WITH SOLUTIONS

Physics PhD Qualifying Examination
Part I – Friday, January 14, 2005

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	6	
	7	
	8	
	9	
	10	

Student's initials
problems handed in:
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism), **and with at least one problem from problems 5 or 10 (Special Relativity).**
   DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

[I-1] **[10]**

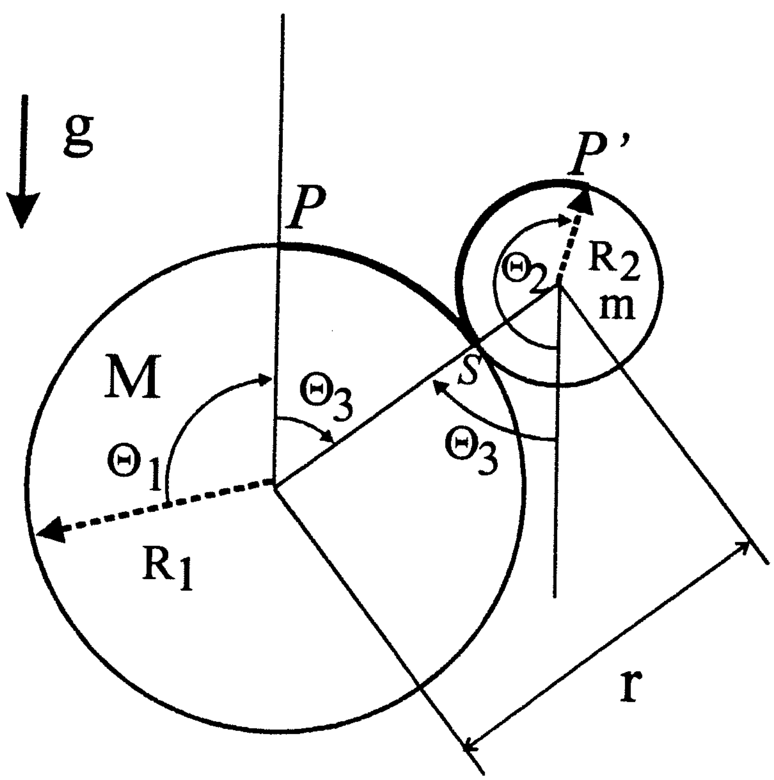
Derive expressions for **the velocity and acceleration** in a stationary system of coordinates in term of the corresponding quantities in a rotating system of coordinates. Assume that the moving system of coordinates rotates with a non uniform angular velocity, the instantaneous angular velocity and acceleration being $\vec{\omega}(t)$ and $\vec{\alpha}(t)$, respectively. Assuming that the force F acting on a point particle is known, **write down an equation of motion as seen in the rotating system of coordinates.**

[I-2] **[5, 5]**

A cylinder of Radius R_2 and mass m is being placed on another cylinder of radius R_1 (with $R_1 > R_2$) and mass M (see drawing). The cylinders roll on each other without slipping. At time $t=0$ the system be at rest and the small cylinder be located at an angle Θ_{30} versus the vertical axis. At which angle Θ_3 will the upper cylinder fall off in case

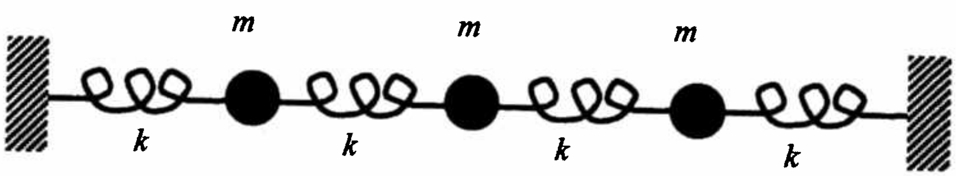
a) the lower cylinder can rotate around its fixed axis without friction.
b) the lower cylinder is held in place and cannot rotate

Hint: Calculate the Lagrange function for the degrees of freedom Θ_1, Θ_2 and Θ_3 and use the boundary conditions $r = R_1 + R_2$ and $R_1(\Theta_1 + \Theta_3) = R_2(\Theta_2 - \Theta_3)$.



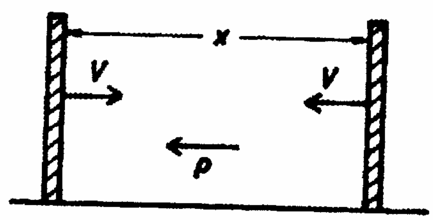
[I-3]   [3,7]

Consider the coupled mass-spring system sketched below. The masses can only move horizontally. The springs are relaxed at equilibrium.
(a) Obtain the normal frequencies.
(b) Find the corresponding normal modes (eigenvectors) explicitly (i.e., not just a sketch).



[I-4]   [6, 4]

A particle is confined inside a box and can move only along the x-axis. The ends of the box move toward the center with a speed small compared with the particle's speed. See figure below.



(a) If the momentum of the particle is p when the walls of the box are a distance x_o apart, find the momentum of the particle at any time later. Collisions with the walls are perfectly elastic. Assume that at all times the speed of the particle is much less than the speed of light. (You may use Newtonian formalism for (a).)

(b) When the walls are a distance x apart, what average external force must be applied to each wall in order to move it at constant speed V? (You may use Hamiltonian formalism for part (b).)

[I-5] **[3, 3, 3, 1]**

The refractive index of a material when at rest is "n". Assume that this material moves with a velocity **v** in the same direction as a beam of light.

(a) What is the velocity **V** of this beam in the stationary frame?
(b) Obtain an approximate expression for **V** for **v** << **c** using special relativity.
(c) How could you use the previous results to determine if Newtonian or relativistic mechanics is valid?
(d) What is the significance of the year **1905**?

[I-6] **[10]**

Find the series solution for the potential Φ (r, z) inside a cylinder of radius r_0 and length L with zero potential end plates at z=0 and z=L, provided the potential is specified at the cylindrical surface as Φ=f(z) at $r=r_0$.

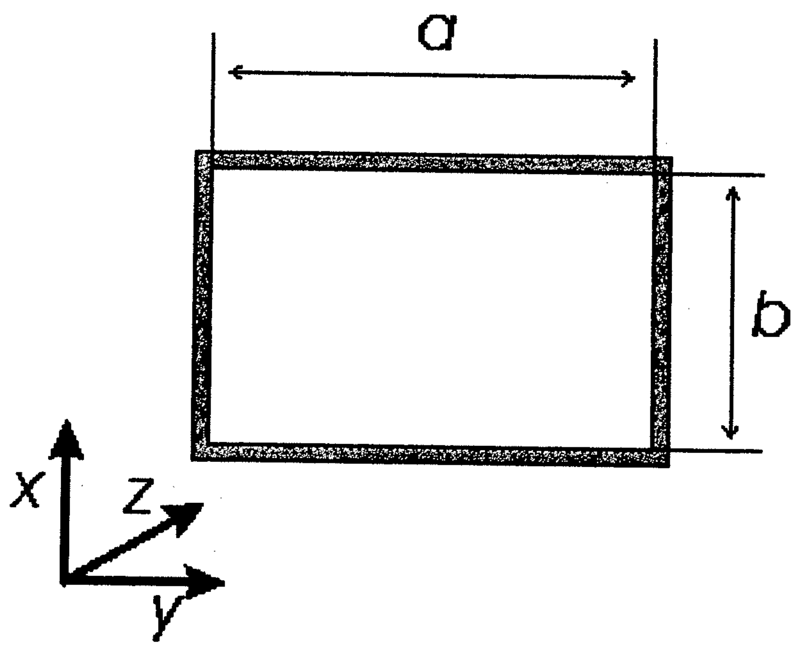
[I-7] [3, 4, 3]

Consider a rectangular wave guide, infinitely long in the z direction, with a width (x direction) of a and a height (y direction) of b ($a > b$). The walls are perfect conductors.

a) What are the boundary conditions on the components of **B** and **E** at the walls?
b) Derive and solve the wave equations which describe the **E** and **B** Fields of the lowest mode.

Hint: The lowest mode has the electric field in the y direction only.

c) For the lowest mode that can propagate, find the phase velocity and the group velocity.



[I-8] [4, 4 3]

A parallel-plate capacitor with plates having the shape of circular disks has the region between the plates filled with a dielectric material of permittivity "ε". The dielectric is imperfect, having a conductivity "σ". The capacitance of the capacitor is "C". The capacitor is charged to a potential difference ΔU and isolated.

(i) Find the charge on the capacitor as a function of time.
(ii) Find the displacement current in the dielectric.
(iii) Find the magnetic field in the dielectric.

[I-9] **[10]**

Given is a medium in which $\rho=0$, $\mathbf{j}=0$, $\varepsilon=\varepsilon_0$, but where the magnetization $\mathbf{M}(\mathbf{x},t)$ is a given function. Show that Maxwell's equations are correctly obtained from a single vector function $\mathbf{Y}$ where $\mathbf{Y}$ satisfies the equation

$$\nabla^2\vec{Y} - \frac{1}{c^2}\frac{\partial^2\vec{Y}}{\partial t^2} = -\mu_0\vec{M}$$

and where $\vec{\boldsymbol{B}} = \vec{\nabla}\times\vec{\nabla}\times\vec{\boldsymbol{Y}}$ and $\boldsymbol{E} = -\vec{\nabla}\times\frac{\partial\vec{\boldsymbol{Y}}}{\partial t}$.

[I-10] **[10]**

A stationary observer detects and observes an incident plane wave with frequency ω striking a *moving* mirror in vacuum. The constant velocity of the mirror $\mathbf{v}$ is parallel to the wave number of the plane wave $\mathbf{k}$. The plane of the mirror is perpendicular to $\mathbf{k}$. *Using the boundary conditions that the electromagnetic waves must satisfy at the plane of the mirror*, find the frequency of the reflected wave ω' detected by the stationary observer.

I-1. Solution

For a rotating system of coordinates

$$\left(\frac{d}{dt}\right)_{space} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times$$

Here $\vec{\omega}$ is the velocity rotation thus

$$(v)_{space} = \frac{d}{dt}(\vec{r})_{space} + \vec{\omega}\times\vec{r}_{body}$$

and

$$\vec{a}_{space} = \left(\frac{d}{dt}\vec{v}\right)_{body} + \vec{\omega}\times\vec{v}$$

$$\vec{a}_{space} = \left(\frac{d^2\vec{r}}{dt}\right)_{bod} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_b) + 2(\vec{\omega}\times\vec{v}_b)$$

$$+ \frac{d\vec{\omega}}{dt}\times\vec{r}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

$$(F)_{space} = m\vec{a}_{sp}$$

$$m(\ddot{\vec{r}})_{body} = \vec{F}_{space} - 2m(\vec{\omega}\times\vec{v}_b) - m\vec{\omega}\times(\vec{\omega}\times\vec{r}_b)$$

$$- m\vec{\alpha}\times\vec{r}$$

Thus the equation in the ... [illegible]

$\boxed{\text{I-2}}$ Lagrangian Mechanics

$$L = T - V$$

$$T = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}_3^2\right) + \frac{1}{2}\left(\frac{1}{2}mR_2^2\right)\dot{\theta}_2^2 + \frac{1}{2}\left(\frac{1}{2}MR_1^2\right)\dot{\theta}_3^2$$

$$V = mgr\cos\theta_3$$

constraint functions $\qquad r = R_1 + R_2 \qquad (N_1)$

(corrected during test!) $\Rightarrow \quad R_1\theta_1 + R_1\theta_3 = R_2(\theta_2 - \theta_3) \quad (N_2)$

The purpose of the Lagrange method is to introduce additional parmeters using the constraint functions. Therefore we should complete the Lagrange method before eliminating any parameters.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \sum \lambda_i \frac{\partial N_i}{\partial r}$$

$\Rightarrow$ apply to $r \quad \Rightarrow m(\ddot{r} - r\dot{\theta}_3^2 + g\cos\theta_3) = \lambda_1 \qquad (1)$

apply to $\theta_3 \Rightarrow \frac{d}{dt} mr^2\dot{\theta}_3 - mgr\sin\theta_3 = (R_1 + R_2)\lambda_2 \qquad (2)$

apply to $\theta_2 \Rightarrow \frac{m}{2}R_2^2\ddot{\theta}_2 = -R_2\lambda_2 \qquad (3)$

apply to $\theta_1 \Rightarrow \frac{M}{2}R_1^2\ddot{\theta}_1 = R_1\lambda_2 \qquad (4)$

Now we can assume $\dot{r} = \ddot{r} = 0$

From (1) $\Rightarrow$ $-m(R_1+R_2)\dot{\theta}_3^2 + mg\cos\theta_3 = \lambda_1$

From second constraint (N_2)

$$R_1\ddot{\theta}_1 + (R_1+R_2)\ddot{\theta}_3 - R_2\ddot{\theta}_2 = 0 \qquad (5)$$

From (3) using (5)

$$\frac{m}{2}R_2^2\ddot{\theta}_2 = \cancel{-R_2\lambda_2} = \frac{m}{2}R_2^2\left\{\frac{R_1}{R_2}\ddot{\theta}_1 + \frac{R_1+R_2}{R_2}\ddot{\theta}_3\right\} = -R_2\lambda_2$$

$$\frac{m}{2}(R_1+R_2)\ddot{\theta}_3 = -\lambda_2\left(1+\frac{m}{M}\right)$$

$$m(R_1+R_2)\ddot{\theta}_3 - mg(R_1+R_2)\sin\theta_3 = -(R_1+R_2)^2\frac{m}{2}\left(\frac{M}{m+M}\right)\ddot{\theta}_3$$

$$\ddot{\theta}_3(R_1+R_2)\left(1+\frac{1}{2}\frac{M}{m+M}\right) - g\sin\theta_3 = 0 \qquad \Big| \text{ Multiply by } \dot{\theta}_3 \Rightarrow$$

$$\frac{1}{2}\frac{d}{dt}\dot{\theta}_3^2(R_1+R_2)\left(1+\frac{1}{2}\frac{M}{m+M}\right) = g\sin\theta_3\cdot\dot{\theta}_3$$

$$= -g\frac{d}{dt}\cos\theta_3 \qquad \Big| \text{ Integrate over time}$$

$$\Rightarrow \dot{\theta}_3^2(R_1+R_2)\left(1+\frac{1}{2}\frac{M}{M+m}\right) = 2g(\cos\theta_{30} - \cos\theta_3)$$

For lift-off constraint condition $N_1 = \lambda_1 = 0$

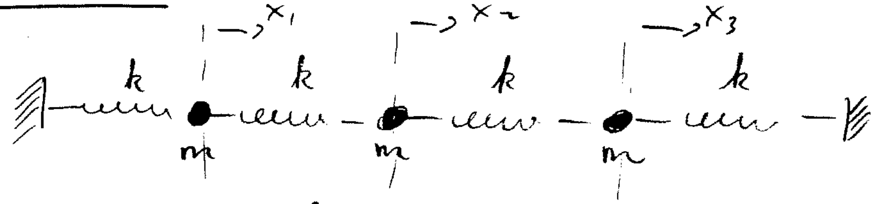
$$\hookrightarrow g\cos\theta_3\left(1+\frac{1}{2}\frac{M}{M+m}\right) + 2g(\cos\theta_3 - \cos\theta_{30}) = \frac{\lambda_1}{m}\left(1+\frac{1}{2}\frac{M}{m+M}\right) = 0$$

$$\Rightarrow \cos\theta_3 = \frac{4\cos\theta_{30}}{6+\frac{M}{m+M}}$$

b) Lower cylinder is fixed = equivalent to $M\to\infty$

$$\hookrightarrow \cos\theta_3 = \frac{4}{7}\cos\theta_{30}$$

$\boxed{\text{I-3}}$ Solution



$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2} k x_1^2 - \frac{1}{2} k (x_2 - x_1)^2 - \frac{1}{2} k (x_3 - x_2)^2 - \frac{1}{2} k x_3^2$$

$$m \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\begin{pmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\bar{x}(t) = \bar{a}\, e^{\pm i\omega t}$ $\qquad$ $\bar{a}$: normal mode with frequency ω

(1) $$\begin{pmatrix} 2k - m\omega^2 & -k & 0 \\ -k & 2k - m\omega^2 & -k \\ 0 & -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0$$

"symmetric" solutions: $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ a \end{pmatrix}$

(2) $$\begin{cases} (2k - m\omega^2)\, a - k b = 0 \\ -2k a + (2k - m\omega^2)\, b = 0 \end{cases}$$

$$(2k - m\omega^2)^2 - 2k^2 = 0$$

$$2k - m\omega^2 = \pm\sqrt{2}\, k \quad \Rightarrow \quad \omega_{1,2}^2 = \frac{k}{m}\left(2 \pm \sqrt{2}\right)$$

corresponding normal modes from (2):

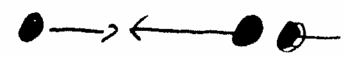
$$b = \mp\sqrt{2}\, a$$

(I-3) continued.

Thus,

$$\omega_1 = \sqrt{\frac{k}{m}}\sqrt{2+\sqrt{2}} \qquad \bar{a}_1 = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

(up to normali



$$\omega_2 = \sqrt{\frac{k}{m}}\sqrt{2-\sqrt{2}} \qquad \bar{a}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

"antisymmetric" mode $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} c \\ 0 \\ -c \end{pmatrix}$

from Eq.(1):

$$(2k - m\omega^2)\, c = 0$$

$$\omega_3 = \sqrt{\frac{2k}{m}} \qquad \bar{a}_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

2)

[I-4] Solution

(a) Consider a collision of the particle with one wall. The collision is perfectly elastic, hence, the relative speeds before and after the collision are equal. If the particle is incident with speed v and reflected back with speed v' and the wall has speed V towards the particle we have:

$$v + V = v' - V \Rightarrow v' = v + 2V$$

Thus, after each collision, the magnitude of the particle momentum gains an amount $2mV$, with m being the mass of the particle. When the walls are at a distance x apart, as V is much smaller than the speed of the particle, the interval between two consecutive collisions is

$$T = \frac{x}{(p/m)} = \frac{xm}{p}$$ with p being

the particle momentum. Now the change in the momentum in time dt is

$$dp = 2mV\frac{dt}{T} = \frac{2Vp\,dt}{x}$$

As the walls move toward each other with speed V,
$$x = x_0 - 2Vt$$
with x_0 being a reference point:

[I-4] Solution - continued.

then: $dp = -p\dfrac{dx}{x}$

and for $p = p_0$ when $x = x_0$,

we obtain $\boxed{p = \dfrac{p_0 x_0}{x} = \dfrac{p_0 x_0}{x - 2Vt}}$

(b.) Use a reference frame attached to one of the walls, i.e. left-hand wall, the particle has velocity $\left(-\dfrac{p}{m} - V\right)$.
the Hamiltonian is:

$$H = \frac{1}{2}m\left(\frac{p}{m} + V\right)^2$$

$$H = \frac{1}{2m}(p + mV)^2 \approx \frac{p^2}{2m} = \frac{p_0^2 x_0^2}{2mx^2}$$

the force on the particle is $\dot{p}$ which is given by Hamilton's equation:

$$\boxed{\dot{p} = -\frac{\partial H}{\partial x} = \frac{p_0^2 x_0^2}{mx^3}}$$

$\boxed{\text{I-5}}$ Solution

$$V = \frac{\frac{c}{n} + v}{1 + v\frac{c}{\cancel{n}}\frac{1}{c^2}} = \frac{\frac{c}{n} + v}{1 + \frac{v}{cn}}$$

$$V = \frac{c}{n}\left(1 + \frac{vn}{c}\right)\left(1 + \frac{v}{nc}\right)^{-1}$$

This is exact. In order to compare with experiment we have to take the first two terms in v/c. Thus

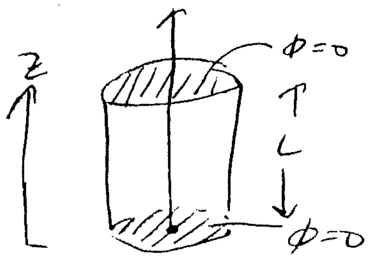
$$V \approx \frac{c}{n}\left(1 + \frac{vn}{c}\right)\left(1 - \frac{v}{c}\frac{1}{n} + \cdots\right)$$

$$\approx \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)v$$

This result is consistant with experiment, but not with Newtonian. Prior to this disagreement was mysterious and taking strange ~~e~~reasoning. ~~T~~aking just the lowest term in $(v/c)^0$ is not sufficient

4. $\boxed{\text{I-6}}$ **Solution**

$$\phi = R(r)\,\xi(z)$$



ϕ is periodic in z,

So Soln is $\xi(z) = A\sin kz + B\cos kz$.

Boundary Conditions:

$\phi = 0$ at $z = 0, L$

$\therefore$ $B = 0$ and $n\pi = kL$

$\therefore$ $\xi(z) = A\sin\frac{n\pi z}{L}$

$R(r)$ use Bessel Functions:

$$R(r) = C\,I_0(kr) + D\,K_0(kr)$$

But R is finite at $r=0$,

So $R(r) = C\,I_0(kr)$

$\Rightarrow$

$$\phi = \sum_{n=0}^{\infty} A_n\, I_0\!\left(\frac{n\pi}{L}r\right)\sin\!\left(\frac{n\pi z}{L}\right)$$

but $\phi(r=a) = f(z)$

[I-6] continued.

So

$$f(z) = \sum_n A_n I_0\left(\frac{n\pi}{L} a\right) \sin \frac{n\pi z}{L}$$

Use orthogonality:

$$\int_0^L f(z) \sin \frac{m\pi z}{L} \, dz = \sum_n \int_0^L A_n \sin \frac{n\pi z}{L} \sin \frac{m\pi z}{L} I_0\left(\frac{n\pi a}{L}\right) dz$$

$$= A_n \frac{L}{2} I_0\left(\frac{n\pi a}{L}\right)$$

$$\Rightarrow \quad A_n = \frac{2}{L\, I_0\left(\frac{n\pi a}{L}\right)} \int_0^L f(z) \sin \frac{n\pi z}{L} \, dz$$

$\therefore$

$$\boxed{\phi(r,z) = \sum_{n=0}^{\infty} \frac{2}{L\, I_0\left(\frac{n\pi a}{L}\right)} I_0\left(\frac{n\pi}{L} r\right) \sin\left(\frac{n\pi z}{L}\right) \int_0^L f(z) \sin\left(\frac{n\pi z}{L}\right) dz}$$

$\boxed{\text{I-7}}$ Solution

Maxwell's Equations

a) Boundary Condition: Because the walls are perfectly conducting, we have for $\vec{E}$ and $\vec{B}$ the boundary conditions

$$\vec{n} \times \vec{E} = 0 \quad , \quad \vec{n} \cdot \vec{B} = 0$$

where $\vec{n}$ is normal to the wall, or in terms of E_z and B_z (z is the direction of wave propagation)

$$E_z \Big|_S = 0 \; ; \quad \frac{\partial B_z}{\partial n} \Big|_S = 0 \qquad (1)$$

b.) Starting from the sourceless Maxwell equations in vacuum

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \; (2) \; , \quad \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \; (3)$$

$$\nabla \cdot \vec{B} = 0 \quad (4) \quad , \quad \nabla \cdot \vec{E} = 0 \quad (5)$$

and substituting $\vec{E}(\vec{r},t) = \vec{E}(\vec{r}) \exp(-i\omega t)$ and the equivalent for $\vec{B}$ we get

$$\underset{(6)}{\nabla \times \vec{E} = \frac{i\omega}{c} \vec{B}} \; ; \quad \underset{(7)}{\nabla \times \vec{B} = -\frac{i\omega}{c} \vec{E}} \; ; \quad \underset{(8)}{\nabla \cdot \vec{B} = 0} \; ; \quad \underset{(9)}{\nabla \cdot \vec{E} = 0} .$$

The field dependence on z may be written in the form $f(z) = \exp[i(kz - \omega t)]$ where k is the wave vector for the wave transmitted in the z direction. For $a > b$ the lowest mode has an electric field in the x-direction only and vice versa. For ~~Ey We get~~ $b > a$ we find for E_y from (6), (7)

$$-ikE_y = \frac{i\omega}{c}B_x \quad (10) \; ; \quad 0 = B_y \quad (11) \; ; \quad \frac{\partial B_z}{\partial y} = 0 \quad (12) \; ; \quad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c}E_y \quad (13)$$

From (12) $B_z = B(x)\exp(ikz)$, and substituting (10) into (13) we get

$$B_x = \frac{ik}{\gamma^2}\frac{\partial B_z}{\partial x} \quad (14) \quad , \quad E_y = -\frac{i\omega}{c\gamma^2}\frac{\partial B_z}{\partial x} \quad (15)$$

where $\gamma^2 \equiv \frac{\omega^2}{c^2} - k^2$ (16)

Using $\nabla\cdot\vec{B} = 0$ and B_x from (14), we get a differential equation for B_z.

$$\frac{ik}{\gamma^2}\frac{\partial^2 B_z(x)}{\partial x^2} + ikB_z(x) = 0 \qquad (17)$$

or

$$\frac{\partial^2 B_z(x)}{\partial x^2} + \gamma^2 B_z(x) = 0 \qquad (18)$$

The solution of this equation satisfying the boundary conditions

$\left.\frac{\partial B_z}{\partial n}\right|_S = 0$ is $B_z = B_0 \cos\gamma x$ with $\gamma = \frac{\pi}{a}$.

So the field in the wave guide in this mode from (14), (15)

$$B_z = B_0\cos\left(\frac{\pi}{a}x\right)\cdot\exp(ikz - i\omega t) \quad (19)$$

$$B_x = -\frac{ika}{\pi}B_0\sin\left(\frac{\pi}{a}x\right)\cdot\exp(ikz - i\omega t) \quad (20)$$

$$E_y = \frac{i\omega a}{c\pi}B_0\sin\left(\frac{\pi}{a}x\right)\cdot\exp(ikz - i\omega t) \quad (21)$$

c) The dispersion relation for the lowest mode is found from (16)

$$\omega = c\sqrt{k^2 + \pi^2/a^2} \qquad (22)$$

The phase velocity v is

$$v = \frac{\omega}{k} = c\sqrt{1 + \pi^2/(k^2a^2)} \qquad (23)$$

The group velocity is

$$u = \frac{\partial \omega}{\partial k} = \frac{ck}{\sqrt{k^2 + \pi^2/a^2}} \qquad (24)$$

$\square$

Q.E.JAug 2005 (30 Nov 2004) John Schroeder

[4,3,3] (E/M) - Simple Electro-magnetism (Faraday, Lenz)

I-8 <u>Problem</u>: A parallel-plate capacitor with plates having the shape of circular disks has the region between the plates filled with a dielectric material of permittivity "ϵ". The dielectric is imperfect, having a conductivity "g". The capacitance of the capacitor is "C". The capacitor is charged to a potential difference ΔU and isolated.

(i) Find the charge on the capacitor as a function of time.

(ii) Find the displacement current in the dielectric

(iii) Find the magnetic field in the dielectric

<u>Solution</u>: Let "σ" be surface charge density on the positive plate. Current flows away from the positive plate such that the current density will be given $j = -\frac{\partial \sigma}{\partial t}$.

We also have for the parallel plate capacitor, $E = \sigma/\epsilon$ and from Ohm's law $j = gE$, $j = (g/\epsilon)\sigma$. We thus obtain:

$$\frac{\partial \sigma}{\partial t} = -\frac{g}{\epsilon}\sigma \Rightarrow \sigma = \sigma_0 e^{-gt/\epsilon}$$

Hence: (i) $Q = A\sigma = A\sigma_0 e^{-gt/\epsilon} = Q_0 e^{-gt/\epsilon}$

$$Q = C\Delta U e^{-gt/\epsilon}$$

(I-8) continued.

(ii) displacement current

$$I_D = A\frac{\partial D}{\partial t} = A\epsilon\frac{\partial E}{\partial t} = A\frac{\partial \sigma}{\partial t} = \frac{\partial Q}{\partial t} = -\frac{g}{\frac{g}{\epsilon}} C\Delta V e^{-g}$$

(iii) $$I_{Total} = I + I_D = -A\frac{\partial \sigma}{\partial t} + A\frac{\partial \sigma}{\partial t} = 0$$

Thus, j_{total} is zero, $\therefore$ zero magnetic field in dielectric.

$\boxed{\text{I-9}}$ **Solution**

Feb 2005

Maxwell's equation

$$\vec{\nabla}\cdot\vec{D} = \rho$$
$$\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{H} = \vec{j} + \dot{\vec{D}}$$

with $\rho = 0 \quad \vec{j} = 0$

$$\vec{\nabla}\cdot\vec{E} = 0 \;,\quad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}} \;,\quad \vec{\nabla}\cdot\vec{B} = 0 \;,\quad \vec{\nabla}\times\vec{H} = \dot{\vec{D}} = \varepsilon_0\dot{\vec{E}}$$

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\left(\vec{\nabla}\times\frac{\partial\vec{Y}}{\partial t}\right) = 0 \qquad \text{"div rot} = 0\text{"}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \quad \vec{\nabla}\cdot(\vec{\nabla}\times(\vec{\nabla}\times\vec{Y})) = 0 \qquad \text{"div rot"} = 0$$
$$\vec{\nabla}\times\vec{E} = \vec{\nabla}\left(-\vec{\nabla}\times\frac{\partial\vec{Y}}{\partial t}\right) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{\nabla}\times\vec{Y}) = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{B} = \mu_0(\vec{H}+\vec{M}) \Rightarrow \mu_0\vec{H} = \vec{B}-\mu_0\vec{M}$$
$$\mu_0\,(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\times\vec{B} - \mu_0\,(\vec{\nabla}\times\vec{M}) = \vec{\nabla}\times(\vec{B}-\mu_0\vec{M})$$

$$\vec{B} = \vec{\nabla}\times\vec{\nabla}\times\vec{Y} = (\vec{\nabla}\,(\vec{\nabla}\cdot\vec{Y}) - \vec{\nabla}^2\vec{Y} = -\vec{\nabla}^2\vec{Y}$$
$$\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = -\frac{1}{c^2}\vec{\nabla}\times\frac{\partial^2\vec{Y}}{\partial t^2} = \vec{\nabla}\times(-\mu_0\vec{M} - \vec{\nabla}^2\vec{Y}) \qquad (1)$$

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{\nabla}\times\vec{Y}) = \vec{\nabla}\times(\vec{\nabla}\times\vec{V}) \qquad \vec{V} = \vec{\nabla}\times\vec{Y}$$
$$= \vec{\nabla}\,(\vec{\nabla}\cdot\vec{V}) - \vec{\nabla}^2\vec{V} = -\vec{\nabla}^2\vec{V} = \vec{\nabla}\times(-\vec{\nabla}^2\vec{Y})$$

$$\vec{\nabla}\times(\vec{B}-\mu_0\vec{M}) = \vec{\nabla}\,(-\vec{\nabla}^2 Y - \mu_0\vec{M}) \qquad (2)$$

compare (1) & (2) $\quad \vec{\nabla}\times\vec{H} = \dot{\vec{D}}$

Solution

I-10

~~in the frame~~, where the mirror moves with velocity v
incident electromagnetic plane wave $\parallel \hat{x}$

e.g. $\bar{E} = \bar{E}_o \cos(\omega t - kx)$ incident wave

$\bar{E}' = \bar{E}_o' \cos(\omega' t + k' x)$ reflected wave

at $x = vt$ the following boundary conditions must be satisfied:

$\bar{E} + \bar{E}' = \bar{0} \qquad \forall t$

i.e., $\bar{E}_o \cos(\omega t - kx)\Big|_{x=vt} = -\bar{E}_o' \cos(\omega' t + k' x)\Big|_{x=vt} \qquad \forall t$

$\bar{E}_o \cos(\omega t - kvt) = -\bar{E}_o' \cos(\omega' t + k' vt)$

$\omega = kc$ and $\omega' = k' c$

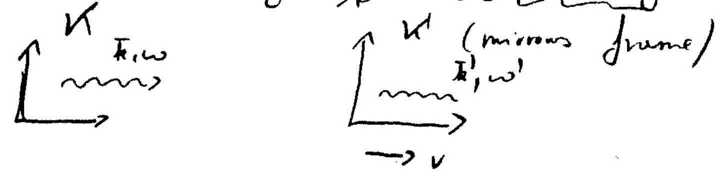
$\Rightarrow \bar{E}_o' = -\bar{E}_o$ and $\omega t - kvt = \omega' t + k' vt \qquad \forall$

$\omega\left(1 - \frac{v}{c}\right) = \omega'\left(1 + \frac{v}{c}\right)$

$$\boxed{\omega' = \omega \frac{1 - v/c}{1 + v/c}}$$

Alternative Solution: tr $\boxed{\text{I}-10}$

Solution using special relativity (relativistic Doppler Effect)



light: $\quad |\bar{k}| = k_0 = \frac{\omega}{c}$

Lorentz tr. for k_0:

I. $\quad k_0' = \gamma\,(k_0 - \bar{\beta}\cdot\bar{k}) \qquad$ here $\bar{\beta} = \frac{\bar{v}}{c} \parallel \bar{k}$

$$k_0' = \gamma\left(k_0 - \frac{v}{c}\cdot\frac{\omega}{c}\right)$$

$$\omega' = \gamma\left(\omega - \frac{v}{c}\,\omega\right) = \gamma\,(1 - v/c)\,\omega = \frac{1}{\sqrt{1 - v^2/c^2}}\,(1 - v/c)\,\omega = \sqrt{\frac{1 - v/c}{1 + v/c}}\;\omega$$

II. after reflection: $\bar{k}' \to -\bar{k}'$ and $\bar{v} \to -\bar{v}$ in inverse transformation (i.e. $\bar{\beta} \parallel \bar{k}'$)

$\omega'_{refl} = \omega'$

$$\Rightarrow \quad \omega_{refl} = \gamma\left(\omega'_{refl.} - \frac{v}{c}\,\omega'_{ref}\right) = \sqrt{\frac{1 - v/c}{1 + v/c}}\;\omega'_{refl} =$$

$$= \sqrt{\frac{1 - v/c}{1 + v/c}}\cdot\omega' = \sqrt{\frac{1 - v/c}{1 + v/c}}\,\sqrt{\frac{1 - v/c}{1 + v/c}}\;\omega =$$

$$= \frac{1 - v/c}{1 + v/c}\,\omega = \frac{c - v}{c + v}\,\omega$$

WITH SOLUTIONS

Physics PhD Qualifying Examination
Part II – Wednesday, January 19, 2005

Name:____

(please print)

Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

\# problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). DO NOT HAND IN MORE THAN EIGHT PROBLEMS.

[II-1] **[10]**

A spin ½ particle interacts with a magnetic field **B** $= B_o\,\hat{z}$ through the Pauli interaction $H = \mu\,\vec{\sigma}\cdot\vec{B}$, where μ is the magnetic moment, σ are the Pauli spin matrices (σ_x, σ_y, σ_z) and H is the Hamiltonian. At $t = 0$ a measurement determines that the spin is pointing along the positive x-axis. What is the probability that it will be pointing along the negative y-axis at a later time "t".

[II-2] **[10]**

Using the Born approximation, evaluate the differential scattering cross section for scattering of particles of mass m and incident energy" E " by the repulsive spherical well with potential

$V(r) = V_0 \quad 0 < r < a$ and $V(r) = 0$ *for* $r > a$.

[II-3] **[4,6]**

The Hamiltonian of an unperturbed charged particle q is $H = \dfrac{p^2}{2m} + m\,\dfrac{\omega^2}{2}x^2$.

Now a weak perturbing electric field $\vec{E}$ is imposed.

(a) Obtain an exact expression for the energy of this charged particle in the absence of the electric field.
(b) Now consider the energy due to the interaction with the electric field as a perturbation. Write down expressions for corrections in the energy up to second order and for the wave function up to first order.

[II-4] **[4, 6]**

Consider two electrons bound to a proton by Coulomb interaction. Neglect the Coulomb repulsion between the two electrons.

(a) What are the ground state energy and wave function for this system?

(b) Consider that a weak potential exists between the two electrons of the form

$$V(\vec{r}_1 - \vec{r}_2) = V_0 \delta^3(\vec{r}_1 - \vec{r}_2)\ s_1 \cdot s_2 \quad ,$$

Where V_o is a constant and s_j is the spin operator for electron j (neglect the spin-orbit interaction). Use first-order perturbation theory to estimate how this potential alters the ground state energy.

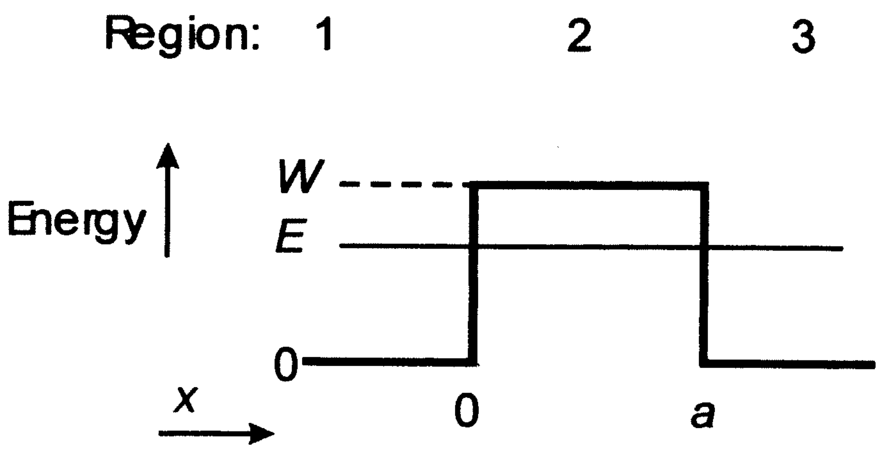
[II-5] **[10]**

A stream of particles of mass m and energy E is incident in region (1) on a potential barrier given by

$$V(x) = \begin{cases} 0 & for\ x < 0 \\ W & for\ 0 < x < a \\ 0 & for\ x > 0 \end{cases}$$

where $W > E$ (see figure).

Calculate the fraction of the stream of particles that is transmitted from region (1) to region (3).



[II-6] **[10]**

Consider an electron in a uniform constant magnetic field **B** pointing to the $+z$ direction, $\mathbf{B} = B\hat{\mathbf{e}}_z$. Initially the electron is in the $s_z = \hbar/2$ state. At $t = 0$ a small x-component of the magnetic field, $\Delta\mathbf{B} = \Delta B\hat{\mathbf{e}}_x$ is turned on. Using first-order *time-dependent perturbation theory*, calculate the probability that at time t, (as a result of a direct transition) the electron is in the state $s_z = -\hbar/2$. Employing the usual s_z-representation, the expressions below for the electron spin operator with the Pauli matrices will be useful:

$$\mathbf{s} = \frac{\hbar}{2}\boldsymbol{\sigma}, \qquad \boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z),$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[II-7] **[4, 4, 2]**

The equation of state for a Van der Waals gas is

$$\mathbf{(P + a/V^2)(V - b) = RT}$$

(a) What is the origin of this equation and what role do the coefficients a and b play?
(b) If the gas expands isothermally from volume $\mathbf{V_1}$ to volume $\mathbf{V_2}$, what is the change in the Helmholtz free energy?
(c) What is the corresponding internal energy change?

[II-8]   [5, 5]

Suppose the equation of state for some system is

$$P^2 T^{-1/3} e^{aV} = b\,.$$

(a) Calculate the isothermal compressibility κ of this system.
(b) Calculate the coefficient of volume expansion β at constant pressure.

[II-9]   [4, 3, 3]

Assume we have a quantum mechanical system that has two energy levels: E_1 and $E_2 = E_1 + \Delta E$. This could represent the electronic levels of an atom or the orientation of an electron spin parallel or anti-parallel to the magnetic field or a molecule that can take two different orientations in a lattice. Also w-fold degeneracy exists in each level (i.e. w_1 in E_1 and w_2 in E_2).

(i) Obtain the partition function "Z" for the above system, then calculate the internal energy "U" of the system and determine the specific heat "$C_v(T)$" for this system.

(ii) Discuss the function $C_v(T)$ for very high and very low temperatures. Sketch C_v/R as a function of (T/θ), where $\theta = \Delta E/k$ and mark all important features (assume $w_1 = w_2$). Note "k" is Boltzmann's constant.

(iii) Sketch "U" versus (T/θ). Discuss the function $U(T/\theta)$, for $(T/\theta) \ll 1$ and $(T/\theta) \gg 1$, with respect to the occupation of lower and upper levels. Explain what it means for a system to have "negative temperatures."

[II-10] [10]

Consider the three-dimensional non-interacting electron gas. The energy-momentum relation is $\varepsilon(p) = \dfrac{p^2}{2m}$.

What is the pressure of this system at $T = 0$ (zero temperature)?

(You must *derive* an expression for the pressure and express your answer in terms of the density of the electrons $n = N/V$ (N is the number of electrons, V is the volume).

Q.E. January 2005

John Schroeder
30·XI·2004.

[10] Quantum Mechanics

II-1 **Problem**: A spin $\frac{1}{2}$ particle interacts with a magnetic field $\vec{B} = B_0 \hat{z}$ through the Pauli interaction $H = \mu \vec{\sigma} \cdot \vec{B}$, where μ is the magnetic moment, $\vec{\sigma}$ are the Pauli spin matrices $(\sigma_x, \sigma_y, \sigma_z)$ and H is the Hamiltonian. At $t=0$ a measurement determines that the spin is pointing along the positive x-axis. What is the probability that it will be pointing along the negative y-axis at a later time "t".

$\boxed{\text{II-1}}$ **Solution**: Let us quantize the spin state along the z-axis so that spin up and spin down are denoted by

$$\alpha \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \beta \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H\alpha = \hbar\omega\alpha \quad , \quad H\beta = -\hbar\omega\beta$$

$$\hbar\omega = \mu B_0$$

The eigenstates of σ_x are ψ_x for pointing along the $+x$-axis and $\psi_{\bar{x}}$ for the $-x$-axis

$$\psi_x = \frac{1}{\sqrt{2}}(\alpha + \beta) \qquad \sigma_x \psi_x = \psi_x$$

$$\psi_{\bar{x}} = -\frac{1}{\sqrt{2}}(\alpha - \beta) \qquad \sigma_x \psi_{\bar{x}} = -\psi_{\bar{x}}$$

$$\psi_y = \frac{1}{\sqrt{2}}(\alpha + i\beta) \qquad \sigma_y \psi_y = \psi_y$$

continued:

$$\psi_{\bar{y}} = \frac{1}{\sqrt{2}}(\alpha - i\beta) \qquad \sigma_y \psi_{\bar{y}} = -\psi_{\bar{y}}$$

At time $t=0$ we start in state ψ_x. Later this state becomes

$$\Psi_x(t) = \frac{1}{\sqrt{2}}\left(\alpha e^{-i\omega t} + \beta e^{i\omega t}\right)$$

The amplitude for pointing in the negative y-direction is found by taking the matrix element with $\psi_{\bar{y}}$. The probability is the square of the absolute magnitude of this amplitude:

$$\langle \psi_{\bar{y}} | \Psi_x(t) \rangle = \frac{1}{2}\left(e^{-i\omega t} + i e^{i\omega t}\right)$$

$$P_{x\bar{y}} = \cos^2\left(\omega t + \frac{\pi}{4}\right)$$

$\boxed{\text{II-2}}$ Solution

Feb 2005

$$f(\theta) = -\frac{2m}{\hbar^2 k}\int_0^{a} r\,V(r)\sin kr\,dr$$

$$= -\frac{2m}{\hbar^2 k}V_0\int_0^{a} r\sin kr\,dr$$

$$= -\frac{2mV_0}{\hbar^2 k}\left[\frac{\sin kr}{k^2} - \frac{r\cos kr}{k}\right]_0^{a}$$

$$= -\frac{2mV_0}{\hbar^2 k}\left[\frac{\sin ka}{k^2} - \frac{a\cos ka}{k}\right]$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

$\boxed{\text{II-3.}}$ **Solution**

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2 + qEx$$

Exact

$$H = \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x^2 + \frac{2qE}{m\omega^2}x\right)$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}\left(x + \frac{qE}{m\omega^2}\right) - \frac{m\omega^2}{2}\frac{q^2E}{m^2\omega^4}$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2}X^2 - \frac{q^2E^2}{2m\omega^2}$$

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right) - \frac{q^2E^2}{2m\omega^2}$$

$\boxed{\text{II-3}}$ continued.

Pert

$$H = H_0 + \lambda H_1 \qquad \lambda \ll 1$$

$$|n\rangle = |n^0\rangle + \lambda |n^1\rangle + \lambda^2 |n^2\rangle$$

$$E_n = E_n^0 + \lambda E_n^1 + \cdots$$

$$|n^1\rangle = \sum C^1_{nm} |m^0\rangle$$

$$|n^2\rangle = \sum C^2_{nm} |m^0\rangle$$

$$E_n^{(1)} = \langle n^0 | H^1 | n^0 \rangle$$

$$c_m^1 = \frac{\langle m^0 | H^1 n^0 \rangle}{E_n^0 - E_m^0}$$

$$E^2 = \sum_{m \neq n} \frac{\langle m^0 | H^1 | n^0 \rangle \langle n^0 | H^1 | n^0 \rangle}{E_n^0 - E_m^0}$$

In our case

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2}{2} = \frac{1}{2} a a^\dagger + a^\dagger a$$

$=$

[II-4] Solution (Interacting Electrons)

(a) The wave function for a single electron bound to a proton is that of a hydrogen atom, which is

$$\psi_0(\vec{r}) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0},$$

where a_0 is the Bohr radius. When one can neglect the Coulomb repulsion between the two electrons, the ground state energy and eigenfunctions are

$$E_0 = -2E_R$$

$$\psi_0(r_1, r_2) = \psi(r_1)\psi(r_2)\chi_0 \quad \text{with}$$

$$\chi_0 = \frac{1}{\sqrt{2}}[\alpha_1\beta_2 - \alpha_2\beta_1]$$

and the last factor in the above equation is the spin-wave function for the singlet $S=0$ (for the) in terms of up α and down β spin states. Since the spin state has odd parity the orbital state has even parity, and a simple product function $\psi_0(r_1)\psi_0(r_2)$ is correct. The eigenvalue is twice the Rydberg energy E_R.

(b) The change in energy in first-order

[II-4] continued

perturbation theory is $\delta E = \langle i | V | i \rangle$. The orbital part of the matrix element is

$$\langle V \rangle_0 = \int d^3r_1 \, d^3r_2 \, \psi_0^2(\vec{r}_1) V_0 \, \delta^3(\vec{r}_1 - \vec{r}_2) \psi_0^2(\vec{r}_2)$$

$$\langle V \rangle_0 = V_0 \int d^3r \, \psi_0^4(r) = \frac{4\pi V_0}{\pi^2 a_0^3} \int_0^\infty dx \, x^2 e^{-4x}$$

$$\langle V \rangle_0 = \frac{V_0}{8\pi a_0^3} \quad \text{with: } (x = r/a_0).$$

Next we evaluate the spin part of the matrix element. The easiest way is to use the definition of the total spin, $S = s_1 + s_2$ to derive

$$S \cdot S = 2 s_1 \cdot s_2 + s_1 \cdot s_1 + s_2 \cdot s_2 \,, \quad \text{and}$$

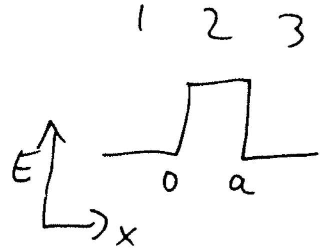
$$\langle s_1 \cdot s_2 \rangle = \frac{1}{2}\left[S(S+1) - \frac{3}{2} \right]$$

where for spin $\frac{1}{2}$ particles, such as electrons, $s_1 \cdot s_1 = s(s+1) = 3/4$. Since the two spins are in the $S = 0$ state, the expectation value $\langle s_1 \cdot s_2 \rangle = -3/4$. Combining this with the orbital contribution, we estimate the perturbed ground state energy $\bar{E}$ to be,

$$\left\{ \bar{E} = -2E_R - \frac{3V_0}{32\pi a_0^3} \right\}$$

II-5 Quantum Mechanics: Solution

Schrödinger equation has the form



$$u'' + k^2 u = 0 \quad \text{region 1 and 3} \qquad (1)$$
$$u'' - \gamma^2 u = 0 \quad \text{region 2}$$

Solutions have the form $\quad k^2 = 2m\frac{E}{\hbar^2}$; $\gamma^2 = 2m(W-E)/\hbar^2$

$$u_1 = \exp(ikx) + R\exp(-ikx)$$
$$u_2 = A\exp(\gamma x) + B\exp(-\gamma x) \qquad (2)$$
$$u_3 = T\exp(ikx)$$

Region 1: incident wave normalized to amplitude 1.
Reflection possible at interface 1→2 therefore $R \neq 0$

Region 2: incident wave (A) and refletion possible at interface 2→3.
therefore $B \neq 0$

Region 3: only out going wave (T), no reflected wave possible.

Boundary conditions in $x=0$ $\quad u$ and u' are continuous ⇒

$$1 + R = A + B \qquad (3)$$
$$1 - R = -i\rho(A-B) \quad \text{with } \rho = \frac{\gamma}{k} \qquad (4)$$

Boundary conditions in $x=a$ $\quad u$ and u' are continuous ⇒

$$A\exp(\theta) + B\exp(-\theta) = T\exp(i\delta) \qquad (5)$$
$$A\exp(\theta) - B\exp(-\theta) = i\frac{T}{\rho}\exp(i\delta) \quad \text{with } \delta = ka,\ \theta = \gamma a. \qquad (6$$

This in principle solves the problem. The rest is manipulation of Eqn (3) to (6) to eliminate R, A, B and to obtain an expression for T

Transmitted flux = $|T^2|$

By first adding and then subtracting (5) and (6) we obtain respectively:

$$A = \tfrac{1}{2} T \exp(i\delta - \theta)\left(1 + \frac{i}{\rho}\right), \qquad (8)$$

$$B = \tfrac{1}{2} T \exp(i\delta + \theta)\left(1 - \frac{i}{\rho}\right) \qquad (9)$$

Add (3) and (4) to eliminate R, and replace A and B in the resulting equation by using (8) and (9). This gives

$$T = \frac{\exp(-i\delta)}{P}$$

with

$$P = \tfrac{1}{4}\left\{\left(1+\frac{i}{\rho}\right)\left(1+\frac{\rho}{i}\right)\exp(-\theta) + \left(1-\frac{i}{\rho}\right)\left(1-\frac{\rho}{i}\right)\exp(\theta)\right\} \qquad (10)$$

$$= \cosh\theta + \tfrac{1}{2} i\left(\rho - \frac{1}{\rho}\right)\sinh\theta \qquad (11)$$

Therefore flux $= |T|^2 = \dfrac{1}{|P|^2}$ (12) where $|P|^2 = 1 + \dfrac{(1+\rho^2)^2}{4\rho^2}\sinh^2\theta$ (13)

in (11) and (13) we used the relations

$$\cosh\theta = \tfrac{1}{2}\{\exp(\theta) + \exp(-\theta)\}\ ; \ \sinh = \tfrac{1}{2}\{\exp(\theta) - \exp(-\theta)\} \qquad (14)$$

$$\cosh^2\theta = 1 + \sinh^2\theta$$

from the definition of ρ we have $\rho^2 = \dfrac{\gamma^2}{k^2} = \dfrac{W-E}{E}$ (15)

use this in (13)

$$|T|^2 = \frac{1}{1 + \dfrac{W^2}{4E(W-E)}\sinh^2\theta} \ ; \ \forall\ \theta = \gamma a,\ \gamma = \frac{\sqrt{2m(W-E)}}{\hbar}$$

□

$\boxed{\text{II}-6}$ Solution.

$\hat{H}_0 |s\rangle = E|s\rangle$ $\qquad B > 0$

$\hat{H}_0 = -\frac{e\hbar}{2mc} B\hat{\sigma}_z$ $\qquad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Pauli matrix

groundstate: $|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ("+" state) $\qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $E_+ = -\frac{1}{2}$

$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $E_- = +\frac{1}{2}$

$t=0$: small B_x is turned on: $\quad B_x = \Delta B\, \hat{e}_x$

$\hat{H} = \hat{H}_0 - \frac{e\hbar}{2mc}\Delta B\hat{\sigma}_x$

$\hat{V} \equiv -\frac{e\hbar}{2mc}\Delta B\hat{\sigma}_x$ as time dependent perturbation

$$W_i(+\to-) = \frac{1}{\hbar^2}\left|\int_0^t \langle - | \hat{V} | + \rangle e^{\frac{i}{\hbar}(E_- - E_+)t'} dt'\right|^2$$

$$= \frac{1}{\hbar^2}\left|\int_0^t \langle - | -\frac{e\hbar}{2mc}\Delta B\hat{\sigma}_x | + \rangle e^{\frac{i}{\hbar}\cdot\frac{e\hbar B}{mc}t'} dt'\right|^2$$

$$= \frac{e^2}{4m^2c^2}(\Delta B)^2 \left|\int_0^t \langle - | \sigma_x | + \rangle e^{i\frac{eB}{mc}t'} dt'\right|^2$$

$$\left[\langle - | \sigma_x | + \rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\right]$$

$$= \frac{e^2(\Delta B)^2}{4m^2c^2}\left|\int_0^t e^{i\frac{eB}{mc}t'} dt'\right|^2 = \frac{e^2(\Delta B)^2}{4m^2c^2}\frac{1}{\left|i\frac{eB}{mc}\right|^2}\left|e^{i\frac{eB}{mc}t} - 1\right|^2$$

$$= \frac{1}{4}\left(\frac{\Delta B}{B}\right)^2\left|e^{i\frac{eB}{mc}t} - 1\right|^2 = \boxed{\left(\frac{\Delta B}{B}\right)^2 \sin^2\left(\frac{eB}{2mc}t\right)}$$

II-7 Solution

The ideal gas equation of state is

$$pV = nRT$$

However when the volume which the molecules occupy is of the order magnitude of the ~~...~~ the volume, for ~~...~~ the equation of stat is smaller than by the number proportional to the volume of all the molecules hence $V \to V' = V - b$

Also the pressure is ~~...~~ reduced by a factor proportional to the square of the molecules but this number is proportional to $1/V$ hence

$$p \to p' \to \frac{a}{V^2}$$

thus the equation of state becomes

$$\left(p + \frac{a}{V^2}\right)(V - b) = nRT$$

$$p = \frac{kT}{V-b} - \frac{a}{V^2}$$

$$\Delta F = -\int_{V_1}^{V_2} p\,dV = -kT \ln \frac{V_2 - b}{V_1 - b} + \frac{a}{V_1} - \frac{a}{V_2}$$

for isothermal process ($dT = 0$)

$\boxed{\text{II-7}}$ continued.

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

From thermodynamics

$$\left(\frac{\partial U}{\partial T}\right)_V = T\left(\left(\frac{\partial P}{\partial T}\right)_V - P\right)$$

$$dU = \int_{V_1}^{V_2} \frac{a}{V^2}\, dV = \frac{a}{V_1} - \frac{a}{V_2}$$

$\boxed{\text{II-8}}$ Solution

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$$p^2 T^{-1/3} e^{aV} = b$$

$$dV = \left(\frac{\partial V}{\partial p}\right)_T dp + \left(\frac{\partial V}{\partial T}\right)_p dT$$

$$e^{aV} = b\, p^{-2}\, T^{1/3}$$

$$aV = \ln b - 2 \ln p + \frac{1}{3} \ln T$$

$$V = \frac{1}{a} \left(\ln b - 2 \ln p + \frac{1}{3} \ln T \right)$$

$$\left(\frac{\partial V}{\partial p}\right)_T = -\frac{2}{ap}$$

$$\left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{3Ta}$$

$$dV = -\frac{2}{ap} dp + \frac{1}{3aT} dT$$

$$\varkappa = -\frac{1}{V} \left(\frac{\partial V}{\partial p}\right)_T = +\frac{1}{V} \left(\frac{2}{ap}\right)$$

$$\beta = \frac{1}{V} \left(\frac{\partial V}{\partial T}\right)_p = \frac{1}{V} \left(\frac{1}{3aT}\right)$$

$\boxed{\text{II-9}}$ Solution

(i) $Z = w_1 e^{-E_1/kT} + w_2 e^{-E_2/kT}$

$$Z = e^{-E_1/kT}\left[w_1 + w_2 e^{-\Delta E/kT}\right]$$

with: $E_2 = E_1 + \Delta E$

now $U = \langle E \rangle = F - T\left(\frac{\partial F}{\partial T}\right)_V$ we find

$$U = E_1 + w_2 \frac{\Delta E\, e^{-\Delta E/kT}}{w_1 + w_2 e^{-\Delta E/kT}}$$ we could

also have used the concept of a statistical average

$$U = \langle E \rangle = \frac{\sum_i w_i E_i e^{-E_i/kT}}{\sum_i w_i e^{-E_i/kT}}$$ or

$$U = \frac{E_1 + \left(\frac{w_2}{w_1}\right) E_2 e^{-\Delta E/kT}}{1 + \left(\frac{w_2}{w_1}\right) e^{-\Delta E/kT}}$$ now let

us simplify:

$$\therefore \boxed{U = \langle E \rangle = E_1 + \frac{\Delta E}{1 + \left(\frac{w_1}{w_2}\right) e^{\Delta E/kT}}}$$ nt

$$C_V(T) = \frac{-(\Delta E)\left(\frac{w_1}{w_2}\right)\left(-\frac{\Delta E}{kT^2}\right) e^{\Delta E/kT}}{\left[1 + (w_1/w_2)\, e^{\Delta E/kT}\right]^2}$$

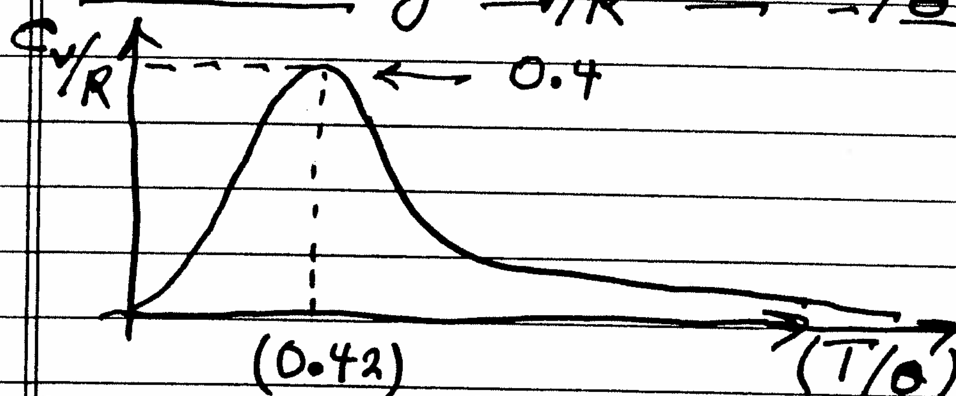
[II-9] continued.

$$C_V(T) = \left(k\frac{w_1}{w_2}\right)\frac{(\theta/T)^2 e^{\theta/T}}{\left[1+\left(\frac{w_1}{w_2}\right)e^{\theta/T}\right]^2}$$

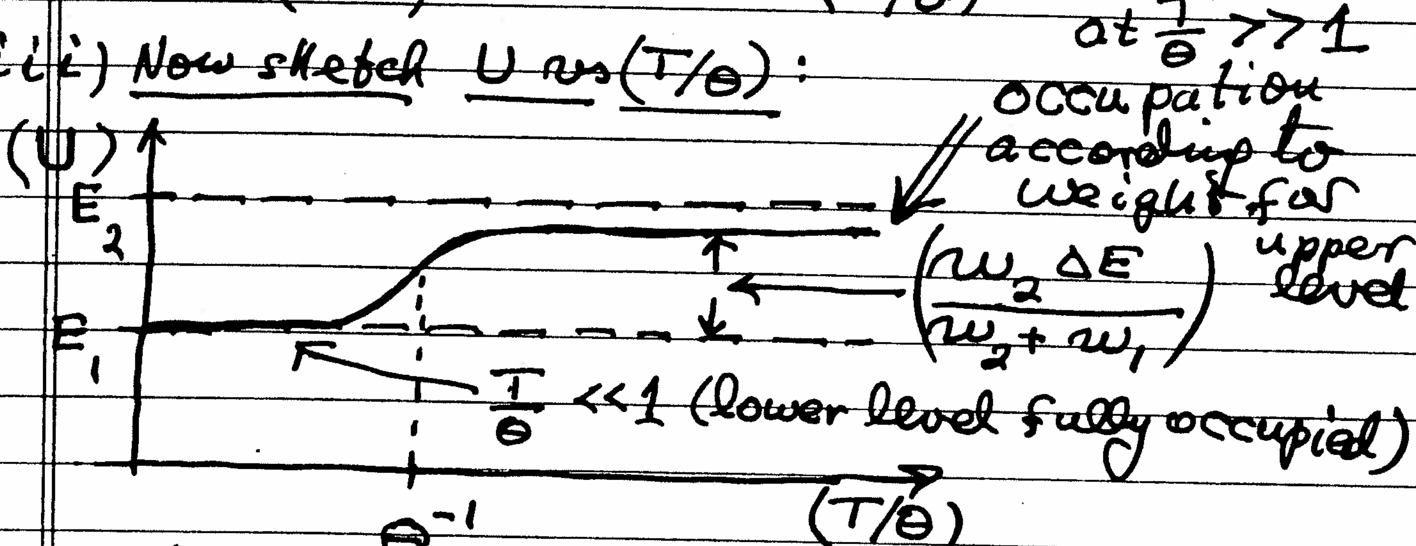
with: $\Theta = \Delta E/k$.

(ii.)
$C_V(T)$ will be zero for very low and very high temperatures. If $w_1 = w_2$, $C_V(T)$ has a maximum of $0.4R$ at $\theta/T = 2.4$. One may use this as an experimental method to determine Θ and ΔE.

A sketch of C_V/R vs. T/Θ:



(iii) Now sketch U vs (T/Θ):



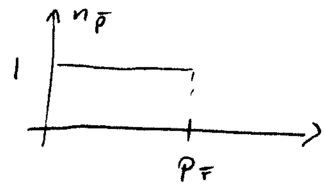
Only in exceptional cases is the upper level more populated than the lower level and that is called "negative temperature

$\boxed{\text{II}-10}$ 3-D ideal (non-interacting) e^- gas $\quad (s=1/2)$

$\varepsilon(\bar p) = \frac{p^2}{2m}$

$$N = \underset{\text{for spin}}{2} \times \sum_{\bar p} n_{\bar p} = 2 \int n_{\bar p} \frac{L^3}{h^3} d^3p = 2\frac{V}{h^3}\int_0^\infty 4\pi p^2 dp \; n_{\bar p}$$

$T=0$:



$$N = \frac{2V}{h^3} 4\pi \int_0^{p_F} p^2 dp = \frac{8\pi V}{3h^3} p_F^3 \quad \Rightarrow \quad p_F = \left[\frac{3h^3}{8\pi}\left(\frac{N}{V}\right)\right]$$

$$E = 2 \times \sum_{\bar p} \varepsilon_{\bar p} n_{\bar p} = 2\frac{V}{h^3}\int_0^{p_F} \frac{p^2}{2m} 4\pi p^2 dp = \frac{8V\pi}{2mh^3}\int_0^{p_F} p^4 dp$$

$$= \frac{8\pi V}{2mh^3}\frac{1}{5} p_F^5 = \frac{4\pi V}{5mh^3} p_F^5 = \frac{4\pi V}{5mh^3}\left(\frac{3}{8\pi}n\right)^{5/3} \cdot h^5 =$$

$$= \frac{4\pi V h^2}{5m}\left(\frac{3n}{8\pi}\right)^{5/3} \qquad n \equiv \frac{N}{V} \quad e^-\text{-density}$$

regardless of statistics: $\left(\text{for } \varepsilon(p) = \frac{p^2}{2m}\right)$ $\qquad PV = \frac{2}{3}E \quad \Rightarrow$

$$P = \frac{2}{3}\frac{E}{V} = \frac{8}{15}\frac{\pi h^2}{m}\left(\frac{3n}{8\pi}\right)^{5/3} \qquad \text{(ground-state pressure}$$

(We can see, that as a result of the Pauli principle free electrons cannot settle into the $\bar p = 0$ single-energy state. They occupy all $\bar p$ states up to p_F with degeneracy 2 due to spin. In turn, the pressure at $T=0$ is not zero!)