l\dot{\theta}$, hence

$$F = \frac{m(l\dot{\theta})^2}{l} + mg\cos\theta \approx ml\dot{\theta}^2 + mg - \tfrac{1}{2}mg\theta^2.$$

For harmonic motion $\theta = A\cos\omega t$. Substituting this expression we obtain

$$F = mlA^2\omega^2\sin^2\omega t + mg - \tfrac{1}{2}mgA^2\cos^2\omega t$$

Averaging over time yields, using $\omega^2 = \frac{g}{l}$,

$$\bar{F} = \tfrac{1}{2}mlA^2\omega^2 + mg - \tfrac{1}{4}mgA^2 = \tfrac{1}{4}mgA^2 + mg$$

Hence the work done to pull up the wire over a distance dl is given by

$$dW = \bar{F}|dl| = F(-dl) = -mg\,dl - \tfrac{1}{4}mg\,A^2\,dl \tag{1}$$

which is clearly independent of the time.

b) The change in the oscillation energy is clearly given by the second term in (1) because the first term gives the change in the potential energy of the pendulum at rest. Hence

$$dE = -\tfrac{1}{4}mgA^2dl$$

The oscillation energy E is the maximal kinetic energy, hence $E = \frac{1}{2}ml^2\dot{\theta}^2_{max} = \frac{1}{2}ml^2\omega^2A^2 = \frac{1}{2}mlgA^2$ and $\frac{dE}{E} = -\frac{1}{2}\frac{dl}{l}$.

c) The relations $E = \frac{1}{2}mlgA^2$ and $dE = -\frac{1}{4}mgA^2dl$ yield the dependence of the amplitude of angular deviations on l

$$\begin{aligned} dE = \tfrac{1}{2}mgA^2dl + \tfrac{1}{2}mlgdA^2 &= -\tfrac{1}{4}mgA^2dl \\ ldA^2 = -\tfrac{3}{2}A^2dl \quad &\Rightarrow \quad A^2(l) = l^{-3/2}A^2(t=0) \end{aligned}$$

The amplitudes for linear displacements is given by $B = Al$, and its dependence on l is

$$B(l) = l^{1/4}B(t=0)\,.$$

Electromagnetism 1

A time dependent dipole

Consider an electric dipole at the spatial origin ($\boldsymbol{x} = 0$) with a time dependent electric dipole moment oriented along the z-axis, *i.e.*

$$\boldsymbol{p}(t) = p_o \cos(\omega t)\hat{\boldsymbol{z}}\,, \tag{1}$$

where $\hat{\boldsymbol{z}}$ is a unit vector in the z direction.

a) [4 pts.] Recall that the near and far fields of the time dependent dipole are qualitatively different. Estimate the length scale that separates the near and far fields.

b) [1 pt.] In the far field, how do the magnitude of the field strengths decrease with radius?

c) [2 pts.] Using a system of units where $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units (such as Gaussian or Heaviside-Lorentz), determine the ratio of E/B at a distance r in the far field*

d) [3 pts.] Estimate the total power radiated in a dipole approximation. How does this power depend on the dipole amplitude p_o, the oscillation frequency ω, and the speed of light†

e) [3 pts.] In the near field regime, *estimate* how the electric and magnetic field strengths decrease with the radius r. (r is the distance from the origin to the observation point.)

f) [3 pts.] Using a system of units where $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same units (such as Gaussian or Heaviside-Lorentz), *estimate* the ratio E/B at a distance r in the near field‡. Is this ratio large or small?

g) [4 pts.] Determine the electric and magnetic fields to the lowest non-trivial order in the near field (or quasi-static) approximation.

Solution

a) The speed of light and the frequency define a length scale

$$1/(R_o) = \omega/c$$

*In SI units this question reads, "Estimate the ratio E/cB at a distance r in the far field."

†In SI units this question reads, "How does the power depend on p_o, ω, c and ϵ_o?"

‡In SI units this question reads, "Estimate the ratio E/cB at a distance r in the near field."

For distances less than R_o a quasi-static approximation may be used. For distances greater than R_o the finite speed of light must be considered to calculate the radiation fields

b) In the far field both field strengths decrease as

$$E \propto \frac{1}{r} \tag{2}$$

$$B \propto \frac{1}{r} \tag{3}$$

c) The magnitudes are equal in a radiation field

d) The power radiated from the Larmour is

$$P \propto \frac{\omega^4}{c^3} p_o^2 \tag{4}$$

e) Standard dipole counting

$$E \propto \frac{1}{r^3} \tag{5}$$

For the magnetic field

$$\nabla \times B = \frac{1}{c} \partial_t E \tag{6}$$

suggests that

$$B \propto \frac{\omega}{c} \frac{1}{r^2} \sim \frac{1}{R_o r^2} \tag{7}$$

f) Then, using the logic of the previous paragraph we see that

$$\frac{E}{B} \sim \frac{R_o}{r} \gg 1 \tag{8}$$

g) There are various ways to do this. Perhaps the most direct is to use the gauge potentials in the lorentz gauge. We will not do this, but use the Maxwell equations directly.

The electric field in the near field region is just the field of a dipole

$$\boldsymbol{E} = \frac{1}{4\pi r^3} \left[3(\boldsymbol{p} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p} \right] \tag{9}$$

Clearly $\boldsymbol{E}$ lies in $\hat{r}, \hat{\theta}$ plane. So

$$\boldsymbol{E} = \frac{1}{4\pi r^3} \left[\left(2p_o(t)\cos\theta\right) \hat{\boldsymbol{r}} + \left(p_o(t)\sin\theta\right) \hat{\boldsymbol{\theta}} \right] \tag{10}$$

where $p_o(t) = p_o \cos(\omega t)$

Since

$$\nabla \times \boldsymbol{B} = \frac{1}{c} \partial_t \boldsymbol{E} \tag{11}$$

We try $\boldsymbol{B}$ in the ϕ direction, with $B_\phi(r,\theta)$. Then

$$(\nabla \times \boldsymbol{B})_\theta = -\frac{1}{r}\partial_r(rB_\phi) = \frac{1}{4\pi r^3}(\partial_t p_o)\sin\theta \tag{12}$$

Integrating with respect to r we find

$$B_\phi = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta + \frac{f(\theta)/R_o^2}{r} \tag{13}$$

Where $f(\theta)$ is a dimensionless integration constant, and we have inserted factors of R_o to make up the dimensions. The terms proportional to $1/r$ can be dropped in the near field regime since it is smaller by r/R_o than the $\frac{1}{r^2}$ term. Thus

$$B_\phi = \frac{1}{4\pi r^2 c}(\partial_t p_o)\sin\theta\,. \tag{14}$$

Then one verifies that

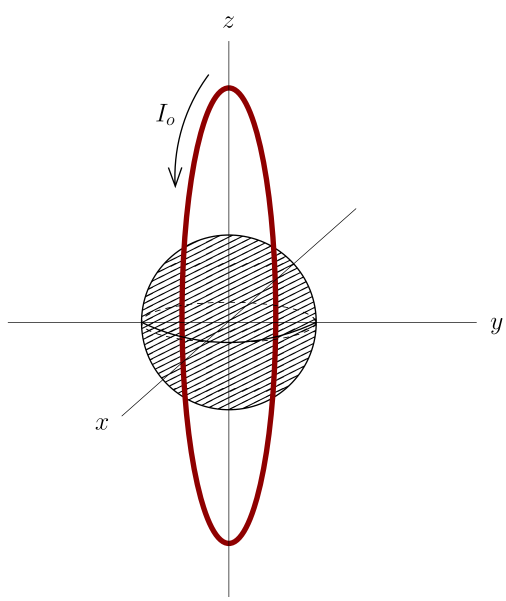
$$(\nabla \times \boldsymbol{B})_r = \frac{1}{r\sin\theta}\partial_\theta(\sin\theta B_\phi) = \frac{1}{4\pi r^3 c}(\partial_t p_o)\,2\cos\theta = \frac{1}{c}\partial_t E_r \tag{15}$$

showing that B_ϕ satisfies the Maxwell equations

Electromagnetism 2

A magnetized sphere and a circular hoop

A uniformly magnetized sphere of radius a centered at origin has a permanent total magnetic moment $\mathbf{m} = m\,\hat{\boldsymbol{z}}$ pointed along the z-axis (see below). A circular hoop of wire of radius b lies in the xz plane and is also centered at the origin. The hoop circles the sphere as shown below, and carries a small current I_o (which does not appreciably change the magnetic field). The direction of the current I_o is indicated in the figure.



a) [5 pts.] Determine the magnetic field $\boldsymbol{B}$ inside and outside the magnetized sphere.

b) [5 pts.] Determine the bound surface current on the surface of the sphere.

c) [5 pts.] What is the direction of the net-torque on the circular hoop? Indicate on the figure how the circular hoop will tend to rotate and explain your result.

d) [5 pts.] Compute the net-torque on the circular hoop.

Solution

a) The magnetic field outside is one of a magnetic dipole, where all of magnetic moment is placed at the origin

$$\boldsymbol{B} = \frac{1}{4\pi r^3}\left[3(\mathbf{m}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \mathbf{m}\right] \tag{1}$$

Inside sphere, the magnetic field is constant

$$\boldsymbol{B} = B_o\,\hat{\boldsymbol{z}} \tag{2}$$

The constant B_o can be picked off from the boundary conditions.

The boundary conditions read

$$\boldsymbol{n} \times (\boldsymbol{B}_2 - \boldsymbol{B}_1) = \frac{\boldsymbol{K}_b}{c} \tag{3}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{4}$$

Then from the boundary conditions at $r = a$

$$B_r|_{\rm out} = B_r|_{\rm in}\ . \tag{5}$$

With the magnetic field outside the sphere

$$B_r|_{\rm out} = \frac{1}{4\pi r^3} 2m\cos\theta\,, \tag{6}$$

and inside the sphere

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{B}|_{\rm in} = B_o\,\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{z}} = B_o\cos\theta\,, \tag{7}$$

comparison at $r = a$ gvies

$$B_o = \frac{1}{4\pi a^3} 2m\,. \tag{8}$$

For later reference we note that with $M = m/(4\pi a^3/3)$

$$H_o = B_o - M = -\frac{m}{4\pi a^3} \tag{9}$$

b) The surface current is in the azimuthal direction

$$\boldsymbol{K} = K_o\,\hat{\boldsymbol{\phi}} \tag{10}$$

Inside we have

$$\boldsymbol{B} = B_o\hat{\boldsymbol{z}} = B_o\cos\theta\,\hat{\boldsymbol{r}} - B_o\sin\theta\,\hat{\boldsymbol{\theta}}\,, \tag{11}$$

while outside we have

$$\boldsymbol{B} = \frac{1}{4\pi r^3} 2m\cos\theta\,\hat{\boldsymbol{r}} + \frac{1}{4\pi r^3} m\sin\theta\,\hat{\boldsymbol{\theta}}\,. \tag{12}$$

Then the jump condition reads

$$B_{\theta,\rm out} - B_{\theta,\rm in} = \frac{K_o}{c}\,. \tag{13}$$

Thus

$$K_o = c\left(\frac{1}{4\pi a^3} m + B_o\right)\sin\theta = \frac{3c}{4\pi a^3} m\sin\theta \tag{14}$$

One can verify using eq. (9)

$$H_{\theta,\text{out}} - H_{\theta,\text{in}} = \left(\frac{1}{4\pi r^3} m \sin\theta + H_o \sin\theta\right) = 0 \tag{15}$$

as should be the case since H is continuous in the absence of external macroscopic currents.

c) To compute the torque we first compute the lorentz force on a element of length $d\ell = bd\theta$.

$$\begin{aligned} dF =& \frac{I_o}{c} d\ell B_\perp & (16) \\ =& \frac{I_o}{c} bd\theta \; B_r & (17) \\ =& \frac{I_o}{c} bd\theta \; \frac{2m\cos\theta}{4\pi b^3} & (18) \end{aligned}$$

The right hand rule indicates that the force is in the $-\hat{\boldsymbol{y}}$ direction in the upper hemisphere, and in the positive $\hat{\boldsymbol{y}}$ direction in the lower hemisphere. This implies that the net torque points along the x-axis. This can be intuited by noting that the magnetic moment of the hoop tends to align with the magnetic field from the sphere

d) The torque around the x-axis

$$\begin{aligned} \tau =& \int d\tau = \int b\cos\theta \, dF & (19) \\ =& 2\int_0^\pi b\cos\theta \; \frac{I_o}{c} bd\theta \; \frac{2m\cos\theta}{4\pi b^3} & (20) \\ =& \frac{4m(I_o/c)b^2}{4\pi b^3} \int_0^\pi d\theta \cos^2\theta & (21) \\ =& \frac{4m(I_o/c)b^2}{4\pi b^3} \frac{\pi}{2} & (22) \\ =& \frac{2m}{4\pi b^3} \left[\frac{I_o}{c}\pi b^2\right] & (23) \end{aligned}$$

Electromagnetism 3

EM fields of a moving charged particle

Consider a particle of charge q moving along the x-axis with a constant velocity v in such a way that time $t = 0$ when the particle is at the point $(0, 0, 0)$.

A. [6 pts.] Determine all components of the electric and magnetic fields at the point $(0, b, 0)$ in terms of q, v, t, b, the velocity of the particle $\beta = v/c$ relative to the speed of light c, and the Lorentz factor $\gamma = (1 - \beta^2)^{-1/2}$.

B. [6 pts.] Show that in the highly-relativistic limit $\beta \approx 1$ and $\gamma \gg 1$, the peak transverse electric field $E_{y_{\max}}$ is

$$E_{y_{\max}} = \frac{\gamma q}{b^2} \tag{1}$$

and the peak longitudinal electric field $E_{x_{\max}}$ is

$$E_{x_{\max}} = \sqrt{\frac{4}{27}}\frac{q}{b^2} \tag{2}$$

and thus that

$$E_{y_{\max}} \gg E_{x_{\max}}. \tag{3}$$

C. [4 pts.] Show that in the highly-relativistic limit, the transverse electric field E_y is appreciable only over a time interval Δt centered on $t = 0$ given by

$$\Delta t \approx \frac{b}{\gamma v}. \tag{4}$$

D. [2 pts.] Now consider a second particle also of charge q initially at rest at the point $(0, b, 0)$. Under the "impulse approximation," the second particle is affected by the impulse produced by fields associated with the first, moving particle. Write down a condition on the mass m of the second particle in terms of the other parameters of the problem in order for the impulse approximation to be valid in the highly-relativistic limit.

E. [2 pts.] Determine the velocity of the second particle after passage of the first particle under the impulse approximation in the highly-relativistic limit.

Solution

A. Consider a primed frame moving with a boost v along the x and x' directions with respect to an unprimed frame, and take all respective coordinate axes to be concident at time $t = t' = 0$. In the primed frame, the particle is at rest and produces only an electric field, and the electric and magnetic fields at the point under consideration are

$$E'_x = \frac{q}{b^2 + v^2 t'^2} \frac{vt'}{(b^2 + v^2 t'^2)^{1/2}} = \frac{qv\gamma t}{(b^2 + v^2\gamma^2 t^2)^{3/2}}, \tag{5}$$

$$E'_y = -\frac{q}{b^2 + v^2 t'^2} \frac{b}{(b^2 + v^2 t'^2)^{1/2}} = -\frac{qb}{(b^2 + v^2\gamma^2 t^2)^{3/2}}, \tag{6}$$

and

$$E'_z = B'_x = B'_y = B'_z = 0. \tag{7}$$

The Lorentz transformation A relating coordinates x' to coordinates x under a boost along the x direction is

$$x' = Ax \tag{8}$$

where

$$A = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

The inverse transformation is obtained by putting $\beta \to -\beta$, so

$$x = A'x' \tag{10}$$

where

$$A' = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

In the primed frame, the particle is at rest and produces only an electric field, and the field-strength tensor F' is

$$F' = \begin{pmatrix} 0 & -E'_x & -E'_y & 0 \\ E'_x & 0 & 0 & 0 \\ E'_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{12}$$

In the unprimed frame, the field-strength tensor F is given by

$$F = A'F'\tilde{A}'. \tag{13}$$

Carrying out the appropriate multiplications,

$$A'F' = \begin{pmatrix} \gamma\beta E'_x & -\gamma E'_x & -\gamma E'_y & 0 \\ \gamma E'_x & -\gamma\beta E'_x & -\gamma\beta E'_y & 0 \\ E'_y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{14}$$

and

$$F = A'F'\tilde{A}' = \begin{pmatrix} 0 & \gamma^2\beta^2 E'_x - \gamma^2 E'_x & -\gamma E'_y & 0 \\ \gamma^2 E'_x - \gamma^2\beta^2 E'_x & 0 & -\gamma\beta E'_y & 0 \\ \gamma E'_y & \gamma\beta E'_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

In general, the field-strength tensor F is

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{16}$$

and matching terms between equations (15) and (16) and noting that $\gamma^2 = (1-\beta^2)^{-1}$ yields

$$E_x = \frac{q\gamma v t}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \tag{17}$$

$$E_y = -\frac{q\gamma b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}, \tag{18}$$

$$B_z = \gamma E_y, \tag{19}$$

and

$$E_z = B_x = B_y = 0. \tag{20}$$

B. $E_{y_{\max}}$ clearly occurs at $t = 0$, so

$$E_{y_{\max}} = -\frac{q\gamma}{b^2}. \tag{21}$$

To determine the time at which $E_{x_{\max}}$ occurs, set the derivative of E_x with respect to t to zero and evaluate at $t = t_{\max}$ to get

$$\left.\frac{dE_x}{dt}\right|_{t_{\max}} = \frac{q\gamma v(b^2 + \gamma^2 v^2 t_{\max}^2)^{3/2} - q\gamma v t_{\max}\frac{3}{2}(b^2 + \gamma^2 v^2 t_{\max}^2)^{1/2} 2\gamma^2 v^2 t_{\max}}{(b^2 + \gamma^2 v^2 t_{\max}^2)^3} = 0. \tag{22}$$

Solve equation (22) to get

$$t_{\max} = \sqrt{\frac{1}{2}}\frac{b}{\gamma v}. \tag{23}$$

Evaluate E_x at the time $t_{\rm max}$ of equation (23) to get

$$E_x = \sqrt{\frac{4}{27}}\frac{q}{b^2}. \tag{24}$$

C. By inspection of equations (17) and (18), the field is appreciable only when

$$\gamma^2 v^2 t^2 \lesssim b^2 \tag{25}$$

or

$$t \lesssim \frac{b}{\gamma} v \tag{26}$$

or over

$$\Delta t \approx \frac{b}{\gamma} v \tag{27}$$

The field strength grows like γ, but the duration of an appreciable field shrinks like $1/\gamma$, so the impulse is independent of γ (but depends on v).

D. The impulse approximation is valid if the distance traveled by the second particle in the time interval Δt over which the field is appreciable is small compared to b, or in other words if

$$u\Delta t \lesssim b, \tag{28}$$

where u is the velocity of the second particle after the impulse. The x component of the electric field is anti-symmetric in time and produces no net impulse, so the net impulse is in the y direction. So in terms of the force $F = qE_{y_{\rm max}}$ exerted on the second particle by the first particle,

$$\frac{F\Delta t}{m}\Delta t \lesssim b, \tag{29}$$

or

$$m \gtrsim \frac{F\Delta t^2}{b} = \frac{q\gamma q}{b^2}\frac{b^2}{\gamma^2 v^2}\frac{1}{b} = \frac{q^2}{\gamma v^2 b}. \tag{30}$$

E. The velocity u of the second particle after passage of the first particle under the impulse approximation in the highly-relativistic limit is

$$u_y = \frac{qE_{y_{\rm max}}\Delta t}{m} = \frac{q\gamma q}{b^2}\frac{b}{\gamma v}\frac{1}{m} = \frac{q^2}{bvm}. \tag{31}$$

Quantum Mechanics 1

Scattering of a particle from a 3D radial potential

A particle of mass m and energy $E = \hbar^2 k^2/2m$ scatters off a 3-dimensional radial potential:

$$V(r) = -V_0 \qquad a < r$$
$$0 \qquad r \geq a \tag{1}$$

a) [4 pts.] Why does the $l = 0$ partial wave dominate the scattering near threshold (zero energy)?

b) [8 pts.] Derive an expression for the S-wave phase shift $\delta_{l=0}$ by matching at $r = a$ the $l = 0$ radial waves.

c) [8 pts.] What is the threshold cross section?

Note: It is useful to define $\hbar^2/2m(k_1^2 = k^2 + k_0^2)$ with $\hbar^2 k_0^2/2m = V_0$.

Solution

a) At threshold with $k \approx 0$, the partial wave amplitude $f_l(k) \approx k^{2l}$, so the $l = 0$ dominates the cross section.

b) Inside and outside the well the reduced radial wavefunctions are

$$u_<(r) = rR_0 = A\sin(k_1 r) \qquad \frac{u_<'}{u_<} = k_1\text{cotan}(k_1 r)$$
$$u_>(r) = B\sin(kr + \delta_0) \qquad \frac{u_>'}{u_>} = k\text{cotan}(kr + \delta_0) \tag{2}$$

with $\hbar^2/2m(k_1^2 = k^2 + k_0^2)$ and $\hbar^2 k_0^2/2m = V_0$. Matching the logarithmic derivatives yields

$$k_1\text{cotan}(k_1 a) = k\text{cotan}(ka + \delta_0) \tag{3}$$

which can be rewritten as

$$\text{cotan}(\delta_0) = \frac{k\sin(ka) + k_1\text{cotan}(k_1 a)\cos(ka)}{k\cos(ka) - k_1\text{cotan}(k_1 a)\sin(ka)} \tag{4}$$

c) The total cross section involves the partial wave amplitude $f_l(k) = \sin(\delta_l(k))/k$. Near threshold

$$\sigma(k) = 4\pi \sum_{l=0}^{\infty} (2l+1) \frac{\sin^2 \delta_l(k)}{k} \approx \frac{\delta_0^2(k)}{k^2} \tag{5}$$

From (4) it follows

$$\frac{\delta_0(k)}{k} \approx \frac{1}{k\mathrm{cotan}(\delta_0)} = a - \frac{\tan(k_0 a)}{k_0} \tag{6}$$

Thus

$$\sigma(k=0) = 4\pi \left(a - \frac{\tan(k_0 a)}{k_0} \right)^2 \tag{7}$$

Quantum Mechanics 2

Particle with EDM moving in an electrostatic potential

Consider a particle of mass m and zero charge but an **electric dipole moment** $\vec{d} = d\vec{s}$, with $\vec{s}$ the spin of the particle. Assume that the particle moves in a spherically symmetric electro-static potential $\varphi(r)$ with $\vec{r} = (x, y, z)$

a) [4 pts.] Write down the corresponding Hamiltonian for this particle.

b) [3 pts.] Is this Hamiltonian invariant under: a) Space rotations; b) Parity; c) Time-reversal. Justify your answers.

Now assume that the particle has spin 1/2 and is confined to move between two parallel planes at $x = \pm L/2$ of a capacitor with an electric potental $\varphi(\vec{r}) = Ez$.

c) [6 pts.] Find the energies and wave functions of this particle.

d) [4 pts.] Consider the lowest energy state with momentum $p_y = 0$ and $p_z = p$. Write the corresponding wave function and the wave function you get by rotating the state an angle $\vec{\theta} = \frac{\pi}{4}\,\hat{x}$.

e) [3 pts.] Let $E \to E(x)$ which is slowly varying over the size of the box, i.e. $\varphi(\vec{r}) \approx (E(0) + x\partial E/\partial x + ...)z$. Calculate the change in the energy levels to first order in the small parameter $\partial E/\partial x$.

Solution

tba.

Quantum Mechanics 3

Two indistinguishable particles in a square potential well.

Consider a 1D system of two indistinguishable particles of mass m confined to an infinitely deep square potential well, $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ otherwise.

a) [4 pts.] Write down the general structure of the two-particle spatial wave function $\psi(x_1, x_2)$ and find the energy spectrum, assuming that the particles do not interact.

b) [6 pts.] Find the spatial wave function $\psi(x_1, x_2)$ for the ground state of the system. Do this for the case that the two particles each are (a) bosons with spin 0, and (b) fermions with spin-1/2. Where in the (x_1, x_2) plane are the nodes of the wave function? Explain your answer.

c) [5 pts.] Now assume that the particles are weakly interacting through the contact interaction $H' = g\delta(x_1 - x_2)$. Calculate the change to the ground-state energy to first order, again for (a) bosons with spin 0, and (b) fermions with spin-1/2. Explain your answer.

d) [5 pts.] For a system of three non-interacting spin-1/2 particles, what are the energies of the ground state and first excited state?

Solution

a) The wave function in the box is given by

$$\psi(x_1, x_2) \propto \sin\left(\frac{n_1 \pi x_1}{L}\right) \sin\left(\frac{n_2 \pi x_2}{L}\right) \pm \sin\left(\frac{n_1 \pi x_2}{L}\right) \sin\left(\frac{n_2 \pi x_1}{L}\right), \tag{1}$$

with corresponding energy

$$E = \frac{\hbar^2}{2m} \frac{\pi^2}{L^2} \left(n_1^2 + n_2^2\right) \tag{2}$$

b) For fermions (bosons) the the total wave function needs to be anti-symmetric (symmetric) under particle exchange. The symmetry of the spatial wave function thus depends on the spin configuration of the two particles.

- For total spin $S = 0$ (i.e. bosons and fermionic singlet), the spatial wave function of the ground state thus needs to be symmetric,

$$\psi(x_1, x_2) = \frac{2}{L} \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{\pi x_2}{L}\right) \tag{3}$$

with

$$E \propto 1^2 + 1^2 = 2 \tag{4}$$

and with nodes at the walls of the box. The particles share the same spatial wave function (occupied spin-up, spin-down in the fermionic case).

- For total spin $S = 1$ (fermionic triplet), the spatial wave function of ground state needs to be anti-symmetric,

$$\psi(x_1, x_2) \propto \sin\left(\frac{\pi x_1}{L}\right) \sin\left(\frac{2\pi x_2}{L}\right) - \sin\left(\frac{\pi x_2}{L}\right) \sin\left(\frac{2\pi x_1}{L}\right) \tag{5}$$

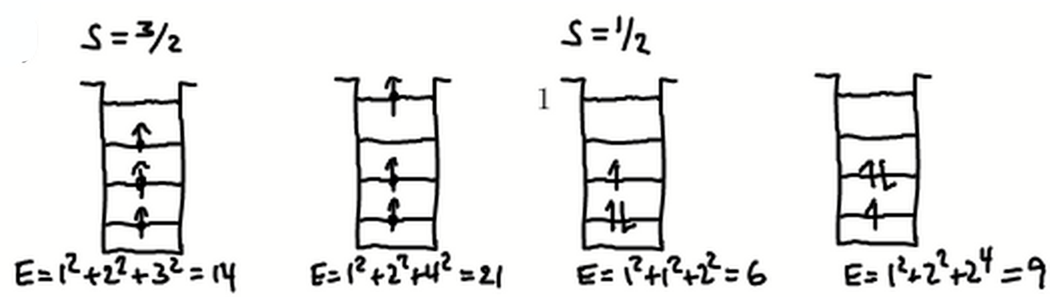
with

$$E \propto 1^2 + 2^2 = 5 \tag{6}$$

and with nodes at the walls of the box and for $x_1 = x_2$. The two particles occupy different spatial wave functions - one is in the ground state of the well, and the other one in the first excited state (i.e. the "Fermi level"). The example clearly illustrates that that fermions of the same spin cannot occupy the same spatial wave function (Pauli principle).

c) A straightforward calculation of the ground-state energy shift $\Delta E = \int\int \psi^* H' \psi dx_1 dx_2$ yields $\Delta E = \frac{3}{2}\frac{g}{L}$ for $S = 0$ and $\Delta E = 0$ for $S = 1$. The latter result shows that two identical fermions (i.e. same spin) cannot interact via the contact interaction; the wave function vanishes for $x_1 = x_2$.

d) By extending the above considerations to three particles, it is easy to find the energies using the following diagrams:



Statistical Mechanics 1

Spinless fermions populating two degenerate energy levels.

Consider a system with two energy levels, one with energy 0 and the other with energy $\Delta > 0$. Both levels are N-fold degenerate, and the system is in equilibrium at temperature T. There are N non-interacting and effectively spinless fermions in the system.

a) [6 pts.] Assume the grand canonical ensemble with chemical potential μ to describe the system. Write down the condition that determines μ, solve it for μ, and find the occupation probabilities f and g of the upper and lower energy levels, respectively.

b) [6 pts.] Now, describe the system using the canonical ensemble. Write down the partition function and find the occupation probabilities f and g in the thermodynamic limit $N \to \infty$ [Hint: $n! \approx (n/e)^n$].

c) [4 pts.] Also within the canonical ensemble, find f and g in the low-temperature limit $T \to 0$. Compare the results to part (a) in the same low-temperature limit.

d) [4 pts.] Use the previous results to find the condition of applicability of the grand canonical ensemble to the system with a fixed number of particles N.

Solution

a) In the grand canonical ensemble, for the system considered, the self-consistency condition for μ is:

$$N = \frac{N}{e^{-\mu/k_BT} + 1} + \frac{N}{e^{(\Delta-\mu)/k_BT} + 1} .$$

This equation for μ gives:

$$e^{(\Delta-2\mu)/k_BT} = 1 , \quad \text{i.e.,} \quad \mu = \Delta/2,$$

and the occupation probabilities f and g of the upper and lower energy levels are:

$$f = \frac{1}{e^{\Delta/2k_BT} + 1} , \quad g = \frac{1}{1 + e^{-\Delta/2k_BT}} .$$

b) In the canonical ensemble, the total energy $E_n = n\Delta$ of the system, and therefore the partition function Z, can be expressed in terms of the number n of the particles in the upper energy states. Counting the number of way of taking n particles from the lower

N levels and distributing them over the upper N levels, one finds

$$Z = \sum_{n=0}^{N} \left[\frac{N!}{n!(N-n)!} \right]^2 e^{-n\Delta/k_B T} .$$

To find the occupation probabilities, one notices that, in the thermodynamic limit $N \to \infty$, the sum over n in Z is dominated by the largest term, which can be found by using Stirling approximation for the factorials in the sum, e.g., $n! \simeq (n/e)^n$. In this way, one finds that the largest term corresponds to

$$n = \frac{N}{e^{\Delta/2k_B T} + 1} ,$$

i.e.,

$$f = n/N = \frac{1}{e^{\Delta/2k_B T} + 1} , \quad g = (N-n)/N = \frac{1}{1 + e^{-\Delta/2k_B T}} ,$$

in agreement with the results of the grand canonical ensemble.

c) In the low-temperature limit $T \to 0$, however, the partition function Z is dominated by the first two terms in the sum over n:

$$Z \simeq 1 + N e^{-\Delta/k_B T} ,$$

so that

$$f = e^{-\Delta/k_B T} , \quad g = 1 - e^{-\Delta/k_B T} .$$

We see that in the low-temperature limit, the occupation probabilities have different temperature dependence that the one that agrees with the grand canonical ensemble.

d) All this means that for the system with fixed number of particles N, the grand canonical ensemble with its Fermi distribution can be used to describe the occupation probabilities only at not-too-low temperatures, when the number of excited particles is large:

$$N e^{-\Delta/k_B T} \gg 1 , \quad \text{i.e.,} \quad k_B T \gg \Delta / \ln N .$$

Although formally this condition is satisfied in the thermodynamic limit $N \to \infty$ for all temperatures, logarithm is a very slow function, and the regime of the small number of excitations can be important experimentally.

Statistical Mechanics 2

Magnetic system in a fixed magnetic field

Consider an equilibrium magnetic system in fixed magnetic field $B = 0$. The free energy $G(m, T)$ of the system as a function of magnetization m can be written as:

$$G(m, T) = a + \frac{b}{2}m^2 + \frac{c}{4}m^4 + \frac{d}{6}m^6.$$

In some relevant range of temperatures T, the coefficients b and d can be taken to be positive constants, $b, d > 0$, while c goes through 0 at some temperature T^* in this range:

$$c(T) = c_0(T - T^*)\,, \quad c_0 > 0\,.$$

a) [7 pts.] The free energy G describes a phase transition, in which the system goes from the state with no magnetization, $m = 0$, to the magnetized state $m = m_0 \neq 0$ at some temperature T_0. Find T_0.

b) [5 pts.] Find the magnitude of the magnetization m_0 appearing at the transition temperature T_0. What is the type of this phase transition?

c) [8 pts.] Calculate the latent heat L of the transition. State qualitatively, for what direction of the temperature change, this heat is absorbed/released by the system.

Solution

a) The free energy G reaches minimum in equilibrium. When all coefficients, $b, c, d > o$, the minimum corresponds to $m = 0$. Transition happens at the temperature T_0, when another minimum of G, at $m = m_0$, becomes smaller than the minimum at $m = 0$, i.e. T_0 is found from the equation

$$G(0, T_0) = G(m = m_0, T_0)\,.$$

This equation can be rewritten as quadratic equation for m^2:

$$\frac{d}{3}m^4 + \frac{c}{2}m^2 + b = 0\,,$$

and gives:

$$m^2 = -\frac{3c}{4d} \pm \Big[\big(\frac{3c}{4d}\big)^2 - \frac{3b}{d}\Big]^{1/2}.$$

Another (besides $m = 0$) minimum of G appears when this solution gives a real value to m, i.e., when

$$\left(\frac{3c}{4d}\right)^2 = \frac{3b}{d}\,.$$

This gives

$$c = 4\sqrt{\frac{bd}{3}}\,, \quad \text{or} \quad T_0 = T^* - \frac{4}{c_0}\sqrt{\frac{bd}{3}}\,.$$

b) From the equation derived in part (a), we see that at the transition temperature T_0

$$m^2 = -\frac{3c}{4d}\,,$$

i.e.,

$$m^2 = \sqrt{\frac{3b}{d}}\,.$$

Since a finite non-vanishing value of magnetization is appearing right at the transition temperature, this is the first-order phase transition.

c) The latent heat of the first-order phase transition is determined by the entropy change ΔS at the transition temperature,

$$L = T_0 \Delta S\,.$$

To find ΔS, we use the general relation between the free energy and entropy:

$$S = -\left(\frac{\partial G}{\partial T}\right)_B\,,$$

from which (dropping the subscript B and the sign of partial derivative indicating constant magnetic field B)

$$\Delta S = \frac{dG(m,T)}{dT}\Big|_{m=m_0} - \frac{dG(0,T)}{dT}\,.$$

It is convenient to calculate the first, main, derivative by breaking it into two group of terms to get:

$$\Delta S = \frac{\partial G(m,T)}{\partial m}\Big|_{m=m_0}\frac{dm_0}{dT} + \frac{m_0^4}{4}\frac{dc}{dT}\,.$$

The fact that both initial and final states of the phase transition are equilibrium states where G is minimum, implies, in particular, that

$$\frac{\partial G(m,T)}{\partial m}\Big|_{m=m_0} = 0\,.$$

Then

$$\Delta S = \frac{m_0^4}{4}c_0\,,$$

and finally,

$$L = T_0 c_0 \frac{3b}{4d}\,.$$

We see that $L > 0$, the sign consistent with the natural requirement that the heat is released by the system when the system is cooled down and absorbed when it is heated up.

Statistical Mechanics 3

Ising chain in zero magnetic field

Consider the Hamiltonian for the Ising model on a one-dimensional lattice without external magnetic field, which may be written as

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{1}$$

where the classical Ising spin variable $\sigma_i = \pm 1$ on each site i, and $\langle ij \rangle$ denotes nearest-neighbor pairs of sites. Consider this model in thermal equilibrium at temperature T in the thermodynamic limit. Take the ferromagnetic case, $J > 0$. Derive exact expressions for

a) [8 pts.] the specific heat per spin, C

b) [7 pts.] the spin-spin correlation function $\langle \sigma_0 \sigma_r \rangle$, where r denotes a lattice site.

c) [5 pts.] the (zero-field) magnetic susceptibility χ per spin

Solution

a) Let $\beta = 1/(k_B T)$ and denote $\beta J \equiv K$ and the total number of sites as N. (i) The partition function for this model is

$$Z = \sum_{\sigma_n} e^{-\beta \mathcal{H}} = \sum_{\sigma_n} \prod_{ij} e^{K \sigma_i \sigma_j} \tag{2}$$

and the free energy per site is $A = -k_B T f$, where f is the dimensionless quantity

$$f = \lim_{N \to \infty} \frac{1}{N} \ln Z \tag{3}$$

Using the identity $e^{K\sigma_i\sigma_j} = \cosh K(1 + v\sigma_i\sigma_j)$, where $v \equiv \tanh K$, we can write Z as

$$Z = (\cosh K)^N \sum_{\sigma_n} \prod_{ij} (1 + v\sigma_i\sigma_j) \tag{4}$$

By an explicit calculation,

$$Z = (\cosh K)^N (1 + v^N) = (\cosh K)^N + (\sinh K)^N \tag{5}$$

A different way to get this result is via a transfer matrix method. In a spin basis $(+, -)$ the transfer matrix $\mathcal{T}$ is

$$\mathcal{T} = \begin{pmatrix} e^K & e^{-K} \\ e^{-K} & e^K \end{pmatrix} \tag{6}$$

Here we assume periodic boundary conditions (the thermodynamic limit is independent of boundary conditions). Then

$$Z = \mathrm{Tr}(\mathcal{T}^N) = \lambda_1^N + \lambda_2^N \tag{7}$$

where λ_j, $j = 1, 2$ are the eigenvalues of $\mathcal{T}$. We calculate

$$\lambda_1 = \cosh K, \quad \lambda_2 = \sinh K \tag{8}$$

which yields the same result as in Eq. (5). Now taking $N \to \infty$ and using the fact that $0 < v < 1$ for finite temperature, we have

$$f = \ln(\cosh K) \tag{9}$$

(Zero or infinite temperatures can be approached as a limit from finite temperatures.)

The internal energy per site is

$$U = -\frac{\partial f}{\partial \beta} = -J \tanh K \tag{10}$$

The specific heat per site here is

$$C = \frac{dU}{dT} = -k_B \beta^2 \frac{dU}{d\beta} = \frac{k_B K^2}{\cosh^2 K} \tag{11}$$

b) The spin-spin correlation function is

$$\langle \sigma_0 \sigma_r \rangle = Z^{-1} \sum_{\sigma_n} \sigma_0 \sigma_r e^{-\beta \mathcal{H}} = \frac{\sum_{\sigma_n} \sigma_0 \sigma_r \prod_{ij}(1 + v\sigma_i\sigma_j)}{\sum_{\sigma_n} \prod_{ij}(1 + v\sigma_i\sigma_j)} \tag{12}$$

An explicit evaluation yields $\langle \sigma_0 \sigma_r \rangle = v^r + v^{N-r}$. Taking $N \to \infty$ and using the fact that $0 \le v < 1$, we find

$$\langle \sigma_0 \sigma_r \rangle = v^r = (\tanh K)^r \tag{13}$$

c) There are two ways to do this problem. One is to insert a magnetic field term in the Hamiltonian, so that $\mathcal{H} \to \mathcal{H} - H \sum_i \sigma_i$, where H denoes the external magnetic field. Denote $\beta H \equiv h$. Then calculate the transfer matrix, which is

$$\mathcal{T} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix} \tag{14}$$

with eigenvalues

$$\lambda_{1,2} = e^K \left[\cosh h \pm (\sinh^2 h + e^{-4K})^{1/2} \right] \tag{15}$$

Then we calculate $Z = \mathrm{Tr}(\mathcal{T}^N)$ and f as before, and then compute the magnetization $M(H) = -\partial f/\partial h$. Finally, one calculates the zero-field susceptibility $\chi \equiv \lim_{H\to 0} \partial M/\partial H$. Following this procedure, we obtain

$$M(H) = \frac{\sinh h}{[\sinh^2 h + e^{-4K}]^{1/2}} \tag{16}$$

and

$$\chi = \beta e^{2K} \tag{17}$$

The other way to do the problem is calculate χ as the normalized sum over all spin-spin correlation functions,

$$\begin{aligned} \beta^{-1}\chi &= \sum_r \langle \sigma_0 \sigma_r \rangle \\ &= 1 + 2\sum_{r=1}^{\infty} v^r = -1 + 2\sum_{r=0}^{\infty} v^r \\ &= -1 + \frac{2}{1-v} = \frac{1+v}{1-v} = e^{2K} \end{aligned} \tag{18}$$

which yields the same result as in (17).