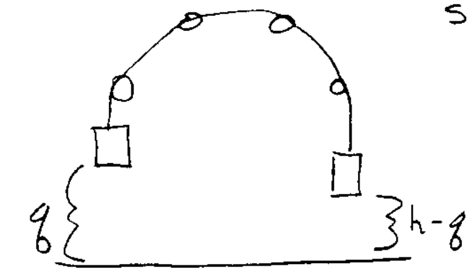
PART I SOLUTIONS Feb. 98

14 Points Total

1.1.1



$$\mathcal{L} = T - V$$
$$= \left(\tfrac{1}{2}m\dot{z}^2 + 4\cdot\tfrac{1}{2}I\left(\underbrace{\tfrac{\dot{z}}{R}}_{\omega}\right)^2 + \tfrac{1}{2}(2m)\dot{z}^2\right) - \left(mgz + 2mg(h-z)\right)$$

height of m_2 when m_1 is on the ground

$$= \tfrac{1}{2}m\dot{z}^2 + 4\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}m\cancel{R^2}\tfrac{\dot{z}^2}{\cancel{R^2}} + \tfrac{1}{2}\cdot 2m\dot{z}^2 + mgz - 2mgh$$

$$\boxed{\mathcal{L} = \tfrac{5}{2}m\dot{z}^2 + mgz - 2mgh}$$ 5pts

1.1.2 ~~4pts~~

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z}$$

$$\frac{d}{dt}(5m\dot{z}) = mg$$

$$5m\ddot{z} = mg \qquad \Rightarrow \boxed{\ddot{z} = \tfrac{1}{5}g}$$ 4pts

1.1.3 Newton's Law on m_1:

$$m\ddot{z} = T_1 - mg \implies T_1 = mg\cdot\tfrac{6}{5} = 1.2\cdot mg$$

At each Pulley

$$I\ddot{\theta} = \sum \tau = RT_{n+1} - RT_n$$

$$\tfrac{1}{2}m\cancel{R^2}\left(\tfrac{\ddot{z}}{\cancel{R}}\right) = \cancel{R}(T_{n+1} - T_n)$$

$$\frac{mg}{10} = T_{n+1} - T_n$$

$$\Rightarrow \boxed{\begin{aligned} T_1 &= (1.2)mg \\ T_2 &= (1.3)mg \\ T_3 &= (1.4)mg \\ T_4 &= (1.5)mg \\ T_5 &= (1.6)mg \end{aligned}}$$ 5pts

Check: Newton's Law on m_2: $m_2\ddot{z} = 2mg - T_5 \Rightarrow T_5 = 2 - \tfrac{2}{5} = 1.6$ ✓

$2m\ddot{z} = 2mg - T_5$

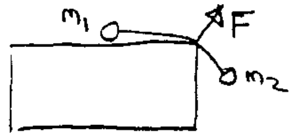
-6 to 12 pts <u>total</u>

1.2.1

$$P \equiv \frac{d}{dt}\left(\tfrac{1}{2} m u^2\right) = \alpha u$$

$$m u \cdot \dot{u} = \alpha u \Rightarrow \dot{u} = \frac{\alpha}{m} = \text{constant} \Rightarrow \boxed{b}$$

1.2.2



Center of mass must move to the right, thus, when m_2 hits, m_1 <u>must be beyond</u> the edge of the table $\Rightarrow \boxed{c}$

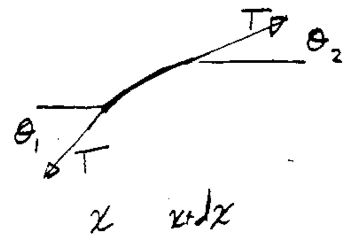
1.2.3

In the boat, the brick displaces a volume of fluid with mass equal to that of the brick.
At the bottom of the lake, the brick displaces its own volume of water.
The brink sinks, thus it's denser than water and displaces more volume when in the boat.
There is less displacement at the finish, so the level must go <u>down</u>. $\Rightarrow \boxed{c}$

(4 points <u>each</u> for correct response
0 " " for no answer
-2 " " for incorrect answer)

14 points total

1.3.1



$$\sum F_y = m a_y$$

$$T\sin\theta_2 - T\sin\theta_1 = \lambda dx \cdot \frac{\partial^2 u}{\partial t^2}(x,t)$$

$$= T\frac{\partial u}{\partial x}(x+dx,t) - T\frac{\partial u}{\partial x}(x,t) \quad \text{(small } \theta\text{'s)}$$

$$= T\frac{\partial^2 u}{\partial x^2}dx$$

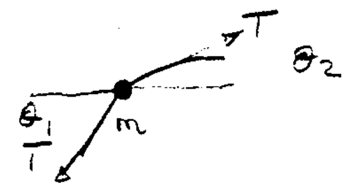
$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} - \frac{\lambda}{T}\frac{\partial^2 u}{\partial t^2} = 0} \quad \text{2pts}$$

If $u(x,t) = e^{i(kx-\omega t)}$ $\Rightarrow -k^2 - \frac{\lambda}{T}(-\omega^2) = 0$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{T}{\lambda}}\,k, \quad c = \sqrt{\frac{T}{\lambda}}} \quad \text{2pts}$$

1.3.2. $\boxed{\text{at } x=a \qquad u(a,t) = 0}$ 1 pt

When $m \neq 0$, at $x=0$ we have:



$$\sum F_y = m a_y$$

$$T\sin\theta_2 - T\sin\theta_1 = m\frac{\partial^2 u}{\partial t^2}(0,t)$$

$$= T\frac{\partial u}{\partial x}(0^+,t) - T\frac{\partial u}{\partial x}(0^-,t) \quad \text{(small } \theta\text{'s)}$$

$$\therefore \boxed{T\,\Delta\frac{\partial u}{\partial x}\Big|_{x=0} = m\frac{\partial^2 u}{\partial t^2}(0,t) \quad \text{Discontinuity in slope at } x=0.} \quad \text{2pts}$$

1.3.3 For odd solutions, $u(0,t) = 0 \Rightarrow \Delta\frac{\partial u}{\partial x} = 0$. Thus, there is no effect from the mass & we have the usual solutions

$u(x) = A\sin ka$ where $u(a) = 0 \Rightarrow ka = n\pi$, $k = \frac{\pi}{a}n$.

Thus, $\boxed{\omega_n = \sqrt{\frac{T}{\lambda}}\frac{\pi}{a}n \quad n = 1, 2, 3 \ldots \quad \text{for odd solutions}}$ 3pt

1.3.4. For even solutions, $u(x,t) = \mathrm{Re}[e^{i\omega t}u(x)]$, the displacement obeys

(1) $\frac{d^2u}{dx^2} + \frac{\lambda\omega^2}{T}u = 0$ (3) $u(a) = 0$

(2) $\Delta\frac{du}{dx}\Big|_0 = \frac{-m\omega^2}{T}u(0)$ (4) $u(x) = u(-x) \Rightarrow \frac{du}{dx}(x) = -\frac{du}{dx}(-x)$.

conditions (1) + (3) give us $u = A\sin k(x-a)$; $\omega^2 = \frac{T}{\lambda}k^2$

From (2) & (4) we have

$$\Delta \frac{du}{dx}\Big|_0 = 2\frac{du}{dx}(0^+) = \frac{-m\omega^2}{T}u(0)$$

$$2kA\cos k(-a) = +\frac{m\omega^2}{T}A\sin k(-a) = \frac{m}{\lambda}k^2 \sin ka$$

$$\boxed{\frac{\lambda}{m} = \frac{k}{2}\tan ka}$$ 4pts

10 points total

2.1.1.

Total # of states with energy less than $\epsilon = \hbar\omega = \alpha\hbar k^2$

$$N(\epsilon) = \underbrace{\frac{4\pi}{3}\left(\sqrt{\frac{\epsilon}{\alpha\hbar}}\right)^3}_{\substack{k\text{-Volume of sphere} \\ \text{of } k\text{'s of radius} \\ k=\sqrt{\epsilon/\alpha\hbar}}} \Big/ \underbrace{\frac{(2\pi)^3}{V}}_{\substack{\text{Volume of} \\ \text{one point in} \\ k\text{-space}}} = V \cdot \frac{4\pi}{3} \frac{1}{8\pi^3} \frac{\epsilon^{3/2}}{(\alpha\hbar)^{3/2}}$$

Number of states in energy interval $(\epsilon, \epsilon+d\epsilon)$ per unit volume

$$g(\epsilon)\,d\epsilon = \frac{N(\epsilon+d\epsilon)-N(\epsilon)}{V} = \frac{1}{V} N'(\epsilon)\,d\epsilon$$

4pts

$$g(\epsilon) = \frac{1}{V} N'(\epsilon) = \frac{\cancel{3}}{2} \cdot \frac{4\pi}{\cancel{3}} \cdot \frac{1}{8\pi^3} \frac{\epsilon^{1/2}}{(\alpha\hbar)^{3/2}} = \boxed{\frac{1}{4\pi^2} \frac{1}{(\alpha\hbar)^{3/2}} \epsilon^{1/2} = g(\epsilon)}$$

2.1.2

$$u(T) = \int \underbrace{g(\epsilon)d\epsilon}_{\substack{\text{Number of} \\ \text{states per} \\ \text{unit} \\ \text{volume}}} \cdot \underbrace{\epsilon}_{\substack{\text{Energy} \\ \text{per} \\ \text{excitation}}} \cdot \underbrace{\frac{1}{e^{\beta\epsilon}-1}}_{\substack{\text{Number of} \\ \text{Bose excitations}}} = \frac{1}{(2\pi)^2(\alpha\hbar)^{3/2}} \int \frac{\epsilon^{3/2}d\epsilon}{e^{\beta\epsilon}-1}$$

$$= \frac{1}{(2\pi)^2(\alpha\hbar)^{3/2}} \left(\frac{1}{\beta}\right)^{5/2} \int \frac{u^{3/2}du}{e^{u}-1} \qquad u \equiv \beta\epsilon$$

$$\boxed{u(T) = \frac{1}{(2\pi)^2}\left(\frac{kT}{\alpha\hbar}\right)^{3/2} \cdot kT\, \Gamma(5/2)\zeta(5/2).}$$

3pts

2.1.3 $$du = T\,ds = \frac{\Gamma(5/2)\zeta(5/2)}{(2\pi)^2} \frac{5}{2}\left(\frac{kT}{\alpha\hbar}\right)^{3/2} k\,dT$$

$$ds = \frac{\Gamma(5/2)\zeta(5/2)}{(2\pi)^2} \frac{5}{2}\left(\frac{k}{\alpha\hbar}\right)^{3/2} k\,T^{1/2}\,dT$$

$$S = \frac{\Gamma(5/2)\zeta(5/2)}{(2\pi)^2} \frac{5}{2}\left(\frac{k}{\alpha\hbar}\right)^{3/2} k\,\frac{2}{3}\,T^{3/2}$$

$$\boxed{S = \frac{\Gamma(5/2)\zeta(5/2)}{(2\pi)^2} \frac{5}{3}\left(\frac{kT}{\alpha\hbar}\right)^{3/2} \cdot k}$$

3pts

10 pts total

2.2.1.

$$\omega = \binom{N}{N_\phi \; N_b} = \frac{N!}{N_\phi!\,N_b!\,(N-(N_\phi+N_b))!} = \frac{N!}{N_\phi!\,N_b!\,(N-N_d)!}$$

$$S = k\ln\omega = k\left[N\ln N \cancel{-N} - N_\phi\ln N_\phi \cancel{+N_\phi} - N_b\ln N_b \cancel{+N_b}\right.$$

$$\left. - (N-N_d)\ln(N-N_d) \cancel{+N-N_d}\right]$$

$$S(N_\phi) = \underbrace{k\left[N\ln N - (N-N_d)\ln(N-N_d)\right]}_{S_o} - k\left[N_\phi\ln N_\phi - (N_d-N_\phi)\ln(N_d-N_\phi)\right]$$

$$F = E - TS = -\Delta(N_\phi - N_b) - TS$$

$$= -\Delta(2N_\phi - N_d) - TS_o + kT\left[N_\phi\ln N_\phi - (N_d-N_\phi)\ln(N_d-N_\phi)\right]$$

5 pts

$$\boxed{F(N_\phi,T) = \underbrace{\Delta N_d - TS_o}_{F_o} - 2\Delta N_\phi + kT\left[N_\phi\ln N_\phi - (N_d-N_\phi)\ln(N_d-N_\phi)\right]}$$

2.2.2.

$$0 = \frac{\partial F}{\partial N_\phi} = -2\Delta + kT\left[\ln N_\phi \cancel{+1} - \ln(N_d-N_\phi) \cancel{-1}\right]$$

$$\frac{2\Delta}{kT} = \ln\frac{N_\phi}{N_b}, \quad \boxed{\frac{N_\phi}{N_b} = e^{2\Delta/kT}}$$

5pts

2.3.1 10 pts total

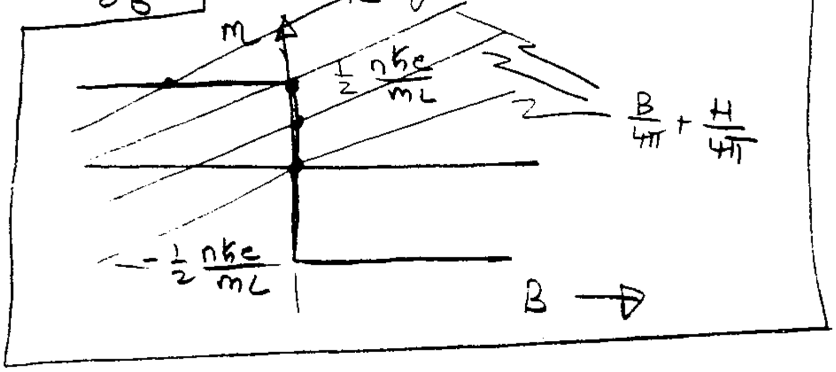
As $T \to 0$ all particles go in ground state

$$U = N \cdot \frac{1}{2} \frac{\hbar e}{mc} |B|$$

$$\boxed{u = n \cdot \frac{1}{2} \frac{\hbar e}{mc} |B|}$$ 3pts

2.3.2.

$$m = -\frac{\partial u}{\partial B} = -\frac{1}{2} \frac{n\hbar e}{mc} \,\mathrm{sgn}\, B = m$$



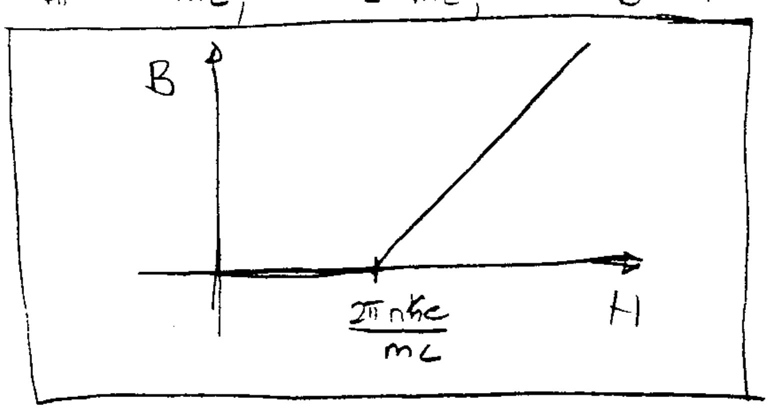
4pts

2.3.3. From given eqn 3, we also have $m = \frac{B}{4\pi} + \frac{H}{4\pi}$.

For $H = 0$, this intersects $m(B)$ at $B = 0$, so $B = 0$

$$\boxed{\text{For } \frac{|H|}{4\pi} < \frac{1}{2}\frac{n\hbar e}{mc}, \text{ this also intersects } m(B) \text{ at } B=0, \text{ so } B=0}$$

For $\frac{H}{4\pi} > \frac{1}{2}\frac{n\hbar e}{mc}$, $m = \frac{1}{2}\frac{n\hbar e}{mc}$, so $B = H - 2\pi \frac{n\hbar e}{mc}$



3pts

10 points total

2.4.1.

a) For these points there are two possibilities, so $\omega_{n+1} = 2 \cdot \omega_n$.

Thus, $S_{n+1} = k \ln \omega_{n+1} = k \ln \omega_n + k \ln 2 = S_n + k \ln 2$

$$\boxed{\Delta S = k \ln 2}$$ 2pts

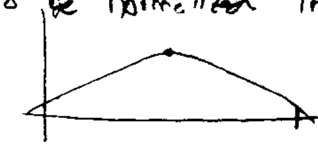
b) For these points, there is only one possibility, so $\omega_{n+1} = \omega_n$

$$\boxed{\Delta S = 0}$$ 2pts

2.4.2.

$$\bar{S} = k \ln 2 \cdot \bar{P}(0<y<w) + 0 \cdot \bar{P}(y=0 \text{ or } y=w)$$

To get $P(y=0)$, we note that $P(n)$ is heading linearly toward zero. For P to be normalized the height must reach



W

$I \sim 1/W$

approximately $1/W$ at the center after $\frac{W}{2}$ steps. The slope must be therefore $\frac{1/W}{W/2} \simeq \frac{2}{W^2}$, thus $P(y=0) \simeq \frac{2}{W^2}$.

$$\boxed{S \simeq k \ln 2 \left(1 - \frac{4}{W^2}\right)}$$ 3pts

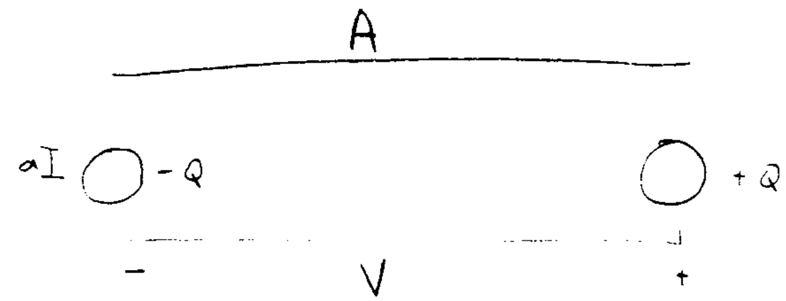
2.4.3 $f = -\frac{\partial (F/L)}{\partial W} = -\frac{\partial}{\partial \omega}(\overset{0}{E} - TS) = +kT\frac{\partial}{\partial \omega} \ln 2\left(1 - \frac{4}{W^2}\right)$

$$= \boxed{+ kT \ln 2 \cdot \frac{8}{W^3} = F}$$ 3pt.

Check: $F > 0$, the chain pushes the wall outward to increase its entropy

10 pts total

3.1.



If $A >> a$, then the potential on each sphere is

$$\phi = \pm \frac{Q}{a} \mp \frac{Q}{A} \approx \pm \frac{Q}{a}\left(1 + \mathcal{O}(a/A)\right) \Rightarrow V = \frac{2Q}{a}, \quad Q = \frac{Va}{2}$$

potential from other sphere

potential from self. Because the other sphere is far away, the charge distribution is nearly spherical to at least $\mathcal{O}(a/A)$

To the same approximations, the electric field outside the positive sphere is

$$\vec{E} = \frac{Q}{a^2}\hat{r}$$

and so the current density is

$$\vec{J} = \sigma \vec{E} = \frac{Q\sigma}{a^2}\hat{r}$$

and the net current is

$$I = 4\pi a^2 \cdot \frac{Q\sigma}{a^2} = 4\pi Q\sigma$$

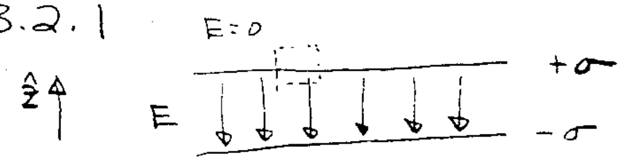
Thus,

$$R = \frac{V}{I} = \frac{2Q/a}{4\pi Q\sigma} = \boxed{\frac{1}{2\pi\sigma a} = R}$$

10pts

10 pts total

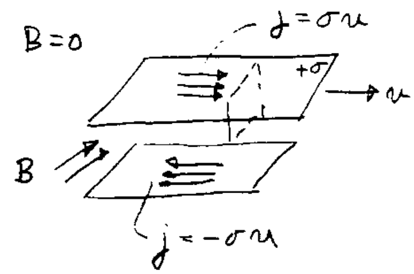
3.2.1



$$\oint\!\!\oint \vec{E}\cdot d\vec{a} = 4\pi Q$$

2 pts

$$E\cdot A = 4\pi\sigma A \Rightarrow E = 4\pi\sigma \quad \boxed{\vec{E} = -4\pi\sigma\,\hat{z} \text{ (inside)},\ \vec{E} = 0 \text{ (outside)}}$$



$$\oint \vec{B}\cdot d\vec{l} = \frac{4\pi}{c} I$$

$$B l = \frac{4\pi}{c} j l$$

$$B = \frac{4\pi}{c}\sigma u$$

3 pts $\boxed{\vec{B} = \frac{4\pi}{c}\sigma u\,\hat{y} \text{ (inside)};\ \vec{E} = 0 \text{ (outside)}}$

3.2.2.

$$U = V\cdot\frac{E^2+B^2}{8\pi} = V\cdot 2\pi\sigma^2 + V\cdot\frac{2\pi\sigma^2}{c^2}u^2$$

V — volume between plates

$$\boxed{2\pi\sigma^2 V + \underbrace{\left(\frac{1}{2}\cdot\frac{4\pi\sigma^2}{c^2}\cdot V\right)}_{M} u^2 = U}$$ 3 pts

$$\vec{J} = V\cdot\frac{\vec{E}\times\vec{B}}{4\pi c} = V\cdot\frac{4\pi\sigma\left(\frac{4\pi}{c}\sigma u\right)}{4\pi c}\hat{x}$$

$$= \boxed{\left(\frac{4\pi\sigma^2}{c^2}\cdot V\right)u = \vec{P}}$$ 2 pts

10 pts total

3.3.1

Planewave: $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$, $\vec{B} = \vec{B}_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$$\nabla \cdot \vec{E} = 4\pi\rho = 0 \Rightarrow i\vec{k}\cdot\vec{E}_0 = 0 \Rightarrow \boxed{\vec{k}\cdot\vec{E}_0 = 0} \checkmark$$

$$\nabla \cdot \vec{B} = 0 \Rightarrow i\vec{k}\cdot\vec{B}_0 = 0 \Rightarrow \boxed{\vec{k}\cdot\vec{B}_0 = 0} \checkmark$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \Rightarrow i\vec{k}\times\vec{E}_0 = \frac{i\omega}{c}\vec{B}_0 \Rightarrow \boxed{\vec{k}\times\vec{E}_0 = \frac{\omega}{c}\vec{B}_0} \checkmark$$

5 pts

3.3.2 The final Maxwell Equation is

$$\nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi\sigma}{c}\vec{E} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow i\vec{k}\times\vec{B}_0 = \frac{4\pi\sigma}{c}\vec{E}_0 - \frac{i\omega}{c}\vec{E}_0 = \left(\frac{4\pi\sigma}{c} - i\frac{\omega}{c}\right)\vec{E}_0 \qquad (*)$$

Do $\vec{k}\times(*)$:

$$i\vec{k}\times(\vec{k}\times\vec{B}_0) = \left(\frac{4\pi\sigma}{c} - i\frac{\omega}{c}\right)\vec{k}\times\vec{E}_0$$

$$i\left[\vec{k}(\vec{k}\cdot\vec{B}_0)^{\,0} - \vec{B}_0(\vec{k}\cdot\vec{k})\right] = \left(\frac{4\pi\sigma}{c} - i\frac{\omega}{c}\right)\frac{\omega}{c}\vec{B}_0$$

$$k^2\vec{B}_0 = \frac{\omega}{c}\left(\frac{\omega}{c} + \frac{4\pi i\sigma}{c}\right)\vec{B}_0$$

$$k^2 = \frac{\omega^2}{c^2}\left(1 + 4\pi i\frac{\sigma}{\omega}\right)$$

$$k = \frac{\omega}{c}\sqrt{1 + 4\pi i\frac{\sigma}{\omega}} = \frac{\omega}{c}e^{i\cdot\frac{1}{2}\tan^{-1}\frac{4\pi\sigma}{\omega}}$$

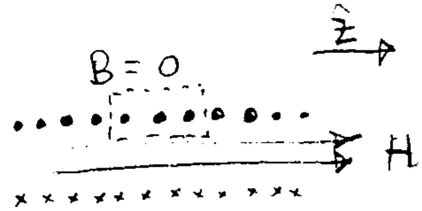
So, $e^{ikx} = e^{i(k_r + ik_i)x} = e^{ik_r}e^{-k_i x}$

Thus, $$\boxed{\lambda = \frac{1}{\operatorname{Im} k} = \frac{1}{\frac{\omega}{c}\sin\left(\frac{1}{2}\tan^{-1}\frac{4\pi\sigma}{\omega}\right)}}$$

5 pts

10 pts total

3.4.1.



$$\oint \vec{H} \cdot d\vec{l} = \frac{4\pi}{c} \sum I$$

$$H \cdot l = \frac{4\pi}{c} I n l \qquad \boxed{\vec{H} = \hat{z} \cdot \frac{4\pi}{c} I n} \quad 4\text{pts}$$

3.4.2.

Potential around one loop of the coil:

$$\phi = -\oint \vec{E} \cdot d\vec{l} = -\iint \left(\frac{-1}{c} \frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{A} = \frac{\pi R^2}{c} \frac{dB}{dt}$$

Potential from bottom to top

$$\Phi_{\text{bottom}}^{\text{top}} = N \phi = \frac{n}{c}(L \pi R^2) \frac{dB}{dt} = \boxed{\frac{n}{c} \cdot V \cdot \frac{dB}{dt} = \Phi,\ V = \text{volume of coil}} \quad 3\text{pts}$$

Potential increases with $\hat{z}$!

3 4 3

$$\frac{W}{V} = \frac{\int \Phi \cdot I \, dt}{V} = \frac{\int n \cancel{V} \frac{1}{c} \frac{dB}{dt} I \, dt}{\cancel{V}} = \frac{n}{c} I \Delta B$$

$$\boxed{\frac{W}{V} = \frac{H \Delta B}{4\pi}} \quad 3\text{pts}$$

10 pts total

4.1.1

$$\boxed{\psi(x) = \begin{cases} e^{ik_1x} + Re^{-ik_1x} & x<0 \\ Te^{ik_2x} & x>0 \end{cases}}$$

B.C.'s

$\psi(0^-) = \psi(0^+) \Rightarrow 1+R=T$

$\psi'(0^-) = \psi'(0^+) \Rightarrow k_1(1-R) = k_2T$

$2 = (1+\frac{k_2}{k_1})T \Rightarrow T = \frac{2k_1}{k_1+k_2}$

$R = T-1 = \frac{k_1-k_2}{k_1+k_2}$

where...

$$\boxed{T = \frac{2k_1}{k_1+k_2} \qquad R = \frac{k_1-k_2}{k_1+k_2}}$$

3pts

4.1.2.

$j_1 = \frac{\hbar k_1}{m}|1|^2 = \frac{\hbar k_1}{m}$, $j_1' = \frac{\hbar k_1}{m}\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 = \boxed{\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 \cdot j_1 = j_1'}$

$j_2 = \frac{\hbar k_2}{m}\left(\frac{2k_1}{k_1+k_2}\right)^2 = \frac{\hbar k_1}{m} \cdot \frac{4k_1k_2}{(k_1+k_2)^2} = \boxed{\frac{4k_1k_2}{(k_1+k_2)^2} \cdot j_1 = j_2}$

check:

$j_1' + j_2 = j_1 \frac{(k_1-k_2)^2+4k_1k_2}{(k_1+k_2)^2} = j_1 \frac{\cancel{(k_1+k_2)^2}}{\cancel{(k_1+k_2)^2}} = j_1$ ✓

3pts

4.2.2

$\boxed{f_1 = \hbar k_1 \cdot j_1}$

Momentum per particle

Flux of particles

$\boxed{f_1' = -\hbar k_1\left(\frac{k_1-k_2}{k_1+k_2}\right)^2 \cdot j_1}$

$\boxed{f_2 = +\hbar k_2 \frac{4k_1k_2}{(k_1+k_2)^2} \cdot j_1}$

2pts

4.2.3 $\boxed{\text{$f_1$ will be greater than $f_1' + f_2$ because the force from the step is opposite to the incoming momentum, \underline{reducing} the net momentum flux.}}$

2pts

4.2 10 pts total

Minimum width gives $E=0$. So the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\psi'' - \kappa\frac{\hbar^2}{m}\delta(x)\,\psi = 0$$

When $x \neq 0$, we have $\psi''=0 \Rightarrow \psi = Ax+b$

Near $x=0$, we have

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{\epsilon}\psi''\,dx - \kappa\frac{\hbar^2}{m}\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx = 0$$

$$-\frac{\hbar^2}{2m}\Delta\psi' - \frac{\kappa\hbar^2}{m}\psi(0) = 0$$

$$\underline{\tfrac{1}{2}\Delta\psi' = -\kappa\,\psi(0).}$$

$\psi = Ax+b$ $\quad a = \frac{b}{A} = -\frac{a}{b}$

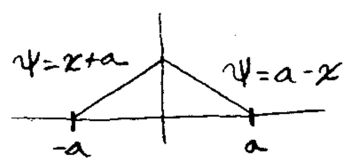
$x = -Aa + b = 0$

$x = c$

$Bc + b = 0$

$A = -B$

So, the solution must look like



$A = -B$

with $\quad \tfrac{1}{2}\Delta\psi' = -1$
$\quad -\kappa\,\psi(0) = -\kappa a$ $\quad \Rightarrow -1 = -\kappa a \Rightarrow a = 1/\kappa \Rightarrow \boxed{W = 2/\kappa}$ 10pts

$A = -\kappa b$

$2A = 2b$

$\kappa = \frac{A}{b} = \frac{1}{a}$

 10 pts total

4.3.1.

$$\mathcal{H} = \alpha \vec{L}\cdot\vec{S} = \frac{\alpha}{2}(J^2 - L^2 - S^2), \text{ where } \vec{J} \equiv \vec{L} + \vec{S}$$

When adding $s = 1/2$ and $\ell = 1$, we have $j = 3/2$, $j = 1/2$ with $(2j+1)$ 4 and 2 states, respectively.

$j = 3/2$ (4 states)

$$\boxed{\mathcal{H} = \frac{\hbar^2\alpha}{2}\left(\frac{3}{2}\cdot\frac{5}{2} - 1\cdot 2 - \frac{1}{2}\cdot\frac{3}{2}\right) = \frac{\hbar^2\alpha}{2}}$$

$j = 1/2$ (2 states)

$$\boxed{\mathcal{H} = \frac{\hbar^2\alpha}{2}\left(\frac{1}{2}\cdot\frac{3}{2} - 1\cdot 2 - \frac{1}{2}\cdot\frac{3}{2}\right) = -\hbar^2\alpha}$$

5 pts

4.3.2 The low energy states are the $j = 1/2$ states. We must find these in the L, S representation:

$$|\tfrac{3}{2}\tfrac{3}{2}\rangle = |11\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

$$J_-|\tfrac{3}{2}\tfrac{3}{2}\rangle = (L_-|11\rangle)|\tfrac{1}{2}\tfrac{1}{2}\rangle + |11\rangle(S_-|\tfrac{1}{2}\tfrac{1}{2}\rangle) \quad J_- = L_- + S_-$$

$$\cancel{\hbar}\sqrt{\tfrac{3}{2}\cdot\tfrac{5}{2} - \tfrac{3}{2}\cdot\tfrac{1}{2}}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle = \cancel{\hbar}\sqrt{1\cdot 2 - 1\cdot 0}\,|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + |11\rangle\left(\cancel{\hbar}\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2} - \tfrac{1}{2}(-\tfrac{1}{2})}\,|\tfrac{1}{2}-\tfrac{1}{2}\rangle\right)$$

$$\sqrt{3}\,|\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{2}\,|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + |11\rangle\cancel{\sqrt{1}}\,|\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

$$|\tfrac{3}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|11\rangle|\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

Note: $|\tfrac{1}{2}\tfrac{1}{2}\rangle$ must be orthogonal to $|\tfrac{3}{2}\tfrac{1}{2}\rangle$, so up to a phase —

$$|\tfrac{1}{2}\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|10\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|11\rangle|\tfrac{1}{2}-\tfrac{1}{2}\rangle$$

by $z \to -z$ symmetry

$$|\tfrac{1}{2}-\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}\,|10\rangle|\tfrac{1}{2}-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|1-1\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle$$

In both cases $P(L_z \neq 0) = \left|\sqrt{\tfrac{2}{3}}\right|^2 = \tfrac{2}{3}$, so $\boxed{P = \tfrac{2}{3}}$ 5 pts

4.4 10 pts total

$$i\hbar\partial_t|\Psi(t)\rangle = e^{iH_0t/\hbar}Ve^{-iH_0t/\hbar}|\Psi(t)\rangle, \text{ where } V = -F\hat{x}\sin\omega t$$
$$= -F\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger+a)\sin\omega t$$

To lowest order in V, we take $|\Psi(t)\rangle$ on the right to be $|\Psi(t)\rangle \simeq |\Psi(0)\rangle = |0\rangle$. Thus,

$$i\hbar\partial_t|\Psi(t)\rangle \simeq e^{iH_0t/\hbar}\left(-F\sqrt{\frac{\hbar}{2m\omega}}\sin\omega t\right)\underbrace{(a^\dagger+a)\underbrace{e^{-iH_0t/\hbar}|0\rangle}_{e^{-iE_0t/\hbar}|0\rangle}}_{e^{-iE_0t/\hbar}\sqrt{1}|1\rangle+0}$$

$$\simeq e^{iE_1t/\hbar}\left(-F\sqrt{\frac{\hbar}{2m\omega}}\sin\omega t\right)e^{-iE_0t/\hbar}|1\rangle$$

So,

$$|\Psi(t)\rangle \simeq |\Psi(0)\rangle + \frac{1}{i\hbar}\int_0^t\left(-F\sqrt{\frac{\hbar}{2m\omega}}\right)\sin\omega t'\, e^{i\overbrace{(E_1-E_0)}^{E_1-E_0=\hbar\omega}t'/\hbar}dt'|1\rangle$$
$$= |0\rangle + \frac{1}{i\hbar}\left(-F\sqrt{\frac{\hbar}{2m\omega}}\right)\int_0^t\frac{e^{i\omega t'}-e^{-i\omega t'}}{2i}e^{i\omega t'}dt'\cdot|1\rangle$$
$$= |0\rangle + \frac{1}{i\hbar}\left(-F\sqrt{\frac{\hbar}{2m\omega}}\right)\frac{1}{2i}\left(\int_0^t \underbrace{e^{2i\omega t'}}_{\text{oscillates}}\underbrace{-1\,dt}_{\text{builds up with } t}\right)|1\rangle$$

$$\simeq |0\rangle - \frac{F}{2\sqrt{2m\omega\hbar}}t\cdot|1\rangle$$

Thus, $\boxed{P_m = \frac{F^2t^2}{8m\omega\hbar}\delta_{m1}}$ 10pts