STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Comprehensive Examination

January 29, 2013, 5:30pm-10:00pm

General Instructions: Twelve problems are given; you should solve four problems. If you do more than four problems, you must choose which four should be graded, and only submit those four. You may do two problems from the same field only once, except for Astronomy, for which you can do up to four problems.

Each problem counts 20 points, and the solution should typically take less than 45 minutes.

Some of the problems cover multiple pages. Make sure you do all the parts of each problem you choose.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

Some potentially useful information:

The atomic mass of hydrogen is 1.00794 amu.
The atomic mass of helium is 4.002602 amu.
1 amu is 1.66×10^{-27} kg.
$c = 2.998 \times 10^{8}$ ms^{-1}.
$G = 6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$.
The solar luminosity is 3.85×10^{26} W.
The mass of the Sun is 1.989×10^{30} kg
The radius of the Sun is 7.00×10^{8} meters
$\hbar = 1.055 \times 10^{-34}$ J
$e = 1.602 \times 10^{-19}$ C
$k_B = 1.38 \times 10^{-23}$ J/K
$N_A = 6.022 \times 10^{23}$ mol^{-1}
$mc^2 \simeq 0.5$ MeV
$\hbar c \simeq 197$ MeV-fm.

Astronomy 1

Consider the disk of the Milky Way galaxy to be a smooth, axisymmetric mass distribution with gravitational potential $\Phi(R, z)$ in cylindrical coordinates (R, ϕ, z).

a. (8 pts.) Starting from the expressions for the acceleration

$$\ddot{\mathbf{r}} = (\ddot{R} - R\dot{\phi}^2)\hat{\mathbf{e}}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\hat{\mathbf{e}}_\phi + \ddot{z}\hat{\mathbf{e}}_z \tag{1}$$

and gradient operator

$$\nabla = \hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \frac{\hat{\mathbf{e}}_\phi}{R}\frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \tag{2}$$

in cylindrical coordinates, show that in the vertical (i.e. z) direction, a star that is slightly perturbed from a circular orbit exhibits simple harmonic motion, and derive the vertical oscillation frequency ν in terms of the potential $\Phi(R, z)$.

b. (5 pts.) Determine the oscillation period of the Sun through the Galactic plane, assuming a local mass density in the solar neighborhood of 0.2 $M_\odot$ pc^{-3} extending well above and below the maximum excursion height of the Sun. Take the gravitaional constant to be $G = 4.5 \times 10^{-3}$ pc^3 $M_\odot^{-1}$ Myr^{-2}.

c. (5 pts.) Assuming that the Sun is currently passing through the midpoint of the Galactic plane with a vertical velocity of $v_z = 7$ km s^{-1}, determine the maximum excursion height of the Sun above and below the plane. Note that 1 pc $= 3.1 \times 10^{13}$ km.

d. (2 pts.) It has been claimed that over at least the past 250 Myr, mass extinctions on Earth occur periodically with a period of around 30 Myr. Comment on the correspondence between this period and the period determined in part C above. What effect might account for this correspondence?

Solution

a. Consider the equation of motion

$$\ddot{\mathbf{r}} = -\nabla\Phi(R, z) \tag{3}$$

and the vertical component of the equation of motion of the star

$$\ddot{z} = -\frac{\partial \Phi}{\partial z}. \tag{4}$$

Considering only small excursions from the plane and noting that

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = 0 \tag{5}$$

(since the disk is symmetric about $z = 0$), expand linearly to obtain

$$\ddot{z} = -\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} - z\left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{z=0} = -z\left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{z=0} = -\nu^2 z. \tag{6}$$

This yields simple harmonic motion with frequency

$$\nu^2 = \left(\frac{\partial^2 \Phi}{\partial z^2}\right)_{z=0}. \tag{7}$$

b. Use Gauss' law to write

$$g_z = -\frac{\partial \Phi}{\partial z} = -4\pi G z \rho. \tag{8}$$

Then

$$\frac{\partial^2 \Phi}{\partial z^2} = \nu^2 = 4\pi G \rho = 0.011 \text{ Myr}^{-2} \tag{9}$$

and

$$\nu = 0.11 \text{ Myr}^{-1} \tag{10}$$

and

$$T = \frac{2\pi}{\nu} = 59 \text{ Myr}. \tag{11}$$

c. Write

$$z = Z \sin(\nu t) \tag{12}$$

and

$$\dot{z} = Z\nu \cos(\nu t), \tag{13}$$

so

$$Z = \frac{(\dot{z})_{\mathrm{z=0}}}{\nu} = 2.0 \times 10^{15} \text{ km} = 65 \text{ pc}. \tag{14}$$

d. The oscillation period of the Sun through the Galactic plane is roughly twice the claimed period of mass extinctions on Earth, causing some to assert that mass extinctions on Earth are trigged when the Sun passes through the midpoint of the plane, perhaps because these passages somehow perturb comets in the Oort cloud into orbits directed toward the inner solar system.

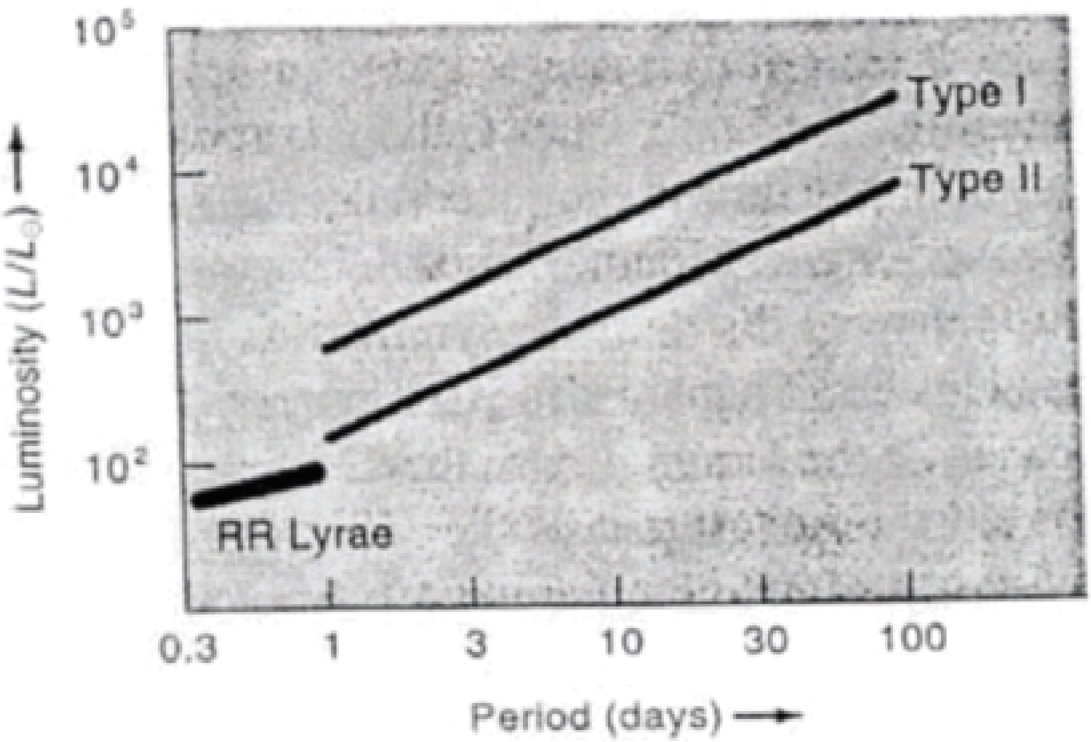
Astronomy 2

Cepheid variable stars are variable stars that pulsate in brightness in a regular fashion according to a period-luminosity relation.

a) (5 points) Using the period-luminosity relation reproduced below, estimate the peak absolute magnitude of a Type I Cepheid variable star of pulsation period 30 days. (Recall that the absolute magnitude of the Sun is $M = 5$.)

b) (5 points) Write down the Hubble relation between redshift z and distance d that applies in the local universe (i.e. for $z \ll 1$) in terms of the Hubble constant H_0 and the speed of light c.

c) (10 points) Say that a particular Type I Cepheid variable star is observed with pulsation period 30 days (negligible uncertainty), redshift $z = 0.002$ (negligible uncertainty), and apparent magnitude $m = 25 \pm 0.2$. Using results of parts A and B, estimate the value and uncertainty of the Hubble constant.



Solution
STILL MISSING

Astronomy 3

Sirius, with a V magnitude $= -1.4$, is the brightest star is the sky (aside from the Sun). In 1862 Alvan Clark, while testing one of his refractors, discovered that Sirius was a visual pair. It is now known that the system is a physical binary, with a semi-major axis of 7.5 arcsec and a period of 50 years. The parallax is 0.37 arcsec. Sirius A is a main sequence star, spectral type A0V, with a mass of 2.2 $M_\odot$ and an effective temperature of 10,000K. Sirius B has an effective temperature of about 30,000K, and a spectral type of B0. Its V magnitude is 8.6.

The numbers above are approximated to simplify the calculations.

Useful information:

$M_\odot = 2 \times 10^{33}$ gm

$R_\odot = 5 \times 10^{10}$ cm

$L_\odot = 4 \times 10^{33}$ erg/s

$AU = 1.5 \times 10^{13}$ cm

$R_\oplus = 6.4 \times 10^{8}$ cm

$V = 0$ corresponds to 996 photons/cm^2/s/Å at 5400Å.

a. (5 pts.) What is the mass of Sirius B?

b. (5 pts.) What is the radius of Sirius B?

c. (5 pts.) What is the mean density of Sirius B? Compare this to the density of the Sun. Why did Sir Arthur Eddington refuse to believe that this could possibly be true?

d. (5 pts.) Sirius B is a white dwarf. Subrahmanyan Chandrasekhar showed that white dwarfs were supported against gravitational collapse by degenerate electron pressure. He also showed that there was a maximum mass for White dwarfs. Starting with the relation between pressure P and the density of an ideal gas, P = nvp, where p is the momentum and v is the velocity of the particles, show the $P_e = n_e p^2/m_e$. Making use of Heisenberg's uncertainty principle, show that when the electron gas is degenerate $P_e \sim h^2 m_e^2 / n_e^{5/3}$.

In hydrostatic equilibrium the mean pressure in a self-gravitating gas is $\propto \frac{M^2}{R^4}$. Derive the functional form of the mass-radius relation for white dwarfs. Compare it to that of main sequence stars.

Derive an expression for P_e in the limit that the degenerate gas decomes relativistic ($v \rightarrow c$). What happens to the mass-radius relation in this case?

Solution

a. The distance = 1/0.37 = 2.7 parsecs.

The semimajor axis is $7.5 \times 2.7 = 20$ AU. *from the definition of the parsec as the distance at which the radius of the Earth's orbit around the Sun subtends an angle of 1 arcsec*

Use Kepler's 3^{rd} law: $a^3=P^2(M_A+M_B)$. $M_B=1.0\ M_\odot$. Note that when a is in AU and P is in years, the value of $G/4\pi^2=1$. Of course, it is possible to do the calculation in CGS or MKS units and get the same answer.

b. The spectral energy distribution of a star can be approximated as a blackbody. At temperatures of 10,000 and 30,000 K, both stars are on the Rayleigh-Jeans tail of the blackbody, so the brightness $\propto R^2T$. The ratio of radii is therefore

$$\frac{R_B}{R_A} = \sqrt{(\frac{B_B}{B_A})(\frac{T_A}{T_B})}$$

where R, B, and T are radii, brighness, and temperature of the star. The brightness difference is $8.6 - -1.4$ mag, or a factor of $= 10^4$ (*from the definition of a magnitude*). Therefore,

$$\frac{R_B}{R_A} = \sqrt{(10^{-4})(\frac{1}{3}} \sim 170$$

Near 1 $M_\odot$ on the main sequence the radius of a star scales with its mass, so the radius of Sirius B is about $2.2/170 = 0.013\ R_\odot$, or about 1.4 $R_\oplus$.

c. The density is about 6×10^5 gm/cm^3. For comparison, the mean density of the Sun is 1.4 gm/cm^3, and the density of the densest known metal (at STP) is about 20 gm/cm^3

d. $P = nvp$. $p=mv$. $P = np^2/m$

Plug in $\Delta p \sim h/\Delta x$ (or $P=h/x$) to get the desired result.

$P\propto \frac{M^2}{R^4} \sim h^2 m_e^2/n_e^{5/3}$. Simplify to find $R\propto M^{-\frac{1}{3}}$. As the mass increases, the radius decreases. This is opposite to the behavior of stars on the main sequence (or any ideal classical gas).

In the relativistic limit, ($v\rightarrow c$), and $P_e \sim ch^2 n_e^{4/3}$. The mass-radius relation is single-valued, since the radius cancels. The numerical value for the mass is the Chandrasekhar mass, of 1.42 $M_\odot$ for a solar-composition gas.

Astronomy 4

In stars about 1.5 solar masses or higher, the CNO cycle dominates the pp-chain in providing the energy from H burning.
In this problem, we explore the temperature sensitivity of the CNO cycle, by constructing an expression for a non-resonant nuclear reaction in the interior of stars. We won't worry about constant factors (and they will be provided when needed for a numerical result)

a. (5 pts.) Explain qualitatively why more massive stars are required for the CNO cycle to take place and why the pp-chain is slower than the CNO cycle in these stars.

b. (5 pts.) The cross-section for a non-resonant reaction rate is usually written as:

$$\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}} \tag{1}$$

where $b = 31.28 Z_\alpha Z_\beta \mu^{1/2}$ (keV)$^{1/2}$ with reduced mass μ measured in amu, and $S(E)$ is a slowly varying function that depends on the nuclear properties. Typically the cross section is measured in barns and the energy in keV, so $S(E)$ has units of barn keV.

What is the physical intrepretation of the factors, E^{-1} and $\exp\{-bE^{-1/2}\}$?

c. (5 pts.) The total reaction rate for two nuclei, α and β is written as

$$r_{\alpha\beta} = n_\alpha n_\beta \langle \sigma v \rangle_{\alpha\beta} \tag{2}$$

where n is the number density and $\langle \sigma v \rangle$ represents the average value of the product of the cross-section and the relative velocity between nuclei. In constructing this average, we take the relative velocity of the nuclei to obey a Maxwell-Boltzmann distribution.

Show that this quantity can be written as:

$$\langle \sigma v \rangle \propto (kT)^{-3/2} \int_0^\infty S(E) e^{-E/kT - bE^{-1/2}} dE \tag{3}$$

where T is the temperature of the gas of nuclei.

To good approximation, we can take $S(E) \approx S_0$ to be constant. With this simplification, find the energy, E_0, where the reaction rate is maximum? (This is called the Gamow peak).

d. (5 pts.) A standard practice is to approximate this integral by, first, replacing the integrand with a Gaussian, and, second, extending the lower integration limit to $-\infty$.

In particular, we write

$$e^{-E/kT - bE^{-1/2}} \approx C e^{-[(E-E_0)/(\Delta/2)]^2} \tag{4}$$

Find the C such that the maxima of the two functions agree, and the Δ such that the two functions have the same curvative at $E = E_0$.

Perform the integral with these approximations and show that the rate can be written as:

$$r \approx n_\alpha n_\beta S_0 \frac{e^{-aT^{-1/3}}}{T^{2/3}} \tag{5}$$

(you may wish to recall that $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\pi/a}$.

In the CNO cycle, the reaction $^{14}\mathrm{N}(p,\gamma)^{15}\mathrm{O}$ is the rate-limiting step, so we can take the overall rate for this cycle to be that of this single reaction. Using the formalism above, express the rate for this reaction as

$$r \approx r_0 (T/T_0)^\nu, \tag{6}$$

and find ν for a temperature consistent with the point where the CNO cycle begins to dominate over the pp-chain in a main-sequence star.

(If you express T in units of $T_6 = T/10^6$ K, then $a = 42.49(Z_\alpha^2 Z_\beta^2 \mu)^{1/3}$).

Solution

a. Massive stars achieve higher central temperatures, from the weight of the overlying layers. These higher temperatures allow for heavier nuclei to overcome the Coulomb barrier, so the reactions like $^{12}\mathrm{C} + \mathrm{p}$ can take place.

   The pp-chain is limited by the first step $\mathrm{p} + \mathrm{p} \to \mathrm{d} + \mathrm{e}^+ + \nu$. This involves the weak interaction and is slow, so if one can overcome the Coulomb barrier for CNO to take place, it will outpace the pp-chain.

b. $e^{-bE^{-1/2}}$ represents the Coulomb barrier penetration factor due to tunnelling. E^{-1} is the physical cross section using the de Broglie wavelength.

c. The M-B distribution in terms of the relative velocity appears as

$$\phi(v)dv = 4\pi v^2 \left(\frac{\mu}{2\pi kT}\right)^{3/2} e^{-\mu v^2/2kT} dv \tag{7}$$

   Taking $E = \mu v^2/2$, we can write this in terms of center-of-mass energy as

$$\phi(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} dE \tag{8}$$

   and we have

$$\begin{aligned} \langle \sigma v \rangle &= \int_0^\infty \sigma(E) v(E) \phi(E) dE && (9) \\ &= \int_0^\infty \left[\frac{S(E)}{E} e^{-bE^{-1/2}}\right] \left(\frac{2E}{\mu}\right)^{1/2} \left[\frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT}\right] dE && (10) \\ &\approx \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{-E/kT - bE^{-1/2}} dE && (11) \end{aligned}$$

Look at the integrand and find the maximum:

$$\frac{d}{dE}\left(e^{-E/kT-bE^{-1/2}}\right) = \left(-\frac{1}{kT}+\frac{b}{2}E^{-3/2}\right)e^{-E/kT} = 0 \tag{12}$$

This gives

$$E_0 = \left(\frac{bkT}{2}\right)^{2/3} \tag{13}$$

d. Taking

$$e^{-E/kT-bE^{-1/2}} \approx Ce^{-[(E-E_0)/(\Delta/2)]^2} \tag{14}$$

For the maxima to agree at $E = E_0$ we need

$$C = e^{-E_0/kT-bE_0^{-1/2}} = e^{-3E_0/kT} \tag{15}$$

The second derivatives at $E = E_0$ are:

$$\frac{d^2}{dE^2}(\text{Gamov}) = e^{-3E_0/kT}\left(-\frac{8}{\Delta^2}\right) \tag{16}$$

and

$$\frac{d^2}{dE^2}(\text{Gaussian}) = -\frac{3}{2}\frac{1}{kTE_0} \tag{17}$$

Equating gives:

$$\Delta = \frac{4}{\sqrt{3}}(E_0kT)^{1/2} \tag{18}$$

Our integral is now

$$r_{\alpha\beta} \approx n_\alpha n_\beta \frac{1}{(kT)^{3/2}} S_0 e^{-3E_0/kT}\int_{-\infty}^{\infty} e^{-[4(E-E_0)^2/\Delta^2]}dE \tag{19}$$

$$= n_\alpha n_\beta \frac{1}{(kT)^{3/2}} S_0 e^{-E_0/kT}(E_0kT)^{1/2} \tag{20}$$

Substituting in $E_0 = (bkT/2)^{2/3}$, we find

$$r_{\alpha\beta} \approx n_\alpha n_\beta \frac{S_0}{(kT)^{3/2}}\frac{e^{-aT^{-1/3}}}{T^{2/3}} \tag{21}$$

with $a = 3b^{2/3}/(4k)^{1/3}$.

$$\nu = \left.\frac{\partial \ln r}{\partial \ln T}\right|_\rho = \frac{\partial}{\partial \ln T}\left[-\frac{2}{3}\ln T - aT^{-1/3}\right] = -\frac{2}{3} + \frac{a}{3T^{1/3}} \tag{22}$$

Taking $Z_\alpha = 7$, $Z_\beta = 1$ for our reaction and $\mu = 14/15$, we find $\nu = 18$ for a temperature $T_6 = 20$ typical of the CNO cycle.

AMO 1

Stimulated Adiabatic Rapid Passage (STIRAP) is an attractive, counterintuitive technique to achieve a complete transfer of atomic or molecular population to thermally unpopulated levels using two lasers. Consider the three-level system shown in figure 1, with eigenstates $|1\rangle$, $|2\rangle$ and $|3\rangle$, and in which initially only $|1\rangle$ is populated. The task is to transfer the population from $|1\rangle$ to $|3\rangle$, using two laser fields. The first field ("pump laser") couples states 1 and 2, and the second one ("Stokes laser") couples states 2 and 3. The coupling strength associated with each laser field varies with time and is given by the respective Rabi frequency $\Omega_k(t) = \mu_k E_k(t)/\hbar$, where μ_k is the relevant dipole matrix element and $E_k(t)$ is the time-dependent amplitude of the Stokes (k=s) or Pump (k=p) laser field: $E_k(t)\frac{(e^{i\omega_k t}+e^{-i\omega_k t})}{2}$. Consider the case of 'two photon resonance', for which the laser detuning for both transitions is the same and is given by Δ.

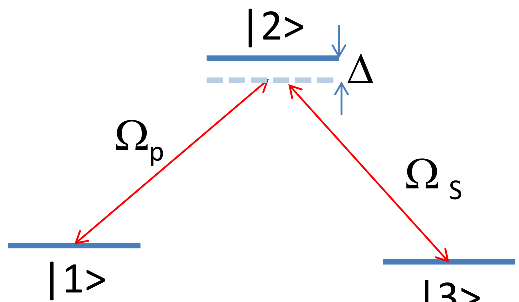


Figure 1:

a. (5 pts) If the energies of the three states are given by E_1, E_2 and E_3, write down a 3×3 Hamiltonian H describing the coupled three-level system.

b. (5 pts) One can simplify the problem by transforming to a rotating frame, in which the Hamiltonian contains only Ω_1,Ω_2 and Δ. Derive this Hamiltonian by making use of the rotating wave approximation and the time-dependent unitary transformation:

$$\mathbf{U} = \begin{pmatrix} e^{i\omega_p t}e^{-i\omega_2 t} & 0 & 0 \\ 0 & e^{-i\omega_2 t} & 0 \\ 0 & 0 & e^{i\omega_s t}e^{-i\omega_2 t} \end{pmatrix}. \tag{1}$$

c. (5 pts) In the rotating frame, what are the eigenstates of the coupled Hamiltonian for the resonant case $\Delta = 0$?

d. (5 pts) To achieve the desired population transfer, naively one would start by applying the pump laser to transfer population to the intermediate level 2 (e.g. using a π pulse), followed by the Stokes laser to complete the transfer to the level 3. However, it turns out that a much more efficient strategy is to apply the fields in *reverse order*, with the Stokes laser pulse preceding (albeit overlapping) the pump laser. Discuss why this counterintuitive strategy works, based on the adiabatic time evolution of the coupled system. Why is the transfer insensitive to the shape and duration of the pulses?

Solution

a.

$$\mathbf{H} = \begin{pmatrix} E_1 & -\frac{\Omega_p e^{i\omega_p t}+c.c}{2} & 0 \\ -\frac{\Omega_p e^{-i\omega_p t}+c.c.}{2} & E_2 & -\frac{\Omega_s e^{-i\omega_s t}+c.c.}{2} \\ 0 & -\frac{\Omega_s e^{i\omega_s t}+c.c.}{2} & E_3 \end{pmatrix}. \tag{2}$$

b.

$$\mathbf{H_I} = \mathbf{U^{-1}HU} - i\hbar\mathbf{U^{-1}\dot{U}} = \frac{-\hbar}{2}\begin{pmatrix} -2\Delta & \Omega_p & 0 \\ \Omega_p & 0 & \Omega_s \\ 0 & \Omega_s & -2\Delta \end{pmatrix} \tag{3}$$

c.

$$|\mathbf{H} - \lambda\mathbf{I}| = 0 \tag{4}$$

For Δ=0,$\lambda = 0, \pm\hbar/2\sqrt{\Omega_s^2 + \Omega_p^2}$

$$\begin{pmatrix} 0 & \Omega_p & 0 \\ \Omega_p & 0 & \Omega_s \\ 0 & \Omega_s & 0 \end{pmatrix} \tag{5}$$

$$\overrightarrow{V} = \lambda\overrightarrow{V} \tag{6}$$

$$\overrightarrow{V} = \frac{1}{\sqrt{\Omega_p^2 + \Omega_s^2}}(\Omega_s|1\rangle - \Omega_p|3\rangle) \tag{7}$$

$$\frac{1}{\sqrt{2(\Omega_p^2 + \Omega_s^2)}}(\Omega_p|1\rangle - \Omega_s|3\rangle) \mp \frac{|2\rangle}{\sqrt{2}} \tag{8}$$

d. If one starts with the Stokes pulse first, then state $|1\rangle$ corresponds to the first dressed state. Then if one adiabatically turns on the pump pulse while the Stokes is on, then one stays in this first dressed state which is now a mixture of $|1\rangle$ and $|3\rangle$. If one adiabatically turns off the Stokes pulse, then this dressed state evolves to be state $|3\rangle$, which means that the systems has evolved from state $|1\rangle$ to state $|3\rangle$ with 100% efficiency without significant probability of being in state $|2\rangle$. This process is not sensitive to the exact values of the Stokes and Pump fields because as long as one maintains adiabaticity, the system stays in the same dressed state and always ends up in state $|3\rangle$ if it started in $|1\rangle$.

AMO 2

The interaction between laser light and atoms is often described in terms of the electric dipole approximation.

a. (4 pts.) Give a short qualitative description of the dipole approximation. Verify that the dipole approximation is justified, taking the example of radiation corresponding to the Lyman series incident on ground-state hydrogen (Rydberg constant: $R_\infty \approx 1.097 \times 10^7 \mathrm{m}^{-1}$).

b. (8 pts.) In the Hamiltonian for the free hydrogen atom $\hat{H}_0 = \frac{1}{2m}\hat{\vec{p}}^2 + V(r)$, explicitly include the interaction with a weak, monochromatic plane electromagnetic wave $(\omega, \vec{k})$ and perform the dipole approximation. Derive the electric-dipole Hamiltonian $\hat{H}'(t)$ and express its matrix elements in terms of the transition dipole matrix elements and the electric field of the wave.

c. (4 pts.) The dipole approximation does not permit all transitions among the sublevels of two states even if the light frequency satisfies conservation of energy between them. Why not? What kind of transitions are permitted? Consider the $n = 2 \to n = 3$ transition of hydrogen. Draw an energy level diagram and indicate which transitions are allowed. You can neglect only the role of the nucleus.

d. (4 pts.) Give a rough estimate for the magnitude of the transition dipole matrix element for the $1S \to 2P$ (Lyman-α) transition of hydrogen at $\lambda = 121.6$ nm (no calculation needed). With recent laser technology, it is possible to generate continuous-wave Lyman-α light at powers ~ 10 nW (cf. eg. J. Walz et al, Nuclear Physics A **692**, 163c (2001)). Assuming that the light can be focused to a Gaussian spot with a $1/e^2$ radius of 100 μm, what resonant Rabi frequency would this result in for a trapped hydrogen atom located at the center of the beam?

Solution

a. The wavelength of light is very much larger than the spatial extent of the wave function of an atomic electron, so the variation of the optical field over the atomic coordinates is negligibly small. The approximation consists of assuming the field to be uniform. For case of hydrogen, we have Bohr radius $a_0 = 0.5\ n$m and $\lambda > 91.2$nm for the Lyman series (use Rydberg formula, with $n' = 1$), so the assumption is well justified.

b. The plane-wave description of the e.m. wave is linked to the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, with vector potential $A = \vec{\epsilon} A_0 \exp[i(\vec{k} \cdot \vec{r} - \omega t) + cc.]$. The interaction with the wave is included in the Hamiltonian by replacing $\hat{\vec{p}} \to \hat{\vec{p}} - q\vec{A}$, where $q = -e$ is the charge of the bound electron. Expanding to first order in $|A|$ leads to $\hat{H} = \hat{H}_0 + \hat{H}'(t)$, with

$$\hat{H}' = \frac{-q}{2m}\left(\hat{\vec{p}} \cdot \vec{A} + \vec{A} \cdot \hat{\vec{p}}\right) = \frac{-q}{m}\, \hat{\vec{p}} \cdot \vec{A} \qquad (1)$$

where the second equality sign comes from $\vec{\nabla} \cdot \vec{A} = \vec{A} \cdot \vec{\nabla}$ in the Coulomb gauge. The dipole approximation consists of setting $\exp[i(\vec{k} \cdot \vec{r})] \approx 1$ (the COM of the atom is at the origin). Furthermore, from $\vec{E} = -\partial_t \vec{A}$, we have $A_0 = iE_0/2\omega$, and thus

$$\hat{H}' = \frac{-iqE_0}{2m\omega}\ \vec{\epsilon} \cdot \hat{\vec{p}}\ \left(e^{-i\omega t} - e^{i\omega t}\right) = \frac{qE_0}{2m\omega}\ \vec{\epsilon} \cdot \hat{\vec{p}}\ \sin\omega t \tag{2}$$

To evaluate the matrix elements of H', we exploit the Heisenberg equation of motion, $i\hbar\hat{\vec{p}}/m = [\vec{r}, \hat{H}_0]$, with which we obtain $\langle k|\hat{\vec{p}}|l\rangle = im\omega_{lk}\langle k|\vec{r}|l\rangle$, where $\hbar\omega_{lk} \equiv (E_l - E_k)$. therefore,

$$H'_{kl} = -i\frac{\omega_{lk}}{\omega}\langle k|\vec{\epsilon} \cdot q\vec{r}|l\rangle E_0\ \sin\omega t \tag{3}$$

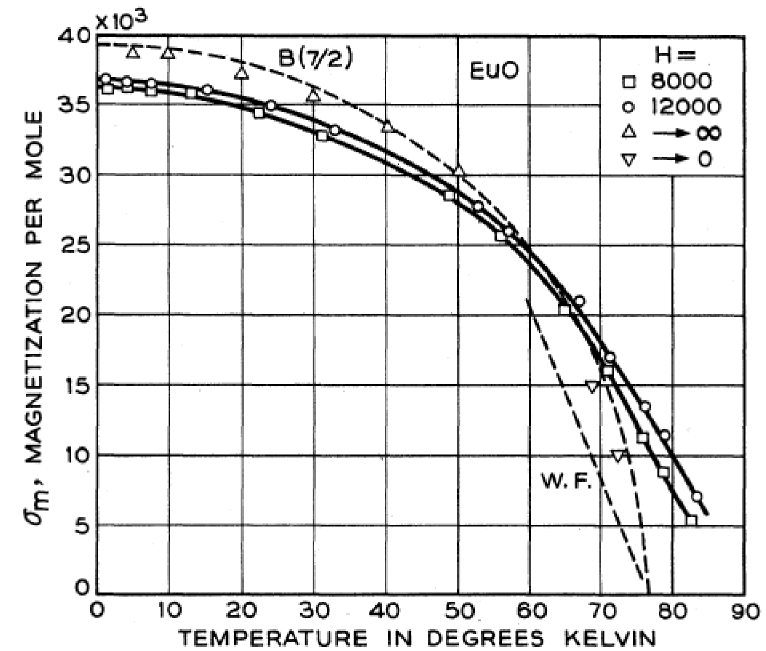
where $\hat{\vec{d}} = q\vec{r}$ is the dipole operator, and $d_{kl} \equiv \langle k|\vec{\epsilon} \cdot q\vec{r}|l\rangle$ the transition dipole matrix element.

c. The selection rules are $\Delta\ell = \pm 1$ and $\Delta m = \pm 1, 0$ depending on the polarization with respect to the chosen z-axis. When there is fine structure, as there is in this question, $\Delta J = \pm 1, 0$ as well as ΔM_J ($M = 0- > 0$ is forbidden for $\Delta J = 0$). etc. I'll draw and scan the diagram if the problem is accepted.

d. The matrix element $d_{12} \sim ea_0$. For a Gaussian beam, we have an intensity of $I_0 = 2P/w_0^2$ on axis, where w_0 is the $1/e^2$ radius, and a corresponding electric-field amplitude E_0 given by $I_0 = (1/2)\epsilon_0 c E_0^2$. The Rabi frequency is given by $\Omega = d_{12}E_0/\hbar$. Plugging in numbers yields $\Omega/(2\pi) \sim 50$ kHz.

Magnetism in EuO

a. (5 pts) By reference to an isolated Eu^{2+} ion, explain why you might guess for EuO a magnetization of 7 μ_B per molecule (μ_B = 0.927 x 10^{-20} erg/gauss or 0.927 x 10^{-23} J/T is the Bohr magneton.) Compare this guess with the measured numerical value of the T=0 magnetization.



This graph is magnetization versus temperature for europium oxide, measured by B. T. Matthias, R. M. Bozorth, and J. H. Van Vleck, Phys. Rev. Letters **7**, 160 (1961). The vertical axis is in cgs units [erg per Gauss per mole of EuO]. EuO has the NaCl crystal structure (cubic) with a lattice constant of 5.14 Å. It is an electrical insulator. The electron configuration of a free Eu atom is $(Xe)4f^7 5d^0 6s^2$.

b. (5 pts) According to the simplest model – microscopic mean field ("Weiss molecular field") or macroscopic Landau – what is the expected temperature-dependence of the magnetic susceptibility χ, for temperatures higher than the Curie temperature T_c? Explain the basic idea of the microscopic mean field approach.

c. (5 pts) One interaction that couples europium spins is the magnetic dipole-dipole interaction. Calculate the energy of interaction of a nearest neighbor Eu-Eu pair in EuO. Compare the value with $k_B T_c$ and discuss.

d. (5 pts) The expected temperature dependence for the low-T specific heat of EuO is $C(T)=AT^{3/2}$. Explain in detail how this 3/2 power law can be explained as thermal excitation of spin waves with dispersion law $\omega \propto Q^2$.

Solution

a. *By reference to an isolated Eu^{2+} ion, explain why you might guess for EuO a magnetization of 7 μ_B per molecule (μ_B = 0.927 x 10^{-20} erg/gauss or 0.927 x 10^{-24} J/T is the Bohr magneton.)*

The free ion Eu^{2+} has configuration $4f^7$, that is, a half-filled f-shell. Hund's rule tells us that these 7 f-electrons strongly prefer parallel spins, so S=7/2. There is a significant energy gap to the excited state with S=5/2. Using g=2, one guesses a moment of *7* μ_B.

Compare this guess with the measured numerical value of the T=0 magnetization.

One mole will then have magnetization $7N_A\mu_B$, where N_A is Avogadro's number, 6.02 x 10^{23} Eu atoms per mole EuO. This gives the molar magnetization (denoted σ_m in the figure) to be 40,000 erg/(Gauss mole), closely agreeing with the measurement.

b. *According to the simplest model – microscopic mean field ("Weiss molecular field") or macroscopic Landau – what is the expected temperature-dependence of the magnetic susceptibility χ, for temperatures higher than the Curie temperature T_c? Explain the basic idea of the microscopic mean field approach.*

The Weiss molecular field theory starts with the picture that there is a near-neighbor coupling of moments, $-J\mu\mu'$ or $-J\vec{\mu}\cdot\vec{\mu}'$ (vector Heisenberg or scalar Ising is not important at this level.) The key idea of mean-field theory is to ignore fluctuations. At any particular time, a spin moment μ feels an effective field from its neighbors' spins, and mean-field theory uses only the mean value of this effective field. The Hamiltonian becomes $H_{\rm MF} = -\mu B_{\rm eff} = -zJ\mu\langle\mu\rangle$ where z is the number of near neighbor spins. For a free moment in a magnetic field, elementary statistical mechanics gives the Curie law $\chi = C/T$. When the effective field term is added, the same math applies, and changes the result to $\chi = C/(T - T_c)$ at temperatures $T > T_c$. In words, the inverse susceptibility is predicted to go to zero linearly with temperature, vanishing at T_c.

c. *One interaction that couples europium spins is the magnetic dipole-dipole interaction. Calculate the energy of interaction of a nearest neighbor Eu-Eu pair in EuO. Compare the value with k_BT_c and discuss.*

Omitting angular dependence, the magnetic dipole-dipole interaction in cgs units will lower the system's energy by $\Delta E \approx -z\mu\mu/R^3$ where R is the nearest neighbor spacing, $5.14\,\mathring{A}/\sqrt{2}$. [In SI units, use $\Delta E \approx -(\mu_0/4\pi)z\mu\mu/R^3$.] There are 6 nearest neighbors, so $\Delta E \approx$-1.1x10^{-17} erg. This corresponds to a temperature about 8K, 10 times lower than the measured Curie temperature. And, in fact, when angular dependence is included, the energy is much reduced, even when correcting for the long range of the interaction. Dipole-dipole interactions have marginal importance.

The true mechanism is not obvious, and may not be totally understood, but is definitely connected with the Pauli principle, the Coulomb interaction, and quantum deconfinement, but not as a simple extension of Hund's rules. The term "exchange interaction" is widely used, but doesn't really account for the physics except in ferromagnetic metals.

d. *The expected temperature dependence for the low-T specific heat of EuO is* $C(T)=AT^{3/2}$. *Explain in detail how this 3/2 power law can be explained as thermal excitation of spin waves with dispersion law* $\omega \propto Q^2$.

The usual low T specific heat of solids is $C(T) = AT + BT^3$. The term AT comes from electrons excited from below to above the Fermi level. But EuO is not a metal, so this term should be zero. The BT^3 term comes from excitation of long-wavelength phonons, with an energy spectrum $\omega = v|Q|$. This term must also occur in EuO, but at low T, apparently there is a bigger term. This must come from a different kind of excitation, spin waves. You should derive the formula for bosonic excitations like phonons and spin waves:

$$C(T) = \int_0^{\omega_{\max}} \mathrm{d}\omega D(\omega) c(\hbar\omega / k_B T),$$

where $c(\hbar\omega/k_B T)$ is the specific heat of a boson mode with frequency ω, and $D(\omega)$ is the density of states of the bosonic excitation. To be completely specific, the formula is $c(x) = k_B[(x/2)/\sinh(x/2)]^2$, but a full derivation is not necessary. The value of $c(x)$ is exponentially small except for values of $\hbar\omega$ smaller than $2k_BT$. Therefore, for low T, we need to know only the low frequency behavior of $D(\omega)$. To derive the density of states from the dispersion law, you can use

$$D(\omega) = \sum_Q \delta(\omega - \omega_Q) = \frac{V}{(2\pi)^3} \int d^3Q\, \delta(\omega - \omega_Q).$$

Then using $\omega \propto Q^2$, you can find $D(\omega) \propto \omega^{1/2}$. This immediately gives the 3/2 power of T for the low T specific heat.

Condensed Matter 2

Plutonium is one of the most complex metals in the periodic table, due to its 5-f electrons and their strong electronic correlations. The phase diagram of Pu presents six different phases as a function of temperature, at ambient pressure. The ground state phase, so called $\alpha - Pu$, is monoclinic, and has the lowest symmetry for a metal in the periodic table. The $\delta - Pu$ phase is stable in the range $310 < T < 450$ K, but can be stabilized at room T when alloyed with a small amount of Ga. Its crystalline structure is FCC.

a. (5 pts.) Figure 1 shows the phonon density of states for both $\alpha - Pu$ and $\delta - Pu$ at room temperature. Estimate the molar heat capacity for both systems. (Note: although there is 2% of Ga in $\delta - Pu$ you may treat it as pure Ga).

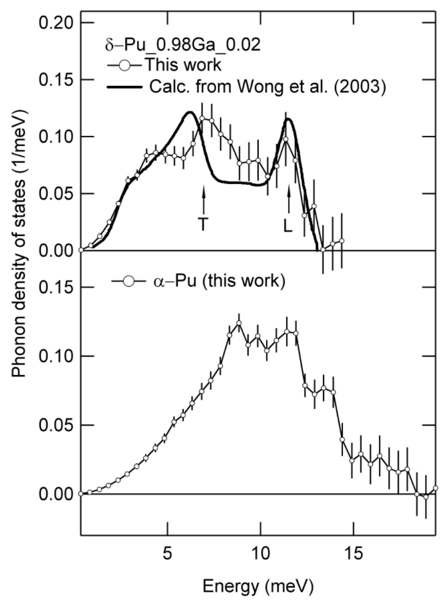


Figure 1: Room T phonon density of states for $\alpha - Pu$ and $\delta - Pu$. From *Phys. Rev. B* **79**, 052301 (2009)

b. (5 pts.) Derive the following equation, the thermal energy density of the ideal electron gas,

$$u = u_o + \frac{\pi^2}{6}(k_B T)^2 g(e_F) \tag{1}$$

where u_0 is the energy density of the ideal gas at T=0 and $g(e_F)$ is the density of states at the Fermi level. The equation is needed for the following part of the problem.

Hint: The Sommerfeld expansion is:

$$\int_0^\infty H(e)f(e)de = \int_0^\mu H(e)de + \frac{\pi^2}{6}(k_B T)^2 H'(\mu) + O(T^4) \tag{2}$$

where μ is the chemical potential and $f(e)$ is the Fermi-Dirac distribution.

c. (5 pts.) Compute the electronic density of states at the Fermi level for both $\alpha - Pu$ and $\delta - Pu$ using the information provided in Figure 2. The units should be (eV atom)$^{-1}$.

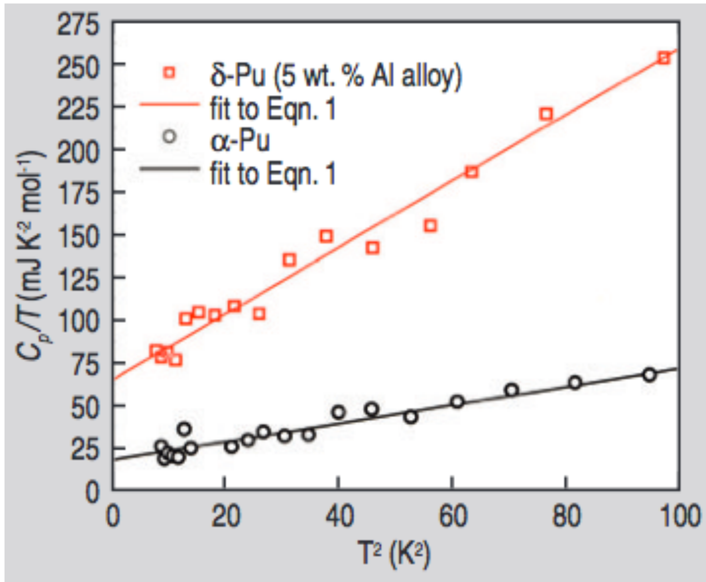


Figure 2: Low T specific heat for both $\alpha - Pu$ and $\delta - Pu$. Note the scale for Cp (mJK^2/mol). From *Journal of the Minerals, Metals and Materials Society*, **55**, pp 34-37, (2003)

d. (5 pts.) Pu is a divalent metal (2 valence electrons per atom). Assume that the free electron model can be applied to both $\alpha - Pu$ and $\delta - Pu$, but using an effective mass m^* for the electrons. Compute the ratio $\frac{m}{m^*}$ for both $\alpha - Pu$ and $\delta - Pu$. According to your results, which allotrope is closer to the free electron model? Data: $\rho_{\delta - Pu} = 16$ g/cm^3, $\rho_{\alpha - Pu}$=20 g/cm^3, $M_{Pu} = 244$ g/mol.

Solution

a. Figure 1 shows that the Debye temperature of each of these two allotropes is certainly bellow the maximum phonon energy, which is under 15 meV in both. $\Theta_D < 15$ meV/k_B = 170 K. Therefore both systems are fairly classical at room T. This means that we can estimate the specific heat using the Dulong-Petit formula. Ignoring anharmonicity effects (thermal expansion) and electronic contributions, we should expect that the specific heat should be similar for both and of of the order of $C_p \approx 3R \approx 25$ J/mol K for each of them. (I used $R = 8.31$ J/mol K). Indeed the actual result at room T is $C_p(\alpha - Pu) \approx 35$ J/mol K and $C_p(\alpha - Pu) \approx 30$ J/mol K.

b. To derive the requested equation all we need to do is to apply the Sommerfield expansion to:

$$u = \int_{\infty}^{\infty} g(e) e f(e) d(e) \tag{3}$$

with $g(e) = \frac{3n}{2e_F^{3/2}}\sqrt{e}$.

We also need to compute the temperature dependence of the chemical potential using:

$$n = \int_0^\infty g(e)f(e)d(e) = \tag{4}$$

$$\int_0^\mu g(e)de + \frac{\pi^2}{6}(k_BT)^2 g'(e_F) = \tag{5}$$

$$\int_0^{e_F} g(e)de + \int_{e_F}^\mu g(e)de + \frac{\pi^2}{6}(k_BT)^2 g'(e_F) = \tag{6}$$

$$n + (\mu - e_F)g(e_F) + \frac{\pi^2}{6}(k_BT)^2 g'(e_F). \tag{7}$$

$$\mu = e_F + \frac{\pi^2}{6}(k_BT)^2 \frac{g'(e_F)}{g(e_F)} = \tag{8}$$

$$\mu = e_F - \frac{\pi^2}{12e_F}(k_BT)^2. \tag{9}$$

Note that in the terms which depend on T^2, we can safely replace already μ by e_F. As the chemical potential μ differs by its $T = 0$ value (e_F) by terms of the order T^2, the Sommerfield integral can be approximated by:

$$\int_0^\infty H(e)f(e)de \approx \int_0^{e_F} H(e)d(e) + (\mu - e_F)H(e_F) + \frac{\pi^2}{6}(k_BT)^2 H'(\mu)|_{e=e_F}. \tag{10}$$

Finally:

$$u = \int_0^\infty g(e)ef(e)de = \tag{11}$$

$$\int_0^\infty e\frac{3n}{2E_F^{3/2}}\sqrt{e}f(e)de = \tag{12}$$

$$= \frac{3n}{2e_F^{3/2}}(\frac{2e_F^{5/2}}{5} + (\mu - e_F)e_F^{3/2} + \frac{\pi^2}{6}(k_BT)^2\frac{3}{2}\sqrt{e_F}) = \tag{13}$$

$$= \frac{3}{5}ne_F + \frac{3n}{2e_F^{3/2}}(-\frac{\pi^2}{12e_F}(k_BT)^2 e_F^{3/2} + \frac{\pi^2}{4}(k_BT)^2\sqrt{e_F}) = \tag{14}$$

$$= \frac{3}{5}ne_F + \frac{n\pi^2}{4e_F}(k_BT)^2 = u_0 + \frac{\pi^2}{6}(k_BT)^2 g(e_F). \tag{15}$$

c. At low temperatures $C_p = \gamma T + \alpha T^3$, with the first term being the electronic contibution and the second the lattice contribution. Therefore the intersect of the two lines in figure 2 will give us γ.

To calcuate the relationship between γ and $g(e_F)$, we need to derive the specific heat as $c_p = (\frac{\partial u}{\partial T})_n$. We obtain: $c_v = \frac{\pi^2}{3}k_B^2 T g(e_F)$.

- $\alpha{-}Pu$ $\gamma \approx 20$ mJ/(mol K^2). We solve for $g(e_F) = \frac{c_p}{T}\frac{\pi^2}{3k_b^2} = \frac{20\text{mJ/(molK}^2)}{300\text{K}*1.38*10^{-23}\text{J/K}}\frac{\pi^2}{3*(1.38*10^{-23}\text{J/K})^2} \approx 3.3*10^{43}\text{mol}^{-1}\text{J}^{-1}$.
- $\delta{-}Pu$ $\gamma \approx 65$mJ/(molK^2). We solve for $g(e_F) = \frac{c_p}{T}\frac{\pi^2}{3k_b^2} = \frac{65\text{mJ/(molK}^2)}{300\text{K}*1.38*10^{-23}\text{J/K}}\frac{\pi^2}{3*(1.38*10^{-23}\text{J/K})^2} \approx 10^{44}\text{mol}^{-1}\text{J}^{-1}$.

To convert to eV/atom we multiply by $e = 1.6*10^{-19}$eV/J and divide by $N_A = 6.02*10^{23}$ atom/mol to obtain: $g(e_F)$, $\alpha - Pu = 9.3$ eV^{-1} atom^{-1} and $g(e_F)$, $\delta - Pu = 28$ eV^{-1} atom^{-1}

d. We use the well known equations of the free electron model, replacing m by m^* (they are trivial to derive) $g(e_F) = \frac{3}{2}\frac{n}{e_F}$ and $e_F = \frac{\hbar^2 k_F^2}{2m^*} = \frac{\hbar^2}{2m^*}(3\pi^2 n)^{2/3}$. We need to obtaing n, the density of free electrons, knowing that we have $Z = 2$ valence electrons/atom.

   To compute $n = ZN/V$ we use $n = \frac{ZNa\rho}{M}$. So $e_F = \frac{\hbar^2 k_F^2}{2m^*} = \frac{\hbar^2}{2m^*}(3\pi^2 Z\frac{Na\rho}{M})^{2/3}$.

   So the solution is $\frac{m^*}{m} = \frac{2\hbar^2 g(e_F)}{6Zm}(3\pi^2 Z\frac{Na\rho}{M})^{2/3}$, which results in appromimately 60 for $\delta - Pu$ and 25 for $\alpha - Pu$.

   Both allotropes have very heavy fermions, and are very far from the free electron model, but $\alpha - Pu$ is closer.

Nuclear 1

Part A.
In a heavy-ion collision of the type generated at RHIC and now at LHC, the Quark-Gluon Plasma (QGP) is produced by colliding two large nuclei at very high energy, with $\gamma = E/m_p c^2 \gg 1$. A simple model of the QGP consists of a weakly coupled gas of deconfined massless quarks (u, d, s) and gluons. During the nuclear collision the initially deconfined medium rapidly expands and cools, and ultimately changes phase back to its confined hadronic form consisting of mostly pions.

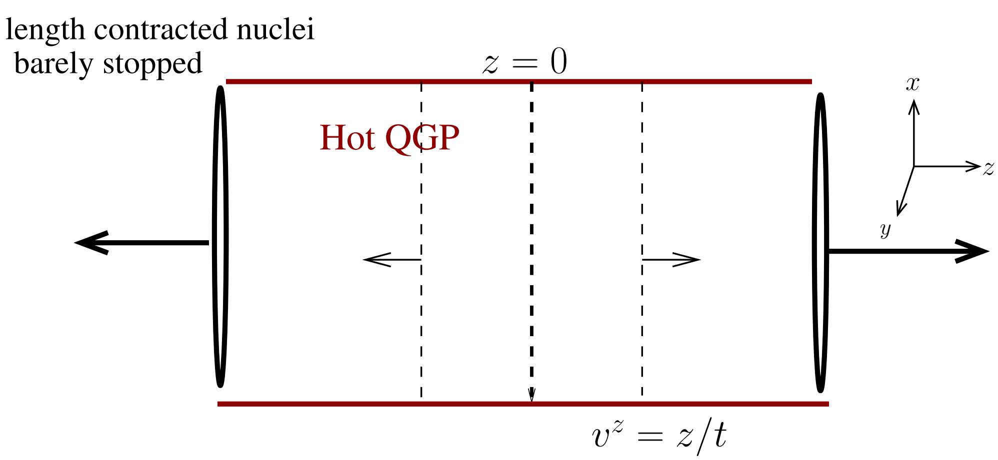
a. (5 points) Write down the pressure p, the energy-density e, and entropy-density s, for both the QGP and the pionic gas assuming the pions to be massless. (We recall that the pressure for a single component massless boson and fermion at finite temperature are $p_B = (\pi^2/90)\,T^4$ and $p_F = \frac{7}{8}(\pi^2/90)\,T^4$, respectively.)

b. (2 points) State an order of magnitude estimate for the critical temperature T_c for the hadronic-to-QGP transition.

Part B.
In the Bjorken model for the longitudinal expansion of colliding nuclei, two length contracted nuclei (of infinite transverse extent) pass through each other unstopped, leaving behind a hot QGP, which is described by ideal hydrodynamics (see figure). Close to the center of the collision zone, energy density is approximately independent of z, and is thus a function of time only, $e(t)$. The velocity of the material at position z and time t is

$$v^z = \frac{z}{t}\,,$$

and may be treated as non-relativistic close to the center.



a. (5 points) In ideal hydrodynamics the ideal stress tensor is characterized by a locally conserved energy momentum tensor defined as

$$T^{\mu\nu} = (e + p)\,u^\mu u^\nu + g^{\mu\nu}\,p\,, \tag{1}$$

where $u^\mu = (\gamma, \gamma v)$ is the four velocity of the fluid, and e and p are the energy-density and pressure. Using the equations of ideal hydrodynamics, determine how the energy density decreases with time.

b. (4 points) Determine how the temperature (T) and entropy density (s) decrease with time.

c. (4 points) Give a physical interpretation of the scalar quantity, $\partial_\mu u^\mu$, and compute this quantity for the Bjorken expansion.

Solution

Part A.

a. Using $p_B = (\pi^2/90)T^4$ as the black-body pressure for a massless boson, and counting the degrees of freedom in the quark gluon plasma, we have:

$$p_{qgp} \quad = \quad (2 \times 8 + \frac{7}{8} \times 2 \times 3 \times 2 \times 3)\, p_B = 47.5\, p_B\,. \tag{2}$$

Here the factors 2×8 are the spin$\times$color degrees of freedom for the gluons; $7/8$ accounts for the Fermi statistics of the quarks; $2 \times 3 \times 2 \times 3$ accounts for the spin of the quark, the three quark flavors (u, d, s), the anti-quarks which double the fermionic degrees of freedom, and the three colors of the quarks.

With the pressure, the energy and entropy densities follow from thermodynamics. For a massless gas, $e = 3p$, and we employ $s = dp/dT$ at zero μ:

$$e_{qgp} = 3\, p_{qgp}\,, \tag{3}$$
$$s_{qgp} = dp_{qgp}/dT = 190\, p_B/T\,. \tag{4}$$

Note, one can easily verify consistency the Gibbs-Duhem relation at zero chemical potential

$$Ts = e + p\,. \tag{5}$$

The pion gas works similarly. Here the pressure is

$$p_\pi = 3\, p_B\,, \tag{6}$$

where the factor of three accounts for the three pion species, π^+, π^-, π^0. The energy and entropy densities are

$$e_\pi = 9 p_B\,,$$
$$s_\pi = dp_\pi/dT = 12 p_B/T\,. \tag{7}$$

b. We estimate that the phase transition temperature is of order $\Lambda_{QCD} \sim 200\,\text{Mev}$. Note that the QGP pressure in Eq. 2 is higher than the corresponding pion pressure, Eq. 3. This is because we have neglected the strong interactions between the quark and gluon degrees of freedom, which reduces the pressure relative to the free gas expectation.

Part B.

a. Using $\partial_\mu T^{\mu\nu} = 0$ we have from $\nu = 0$:

$$\partial_t e + (e+p)\partial_z v^z = 0\,, \tag{8}$$

$$\partial_t e = -\frac{(e+p)}{t}\,, \tag{9}$$

where we have approximated $u^z = \gamma v^z \simeq v^z = z/t$ for non-relativistic motion.

Solving this equation with equation of state $p = \frac{1}{3}e$ (as appropriate for a massless gas) gives

$$e = e_o \left(\frac{t_o}{t}\right)^{4/3}\,. \tag{10}$$

b. This implies (since $e \propto T^4$) that the temperature decreases as

$$T = T_o \left(\frac{t_o}{t}\right)^{1/3}\,, \tag{11}$$

and the entropy density (since $s \propto T^3$) decreases as

$$s = s_o \frac{t_o}{t}\,. \tag{12}$$

c. $\partial_\mu u^\mu$ is the fractional change in volume per time, or the expansion rate. Close to the rest frame $\partial_\mu u^\mu \simeq \partial_i v^i$, and a simple picture of an expanding volume shows that

$$\partial_\mu u^\mu = \frac{1}{V}\frac{dV}{dt}\,. \tag{13}$$

For the Bjorken expansion this is

$$\partial_\mu u^\mu \simeq \partial_z v^z = \frac{1}{t}\,. \tag{14}$$

The combination of Eq. 14 and Eq. 12 shows that the total entropy $s\,V$ is constant

$$\frac{d(sV)}{dt} = 0\,, \tag{15}$$

as should be expected in ideal hydrodynamics.

Nuclear 2

The experimentally measured intrinsic magnetic dipole moments of the proton and the neutron are:

$$\mu_p = 2.792847356(23)\mu_N$$
$$\mu_n = -1.91304272(45)\mu_N$$

where $\mu_N = \frac{e\hbar}{2M_p} = 5.05078324(13) \times 10^{-27}[J/T]$ is the nuclear magneton (SI units). The content e denotes the electron charge, $\hbar$ is the Planck constant, and M_p is the proton mass.

a. (6 pts.)

(a) Calculate the magnetic dipole moment ratio in the quark model assuming SU(2) isospin symmetry and the spin/flavor wave function for a proton with spin 'up':

$$\begin{aligned} \left|p^{\uparrow}\right\rangle \sim\ & 2(|u\uparrow u\uparrow d\downarrow\rangle + |u\uparrow d\downarrow u\uparrow\rangle + |d\downarrow u\uparrow u\uparrow\rangle) \\ & -(|u\uparrow u\downarrow d\uparrow\rangle + |u\uparrow d\uparrow u\downarrow\rangle + |d\uparrow u\uparrow u\downarrow\rangle) \\ & -(|u\downarrow u\uparrow d\uparrow\rangle + |u\downarrow d\uparrow u\uparrow\rangle + |d\uparrow u\downarrow u\uparrow\rangle) \end{aligned}$$

Explain the form of this wave function.

(b) Compare the calculated - in (a) - value of μ_n/μ_p with the value determined from experimental measurements. Discuss the possible origin of a difference.

b. (6 pts.) Calculate the magnetic dipole moment of the deuteron knowing that the deuteron ground state is a mixture of an S state (96%) and a D state (4%).

c. (2 pts.) Describe the nuclear shell model. Discuss experimental evidence(s) to support the shell model approach.

d. (6 pts.) In the nuclear shell model, consider a nuclear level corresponding to a closed shell plus a single proton in a state with the angular momentum quantum numbers l and j ($j = l \pm 1/2$). Let g_p be the g-factor of a free proton. Compute the g-factor for the level in question, for each of the two cases:
(a) $j = l + 1/2$
and
(b) $j = l - 1/2$.

Solution

The magnetic dipole moments (intrinsic spin and orbital) of a nucleon can be expressed in terms of its gyromagnetic ratio or a 'g-factor':

$$\vec{\mu}^s = g_s\mu_N\frac{\vec{S}}{\hbar},\ \vec{\mu}^l = g_l\mu_N\frac{\vec{L}}{\hbar}$$

The gyromagnetic ratios for a proton and a neutron are:
a) intrinsic spin $g_s^p = 2\mu_p/\mu_N, \ g_s^n = 2\mu_n/\mu_N$
and
b) orbital momentum $g_l^p = 1, \ g_l^n = 0$

a. **Magnetic moment** μ_n/μ_p
The magnitude of the magnetic moment of a proton is the value of the z-component of μ in the spin "up" state, for which $S_z = \hbar/2$:

$$\mu_p = \left\langle p\uparrow \left| (\mu_p)_z \right| p\uparrow \right\rangle \tag{1}$$

In the quark model, the magnetic moment of a proton comes from the intrinsic dipole moments of the quarks (the dipole moment coming from the orbital motion of the quarks is zero, because l=0 in the ground state):

$$\mu_p = \langle p\uparrow |(\mu_u + \mu_u + \mu_d)_z| \, p\uparrow\rangle \tag{2}$$

where

$$\mu_u = \frac{2}{3}\frac{e\hbar}{2m_u}, \quad \mu_d = -\frac{1}{3}\frac{e\hbar}{2m_d}, \quad \text{and } m_u \simeq m_d$$

$$\begin{aligned} \mu_p &= \frac{2}{\hbar}\langle p\uparrow |(\mu_u S_{u,z} + \mu_u S_{u,z} + \mu_d S_{d,z})| \, p\uparrow\rangle \\ &= \frac{2}{\hbar}\left\langle p\uparrow \left| \left(\mu_u\frac{\hbar}{2} + \mu_u\frac{\hbar}{2} - \mu_d\frac{\hbar}{2}\right) \right| p\uparrow \right\rangle \end{aligned} \tag{3}$$

After using the given expression for the proton wave function, Eq. 3 becomes:

$$\mu_p = \frac{1}{2}\left(4\mu_u - \mu_d\right) \tag{4}$$

The neutron wave function can be written down from that for a proton by substituting all u quarks by d quarks and vice versa. The neutron magnetic moment can be calculated in a similar way as the proton magnetic moment:

$$\mu_n = \frac{1}{2}\left(4\mu_d - \mu_u\right) \tag{5}$$

In quark model, the neutron and proton magnetic moment ratio is:

$$\frac{\mu_n}{\mu_p} = -\frac{2}{3} \simeq -0.667 \tag{6}$$

and the experimentally measured value is:

$$\frac{\mu_n}{\mu_p} = -\frac{1.91304}{2.79285} \simeq -0.685 \tag{7}$$

There is a very small difference of 2.7% between quark model prediction and the experimental measurement. In the quark model calculation we have made a few assumptions which are only approximately correct. *One is the assumption that u and d quarks have the same mass. Another one is the assumption to neglect $l > 0$ states in the nucleon wave function (l=0 is true only if orbital momentum is conserved by the interactions between quarks). (check this!)*

Proton wave function

Proton spin and parity is $1/2^+$, orbital momentum is 0. In quark model, proton consists of 3 quarks (u,u,d). The proton wave function should be symmetrical under interchange of both spin and flavor between any two quarks.

$$\left| p^\uparrow \right\rangle = A\left[\psi_{12}(spin)\psi_{12}(flavor) + \psi_{23}(spin)\psi_{23}(flavor) + \psi_{13}(spin)\psi_{13}(flavor)\right] \tag{8}$$

where $\psi_{ij}(spin)$ and $\psi_{ij}(flavor)$ are antisymmetric functions in i and j :

$$\psi_{ij} = \frac{1}{\sqrt{2}}\left(\xi(i)\chi(j) - \chi(i)\xi(j)\right) \tag{9}$$

ξ and χ are single-particle wave functions.

$$\begin{aligned} \left|p^\uparrow\right\rangle \sim & \frac{1}{\sqrt{2}}\left[\uparrow(1)\downarrow(2) - \downarrow(1)\uparrow(2)\right]\uparrow(3) \times \frac{1}{\sqrt{2}}\left[u(1)d(2) - d(1)u(2)\right]u(3) \\ & + \uparrow(1)\frac{1}{\sqrt{2}}\left[\uparrow(2)\downarrow(3) - \downarrow(2)\uparrow(3)\right] \times u(1)\frac{1}{\sqrt{2}}\left[u(2)d(3) - d(2)u(3)\right] \\ & + \uparrow(2)\frac{1}{\sqrt{2}}\left[\uparrow(1)\downarrow(3) - \downarrow(1)\uparrow(3)\right] \times u(2)\frac{1}{\sqrt{2}}\left[u(1)d(3) - d(1)u(3)\right] \end{aligned} \tag{10}$$

$$\begin{aligned} \left| p^\uparrow \right\rangle \sim & \, u(1)u(2)d(3)\left[2\uparrow(1)\uparrow(2)\downarrow(3) - \uparrow(1)\downarrow(2)\uparrow(3) - \downarrow(1)\uparrow(2)\uparrow(3)\right] \\ & + u(1)d(2)u(3)\left[2\uparrow(1)\downarrow(2)\uparrow(3) - \downarrow(1)\uparrow(2)\uparrow(3) - \uparrow(1)\uparrow(2)\downarrow(3)\right] \\ & + d(1)u(2)u(3)\left[2\downarrow(1)\uparrow(2)\uparrow(3) - \uparrow(1)\downarrow(2)\uparrow(3) - \uparrow(1)\uparrow(2)\downarrow(3)\right] \end{aligned} \tag{11}$$

$$\left| p^\uparrow \right\rangle \sim \left[2u\uparrow(1)u\uparrow(2)d\downarrow(3) - u\uparrow(1)u\downarrow(2)d\uparrow(3) - u\downarrow(1)u\uparrow(2)d\uparrow(3) + permutations\right] \tag{12}$$

b. The magnetic dipole moment of a deuteron arises from the intrinsic spin magnetic moments of the proton and the neutron and orbital magnetic moment of the proton (only proton contributes to the orbital magnetic moment because neutron does not have a charge):

$$\vec{\mu}_d = g_s^p \mu_N \frac{\vec{S}_p}{\hbar} + g_s^n \mu_N \frac{\vec{S}_n}{\hbar} + g_l^p \mu_N \frac{\vec{L}_p}{\hbar} \tag{13}$$

Using $g_l^p = 1$, the spin of deuteron $\vec{S} = \vec{S}_p + \vec{S}_n$ and assuming that each of the two nucleons carries 50% of the orbital momentum $\vec{L}_p = \frac{1}{2}\vec{L}$, Eq. 13 becomes:

$$\vec{\mu}_d = \frac{\mu_N}{\hbar}\left[\frac{1}{2}\left(g_s^p + g_s^n\right)\vec{S} + \frac{1}{2}\left(g_s^p - g_s^n\right)\left(\vec{S}_p - \vec{S}_n\right) + \frac{1}{2}\vec{L}\right] \tag{14}$$

$$\begin{aligned} \langle j,m\left|\mu_{d,z}\right|j,m\rangle &= \frac{\mu_N}{\hbar^2}\frac{m}{2j(j+1)}\left\langle j,m\left|\left[\left(g_s^p + g_s^n\right)\vec{S} + \left(g_s^p - g_s^n\right)\left(\vec{S}_p - \vec{S}_n\right) + \vec{L}\right]\cdot\vec{J}\right|j,m\right\rangle = \\ &= \frac{\mu_N}{\hbar^2}\frac{m}{2j(j+1)}\left\langle j,m\left|\left[\left(g_s^p + g_s^n\right)\vec{S}\cdot\vec{J} + \vec{L}\cdot\vec{J}\right]\right|j,m\right\rangle \end{aligned} \tag{15}$$

Using:

$$\vec{S}\cdot\vec{J} = \vec{S}\cdot\left(\vec{L}+\vec{S}\right) = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 + \vec{S}^2\right), \vec{L}\cdot\vec{J} = \vec{L}\cdot\left(\vec{L}+\vec{S}\right) = \frac{1}{2}\left(\vec{J}^2 + \vec{L}^2 - \vec{S}^2\right) \tag{16}$$

$$\begin{aligned} \vec{J}^2|j,m> &= j(j+1)\hbar^2|j,m> \\ \vec{L}^2|j,m> &= l(l+1)\hbar^2|j,m> \\ \vec{S}^2|j,m> &= s(s+1)\hbar^2|j,m> \end{aligned} \tag{17}$$

$$\begin{gathered} \mu_d = \langle j,m=j\left|\mu_{d,z}\right|j,m=j\rangle = \frac{\mu_N}{\hbar^2}\frac{1}{2(j+1)}\frac{\hbar^2}{2}\times \\ \left[\left(g_s^p + g_s^n\right)\left(j(j+1) - l(l+1) + s(s+1)\right) + \left(j(j+1) + l(l+1) - s(s+1)\right)\right] \end{gathered} \tag{18}$$

Using Eq.18 we can calculate the magnetic moment of a deuteron S state (l=0, s=1, j=1) :

$$\mu_d\left({}^3S_1\right) = \frac{\mu_N}{8}\left[\left(g_s^p + g_s^n\right)(2+2) + (2-2)\right] = \frac{1}{2}\left(g_s^p + g_s^n\right)\mu_N = \mu_p + \mu_n \tag{19}$$

Using Eq.18 we can calculate the magnetic moment of a deuteron D state (l=2, s=1, j=1) :

$$\mu_d\left({}^3S_1\right) = \frac{\mu_N}{8}\left[\left(g_s^p + g_s^n\right)(-2) + 6\right] \simeq 0.310\mu_N \tag{20}$$

The wave function of a deuteron is a combination of S and D states:

$$\psi_d = a\left|{}^3S_1\right\rangle + b\left|{}^3D_1\right\rangle, \ a^2 + b^2 = 1 \tag{21}$$

Since there are no off-diagonal matrix element of $\vec{mu}$ between ${}^3S_1$ and ${}^3D_1$ states, the deuteron magnetic moment is given by:

$$\mu_d = a^2\mu_d\left({}^3S_1\right) + b^2\mu_d\left({}^3D_1\right) = 0.96\left(\mu_p + \mu_n\right) + 0.04 \cdot 0.310\mu_N \simeq 0.857\mu_N \tag{22}$$

c. (Any introductory nuclear physics textbook)

d. According to the shell model, the total angular momentum of the nucleons in a closed shell is zero, so is the magnetic moment. The magnetic moment and angular momentum of the nucleus are determined by the single proton outside the closed shell.

$$\vec{\mu}_j = \vec{\mu}_l + \vec{\mu}_s = \frac{\mu_N}{\hbar}\left(g_j\vec{J}\right) = \frac{\mu_N}{\hbar}\left(g_s\vec{S} + g_l\vec{L}\right) \tag{23}$$

$$g_j\vec{J}\cdot\vec{J} = g_s\vec{S}\cdot\vec{J} + g_l\vec{L}\cdot\vec{J}$$

Using Eq.(16) and (17):

$$g_j j(j+1) = g_l\frac{1}{2}\left(j(j+1) + l(l+1) - s(s+1)\right) + g_s\frac{1}{2}\left(j(j+1) - l(l+1) + s(s+1)\right) \tag{24}$$

For a proton: $g_l = 1$, $g_s = g_p$, $j = l \pm 1/2$
a) $j = l + 1/2$

$$g_j = \frac{2j-1}{2j} + \frac{g_p}{2j}$$

b) $j = l - 1/2$

$$g_j = \frac{1}{j+1}\left(j + \frac{3}{2} - \frac{g_p}{2}\right)$$

High Energy 1

a. (5 pts.) Write down the lowest-order Feynman diagram(s) for the reaction

$$\nu_e + e \to \nu_e + e,$$

clearly labelling the particles on each line. Give an approximate analytic expression and numerical evaluation of the total cross section for this reaction if the ν_e has a lab energy of 10 MeV, explaining the reasoning that you use. (This problem does not require detailed calculation.)

b. (5 pts.) (1) Give two pieces of experimentally based evidence that show that the color gauge group is SU(N_c) with $N_c = 3$. (2) How do we know that the color gauge group is SU(3) instead of U(3)? (3) Give a theoretical argument and an experimental proof that the electroweak gauge group is $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and not $\mathrm{SU}(2)_L$.

c. (5 pts.) Consider massive neutrinos and pretend that, for simplicity, there are only two generations, the second and third. Denote the relevant mixing angle as θ, so that $|\nu_\mu\rangle = \cos\theta|\nu_2\rangle + \sin\theta|\nu_3\rangle$ and $|\nu_\tau\rangle = -\sin\theta|\nu_2\rangle + \cos\theta|\nu_3\rangle$. Consider a beam of relativistic ν_μs produced at $t = x = 0$. Calculate an expression for the probability that one will observe a μ produced via a charged-current neutrino reaction at a detector located at a distance $x = L$ from the point where the ν_μ is produced. Assume that the neutrinos propagate through vacuum.

d. (5 pts.) (1) Explain what Bjorken scaling is in deep inelastic scattering (DIS) and why it was important for the understanding of quantum chromodynamics (QCD). Experimentally, small scaling violations are observed. (2) What is the relation between these scaling violations and the running of the coupling constant of QCD? (3) Which Feynman diagrams must be calculated to explain this running?

Solution

a. See the figures below for Feynman diagrams. There are two lowest-order diagrams, both of which are tree-level: (i) a CC (charged-current) diagram with exchange of a virtual W boson in the t channel, and (ii) a NC (neutral-current) diagram with exchange of a virtual Z boson, also in the t channel. Since the momentum transfer squared is $Q^2 << m_W^2,\ m_Z^2$, the W and Z propagators can be quite accurately approximated as constants. Let us denote the gauge couplings for the electroweak (EW) isospin $\mathrm{SU}(2)_L$ and hypercharge $\mathrm{U}(1)_Y$ gauge groups as g and g', in a normalization of Y such that the electric charge $Q_{em} = T_{3L} + (Y/2)$. The CC vertex is $(g/\sqrt{2})\gamma_\mu P_L$, where $P_L = (1-\gamma_5)/2$ is the left-handed chiral projection operator. Then the contribution to the amplitude from the CC diagram involves a prefactor is

$$\frac{g^2}{2m_W^2}[\bar{e}_L\gamma_\mu\nu_{eL}][\bar{\nu}_{eL}\gamma^\mu e_L] = \frac{g^2}{8m_W^2}[\bar{e}\gamma_\mu(1-\gamma_5)\nu_e][\bar{\nu}_e\gamma^\mu(1-\gamma_5)e]\ . \tag{1}$$

Now,

$$\frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}} \tag{2}$$

where $G_F = 1.166 \times 10^{-5}$ GeV^{-2} is the Fermi weak interaction constant. Similarly, the contribution to the amplitude from the NC diagram involves an overall prefactor $(g^2 + g'^2)/(8m_Z^2) = G_F/\sqrt{2}$. The cross section for $\nu_e + e \to \nu_e + e$ is therefore $\sigma \propto G_F^2$. Because the lab energy $E_{lab} = 10$ MeV is $>> m_e$, one can estimate the cross section simply using scaling arguments. Since one has integrated over the momentum transfer $t \equiv q^2$, the cross section can only depend on the total center-of-mass energy squared, s. Now the dimensions of the cross section are area, or equivalently, $1/(\text{mass})^2$. The dimensions of G_F^2 are $1/(\text{mass})^4$, so σ must be proportional to $G_F s$. Since this is a $2 \to 2$ reaction, the basic formula for the differential cross section involves a factor of $1/(2\pi)^2$ times the azimuthal integration, and hence a factor of $1/(2\pi)$. So we can estimate

$$\sigma \sim \frac{G_F^2 s}{2\pi} \tag{3}$$

Evaluating s, we have $s = (p_\nu + p_e)^2 = m_e^2 + 2m_e E_{lab} \simeq 2m_e E_{lab}$. Hence,

$$\sigma \simeq \frac{G_F^2 m_e E}{\pi} \tag{4}$$

Evaluating this numerically using the conversion factor $\hbar c = 2 \times 10^{-14}$ GeV-cm, we get

$$\sigma \simeq 0.9 \times 10^{-43} \text{ cm}^2 \tag{5}$$

This result is actually quite close to the actual cross section, as tabulated, e.g., in Table 8.8, p. 224, in the book J. Bahcall, *Neutrino Astrophysics*.

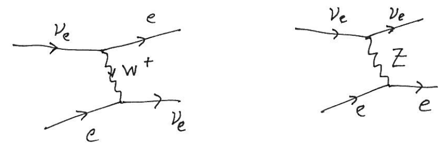
b. Part (1): A first piece of evidence comes from baryon spectroscopy. In the 1960s, one knew that the quarks forming the hadrons, and baryons in particular, were u, d, and s. The ground-state baryons included the baryon octet (adjoint representation of flavor SU(3)) including the nucleons and hyperons with spin $J = 1/2$, and the baryon decuplet (symmetric rank-2 tensor representation of flavor SU(3)) with spin $J = 3/2$. Let us focus on, say, $\Delta^{++} = (uuu)$ in the decuplet. In the simple quark model, the wavefunction for $\Delta^{++} = (uuu)$ is of the form

$$\psi = \psi_{flav.} \psi_S \psi_L \psi_C \tag{6}$$

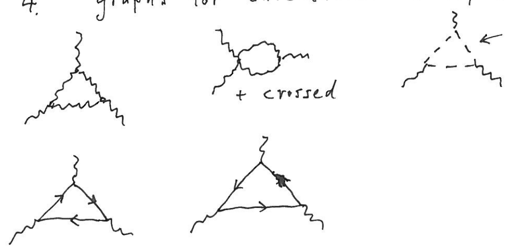
where $\psi_{flav.}$ is for flavor, ψ_S is for spin, ψ_L is for orbital angular momentum, and ψ_C is for color. Now, since the quark content is uuu, the flavor wavefunction is obviously symmetric under interchange of quarks. Since the spin is $J = 3/2$, the spin wavefunction is also clearly symmetric under interchange of quarks. Owing to the fact that the decuplet is a ground-state baryon, the orbital angular wavefunction has $L = 0$ and hence is also symmetric. But the total wavefunction ψ must be antisymmetric under exchange of any two fermions. This implies that ψ_C must be antisymmetric. The only

FIGURES FOR SOLUTIONS

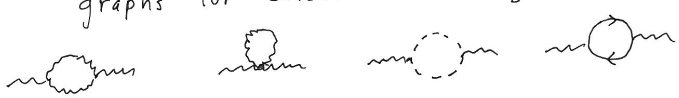
1.



4. graphs for calculation of Z_1 :



graphs for calculation of Z_3 :



$\beta = \frac{dg}{d\ln\mu}$; $g = \frac{Z_3^{3/2}}{Z_1} g_0$

antisymmetric color wavefunction that one can construct for a three-fermion bound state requires that they transform under color SU(3), and it has the form

$$\psi_C \propto \epsilon_{abc} q^a q^b q^c \tag{7}$$

where $a,\ b,\ c$ are SU(3) color indices and ϵ_{abc} is the totally antisymmetric tensor density for SU(3). This wavefunction is invariant under color SU(3) gauge transformations. Note that it would *not* be invariant under the gauge transformation of the group U(3) (which is the answer to part (3).)

A second piece of evidence that $N_c = 3$ can be explained as follows. For center-of-mass energy $\sqrt{s} >> \Lambda_{QCD}$, where $\Lambda_{QCD} \simeq 0.3$ GeV, the quantum chromodynamic gauge coupling squared α_s has become small enough that one can use perturbation theory for some quantities. One of these quantities is the total inclusive cross section $\sigma(e^+e^- \to$ hadrons) for $\sqrt{s}$ away from particle thresholds. That is, this can be approximated by $\sum_q \sigma(e^+e^- \to \bar{q}q)$, where the sum over q includes all quarks that can be produced consistent with the given $\sqrt{s}$. Henceforth, we assume $\sqrt{s} >> \Lambda_{QCD}$ and $\sqrt{s}$ is not near a particle threshold. It is convenient to consider the ratio of this cross section divided by the total cross section $\sigma(e^+e^- \to \mu^+\mu^-)$, which, for this range of $\sqrt{s}$, is, to lowest order in electroweak interactions,

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha_{em}^2}{3s} \tag{8}$$

The ratio, to lowest order in QCD and electroweak interactions, is

$$R(s) \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_q Q_q^2 \tag{9}$$

where Q_q denotes the electric charge of the quark q. In the early 1970s, experiments at the Cambridge Electron Accelerator (CEA) and at SPEAR at SLAC measured this cross section at $\sqrt{s} \sim 2$ GeV, where the sum $\sum_q$ includes the u, d, and s quarks. The electric charges of these quarks are known from hadron spectroscopy, namely $Q_u = 2/3$ and $Q_d = Q_s = -1/3$. So $\sum_q Q_q^2 = 2/3$. But the experimental data gave 2, not 2/3. This shows that $N_c = 3$.

Part (2): This was explained above; the baryon color wavefunction ψ_C would not be invariant under U(3) color gauge transformations, and hence would be ill-defined for a U(3) color group, but it is invariant under SU(3) color gauge transformations.

Part (3): There are several pieces of evidence that show that the electroweak gauge group is $G_{EW} = \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ and not just $\mathrm{SU}(2)_L$. A theoretical argument is that the electric charge Q_{em} must be expressible as a sum of the diagonal Cartan generator of $\mathrm{SU}(2)_L$, namely T_{3L} and also, necessarily, another quantity, since, for example, a right-handed e, u, or d has $T_{3L} = 0$ but not charge zero. But this automatically implies that the EW gauge group must be G_{EW} as above, and not just $\mathrm{SU}(2)_L$.

A closely related theoretical argument is that $\mathrm{SU}(2)_L$ has order 3 (order = number of infinitesimal generators of the corresponding Lie algebra) and hence 3 gauge bosons. But even before the actual production and detection of the W^+, W^-, and Z, one knew that there were four EW gauge bosons, namely the photon and the above three vector bosons, $W^\pm$ and Z. This was known because there had been observations and measurements of (i) electromagnetic interactions (mediated by the photon), (ii) charged-current weak decays and reactions (mediated by the $W^\pm$), and (iii) as of 1973, neutral-current weak reactions (mediated by the Z. The charged-current and neutral-current interactions involved exchanges of heavy vector bosons, as described by the electroweak theory which later was confirmed as part of the so-called Standard Model. There is no way that $\mathrm{SU}(2)_L$, by itself, could explain all three of these interactions.

The direct experimental proof came in 1982-1983 with the production and observation of the decays of the $W^\pm$ and Z at the CERN $\bar{p}p$ collider.

c. The $|\nu_\mu\rangle$ state at $t = x$ is

$$|\nu_\mu(t)\rangle = \cos\theta e^{-iE_2 t}|\nu_2\rangle + \sin\theta e^{-iE_3 t}|\nu_3\rangle \tag{10}$$

where $E_j = \sqrt{p^2 + m_j^2}$, with m_j denoting the mass of the j'th neutrino mass eigenstate. Here $p = |\vec{p}|$ is effectively the same for all of the mass eigenstates since $p >> m_j$ for all j. For the same reason, we can approximate

$$E_j \simeq p + \frac{m_j^2}{2p} \tag{11}$$

to very good accuracy. Extracting a common factor e^{-ipt}, we then have

$$|\nu_\mu(t)\rangle = e^{-ipt}\left[\cos\theta e^{-im_2^2 t/(2p)}|\nu_2\rangle + \sin\theta e^{-im_3^2 t/(2p)}|\nu_3\rangle\right] \tag{12}$$

We next calculate the Fock-space inner product $\langle\nu_\mu|\nu_\mu(t)\rangle$ and square the modulus to get the probability that at $t = x$, the incident $\nu_\mu(t)$ is $|\nu_\mu\rangle$ and hence will produce a μ via a charged-current reaction. We find, for $x = L$,

$$|\langle\nu_\mu|\nu_\mu(t)\rangle|^2 = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{32}^2 L}{4E}\right) \tag{13}$$

where

$$\Delta m_{jk}^2 \equiv m_j^2 - m_k^2 \tag{14}$$

and we have set $p = E$ in the final expression because of the relativistic nature of the propagation.

d. Bjorken scaling can be explained as follows. Consider the DIS reaction $e+N \to e+X$, with an incident electron of lab energy E and a target nucleon N (p or n). Denote the four-momenta of the incident and scattered electron as ℓ and ℓ', and the four-momentum of the target nucleon as p. Further, denote the momentum transfer four-momentum as $q = \ell - \ell'$. The center-of-mass energy is $s = (\ell + p)^2 = m_N^2 + 2m_N E$ (where we neglect m_e^2 because $E >> m_N >> m_e$). The experiment observes the lab energy E' and scattering angle θ of the outgoing electron. From basic kinematics, one has

$$q^2 = -2\ell\cdot\ell' = -4EE'\sin^2(\theta/2) \tag{15}$$

It is convenient to use $Q^2 \equiv -q^2 > 0$. Further, define the Lorentz-invariant quantities

$$x = -\frac{q^2}{2q\cdot p} = \frac{Q^2}{2m_N\nu} \tag{16}$$

and

$$y = \frac{2q\cdot p}{s} = \frac{\nu}{E} = \frac{E-E'}{E} \tag{17}$$

where

$$\nu \equiv E - E' \tag{18}$$

The differential cross section involves a prefactor $\propto \alpha^2/(q^2)^2$ times a lepton tensor times the hadron tensor,

$$W_{\mu\nu} = (2\pi)^{-1}\int d^4x\, e^{iq\cdot x}\langle N|J_\mu(x)J_\nu(0)|N\rangle = -g_{\mu\nu}W_1 + \frac{p_\mu p_\nu}{m_N^2}W_2 \tag{19}$$

where $W_i = W_i(q^2, q\cdot p)$, $i = 1,2$. Define the dimensionless structure functions

$$F_1 = m_N W_1 \ , \quad F_2 = \nu W_2 \tag{20}$$

Exact Bjorken scaling is the property that the F_j depend on q^2 and $q\cdot p$ only through the dimensionless combination x. Measurements of DIS electron scattering at SLAC in the late 1960s showed that for scattered electrons with E' and θ values yielding different q^2 and ν, but equal x values, these structure functions were, indeed, approximately invariant.

Physically, Bjorken scaling is a consequence of the fact that in the deep inelastic limit $Q^2 >> \Lambda_{QCD}^2$, the photon probes the constituents of the nucleon at momentum scales where the strong coupling has gotten weak, so the quarks interact in a quasi-perturbative manner, i.e., appear to be quasi-free rather than confined. Now consider an elementary process of a quark with momentum fraction xp scattering the virtual photon carrying momentum transfer q and going to a final-state quark q'. Since these quarks have negligible mass, we have $(xp+q)^2 = 0$, i.e., $2xp \cdot q + q^2 = 0$, where the term $(xp)^2$ is negligible and hence is dropped. But this is precisely the condition that $x = -q^2/(2q \cdot p)$. This shows that this scattering process, and hence the structure functions, only depend on q^2 and $p \cdot q$ via this dimensionless kinematic invariant, x.

The experimental observation of approximate Bjorken scaling was very important in the history of particle physics because it showed that QCD must have the property of being an asymptotically free theory to account for this experimental fact. In turn, this led to the calculation of the beta function $\beta_g = dg/d\ln\mu$, and the demonstration that this is negative for a non-Abelian Yang-Mills theory. The DIS experiments at SLAC and Fermilab did not observe exact Bjorken scaling, but instead observed approximate scaling with some small scaling deviations. These scaling deviations show that the strong coupling squared, $\alpha_s(Q^2) = g_s(Q)^2/(4\pi)$ is not negligibly small. Furthermore, these scaling deviations showed that $\alpha_s(Q^2)$ decreases as Q^2 increases. In QCD, to leading order, for a given x, the quark that scatters the photon carries a momentum fraction x of the total proton momentum (where $0 \le x \le 1$). See the figures above for the diagrams that were calculated (by Gross, Wilczek, and Politzer) to get the beta function of QCD.

High Energy 2

While physicists working at Large Hadron Collider at CERN were greatly successful in the discovery of the Higgs boson that seems to fit the Standard Model (SM) well, the search for SuperSymmetry (SUSY) has been without success so far. SUSY adds another symmetry to the SM, postulating scalar bosonic partners for all known SM fermions (s-particles), and spin-$\frac{1}{2}$ partners to all SM spin-1 gauge bosons (gauginos). Supersymmetry would solve a number of theoretical problems with the SM.

The LHC accelerates and collides two high-intensity counter-rotating proton beams, currently at an energy of 4 TeV (4×10^{12} eV) per beam in 2012, giving a total energy of 8 TeV in the center-of-mass (= the Laboratory). At two interaction points, the beams are strongly focussed and collisions occur at high rate (about 10 interactions every 50 ns). The 'beam spot' size (standard variation) at these interaction points is about 10 μm transversely and 6 cm along the beam direction.

a. (5 pts.) Supersymmetric theories solve a number of problems in the Standard Model if the supersymmetric particles it predicts have masses of order 1000 GeV or less. Discuss two of the problems with the Standard Model that are solved by SUSY and explain how.

b. (5 pts.) Most SUSY theories conserve so-called R-parity, a multiplicative quantum number defined as $R = (-1)^{F+3B-L}$, where F is the fermion number, and B and L are the usual baryon and lepton numbers. Thus, $R = +1$ for all SM particles, and $R = -1$ for all superparticles.

   Assume the existence of a 'sbottom' particle $\tilde{b}$ where the ' ' denotes the spin-0 superpartner of the bottom quark b. Assuming R-parity conservation, describe a possible series of sequential decay steps that a putative 1.0 TeV mass $\tilde{b}$ squark might undergo in its transition to stable, long-lived particles. Discuss the experimental signatures of such SUSY decays.

c. (5 pts.) Because the probability of existence of O(TeV) mass spin-0 superpartners of the fermions is diminishing rapidly, a SUSY variant named 'Split-SUSY' is getting more attention recently. In Split-SUSY the spin-0 SUSY particles are at very high mass, O(100 TeV) or more, while the spin-$\frac{1}{2}$ gauginos remain relatively light, O(TeV). A consequence of Split-SUSY is that gauginos will be relatively long-lived and traverse O(cm) distances in the detectors before decaying; searches for long-lived particles produced at the LHC are ongoing. Describe how a long-lived neutral gaugino would be detected in a LHC detector if it decays after 2 cm (still inside the LHC beam pipe). What are the physics and non-physics backgrounds?

d. (5 pts.) In many SUSY models there exists a lightest supersymmetric particle (LSP) which is neutral and stable. Explain how the detection of 'Missing Transverse Energy' is used to trigger and detect evidence of SUSY at the LHC. Assuming an LSP with mass of 500 GeV, discuss the signal for SUSY and the main backgrounds.

Solution

a. (a) From astronomical observations, our universe is measured to have a vacuum energy density of order a few $(\text{meV})^4$ (using $\bar{h}c = 197$ MeV·fm). If there is physics at the Planck scale, $M_P \simeq 10^{19}$ GeV, one calculates that its contribution to the vacuum energy density must dominate and predicts an energy density of $(M_P^2/4\pi)^2$, i.e. many orders of magnitude larger than observed. Even if only the 'low-energy' physics is considered, the top quark alone would contribute $(m_t^2/4\pi)^2 = 5 \times 10^{42}$ eV^4! This is the so-called cosmological constant problem in the SM.

(b) The Higgs mechanism in the SM is used to provide mass terms for all fermions via their coupling to the Higgs field. The Higgs (mechanism) is employed to provide mass terms for the otherwise massless Weak Vector Bosons W and Z by its coupling to the SM gauge fields. Moreover, the interaction of the Higgs field with each fermion in the SM, with a coupling strength proportional to the fermion mass, provides effective mass terms for the fermions in the theory.

However, the Higgs mass is subject (like other particle masses like the W and Z masses) to radiative loop corrections from all particles in the model, and highest-mass particles give the largest corrections. These loop corrections, positive (negative) for fermions (bosons) are growing in magnitude with the particle mass and this is unstable, unless exquisitely finely tuned in all orders. This sometimes called the un-naturalness problem.

SUSY, by providing loop corrections from the supersymmetric partners with opposite statistics and thus opposite sign, provides a trivial way of stabilizing the corrections to all orders. This works unless the mass splitting betwen the SM and SUSY partners becomes too large and fine-tuning is again required to prevent a run-away Higgs mass.

(c) The three fundamental coupling "constants" in the SM are related to the three SM gauge groups U(1), SU(2)$_L$, and SU(3)$_C$. These couplings are changing ('run') in a predictable manner with energy, and depend calculationally on the number of fermions and their masses in the theory. It turns out that the couplings, extrapolated to very high energy are crossing a few orders of magnitude below the Planck mass/energy. This approximate unification of the couplings is of course very attractive from a theoretical standpoint if it can be made exact. SUSY, by providing new particles in the TeV mass range, can provide just the right changes in the extrapolation to make the all couplings unify at a single (very high) energy around 10^{16} GeV..

b. The chain might be: $\tilde{b} \to b + \tilde{\chi}_2^0 \to (c\bar{u}d) + e^+e^-\tilde{\chi}_1^0$, where the $\bar{u}d$ pair comes from the virtual W in the b quark decay, and the e^+e^- pair comes from the virtual Z in the $\tilde{\chi}_2^0$ gaugino decay into the lightest supersymmetric particle $\tilde{\chi}_1^0$. Various other decay chains are possible, always containing one (or three) SUSY particles per decay step, and ending with the lightest neutral SUSY particle that is stable.

c. A long-lived neutral gaugino decaying inside the beam pipe will generally produce several charged secondary particles ('tracks') that are measured in the tracking detectors (typically silicon pixel sensors) surrounding the beam pipe. Tracks are curved in the solenoidal magnetic fields and their momenta can be measured. The collision spot at the LHC is small and well-defined in the transverse direction, and therefore tracks that originate 2 cm away from the primary vertex typically will not extrapolate back to the beam spot in the transverse plane. For two or more tracks a common origin (secondary decay vertex) can be constructed. Backgrounds for long-lived particles are 'ordinary' long-lived SM particles: b and c-hadrons, and the well-known K_S^0 mesons. However, all these are low in mass (O(GeV)) and their decay products will therefore be very collimated if they have significant energy. For the high-mass gaugino this is not the case. Non-physics backgrounds are from the very many tracks in the detector, including secondary tracks from interactions in the beam pipe itself, that gives a non-zero probability that several tracks accidentially extrapolate to a common vertex away from the beam center. This background can be estimated directly from the data or simulated with a well-tuned detector simulation.

d. The LSP will be massive, stable, neutral, and weakly interacting. In this rspect it much resembles a heavy new neutrino. In the production, a pair of SUSY particles will be produced approximately balanced in transverse momentum. The production probability (the cross section) will be the same as for the production of quark or gluon jets, except that the phasespace for production of SUSY particle pairs is squeezed by their very large mass. The SUSY particles subsequently decay, producing quarks, gluons, leptons, and an LSP. Note, that in in LHC interactions, much of the energy flow occurs in the forward directions remaining inside the beam pipe; hence logitudinal momentum conservation is impossible to employ in pp interactions. In SUSY events, the presence of two LSPs, not necessarily balanced in transverse momentum, means that typically a large net missing transverse momentum is present. The only way that this happens in SM interactions is either by a (gross) mis-measurement of the energy flow in the detector, or in events that have neutrinos with significant (net) transverse momentum. The latter occurs in events where Ws (or Zs), either virtual or real, are produced and decay into charged lepton plus neutrino. Examples are top quark pair production and di-boson production (WW, WZ, ZZ). However, the missing transverse energy (MET) in SM events rarely exceeds a hundred GeV, and therefore a an excess of events with high MET over the expectatations from SM simulations, is a sure sign of something like SUSY occuring.