E. Bertschinger

Classical Mechanics I: Binary Scattering

a) $L = \frac{1}{2}M_1\dot{\underline{x}}_1^2 + \frac{1}{2}M_2\dot{\underline{x}}_2^2 + \frac{1}{2}M_3\dot{\underline{x}}_3^2 + \frac{GM_1M_2}{|\underline{x}_1-\underline{x}_2|} + \frac{GM_1M_3}{|\underline{x}_1-\underline{x}_3|} + \frac{GM_2M_3}{|\underline{x}_2-\underline{x}_3|}$

Now change variables: $(\underline{x}_1,\underline{x}_2,\underline{x}_3) \to (\underline{R},\underline{r},\underline{s})$ where

$\underline{R} = M_1\underline{x}_1 + M_2\underline{x}_2$, $\underline{r} = \underline{x}_1 - \underline{x}_2$, $\underline{s} = \underline{x}_3 - \underline{R}$

$\Rightarrow \underline{x}_1 = \underline{R} + \frac{M_2}{M_1+M_2}\underline{r}$, $\underline{x}_2 = \underline{R} - \frac{M_1}{M_1+M_2}\underline{r}$, $\underline{x}_3 = \underline{R}+\underline{s}$.

$\Rightarrow \frac{1}{2}M_1\dot{\underline{x}}_1^2 + \frac{1}{2}M_2\dot{\underline{x}}_2^2 = \frac{1}{2}(M_1+M_2)\dot{\underline{R}}^2 + \frac{1}{2}\mu\dot{\underline{r}}^2$, $\mu \equiv \frac{M_1M_2}{M_1+M_2}$

Also, $\frac{1}{|\underline{x}_1-\underline{x}_3|} = \frac{1}{\left|\underline{s} - \frac{M_2}{M_1+M_2}\underline{r}\right|} = \frac{1}{s}\left[1 + \frac{M_2}{M_1+M_2}\frac{\underline{r}\cdot\underline{s}}{s^2} + O\left(\frac{r}{s}\right)^2\right]$

$\frac{1}{|\underline{x}_2-\underline{x}_3|} = \frac{1}{\left|\underline{s} + \frac{M_1}{M_1+M_2}\underline{r}\right|} = \frac{1}{s}\left[1 - \frac{M_1}{M_1+M_2}\frac{\underline{r}\cdot\underline{s}}{s^2} + O\left(\frac{r}{s}\right)^2\right]$

$$\Rightarrow L = \frac{1}{2}(M_1+M_2)\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}M_3(\dot{R}^2 + 2\dot{\underline{R}}\cdot\dot{\underline{s}} + \dot{s}^2) + \frac{GM_1M_2}{r} + \frac{G(M_1+M_2)M_3}{s}\left[1 + O\left(\frac{r}{s}\right)^2\right]$$

Now perform Legendre transformation:

$\underline{p}_R \equiv \frac{\partial L}{\partial \dot{\underline{R}}} = (M_1+M_2)\dot{\underline{R}} + M_3(\dot{\underline{R}}+\dot{\underline{s}})$

$\underline{p}_r \equiv \frac{\partial L}{\partial \dot{\underline{r}}} = \mu\dot{\underline{r}}$, $\underline{p}_s \equiv \frac{\partial L}{\partial \dot{\underline{s}}} = M_3(\dot{\underline{R}}+\dot{\underline{s}})$

$$h(\underline{R},\underline{r},\underline{s},\dot{\underline{R}},\dot{\underline{r}},\dot{\underline{s}}) \equiv \sum p\dot{q} - L = (M_1+M_2+M_3)\dot{R}^2 + 2M_3\dot{\underline{R}}\cdot\dot{\underline{s}} + \mu\dot{r}^2 + M_3\dot{s}^2 - L$$
$$= \frac{1}{2}(M_1+M_2)\dot{R}^2 + \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}M_3(\dot{R}^2 + 2\dot{\underline{R}}\cdot\dot{\underline{s}} + \dot{s}^2) - \frac{GM_1M_2}{r} - \frac{G(M_1+M_2)M_3}{s}\left[1+O\left(\frac{r}{s}\right)^2\right]$$

Now Solve $p(q,\dot{q}) \to \dot{q}(p,q)$: $\dot{\underline{R}} = \frac{\underline{p}_R - \underline{p}_s}{M_1+M_2}$, $\dot{\underline{r}} = \frac{\underline{p}_r}{\mu}$, $\dot{\underline{s}} = \frac{\underline{p}_s}{M_3} + \frac{\underline{p}_s - \underline{p}_R}{M_1+M_2}$

$\therefore H(\underline{R},\underline{r},\underline{s},\underline{p}_R,\underline{p}_r,\underline{p}_s) = h(\underline{R},\underline{r},\underline{s},\dot{\underline{R}}(\underline{p}_R),\dot{\underline{r}}(\underline{p}_r),\dot{\underline{s}}(\underline{p}_s))$

$$= \frac{1}{2}\frac{|\underline{p}_R - \underline{p}_s|^2}{(M_1+M_2)^2} + \frac{p_r^2}{2\mu} + \frac{p_s^2}{2M_3} - \frac{GM_1M_2}{r} - \frac{G(M_1+M_2)M_3}{s}\left[1+O\left(\frac{r}{s}\right)^2\right]$$

Classical Mechanics I, continued.

b) Note $H = H_r(r, \underline{p}_r) + H_{RS}(\underline{R}, \underline{s}, \underline{p}_R, \underline{p}_s)$ is separable

where $\boxed{H_r = \frac{p_r^2}{2\mu} - \frac{GM_1M_2}{r}}$ $\therefore$ can solve for r-motion independently of $\underline{R}$ and $\underline{s}$. H_r is the Hamiltonian for the Kepler problem, $\therefore$ the trajectories are conic sections $r(\phi) = \frac{a(1-e^2)}{1+e\cos\phi}$.

At $\phi = 0$ and $\phi = \pi$ the motion is entirely tangential,

$\underline{p}_r = \frac{L}{r}\underline{e}_\phi$ for $r = a(1\pm e)$

$$\therefore \quad E = H_r = \frac{p_r^2}{2\mu} - \frac{GM_1M_2}{r} = \frac{L^2}{2\mu r^2} - \frac{GM_1M_2}{r} \quad \text{for } r = a(1\pm e)$$

$$\Rightarrow r^2 + \frac{GM_1M_2}{E}r - \frac{L^2}{2\mu E} = 0 \Rightarrow r = -\left(\frac{GM_1M_2}{2E}\right) \pm \left[\left(\frac{GM_1M_2}{2E}\right)^2 + \frac{L^2}{2\mu E}\right]^{1/2}$$

$$\Rightarrow \boxed{a = -\frac{GM_1M_2}{2E}, \quad e = \left[1 + \frac{2EL^2}{\mu(GM_1M_2)^2}\right]^{1/2}}$$

c) $\frac{\partial H}{\partial \underline{R}} = 0 \Rightarrow \frac{d\underline{p}_R}{dt} = 0$ where $\underline{p}_R = (M_1+M_2)\dot{\underline{R}} + M_3(\dot{\underline{R}} + \dot{\underline{s}}) = M_1\dot{\underline{x}}_1 + M_2\dot{\underline{x}}_2 + M_3\dot{\underline{x}}_3$

$\therefore$ $\underline{R}$-motion is simple; choose $\underline{p}_R = 0$ without loss of generality.

$$\Rightarrow \boxed{H_{RS} = \frac{p_s^2}{2\mu_s} - \frac{G(M_1+M_2)M_3}{s}} \quad \text{where } \boxed{\frac{1}{\mu_s} = \frac{1}{M_1+M_2} + \frac{1}{M_3}}$$

This has the same form as H_r, therefore the $\underline{s}$-motion is also Keplerian. Explicitly,

$$\frac{d\underline{p}_s}{dt} = -\frac{\partial H}{\partial \underline{s}} = -\frac{G(M_1+M_2)M_3}{s^3}\underline{s} \; ; \quad \underline{p}_s = \mu_s\dot{\underline{s}} \Rightarrow \frac{d\underline{p}_s}{dt} = \mu_s\ddot{\underline{s}}$$

$$\Rightarrow \frac{d^2\underline{s}}{dt^2} = -\frac{G(M_1+M_2+M_3)}{s^3}\underline{s}$$

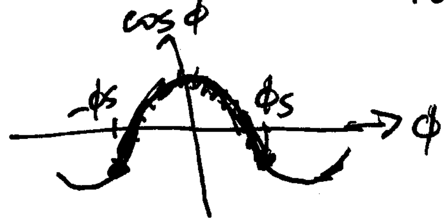
We can get the Kepler parameters a_s and e_s by comparing $E_s = H_{RS}(\underline{p}_R = 0)$ with part b).

$$\therefore \quad \boxed{a_s = -\frac{G(M_1+M_2)M_3}{2E_s}, \quad e_s = \left[1 + \frac{2E_sL_s^2}{\mu_s(G(M_1+M_2)M_3)^2}\right]^{1/2}}$$

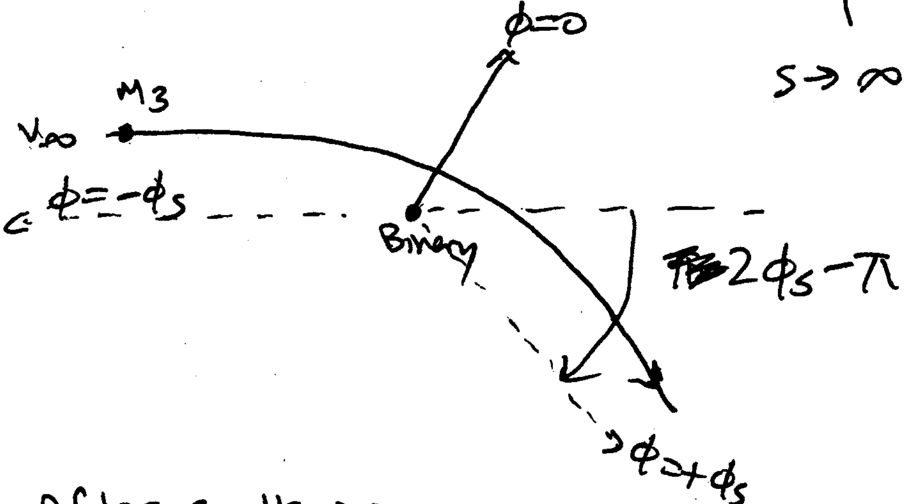
[note $(M_1+M_2)M_3 = (M_1+M_2+M_3)\mu_s$]

d) Initially, $\dot{\underline{s}} = V_\infty \underline{e}_x$ ($\underline{e}_x$ is arbitrary) $\Rightarrow \underline{p}_s = \mu_s V_\infty \underline{e}_x$.

$$s(\phi) = \frac{a_s(1-e_s^2)}{1+e_s\cos\phi}$$



$s \to \infty \Rightarrow \cos\phi_s = -\frac{1}{e_s}$



After scattering,

$$\dot{\underline{s}} = V_\infty\left[\underline{e}_x \cos(2\phi_s-\pi) - \underline{e}_y \sin(2\phi_s-\pi)\right]$$

$$\cos(2\phi_s-\pi) = -\cos 2\phi_s = 1-2\cos^2\phi_s = 1-\frac{2}{e_s^2}$$

$$\sin(2\phi_s-\pi) = -\sin 2\phi_s = -2\sin\phi_s\cos\phi_s = \frac{2}{e_s}\sqrt{1-\frac{1}{e_s^2}}$$

$$\therefore \Delta \underline{p}_s = \underline{p}_s(t\to\infty) - \underline{p}_s(t\to-\infty) = \mu_s V_\infty\left[-\frac{2}{e_s^2}\underline{e}_x - \frac{2}{e_s}\sqrt{1-\frac{1}{e_s^2}}\,\underline{e}_y\right]$$

This impulse is given to the binary:

$$\Delta(M_1\dot{\underline{x}}_1 + M_2\dot{\underline{x}}_2) = (M_1+M_2)\Delta\dot{\underline{R}} = -\Delta\underline{p}_s$$

$\therefore$ the binary recoils (in its initial rest frame) with velocity

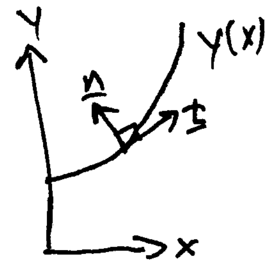
$$\dot{\underline{R}} = \frac{-\Delta\underline{p}_s}{M_1+M_2} = \frac{2\mu_s V_\infty}{M_1+M_2}\left[\frac{1}{e_s^2}\underline{e}_x + \frac{1}{e_s}\sqrt{1-\frac{1}{e_s^2}}\,\underline{e}_y\right]$$

$\therefore$ the final speed is

$$\dot{R} = \frac{2\mu_s V_\infty}{M_1+M_2}\cdot\frac{1}{e_s} = \frac{2\mu_s V_\infty}{e_s(M_1+M_2)} = \boxed{\frac{2M_3 V_\infty}{e_s(M_1+M_2+M_3)}}$$

Classical Mechanics II: Rolling Disk

Will need tangent and normal vectors to the curve:



$$\underline{t} = \frac{d\underline{x}}{ds} = \frac{dx}{ds}\underline{e}_x + \frac{dy}{ds}\underline{e}_y, \quad ds = dx\sqrt{1+(dy/dx)^2}$$

$$y = a\cosh(x/a) \Rightarrow \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right) \rightarrow ds = dx\cosh\left(\frac{x}{a}\right)$$

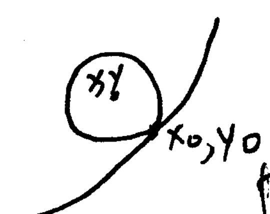
$$\Rightarrow \underline{t} = \frac{1}{\cosh(x_0/a)}\underline{e}_x + \tanh(x_0/a)\underline{e}_y \equiv t_x\underline{e}_x + t_y\underline{e}_y,$$

$$\underline{n} = -t_y\underline{e}_x + t_x\underline{e}_y$$

a) Need to express (x,y) on curve to ϕ; rolling without slipping $\Rightarrow ds = R\,d\phi \Rightarrow R\phi = \int_0^{x_0} dx\cosh(x/a) = a\sinh\left(\frac{x_0}{a}\right)$

$$\phi = \frac{a}{R}\sinh\left(\frac{x_0}{a}\right), \quad x_0 = a\sinh^{-1}\left(\frac{R\phi}{a}\right), \quad y_0 = a\cosh\left(\frac{x_0}{a}\right) = \sqrt{a^2+R^2\phi^2}.$$

Use subscript 0 to denote coordinates of contact point.



$$\underline{X} = \underline{X}_0 + \underline{n}R \qquad t_x = \frac{1}{\cosh(x_0/a)} = \frac{a}{y_0} = \frac{a}{\sqrt{a^2+R^2\phi^2}}$$

$$t_y = \frac{\sinh(x_0/a)}{\cosh(x_0/a)} = \frac{R\phi}{a}t_x = \frac{R\phi}{\sqrt{a^2+R^2\phi^2}}$$

$\Rightarrow$ The center of the disk has coordinates

$$\boxed{\begin{aligned} x &= x_0 + Rn_x = x_0 - Rt_y = a\sinh^{-1}\left(\frac{R\phi}{a}\right) - \frac{R^2\phi}{\sqrt{a^2+R^2\phi^2}} \\ y &= y_0 + Rn_y = y_0 + Rt_x = \sqrt{a^2+R^2\phi^2} + \frac{aR}{\sqrt{a^2+R^2\phi^2}} \end{aligned}}$$

b) $L = \frac{1}{2}M(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\phi}^2 - mgy; \quad \dot{x} = \frac{dx}{d\phi}\dot{\phi}, \quad \dot{y} = \frac{dy}{d\phi}\dot{\phi}, \quad I = \frac{1}{2}MR^2$

$$\phi = \frac{a}{R}\sinh\frac{x_0}{a} \Rightarrow \frac{d\phi}{dx_0} = \frac{1}{R}\cosh\frac{x_0}{a} = \frac{y_0}{aR} = \frac{\sqrt{a^2+R^2\phi^2}}{aR}$$

$$\Rightarrow \frac{dx}{d\phi} = \frac{aR}{\sqrt{a^2+R^2\phi^2}} - \frac{R^2}{\sqrt{a^2+R^2\phi^2}} + \frac{R^4\phi^2}{(a^2+R^2\phi^2)^{3/2}} = \frac{aR}{\sqrt{a^2+R^2\phi^2}}\underbrace{\left[1 - \frac{aR}{a^2+R^2\phi^2}\right]}_{D}$$

$$\frac{dy}{d\phi} = \frac{R^2\phi}{\sqrt{a^2+R^2\phi^2}}\left[1 - \frac{aR}{a^2+R^2\phi^2}\right] \Rightarrow \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = R^2D^2$$

$$\Rightarrow \boxed{\begin{aligned} L &= \frac{1}{2}MR^2\dot{\phi}^2\left(D^2+\frac{1}{2}\right) - mg\left(\sqrt{a^2+R^2\phi^2} + \frac{aR}{\sqrt{a^2+R^2\phi^2}}\right) \\ D &\equiv 1 - \frac{aR}{a^2+R^2\phi^2} \end{aligned}}$$

c) For $R^2\phi^2 \ll a^2$, $D \simeq D_0 \equiv 1 - \frac{R}{a}$, and

$$y \simeq a\left(1 + \frac{1}{2}\frac{R^2\phi^2}{a^2}\right) + R\left(1 - \frac{1}{2}\frac{R^2\phi^2}{a^2}\right) = a + R + \frac{1}{2}\frac{R^2\phi^2}{a} D_0$$

$$\Rightarrow L \simeq \frac{1}{2} M R^2 \left(D_0^2 + \frac{1}{2}\right)\dot{\phi}^2 - mg(a+R) - \frac{1}{2} mg \frac{R^2 D_0}{a}\phi^2$$

or $L = \frac{1}{2} A \dot{\phi}^2 - \frac{1}{2} B \phi^2 + \text{constant}$, simple harmonic oscillator

$$A\ddot{\phi} = -B\phi \Rightarrow \boxed{\omega^2} = \frac{B}{A} = \frac{\cancel{M} g R^2 D_0}{a \cancel{M} R^2 (D_0^2 + \frac{1}{2})} \boxed{= \frac{g}{a}\frac{D_0}{D_0^2 + \frac{1}{2}}}, \; D_0 = 1 - \frac{R}{a}$$

d) $\cancel{m}\ddot{\underline{x}} = -mg\,\underline{e}_y + N\underline{n} + T\underline{t}$ where N, T = normal and tangential contact forces

$$\ddot{x} = \frac{d}{dt}\left(\frac{dx}{d\phi}\dot{\phi}\right) = \frac{dx}{d\phi}\ddot{\phi} + \cancel{\frac{d^2x}{d\phi^2}\dot{\phi}^2} \quad (\dot{\phi} = 0)$$

$$\ddot{y} = \frac{d}{dt}\left(\frac{dy}{d\phi}\dot{\phi}\right) = \frac{dy}{d\phi}\ddot{\phi} + \cancel{\frac{d^2y}{d\phi^2}\dot{\phi}^2}$$

$$m\ddot{x} = n_x N + t_x T = -t_y N + t_x T$$

$$m\ddot{y} = n_y N + t_y T - mg = t_x N + t_y T - mg$$

$$\begin{pmatrix} -t_y & t_x \\ t_x & t_y \end{pmatrix}\begin{pmatrix} N \\ T \end{pmatrix} = m\begin{pmatrix} \ddot{x} \\ \ddot{y} + g \end{pmatrix} = m\begin{pmatrix} \frac{dx}{d\phi}\ddot{\phi} \\ \frac{dy}{d\phi}\ddot{\phi} + g \end{pmatrix}. \quad \begin{pmatrix} -t_y & t_x \\ t_x & t_y \end{pmatrix}^{-1} = \begin{pmatrix} -t_y & t_x \\ t_x & t_y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} N \\ T \end{pmatrix} = m\begin{pmatrix} -t_y & t_x \\ t_x & t_y \end{pmatrix}\begin{pmatrix} \frac{dx}{d\phi}\ddot{\phi} \\ \frac{dy}{d\phi}\ddot{\phi} + g \end{pmatrix} \qquad t_x = \frac{a}{\sqrt{a^2 + R^2\phi^2}}, \; t_y = \frac{R\phi}{\sqrt{a^2 + R^2\phi^2}}$$

$$\frac{N}{m} = -\frac{a R^2 \phi D}{a^2 + R^2\phi^2}\ddot{\phi} + \frac{a R^2 \phi D}{a^2 + R^2\phi^2}\ddot{\phi} + \frac{ag}{\sqrt{a^2 + R^2\phi^2}} = \frac{ga}{\sqrt{a^2 + R^2\phi^2}}$$

$$\frac{T}{m} = \frac{a^2 R D}{a^2 + R^2\phi^2}\ddot{\phi} + \frac{R^3\phi^2 D}{a^2 + R^2\phi^2}\ddot{\phi} + \frac{gR\phi}{\sqrt{a^2 + R^2\phi^2}} = RD\ddot{\phi} + \frac{gR\phi}{\sqrt{a^2 + R^2\phi^2}}$$

Now, for $R\phi \ll a$, $\ddot{\phi} = -\omega^2\phi$ with ω^2 from part c) above

$$\Rightarrow \frac{T}{m} = R\phi\left(\frac{g}{a} - \omega^2 D_0\right) = \frac{g}{a} R\phi\left(1 - \frac{D_0^2}{D_0^2 + \frac{1}{2}}\right) = \frac{g}{a}\frac{R\phi}{2D_0^2 + 1}$$

$$\therefore \boxed{N \simeq mg\left(1 - \frac{1}{2}\frac{R^2\phi^2}{a^2}\right), \; T \simeq mg\frac{R}{a}\frac{\phi}{2D_0^2 + 1}\left[1 + O\left(\frac{R^2\phi^2}{a^2}\right)\right]}$$

$$\boxed{D_0 = 1 - \frac{R}{a}}$$

e) The disk slips if $T > \mu N$

$$\cancel{mg}\frac{R}{a}\frac{\phi}{2D_0^2 + 1} > \frac{1}{2}\cancel{mg} \quad \text{or} \quad \frac{R\phi}{a} > D_0^2 + \frac{1}{2}.$$

Since we assume $\frac{R\phi}{a} \ll 1$, the disk will not slip.

E. Bertschinger

Electromagnetism : Electromagnetic waves

a) $\nabla^\nu F_{\mu\nu} = \nabla^\nu(\nabla_\mu A_\nu - \nabla_\nu A_\mu) = \nabla_\mu(\nabla^\nu A_\nu) - \Box A_\mu = -4\pi J_\mu$.

Applying the Lorentz gauge condition $\nabla^\nu A_\nu = \nabla_\nu A^\nu = 0$ gives the desired wave equation $\Box A^\mu = +4\pi J^\mu$

The Lorentz gauge condition does not completely fix A_μ because we can add $\nabla_\mu \psi$ where $\psi(x)$ is a scalar field satisfying $\Box\psi = 0$. Adding this term has no effect on the Lorentz gauge condition $[\nabla^\nu(A_\nu + \nabla_\nu\psi)\} = \nabla^\nu A_\nu + \Box\psi]$ or on the electromagnetic field: $F_{\mu\nu} = \nabla_\mu(A_\nu + \nabla_\nu\psi) - \nabla_\nu(A_\mu + \nabla_\mu\psi) = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

b) The point charge corresponds to a current density

$J^\mu(\underline{x}, t) = (q, q\underline{v}_q)\,\delta^3(\underline{x} - \underline{x}_q(t))$ where $\underline{v}_q = d\underline{x}_q/dt$.

In the radiative solution, we cannot immediately use the delta function, because t_r depends on $\underline{x}$, hence $\underline{x}_q(t_r)$ depends on $\underline{x}'$. We work around this problem by introducing another integral (time) with its own delta function

$$A^\mu = q\int d^3x' \,\frac{(1, \underline{v}_q)}{|\underline{x} - \underline{x}'|}\,\delta^3(\underline{x}' - \underline{x}_q(t_r))\ , \quad t_r = t - |\underline{x} - \underline{x}'|$$

$$= q\int dt'\int d^3x' \,\frac{(1, \underline{v}_q(t'))}{|\underline{x} - \underline{x}'|}\,\delta^3(\underline{x}' - \underline{x}_q(t'))\,\delta(t' - t + |\underline{x} - \underline{x}'|)$$

Now we can $\int d^3x'$!

$$= q\int dt' \,\frac{(1, \underline{v}_q(t'))}{|\underline{x} - \underline{x}_q(t')|}\,\delta(t' - t + |\underline{x} - \underline{x}_q(t')|)$$

Now we use the relation $\delta(f(t')) = \dfrac{1}{|df/dt'|}\,\delta(t' - \cdots)$

with $\dfrac{d}{dt'}\left[t' - t + |\underline{x} - \underline{x}_q(t')|\right] = 1 - \underline{n}\cdot\underline{v}_q(t')$, $\quad \underline{n} \equiv \dfrac{\underline{x} - \underline{x}_q(t')}{|\underline{x} - \underline{x}_q(t')|}$

$\Downarrow$

$$A^\mu = \left[\frac{q(1, \underline{v}_q)}{|\underline{x} - \underline{x}_q|(1 - \underline{n}\cdot\underline{v}_q)}\right]_{\text{ret}} = \left[\frac{q(1, \underline{v}_q)}{R - \underline{R}\cdot\underline{v}_q}\right]_{\text{ret}}\ , \quad \underline{R} \equiv \underline{x} - \underline{x}_q$$

ret means evaluate $\underline{x}_q$ and $\underline{v}_q$ at retarded time.

/ E & M

Problem 1

a) ~~Start~~ w. Maxwell eqs inside the waveguide

$\nabla\cdot E = 0$ $\quad\quad$ $\nabla\cdot B = 0$

$\nabla\times E = -\frac{1}{c}\dot{B}$ $\quad\quad$ $\nabla\times B = \frac{1}{c}\dot{E}$

Looking for solutions of the form
$E, B \propto e^{-i\omega t + ikx} f(r_\perp)$

Consider TE modes, $E_{\parallel} = 0$

$\nabla\times(\nabla\times B) = \frac{1}{c}\nabla\times\dot{E} = -\frac{1}{c^2}\ddot{B}$

$\nabla(\underbrace{\nabla\cdot B}_{0}) - \nabla^2 B = \frac{\omega^2}{c^2} B$

for $B_{\parallel}$ have $\quad \nabla_\perp^2 B_{\parallel} = \left(k^2 - \frac{\omega^2}{c^2}\right) B_{\parallel}$

Boundary condition $\frac{\partial B_{\parallel}}{\partial n}$ follows from the property of ideal conductor, $E_{\parallel} = 0$ together with $\frac{1}{c}\dot{E} = \nabla\times B$

For TM modes, similarly, have $\nabla_\perp^2 E_{\parallel} = \left(k^2 - \frac{\omega^2}{c^2}\right) E_{\parallel}$ with boundary condition $E_{\parallel} = 0$

To prove that this exausts all modes suppose that $E_{\parallel} = B_{\parallel} = 0$. Then, from Maxwell eqs, $E_\perp = \nabla_\perp \varphi$, $\nabla_\perp^2 \varphi = 0 \rightarrow \varphi = 0$ and similarly, $B_\perp = 0$

Solution

Quantum Mechanics Hydrogen Like Atoms

a) $\vec{L}\psi = 0$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)e^{-\beta r}-\frac{e^2}{r}e^{-\beta r} = E\,e^{-\beta r}$$

$$-\frac{\hbar^2}{2m}\left(\beta^2-\frac{2\beta}{r}\right)-\frac{e^2}{r} = E$$

$$E = -\frac{\hbar^2}{2m}\beta^2 \qquad \frac{\hbar^2\beta}{m} = e^2$$

so $\beta = \frac{me^2}{\hbar^2}$ $\quad E = -\frac{\hbar^2}{2m}\frac{m^2e^4}{\hbar^4} = -\frac{1}{2}\frac{me^4}{\hbar^2}$

b) $\psi = N e^{-\beta r}$

$$1 = 4\pi N^2\int_0^\infty e^{-2\beta r} r^2\,dr$$

$$1 = 4\pi N^2\frac{2}{(2\beta)^3} \qquad 1 = \frac{\pi N^2}{\beta^3} \qquad N = \left(\frac{\beta^3}{\pi}\right)^{1/2}$$

$$\beta_T = \frac{me^2}{\hbar^2} \qquad \beta_{He} = 2\frac{me^2}{\hbar^2}$$

$$\beta_T \equiv \beta \qquad \beta_{He} = 2\beta$$

Solution

$$\psi_T(r) = \left(\frac{\beta_T^3}{\pi}\right)^{1/2} e^{-\beta_T r} = \left(\frac{\beta^3}{\pi}\right)^{1/2} e^{-\beta r}$$

$$\psi_{He}(r) = \left(\frac{\beta_{He}^3}{\pi}\right)^{1/2} e^{-\beta_{He} r} = \left(\frac{8\beta^3}{\pi}\right)^{1/2} e^{-2\beta r}$$

$$\text{overlap} = 4\pi \int_0^\infty r^2 dr \left(\frac{\beta^3}{\pi}\right)^{1/2} \left(\frac{8\beta^3}{\pi}\right)^{1/2} e^{-3\beta r}$$

$$= 4\cdot 8^{1/2} \beta^3 \int_0^\infty r^2 dr\, e^{-3\beta r}$$

$$= 4\cdot 8^{1/2} \beta^3 \frac{2}{(3\beta)^3} = \frac{8\cdot 8^{1/2}}{3^3}$$

$$\text{probability} = \frac{8^3}{3^6} = .7023$$

c) $\vec{L}^2 Y_m^1 = 2\hbar^2 Y_m^1$

$$\cancel{-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) r e^{-\tilde\beta r} + \frac{2\hbar^2}{2mr^2} r e^{-\beta r} - e^2 e^{-\beta r} = E r e^{-\tilde\beta r}}$$

$$\cancel{-\frac{\hbar^2}{2m}\left(-2\tilde\beta + r\tilde\beta^2 + \frac{2}{r}(1-\tilde\beta r)\right) + \frac{\hbar^2 r}{mr^2} - e^2 = Er}$$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right) r e^{-\tilde\beta r} + \frac{1}{2m}\frac{1}{r^2}\, 2\hbar^2 r e^{-\tilde\beta r} - \frac{e^2}{r}\, r e^{-\tilde\beta r} = E r e^{-\tilde\beta r}$$

$$-\frac{\hbar^2}{2m}\left(-2\tilde\beta + r\tilde\beta^2 + \frac{2}{r}(1-\tilde\beta r)\right) + \frac{\hbar^2}{mr} - e^2 = Er$$

$$\frac{1}{r} : \quad -\frac{\hbar^2}{m} + \frac{\hbar^2}{m} = 0 \ \checkmark$$

$$r^0 : \quad 2\frac{\hbar^2}{m}\tilde\beta - e^2 = 0 \qquad \tilde\beta = \frac{me^2}{2\hbar^2}$$

$$E = -\frac{\hbar^2}{2m}\tilde\beta^2 = -\frac{\hbar^2}{2m}\frac{m^2 e^4}{4\hbar^4} = -\frac{me^4}{8\hbar^2}$$

Quantum Mechanics II Solution

a) Expand $|\psi\rangle$ in the energy eigenstates

$$|\psi\rangle = \sum_a \langle E_a|\psi\rangle |E_a\rangle$$

$$\langle\psi|H|\psi\rangle = \sum_a E_a \left|\langle E_a|\psi\rangle\right|^2$$

$$p_a \equiv \left|\langle E_a|\psi\rangle\right|^2 \qquad \sum_a p_a = 1 \qquad p_a \geq 0$$

$$\langle\psi|H|\psi\rangle = \sum_a p_a E_a = \text{average } E$$

average is always bigger than smallest

$$E_0 \leq \langle\psi|H|\psi\rangle$$

b) $\psi(x) = N \sin(\pi x/L) \qquad 0 \leq x \leq L$

$$N^2 \int_0^L \sin^2 \frac{\pi x}{L}\, dx = N^2 \frac{L}{2} = 1 \qquad N = \sqrt{\frac{2}{L}}$$

$$\psi(x) = \sqrt{\frac{2}{L}} \sin(\pi x/L)$$

$$\langle\psi|H|\psi\rangle = \langle\psi|H_0|\psi\rangle + \langle\psi|V|\psi\rangle$$

ψ is an eigenstate of H_0

$$\langle\psi|H_0|\psi\rangle = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2}$$

$$\langle\psi|V|\psi\rangle = \int_a^L V_0 \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$\frac{\pi x}{L} = y$$

$$\langle\psi|V|\psi\rangle = \frac{2V_0}{L}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y \,\frac{L\,dy}{\pi} = \frac{2V_0}{\pi}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y\, dy$$

$$\langle\psi|H|\psi\rangle = \frac{\hbar^2}{2m}\frac{\pi^2}{L^2} + \frac{2V_0}{\pi}\int_{\frac{\pi a}{L}}^{\pi} \sin^2 y\, dy$$

Take $\frac{d}{dL}$: $-\frac{\hbar^2\pi^2}{mL^3} + \frac{2V_0}{\pi}\left(-\sin^2\frac{\pi a}{L}\right)\left(\frac{-\pi a}{L^2}\right) = 0$

$$\frac{\hbar^2\pi^2}{mL} = 2V_0 a \sin^2\left(\frac{\pi a}{L}\right)$$

$$L = a + \Delta$$

$$\frac{\hbar^2\pi^2}{2mV_0 a(a+\Delta)} = \sin^2\left(\frac{\pi}{1+\Delta/a}\right)$$

$$\simeq \sin^2 \pi(1-\Delta/a)$$

$$\frac{\hbar^2\pi^2}{2mV_0 a^2} \simeq \frac{\pi\Delta}{a}$$

$$\Delta \simeq \frac{\hbar^2\pi}{2ma\,V_0}$$

$$\langle\psi|H|\psi\rangle = \frac{\hbar^2\pi^2}{2mL^2} + \frac{2V_0}{\pi}\int_{\pi a/L}^{\pi} \sin^2 y\, dy$$

$$\frac{\pi a}{L} = \frac{\pi}{1+\Delta/a} \simeq \pi(1-\Delta/a)$$

$\int$ is of order Δ^3

$$\frac{\hbar^2\pi^2}{2m(a+\Delta)^2} = \frac{\hbar^2\pi^2}{2ma^2}\frac{1}{(1+\Delta/a)^2}$$

$$\approx \frac{\hbar^2\pi^2}{2ma^2}\left(1-\frac{2\Delta}{a}\right)$$

PHYSICAL ADSORPTION

FIRST DO THE BULK GAS. USE $N_{BULK} = N$ FOR BREVITY

$$\epsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

$$Z(N,T,V) = \frac{1}{N!(\hbar)^{3N}} \underbrace{\left[\int_{-\infty}^{\infty} e^{-\frac{p_x^2}{2mkT}} dp_x \right]}_{\sqrt{mkT}\,\sqrt{2\pi}\,\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-s^2/2} ds}_{1}}{}^{3N} V^N = \frac{(2\pi mkT)^{\frac{3N}{2}} V^N}{N!\,(\hbar)^{3N}}$$

USE $N! \approx (N/e)^N$

$$Z(N,T,V) \approx \left\{ \frac{e}{N} \frac{1}{\hbar^3} (2\pi mkT)^{3/2} V \right\}^N$$

$$F = -kT \ln Z = -NkT \ln \left\{ \frac{e}{\hbar^3} (2\pi mkT)^{3/2} \frac{V}{N} \right\}$$

$$\mu = \left.\frac{\partial F}{\partial N}\right|_{T,V} = -kT \ln \left\{ \frac{e}{\hbar^3} (2\pi mkT)^{3/2} \frac{V}{N} \right\} \underbrace{- NkT \frac{\{\,\}(-1/N)}{\{\,\}}}_{+kT}$$

NOW DO THE SURFACE. AGAIN USE $N_{SURFACE} = N$ TEMPORARILY

$$\epsilon = -E_0 + \frac{p_x^2 + p_y^2}{2m}$$

$$Z(N,T,A) = \frac{1}{N!\,\hbar^{2N}} \left[e^{E_0/kT} \underbrace{\left(\int_{-\infty}^{\infty} e^{-p_x^2/2mkT} dp_x \right)^2}_{(2\pi mkT)} A \right]^N$$

$$Z(N,T,A) = \frac{1}{N!\,\hbar^{2N}} \left[e^{E_0/kT} 2\pi mkT A \right]^N \approx \left\{ \frac{e}{\hbar^2} (2\pi mkT) \frac{A}{N} e^{E_0/kT} \right\}^N$$

$$F = -kT \ln Z = -NkT \ln \left\{ \frac{e}{\hbar^2} (2\pi m k T) \frac{A}{N} e^{E_0/kT} \right\}^N$$

$$\mu = -kT \ln \left\{ \frac{e}{\hbar^2} (2\pi m k T) \frac{A}{N} e^{E_0/kT} \right\} + kT$$

NOW SET $\mu_{BULK} = \mu_{SURFACE}$ AND USE

$$N_{BULK}/V \equiv n \ , \qquad N_{SURFACE}/A \equiv \sigma$$

$$\ln \left\{ \frac{e}{\hbar^3} (2\pi m k T)^{3/2} \frac{1}{n} \right\} = \ln \left\{ \frac{e}{\hbar^2} (2\pi m k T) \frac{1}{\sigma} e^{E_0/kT} \right\}$$

$$\frac{1}{\hbar} \sqrt{2\pi m k T} \, \frac{1}{n} = e^{E_0/kT} \frac{1}{\sigma}$$

$$\underline{\underline{\sigma(n,T) = n \, e^{E_0/kT} \frac{\hbar}{\sqrt{2\pi m k T}}}}$$

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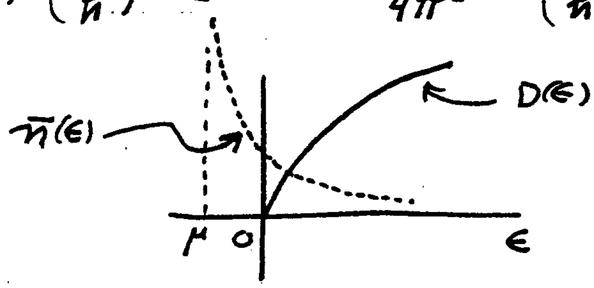
BOSE-EINSTEIN CONDENSATION

a) $D(k) = (2S+1)\dfrac{V}{(2\pi)^3}$ AND $\epsilon = \dfrac{\hbar^2 k^2}{2m}$

$$\#(\epsilon) = D(k)\,\frac{4}{3}\pi k^3 = D(k)\,\frac{4\pi}{3}\left(\frac{2m\epsilon}{\hbar^2}\right)^{3/2}$$

$$D(\epsilon) = \frac{d\#}{d\epsilon} = 2\pi\, D(k)\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2} = \frac{2S+1}{4\pi^2}\,V\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon^{1/2}$$

$$\bar{n}(\epsilon) = \frac{1}{e^{\frac{\epsilon-\mu}{kT}}-1}$$



$$N = \int_0^\infty \frac{D(\epsilon)}{e^{(\epsilon-\mu)/kT}-1}\,d\epsilon \quad\Rightarrow\quad \mu \leq 0 \text{ OR } \int_0^\infty \text{ DIVERGES}$$

μ BECOMES PINNED AT $\underline{\mu = 0}$
AT $T = T_c$

$$N = \frac{2S+1}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} V \int_0^\infty \frac{\epsilon^{1/2}}{e^{\epsilon/kT}-1}\,d\epsilon = \frac{2S+1}{4\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2} V \underbrace{\int_0^\infty \frac{x^{1/2}}{e^{x}-1}\,dx}_{\equiv I_c}$$

$$\frac{4\pi^2 n}{(2S+1)\,I_c} = \left(\frac{2mkT_c}{\hbar^2}\right)^{3/2} \quad\Rightarrow\quad kT_c = \frac{\hbar^2}{2m}\left(\frac{4\pi^2 n}{(2S+1)\,I_c}\right)^{2/3}$$

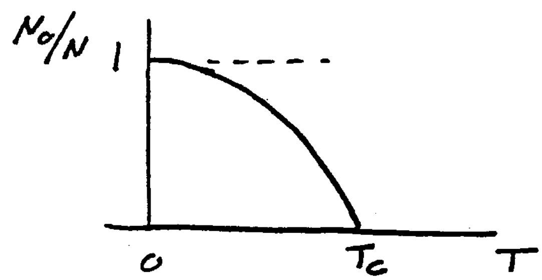
NOTE THAT kT_c IS EQUAL TO THE ENERGY OF A SINGLE PARTICLE WITH A WAVEVECTOR COMPARABLE TO OVER THE MEAN SPACING BETWEEN PARTICLES

b) BELOW T_c $\bar{n}(\epsilon)$ WITH $\mu = 0$ APPLIES TO ALL STATES EXCEPT THE GROUND STATE. THUS

$$N = N_0(T) + \underbrace{\frac{2S+1}{4\pi^2}}_{\frac{n}{\left(\frac{2mkT_c}{\hbar^2}\right)^{3/2}}} \left(\frac{2mkT}{\hbar^2}\right)^{3/2} V \, I_c$$

FROM LAST LINE IN a)

$$N = N_0(T) + \underbrace{nV}_{N} \left(\frac{T}{T_c}\right)^{3/2} \Rightarrow \frac{N_0(T)}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$



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