University of Illinois at Chicago
Department of Physics

Electricity and Magnetism
PhD Qualifying Examination

January 6, 2015
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to <u>4 questions</u>. If the student attempts all 5 questions, all of the answers will be graded, and the <u>top 4 scores</u> will be counted toward the exam's total score.

1. Boundary value

The electrostatic potential on a spherical surface of radius R is given by

$$\Phi(R, \theta, \phi) = V_0 + V_1 \sin\theta \cos\phi + V_2 \cos 2\theta.$$

where θ and ϕ are the polar and azimuthal angles in the spherical coordinate system with the origin at the center of the sphere. There are no charges outside of this spherical surface.

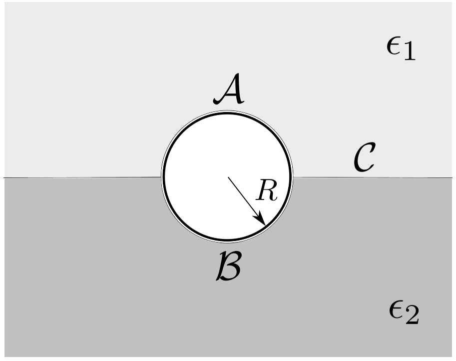
(a) Find the potential $\Phi(r, \theta, \phi)$ outside of the spherical surface as a function of the distance r from the center and the spherical angles θ and ϕ.

(b) Find the total charge Q inside or on the spherical surface.

(c) Now consider a different problem: a thin sheet lying in xy plane at $z = 0$ and carrying surface charge density $\sigma = \sigma_0 \sin(kx)$. There are no other charges.

Find the potential and the electric field at all points $z > 0$.

2. Conducting sphere surrounded by dielectric



The center of a conducting sphere of radius R is located on the flat boundary between two dielectrics each filling half of the whole space outside the sphere. The dielectric permittivities are ε_1 and ε_2. The conducting sphere is held at potential V. Consider the space outside of the conducting sphere.

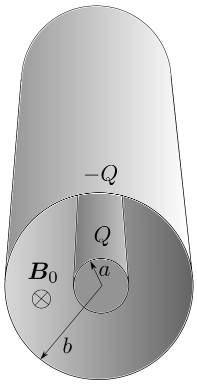
(a) Show that the potential $\Phi = VR/r$ satisfies the required boundary conditions on the plane $\mathcal{C}$ separating dielectrics as well as on the sphere.

(b) Find the free charge density σ on the surface of the conducting sphere and the total amount of free charge Q on it.

(c) Find the bound charge densities σ_b on the spherical boundaries $\mathcal{A}$ and $\mathcal{B}$ of the dielectrics.

(d) Find the bound charge density σ_b on the flat boundary $\mathcal{C}$ between the dielectrics.

3. Cylindrical capacitor in a magnetic field

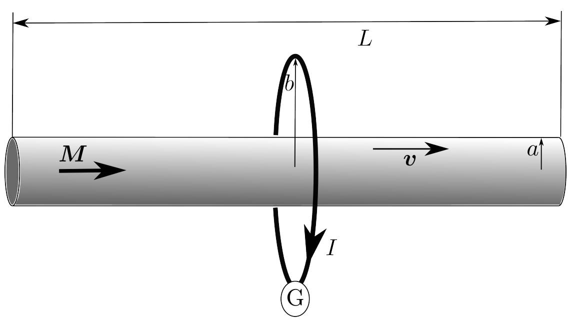


A capacitor is made out of two concentric conducting cylindrical surfaces of radii a and $b > a$. The charge on the inner conductor is $Q > 0$ and on the outer $-Q$. Uniform external magnetic field is applied in the direction of the axis of the cylinders. The magnetic flux density $\boldsymbol{B}$ is increasing with time from 0 to $\boldsymbol{B}_0$ along the axis of the cylinder. The time dependence of its magnitude is given: $B(t)$.
The magnetic field created by the currents on the cylinders is negligible.

(a) Find the magnitude of the torque experienced by the capacitor at time t and describe its direction relative to the direction of the magnetic field $\boldsymbol{B}_0$ (same or opposite).

(b) Find the magnitude of the total angular momentum that the capacitor receives as a result by the time the magnitude of the magnetic field reaches B_0 and compare with the magnitude of the total angular momentum of the electromagnetic field using the fact that the field carries the momentum density $\boldsymbol{\Pi} = \boldsymbol{S}/c^2 = \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$. Compare the directions of these angular momenta.

4. Permanent magnet and a ring



A uniform permanent magnet in the shape of a long cylinder of radius a and length L carries magnetization M directed along its axis. The magnet is moving with constant velocity v along its axis. A ring of radius b of conducting wire is placed around the cylinder ($b > a$) in the plane perpendicular to its axis. A galvanometer with internal resistance R measures the current I through the wire of the ring.

The magnetic field created by the current in the wire, the thickness and the resistance of the wire are negligible. At time $t = 0$ the center of mass of the cylinder and the center of the conducting ring coincide.

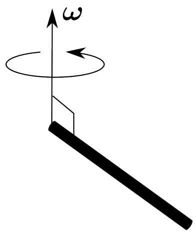
(a) At $t = 0$, find magnetic flux density B in the plane of the ring (inside and outside of the cylinder) assuming $L \gg a$.

(b) Find the total amount Q of the charge that will flow through the galvanometer between the time $t = 0$ and $t = \infty$.

(c) Sketch on the graph below (I vs vt) the time dependence of the current measured by the galvanometer. At what time is the current maximal?

(d) By considering the characteristic time during which the current is significant, estimate the maximum value of the current I_{max}.

5. Radiation from a rotating rod



An infinitezimally thin uniformly charged rod of length L is rotated around the axis perpendicular to it going through its end with the angular frequency $\omega \ll c/L$. The total charge on the rod is Q.

(a) Find the electric dipole moment of the rotating rod.

(b) Find the averaged electric dipole energy radiated by the rod per unit of time.

(c) Find the magnetic dipole moment and the magnetic dipole energy radiated by the rod per unit of time.

Equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{D}=\rho;\quad \boldsymbol{\nabla}\times\boldsymbol{E}=-d\boldsymbol{B}/dt;\quad \boldsymbol{\nabla}\times\boldsymbol{H}=\boldsymbol{J}+d\boldsymbol{D}/dt;\quad \boldsymbol{\nabla}\cdot\boldsymbol{B}=0;$$

$$\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P};\quad \boldsymbol{H}=\boldsymbol{B}/\mu_0-\boldsymbol{M};$$

$$\boldsymbol{E}=-\boldsymbol{\nabla}\Phi-\partial\boldsymbol{A}/\partial t;\quad \boldsymbol{B}=\boldsymbol{\nabla}\times\boldsymbol{A};$$

$$U=\int d^3\boldsymbol{r}\,\rho\,\Phi$$

$$\boldsymbol{p}=\int d^3\boldsymbol{r}\,\rho\boldsymbol{r};\quad \boldsymbol{m}=\frac{1}{2}\int d^3\boldsymbol{r}\,\boldsymbol{r}\times\boldsymbol{J};$$

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}=\sum_l\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\Omega_{\boldsymbol{x}})Y_{lm}^*(\Omega_{\boldsymbol{y}})$$

$$\begin{array}{lll} Y_{00}\sim 1; & Y_{10}\sim\cos\theta; & Y_{20}\sim 3\cos^2\theta-1; \\ & Y_{11}\sim\sin\theta\,e^{i\phi}; & Y_{21}\sim\cos\theta\sin\theta\,e^{i\phi}; \\ & & Y_{22}\sim\sin^2\theta\,e^{2i\phi}. \end{array}$$

$$\frac{d\mathcal{E}}{dt}=\frac{c}{12\pi\varepsilon_0}k^4|\boldsymbol{p}_\omega|^2\qquad\text{for}\quad \boldsymbol{p}(t)=\mathrm{Re}\,\left(\boldsymbol{p}_\omega e^{-i\omega t}\right)$$

$$\frac{d\mathcal{E}}{dt}=\frac{\mu_0 c}{12\pi}k^4|\boldsymbol{m}_\omega|^2\qquad\text{for}\quad \boldsymbol{m}(t)=\mathrm{Re}\,\left(\boldsymbol{m}_\omega e^{-i\omega t}\right)$$