Part I - Fall, 1998

SECTION I SOLUTIONS

1. $\underline{\omega = \sqrt{g/L}}$ ONE HAS INDEPENDENT HARMONIC MOTION OF THE SAME FREQUENCY IN TWO ORTHOGONAL DIRECTIONS.

$$E_{n,m} = \hbar\omega(n+\tfrac{1}{2}) + \hbar\omega(m+\tfrac{1}{2}) \qquad m, n = 0, 1, 2 \ldots$$

GROUND STATE ENERGY $= E_{0,0} = \underline{\hbar\omega}$ $(n=0, m=0)$

NEXT ENERGY $\underline{2\hbar\omega}$, DEGENERACY $\underline{2}$ $\begin{pmatrix} n=1, m=0 \\ n=0, m=1 \end{pmatrix}$

NEXT ENERGY $\underline{3\hbar\omega}$, DEGENERACY $\underline{3}$ $\begin{pmatrix} n=2, m=0 \\ n=1, m=1 \\ n=0, m=2 \end{pmatrix}$

2. HYDROSTATIC APPROACH:

$$dP/dy = -\rho g$$

$$\rho = m\frac{N}{V} = m\left(\frac{P}{kT}\right)$$

$$dP/dy = -\frac{mg}{kT}P \Rightarrow P(y) = P(0)e^{-\frac{mg}{kT}y}$$

$$\underline{P(h)/P_0 = e^{-\frac{mgh}{kT}}}$$

STATISTICAL MECHANICAL APPROACH:

$$\mathcal{H} = \frac{\vec{p}^{\,2}}{2m} + V(y) \qquad V(y) = mgy$$

CANONICAL ENSEMBLE $\Rightarrow p(y) \propto e^{-mgy/kT}$

$$P \propto n \propto p(y) \Rightarrow P(h)/P(0) = \underline{e^{-mgh/kT}}$$

NOTE: THE BEST STUDENTS MAY POINT OUT THAT THIS MODEL FAILS FOR THE EARTH AS A WHOLE (SPHERE IN 3-DIMENSIONS) AND OUR ATMOSPHERE WOULD NOT BE BOUND.

3. $\overset{m \quad -e}{\bullet}$ $\xrightarrow{\quad} E$ $\qquad F_x = -Ee = -eE_0 e^{-i\omega t} = m\ddot{x}$

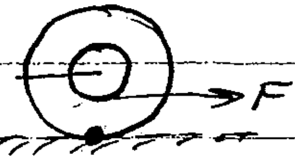
$$x = \frac{eE_0}{m\omega^2} e^{-i\omega t}$$

$$P = n_0 p = n_0(-ex) = -\frac{n_0 e^2}{m\omega^2} E_0 e^{-i\omega t}$$

$$n = \sqrt{\epsilon} = \left(1 - 4\pi \frac{P}{E}\right)^{1/2} = \underline{\left(1 - \frac{4\pi n_0 e^2}{m\omega^2}\right)^{1/2}}$$

$\Rightarrow$ TOTAL REFLECTION FOR FREQUENCIES BELOW $\underline{\omega = \left(\frac{4\pi n_0 e^2}{m}\right)^{1/2}}$

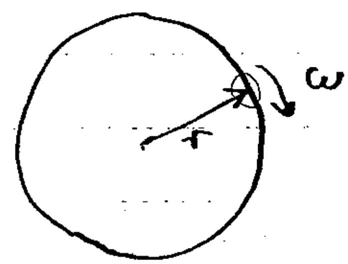
4. a) CALCULATE TORQUE ABOUT POINT OF CONTACT: $\vec{\tau} = \vec{r} \times \vec{F}$ IS INTO PAPER



$\Rightarrow \frac{d\vec{L}}{dt}$ IS POSITIVE INTO PAPER

$\Rightarrow$ YO-YO ROLLS TO THE RIGHT, IN THE DIRECTION OF F.

b) IN THE COORDINATE SYSTEM ROTATING AT ω THE OUTWARD ACCELERATION a_r IS GIVEN BY



$$a_r(r) = \omega^2 r - \frac{M_S G}{r^2}$$, THE EARTH'S C.M. IS AT r_0

$$a_r(r=r_0) = 0 \Rightarrow \omega^2 = \frac{M_S G}{r_0^3}$$

LET $r = r_0 + \delta$, $a_r(\delta) = \omega^2(r_0+\delta) - \frac{M_s G}{(r_0+\delta)^2}$

$$\approx \underbrace{(1+3)\,\omega^2 \delta}_{\text{ODD IN } \delta}$$

$\Rightarrow$ EQUAL TIDAL FORCES (IN OPPOSITE DIRECTIONS) ON OPPOSITE SIDES OF THE EARTH.

SECTION II SOLUTIONS

1. [NOTE: THE STUDENT IS EXPECTED TO KNOW THAT $\alpha \equiv \frac{e^2}{\hbar c}$ AND THAT α IS SMALL.]

$$mc^2 = \frac{p^2}{2m} \rightarrow \frac{(\Delta p)^2}{2m}$$ WHERE $\Delta p \Delta x \sim \hbar$ AND $\Delta x \sim d$

$$mc^2 \sim \frac{1}{m}\left(\frac{\hbar}{d}\right)^2 \Rightarrow d \sim \frac{\hbar}{mc}$$

$$E_{COULOMB} = \frac{e^2}{d} \sim \frac{e^2 mc}{\hbar}$$

$$\underline{E_{COULOMB}/E_{REST} \sim \frac{e^2 mc}{\hbar}\frac{1}{mc^2} = \frac{e^2}{\hbar c} \equiv \alpha}$$

$E_c/E_R \sim \alpha$, SINCE $\alpha < 10^{-2}$ ONE CAN SAY THAT THE EM INTERACTION IS WEAK COMPARED TO NUCLEAR FORCES

2. BERNOULLI'S LAW: $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$

[NOTE: APPLIES ONLY TO AN INCOMPRESSABLE FLUID, BUT WATER IS EFFECTIVELY INCOMPRESSAB

$$v_2 A_2 = v_1 A_1 \Rightarrow v_2 = v_1 \frac{A_1}{A_2}$$

$$P_1 - P_2 = \rho g h = \frac{1}{2}\rho(v_2^2 - v_1^2) = \frac{1}{2}\rho\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]v_1^2$$

$$\underline{v_1 = \left(2gh / \left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]\right)^{1/2}}$$

3. $2\pi r\, B(r) = \frac{4\pi}{c} N_0 I \Rightarrow B(r) = \frac{2N_0 I}{c}\frac{1}{r}$

$U \equiv \frac{1}{2}LI^2$, ALSO $= \int \frac{B^2}{8\pi}\, dV$

$$\frac{1}{2}LI^2 = \frac{1}{8\pi}\int_{r_a}^{r_b} B^2(r)\,(2\pi r h)\, dr$$

$$= \frac{1}{4}h\,\frac{4N_0^2 I^2}{c^2}\int_{r_a}^{r_b}\frac{1}{r}\,dr = \frac{N_0^2 h \ln(r_b/r_a)}{c^2} I^2$$

$$\Rightarrow \underline{L = \frac{2N_0^2 h}{c^2}\ln(r_b/r_a)}$$

4. CONSERVATION OF ENERGY

$$(\vec{p}_X^{\,2} + M_X^2)^{1/2} = (\vec{p}_1^{\,2} + M_\pi^2)^{1/2} + (\vec{p}_2^{\,2} + M_\pi^2)^{1/2}$$

$$\Rightarrow \vec{p}_X^{\,2} + M_X^2 = \vec{p}_1^{\,2} + \vec{p}_2^{\,2} + 2M_\pi^2 + 2(\vec{p}_1^{\,2} + M_\pi^2)^{1/2}(\vec{p}_2^{\,2} + M_\pi^2)^{1/2}$$

CONSERVATION OF MOMENTUM

$$\vec{p}_X = \vec{p}_1 + \vec{p}_2$$

$$\Rightarrow \vec{p}_X^{\,2} = \vec{p}_1^{\,2} + \vec{p}_2^{\,2} + 2\vec{p}_1\cdot\vec{p}_2$$

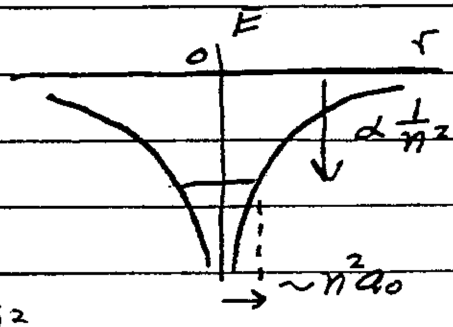
SUBTRACT ONE "SQUARED" EQ. FROM THE OTHER

$$\underline{M_X^2 = 2M_\pi^2 - 2\vec{p}_1\cdot\vec{p}_2 + 2(\vec{p}_1^{\,2} + M_\pi^2)^{1/2}(\vec{p}_2^{\,2} + M_\pi^2)^{1/2}}$$

[UNITS: $M \to Mc^2$, $\vec{p} \to \vec{p}c$]

GROUP III SOLUTIONS

1.



EQUATIONS:

$$E_n = \frac{e^2}{2a_0}\frac{1}{n^2}$$

$$r_n = a_0 n^2$$

FOR IONIZATION $\quad e r_n \mathcal{E} \sim E_n$

$$\mathcal{E} \sim \frac{e^2}{2a_0 n^2}\frac{1}{e a_0 n^2}$$

$$\sim \underline{\frac{e}{2a_0^2}\frac{1}{n^4}}$$

NUMBERS:

$$E_n = \frac{13.6\,eV}{n^2}$$

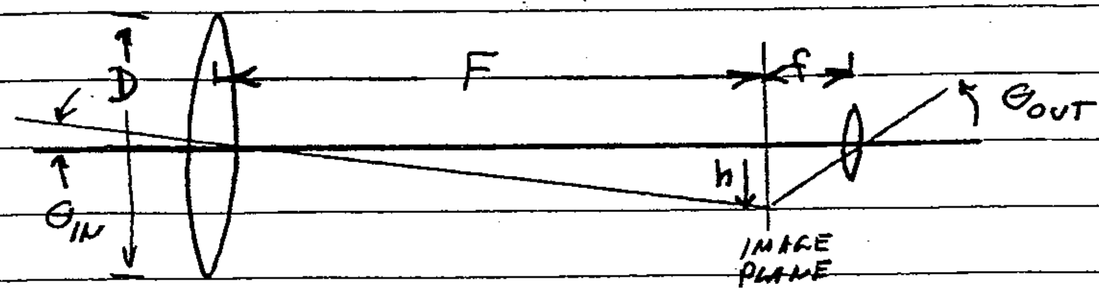
$$r_n \sim n^2 a_0 = n^2 (0.5\text{Å})$$

$$\mathcal{E} \sim \frac{13.6\,eV}{(e)\,n^4\,(0.5\text{Å})}$$

$$\sim \underline{\frac{1}{n^4} \times 3 \times 10^{9}\ \frac{vo}{cm}}$$

EITHER WAY OF PRESENTING THE ANSWER IS ACCEPTABLE

2.



a) $\theta_{IN} = \frac{h}{F}$ $\theta_{OUT} = \frac{h}{f}$

ANGULAR MAGNIFICATION $= \frac{\theta_{OUT}}{\theta_{IN}} = \frac{F}{f}$

b) DIFFRACTION LIMITED $\Delta\theta_{IN} \sim \frac{\lambda}{D}$

c) TWO SEPARATE GLASSY MATERIALS WITH DIFFERENT DISPERSION, $n(\vartheta)$, ARE CEMENTED TOGETHER TO MINIMIZE CHROMATIC ABERRATION. THE RESULTING LENS IS CALLED AN "ACHROMAT."

3. a) $\Delta x \Delta p \sim \hbar$

AS Δx USE THE MEAN SPACING BETWEEN PARTICLES $\Delta x \sim (V/N)^{1/3}$

THEN $\epsilon_F \sim \frac{\Delta p^2}{2m} = \frac{1}{2m}\hbar^2 \left(\frac{N}{V}\right)^{2/3} \sim \underline{\frac{\hbar^2}{m}(N/V)^{2/3}}$

b) THERMODYNAMICALLY FOR A NON-INTERACTING GAS

$$p = \frac{2}{3}U/V \sim \frac{N\epsilon_F}{V} \sim \underline{\frac{\hbar^2}{m}(N/V)^{5/3}}$$

ALTERNATIVE APPROACH

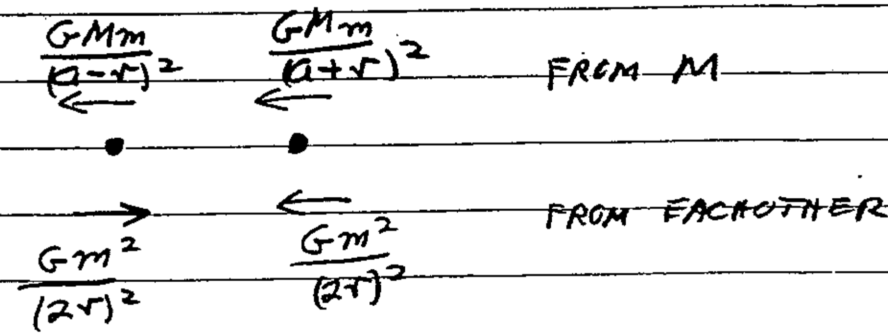
FIRST LAW $dE = T\underbrace{\cancel{dS}}_{=0 \text{ AT } T=0} - p\,dV$

$\Rightarrow p = -\left.\frac{\partial E}{\partial V}\right|_N$ BUT $E \sim N\epsilon_F = \frac{N\hbar^2}{m}(N/V)^{2/3}$

$$\left.\frac{\partial E}{\partial V}\right|_N = -\frac{2}{3}\frac{N\hbar^2}{m}(N/V)^{2/3}\,{}^{1}/_{V}$$

$$\underline{p \sim \frac{\hbar^2}{m}(N/V)^{5/3}}$$

4.



FOR STABILITY, FORCE TO THE LEFT ON THE LEFT OBJECT MUST BE <u>LESS</u> THAN THE FORCE TO THE LEFT ON THE RIGHT OBJECT

$$\frac{GMm}{(a-r)^2} - \frac{Gm^2}{4r^2} < \frac{GMm}{(a+r)^2} + \frac{G\cancel{Mm}\,m^2}{4r^2}$$

$$\frac{M}{(a^2-r^2)^2}(a+r)^2 - \frac{M}{(a^2-r^2)^2}(a-r)^2 < \frac{m}{2r^2}$$

$$4ar\,\frac{M}{a^4} < \frac{m}{2r^2}$$

$$\frac{8Mr^3}{a^3} < m = \frac{4}{3}\pi r^3 \rho$$

$$\underline{\underline{\frac{6}{\pi}\,\frac{M}{a^3} < \rho}}$$

SECTION IV SOLUTIONS

1. $$v_p = c/n = \frac{c}{(1-\omega_p^2/\omega^2)^{1/2}} = \frac{\omega c}{(\omega^2-\omega_p^2)^{1/2}} = \omega/k$$

$$\Rightarrow \quad k = \frac{1}{c}(\omega^2-\omega_p^2)^{1/2}$$

$$dk = \frac{1}{c}\,\frac{\omega}{(\omega^2-\omega_p^2)^{1/2}}\,d\omega \Rightarrow v_g \equiv \frac{d\omega}{dk} = c\,\frac{(\omega^2-\omega_p^2)^{1/2}}{\omega}$$

$$v_p v_g = \frac{c\,\omega}{\sqrt{}} \times c\,\frac{\sqrt{}}{\omega} = \underline{\underline{c^2}}$$

2. THERE ARE THREE NORMAL MODES OF LINEAR MOTION. [STUDENTS SHOULD <u>NOT</u> LOSE CREDIT FOR NEGLECTING THE TRANSLATIONAL MODE.]

UNIFORM TRANSLATION
NO DIPOLE MOMENT — DOES NOT COUPL

SYMMETRIC OSCILLATION
NO DIPOLE MOMENT — DOES NOT COUPL

ASYMMETRIC OSCILLATION
<u>HAS</u> A DIPOLE MOMENT — <u>COUPLES</u>

3. MICROCANONICAL APPROACH:

$$\Omega = \left[\int dp^3 \int dr^3\right]^N = \underbrace{\left[\int dp^3\right]^N}_{\text{DOES NOT CHANGE;}} \underbrace{\left[\int dr^3\right]^N}_{\text{DOUBLES}}$$

$$\Delta U = \underbrace{\cancel{\Delta Q}}_{\substack{0\\ \text{NO HEAT ADDED}}} + \underbrace{\cancel{\Delta W}}_{\substack{0\\ \text{NO WORK}}} = 0$$

$$S = k \ln \Omega$$

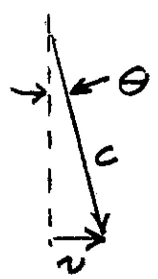
$$\underline{\Delta S = Nk \ln 2}$$

THERMODYNAMIC APPROACH: S IS A STATE FUNCTION SO WE CAN COMPUTE ITS CHANGE BY FOLLOWING A DIFFERENT (EQUILIBRIUM) PATH BETWEEN THE TWO STATES.

SINCE $dQ = TdS = pdV$ SINCE $dU = 0$

$$\Delta S = \int_V^{2V} \frac{p}{T}\,dV = \int_V^{2V} Nk\,\frac{1}{V}\,dV = \underline{Nk \ln 2}$$

4a THIS IS NOT PARALAX, SINCE EVEN THE MOST DISTANT STARS APPEAR TO MOVE. IT IS THE RESULT OF THE EARTH'S VELOCITY. IT IS KNOWN BY THE TERM "ABERRATION OF STARLIGHT".



θ = SHIFT IN APPARENT DIRECTION OF STAR

$\approx v/c$

$v = 2\pi\,(1.5\times 10^{11}) / (60\times 60\times 24\times 365)$ m/s

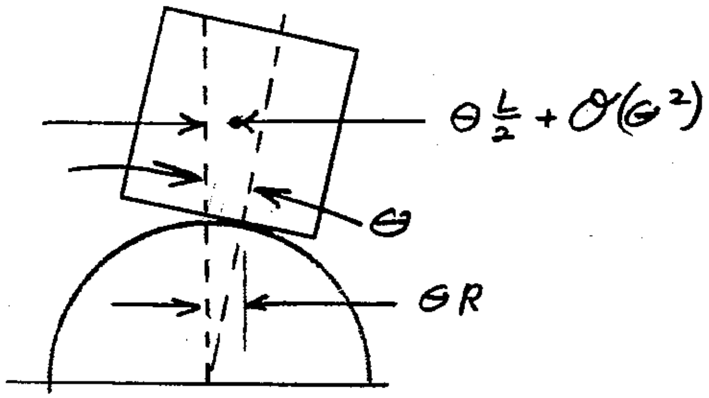
$\sim 3\times 10^{4}$ m/s

$$\theta \approx \frac{3\times 10^{4}}{3\times 10^{8}} = 10^{-4} \text{ RADIANS}$$

ANGULAR DIAMETER OF CIRCLES $\sim 2\times 10^{-4}$ RAD.

SECTION V SOLUTIONS

1. CENTER OF MASS OF THE BLOCK MUST REMAIN CLOSER TO THE AXIS THAN THE POINT OF CONTACT.



$$\theta \frac{L}{2} < \theta R$$

$$\Rightarrow \underline{L < 2R}$$

2. WHEN $\vec{F}$ COMES CLOSE TO COMMUTING WITH $\mathcal{H}_0$ BUT $\vec{J}$ DOES NOT, $\vec{J}$ WILL "ROTATE AROUND $\vec{F}$" AND ONLY THE COMPONENT OF $\vec{J}$'S MOMENT ALONG $\vec{F}$ WILL CONTRIBUTE TO THE TOTAL MAGNETIC MOMENT. THUS ONE MUST FIND THE PROJECTION OF $\vec{J}$ ALONG $\vec{F}$:

$$E \approx -g_J \mu_0 \frac{\langle \vec{J}\cdot\vec{F} \rangle}{\langle \vec{F}\cdot\vec{F} \rangle} \vec{F}\cdot\vec{B} \quad \Rightarrow \quad g_F = g_J \frac{\langle \vec{J}\cdot\vec{F} \rangle}{\langle \vec{F}\cdot\vec{F} \rangle}$$

$$\vec{I} = \vec{F} - \vec{J} \qquad \vec{I}\cdot\vec{I} = \vec{F}\cdot\vec{F} + \vec{J}\cdot\vec{J} - 2\vec{J}\cdot\vec{F}$$

$$\text{OR} \quad \vec{J}\cdot\vec{F} = \frac{1}{2}\left(\vec{F}\cdot\vec{F} + \vec{J}\cdot\vec{J} - \vec{I}\cdot\vec{I}\right)$$

$$\underline{g_F = g_J \left(\frac{1}{2} \; \frac{F(F+1) + J(J+1) - I(I+1)}{F(F+1)} \right)}$$

4b BALANCE HEATING BY SUN AGAINST THERMAL RADIATION TO EMPTY SPACE.

$$E_{IN} = E_{SUN} \frac{\pi R^2}{4\pi r^2}$$

PLANET INTERCEPTS ONLY A FRACTION OF SUN'S ENERGY OUTPUT

$$E_{OUT} = \sigma 4\pi R^2 T^4$$

STEFAN-BOLTZMANN LAW

$$E_{SUN} \frac{\cancel{\pi R^2}}{4\pi r^2} = \sigma 4\cancel{\pi} \cancel{R^2} T^4 \Rightarrow T \propto r^{-1/2}$$

SO $\delta = -1/2$

3a. i) COULOMB REPULSION BETWEEN THE FRAGMENTS ACCELERATES THEM APART AFTER FISSION.

ii) THE FRAGMENTS ARE NEUTRON RICH (FROM FAMILIAR N vs Z OR A vs Z PLOTS) SO THEY ARE LIKELY TO BE β^- EMITTERS, NOT β^+ EMITTERS. [γ RAYS ARE ALSO COMMON]

3b. i) THE PARITY OPERATOR CAUSES INVERSION THROUGH THE COORDINATE ORIGIN, SO $P\psi(\vec{r}) = \psi(-\vec{r})$

ii) ELECTROMAGNETIC AND STRONG INTERACTIONS CONSERVE PARITY; WEAK INTERACTIONS DO NOT.

4. NOTE: σ IS THE SAME BEFORE AND AFTER, V IS <u>NOT</u>.

GAUSS'S LAW AT BOUNDARY $\Rightarrow$ $\sigma = \epsilon E = \epsilon \frac{V}{d}$

$$Q = \sigma A = \left(\frac{\epsilon A}{d}\right) V \equiv CV \Rightarrow C = \frac{\epsilon A}{d}$$

$U = \frac{1}{2}CV^2 = \frac{1}{2}\frac{dA}{\epsilon}\sigma^2$ BEFORE, $\frac{1}{2}dA\sigma^2$ AFTER

WORK DONE ON SYSTEM = $U_{AFTER} - U_{BEFORE} = \underline{\frac{1}{2}dA\sigma^2\left(1-\frac{1}{\epsilon}\right)}$

$\text{WORK}_{\text{ON SYSTEM}} > 0 \Rightarrow$ HAD TO <u>PULL</u> SLAB OUT.