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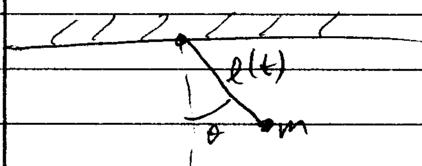
Classical Mechanics: Plane Pendulum



a) $L = T - V = \boxed{\frac{1}{2}m(\dot\ell^2 + \ell^2\dot\theta^2) + mg\ell\cos\theta = L}$

$P_\theta = \frac{\partial L}{\partial\dot\theta} = m\ell^2\dot\theta \Rightarrow \dot\theta = \frac{P_\theta}{m\ell^2}$

$H = \dot\theta P_\theta - L = \boxed{\frac{1}{2}\frac{P_\theta^2}{m\ell^2} - \frac{1}{2}m\dot\ell^2 - mg\ell\cos\theta = H}$

b) $E = T + V = \frac{1}{2}m(\dot\ell^2+\ell^2\dot\theta^2) - mg\ell\cos\theta = H + m\dot\ell^2 \neq H$

$\frac{dH}{dt} = \frac{\partial H}{\partial t} \neq 0$ if $\dot\ell \neq 0 \Rightarrow$ H is not conserved

$\frac{dE}{dt} = \frac{dH}{dt} + 2m\dot\ell\ddot\ell = -\frac{P_\theta^2}{m\ell^2}\frac{\dot\ell}{\ell} + m\dot\ell\ddot\ell - mg\dot\ell\cos\theta \neq 0$ in general

$\Rightarrow$ E isn't conserved, either. (Work must be done to change ℓ given the tension in the string)

c) $\dot\theta = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{m\ell^2}$, $\dot P_\theta = -\frac{\partial H}{\partial\theta} = -mg\ell\sin\theta$

$\Rightarrow \frac{d}{dt}(m\ell^2\dot\theta) + mg\ell\sin\theta = 0$, $\boxed{\ddot\theta + 2\frac{\dot\ell}{\ell}\dot\theta + \frac{g}{\ell}\sin\theta = 0}$

$\dot\ell = 0 \Rightarrow \ddot\theta + \frac{g}{\ell}\sin\theta = 0 \simeq \ddot\theta + \frac{g}{\ell}\theta$. For $\theta^2 \ll 1$,

$\theta \propto \cos(\omega t+\phi)$, $\boxed{\omega = \sqrt{\frac{g}{\ell}}}$

d) For $\left(\frac{\dot\ell}{\ell}\right)^2 \ll \omega^2$, $I_\theta = \oint P_\theta\, d\theta$ is an adiabatic invariant

$I_\theta = \int m\ell^2\dot\theta\, d\theta \simeq m\ell^2\int_0^{2\pi/\omega}\dot\theta^2\, dt$

Try $\theta = A(t)\cos(\omega t+\phi)$ where $\left(\frac{\dot A}{A}\right)^2 \ll \omega^2$ - slowly varying amplitude

$\Rightarrow I_\theta = m\ell^2A^2\omega^2\frac{1}{2}\frac{2\pi}{\omega} = \pi m\ell^2\omega A^2 = \pi m\ell^2\sqrt{\frac{g}{\ell}}A^2 \propto \ell^{3/2}(t)A^2(t)$

$\Rightarrow \boxed{A(t) \propto \ell^{-3/4}(t)}$ Can also obtain this result by applying the WKB method to the differential equation.

Note: I worry that this problem is too easy. Any ideas for how to make it a little harder? Maybe let the mass swing in a circle ⊥ the page?

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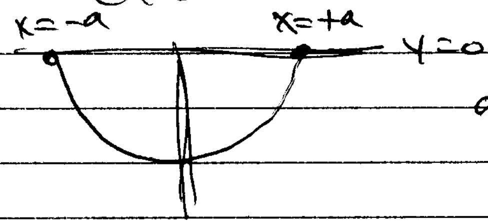
Classical Mechanics: Catenary



a) The equilibrium shape is the one that minimizes the potential energy

$$U = g\int \mu y \, dl = \mu g \int_{-a}^{a} y\sqrt{1+(y')^2}\, dx \qquad \left(y' = \frac{dy}{dx}\right)$$

subject to the constraint

$$\int_{-a}^{a} dl = \int_{-a}^{a} \sqrt{1+(y')^2}\, dx = l.$$

Using a Lagrange multiplier to enforce the constraint,

$$\boxed{I[y(x)] = U - \lambda l = \int_{-a}^{a} (\mu g y - \lambda)\sqrt{1+(y')^2}\, dx}$$

b) Let $L(y, y', x) \equiv (\mu g y - \lambda)\sqrt{1+(y')^2}$.

Then $\delta I = 0 \Rightarrow \dfrac{d}{dx}\left(\dfrac{\partial L}{\partial y'}\right) - \dfrac{\partial L}{\partial y} = 0$, or

$$\frac{d}{dx}\left[\frac{(\mu g y - \lambda) y'}{\sqrt{1+(y')^2}}\right] - \mu g\sqrt{1+(y')^2} = 0.$$

λ will be determined by enforcing the constraint.

Check the given solution: $y = A\cosh(kx+\phi) + B$.

$y' = kA\sinh(kx+\phi)$, $\sqrt{1+(y')^2} = \sqrt{1+k^2A^2\sinh^2(kx+\phi)}$

$$\frac{d}{dx}\left[\frac{(\mu g A\cosh(kx+\phi) + \mu g B - \lambda)\, kA\sinh(kx+\phi)}{\sqrt{1+k^2A^2\sinh^2(kx+\phi)}}\right] = \mu g\sqrt{1+k^2A^2\sinh^2(kx+\phi)}$$

Requires $kA = 1$ and $\mu g B = \lambda$, then $\sqrt{1+(y')^2} = \cosh(kx+\phi)$

and $\dfrac{d}{dx}\left[\mu g A\sinh(kx+\phi)\right] = \mu g\cosh(kx+\phi)$ ✓

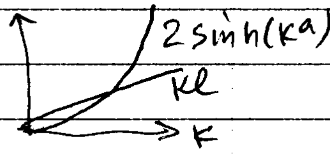
$\therefore \boxed{A = k^{-1}}$, $\boxed{B = \dfrac{\lambda}{\mu g}}$ where k, λ remain to be determined.

Catenary - continued, part b

~~Enforcing boundary conditions at $x = \pm a$: $y = 0$~~

$$\Rightarrow 0 = k^{-1}\cosh(\pm ka + \phi) + \frac{\lambda}{\mu g} \Rightarrow \boxed{\phi = 0},\ \boxed{k^{-1}\cosh(ka) = -B = -\frac{\lambda}{\mu g}}$$

~~Now get λ by enforcing constraint:~~



$$\ell = \int_{-a}^{a} \cosh(kx)\,dx = \boxed{2k^{-1}\sinh(ka) = \ell}$$

The last equation is an implicit equation for $k(\ell)$; as the sketch shows it has a unique positive root.

~~Once k is known, B & λ follow from $B = \frac{\lambda}{\mu g} = -k^{-1}\cosh(ka)$~~ and $A = k^{-1}$.

c) The tension follows by considering adding a section of length $d\ell$ to the chain at some location (x,y). The potential energy of the chain increases by $\mu g y\, d\ell$, and making a gap of size $d\ell$ releases energy $T d\ell$. Thus, the change in potential energy of the chain is $(\mu g y - T)\,d\ell$. But if the chain remains a catenary (minimum potential energy), then

$$dI = \left(\frac{\partial I_{min}}{\partial \ell}\right) d\ell = \lambda\, d\ell = (\mu g y - T)\, d\ell$$

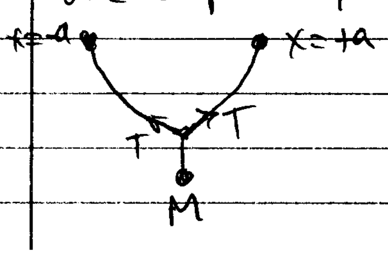
$\uparrow$ using result of part a)

$$\Rightarrow T(x) = \mu g y(x) - \lambda = \mu g\left[y(x) - B\right] = \frac{\mu g}{k}\cosh(kx).$$

d) Adding a mass: either apply Newton's Law $\Sigma F = 0$ at the mass, or modify the variational principle. I'll show the latter:

$$I[y(x)] = \int_{-a}^{a} L(y)\,dx + Mg\,y(0), \quad L(y) = (\mu g y - \lambda)\sqrt{1 + y'^2}.$$

~~We expect y' to be discontinuous at $x = 0$, so the variational principle must exercise care at $x = 0$.~~



Catenary - part d, continued.

$y(x) \Rightarrow y(x) + \delta y(x) \Rightarrow$ Taylor series

$$\delta I = \int_{-a}^{0} \left[\frac{\partial L}{\partial y}\delta y(x) + \left(\frac{\partial L}{\partial y'}\right)\frac{d}{dx}\delta y(x)\right]dx$$
$$+ \int_{0}^{a} \left[\frac{\partial L}{\partial y}\delta y(x) + \left(\frac{\partial L}{\partial y'}\right)\frac{d}{dx}\delta y(x)\right]dx + Mg\,\delta y(0)$$
$$= \int_{-a,\ x\neq 0}^{a} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\delta y(x)\,dx$$
$$+ \int_{-a}^{0} \frac{d}{dx}\left[\frac{\partial L}{\partial y'}\delta y(x)\right]dx + \int_{0}^{a}\frac{d}{dx}\left[\frac{\partial L}{\partial y'}\delta y(x)\right]dx + Mg\,\delta y(0)$$
$$= \int_{-a,\ x\neq 0}^{a} \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]\delta y(x)\,dx + \left[\left(\frac{\partial L}{\partial y'}\right)_{x=0^-} - \left(\frac{\partial L}{\partial y'}\right)_{x=0^+} + Mg\right]\delta y(0)$$

$\Rightarrow \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0$ as before (at $x \neq 0$)

and $Mg = \left[\frac{\partial L}{\partial y'}\right]_0$, the jump at $x=0$

Now, we know from b) that $y(x) = A\cosh(kx+\phi) + B$ satisfies the Euler-Lagrange equation if $\boxed{A = k^{-1},\ B = \frac{\lambda}{\mu g}}$ (4)

But now ϕ is different for $x<0$ ($\phi = \phi_-$) and $x>0$ ($\phi = \phi_+$)

BC: $y=0$ at $x = \pm a \Rightarrow k^{-1}\cosh(\pm ka + \phi_\pm) = -B$

$\Rightarrow \phi_+ = -\phi_- = \phi_0$, $\boxed{k^{-1}\cosh(ka+\phi_0) = -B = -\frac{\lambda}{\mu g}}$ (3)

$\left[\frac{\partial L}{\partial y'}\right]_0 = Mg = \frac{\mu g}{k}\left[\sinh\phi_0 - \sinh(-\phi_0)\right] \Rightarrow \boxed{\sinh\phi_0 = \frac{kM}{2\mu}}$ (1)

Finally, $\frac{l}{2} = \int_0^a dx\sqrt{1+(y')^2} = \int_0^a \cosh(kx+\phi_0)\,dx = \boxed{k^{-1}\left[\sinh(ka+\phi_0) - \sinh\phi_0\right] = l/2}$ (2)

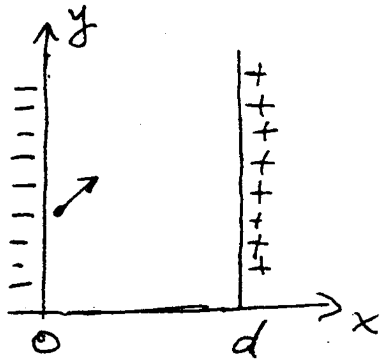
Solve (1) and (2) simultaneously for k, ϕ_0

Then (3) $\Rightarrow B$ and (4) $\Rightarrow A$.

Solutions E and M

#1

a)



$$\frac{d\vec{p}}{dt} = e\mathcal{E}\hat{x} \qquad \mathcal{E} = \frac{V}{d}$$

conservation laws:

$$p_y(t) = p_y^0 \quad \text{(momentum)}$$

$$E = \frac{mc^2}{\sqrt{1-v^2/c^2}} - e\mathcal{E}x = E^0 \quad \text{(energy)}$$

b) initial conditions: $x = y = 0$ $(t=0)$

$\vec{p}(t=0) = 0$

Solving eqs. of motion:

$p_x(t) = e\mathcal{E}t$

$p_y(t) = 0$

from energy conservation: $mc^2 = \sqrt{(mc^2)^2 + c^2 p_x^2(t)} - e\mathcal{E}x$

$$\boxed{x(t) = \left(\sqrt{(mc^2)^2 + (e\mathcal{E}ct)^2} - mc^2\right)/e\mathcal{E}}$$

$x(t_*) = d$

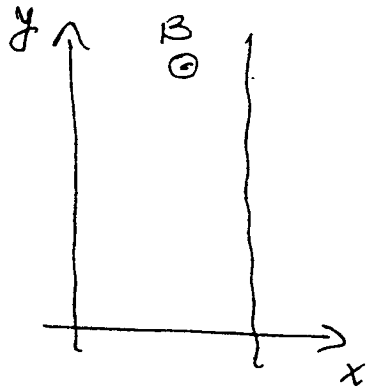
Solving for d:

$$(e\mathcal{E}ct_*)^2 = (mc^2 + e\mathcal{E}d)^2 - (mc^2)^2$$

$$\boxed{t_* = \frac{1}{e\mathcal{E}c}\sqrt{e\mathcal{E}d(2mc^2 + e\mathcal{E}d)}}$$

$e\mathcal{E}d \ll mc^2 \qquad t_* = \sqrt{\frac{2md}{e\mathcal{E}}}$

$e\mathcal{E}d \gg mc^2 \qquad t_* = d/c$



$$\frac{d\vec{p}}{dt} = e\mathcal{E}\hat{x} + \frac{e}{c}\vec{v}\times\vec{B}$$

$$\vec{A} = Bx\,\hat{y}$$

conservation laws:

$$p_y^{\text{canonical}} = \frac{m v_y}{\sqrt{1-v^2/c^2}} + \frac{e}{c}A_y = p_y^{(0)}$$

$$\frac{m v_y}{(1-v^2/c^2)^{1/2}} + \frac{eB}{c}x = 0$$

energy:

$$\frac{mc^2}{\sqrt{1-v^2/c^2}} - e\mathcal{E}x = mc^2$$

d) Solving eqs. of motion:

from energy conservation:

$$\gamma = (1-v^2/c^2)^{-1/2} = 1 + \frac{e\mathcal{E}x}{mc^2}$$

$$v^2 = c^2\left(1 - \frac{1}{\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^2}\right)$$

from momentum conservation:

$$m v_y\left(1+\frac{e\mathcal{E}x}{mc^2}\right) = -\frac{eBx}{c}$$

$$v_y = -\frac{eB}{mc}\,x\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^{-1} \qquad \omega_B = \frac{eB}{mc}$$

$$v_x^2 = v^2 - v_y^2 = c^2\left(1 - \frac{1}{\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^2}\right) - \frac{\omega_B^2 x^2}{\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^2}$$

$$v_x^2 = c^2 - \frac{c^2+\omega_B^2 x^2}{\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^2}$$

parameterize motion by the proper time:

$$\frac{dx}{d\tau} = \gamma v_x = \left(c^2\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^2 - c^2 - \omega_B^2 x^2\right)^{1/2}$$

$$\frac{dy}{d\tau} = \gamma v_y = -\omega_B x$$

$$\frac{dt}{d\tau} = \gamma = 1 + \frac{e\mathcal{E}x}{mc^2}$$

$$c\tau = \int_0^{\bar{x}} \frac{dx}{\left(\left(1+\frac{e\mathcal{E}x}{mc^2}\right)^2 - 1 - \frac{\omega_B^2 x^2}{c^2}\right)^{1/2}} = \int_0^{\bar{x}} \frac{dx}{(2ax - b^2x^2)^{1/2}}$$

where $a = \frac{e\mathcal{E}}{mc^2}$ $\quad b^2 = \frac{\omega_B^2}{c^2} - \left(\frac{e\mathcal{E}}{mc^2}\right)^2 = \left(\frac{e}{mc^2}\right)^2(B^2 - \mathcal{E}^2)$

integrate:

$$\int \frac{dx}{(2ax - b^2x^2)^{1/2}} = \int \frac{dx}{\sqrt{\left(\frac{a}{b}\right)^2 - \left(bx - \frac{a}{b}\right)^2}} = \frac{1}{b}\sin^{-1}\left(\frac{b}{a}\left(bx - \frac{a}{b}\right)\right)$$

$$= \frac{1}{b}\sin^{-1}\left(\frac{b^2}{a}x - 1\right)$$

$$c\tau = \frac{1}{b}\sin^{-1}\left(\frac{b^2}{a}x(\tau) - 1\right) - \frac{1}{b}\underbrace{\sin^{-1}(-1)}_{= -\frac{\pi}{2}} = \frac{1}{b}\left(\frac{\pi}{2} - \sin^{-1}\left(1 - \frac{b^2x(\tau)}{a}\right)\right)$$

$$\cos(bc\tau) = 1 - \frac{b^2x(\tau)}{a}$$

$\omega_B = \frac{eB}{mc}$

$a = \frac{e\mathcal{E}}{mc^2}$

$b = \frac{|e|}{mc^2}(B^2 - \mathcal{E}^2)^{1/2}$

$$x(\tau) = \frac{a}{b^2}\left(1 - \cos(bc\tau)\right)$$

$$y(\tau) = \int_0^{\tau} -\omega_B x(\tau)\, d\tau = -\frac{\omega_B a}{b^2}\left[\tau - \frac{1}{bc}\sin bc\tau\right]$$

$$t(\tau) = \int_0^{\tau}\left(1 + \frac{e\mathcal{E}x(\tau)}{mc^2}\right)d\tau = \left(1 + \frac{a^2}{b^2}\right)\tau - \frac{a^2}{b^3c}\sin bc\tau$$

In order that $x(\tau) < d$ at all times, must have

$$\frac{2a}{b^2} < d \rightarrow \frac{2e\mathcal{E}}{mc^2} < d\left(\frac{e}{mc^2}\right)^2(B^2 - \mathcal{E}^2)$$

$$B^2 > \mathcal{E}^2 + \frac{2mc^2\mathcal{E}}{ed} = (V^2 + 2mc^2V/e)/d^2$$

Problem 2 :

(a) $\vec{B} = B_0 \hat{k}$ $r<a$

$\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3\hat{n}(\hat{n}\cdot\vec{m})-\vec{m}}{r^3}\right]$ $r>a$

B_{normal} continuous at poles $\Rightarrow$

$$\frac{\mu_0}{4\pi}\frac{2m}{a^3} = B_0$$

$B_{tangential}$: $\oint \vec{B}\cdot d\vec{l} = \mu_0 I$ at equator

$$\frac{\mu_0}{4\pi}\frac{m}{a^3} + B_0 = \mu_0 \sigma \omega a$$

$\Rightarrow$

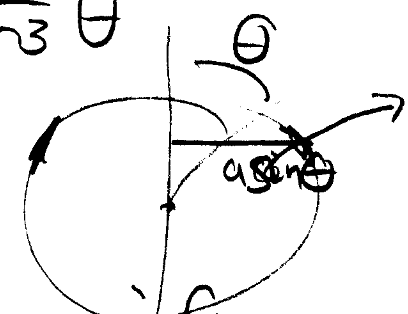
$$\boxed{m = \frac{4\pi}{3}\sigma a^4 \omega}$$

$$\boxed{B_0 = \frac{2\mu_0}{3}\sigma a \omega}$$

$\vec{B} = \frac{2\mu_0}{3}\sigma a \omega \hat{z}$ $r<a$

$\vec{B} = \frac{\mu_0}{3}\sigma \omega \frac{a^4}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$ $r>a$

$$\vec{B} = \frac{m\mu_0}{4\pi}\frac{2\cos\theta}{r^3}\hat{r} + \frac{m\mu_0}{4\pi}\frac{\sin\theta}{r^3}\hat{\theta} \quad r>a$$



Can also get m from

$dm = $ area dI

$= \pi a^2 \sin^2\theta \; a\,d\theta \; \sigma \omega a \sin\theta$

$m = \pi \sigma a^4 \omega \int_{-1}^{1}(1-\cos^2\theta)\,d(\cos\theta)$

$m = \qquad\qquad \left[2 - \frac{2}{3}\right]$

$$\boxed{m = \frac{4\pi}{3}\sigma a^4 \omega \; ✓}$$

(b) by inspection $m = \frac{4\pi}{3}\sigma a^4 \omega$ $\boxed{2}$

(c) $\vec S = \frac{\vec E \times \vec B}{\mu_0}$ $\quad r>a \quad$ $\vec E = \frac{1}{4\pi\epsilon_0}\frac{\sigma 4\pi a^2}{r^2}\hat r$

$$\vec S = \frac{1}{\cancel{\mu_0}}\frac{1}{\cancel{4\pi}\epsilon_0}\frac{\sigma \cancel{4\pi} a^2}{r^2}\,\hat r \times \hat\theta\,\frac{m\cancel{\mu_0}}{4\pi}\frac{\sin\theta}{r^3}$$

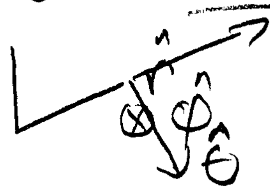
$$\boxed{\vec S = \hat\phi\,\frac{1}{\epsilon_0}\frac{\sigma m}{4\pi}\frac{a^2}{r^5}\sin\theta = \hat\phi\,\frac{\sigma^2 a^6 \omega \sin\theta}{3\epsilon_0 r^5}}$$

$$= \hat\phi\,\frac{1}{\epsilon_0}\,\sigma\,\frac{1}{3}\sigma a^4\omega\,\frac{a^2}{r^5}\sin\theta$$

dimension $= \hat\phi\,\frac{\sigma^2 a \omega}{\epsilon_0}$

$= \hat\phi\,\epsilon_0\left[\frac{\sigma}{\epsilon_0}\right]^2 v$ = energy density $\times\, v$ ✓

(d)



$$d\vec F = I\,d\vec l \times \vec B$$

$$= I\,dl\,\hat\phi \times \left[\frac{m\mu_0}{4\pi r^3}\right]\left[2\cos\theta\,\hat r + \sin\theta\,\hat\theta\right]$$

$$= I\,dl\,\frac{m\mu_0}{4\pi r^3}\left[2\cos\theta\,\hat\phi\times\hat r + \sin\theta\,\hat\phi\times\hat\theta\right]$$

$$= I\,dl\,\frac{m\mu_0}{4\pi r^3}\left[2\cos\theta\,\hat\theta - \sin\theta\,\hat r\right]$$

$$\boxed{\begin{aligned}\hat\phi\times\hat r &= \hat\theta\\ \hat\phi\times\hat\theta &= -\hat r\end{aligned}}$$

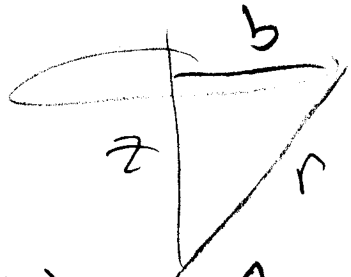
$\hat\theta = -\hat z\sin\theta + \hat\rho\cos\theta$

$\hat r = +\hat z\cos\theta + \hat\rho\sin\theta$

ignore $\hat\rho$ component

$$d\vec{F} = \hat{z} I \, dl \frac{m\mu_0}{4\pi r^3}\left[-2\cos\theta\sin\theta - \sin\theta\cos\theta\right] \boxed{3}$$

$$\vec{F} = -\hat{z} I \, 2\pi b \, 3 \frac{m\mu_0}{4\pi r^3} \cos\theta \sin\theta$$



$$\sin\theta = \frac{b}{r} \qquad \cos\theta = \frac{\sqrt{r^2-b^2}}{r}$$

$$\vec{F} = -\hat{z} I \, 2\pi b 3 \frac{m\mu_0}{4\pi r^3} \frac{b\sqrt{r^2-b^2}}{r^2}$$

let $r \gg b$, Then

$$\vec{F} = -\hat{z} 3 I \, 2\pi b^2 \frac{m\mu_0}{4\pi r^4}$$

$$= -\hat{z} 3 M_b \frac{2m\mu_0}{4\pi r^4}$$

$$= +\hat{z} \, \vec{M}_b \cdot \vec{\nabla} \frac{\mu_0}{4\pi} \frac{2m}{r^3}$$

$$= M \cdot \nabla B$$ checks in limit

In terms of z

$$\vec{F} = -\hat{z} \frac{3I(2\pi b)}{4\pi r^3} m\mu_0 \frac{bz}{r^2}$$

$$\boxed{\vec{F} = -\hat{z} \frac{z 3 I b^2 m\mu_0}{2 r^5}}$$

$\hookrightarrow r^2 = z^2 + b^2$

P1 solution STAT Mech

1) From $dF = -S\,dT - P\,dV$

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V = \frac{\alpha}{V^2}$$

2) $C_V = T\left(\frac{\partial S}{\partial T}\right)_V = A(V)\,T$ with $S = S(T,V)$

$$\frac{\partial A}{\partial V} = \frac{\partial}{\partial V}\left(\frac{\partial S}{\partial T}\right)_V = \frac{\partial}{\partial T}\left(\frac{\partial S}{\partial V}\right)_T = \frac{\partial}{\partial T}\,\frac{\alpha}{V^2} = 0$$

3) $$S(T,V) = S(T_0,V_0) + \int_{T_0}^{T} \frac{\partial S}{\partial T}(V_0,T)\,dT + \int_{V_0}^{V} \frac{\partial S}{\partial V}(V,T)\,dV$$

$$= S(T_0,V_0) + \int_{T_0}^{T} A\,dT + \int_{V_0}^{V} \frac{\alpha}{V^2}\,dV = S_0 + A(T-T_0) + \alpha\left(\frac{1}{V_0^2} - \frac{1}{V^2}\right)$$

4) $$C_p = T\left(\frac{\partial S}{\partial T}\right)_p = T\left(\left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p\right) =$$

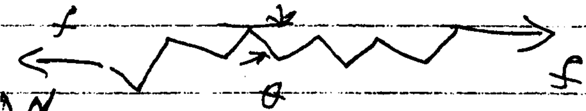
$$= T\left[A + \frac{\alpha}{V^2}\,\frac{\alpha}{2PV}\right] = T\left[A + \frac{\alpha}{2VT}\right]$$

$$\underline{C_p = C_V + \frac{\alpha}{2V}}$$

Problem 2

a) $\langle r_{12}^2 \rangle = N d^2$, $\langle r_{12} \rangle = 0$

b) Partition function for a chain stretched with force f



$$Z = \left(\int_0^{\pi} 2\pi \, d\theta \, \sin\theta \, e^{\beta f d \cos\theta} \right)^N$$

Free energy $F = -NT \ln\left(4\pi \frac{\sinh \beta f d}{\beta f d}\right)$

Average displacement L can be found by minimizing $F + fL$:

$$L = -\frac{\partial F}{\partial f} = N\left(d \coth \beta f d - \frac{T}{f}\right)$$

At $fd \ll T$, $L = Nd \frac{\frac{1}{3}(\beta f d)^2}{\beta f d} = \frac{1}{3} \frac{N d^2}{T} f$

At $fd \gg T$, $L = Nd$

c) $$S = -\left(\frac{\partial F}{\partial T}\right)_f = N \ln\left(4\pi \frac{\sinh \beta f d}{\beta f d}\right) + NT \frac{\partial}{\partial T} \ln\left(4\pi \frac{\sinh \beta f d}{\beta f d}\right)$$

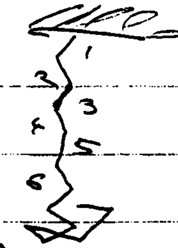
$$= N \ln\left(4\pi \frac{\sinh \beta f d}{\beta f d}\right) + N - N \beta f d \coth \beta f d$$

$$S = N \ln\left(4\pi e \frac{\sinh \beta f d}{\beta f d}\right) - N \beta f d \coth \beta f d$$

$$S(\beta f d \ll 1) = N \ln 4\pi$$

$$S(\beta f d \gg 1) = N \ln\left(\frac{2\pi e}{\beta f d}\right)$$

d) The force applied to m-th link is $f_m = mg(N-m)$



Thus $$Z = \prod_{m=1}^{N} \int_0^{\pi} 2\pi \, d\theta \sin\theta \, e^{\beta f_m d \cos\theta} =$$

$$= \prod_1^N 4\pi \frac{\sinh \beta d f_m}{\beta d f_m} = (4\pi)^N \exp \sum_{m=1}^{N} \ln\left(\frac{\sinh \beta d f_m}{\beta d f_m}\right)$$

Replacing the sum by an integral:

$$F = -NT \ln 4\pi - T \int_0^N \ln \frac{\sinh \alpha u}{\alpha u} \, du \qquad \alpha = mg\beta d$$

$$L = -\frac{\partial F}{\partial mg} = d \int_0^N \frac{\partial}{\partial \alpha} \ln \frac{\sinh \alpha u}{\alpha u} \, du =$$

$$= d \int_0^N \frac{\partial}{\partial u} \left(\ln \frac{\sinh \alpha u}{\alpha u}\right) du =$$

$$= d \ln \frac{\sinh \alpha N}{\alpha N} = d \ln \left(\frac{\sinh (mg\beta d N)}{mg\beta d N}\right)$$

Solution to Q.M. 1.

a) Let $|\theta\rangle = \sum_n e^{in\theta} |n\rangle$

$$\hat{T}|\theta\rangle = \sum_n e^{in\theta} |n+1\rangle$$

$$= \sum_n e^{i(n-1)\theta} |n\rangle = e^{-i\theta} |\theta\rangle$$

b)

$$\hat{H}|\psi_k\rangle = E_k |\psi_k\rangle$$

$$\langle n|\hat{H}|\psi_k\rangle = E_k \langle n|\psi_k\rangle$$

$$-\frac{1}{2\Delta^2}\left\{ \langle n+1|\psi_k\rangle + \langle n-1|\psi_k\rangle - 2\langle n|\psi_k\rangle \right\}$$

$$= E_k \langle n|\psi_k\rangle$$

$$-\frac{1}{2\Delta^2}\left\{ e^{ik\Delta} + e^{-ik\Delta} - 2 \right\} = E_k$$

$$\frac{1}{\Delta^2}(1 - \cos k\Delta) = E_k$$

$$\Delta \to 0 \qquad E_k \to \frac{k^2}{2}$$

3c) Scattering State $|S_k\rangle$ has energy E_k

$$n<0 \qquad \langle n|S_k\rangle = e^{ikn\Delta} + R e^{-ikn\Delta}$$

$$n>0 \qquad \langle n|S_k\rangle = T e^{ikn\Delta}$$

$$n=0 \qquad 1+R=T$$

$$\hat{H}|S_k\rangle = E_k|S_k\rangle$$

$$\langle 0|\hat{H}|S_k\rangle = E_k \langle 0|S_k\rangle$$

$$-\frac{1}{2\Delta^2}\left\{\langle 1|S_k\rangle + \langle -1|S_k\rangle - 2\langle 0|S_k\rangle\right\} + V\langle 0|S_k\rangle$$

$$= E_k\langle 0|S_k\rangle$$

$$-\frac{1}{2\Delta^2}\left\{Te^{ik\Delta} + e^{-ik\Delta} + Re^{ik\Delta} - 2T\right\} + VT =$$

$$-\frac{1}{2\Delta^2}\left\{e^{ik\Delta} + e^{-ik\Delta} - 2\right\}T$$

$$e^{-ik\Delta} + (T-1)e^{ik\Delta} - 2\Delta^2 VT = e^{-ik\Delta}T$$

$$T\left\{e^{ik\Delta} - e^{-ik\Delta} - 2\Delta^2 V\right\} = e^{ik\Delta} - e^{-ik\Delta}$$

$$T\left\{-2i\sin k\Delta - 2\Delta^2 V\right\} = -2i\sin k\Delta$$

$$|T|^2 = \frac{\sin^2 k\Delta}{\sin^2 k\Delta + \Delta^4 V^2}$$

a) $[\hat{\pi}_x, \hat{\pi}_y]$

$$= [\hat{p}_x + \frac{q}{c}\hat{A}_x, \hat{p}_y + \frac{q}{c}\hat{A}_y]$$

$$= \frac{q}{c}\left([\hat{A}_x, \hat{p}_y] + [\hat{p}_x, \hat{A}_y]\right)$$

$$= \frac{q}{c}\left(-\frac{\hbar}{i}\frac{\partial}{\partial y}A_x + \frac{\hbar}{i}\frac{\partial}{\partial x}A_y\right)$$

$$= \frac{\hbar q}{ci}B \qquad \text{Let } \omega = \frac{qB}{mc}$$

$$= -i\hbar m\omega$$

Let $\hat{\pi}_x \equiv \hat{P}$ $\quad \hat{\pi}_y \equiv m\omega\hat{Q}$

$$[\hat{Q}, \hat{P}] = \frac{1}{m\omega}[\hat{\pi}_y, \hat{\pi}_x] = i\hbar$$

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2m}m^2\omega^2\hat{Q}^2$$

$$= \frac{1}{2m}\hat{P}^2 + \frac{m\omega^2}{2}\hat{Q}^2$$

Eigenvalues $\hbar\omega(n + 1/2)$

b) $A_x = 0 \quad A_y = Bx$

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\left(\hat{p}_y + \frac{qB}{c}\hat{x}\right)^2$$

$[\hat{p}_y, \hat{H}] = 0$ because no y in $\hat{H}$

Can take $\hat{p}_y$ to equal its eigenvalue p_y

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\left(p_y + \frac{qB}{c}\hat{x}\right)^2$$

Shift by p_y.

$$\hat{H} = \frac{1}{2m}\hat{p}_x^2 + \frac{1}{2m}\frac{q^2B^2}{c^2}\hat{x}^2$$

$\omega = \frac{qB}{mc}$ Eigenvalues $\hbar\omega(n+\tfrac{1}{2})$ ✓