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# DOCTORAL GENERAL EXAMINATION

## PART 1

FEBRUARY 3, 2000

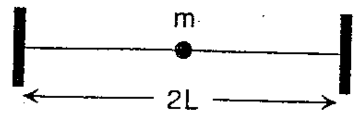
FIVE HOURS

1. This examination is divided into five sections, each consisting of four problems. Answer all the problems. Each problem is worth 5 points, thus the maximum score for the exam is 100.
2. Use a separate fold of paper for each problem. Write your name and the problem number (IV-3 for example) on each fold. A diagram or sketch as part of the answer is often useful, particularly when a problem asks for a quantitative response.
3. Read the problem carefully and do not do more work than is necessary. For example "give" and "sketch" do not mean "derive".
4. Calculators may be used but are not necessary.
5. No books, notes or reference materials may be used.

# GROUP I

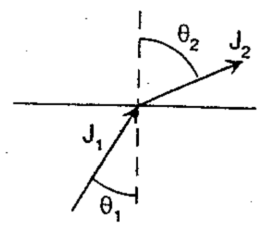
## I-1. Bead on a String

A bead of mass $m$ is fixed to the midpoint of a string of length $2L$. The mass per unit length of the string is $\mu$ and it is stretched between two fixed points under tension $T$. Ignoring the effects of gravity, find an equation whose solution yields the mass-dependent oscillation frequencies of the system.




## I-2. Current across an Interface

Two conducting slabs of conductivities $S_1$ and $S_2$ and dielectric constant one ($\epsilon = \epsilon_0$) are in contact on a plane. A current density of magnitude $J_1$ flows toward the interface at an angle of $\theta_1$ to the normal.




a) Find $\theta_2$, the angle the current density in slab 2 makes with the normal.

b) Find an expression for the surface charge density on the interface, $\sigma$, in terms of $S_1$, $S_2$, $\epsilon_0$, $J_1$ and $\theta_1$.

## I-3. Quantum Mechanics in a Magnetic Field

The non-relativistic Hamiltonian for a hydrogen atom, neglecting spin, can be taken to be

$$H_0 = \vec{p}\cdot\vec{p}/2m - e^2/|\vec{r}|.$$

A magnetic field $\vec{B} = B\hat{z}$ is applied along the $z$ axis. Find the new Hamiltonian, and identify the new term(s). Note that in these cgs units, the Lorentz force is $Q\vec{v}/c \times \vec{B}$, and the Bohr magneton is $\mu_B = e\hbar/2mc$. Also, the vector potential can be written $\vec{A} = (1/2)\vec{r}\times\vec{B}$.

## I-4. Absorption and Radiation

An atom of mass $m$ has an excited state with energy $\Delta E$. While in its ground state and moving with an initial momentum $\vec{p}_i$ it absorbs a photon of frequency $\nu$ propagating in the direction $\vec{n}$. You may treat the motion of the atom classically.

a) Find an expression for the absorption frequency $\nu$.

b) Will an atom at rest absorb radiation at the same frequency at which it would radiate? Explain why in simple physical terms.

## GROUP II

### II-1. Ocean Waves

The dispersion relation for deep water gravity waves is $\omega(k) = \sqrt{gk}$. After an earthquake occurred beneath the ocean far from land, a shore station recorded the arrival of waves of frequency 0.1 Hz at 6 am, and waves of frequency 0.2 Hz at 12 noon. At what time did the earthquake occur?

### II-2. Stability of a Charge

Four equal positive charges are fixed at the corners of a tetrahedron.

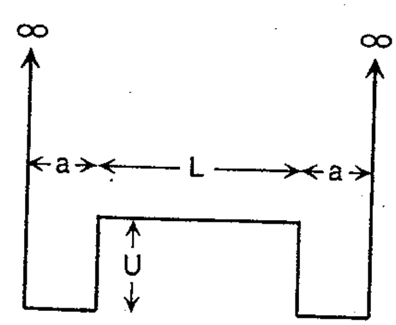
a) A negative charge is placed at the center of the tetrahedron. Is it a stable equilibrium position? If not, what happens to the charge?

b) A positive charge is placed at the center of the tetrahedron. Is it a stable equilibrium position? If not, what happens to the charge?

c) Explain whether or not it is possible to trap a charge by a static configuration of other charges.

### II-3. Double Potential Well

A quantum particle of mass $m$ is in its ground state in the potential shown in the figure. What must be the relation between $U$ and $a$ in the special case that the ground state energy is independent of the separation $L$. [Note that we assume the wave function of the particle extends over the entire distance $L + 2a$.]




### II-4. Ideal Gas Mixture

The following *non-interacting* particles are placed in a box of volume $V$: $N_1$ point particles of mass $m_1$, $N_2$ linear molecules of mass $m_2$ and moment of inertia $I_2$, and $N_3$ relativistic point particles ($m_3c^2 \ll k_BT$). The particles are in equilibrium at a temperature $T$ which is high enough for quantum effects to be unimportant.

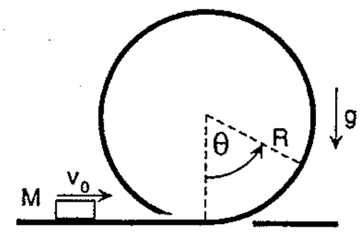
a) Calculate the pressure $P$ of this ideal gas.

b) Calculate the internal energy $E$ of the gas.

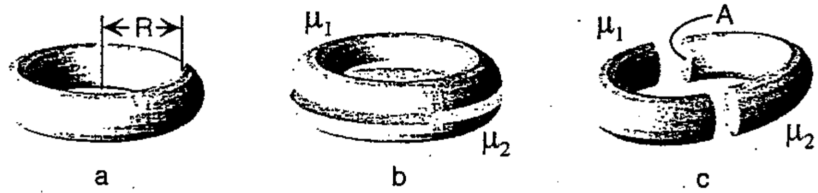
## GROUP III

### III-1. Car on a Loop-the-Loop

A car of mass $M$ attempts to drive up a loop-the-loop of radius $R$ at constant speed $v_0$. It is not going fast enough to make it to the top of the loop. The car starts to slip at some angle $\theta$. The coefficient of friction between the tire and the road is $\mu$. Find $\mu$.




### III-2. Split Core Toroids




A toroid of radius $R$ and cross section $A$ is wound uniformly with $N$ turns of wire. For simplicity treat the field in the toroid as if it were the field of a long solenoid.

a) Find the inductance of the toroid, $L_0$.

b) The toroid is now filled with magnetic material. It is composed of two halves with permeabilities $\mu_1$ and $\mu_2$, separated by a plane perpendicular to the axis. Find the inductance in terms of $L_0$, $\mu_1$ and $\mu_2$.

c) Now the two regions of different permeability are separated by a plane through the axis of the toroid. Find the inductance for this geometry.

### III-3. Kaon Beam

A beam of high energy neutral kaons travels in vacuum at a speed close to $c$. At the source the beam consists entirely of $K^\circ$ mesons. Measurements further from the source find a mixture of $K^\circ$, and its antiparticle $\overline{K}^\circ$, oscillating (and decaying) as a function of distance. Explain the implication of this observation.

### III-4. Rectangular Lattice

The atoms of a two-dimensional crystal form a rectangular lattice with unit vectors of length $a$ and $b < a$. Each atom contributes a single electron to the conduction band, which can be treated as a collection of non-interacting free particles of energy $\epsilon(k_x, k_y) \propto (k_x^2 + k_y^2)$.

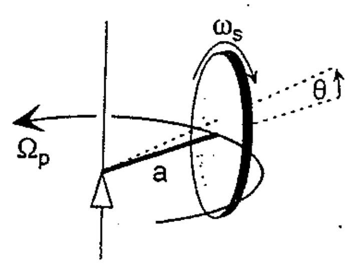
a) Indicate the shapes of the Brillouin zone and the Fermi sea.

b) For what values of $b/a$ does the Fermi sea intersect the Brillouin zone?

## GROUP IV

### IV-1. Gyroscope Motion

A gyroscope consists of a thin disc of mass $M$ and radius $R$, spinning at one end of a massless axle of length $a$. The other end of the axle is pivoted so that it can rotate freely in any direction. Gravity is pointing down. The disc is spinning with angular speed $\omega_s$ and the axle is horizontal.

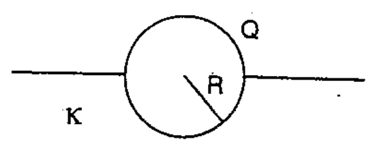



a) Find the gyroscope's rate of precession $\Omega_p$ about the vertical axis.

b) The gyroscope, still spinning at $\omega_s$, is held with its axle at an angle $\theta$ above the horizontal. When the axle is released the gyroscope is observed to wobble (nutate) for a short time, and then settle into uniform precession at the same rate as before, $\Omega_p$, with the axle horizontal. Find $\theta$ assuming that no angular momentum is lost while the gyroscope settles down.

### IV-2. Sphere in a Dielectric

A conducting sphere of radius $R$ is half immersed in a material with dielectric constant $\kappa$ ($\epsilon = \kappa\epsilon_0$). The sphere carries charge $Q$, and its electric field is radial. Find the potential of the sphere assuming $V(\infty) = 0$.




### IV-3. Origin of Spin-Orbit Coupling

The spin-orbit interaction in an atom is of the form $X\vec{S}\cdot\vec{L}$. Explain in the simplest terms how this arises, and find an expression whose value gives $X$. Take $\vec{S}$ and $\vec{L}$ to be dimensionless and neglect the relativistic Thomas correction.

### IV-4. Elementary Excitations

Early experiments on superfluid $He^4$ indicated that at temperatures below $1^oK$ its heat capacity is given by $C_V = 20.4T^3\,\mathrm{J\,kg^{-1}\,{}^oK^{-1}}$. Landau concluded that the low energy elementary excitations (massless bosons) in the superfluid have energies of the form $\epsilon(\vec{k}) = a|\vec{k}|^n$. Find $n$ and provide an interpretation of the nature of these excitations.

## GROUP V

### V-1. Neutrino Capture

The cross section for the capture of a neutrino by a nucleon is approximately $\sigma \approx 10^{-45} m^2$.

a) Which force is responsible for this process?

b) Estimate the mean free path of a neutrino in lead (atomic number 207, density of $3.2 \times 10^{28}$ atoms per cubic meter).

c) Estimate the mean free path of neutrinos in a neutron star.

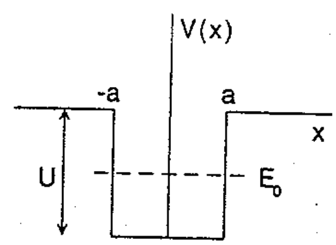
### V-2. Black Hole Temperature

a) Find the relation between the mass, $M$, and the radius, $R$, of a black hole. (Hint: equate the classical escape velocity to the speed of light.)

b) The internal energy of the black hole is $E = Mc^2$, and its entropy is linearly related to its area, $A = 4\pi R^2$, by $S = (k_B c^3/4G\hbar)A$. Find the temperature of the black hole in terms of its mass.

### V-3. Quantum Well

A particle of mass $m$ moves in a one dimensional square-well potential of depth $U$ and width $2a$. The ground state energy of the particle is exactly one half the depth of the well. Find $U$ in terms of $m$ and $a$.




### V-4. Hard Core Gas

The pressure $P$ of a gas of hard core particles is related to its temperature $T$ and volume $V$ by the equation of state $P(V - Nb) = Nk_BT$ where $N$ is the number of particles and $b$ is a parameter with the dimensions of volume. For this gas show that the internal energy $E$ at fixed $N$ is a function of $T$ alone; that is, show that $(\partial E(T,V)/\partial V)_T = 0$.