Physics PhD Qualifying Examination : With Solutions

Part I – Saturday, August 21, 2004

Name:____

(please print)

Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
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	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism), **and with at least one problem from problems 5 or 10 (Special Relativity).**
   DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

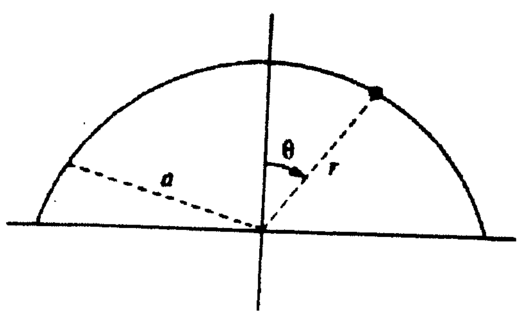
I-1. [3,4,3]

Consider the motion of a particle in a potential $U(\mathbf{r})$ in a *rotating* frame. ω, the angular velocity of the rotation with respect to an inertial frame, is constant in time.

(a) Construct the Lagrangian for the particle in the rotating frame.
(b) Obtain the equation of motion in the rotating frame.
(c) Obtain the Hamiltonian of the particle in the rotating frame.

I-2. [10]

A particle of mass m starts at rest on top of a smooth fixed hemisphere of radius a. Find the force of constraint, and determine the angle at which the particle leaves the hemisphere.

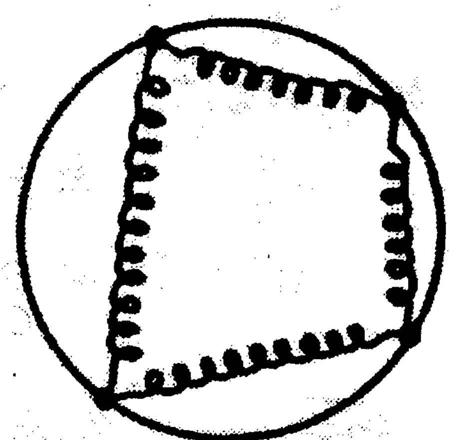


Problem I-2.

I-3. [3, 7]

Four identical masses are connected by four identical springs and constrained to move on a frictionless circle of radius "b" as shown below in the figure.

Problem I-3.



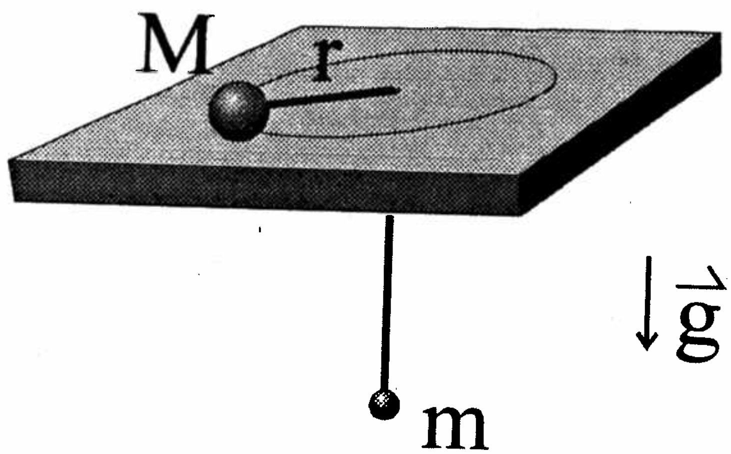
(a) How many normal-modes of oscillations are there?
(b) What are the frequencies of small oscillations?

I-4. [5, 5]

A particle of mass M is constrained to move on a horizontal plane in a field of gravity g. A second particle of mass m is constrained to a vertical line. The two particles are connected by a massless string which passes through a hole in the plane. The motion is frictionless.

a) Find the Lagrangian of the system and derive the equations of motion.

b) Show that the object is stable in radius r_0 with respect to small changes in the radius, and find the frequency of small oscillations.



I-5. [5, 5]

A relativistic particle moves from rest at the origin at $t = 0$ under a constant force $\vec{F} = m_o g\,\hat{i}$.

(a) Calculate the velocity and the position of the relativistic particle at time t.
(b) Obtain expressions for the position and velocity at time t for ($v/c << 1$). Would you anticipate this result. Explain.

I-6. [2,3,5]

Consider a medium with nonzero conductivity σ [($J = \sigma E$) gives the current density] and no net charge ($\rho = 0$).

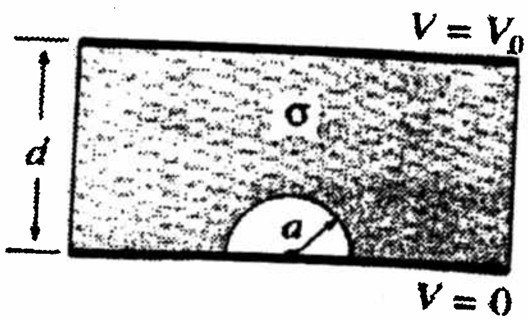
(a) Write down the set of Maxwell's equations appropriate for this medium.
(b) Derive the wave equation for E in this medium.
(c) Consider a monochromatic wave moving in the +x direction with E_y (or E_z) given by

$$E_y = \psi = \psi_o e^{i(kx-\omega t)}$$

Show that this wave has an amplitude which decreases exponentially; find the attenuation length.

I-7. [10]

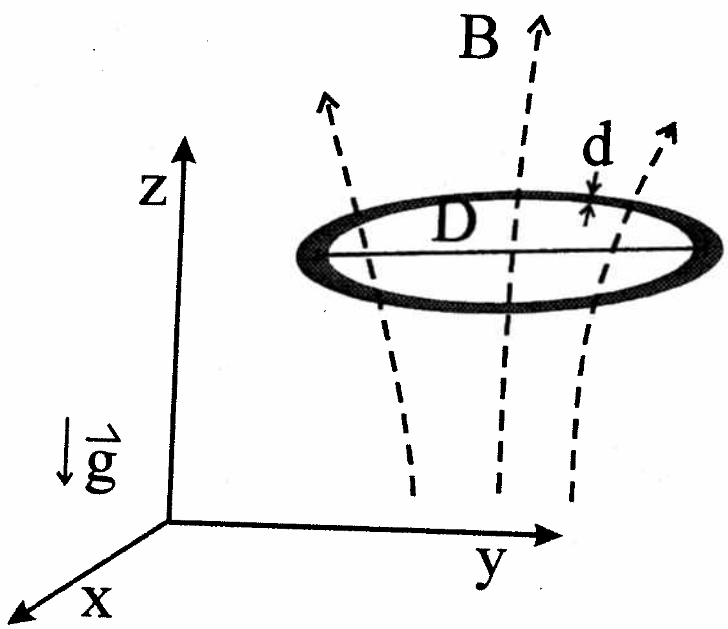
Two very large metal plates are held a distance d apart, one at potential zero, the other at potential V_0. A metal sphere of radius a ($a \ll d$) is sliced in two, and one hemisphere is placed on the grounded plate so that its potential is likewise zero. If the region between the plates is filled with weakly conducting materials of uniform conductivity σ, what current flows to the hemisphere?



Problem I-7.

I-8. [10]

A conducting circular loop made of wire of diameter d, resistivity ρ, and a mass density ρ_m is falling from a great height h in a magnetic field with a component $B_z = B_0 (1 + \kappa z)$, where κ is some constant.
The loop of diameter D is always parallel to the x-y plane. Disregarding air resistance, find the terminal velocity of the loop.



I-9. [10]

Electromagnetic radiation with $\vec{E} = E_0\, exp(i(kz - \omega t))\hat{y}$ is incident on an atom of polarizability α at position (0, 0, 0). [Treat the polarized atom as a dipole] .

(a) Find the electric and magnetic fields of the radiated wave at large distance D at the following points (i) on the y-axis, (ii) on the x-axis.

(b) Find the total time-averaged power radiated by the polarized atom.

I-10. [4, 6]

A field $\vec{E}$ in spherical coordinates $\vec{E}(\vec{r}, t)$ has the form:

$$\vec{E} = A\frac{\sin\theta}{r}\left[\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right]\hat{\varphi}\ .$$

(a) Show that this is a valid expression for the electric part of the electromagnetic wave.

(b) Obtain an expression for the energy per unit time carried by this wave through the surface of a sphere of radius *R*. Does the answer depend on *R*. Explain.

I-1. $\bar{\omega}$ is the angular velocity of the rotating frame with respect to an inertial frame. $\bar{\omega} = \text{const}$.
It is NOT the angular velocity of the particle!

What we are considering here is the generalization of the Lagrangian formalism in rotating frames.

The motion of the particle is completely general, described by the 3-dimensional vectors $\bar{r}(t), \bar{V}(t)$.

a) $L = \frac{1}{2} m \bar{V}_i^2 - U(\bar{r})$ in an inertial frame

$\bar{V}_i = \bar{V}_r + \bar{\omega} \times \bar{r}$ ($\bar{r} = \bar{r}_r$ since only rotating motion is considered between frames)

$\bar{V}_r$ – velocity in rotating frame
$\bar{V}_i$ – velocity in inertial frame

$$L = \frac{1}{2} m (\bar{V}_r + \bar{\omega} \times \bar{r})^2 - U(\bar{r}) =$$

$$\boxed{= \frac{1}{2} m \bar{V}_r^2 + \frac{1}{2} m (\bar{\omega} \times \bar{r})^2 + m \bar{V}_r \cdot (\bar{\omega} \times \bar{r}) - U(\bar{r})}$$

b) to obtain the equation of motion in the rotating frame

$$\frac{\partial L}{\partial \bar{V}_r} = m \bar{V}_r + m \bar{\omega} \times \bar{r}$$

$$\frac{\partial L}{\partial \bar{r}} = \frac{\partial}{\partial \bar{r}} \left(\frac{m}{2} (\bar{\omega} \times \bar{r})^2 + m \bar{V}_r \cdot (\bar{\omega} \times \bar{r}) - U(\bar{r}) \right)$$

to obtain the derivative of the first term, consider the differential of this term with respect to $\bar{r}$:
($\bar{\omega} = \text{const.}$)

$$d(\bar{\omega}\times\bar{r})^2 = 2(\bar{\omega}\times\bar{r})\cdot(\bar{\omega}\times d\bar{r}) = 2[(\bar{\omega}\times\bar{r})\times\bar{\omega}]\cdot d\bar{r}$$

$$\Rightarrow \frac{\partial(\bar{\omega}\times\bar{r})^2}{\partial\bar{r}} = 2(\bar{\omega}\times\bar{r})\times\bar{\omega} = -2\bar{\omega}\times(\bar{\omega}\times\bar{r})$$

and $\frac{\partial}{\partial\bar{r}}[\bar{v}_r\cdot(\bar{\omega}\times\bar{r})] = \frac{\partial}{\partial\bar{r}}[(\bar{v}_r\times\bar{\omega})\cdot\bar{r}] = \bar{v}_r\times\bar{\omega} = -\bar{\omega}\times\bar{v}_r$

Combined:

$$\frac{\partial L}{\partial\bar{r}} = -m\,\bar{\omega}\times(\bar{\omega}\times\bar{r}) - m\,\bar{\omega}\times\bar{v}_r - \frac{\partial U}{\partial\bar{r}}$$

E-L equations:

$$\frac{d}{dt}\frac{\partial L}{\partial\bar{v}_r} = \frac{\partial L}{\partial\bar{r}}$$

$$\frac{d}{dt}\left(m\bar{v}_r + m\,\bar{\omega}\times\bar{r}\right) = -m\,\bar{\omega}\times(\bar{\omega}\times\bar{r}) - m\,\bar{\omega}\times\bar{v}_r - \frac{\partial U}{\partial\bar{r}}$$

using $\frac{d\bar{\omega}}{dt}=0$ and rearranging: $\left(\frac{d\bar{r}}{dt}=\bar{v}_r\right)$

$$\boxed{m\dot{\bar{v}}_r = -\frac{\partial U}{\partial\bar{r}} - m\,\bar{\omega}\times(\bar{\omega}\times\bar{r}) - 2m\,\bar{\omega}\times\bar{v}_r}$$

Coriolis force

centrifugal force

equation of motion in a rotating frame with $\bar{\omega}=$const.

c) obtain the Hamiltonian.

$$\bar{p}_r \equiv \frac{\partial L}{\partial \bar{V}_r} = m\bar{V}_r + m\bar{\omega}\times\bar{r}$$

$$H = \frac{\partial L}{\partial \bar{V}_r}\cdot\bar{V}_r - L = m\bar{V}_r^2 + m(\bar{\omega}\times\bar{r})\cdot\bar{V}_r - L$$

$$= \frac{1}{2}m\bar{V}_r^2 + U(\bar{r}) - \frac{1}{2}m(\bar{\omega}\times\bar{r})^2$$

$$= \frac{1}{2m}\left(\bar{p}_r - m(\bar{\omega}\times\bar{r})\right)^2 + U(\bar{r}) - \frac{m}{2}(\bar{\omega}\times\bar{r})^2 = H(\bar{p},\bar{r})$$

I-2) generalized coord.: r, θ constraint: $f(r,\theta) = r - a = 0$

$$T = \tfrac{1}{2}m(\dot r^2 + r^2\dot\theta^2)$$
$$U = mgr\cos\theta$$
$$L = T - U = \tfrac{1}{2}m(\dot r^2 - r^2\dot\theta^2) - mgr\cos\theta$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} + \sum_k \lambda_k \frac{\partial f_k}{\partial q} = 0$$

r: $mr\dot\theta^2 - mg\cos\theta - m\ddot r + \lambda = 0$

θ: $mgr\sin\theta - mr^2\ddot\theta - 2mr\dot r\dot\theta = 0$

Constraint $\Rightarrow r = a$, $\dot r = 0 = \ddot r$

$\Rightarrow ma\dot\theta^2 - mg\cos\theta + \lambda = 0$

$mga\sin\theta - ma^2\ddot\theta = 0 \Rightarrow \ddot\theta = \frac{g}{a}\sin\theta$

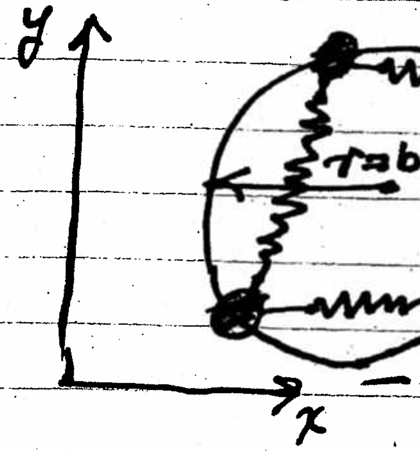
But $\ddot\theta = \dot\theta \frac{d\dot\theta}{d\theta} \Rightarrow \int d\dot\theta\,\dot\theta = \frac{g}{a}\int d\theta \sin\theta$

$\Rightarrow \frac{\dot\theta^2}{2} = -\frac{g}{a}\cos\theta + \frac{g}{a}$ { at $t=0$, $\theta = 0 = \dot\theta$

$\therefore \lambda = mg(3\cos\theta - 2)$ *

falls off when $\cos\theta_0 = \frac{2}{3}$ *

[I-3] Four identical masses are connected by four identical springs and constrained to move on a frictionless circle of radius "b" as shown below in the figure.



(a) How many normal-modes of oscillations are there?
(b) What are the frequencies of small oscillations?

Solutions:

(a) Take lengths of arc s_1, s_2, s_3, s_4; of the four masses from their ~~equilibrium~~ initial equilibrium positions as the generalized coordinates. The kinetic energy of the system is

Mechanics: I-3 continued.

$$T = \frac{1}{2} m (\dot{s}_1^2 + \dot{s}_2^2 + \dot{s}_3^2 + \dot{s}_4^2)$$

As the springs are identical, at equilibrium the four masses are positioned symmetrically on the circle, i.e. the arc between two neighboring masses, the n-th and the $(n+1)$th subtends an angle $\pi/2$ at the center. When the neighboring masses are displaced from the equilibrium positions, the spring connecting them will be extended by

$$2b \sin\left[\frac{1}{2}\left(\frac{s_{n+1}}{b} - \frac{s_n}{b} + \frac{\pi}{2}\right)\right] - 2b \sin\frac{\pi}{4} \approx \frac{1}{\sqrt{2}}(s_{n+1} - s_n),$$

for small oscillations for which s_n are small. ∴ the potential energy is,

$$V = \frac{k}{2}\left[s_1^2 + s_2^2 + s_3^2 + s_4^2 - s_1 s_2 - s_2 s_3 - s_3 s_4 - s_4 s_1\right]$$

This system has four degrees of freedom and hence four normal modes.

(b) The T and V matrices are

$$T = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix}, \quad V = \begin{pmatrix} k & -k/2 & 0 & -k/2 \\ -k/2 & k & -k/2 & 0 \\ 0 & -k/2 & k & -k/2 \\ -k/2 & 0 & -k/2 & k \end{pmatrix}$$

[I-3] continued.

the secular equation becomes

$$\left| V - \omega^2 T \right| = \begin{vmatrix} k - m\omega^2 & -\frac{k}{2} & 0 & -k/2 \\ -k/2 & k - m\omega^2 & -k/2 & 0 \\ 0 & -k/2 & k - m\omega^2 & -k/2 \\ -k/2 & 0 & -k/2 & k - m\omega^2 \end{vmatrix} = 0$$

and the secular equation has four roots

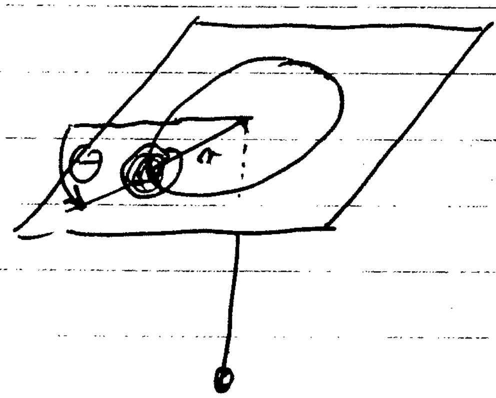
$$\left(0, 0, \sqrt{\frac{k}{m}}, \sqrt{\frac{2k}{m}} \right).$$

Hence, the angular frequencies of small oscillations are:

$$\sqrt{\frac{k}{m}}, \text{ and } \sqrt{\frac{2k}{m}} .$$

I-4) Solution Mechanics.

a) the two discriptive parameters are the length r of the string <u>on</u> the table an the angle Θ



Mass M: Kinetic Energy $\frac{1}{2}M(\dot{r}^2 + r^2\dot{\Theta}^2)$

mass m Kinetic Energy $\frac{1}{2}m\dot{r}^2$

potential energy $+mgr$ (sic +)

$\mathcal{L} = T - V$

resulting in $\mathcal{L} = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\Theta}^2) + \frac{1}{2}m\dot{r}^2 \overset{-}{(+)} mgr$

Equations of motion: Consider acting forces along coordinates:

Radial part: $F = m \cdot a \Rightarrow F_{radial} = (M+m)\ddot{r}$

Centripetal force $F_{cent} = M \cdot a_c =$

$= -M\frac{v^2}{r} = -M\frac{\dot{\Theta}^2 r^2}{r} = -M\dot{\Theta}^2 r$

gravitational force $= -mg$ (sic)

Balance of radial forces $\sum F = 0$

$$\Rightarrow (M+m)\ddot{r} - M\dot{\theta}^2 r - mg = 0$$

Equation of radial motion

Angular Part   angular Momentum $L = m \cdot v \cdot r$

$$- M\dot{\theta} r \cdot r = M\dot{\theta} r^2 = l_0$$

$$\hookrightarrow \dot{\theta} = \frac{l_0}{M r^2}$$

Angular Forces $\frac{dL}{dt} = 0$, since nothing acts on $\dot{\theta}$, not even friction

$\Rightarrow$ equation of angular motion

$$\frac{d}{dt}\left(M\dot{\theta} r^2\right) = 0$$

with Eqs of motion we re-write $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}M\left(\dot{r}^2 + \left(\frac{l_0}{Mr^2}\right)^2 r^2\right) + \frac{1}{2}m\dot{r}^2 + mgr$$

$$= \underbrace{\frac{1}{2}\dot{r}^2(M+m)}_{\text{Kinetic}} + \underbrace{\frac{l_0^2}{2Mr^2} + mgr}_{\text{potential}} \quad \square$$

b) to find Equilibrium in r_0

consider potential part of $\mathcal{L}$

$$U_{eff} = +mgr + \frac{l_0^2}{2Mr^2}$$

Check ~~stability~~ for extremum

should be

$$\frac{\partial U_{eff}}{\partial r} = mg - 2\frac{l_0^2}{2Mr^3} = 0$$

$\Rightarrow$ solve for r $\quad mg = \frac{l_0^2}{Mr^3}$

Equilibrium $\Rightarrow r_0 = \left(\frac{l_0^2}{Mmg}\right)^{1/3}$ $\square$

Stability $\frac{\partial^2 U_{eff}}{\partial r^2} = (-3)(-2)\frac{l_0^2}{2Mr^4} = +3\frac{l_0^2}{Mr^4} > 0$ $= \underline{\text{stable}}$

i.e., potential goes up for $r < r_0$ or $r > r_0$

Freq. of small oscillation $\quad \omega = \sqrt{\frac{D}{m}}$ ← pot. stiffness

$$D = \frac{\partial^2 U}{\partial r^2} \Rightarrow \omega^2 = \frac{1}{M+m}\left.\frac{\partial^2 U}{\partial r^2}\right|_{r=r_0} = \frac{3 l_0^2}{M r_0^4}$$

with $r_0^3 = \frac{l_0^2}{Mmg} \Rightarrow l_0^2 = Mmg\, r_0^3$

$$\Rightarrow \omega^2 = \frac{3}{M+m}\,\frac{r_0^3 g M m}{M r_0^4} = \frac{3gm}{(M+m) r_0} = \frac{3g}{\left(1+\frac{M}{m}\right) r_0}$$ $\square$

[I-5]

$$\frac{d\vec{p}}{dt} = \vec{F} \qquad \vec{p} = \frac{m_0 \vec{v}}{\sqrt{1+(\frac{v}{c})^2}} \qquad \vec{F} = m_0 g \hat{z}$$

$$\frac{d}{dt} \frac{v}{\sqrt{1-(\frac{v}{c})^2}} = g$$

$$\frac{v}{\sqrt{1-(\frac{v}{c})^2}} + v_0 = gt$$

$$v(t=0) = 0 \qquad v_0 = 0$$

$$\begin{cases} v = \dfrac{gt}{\sqrt{1+(\frac{gt}{c})^2}} \\ v = \dfrac{dx}{dt} \end{cases} \qquad \frac{dx}{dt} = \frac{gt}{\sqrt{1+(\frac{gt}{c})^2}}$$

$$x = \int_0^t \frac{g t' dt'}{\sqrt{1+(gt'/c)^2}}$$

$$= \frac{c^2}{g}\left(\sqrt{1+(\frac{gt}{c})^2} - 1\right) + x_0$$

$$x(t=0) = 0$$

$$x = \frac{c^2}{g}\left(\sqrt{1+(\frac{gt}{c})^2} - 1\right)$$

(1-5) continued.

In the nonrelativistic only the leading terms in the expansion in powers of (v/c) ~~have~~ to be included so that

$$v = \frac{gt}{\sqrt{1+(gt/c)^2}} = gt + \text{h.o.t}$$

$$x = \frac{c^2}{g}\left(\sqrt{1+(gt/c)^2} - 1\right) = \frac{gt^2}{2}$$

These values are predicted by Newtonian mechanics

[I-6.]

I-6 Solution

[I-6] Solution:

(a) $\vec{\nabla}\cdot\vec{D} = 4\pi\rho$ (i)

$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$ (ii)

$\vec{\nabla}\cdot\vec{B} = 0$ (iii)

$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$ (iv)

(b) Take the curl of (iv)

$$\vec{\nabla}\times\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{\nabla}\times\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{D}$$

use $(\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}) \Leftarrow$ given

hence we obtain

$$-\vec{\nabla}^2\vec{H} = \frac{4\pi}{c}\vec{\nabla}\times\vec{J} + \frac{1}{c}\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{D})$$ now $\vec{J} = \sigma\vec{E}$ and $\vec{D} = \epsilon\vec{E}$ hence:

$$-\nabla^2\vec{H} = \frac{4\pi}{c}\sigma\vec{\nabla}\times\vec{E} + \frac{\varepsilon}{c}\frac{\partial}{\partial t}\vec{\nabla}\times\vec{E}$$

using equation (ii) and $\vec{B} = \mu\vec{H}$ we find.
Or starting with equ (ii) we can find that:

$$\vec{\nabla}^2 E - \frac{\epsilon\mu}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} - \frac{4\pi\sigma\mu}{c^2}\frac{\partial E}{\partial t} = 0$$

(c) Now from $E_y = \psi_0 e^{i(kx-\omega t)}$ upon substitution into the above equation we have

$$-k^2 + \frac{\varepsilon\mu\omega^2}{c^2} + i\frac{4\pi\sigma\mu\omega}{c^2} = 0$$

$$\therefore k = \sqrt{\epsilon\mu}\,\frac{\omega}{c}\left(1 + i\frac{4\pi\sigma}{\epsilon\omega}\right)^{1/2} \equiv k' + ik''$$

solve for the square root of the complex expression.

$\therefore \left(1+i\dfrac{4\pi\sigma}{\epsilon\omega}\right)^{1/2} \equiv a+ib$, and $k' = \sqrt{\epsilon\mu}\,\dfrac{\omega}{c}\,a$

$k'' = \sqrt{\epsilon\mu}\,\dfrac{\omega}{c}\,b$ now squaring the above eqn.

$a^2 - b^2 = 1$, $2ab = \dfrac{4\pi\sigma}{\epsilon\omega}$ upon substitution

$b = \left(-\dfrac{1}{2} + \sqrt{\dfrac{1}{4} + \left(\dfrac{2\sigma\pi}{\epsilon\omega}\right)^2}\right)^{1/2}$ (+) sign branch of root chosen.

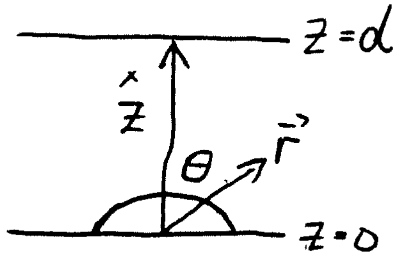
(need to satisfy $\sigma = 0$ (vacuum) $b = k'' = 0$ (no dissipation).

$\therefore k'' = \sqrt{\dfrac{\epsilon\mu}{2}}\,\dfrac{\omega}{c}\left(\sqrt{\left(\dfrac{4\pi\sigma}{\epsilon\omega}\right)^2 + 1} - 1\right)^{1/2}$ $\therefore$ attenuation length (e^{-1} point) for amplitude

$$\delta = (k'')^{-1} = \sqrt{\frac{2}{\epsilon\mu}}\,\frac{c}{\omega}\left(\sqrt{\left(\frac{4\pi\sigma}{\epsilon\omega}\right)^2 + 1} - 1\right)^{-1/2}$$

whereas for the intensity attenuation length is $(\delta/2)$.

[-7)



azimuthal symm.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

B.C.: $z=0$, $V=0$

$V=0$ on surface of hemisphere $(a \ll d)$

$z = d$, $V = V_0$, $V = \frac{V_0 z}{d} = V_0 \frac{r}{d}\cos\theta$

$$z=0: \Rightarrow B_l = -A_l a^{2l+1}$$

$$z=d: \Rightarrow V(r,\theta) = \sum_{l=0}^{\infty} A_l\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta) \to V_0 \frac{r}{d}\cos\theta$$

$\therefore$ $l=1$ only

$$\therefore V(r,\theta) = \frac{V_0}{d}\left(r - \frac{a^3}{r^2}\right)\cos\theta$$

$$I = \int_{\text{hemisphere}} \vec{J}\cdot d\vec{A} = \int \sigma \vec{E}\cdot d\vec{A}$$

close surface in $z=0$ plane + use Gauss' law

$$I = \sigma \int_{\text{sphere}} \vec{E}\cdot dA = \frac{\sigma}{\varepsilon_0} Q_{enc} = \frac{\sigma}{\varepsilon_0}\int \sigma_e \cdot dA$$

But surface chg. $\sigma_e = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a} = 3\varepsilon_0 \frac{V_0}{d}\cos\theta$

$$\therefore I = 3\sigma\pi V_0 \frac{a^2}{d}$$

I-8] Solution

Consider Electromagnetic induction $\mathcal{E}mf = -N\frac{d\phi}{dt}$

with magnetic flux $\Phi = B A$ ← Area of loop
↑ mg field

we have $B_z = B_0(1+kz) \Rightarrow \phi = AB_0(1+kz)$

$\Rightarrow \mathcal{E}mf = -1AB_0\frac{k\,dz}{dt}$, with $\frac{dz}{dt} = v_z$

$\hookrightarrow = -AB_0 k v_z$

Gravity performs $\boxed{\text{power}}$ $F\cdot\frac{\Delta s}{\Delta t} \Rightarrow \boxed{P_1} = m\cdot g\cdot v_z$ + acceleration

resistive Joule heating power $I^2\cdot R = \frac{\mathcal{E}mf^2}{R} = \boxed{P_2}$

with $\underset{\uparrow\text{Resistance}}{R} = \rho\cdot\frac{\text{length}}{\text{crossection}}$

$$= \rho\frac{\pi D}{\pi(d/2)^2} = \rho\frac{4D}{d^2}\,.$$

in terminal situation, ~~acc~~ no acceleration $\Rightarrow P_1 = P_2$

$$m g v_z = \frac{\mathcal{E}mf^2}{R}$$

$$= \frac{A^2B_0^2k^2v_z^2}{R}$$

$$\Rightarrow mgR = (AB_0k)^2 v_z$$

$$v_z = \frac{mgR}{(AB_0k)^2}$$

with $m = \rho_m \cdot Vol = \rho_m \cdot \text{length} \cdot \text{Area}$ 3

$$= \rho_m \pi D \cdot \pi \left(\frac{d}{2}\right)^2 = \rho_m \pi^2 D \frac{d^2}{4}$$

and $R = \rho \frac{4D}{d^2}$ (as above)

$$\Rightarrow v_z = \frac{\rho_m \pi^2 D \frac{d^2}{4} g \rho \frac{4D}{d^2}}{\left(\pi \left(\frac{D}{2}\right)^2 B_0 k\right)^2} = \frac{\rho_m g \rho 4^2}{D^2 B_0^2 k^2}$$

$\Rightarrow$ Insulators fall fester, $\rho \nearrow$

Weaker field $\rightarrow$ fall faste $B_0 \searrow$

smaller rings fall faste $D \searrow$

denser rings fall faste $\rho_m \nearrow$ o.k.

[I-9]

induced dipole moment $\vec{P} = \alpha \vec{E}$

a) scattered fields

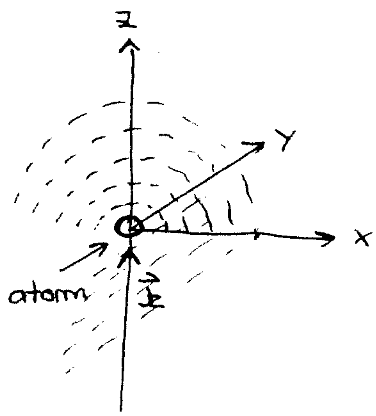
$$\vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r} \left[(\vec{n} \times \vec{p}) \times \vec{n} \right]$$

$$\vec{H}_{sc} = \vec{n} \times \vec{E}_{sc} / Z_0 \qquad \text{or} \qquad \vec{E} = \frac{(\ddot{\vec{p}} \times \vec{n}) \times \vec{n}}{4\pi \varepsilon_0 c^2 R}$$

$$\vec{B} = \frac{\ddot{\vec{p}} \times \vec{n}}{4\pi \varepsilon_0 c^3 R}$$

$$\vec{P} = \alpha E_0 e^{i(kz - \omega t)} \vec{e}_y$$

$\hat{n}$: unit vector between source & observer ~~x axis~~



$$\vec{P} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix}$$

(i) on y-axis

$\vec{n} = \vec{e}_y \qquad r = D$

$$(\vec{n} \times \vec{p}) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow (\vec{n} \times \vec{p}) \times \vec{n} = 0$$

$$\Rightarrow \vec{E}_{sc} = 0 \;, \; \vec{H}_{sc} = 0 \quad \text{on y-axis}$$

(ii) on x-axis $\quad \vec{n} = \vec{e}_x \quad r = D$

$$(\vec{n} \times \vec{p}) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix}$$

$$(\vec{n} \times \vec{p}) \times \vec{n} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ 0 \end{pmatrix} \qquad \vec{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikD}}{D} \alpha E_0 e^{i(kD - \omega t)} \vec{e}_y$$

$$\vec{H}_{sc} = \vec{n} \times \vec{E}_{sc} / Z_0 \qquad \vec{H}_{sc} = H_{sc} \vec{e}_z$$

b) $S = \frac{1}{2} \frac{E^2}{Z_0}$ $\qquad$ power $P = \int dt \int d\Sigma \, S = \frac{\alpha^2 \omega^4 E_0^2}{12\pi \varepsilon_0 c^3}$

[I-10] It can be shown, in more than one way that $\vec{E}$ as given is a good candidat for an electric field.

a) One can show that $\vec{E}$ and $\vec{B}$ satisfy Maxwell's equation

$$\vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\cdot\vec{E}=0$$

$$\nabla \vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

$$\vec{\nabla}\vec{D}-\frac{\partial\vec{H}}{\partial t}=0$$

$\vec{B}$ can be determind from the third equation.

b) Alternatively one show that $\vec{E}$ ad $\vec{B}$ satisfy the wave equations

$$\nabla^2\vec{E}-\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}=0 \qquad \vec{\nabla}^2\vec{B}-\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}=0$$

To calculate the toat energ per unit time one has to integrat the energy flux ove a sphere

$$\frac{dE}{dt}=\iint(\vec{E}\times\vec{H})\cdot\hat{n}\,da$$

The easy way to integrate ove a sphere at infinity and take ~~the~~ leading term

Physics PhD Qualifying Examination : With Solutions

Part II – Wednesday, August 25, 2004

Name:____

(please print)

Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.

PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

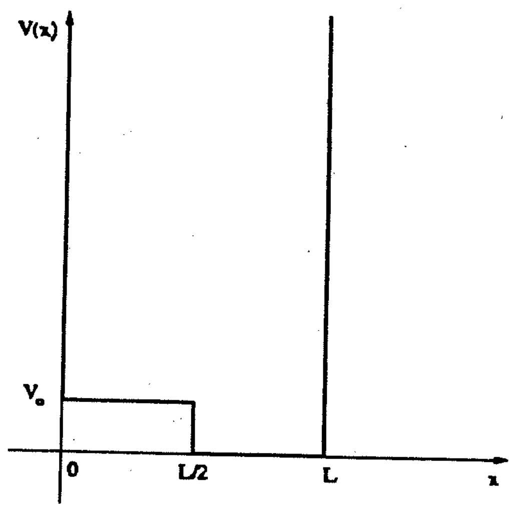
	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

II-1. [10]
A particle of mass m is in an infinitely high one-dimensional potential box of width L. At the bottom of the box, there is a finite bump of height V_o and width $L/2$. Using time-independent perturbation theory, determine the perturbed energy up to second order and the perturbed wavefunction up to first order *of the groundstate*. Give the condition in terms of V_o, m, $\hbar$, and L for the perturbation expansion to be sensible.



II-2. [4,4,2]

A particle with an initial momentum vector k_a is scattered by a potential $V(\vec{r})$ into a state k_b.

(a) Write down an expression for the first term in the amplitude of the scattered wave in the Born approximation. To what does this reduce for elastic scattering in a spherically symmetric potential?
(b) State and and outline how to derive the optical theorem.
(c) A positron is scattered by a nucleus of charge Ze. What are the differential and total cross sections? Can you compare this result to the classical analog?

II-3. [10]

Use $\frac{d}{dt}\langle \hat{Q} \rangle = \frac{i}{\hbar}\langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$, which holds for any observable Q, to prove that, for a particle in a general potential $V(\mathbf{r})$, the rate of change of the expectation value of the orbital angular momentum **L** is equal to the expectation value of the torque:
$\frac{d}{dt}\langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle$, where $\mathbf{N} = \mathbf{r} \times (-\nabla V)$.

Use this to show that $\langle \mathbf{L} \rangle$ is conserved for any spherically symmetric potential.

II-4. [3,4,3]

A particle of mass m moves in a three dimensional central potential $V(\vec{r})$ which vanishes for $r \to \infty$. The particle is non-relativistic.

We know the exact Eigenstate of the particle

$$\Psi(\vec{r}) = C\, r^{\sqrt{3}}\, e^{\alpha r} \cos\Theta$$

Where C and α are constants.

a) What is the angular momentum of this state? Justify your answer.

b) What is the energy, what is the kinetic energy of the particle?

c) What is $V(\vec{r})$? Is the potential attractive or repulsive?

II-5. [10]

Prove the inequality $\| A\phi \| \| B\phi \| \geq \frac{1}{2} | \phi, [A,B]\phi |$.

Where A and B are Hermitian operators and [A,B] = AB-BA is the commutator of A and B. ϕ is a vector in Hilbert space.

II-6. [10]

A linear harmonic oscillator is acted upon by a uniform electric field which is considered to be a perturbation and which depends as follows on the time:

$$\mathscr{E}(t) = A \frac{1}{\sqrt{(\pi)}\,\tau} e^{-(t/\tau)^2},$$

where A is a constant and τ is the characteristic time parameter. Solve this problem by using perturbation techniques. Assuming that when the field is switched on (that is, at $t = -\infty$) the oscillator is in its ground state, evaluate to a first approximation the probability that it is excited at the end of the action of the field (that is, at $t = +\infty$).

II-7. [4,6]

A gas is initially confined to one-half of a thermally isolated container. The other half is empty. The gas is suddenly permitted to expand to fill the entire chamber. Assuming the initial temperature of the gas in the half-container is T_i, find the temperature T_f after the expansion for the following two cases:

(a) Equation of state : $pV = nRT$

(b) Equation of state: b $(p + \frac{a}{V^2}) = nRT$

II-8. [10]

Derive a general expression for the difference between the specific heat at constant pressure C_p and the specific heat at constant volume C_v, $\Delta C = C_p - C_v$, in terms of thermodynamic variables (P, V, T) and their derivatives.

II-9. [4,3,3]

Consider the three-dimensional non-interacting classical ultra-relativistic gas, (e = pc), in the canonical ensemble.

(a) Find the partition function.
(b) Find the equation of state.
(c) Find the internal energy and the specific heat.

II-10. [10]

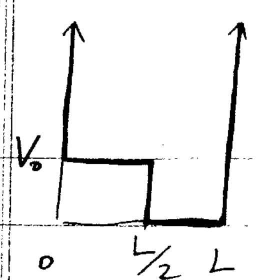
If a magnetic field H is applied to a gas of uncharged particles having spin 1/2 and magnetic moment μ and obeying Fermi-Dirac statistics, the lining up of the spins produces a magnetic moment/volume. Set up general expressions for the magnetic moment/volume at arbitrary T and H.

Then for low enough temperature, determine the magnetic susceptibility of the gas in the limit of zero magnetic field, correct to terms of order T^2. Note the integral

$$\int_0^\infty \frac{\sqrt{E}\,dE}{\exp\left[(E-\xi)/kT\right]+1} = \frac{2}{3}\xi^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\xi}\right)^2+\cdots\right].$$

Here is defined as the chemical potential function in the Fermi-Dirac distribution.

II-1



unperturbed solution:

$$\varphi_n^{(0)}(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$E_n^{(0)} = \frac{\hbar^2 \pi^2}{2mL^2} n^2 = E_1^{(0)} n^2 \qquad n = 1, 2, 3, \ldots$$

perturbation:

$$H'(x) = \begin{cases} V_0 & \text{if} \quad 0 \leq x \leq L/2 \\ 0 & \text{if} \quad L/2 < x \leq L \end{cases}$$

matrix elements needed:

$$H'_{kn} = \int_0^L \varphi_k^{(0)*}(x)\, H'(x)\, \varphi_n^{(0)}(x)\, dx = \int_0^{L/2} \varphi_k^{(0)}(x)\, V_0\, \varphi_n^{(0)}(x)\, dx = \frac{2V_0}{L} \int_0^{L/2} \sin\left(\frac{k\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$= -\frac{V_0}{L} \int_0^{L/2} \left\{ \cos\left(\frac{(k+n)\pi x}{L}\right) - \cos\left(\frac{(k-n)\pi x}{L}\right) \right\} dx = \frac{V_0}{L} \left\{ \int_0^{L/2} \cos\left(\frac{(k-n)\pi x}{L}\right) dx - \int_0^{L/2} \cos\left(\frac{(k+n)\pi x}{L}\right) dx \right.$$

$$= \frac{V_0}{L} \left\{ \frac{L}{(k-n)\pi} \sin\left(\frac{(k-n)\pi x}{L}\right) \Bigg|_0^{L/2} - \frac{L}{(k+n)\pi} \sin\left(\frac{(k+n)\pi x}{L}\right) \Bigg|_0^{L/2} \right\} =$$

$$\frac{V_0}{\pi} \left\{ \frac{1}{k-n} \sin\left(\frac{(k-n)\pi}{2}\right) - \frac{1}{k+n} \sin\left(\frac{(k+n)\pi}{2}\right) \right\}$$

and $H'_{nn} = \dfrac{V_0}{2} \qquad \forall n$

ground state perturbation $\quad n=1$

$k \neq 1$: $\quad H'_{k1} = \frac{V_0}{\pi}\left\{\frac{1}{k-1}\sin\left(\frac{(k-1)\pi}{2}\right) - \frac{1}{k+1}\sin\left(\frac{(k+1)\pi}{2}\right)\right\} =$

$$= \begin{cases} (-1)^{\ell}\frac{V_0}{\pi}\frac{-4\ell}{4\ell^2-1} & \text{if } k=2\ell \text{ (even) } \ell=1,2,3,\ldots \\ \emptyset & \text{if } k=2\ell+1 \text{ (odd) } \ell=1,2,3\ldots \end{cases}$$

Ground-state energy up to 2nd order:

$n=1$

$$E_1 = E_1^{(0)} + H'_{11} + \sum_{k\neq 1}\frac{|H'_{k1}|^2}{E_1^{(0)}-E_k^{(0)}} =$$

$$= E_1^{(0)} + \frac{V_0}{2} + \sum_{\ell=1}^{\infty}\frac{|H'_{2\ell,1}|^2}{E_1^{(0)}-E_{2\ell}^{(0)}} =$$

$$= E_1^{(0)} + \frac{V_0}{2} + \sum_{\ell=1}^{\infty}\frac{\frac{V_0^2}{\pi^2}\frac{16\ell^2}{(4\ell^2-1)^2}}{E_1^{(0)}(1-(2\ell)^2)} =$$

$$= E_1^{(0)} + \frac{V_0}{2} - \sum_{\ell=1}^{\infty}\frac{V_0^2}{E_1^{(0)}\pi^2}\frac{16\ell^2}{(4\ell^2-1)^3} =$$

$$= E_1^{(0)}\left\{1 + \frac{1}{2}\frac{V_0}{E_1^{(0)}} - \frac{1}{\pi^2}\left(\frac{V_0}{E_1^{(0)}}\right)^2\sum_{\ell=1}^{\infty}\frac{16\ell^2}{(4\ell^2-1)^3}\right\}$$

Ground-state wave function up to 1st order

$$\varphi_1(x) = \varphi_1^{(0)}(x) + \sum_{k\neq 1} \frac{H'_{k1}}{E_1^{(0)} - E_k^{(0)}} \varphi_k^{(0)}(x)$$

$$= \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) + \sum_{\ell=1}^{\infty} \left(-\frac{V_0}{\pi}\right)(-1)^{\ell} \frac{4\ell}{4\ell^2-1} \cdot \frac{1}{E_1^{(0)}(1-4\ell^2)} \sqrt{\frac{2}{L}} \sin\left(\frac{2\ell\pi x}{L}\right)$$

$$= \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) + \frac{1}{\pi}\left(\frac{V_0}{E_1^{(0)}}\right)\sqrt{\frac{2}{L}} \sum_{\ell=1}^{\infty} (-1)^{\ell} \frac{4\ell}{(4\ell^2-1)^2} \sin\left(\frac{2\pi\ell x}{L}\right)$$

$\frac{V_0}{E_1^{(0)}} \ll 1$ is needed, i.e.,

$$\frac{V_0}{\frac{\hbar^2\pi^2}{2mL^2}} = \frac{2mV_0L^2}{\hbar^2\pi^2} \ll 1$$

$$\boxed{\frac{2mV_0L^2}{\hbar^2\pi^2} \ll 1}$$

s)

[II-2.] Taking the first two terms of originally plane wave φ is

$$\varphi = e^{ikz} + \frac{f(\theta)\, e^{ikr}}{r}$$

The scattering amplitude $f(\theta)$, in the Born Approximation, can be written as

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{1}{4\pi}\iiint e^{i(\vec{k}-\vec{k}_0)\cdot\vec{r}\,'}\, V(\vec{r}\,')\, d\vec{r}\,'$$

Here $V(\vec{r}\,')$ is the perturbing potential and $q = (\vec{k}-\vec{k}_0)\hbar$ is the change in momentum. For the case of a spherically symmetric potential the expression for $f(\theta)$ can be simplified to

$$f(\theta) = \frac{-2m}{\hbar^2}\frac{1}{4\pi}\iiint e^{iqr'\cos\theta}\, r'^2 dr' \sin\theta' d\theta' d\phi$$

$$f(\theta) = \frac{-2m}{\hbar^2 q}\int_0^\infty r' V(r') \sin q r' \, dr'$$

The optical theorem states

$$\sigma_{total} = \frac{4\pi}{k} \operatorname{Im}\left(f(0)\right)$$

The differential cross in the partial wave expansion is

$$\frac{d\sigma}{d\Omega}(\theta) = |f(\theta)|^2$$

and $f(\theta)$ is given as

$$f(\theta) = \frac{1}{k}\sum(2l+1)\, e^{i\delta_l} \sin\delta_l \, P_l(\cos\theta)$$

$$\sigma_t = \frac{4\pi}{k^2} \sum(2l+1)\sin^2\delta_l$$

Here δ_l is the phase shift of the l_{th} partial wave

We now calculat $\sigma_{tot} = \frac{4\pi}{k}\, \mathrm{Im}\, f(\theta)$

$$\sigma'_{total} = \frac{4\pi}{k}\, \mathrm{Im} f(\theta = 0)$$

$$f'(\theta) \quad \frac{4\pi}{k}\frac{1}{k} \sum(2l+1) \sin\delta_l (\cos\delta_l + i \sin\delta_l)$$

$$= \frac{4\pi}{k^2} \sum(2l+1)\sin^2\delta_l$$

Thus $\sigma'_{total} = \sigma_{tot}$

II-3) (a) $\frac{d}{dt}\langle L_x\rangle = \frac{i}{\hbar}\langle[H,L_x]\rangle = \underbrace{\frac{i}{\hbar}\frac{1}{2m}\langle[p^2,L_x]\rangle}_{\text{show that} = 0} + \frac{i}{\hbar}\langle[V,L_x]\rangle$

$$\Rightarrow [V,(yp_z - zp_y)] = y\, i\hbar \frac{\partial V}{\partial z} - z\, i\hbar \frac{\partial V}{\partial y}$$

$$= i\hbar\, [\vec{r}\times(\vec{\nabla}V)]_x$$

$$\therefore \frac{d\langle\vec{L}\rangle}{dt} = \langle(\vec{r}\times(-\vec{\nabla}V))\rangle$$

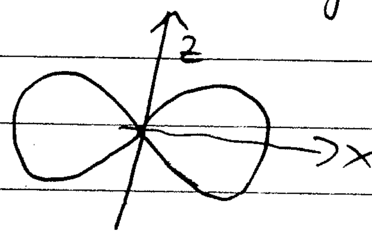
(b) if $V(\vec{r}) = V(r) \Rightarrow \vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r}$; $\vec{r}\times\hat{r} = 0$

$\therefore \langle\vec{L}\rangle$ conserved

[II-4.] Solution

a) the short way: Consider the angular dependence of $\psi(\vec{r})$

$= \cos\Theta$



This is a p_z state with angular momentum $l=1$.

a) the long way

Consider energy operator in spherical polar coordinates:

$$\mathcal{E}\psi = \frac{-\hbar^2}{2m}\nabla^2\psi$$

$$= -\frac{\hbar^2}{2m}\frac{1}{r^2}\Big[\underbrace{\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)}_{A} + \underbrace{\frac{1}{\sin\Theta}\frac{\partial}{\partial\Theta}\Big(\sin\Theta\frac{\partial}{\partial\Theta}\Big)}_{B}$$

$$+ \underbrace{\frac{1}{\sin^2\Theta}\Big(\frac{\partial^2}{\partial\varphi^2}\Big)}_{D}\Big]\psi$$

with a radial component A, an angular component in Θ, B and an angular component in φ, D

Applying $\mathcal{E}$ to $\psi = C\, r^{\sqrt{3}} e^{-\alpha r}\cos\Theta$, (write $s=\sqrt{3}$)

we find $\frac{\partial^2}{\partial\varphi^2}\psi = 0 \Rightarrow D = 0$

B+D is the operator of the angular momentum L^2, (square)

$$\cancel{L^2 \sim B}$$

$$L^2\psi = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi^2}\right]\psi$$

individual step:

$$\frac{\partial}{\partial\theta}\psi = -Cr^5 e^{-\alpha r}\sin\theta$$

$$\sin\theta\frac{\partial}{\partial\theta}\psi = -Cr^5 e^{-\alpha r}\sin^2\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\psi = 2Cr^5 e^{-\alpha r}\cos\theta\sin\theta$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\psi = 2Cr^5 e^{-\alpha r}\cos\theta = \underline{\underline{2\psi}}$$

with $L^2 = l(l+1) \Rightarrow \underline{\underline{l=1}}$

The angular momentum is $\underline{\underline{l=1}}$ ✓

Solution

b) Consider total energy operator using results for angular momentum

$$\mathcal{E}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$

$$= -\frac{\hbar^2}{2m}\frac{1}{r^2}\Big[\underbrace{\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)}_{A} - l(l+1)\Big]\psi$$

apply A: in individual steps

$$\frac{\partial}{\partial r}\psi = \frac{\partial}{\partial r} C r^s e^{-\alpha r}\cos\theta =$$

$$= C\cos\theta\Big[\frac{r^s}{r}e^{-\alpha r} + r^s(-\alpha)e^{-\alpha r}\Big]$$

$$= \Big(\frac{s}{r}-\alpha\Big)\psi.$$

$$\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\Big)\psi = \frac{\partial}{\partial r}\big(rs-\alpha r^2\big)\psi$$

$$= s\psi + rs\Big(\frac{s}{r}-\alpha\Big)\psi - \Big(2\alpha r + \alpha r^2\Big(\frac{s}{r}-\alpha\Big)\Big)\psi$$

$$= (s+s^2-\alpha rs - 2\alpha r - \alpha s r + \alpha^2 r^2)\psi$$

$$= (\alpha^2 r^2 - 2\alpha(r+rs) + s(s+1))\psi$$

$$\Rightarrow \frac{1}{r^2}A\psi = \Big(\alpha^2 - \frac{2\alpha(1+s)}{r} + \frac{s(s+1)}{r^2}\Big)\psi$$

and with L^2

$$\mathcal{E}\psi = -\frac{\hbar^2}{2m}\Big(\alpha^2 - \frac{2\alpha(1+s)}{r} + \frac{s(s+1)}{r^2} - \frac{2}{r^2}\Big)\psi$$

$$= -\frac{\hbar^2}{2m}\Big(\alpha^2 - \frac{2\alpha(1+\sqrt{3})}{r} + \frac{1+\sqrt{3}}{r^2}\Big)\psi$$

Solution

with $\mathcal{E}_{total} = \mathcal{E}_{kinetic} + \mathcal{E}_{potential}$

In the limit of $r \to \infty$, the potenial & potential Energy vanishes

$$\forall\ r \to \infty \quad \mathcal{E}_{kin} = -\frac{\hbar^2}{2m}\alpha^2 \quad \square$$

This is the kinetic energy of the particle.

c) The potential $V(r)$ is the difference of total & kinetic Energy

$$V(r) = \mathcal{E}_{total} - \mathcal{E}_{kin} = -\frac{\hbar^2}{2m}\left(-2\alpha\frac{(1+\sqrt{3})}{r} + \frac{1+\sqrt{3}}{r^2}\right)$$

$$= \frac{\hbar^2}{2m}(1+\sqrt{3})\left(\frac{-2\alpha}{r} + \frac{1}{r^2}\right) \quad \square$$

the potential has an attractive term $\sim \frac{1}{r}$

AND

an repulsive term $\sim \frac{1}{r^2}$

$$\frac{1}{2}\,|\phi, [A,B]\phi| = \frac{1}{2}\,|\phi, (AB-BA)\phi|$$

$$= \frac{1}{2}\,|(\phi, AB\phi) - (\phi, BA\phi)|$$

$$= \frac{1}{2}\,|(A\phi, B\phi) - (B\phi, A\phi)| \qquad \left\{ \begin{array}{l} A = A^\dagger \\ B = B^\dagger \\ \text{Hermitian Ope-} \\ \text{rator} \end{array} \right.$$

$$= \frac{1}{2}\,|(A\phi, B\phi) - \overline{(A\phi, B\phi)}| \qquad (\psi, \varphi) = \overline{(\varphi, \psi)}$$

$$= \frac{1}{2}\,|2\,\mathrm{Im}\,(A\phi, B\phi)| \qquad (x+iy)-(x-iy) = i2y$$

$$= |\mathrm{Im}\,(A\phi, B\phi)| \leq |(A\phi, B\phi)| \leq \|A\phi\|\,\|B\phi\|$$

alternative approach to solve II-5

 From David Park: "Introduction to Quantum Theory" McGraw Hill 1992

To derive the indeterminacy relations between two quantities P and Q in terms of the standard deviations of their statistical distributions, let $\hat{P}$ and $\hat{Q}$ be the corresponding operators in some representation and let their commutator be written as

$$[\hat{P}, \hat{Q}] = i\hat{C} \tag{3.36}$$

where the i has been introduced in order that $\hat{C}$ shall be a hermitian operator. Now examine the real number

$$G = \int |(\hat{P} + i\lambda\hat{Q})\psi|^2$$

considered as a function of the real number λ. Evidently, G must be positive or zero. Writing it out in detail, we have, by (3.28),

$$G = \int [(\hat{P} + i\lambda\hat{Q})\psi]^* . (\hat{P} + i\lambda\hat{Q})\psi$$

$$= \int (\hat{P}^* - i\lambda\hat{Q}^*)\psi^* . (\hat{P} + i\lambda\hat{Q})\psi$$

(the dot, as before, shows where $\hat{P}$ and $\hat{Q}$ stop operating)

$$G = \int \psi^*(\hat{P} - i\lambda\hat{Q})(\hat{P} + i\lambda\hat{Q})\psi \geq 0$$

(Problem 3.20 warns to be careful with hermiticity in this step.) Expanding the product but keeping track of the order of factors and using (3.36) gives

$$G = \langle \hat{P}^2 \rangle - \lambda\langle \hat{C} \rangle + \lambda^2 \langle Q^2 \rangle \geq 0 \tag{3.37}$$

which must be true for any real value of λ. In view of (3.33), G is certainly not negative when λ tends toward 0 or ∞, but even the smaller intermediate values are also not negative. The graph of G as a function of λ must therefore look something like Fig. 3.1.

G is smallest when $dG/d\lambda = 0$, or $\lambda = \langle \hat{C} \rangle / 2\langle \hat{Q}^2 \rangle$, so that

$$G_{\min} = \langle \hat{P}^2 \rangle - \frac{1}{4}\frac{\langle \hat{C} \rangle^2}{\langle \hat{Q}^2 \rangle} \geq 0$$

that is,

$$\langle \hat{P}^2 \rangle \langle \hat{Q}^2 \rangle \geq \tfrac{1}{4}\langle \hat{C} \rangle^2 \tag{3.38}$$

The uncertainty principle is a relation between standard deviations defined as in (3.34), and it follows at once from the general relation (3.38). To derive it replace $\hat{P}$ and $\hat{Q}$ in (3.38) by $\hat{P} - \langle \hat{P} \rangle$ and $\hat{Q} - \langle \hat{Q} \rangle$, respectively. The new $\hat{P}$ and $\hat{Q}$ have the same commutation relation (3.36) as the old, but now comparison with (3.34) gives

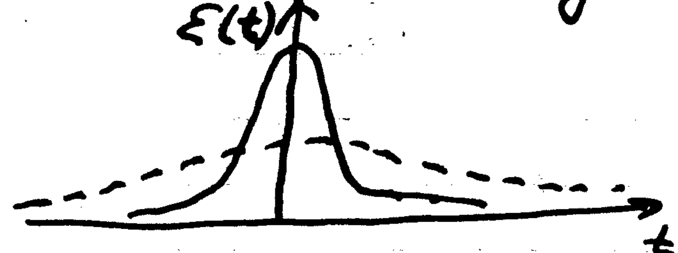
$$\Delta P \, \Delta Q \geq \tfrac{1}{2}|\langle \hat{C} \rangle| \tag{3.39}$$

[II-6] Solution

Let us show that τ enters into the perturbing field in such a way that the total pulse P, which is transferred to the oscillator by the electrical field over the duration of the perturbation, does not depend on τ,

$$P = \int_{-\infty}^{\infty} e\,\mathcal{E}(t)\,dt = \frac{eA}{\sqrt{\pi}\,\tau}\int_{-\infty}^{\infty} \exp\left[-\left(\frac{t}{\tau}\right)^2\right]dt = eA = \text{con}$$

Graphically this means that the area under the curve is the same for all values of τ i.e.



Now the probability for a transition from the n-th stationary state of the discrete spectrum to the k-th is equal to

$$w_{nk} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} V_{kn}\exp(i\omega_{kn}t)\,dt\right|^2$$

with the matrix element for the perturbation $\hat{V}$, $V_{kn} = \int_{-\infty}^{\infty}\psi_k^{(0)*}\hat{V}\psi_n^{(0)}\,dx$, with

$\omega_{kn} = \frac{1}{\hbar}[E_k^{(0)} - E_n^{(0)}]$, where $\psi_k^{(0)}, \psi_n^{(0)}, E_k^{(0)}$ and $E_n^{(0)}$ are the wave functions and energy levels of the corresponding (unperturbed) stationary states.

Let e, μ and ω the charge, mass and eigenfreque...

of the oscillator, and x its displacement from its equilibrium position, then with a uniform field for the perturbation operator,

$$\hat{V}(x,t) = -ex\mathcal{E}(t) \sim x$$

Hence, only the following matrix elements in the energy representation are different from zero,

$$x_{n,n+1} = x_{n+1,n} = \sqrt{[(n+1)\hbar/2\mu\omega]}$$

(oscillator originally in ground state, $n=0$)

$$\therefore \; V_{01} = V_{10} = \frac{-P}{\sqrt{\pi}\,\tau}\sqrt{\frac{\hbar}{2\mu\omega}}\;\exp\left[-\left(t/\tau\right)^2\right]$$

In first order perturbation theory the uniform field can produce a transition of the oscillator only to the first excited state ($k = n+1 = 1$)

To evaluate the probability for a transition to the second excited state we must use second-order perturbation theory ($n=0 \to n=2$), goes through the intermediate state $n=1$ and so on; $\therefore\; 0 \to 1 \to 2 \to 3$. Now $\omega_{kn} = \omega_{10} \equiv \left(\frac{1}{\hbar}\right)\left[E_1^{(0)} - E_0^{(0)}\right] = \omega$ and take $V_{01} = V_{10} = \ldots$, into the expression for w_{nk} above, we obtain the probability for excitation,

$$w_{01} = \frac{P^2}{2\pi\tau^2\mu\hbar\omega}\left|\int_{-\infty}^{\infty}\exp\left[i\omega t-\left(t/\tau\right)^2\right]dt\right|^2$$

from the well known integral relation

$$\int^{\infty}\exp(i\beta x - \alpha x^2)\,dx = \sqrt{(\pi/\alpha)}\,\exp(-\beta^2/4\alpha)\quad [\text{Give}]$$

we obtain

$$w_{01} = \frac{P^2}{2\mu\hbar\omega} \exp\left[-\frac{1}{2}(\omega\tau)^2\right]$$

now for $\tau \gg \frac{1}{\omega}$ (adiabatic perturbation

for $\tau \ll \frac{1}{\omega}$ the probability for excitation is practically constant.

[II-7] Free expansion of a gas.

$$dU = \delta Q + \delta W = 0$$

$\delta Q = 0$ ~~thermally~~ adiabatic
$\delta W = 0$ sudden expansion
$U = U(T,V) = \text{const.}$

$$dU = T\,dS + p\,dV$$

$$= T\left(\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV - p\,\cancel{dV}\right)$$

$c_V = T\left(\frac{\partial S}{\partial T}\right)_V$

$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$

$$= c_V\,dT + \left(T\left(\frac{\partial p}{\partial T}\right)_V - p\right) dV = 0$$

$$\rightsquigarrow \quad dT = \frac{1}{c_V}\left(p - T\left(\frac{\partial p}{\partial T}\right)_V\right) dV$$

(a)

$$pV = nRT$$

$$p = \frac{nRT}{V}$$

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{nR}{V} \quad \Rightarrow \quad dT = \frac{1}{c_V}\left(p - \frac{TnR}{V}\right) dV = 0$$

$$dT = 0 \rightsquigarrow T_F = T_i$$

(b)

$$b\left(p + \frac{a}{V^2}\right) = nRT$$

$$p = \frac{nRT}{b} - \frac{a}{V^2 b}$$

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{nR}{b} \quad \Rightarrow \quad dT = \frac{1}{c_V}\left[\frac{nRT}{b} - \frac{a}{V^2} - \frac{TnR}{b}\right] dV$$

$$dT = -\frac{1}{c_V}\frac{a}{V^2}\,dV$$

$$\int_{T_i}^{T_F} dT = -\frac{1}{c_V}\,a\int_{V_i}^{V_F}\frac{dV}{V^2}$$

$V_i = V_0$
$V_F = 2V_0$

$$T_F - T_i = \frac{a}{c_V}\left(\frac{1}{V_F} - \frac{1}{V_i}\right) = -\frac{a}{c_V V_0}$$

[II-8] Solution

$$C_p\,dT - T\left(\frac{\partial V}{\partial T}\right)_p = C_V\,dT + T\left(\frac{\partial P}{\partial T}\right)_V dV$$

$$dT = \frac{T\left(\frac{\partial P}{\partial T}\right)_V}{C_p - C_V} + \frac{T\left(\frac{\partial V}{\partial T}\right)_p}{C_p - C_V}\,dP$$

Solving for $\left(\frac{\partial T}{\partial V}\right)_p$ and $\left(\frac{\partial T}{\partial P}\right)$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{T\left(\frac{\partial P}{\partial T}\right)_V}{C_p - C_V}$$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{T\left(\frac{\partial V}{\partial T}\right)_p}{C_p - C_V}$$

$$C_p - C_V = T\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial P}{\partial V}\right)_T$$

$$\left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial P}{\partial V}\right)_T$$

$$C_p - C_V = -T\left(\frac{\partial V}{\partial T}\right)_p^2 \frac{\partial P}{\partial V}$$

$$\left(\left(\frac{\partial P}{\partial T}\right)_V = -\left(\frac{\partial V}{\partial T}\right)_p \left(\frac{\partial P}{\partial V}\right)_T \right)$$

Also simpler for ideal gas

$$dQ = C_V \, dT + p\, dV$$

$$pV = nRT$$

$$p\,dV + V\,dp = nR\,dT$$

$$C_p = C_V + nR$$

$$C_p - C_V = nR$$

II - 9 Consider the 3d non-interacting classical ultra-relativistic gas ($\varepsilon = pc$) in the canonical ensemble

a) Find the partition function
b) Find the equation of state
c) Find the internal energy & the specific heat

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a)

$$Z = \frac{1}{N!\,h^{3N}} \int d\Gamma\, e^{-\beta \sum_{i=1}^{N} c p_i} \qquad d\Gamma = dp^{3N} dq^{3N} \qquad p_i = |\bar{p}_i|$$

$$\beta = \frac{1}{k_B T}$$

$$= \frac{1}{N!\,h^{3N}} V^N \left[\int_0^\infty e^{-\beta c p} 4\pi p^2 dp \right]^N$$

$$= \frac{V^N}{N!\,h^{3N}} \left[\frac{8\pi}{(\beta c)^3} \right]^N = \frac{1}{N!} \left[\frac{8\pi V (k_B T)^3}{h^3 c^3} \right]^N$$

b) $F = -k_B T \ln Z$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{k_B T N}{V} \qquad PV = N k_B T$$

c)

$$E = -\frac{\partial}{\partial \beta} \ln Z = 3NkT$$

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = 3Nk$$

[II-10] <u>Solution</u>:

The energy of a particle whose magnetic moment is parallel (antiparallel) to H is given by

$$U_{\mp} = \frac{p^2}{2m} \mp \mu H.$$

Since the energy levels of the system are populated according to the distribution function

$$f(U) = \frac{1}{\exp\left(\frac{[U-\zeta]}{kT}\right) + 1}$$

and the density of levels is given by $(4\pi V/h^3)\, p^2 dp$, the total number of particles N is given by

$$N = \frac{4\pi V}{h^3}\int p^2 dp\,[f(U_-) + f(U_+)] \quad (1)$$

and the magnetization/volume is

$$\frac{M}{V} = \frac{4\pi\mu}{h^3}\int dp\, p^2\,[f(U_-) + f(U_+)]. \quad (2)$$

Equation (1) may be solved in terms of N, T and H, and ζ may then be substituted into equ. (2) to determine M/V as a function of N, T, and H.

[II-10] Solution (continued)

Upon defining a new variable of integration $E' = p^2/2m$ and using the low temperature expansion formula given, we find that equation (2) becomes

$$\frac{M}{V} = \frac{8\pi\mu(2m^3)^{1/2}}{3h^3}\left\{(\zeta+\mu H)^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\zeta+\mu H}\right)^2\right] - (\zeta-\mu H)^{3/2}\left[1+\frac{\pi^2}{8}\left(\frac{kT}{\zeta-\mu H}\right)^2\right]\right\},$$

which after expanding in powers of H and keeping only the leading term, becomes

$$\frac{M}{V} = \frac{8\pi\mu^2(2m^3)^{1/2}}{h^3}\zeta^{1/2}H\left\{1-\frac{\pi^2}{24}\left(\frac{kT}{\zeta}\right)^2+\cdots\right\}$$

+ higher order terms in H^3. Equ (1) for $H=0$, becomes

$$n = \frac{N}{V} = \frac{16\pi}{3h^3}(2m^3)^{1/2}\zeta^{3/2}\left\{1+\frac{\pi^2}{8}\left(\frac{kT}{\zeta}\right)^2+\cdots\right\}$$

Solving for ζ, one obtains

$$\zeta = \zeta_0\left\{1-\frac{\pi^2}{12}\left(\frac{kT}{\zeta_0}\right)^2+\cdots\right\},$$ where

ζ_0 is the Fermi energy at $T = 0°K$ and $\zeta_0^{3/2} = 3h^3 n/16\pi(2m^3)^{1/2}$. The susceptibility becomes then

$$\chi = \frac{M}{VH} = \left(\frac{3\mu^2 n}{2\zeta_0}\right)\left\{1-\frac{\pi^2}{12}\left(\frac{kT}{\zeta_0}\right)^2+\cdots\right\}.$$