Solutions

Physics PhD Qualifying Examination
Part I – Wednesday, January 10, 2007

Name:____
(please print)
Identification Number:________

STUDENT: **Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.**
PROCTOR: **Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.**

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Student's initials
problems handed in:
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INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box_of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[10]**

Consider a projectile of mass m fired from the origin of a coordinate system with initial velocity $\mathbf{v}_0$ in a direction making an angle α with the horizontal.

(a) Calculate the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ of the projectile with respect to the origin of the coordinate system.

(b) Calculate the torque due to gravity $\vec{N} = \vec{r} \times \vec{F}_g$ acting on the projectile with respect to the origin of the coordinate system.

(c) Demonstrate that the torque is equal to the (temporal) rate of change of the angular momentum.

[I-2] **[5, 5]**

A particle is constrained to move in a plane. It is attracted to a fixed point P in this plane; the force is always directed exactly at P and is inversely proportional to the square of the distance from P.

(a) Using polar coordinates, write down the Lagrangian of this particle.

(b) Obtain the Lagrangian equations for this particle and find at least one first integral (constant of motion).

[I-3] **[10]**

Three oscillators of equal mass m are coupled such that the potential energy of the system is given by:

$$U(x_1, x_2, x_3) = \frac{1}{2}\left[k_1(x_1^2 + x_3^2) + k_2 x_2^2 + k_3(x_1 x_2 + x_2 x_3)\right],$$

where $k_3 = \sqrt{2k_1 k_2}$.

(a) Find the eigenfrequencies of the coupled oscillators by solving the secular equation.

(b) What is the physical interpretation of the zero-frequency mode?

[I-4] [10]

An observer measures the period of oscillations of a simple plane pendulum to be T_o on the surface of the Earth. The observer then descends to a depth $R/4$ below the surface, where R is the radius of the Earth. What will be the period of oscillations of the same pendulum at that depth? You must express your answer in terms of T_o.

[I-5] [3, 7]

The news agency Reuters reported: In his acceptance speech for the Royal Society's Copley Medal Nov 30, 2006, cosmologist Prof. Stephen Hawking said that humans must colonize planets in other solar systems or otherwise face extinction by one disaster or another. To travel there they should use a "Star Trek"-style propulsion.

In order to survive, humanity would have to venture off to other hospitable planets orbiting another star, but conventional chemical fuel rockets that took man to the moon on the Apollo mission would take 50,000 years to travel there, he said (see hint below).

"Science fiction has developed the idea of warp drive, which takes you instantly to your destination," he said.

"Unfortunately, this would violate the scientific law which says that nothing can travel faster than light."

However, by using "matter/antimatter annihilation", velocities just below the speed of light could be reached, making it possible to reach the next star in about 6 years.

"It wouldn't seem so long for those on board," he said.

(a) How fast would that spaceship need to travel?
(b) How long would the one-way trip appear to the passengers?

Hint: For the Apollo rocket's constant speed, assume a "first escape velocity" of 10 km/s.

[I-6] **[10]**

A point charge $+q$ is placed at one corner of a cube with edge a. What is the flux through each side of the cube?

[I-7] **[10]**

Starting from Maxwell's equations, demonstrate that the electric field $\vec{E}$ and magnetic field $\vec{H}$ satisfy the following differential equations in a homogenous medium containing charges ρ and currents $\vec{j}$ (ε and μ are the electric and magnetic permeabilities):

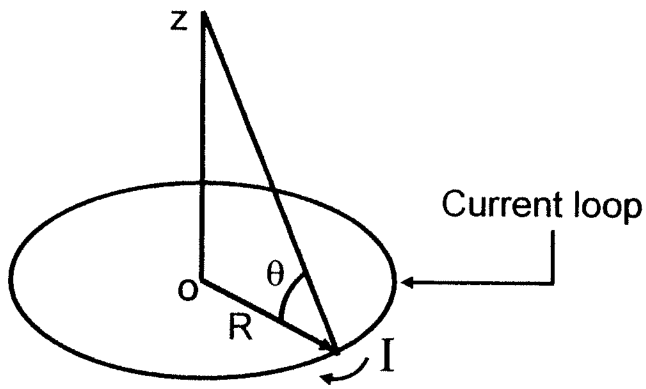
$$\nabla^2\vec{E} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} = \frac{1}{\varepsilon}\vec{\nabla}\rho + \mu\frac{\partial\vec{J}}{\partial t}$$

and

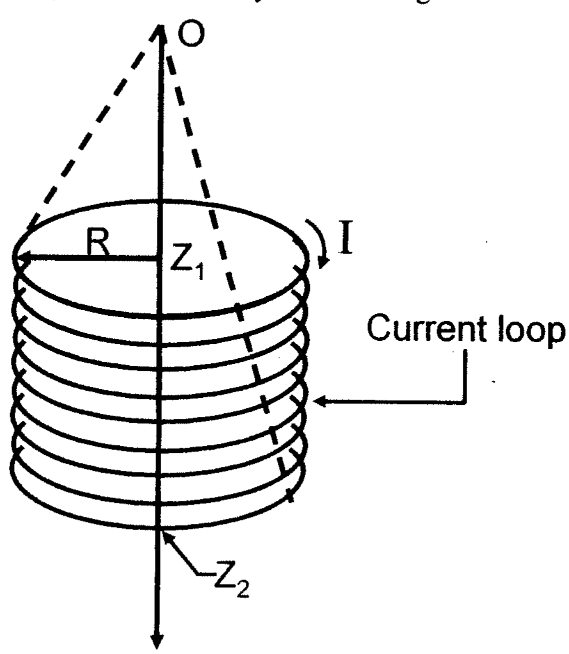
$$\nabla^2\vec{H} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} = -\vec{\nabla}\times\vec{j}\,.$$

[I-8] **[4, 6]**

(a) Find the magnetic field at a distance z above the center (point-O) of a circular loop of radius R, which carries a steady current I. A schematic drawing is shown below.



(b) Find the magnetic field at point-O (the origin of z-axis) on the axis of a tightly wound solenoid consisting of N turns *per unit length* wrapped around a cylindrical tube of radius R and carrying current I. The top and the bottom of the solenoid are at a distance z_1 and z_2 from point-O, respectively (see illustration). You must carry out the integral.



[1-9] **[2, 2, 2, 2, 2]**

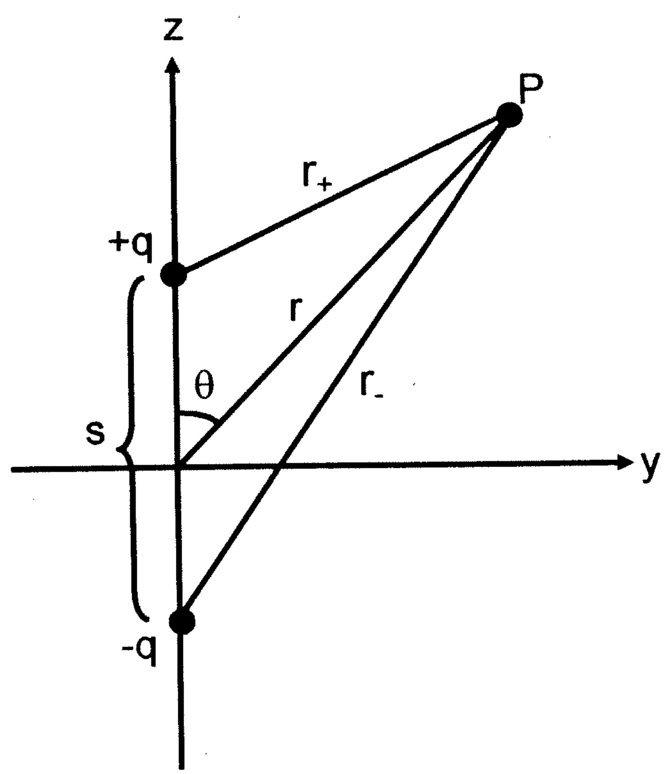
The far field radiation fields due to an oscillating electric dipole is given by:

$$|\vec{E}| = \frac{\mu_0 p_0 \omega^2}{4\pi}\left[\frac{\sin(\theta)}{r}\right]\cos[\omega(t - r/c)]$$

$$|\vec{B}| = \frac{\mu_0 p_0 \omega^2}{4\pi c}\left[\frac{\sin(\theta)}{r}\right]\cos[\omega(t - r/c)].$$

Here, p_0 is the maximum value of the dipole moment and ω the angular frequency of the charge oscillation.

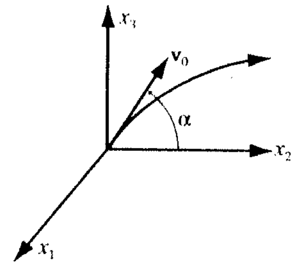
(a) State the definition of "far-field radiation zone", the direction of electric and magnetic fields in the spherical coordinates.
(b) Compute the energy radiated by the oscillating electric dipole.
(c) Compute the time averaged radiation energy over a cycle.
(d) Compute the time averaged total power radiated.
(e) Sketch the radiation profile on the z-y plane.



[I-10] **[10]**

Find the trajectory of a relativistic particle of mass m, charge q in a uniform electric field E. Assume that at t=0 the particle has a velocity v_0 into the x-direction and the electric field is in the y-direction. Provide an expression for the trajectory in the form of $y = f(x)$. Sketch the trajectory in the plane of motion. You must treat this problem *relativistically.*

I-1



Assume a coordinate system in which the projectile moves in the $x_2 - x_3$ plane. Then,

$$\left.\begin{aligned} x_2 &= v_0 t \cos\alpha \\ x_3 &= v_0 t \sin\alpha - \frac{1}{2} g t^2 \end{aligned}\right] \tag{1}$$

or,

$$\begin{aligned} \mathbf{r} &= x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \\ &= \left(v_0 t \cos\alpha\right)\mathbf{e}_2 + \left(v_0 t \sin\alpha - \frac{1}{2} g t^2\right)\mathbf{e}_3 \end{aligned} \tag{2}$$

The linear momentum of the projectile is

$$\mathbf{p} = m\dot{\mathbf{r}} = m\left[\left(v_0 \cos\alpha\right)\mathbf{e}_2 + \left(v_0 \sin\alpha - gt\right)\mathbf{e}_3\right] \tag{3}$$

and the angular momentum is

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} = \left[\left(v_0 t \cos\alpha\right)\mathbf{e}_2 + \left(v_0 t \sin\alpha - g t^2\right)\mathbf{e}_3\right]\times m\left[\left(v_0 \cos\alpha\right)\mathbf{e}_2 + \left(v_0 \sin\alpha - gt\right)\mathbf{e}_3\right] \tag{4}$$

Using the property of the unit vectors that $\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_3\ \varepsilon_{ijk}$, we find

$$\mathbf{L} = \frac{1}{2}\left(mg\, v_0 t^2 \cos\alpha\right)\mathbf{e}_1 \tag{5}$$

This gives

$$\dot{\mathbf{L}} = -\left(mg\, v_0 t \cos\alpha\right)\mathbf{e}_1 \tag{6}$$

Now, the force acting on the projectile is

$$\mathbf{F} = -mg\,\mathbf{e}_3 \tag{7}$$

so that the torque is

$$\begin{aligned} \mathbf{N} = \mathbf{r}\times\mathbf{F} &= \left[\left(v_0 t \cos\alpha\right)\mathbf{e}_2 + \left(v_0 t \sin\alpha - \frac{1}{2} g t^2\right)\mathbf{e}_3\right]\left(-mg\right)\mathbf{e}_3 \\ &= -\left(mg\, v_0 t \cos\alpha\right)\mathbf{e}_1 \end{aligned}$$

which is the same result as in (6).

I-2. Solution:

(a.) Choose polar coordinates with origin at P in the plane in which the particle is constrained to move. The force acting on the particle is

$$\vec{F} = -\frac{k\hat{r}}{r^3}$$

with "k" being a positive constant. It potential energy with respect to infinity is

$$V = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = -\frac{k}{r}$$ and the kinetic

energy of the particle is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}$$

(b) Lagrange's Equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0,$$ which give the equation of motion

$$-mr\dot{\theta}^2 + m\ddot{r} + \frac{k}{r^2} = 0, \quad \frac{d}{dt}(mr^2\dot{\theta}) = 0.$$

The second equation gives immediately a first integral: $mr^2\dot{\theta}$ = constant, which means that the angular momentum with respect to "P" is conserved.

12-21. The tensors $\{\mathbf{A}\}$ and $\{\mathbf{m}\}$ are:

$$\{\mathbf{A}\} = \begin{bmatrix} \kappa_1 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 \end{bmatrix} \tag{1}$$

$$\{\mathbf{m}\} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \tag{2}$$

thus, the secular determinant is

$$\begin{vmatrix} \kappa_1 - m\omega^2 & \frac{1}{2}\kappa_3 & 0 \\ \frac{1}{2}\kappa_3 & \kappa_2 - m\omega^2 & \frac{1}{2}\kappa_3 \\ 0 & \frac{1}{2}\kappa_3 & \kappa_1 - m\omega^2 \end{vmatrix} = 0 \tag{3}$$

from which

$$\left(\kappa_1 - m\omega^2\right)^2 \left(\kappa_2 - m\omega^2\right) - \frac{1}{2}\kappa_3^2 \left(\kappa_1 - m\omega^2\right) = 0 \tag{4}$$

In order to find the roots of this equation, we first set $(1/2)\kappa_3^2 = \kappa_1\kappa_2$ and then factor:

$$\begin{aligned} &\left(\kappa_1 - m\omega^2\right)\left[\left(\kappa_1 - m\omega^2\right)\left(\kappa_2 - m\omega^2\right) - \kappa_1\kappa_2\right] = 0 \\ &\left(\kappa_1 - m\omega^2\right)\left[m^2\omega^4 - \left(\kappa_1 + \kappa_2\right)m\omega^2\right] = 0 \\ &\left(\kappa_1 - m\omega^2\right)m\omega^2\left[m\omega^2 - \left(\kappa_1 + \kappa_2\right)\right] = 0 \end{aligned} \tag{5}$$

Therefore, the roots are

$$\boxed{\begin{aligned} \omega_1 &= \sqrt{\frac{\kappa_1}{m}} \\ \omega_2 &= \sqrt{\frac{\kappa_1 + \kappa_2}{m}} \\ \omega_3 &= 0 \end{aligned}} \tag{6}$$

Consider the case $\omega_3 = 0$. The equation of motion is

$$\ddot{\eta}_3 + \omega_3^2 \eta_3 = 0 \tag{7}$$

so that

$$\ddot{\eta}_3 = 0 \tag{8}$$

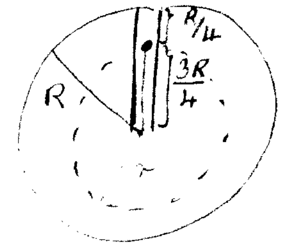
with the solution

$$\eta_3(t) = at + b \tag{9}$$

That is, the zero-frequency mode corresponds to a *translation* of the system with oscillation.

$\boxed{I-4}$ Gauss' law for gravitational field:

$$\oiint \vec{g}(\vec{r})\cdot d\vec{a} = -4\pi G M_{encl.}$$



M

$r \leq R$ $$g(r)\,4\pi r^2 = -4\pi G M \frac{r^3}{R^3}$$

$$g(r) = -GM\frac{r}{R^3}$$

on the surface of the Earth: $g(R) = -GM\frac{1}{R^2}$

at $r = \frac{3R}{4}$: $$g\left(\tfrac{3R}{4}\right) = -GM\frac{\frac{3R}{4}}{R^3} = -\frac{3}{4}G\frac{M}{R^2} = \frac{3}{4}g(R)$$

frequency of oscillations: $\omega^2(r) = \dfrac{|g(r)|}{\ell}$ ← frequency for simple pendulum

$$T(r) = \frac{2\pi}{\omega(r)} = 2\pi\sqrt{\frac{\ell}{|g(r)|}}$$

$$\frac{T\left(\frac{3R}{4}\right)}{T(R)} = \sqrt{\frac{|g(R)|}{\left|g\left(\frac{3R}{4}\right)\right|}} = \sqrt{\frac{|g(R)|}{\frac{3}{4}|g(R)|}} = \frac{2}{\sqrt{3}}$$

$$T\left(\tfrac{3R}{4}\right) = \frac{2}{\sqrt{3}}T(R) \simeq 1.155\,T(R)$$

$$\boxed{T' = \frac{2}{\sqrt{3}}T_0}$$

I-5 Solution

Distance using the speed of Apollo:

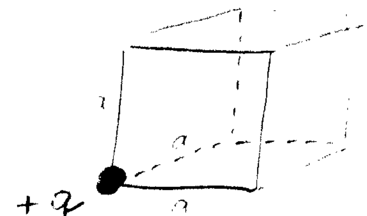
50000 yr * 365.25 d/yr * 24 hr/d * 3600 s/hr *10000 m/s = 1.58 E16 m
= 1.67 light years

v_Startrek = v_Apollo * Time_Apollo / Time_Startrek

= 1E4 m/s * 50000 yr / 6 yr = 8.33E7 m/s = 83300 km/s = 0.2778 * c

T_inside_Startrek = T_Startrek * $(1 - v^2/c^2)^{1/2}$
= 6 yr * $(1 - 0.2778^2)^{1/2}$
= 5.7638 yr

I-6



Gauss' Law: $\oint \vec{E}\cdot d\vec{a} = \frac{q_{enc.}}{\varepsilon_0}$

i) The flux through the three sides that touch the charge is zero, since the electrical field is parallel to each of those sides.

ii) The flux through the other three sides (not touching the charge) can be reasoned using symmetry and Gauss' Law:

The positions/orientations of these three sides, relative to the charge, are equivalent, hence the flux through each of those will be identical.
The total flux passing through these three sides combined is 1/8 of the overall flux emanating from a point charge through a closed surface, i.e., $\frac{1}{8}\frac{q}{\varepsilon_0}$

(Imagine 8 cubes attached and all touching the charge +q at one of their corners, and forming a closed Gaussian surface)

Thus, the flux through each of those sides not touching the charge will be

$$\frac{1}{3}\cdot\frac{1}{8}\frac{q}{\varepsilon_0} = \boxed{\frac{1}{24}\frac{q}{\varepsilon_0}}$$

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon}$ & $\vec{B} = \mu\vec{H}$

$\vec{D} = \varepsilon\vec{E}$

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$

$\vec{\nabla}\times\vec{H} = \vec{J} + \dot{\vec{D}}$

$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = \vec{\nabla}\,\underbrace{(\vec{\nabla}\cdot\vec{H})}_{0} - \Delta\vec{H} = \vec{\nabla}\times\vec{J} + \vec{\nabla}\times\dot{\vec{D}}$

because $\vec{\nabla}\cdot\vec{B} = 0$

$-\Delta\vec{H} - \vec{\nabla}\times\varepsilon\dot{\vec{E}} = \vec{\nabla}\times\vec{J}$

$-\Delta\vec{H} + \varepsilon\ddot{\vec{B}} = \vec{\nabla}\times\vec{J}$ because $\vec{\nabla}\times\dot{\vec{E}} = -\ddot{\vec{B}}$

$-\Delta\vec{H} + \varepsilon\mu\ddot{\vec{H}} = \vec{\nabla}\times\vec{J}$ " $\vec{B} = \mu\vec{H}$

$$\boxed{\Delta\vec{H} - \varepsilon\mu\ddot{\vec{H}} = -\vec{\nabla}\times\vec{J}}$$

$\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \Delta\vec{E} = -\vec{\nabla}\times\dot{\vec{B}}$

$= \frac{1}{\varepsilon}\vec{\nabla}\rho - \Delta\vec{E} = -\vec{\nabla}\times\mu\dot{\vec{H}} = -\mu(\dot{\vec{J}} + \ddot{\vec{D}})$

$= \frac{\vec{\nabla}\rho}{\varepsilon} - \Delta\vec{E} = -\mu\dot{\vec{J}} - \mu\varepsilon\ddot{\vec{E}}$

$\Rightarrow$ $$\boxed{\Delta\vec{E} - \mu\varepsilon\ddot{\vec{E}} = \frac{1}{\varepsilon}\vec{\nabla}\rho + \mu\dot{\vec{J}}}$$

I-8

(a)

The field $d\vec{B}$ due to the segment dl is:

$$dB_z = \frac{\mu_0}{4\pi} I \frac{dl}{r^2} \cos\theta$$

(note: The horizontal components cancel due to symmetry.)

$$B_z = \frac{\mu_0}{4\pi} I \int \frac{dl}{r^2} \cos\theta = \frac{\mu_0}{4\pi} \frac{I}{r^2} 2\pi R \cos\theta$$

$$\therefore B_z = \frac{\mu_0 I}{2} \cdot \frac{R^2}{(R^2+z^2)^{3/2}}$$

(b)

The solenoid has N turns per unit length

$$dB_z = \frac{\mu_0}{2}(dI)\frac{R^2}{(R^2+z^2)^{3/2}}$$

Given $(dI) = I N (dz)$

$$B_z = \frac{\mu_0 N I}{2}\int_{z_1}^{z_2}\frac{R^2\,dz}{(R^2+z^2)^{3/2}}$$

$$B_z = \frac{\mu_0 N I}{2}\left.\frac{R^2 z}{R^2(R^2+z^2)^{1/2}}\right|_{z_1}^{z_2}$$

I-9

(1)(a) far field: $|s| << r$

$\vec{E} \parallel \hat{\theta}$

$\vec{B} \parallel \hat{\phi}$ #

(and also $r >> c/\omega >> |s|$)

(2)(b) $\vec{S} \equiv \frac{1}{\mu_0}(\vec{E} \times \vec{B})$

$$\vec{S} = \frac{\mu_0}{c}\left[\frac{p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos(t - r/c)\right]^2 \hat{r},$$

where $p_0 \equiv q \cdot s$ #

(3)(c) $$\langle \vec{S} \rangle_{time\text{-}average} = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{r}.$$ #

1-9

(4) Total power $\langle p \rangle = \int \langle \vec{S} \rangle \cdot d\vec{a}$

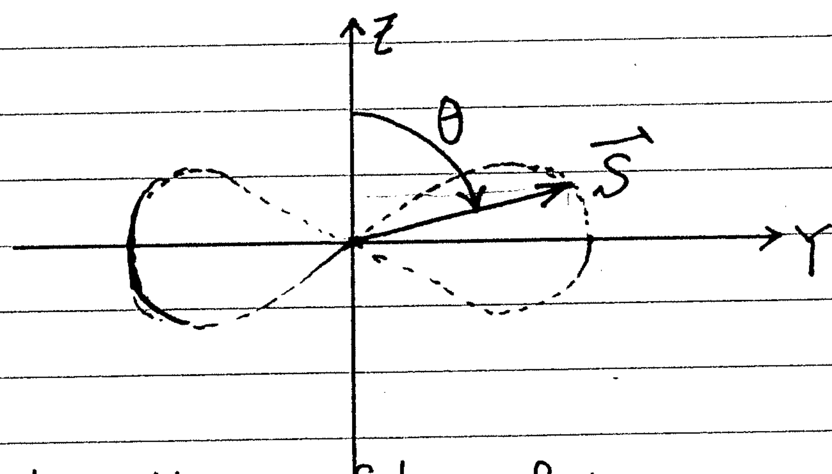
(d)

$$\langle p \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \right) \int \frac{\sin^2\theta}{r^2} \cdot (r^2 \sin\theta \, d\theta \, d\phi)$$

$$\therefore \langle p \rangle = \left(\frac{\mu_0 p_0^2}{12\pi c} \right) \omega^4 = \frac{1}{4\pi\varepsilon_0} \frac{p_0^2 \omega^4}{3c^2}$$

(5)

(e)



Intensity profile for dipole radiation

#

The plane of motion of a particle will be defined by its initial velocity v and the direction of the electric field **E**. Let the initial velocity coincide with the x axis and E with the y axis. We may write the equations of motion for a charge in an electric field

$$\frac{d\vec{p}}{dt} = q\vec{E}$$

where **p** is the momentum. Since there is no force in the z-direction, the particle will move within the x-y plane. We therefore rewrite:

$$\frac{dp_x}{dt} = 0 \qquad \frac{dp_y}{dt} = qE$$

Integration then yields

$$p_x = p_{x0} = p_0 \qquad p_y = qEt$$

The energy of the particle (without potential energy in the field) is given by

$$\mathcal{E} = \sqrt{m^2c^4 + p^2c^2} = \sqrt{m^2c^4 + p_0^2c^2 + c^2q^2E^2t^2} = \sqrt{\mathcal{E}_0^2 + (cqEt)^2}$$

where

$$\mathcal{E}_0 = \sqrt{m^2c^4 + p_0^2c^2}$$

is the initial energy of the particle. The work done by the field changes the energy of the particle

$$\frac{d\mathcal{E}}{dt} = q\vec{E}\cdot\vec{v} = qEv_y = qE\frac{dy}{dt} \quad \text{or} \quad \mathcal{E} = \mathcal{E}_0 + qEy$$

leading to

$$\mathcal{E}_0 + qEy = \sqrt{\mathcal{E}_0^2 + (cqEt)^2} \qquad y = \frac{\mathcal{E}_0}{qE}\left\{\sqrt{1 + (cqEt)^2/\mathcal{E}_0^2} - 1\right\}$$

and

$$t = \frac{\sqrt{(\mathcal{E}_0 + qEy)^2 - \mathcal{E}_0^2}}{cqE}$$

On the other hand,

$$\frac{p_y}{p_x} = \frac{\gamma m v_y}{\gamma m v_x} = \frac{v_y}{v_x} = \frac{dy/dt}{dx/dt} = \frac{dy}{dx}$$

Substituting

$$p_x = p_0 \quad \text{and} \quad p_y = qEt \quad \text{and } t \text{ from above,}$$

we find

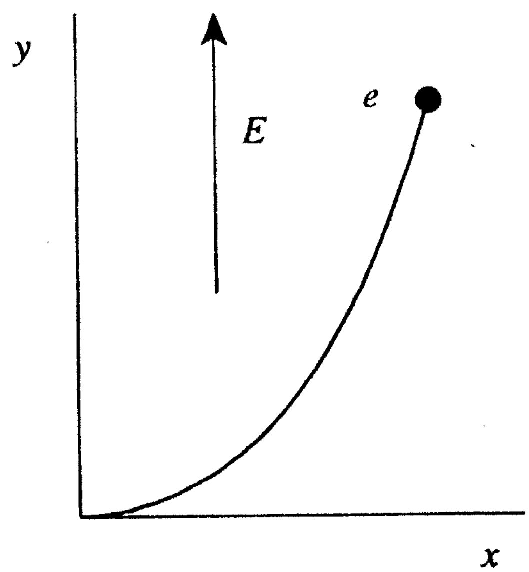
$$\frac{dy}{dt} = \frac{qEt}{p_0} \cdot \frac{\sqrt{(\mathcal{E}_0 + qEt)^2 - \mathcal{E}_0^2}}{p_0 c}$$

After integrating we get

$$\frac{x}{p_0 c} = \int \frac{dy}{\sqrt{(\mathcal{E}_0 + qEy)^2 - \mathcal{E}_0^2}}$$

$$= \frac{1}{qE} \cosh^{-1} \frac{qEy}{\mathcal{E}_0} + const$$

So the particle in a constant electric field moves along a catenary.



If the velocity of the particle $v \ll c$, then $p_0 = m v_0$, $\varepsilon_0 = mc^2$ and expanding the cosh function we obtain

$$y \approx \frac{eE}{2mv_0^2} x^2$$

Which gives the classical result for a charged particle in an electric field.

Solution

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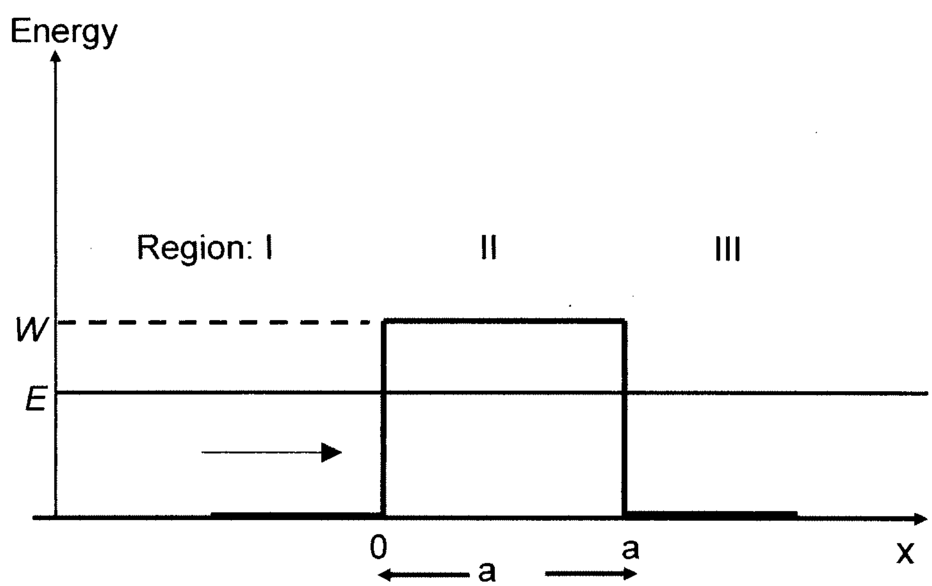
[II-1] [10]

A stream of particles of mass m and energy E is incident in region (I) on a potential barrier given by:

$$V(x) = 0, \ \ for\ x < 0 \ \ and\ for\ x > a$$
$$V(x) = W, \ \ for\ 0 < x < a$$

where $E < W$ (see illustration below).

Calculate the fraction of the stream of particles (transmission flux relative to incident flux) that is transmitted from region-I to region-III.



[II-2] **[6, 4]**

A particle of mass m moves in a one-dimensional simple harmonic potential $V^{(0)} = ½\, k\, x^2$, with angular frequency $\omega = (k/m)^{1/2}$. A small perturbing term $V^{(1)} = ½\, \delta k\, x^2$ is added to $V^{(0)}$.

(a) *Using time-independent perturbation theory*, derive expressions for the energy of the ground state in first- and second- order perturbation.
(b) How do these expressions relate to the exact expression for the energy?

[II-3] **[10]**

Find the state of a system consisting of two spin-1/2 particles which is an eigenstate of each of the two commuting operators, the square and the z-component of the total spin.

[II-4] **[10]**

Using the Born approximation, calculate the differential scattering cross section for scattering of particles of mass m and incident energy E by a repulsive spherical well with potential

$$V(r) = V_0 \qquad \text{for } 0 < r < a$$
$$V(r) = 0 \qquad \text{for } r > a\,.$$

[II-5] **[2, 3, 3, 2]**

Consider the wavefunction $\Psi(x) = N\dfrac{1}{a^2 + x^2}$.

(a) Determine N from the requirement to normalize the probability.

(b) Determine the expectation values for $<\hat{x}^n>$ (n =1,2,...) of the position operator $\hat{x}$, as well as its variance $\Delta x = \sqrt{\langle(\hat{x} - \langle\hat{x}\rangle)^2\rangle}$.

(c) Determine the associated wavefunction in momentum space and calculate the expectation values $<\hat{p}^n>$ (n = 1,2,...) of the momentum operator $\hat{p}$. Determine the variance $\Delta p = \sqrt{\langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle}$.

(d) Determine the product $\Delta x\ \Delta p$.

[II-6] **[4, 4, 2]**

For a two-level system, the two eigenstates of the system are ψ_a and ψ_b, respectively. Suppose we turn on a *time-dependent* perturbation, $H'_{ab} = <\psi_a | H' | \psi_b>$ (and $H'_{aa} = H'_{bb} = 0$ for simplicity). The time-dependent wave-function then can be written as

$$\psi(t) = c_a(t)\psi_a e^{-iE_a t/\hbar} + c_b(t)\psi_b e^{-iE_b t/\hbar},$$

and we also have

$$\frac{dc_a}{dt} = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b;$$

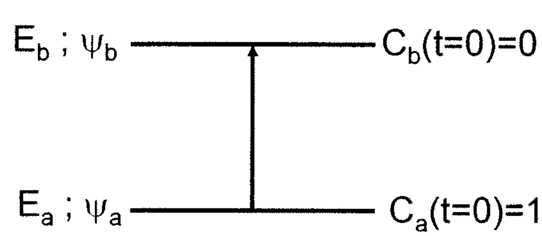
$$\frac{dc_b}{dt} = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a; \quad \text{where } \omega_0 \equiv \frac{E_b - E_a}{\hbar}.$$

Suppose that the particle starts out in the lower state (i.e., c_a=1 and c_b=0 at t=0). Also, suppose that the perturbation has sinusoidal dependence, i.e., $H'_{ab} = V_{ab}\cos(\omega t)$.

(a) Calculate $c_b(t)$ to first order.

(b) Calculate the transition probability, $P_{a\to b}(t)$, from ψ_a to ψ_b near the resonance frequency, i.e. $\omega \approx \omega_0$.

(c) Sketch the transition probability as a function of ω.



[II-7] **[10]**

The specific Gibbs function (Gibbs potential per particle) for a gas is given by

$$g = k_B T \ln(P / P_o) - AP ,$$

where $A = A(T)$ is a function of the temperature T. Find expressions for

(a) the equation of state;

(b) the specific Helmholtz free energy (Helmholtz free energy per particle).

[II-8] **[10]**

Two identical ideal gases with the same pressure P and the same number of particles N, but with different temperatures T_1 and T_2, are confined in two vessels, of volume V_1 and V_2, which are connected. Find the change in entropy after the system has reached equilibrium.

[II-9] **[10]**

In the Einstein model for a three-dimensional solid of N atoms, the system is treated as an ensemble of $3N$ *distinguishable* and *independent* quantum harmonic oscillators with *identical* frequency ω_o (three oscillators for each atoms).

(a) Find the specific heat of this simple model for solids (the Einstein crystal). You must express your answer in terms of N, k_B (Boltzmann's constant), and the ratio θ_E / T, where $\theta_E = \hbar\omega_o / k_B$ is the characteristic Einstein temperature.

(b) Obtain the high-temperature behavior of the specific heat. How does it compare to the classical behavior?

(c) Obtain the low-temperature behavior of the specific heat. What can you say about its limiting value as $T \to 0$?

[II-10] **[10]**

Write the internal energy of a system of fermions as a one-dimensional integral involving the density of states $D(\varepsilon)$, and show that at low temperatures, it is only the density of states at the Fermi energy, and not its derivative, that is important in determining the specific heat.
That is, obtain and express C_V in terms of the density of states.

Hint: (i) The low-temperature expansion for a general Fermi-Dirac integral is given by:

$$\int_0^\infty \frac{g(\varepsilon)d\varepsilon}{e^{(\varepsilon-\mu)/\tau}+1} = \int_0^\mu g(\varepsilon)d\varepsilon + \frac{\pi^2}{6}g'(\mu)\tau^2 + \frac{7\pi^4}{360}g'''(\mu)\tau^4 + \ldots \quad (\tau = k_B T).$$

Here the primes indicate derivatives with respect to ε, evaluated at $\varepsilon = \mu$.

(ii) At fixed N, the chemical potential μ is a function of temperature, T, given by:

$$\mu(T) = \varepsilon_F - \frac{\pi^2 k_B^2 T^2}{6}\frac{D'(\varepsilon_F)}{D(\varepsilon_F)}$$, with $D(\varepsilon)$ being a density of states.

II-1

$$\begin{cases} \dfrac{d^2U}{dx^2} + k^2 U = 0 & \text{for region 1 and 3} \\ \dfrac{d^2U}{dx^2} - \gamma^2 U = 0 & \text{for region 2} \end{cases}$$

where $k^2 = \dfrac{2mE}{\hbar^2}$ and $\gamma^2 = \dfrac{2m(W-E)}{\hbar^2}$

- Region 1 has incident ~~wafe~~ wave and reflected wave. Incident wave was normalize to 1
- Region 3 has transmission wave only.

Therefore, we can write down solutions in Region 1, 2, and 3:

Region 1: $u_1 = e^{ikx} + R e^{-ikx}$ (R: reflection coefficient)

Region 2: $u_2 = A e^{\gamma x} + B e^{-\gamma x}$

Region 3: $u_3 = T e^{ikx}$ (T: transmission coefficient)

consider boundary conditions at $x=0$

$$\Rightarrow \begin{cases} 1 + R = A + B & (1) \\ 1 - R = -i\rho(A-B) \quad \text{where } \rho = \dfrac{\gamma}{K} & (2) \end{cases}$$

consider boundary conditions at $x=a$

$$\Rightarrow \begin{cases} A e^{\gamma a} + B e^{-\gamma a} = T e^{ika} & (3) \\ A e^{\gamma a} - B e^{-\gamma a} = i \dfrac{T}{\rho} e^{ika} & (4) \end{cases}$$

eq (3) + eq. (4)

$$\Rightarrow \quad A = \frac{1}{2} T e^{ika - \gamma a} \left(1 + \frac{i}{\xi}\right) \quad —(5)$$

eq (3) − eq (4)

$$\Rightarrow \quad B = \frac{1}{2} T e^{ika + \gamma a} \left(1 - \frac{i}{\xi}\right) \quad —(6)$$

eq (1) + eq. (2)

$$2 = (1 - i\xi) A + (1 + i\xi) B$$

use eq. (5) and eq. (6)

$$2 = (1 - i\xi) \frac{1}{2} \left(1 + \frac{i}{\xi}\right) e^{ika - \gamma a} T + (1 + i\xi) \frac{1}{2} \left(1 - \frac{i}{\xi}\right) e^{ika + \gamma a} T$$

$$\Rightarrow T = \frac{e^{-ika}}{\frac{1}{4}\left\{\left(1 + \frac{i}{\xi}\right)\left(1 + \frac{\xi}{i}\right) e^{-\gamma a} + \left(1 - \frac{i}{\xi}\right)\left(1 - \frac{\xi}{i}\right) e^{\gamma a}\right\}}$$

Transmission flux $= |T|^2$

$$|T|^2 = \frac{1}{\left|\frac{1}{4}\left\{\left(1 + \frac{i}{\xi}\right)\left(1 + \frac{\xi}{i}\right) e^{-\gamma a} + \left(1 - \frac{i}{\xi}\right)\left(1 - \frac{\xi}{i}\right) e^{\gamma a}\right\}\right|^2}$$

$$= \frac{1}{1 + \frac{1 + \xi^2}{4\xi^2} \sinh^2 \gamma a}$$

$$|T|^2 = \frac{1}{1 + \frac{\left(1+\frac{\gamma^2}{K^2}\right)^2}{4\frac{\gamma^2}{K^2}} \sinh^2 \gamma a}$$

$$(\text{or}) \quad = \frac{1}{1 + \frac{W^2}{4E(W-E)} \sinh^2\left(\sqrt{\frac{2m(W-E)}{\hbar^2}}\, a\right)}$$

II-2 Solution

a) The Hamiltonian is $\mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)}$ $\qquad \mathcal{H}^{(1)} = \frac{1}{2}\delta k x^2$

The basis states are the eigenfunctions $|n\rangle$ of the Hamiltonian for the unperturbed simple harmonic oscillator $\mathcal{H}^{(0)}$.

These functions are orthogonal and normalized, i.e. $\langle n|n'\rangle = \delta_{n'm}$.

The first-order energy term is $E^{(1)} = \langle n|\mathcal{H}^{(1)}|n\rangle = \frac{1}{2}\delta k\langle 0|x^2|0\rangle,$

since the unperturbed state is the ground state $n = 0$.

Then

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\omega}$$

and

$$E^{(1)} = \frac{1}{2}\delta k\,\frac{\hbar}{2m\omega} = \frac{1}{4}\,\frac{\delta k}{k}\,\hbar\omega.$$

The second-order energy term is

$$E^{(2)} = \sum_{j\neq 0}\frac{|\mathcal{H}^{(1)}_{j0}|^2}{E_0 - E_j},$$

$$\mathcal{H}^{(1)}_{j0} = \frac{1}{2}\delta k\langle j|x^2|0\rangle = \frac{1}{2}\delta k\,\frac{\hbar}{2m\omega}\,\langle j|(a+a^\dagger)(a+a^\dagger)|0\rangle.$$

Since j cannot be zero, the ladder properties of the operators a, a^+ show that only $j = 2$ gives the non-zero $\mathcal{H}^{(1)}_{j0}$.

The pair of operators that converts $|0\rangle$ into $|2\rangle$ is a^+a^+. From the relation

$$a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle,$$

We have

$$a^\dagger a^\dagger|0\rangle = \sqrt{2}\,|2\rangle.$$

Thus

$$\mathcal{H}^{(1)}_{20} = \frac{\sqrt{2}}{2}\delta k\,\frac{\hbar}{2m\omega}. \qquad E_0 = \frac{1}{2}\hbar\omega \qquad E_2 = \frac{5}{2}\hbar\omega$$

Inserting those values gives

$$E^{(2)} = -\frac{1}{16}(\delta k)^2 \frac{\hbar}{m^2\omega^3} = -\frac{1}{16}\left(\frac{\delta k}{k}\right)^2 \hbar\omega.$$

b) We do not need perturbation theory to solve this problem. The calculation can be done exactly and more simply. Here however, we had asked for the perturbation theory approach. The ground state energy of the unperturbed oscillator is

$$E_0 = \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\left(\frac{k}{m}\right)^{1/2}.$$

So the energy of the perturbed oscillator is

$$E = \frac{1}{2}\hbar\left(\frac{k+\delta k}{m}\right)^{1/2} = \frac{1}{2}\hbar\omega\left(1+\frac{\delta k}{k}\right)^{1/2}$$

$$= \frac{1}{2}\hbar\omega\left\{1 + \frac{1}{2}\frac{\delta k}{k} - \frac{1}{8}\left(\frac{\delta k}{k}\right)^2 + O\left(\frac{\delta k}{k}\right)^3\right\}.$$

The binominal expansion in the last line is valid only when $\delta k \ll k$, i.e. when $\mathcal{H}^{(1)} \ll \mathcal{H}^{(0)}$.

We see that the terms in $\delta k/k$ and $(\delta k/k)^2$ correspond to the 1st and 2nd order energies in perturbation theory, respectively. The calculation shows that when $\delta k \ll k$, perturbation theory provides a good approximation to the correct energy.

II-3. Solution:

The operator of the square of the total spin is equal to

$$\hat{S}^2 = \hat{s}_1^2 + \hat{s}_2^2 + 2(\hat{s}_1\cdot\hat{s}_2) = \frac{3}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}_1\begin{pmatrix}1&0\\0&1\end{pmatrix}_2$$
$$+\frac{1}{2}\left\{\begin{pmatrix}0&1\\1&0\end{pmatrix}_1\begin{pmatrix}0&1\\1&0\end{pmatrix}_2+\begin{pmatrix}0&-i\\i&0\end{pmatrix}_1\begin{pmatrix}0&-i\\i&0\end{pmatrix}_2+\begin{pmatrix}1&0\\0&-1\end{pmatrix}_1\begin{pmatrix}1&0\\0&-1\end{pmatrix}_2\right\}$$

where the indices 1 and 2 indicate the number of the particles.

We determine the eigenfunctions of this operator first of all in the case where the components of the total spin are equal to zero,

$$\hat{S}^2\left\{a\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+b\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\}=$$
$$=\lambda\left\{a\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+b\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}_2\right\}.$$ We find then

$$(\lambda-1)a-b=0,\quad -a+(\lambda-1)b=0$$

for λ we have two values, $\lambda=2$ and $\lambda=0$. For $\lambda=2$, $a=b$ and for $\lambda=0$, $a=-b$. If we take into account the normalization condition $a^2+b^2=1$, we find the eigenfunctions in the following form:

(II-3) Solution: continued.

$$\frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2+\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\},\ \lambda=2,\ (S=1)$$

$$\frac{1}{\sqrt{2}}\left\{\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2-\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2\right\},\ \lambda=0\ (S=0).$$

One can easily verify that the functions $\begin{pmatrix}1\\0\end{pmatrix}_1\begin{pmatrix}1\\0\end{pmatrix}_2$ and $\begin{pmatrix}0\\1\end{pmatrix}_1\begin{pmatrix}0\\1\end{pmatrix}_2$ are also eigenfunctions of the operator $\hat{S}^2$, where in the first case the z-component of the spin is equal 1 and in the second case to -1. The functions we have found are also eigenfunctions of the operator $(\hat{s}_1\cdot\hat{s}_2)$.

$$F(\theta) = -\frac{2m}{\hbar^2 K}\int_0^a r\,V(r)\sin Kr\,dr$$

$$= -\frac{2m}{\hbar^2 K}V_0\int_0^a r\sin \cancel{K}r\,dr$$

$$= -\frac{2mV_0}{\hbar^2 K}\int_0^a x\sin\alpha x\,dx \qquad \begin{matrix} x = r \\ \alpha = K \end{matrix}$$

$$= -\frac{2mV_0}{\hbar^2 K}\left[\frac{\sin Kx}{K^2} - \frac{x\cos Kx}{K}\right]_0^a$$

$$= -\frac{2mV_0}{\hbar^2 K}\left[\frac{\sin Ka}{K^2} - \frac{a\cos Ka}{K} - (0)\right]$$

check dimensions

$$= -\frac{2mV_0}{\hbar^2 K}\left[\frac{\sin Ka}{K^2} - \frac{a\cos Ka}{K}\right]$$

$$\frac{d\sigma}{d\Omega} = |F(\theta)|^2$$

II-5 Solution

$\psi(x) = N \frac{1}{a^2+x^2}$, assume $a>0$

a) $\int_{-\infty}^{\infty} |N \frac{1}{a^2+x^2}|^2 dx = N^2 \int_{-\infty}^{\infty} (a^2+x^2)^{-2} dx$

$= N^2 \left[\frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3} \arctan \frac{x}{a} \right]_{-\infty}^{\infty} = N^2 \frac{1}{2a^3} \pi \overset{?}{=} 1$

$\Rightarrow N = \sqrt{\frac{2a^3}{\pi}}$ ged

b) $\langle \hat{x}^n \rangle = \int_{-\infty}^{\infty} x^n P(x) dx = \int_{-\infty}^{\infty} x^n \psi(x) \psi^*(x) dx$

$= N^2 \int \frac{x^n}{(a^2+x^2)} dx$

$n=1: \Rightarrow N^2 \left[-\frac{1}{2} \frac{1}{a^2+x^2} \right]_{-\infty}^{\infty} = 0$ Center of Mass is origin

$n=2: \Rightarrow N^2 \left[\frac{-x}{2(a^2+x^2)} + \frac{1}{2a} \arctan \frac{x}{a} \right]_{-\infty}^{\infty} = N^2 \frac{\pi}{2a} = \frac{2a^3}{\pi} \frac{\pi}{2a} = a^2 = \langle x^2 \rangle$

rotational inertia

$n=3: \Rightarrow N^2 \left[\frac{a^2}{2(a^2+x^2)} + \frac{1}{2} \ln x \right]_{-\infty}^{\infty} \Rightarrow \infty$ divergent.

$n=3 \ldots$ divergent

$\Delta x = \sqrt{\langle (x - \langle \hat{x} \rangle)^2 \rangle} = \sqrt{\langle (x-0)^2 \rangle} = \sqrt{a^2} = |a|$

c) $\psi(x) = N\frac{1}{a^2+x^2} = \int \frac{1}{2\pi\hbar}\, \phi(p)\, e^{ipx/\hbar}\, dp$

$$\phi(p) = \int \psi(x)\, e^{-ipx/\hbar}\, dx$$

$$= N \int_{-\infty}^{\infty} \frac{1}{a^2+x^2}\, e^{-ipx/\hbar}\, dx = N\left(\int_{-\infty}^{\infty} \frac{1}{a^2+x^2} \cos\frac{px}{\hbar}\, dx - i\int_{-\infty}^{\infty} \frac{1}{a^2+x^2} \sin\frac{px}{\hbar}\, dx \right)$$

$$= N\sqrt{\frac{\pi}{2}}\, 2\sqrt{\frac{\pi}{2}}\, \frac{e^{-a|p|/\hbar}}{a} = \frac{\pi N}{a}\, e^{-a|p|/\hbar} = e$$

$$= e^{-a|p|/\hbar}\, \frac{\pi}{a}\sqrt{\frac{2a^3}{\pi}} = \sqrt{2a\pi}\; e^{-a|p|/\hbar}$$

$$\langle \hat{p}^n \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp\, N^2 \left(\frac{\pi}{a}\right)^2 e^{-2a|p|/\hbar}\, p^n = \frac{\pi}{2a^2\hbar}\, \frac{2a^3}{\pi} \int_{-\infty}^{\infty} e^{-2a|p|/\hbar}\, p^n\, dp$$

$$= \frac{a}{\hbar} \int_{-\infty}^{\infty} e^{-2a|p|/\hbar}\, p^n\, dp$$

n = odd $\Rightarrow \langle \hat{p}^n \rangle = 0$

n = even $\quad \langle \hat{p}^n \rangle = \frac{2a}{\hbar}\left(\frac{n\hbar}{2a} \int_0^{\infty} e^{-2a|p|/\hbar}\, p^{n-1}\, dp \right)$

$$= \frac{2a}{\hbar}\, \frac{n\hbar}{2a}\, \frac{(n-1)\hbar}{2a} \int_0^{\infty} e^{-2a|p|/\hbar}\, p^{n-2}\, dp$$

$$= \frac{n(n-1)\hbar^2}{4a^2} \langle \hat{p}^{n-2} \rangle$$

$\langle \hat{p}^0 \rangle$ = Normalization $= \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, 2a\pi\, e^{-2a|p|/\hbar} = \frac{2a}{a}\left[\frac{-\hbar}{2}\, e^{-2ap/\hbar} \right]_0^{\infty}$

$$= \frac{a\hbar}{\hbar^2}\, 2\, \frac{\hbar}{2a} = 1$$

$$\langle \hat{p}^2 \rangle = \frac{2 \cdot 1\, \hbar^2}{4a^2} = \frac{\hbar^2}{2a^2}$$

$$\Delta p = \sqrt{\langle (p - \langle \hat{p} \rangle)^2 \rangle} = \sqrt{\langle (p-0)^2 \rangle} = \frac{\hbar}{a\sqrt{2}}$$

d) $\quad \Delta x\, \Delta p = |a| \frac{\hbar}{a\sqrt{2}} = \frac{\hbar}{\sqrt{2}}$

II-6

(a)

Given $C_b(t=0)=0$, $C_a(t=0)=1$, $H'_{ab}=V_{ab}\cos\omega t$

$$C_b(t) \cong -\frac{i}{\hbar}\int_0^t H'_{ab}\, e^{i\omega_0 t'}\, dt'$$

$$C_b(t) \cong -\frac{i}{\hbar} V_{ab}\int_0^t \cos(\omega t')\, e^{i\omega_0 t'}\, dt'$$

$$= -\frac{i}{2\hbar} V_{ab}\int_0^t \left[e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right] dt$$

$$\therefore C_b(t) \cong -\frac{V_{ab}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right] \#$$

(b) near resonane $\Rightarrow$ $\omega_0 \approx \omega$, $\frac{1}{\omega_0-\omega} \gg \frac{1}{\omega_0+\omega}$

$$C_b(t) \cong -\frac{V_{ab}}{2\hbar}\left[\frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right]$$

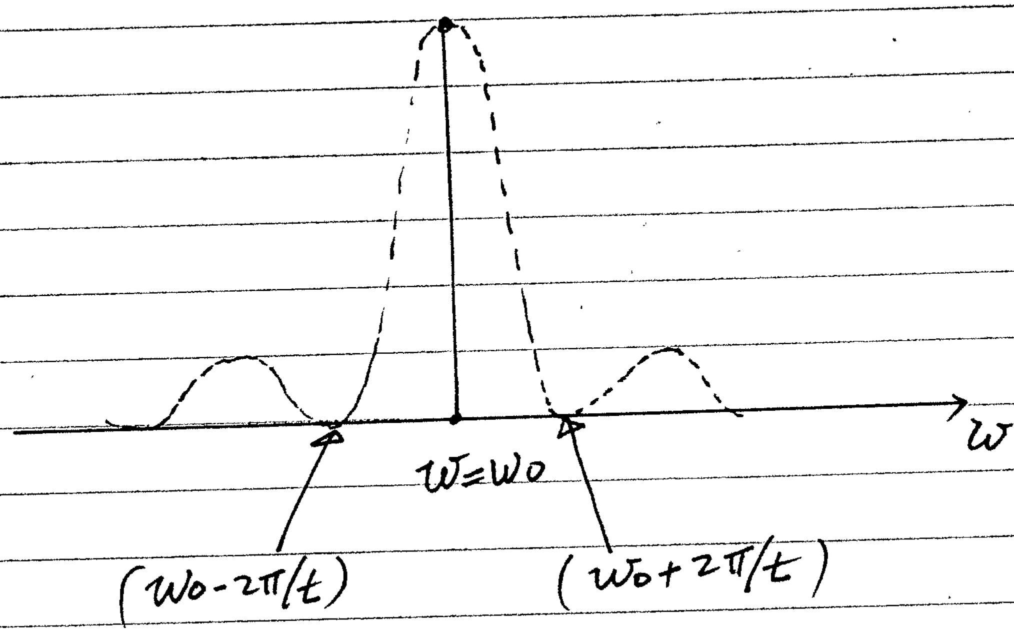
Transition probability $P_{a\to b}(t) = |C_b(t)|^2$

$$\therefore P_{a\to b}(t) \cong \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2} \#$$

(note: $e^{i(\omega_0-\omega)t}-1 = e^{i(\omega_0-\omega)/2t}\,(2i\sin((\omega_0-\omega)t/2))$)

II-6

(c)



$\boxed{\text{II-7}}$ (Specific) Gibbs function (per particle)

$$g = \frac{G}{N} = k_B T \ln\left(\frac{P}{P_0}\right) - AP \qquad \text{, where } A = A(T)$$

$$g = g(T,P)$$

a) equation of state:

$$\frac{V}{N} = (\text{specific volume}) \equiv v = \left(\frac{\partial g}{\partial P}\right)_T = \frac{k_B T}{P} - A(T)$$

$$\frac{V}{N} = \frac{k_B T}{P} - A(T) \quad \longrightarrow \left(\frac{V}{N} + A(T)\right) = \frac{k_B T}{P}$$

$$P\,(V + N A(T)) = N k_B T$$

or

$$\boxed{P\,(v + A(T)) = k_B T}$$

b) Helmholtz free-energy: $\left(f = F/N\right)$

$$f = g - Pv = k_B T \ln(P/P_0) - A(T)P - P\left[\frac{k_B T}{P} - A(T)\right]$$

$$= k_B T \ln(P/P_0) - \cancel{A(T)P} - k_B T + \cancel{P A(T)} =$$

$$= k_B T\left[\ln(P/P_0) - 1\right]$$

$$= k_B T\left[\ln\left(\frac{k_B T}{(v + A(T))\,P_0}\right) - 1\right] = f(T,v)$$

II-8. <u>Solution</u>:

Since the final entropy does not depend o[n] how the final state is reached it will be calculated as if it were reached isobaricall[y]. This is possible because the final pressure is $P_f = P$. Then, for each side separatel[y]

$$T\,dS = C_p\,dT \qquad \text{hence,}$$

$$\Delta S_1 = C_p \log \frac{T_f}{T_1} \quad \text{and} \quad \Delta S_2 = C_p \log \frac{T_f}{T_2}.$$

But $T_f = (T_1 + T_2)/2$ and $C_p = \frac{5}{2} N k$.

Therefore,

$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{5}{2} N k \log\left(\frac{T_f^2}{T_1 T_2}\right)$$

$$\Delta S = 5 N k \log\left\{\frac{(T_1 + T_2)}{2\sqrt{T_1 T_2}}\right\}$$

which vanishes as $T_1 = T_2$ as it should.

$\boxed{\text{II}-9}$ 3-dimensional Einstein solid

ensemble of 3N distinguishable independent quantum oscillator $\varepsilon_i = \hbar\omega_0 (n_i + 1/2)$ $n_i = 0, 1, 2, \ldots$

a) $Z_{3N} = Z_1^{3N}$

$$Z_1 = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega_0 (n+1/2)} = e^{-\beta\frac{\hbar\omega_0}{2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega_0 n} =$$

$$= e^{-\beta\frac{\hbar\omega_0}{2}} \frac{1}{1 - e^{-\beta\hbar\omega_0}} \qquad \beta = \frac{1}{k_B T}$$

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \ln Z_{3N} = -3N \frac{\partial}{\partial\beta} \ln Z_1$$

$$\frac{\partial}{\partial\beta} \ln Z_1 = \frac{\partial}{\partial\beta}\left\{-\beta\frac{\hbar\omega_0}{2} - \ln\left(1 - e^{-\beta\hbar\omega_0}\right)\right\} = -\frac{\hbar\omega_0}{2} - \frac{e^{-\beta\hbar\omega_0} \cdot \hbar\omega_0}{1 - e^{-\beta\hbar\omega_0}}$$

$$\langle E \rangle = \underbrace{\frac{3N}{2}\hbar\omega_0}_{\text{ground-state energy}} + \frac{3N\hbar\omega_0}{e^{\beta\hbar\omega_0} - 1}$$ internal energy of Einstein solid

$$C_N = \left(\frac{\partial\langle E\rangle}{\partial T}\right)_N = \left(\frac{\partial\langle E\rangle}{\partial\beta}\right)_N \frac{\partial\beta}{\partial T} = -\frac{1}{k_B T^2}\left(\frac{\partial\langle E\rangle}{\partial\beta}\right)_N = \frac{3N\hbar\omega_0}{k_B T^2} \frac{\hbar\omega_0 e^{\beta\hbar\omega_0}}{\left(e^{\beta\hbar\omega_0} - 1\right)^2}$$

$$= 3Nk_B \left(\frac{\hbar\omega_0}{k_B T}\right)^2 \frac{e^{\frac{\hbar\omega_0}{k_B T}}}{\left[e^{\frac{\hbar\omega_0}{k_B T}} - 1\right]^2} =$$

$$= 3Nk_B \left(\frac{\Theta_E}{T}\right)^2 \frac{e^{\Theta_E/T}}{\left(e^{\Theta_E/T} - 1\right)^2}$$ where $\Theta_E = \frac{\hbar\omega_0}{k_B}$

b) high-temperature limit $\frac{\Theta_E}{T} \ll 1$

$$e^{\Theta_E/T} \simeq 1 + \frac{\Theta_E}{T} + \dots$$

$$C_N \cong 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 \frac{1+\frac{\Theta_E}{T}+\dots}{\left(1+\frac{\Theta_E}{T}+\dots-1\right)^2} =$$

$$= 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 \frac{1+\Theta_E/T+\dots}{\left(\frac{\Theta_E}{T}\right)^2+\dots} \simeq 3Nk_B + \mathcal{O}\left(\frac{\Theta_E}{T}\right)$$

$$\simeq \boxed{3Nk_B}$$

i.e., for high-temperatures C_N approaches a constant
↳ Dulong-Petit

classical behavior:
3N classical oscillator

equipartition theorem: $2 \times \frac{1}{2}k_BT = \langle E \rangle$ per oscillator
$3Nk_BT = \langle E \rangle$ for overall
$C_N^{classical} = \frac{\partial\langle E\rangle}{\partial T} = 3Nk_BT$

c) low-temperature behavior $\frac{\Theta_E}{T} \gg 1$

$$C_N \simeq 3Nk_B\left(\frac{\Theta_E}{T}\right)^2 e^{-\Theta_E/T}$$

non-analytic at $T=0$
exponential vanishes much faster than power law diverges as $T \to$

Thus

$$C_N \xrightarrow[T\to 0]{} 0$$, specific heat vanishes

II-10. Solution:

The average occupation of a quantum state of energy ϵ is

$$\left\{ e^{[(\epsilon-\mu)/kT]} + 1 \right\}^{-1}.$$

The number of such quantum states within the energy interval $d\epsilon$ is $D(\epsilon)d\epsilon$. Thus the total energy of a fermion system at temperature T is

$$E = \int_0^\infty \frac{\epsilon D(\epsilon) d\epsilon}{e^{(\epsilon-\mu)/kT} + 1}$$

Using the first two terms in the low-temperature expansion for a general Fermi-Dirac integral, as given in Hint: (i), we can write E as

$$E = \int_0^\mu \epsilon D(\epsilon)\, d\epsilon + \frac{\pi^2 k^2 T^2}{6}\left[D(\mu) + \mu D'(\mu)\right].$$

In calculating $\left(\frac{\partial E}{\partial T}\right)$, to obtain the specific heat, we must remember that, at fixed N, the chemical potential μ is a function of T, given in Hint: (ii). Thus,

$$c_V = \frac{\partial E}{\partial T} = \epsilon_F D(\epsilon_F)\frac{\partial \mu}{\partial T} + \frac{\pi^2 k^2}{3} T \left[D(\epsilon_F) + \epsilon_F D'(\epsilon_F)\right]$$

where we have ignored terms that are proportional to $T^2 \mu'(T)$, because they are of order T.

II-10. Solution: continued

Using the fact that

$$\frac{d\mu}{dT} = -\left(\frac{\pi^2 k^2 T}{3}\right)\left[D'(\varepsilon_F)/D(\varepsilon_F)\right]$$

as seen from Hint: (ii), we see that the term involving $D'(\varepsilon_F)$ cancels in the equation for C_V, leaving

$$C_V = \frac{\pi^2 k^2 T}{3} D(\varepsilon_F) \quad .$$