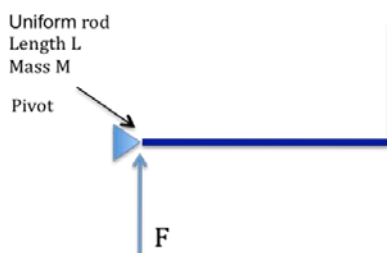


1. An electric charge distribution produces an electric field

$$E = c(1 - e^{-\alpha r}) \frac{\hat{r}}{r^2}$$

where c and α are constants. Find the net charge within the radius $r = 1/\alpha$.

2. Suppose the US and Russia were to jointly build a space station with two identical spherical modules (mass, m) to house their crews, and that they are joined by an effectively mass-less conduit (length, d) for astronauts to visit each other. What is the stable orientation of this satellite? (i.e., determine the angle α between the conduit and the line joining the c.m. of the satellite and the Earth's center). *Hint: The potential energy of a non-spherical satellite varies slightly, depending on its orientation relative to its orbit.*
3. A uniform rod of length L and mass M is pivoted at one end and held up by a thin string at the other end. The moment of inertia of the rod about its end is $(1/3)ML^2$. What is the supporting force to the rod at the pivot in the moment after the string is cut?



4. It is observed that a pulse requires 0.1 s to travel from one end of a stretched string to the other. The tension in the string is provided by passing the string over a pulley to a weight with a mass 40 times the mass of the string.
- What is the length of the string?
 - What is the fundamental frequency of this piece of string?
5. Liquid ^4He is used for experiments in low-temperature physics. Unfortunately, its cost on campus has risen recently to about \$10/liter. Compute the cost of the electrical energy required by the refrigerators that cool the helium gas (starting at 300 K) to produce liquid helium (4.2 K).

Useful facts:

- one liter of liquid ^4He contains 32 moles.
- the ideal gas constant $R = 8.31 \text{ J/mol K}$.
- the heat of vaporization of liquid ^4He is 2.6 J/liter

Assume that:

- the coefficient of performance of the refrigerator is $\text{COP} = K = Q_{\text{Cold}}/W_{\text{in}} = 1\%$
- the cost of electricity is \$0.10/kilowatt-hour
- the volume of the gas remains constant during the cooling process.

6. A $25.0\ \Omega$ resistor, $30.0\ \text{mH}$ inductor and $12.0\ \mu\text{F}$ capacitor are connected in series to a $90.0\ \text{V AC (rms)}$ $500\ \text{Hz}$ source.
- What is the rms current in the circuit?
 - What are the rms voltages across the (i) resistor, (ii) inductor and (iii) capacitor?
 - Does the current lead or lag the voltage? What is the phase angle ϕ (in degrees)?
 - What is the average power dissipated in the circuit?
7. The mass of the heaviest neutrino is approximately $mc^2 \sim 0.03\ \text{eV}$. How much does the speed of a $1\ \text{MeV}$ neutrino differ from the speed of light, i.e. what is $\frac{c-v}{c}$?
8. An electron moves in a two-dimensional rectangular well potential: it satisfies the Schrödinger equation
- $$-\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) + V(x, y)\psi(x, y) = \varepsilon\psi(x, y)$$
- with
- $$V(x, y) = \begin{cases} 0, & \text{if } -L/2 < x < L/2 \text{ and } 0 < y < L/2 \\ \infty, & \text{otherwise} \end{cases}$$
- Give the four lowest energies ε .
 - Linearly polarized light of variable frequency ω is shone on the electron. The potential energy of the electron in the light field is $V_f(x, y, t) = V_0 x \cos(\omega t)$. What are the selection rules for transitions between the four lowest energy levels already found?
9. For a particle of mass m in an infinite square well of length L , we are given that at $t=0$, its wavefunction is a linear combination of the first four energy eigenstates $|\varphi_n\rangle$ of the well: $|\psi(t=0)\rangle = \sum_{n=1}^4 \alpha_n |\varphi_n\rangle$ with $\sum_{n=1}^4 |\alpha_n|^2 = 1$.
- What is the probability of measuring a value of the energy less than $4\pi^2\hbar^2/(mL^2)$?
 - Do you expect this result to change at times $t > 0$? Explain.
10. Consider a sodium atom with a $3d$ valence electron ($n=3, l=2$). Draw a diagram to show the splitting of $\text{Na}(3d)$ due to spin-orbit interaction. Give the quantum numbers for each level and express them in spectroscopic notation.

1. In astrophysical environments beams of photons are often produced by the radiation of highly relativistic electrons.
 - a. Derive an expression for the emission angle of the photons in the frame of the observer (laboratory frame) in terms of the emission angle in the rest frame of the electron.
 - b. Show that half of the photons are emitted in a forward cone with an opening angle of order $1/\gamma$.

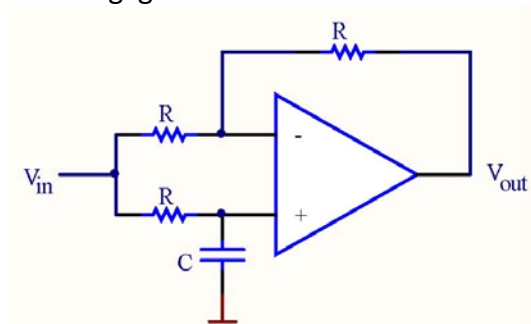
2.
 - a. Give the quantum numbers of a spin-less electron in a solid (at least two).
 - b. Give the mathematical form of the wave function for an electron in a solid. (Assume a crystalline solid.)
 - c. Plot this wave function for a one-dimensional chain of atoms.
 - d. Which symmetries give rise to the quantum numbers in a. ?

3. The wave function of a particle of mass m in one dimension is bound to a fixed center by an attractive δ -function potential $V_\alpha(x) = -\alpha\delta(x)$ with $\alpha > 0$. A wave function is given by

$$\psi_\alpha(x) = \frac{\sqrt{\alpha m}}{\hbar} \exp\left(-\frac{m\alpha|x|}{\hbar^2}\right).$$

Calculate the probability that the particle becomes unbound if the potential is abruptly changed to $V_{new}(x) = -2\alpha\delta(x)$ at time $t=0$.

4. Consider the op-amp circuit shown below. The circuit is driven by a sinusoidal voltage source $V_{in}(t) = V_o e^{j\omega t}$. Calculate the amplitude and phase of the output voltage V_{out} as a function of the driving frequency ω . Recall that negative feedback will maintain equal voltages at the op-amp inputs, and that the inputs draw negligible current.



5. The universe is permeated by cosmic-microwave background photons. Cosmic-ray protons can lose energy through interactions with CMB photons via the process:
 $\gamma_{cmb} + p \rightarrow p + \pi^0$ (or $n + \pi^+$)

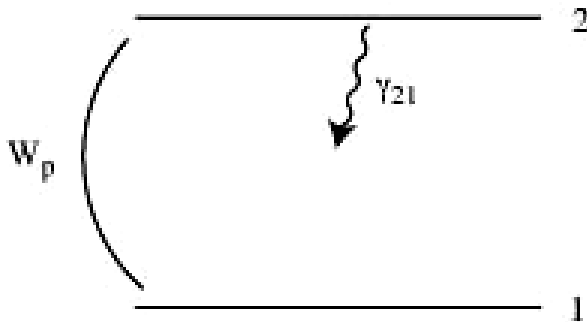
Find the minimum incoming proton energy for which this can happen. This leads to the GZK cutoff in the cosmic-ray spectrum. $m_p c^2 = 939 \text{ MeV}$, $m_\pi c^2 = 139 \text{ MeV}$, $E_{cmb} = 2.6 \times 10^{-4} \text{ eV}$

6. An electromagnetic wave with frequency ω and electric field $E_\omega(x,t) = \text{Re}[E_0 \exp(-i\omega t + ik \cdot x)]$ is launched by an antenna into a dissipative plasma medium with the complex dielectric function

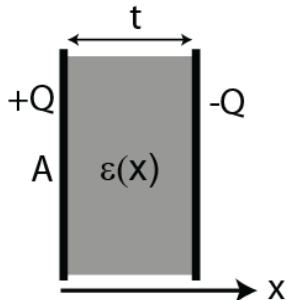
$$\frac{\epsilon(\omega)}{\epsilon_0} = 1 - \frac{\omega_{pe}^2}{\omega^2 + i\gamma\omega}$$

where ω_{pe} is the electron plasma frequency, and γ describes the dissipative processes, $\gamma \ll \omega$. Assume that the permeability of the plasma is $\mu \approx \mu_0$, and that $\omega_{pe} \ll \omega$. Estimate the distance such a wave can propagate inside the plasma before it gets significantly attenuated.

7. It is well-known that to achieve population inversion, a system with more than two levels is required. To check this argument, you will analyze population inversion in a two level system. Consider the following two level atomic system. Assume that there is a light source that pumps between the two states at a rate W_p . Also assume that the only decay process is the decay of state $|2\rangle$ to state $|1\rangle$ with a decay rate γ_{21} .



- Write the rate equations for the number of atoms in each state, N_1 and N_2 .
 - Solve these equations at steady state. Using the conservation condition, $N_1 + N_2 = N$ (N : total number of atoms), please find the population difference as expressed as a fraction of the total number of atoms, i.e. $(N_2 - N_1)/N$.
 - Plot the population difference as a function of the pumping rate W_p . Is there ever population inversion?
8. A dielectric separates two conducting plates with charges Q and $-Q$. The area of the plates is A and they are separated by distance t . Neglect edge effects. The dielectric constant varies linearly as $\epsilon = \epsilon_0 + kx$ $0 \leq x \leq t$.



- Find the electric field E , the displacement field D , and the polarization P between the plates.
- What is the capacitance $C = Q/V$ of the plates?

9. Two identical spin - $\frac{1}{2}$ fermions are confined by an infinite one dimensional square well. The Hamiltonian

$$\mathcal{H} = -\frac{\hbar^2}{2m} \left[\frac{d^2}{dx_1^2} \right] - V(x_1) - \frac{\hbar^2}{2m} \left[\frac{d^2}{dx_2^2} \right] - V(x_2) ,$$

where

$$V(x) = \begin{cases} 0, & \text{for } |x| < L/2 \\ \infty, & \text{otherwise} \end{cases} ,$$

applies to this system.

- Find the ground level energy and wavefunction of this two particle system.
- Find the ground level total spin.
- Find the energy, degeneracy, and the wavefunction(s) for the first excited state.

10. The Hamiltonian for the positronium atom in the 1S state in a magnetic field B is, to a good approximation,

$$H = A \mathbf{S}_1 \cdot \mathbf{S}_2 + \frac{e}{\mu c} \mathbf{B} \cdot (\mathbf{S}_1 - \mathbf{S}_2)$$

where 1 labels the electron and 2 the positron. Here S_1 and S_2 are the spin operators for the two particles and A is a constant coupling. Choose the z-axis along B. Which of the following are constants of the motion?

- S_1^2 and S_2^2
- S_{1z} and S_{2z}
- $S = S_1 + S_2$
- S^2
- $S_z = S_{1z} + S_{2z}$

Explain your answers.

11. Suppose C is the capture rate of dark matter in an astrophysical body. Let C_A be the dark matter annihilation rate per effective volume. Then an approximate Boltzmann equation governing the number N of dark matter particles in the astrophysical body is

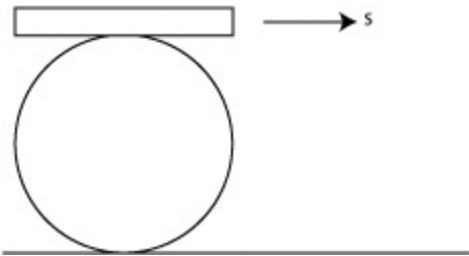
$$dN/dt = C - C_A N^2$$

If initially, $N(0) = 0$, find $N(t)$. Hint: $\int \frac{du}{1-u^2} = \tanh^{-1} u$.

12. The stable isotope ^{206}Pb is at the end of the radiative chain decay of ^{238}U . The half-life time for $^{238}\text{U} \rightarrow ^{206}\text{Pb}$ is $t_{1/2} = 4.47 \times 10^9 \text{ yr}$.

- Assume that the age of the Earth is 4.5 billion years, what should be the ratio of $^{206}\text{Pb}/^{238}\text{U}$ in a uranium-bearing rock as old as the Earth?
- A sample of uranium ore is an admixture with lead and has a ratio of $^{206}\text{Pb}/^{238}\text{U} = 3/5$. How old is the ore?

13. A localized wave disturbance (wave packet) propagates in a dispersive medium with $\omega=\omega(k)$, where $\omega(k)$ is some function of wavenumber k such that frequency increases monotonically with wavenumber. Assume that the initial waveform has a spread of wavenumbers centered about k_0 with a width Δk , such that $k_0 \gg \Delta k$.
- At what velocity does the waveform propagate?
 - Using the smallness of Δk , find a condition such that higher frequencies arrive at a distant location ahead of lower frequencies.
 - After a very long time, what is the approximate width of the waveform in real space x , as a function of time, Δk , and the appropriate measure of dispersion?
14. A uniform long, thin plank with mass m and length L is perfectly balanced on top of a hollow cylinder $L/2$ in radius and mass m , as shown. The cylinder sits initially at rest on a rough, flat surface. A person grabs one end of the plank and rapidly pulls the plank, perfectly horizontally, at a speed s . The plank slides across the top of the cylinder, causing the cylinder to roll without slipping on the horizontal surface. The coefficient of friction between the plank and the cylinder is μ . What is the angular velocity ω of the cylinder about its axis, after the plank is pulled free of the cylinder? You may assume that the distance the cylinder rolls while the plank is in contact is negligible.



15. A conducting wire of length $L=30$ cm is stretched between two points, perpendicular to a uniform and constant magnetic field $B=0.1$ T. It is vibrating in its fundamental mode with frequency 120 Hz in a plane perpendicular to the magnetic field. If the amplitude of the vibration is 1.0 mm, what is the maximum voltage between the two ends of the wire?
16. Given the masses of the Earth (6.0×10^{24} kg) and the Moon (7.35×10^{22} kg), the radius of the Earth (6380 km), and given that 1 sidereal month = 27.3 days = 2.35×10^6 sec,
- Find the Earth-Moon distance.
 - What fraction of the angular momentum of the Earth-Moon system (about its center of mass) is due to the spin of the earth? The moment of inertia of the Earth is $I_E = 0.332 m_E R_E^2$. (You can neglect the contribution from the spin of the Moon)