

Solutions

Physics PhD Qualifying Examination Part I – Wednesday, August 20, 2014

Name: _____

(please print)

Identification Number: _____

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.

PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
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	9	
	10	

Student's initials

problems handed in:

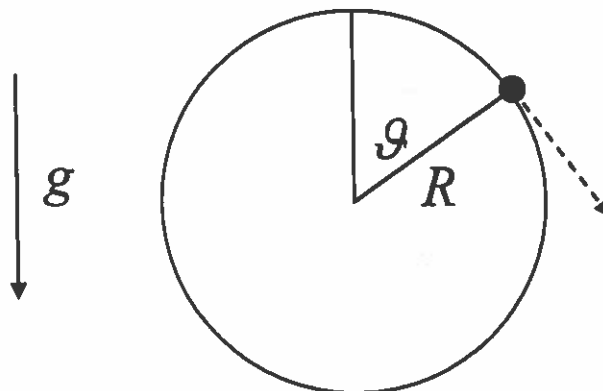
Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least three passed problems (from problems 1-5) for Mechanics and three problems (from problems 6-10) for Electricity and Magnetism.
7. **YOU MUST SHOW ALL YOUR WORK.**

I-1 [10]

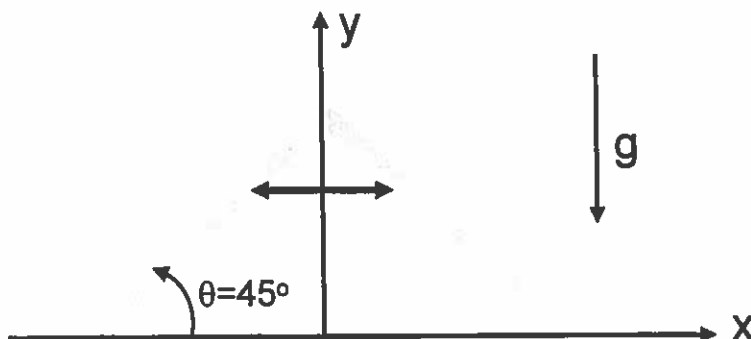
A particle slides freely and without friction on the top surface of a spherically-shaped solid object with radius R . The mass of the particle is m and the magnitude of the gravitational acceleration is g . The particle is initially at the top of the sphere with infinitesimally small velocity. Determine the angle ϑ at which the particle “takes off”, i.e., the angle at which the particle separates from the surface of the sphere. See illustration below.



I-2 [10]

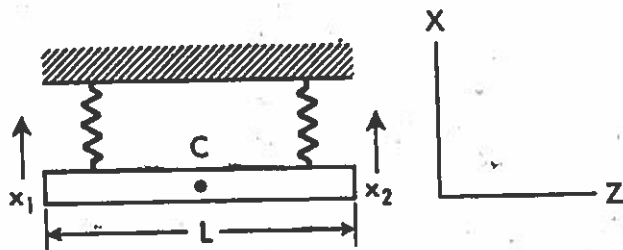
Two thin beams of mass m and length l are connected by a frictionless hinge and thread. The system rests on a smooth surface in the way shown in the figure below. At $t=0$, the thread is cut. In the following, you may neglect the thread and the mass of the hinge.

- Find the angular acceleration of the hinge as a function of angle, θ . Also, explain its physical significance.
- Find the speed of the hinge when it hits the ground, i.e. $\theta=0^\circ$.
- Find the time it takes the hinge to hit the floor, expressing this in terms of a concrete mathematical integral which you need not evaluate explicitly.



I-3 [10]

A rigid uniform bar of mass M and length L is supported in equilibrium in a horizontal position by two mass-less springs attached one at each end, as shown in the figure below.



The springs have the same force constant k . The motion of the center of gravity is constrained to move parallel to the vertical x -axis. Find the normal modes and frequencies of vibration of the system, if the motion is constrained to the xz -plane.

I-4 [10]

A particle of mass m is moving under the influence of a central potential (with a fixed center),

$$U(r) = k \ln(r),$$

where $k > 0$ is a constant. The particle performs circular motion with a radius r_o . Determine the frequency of small oscillations ω_o about this circular orbit. Your answer must be expressed in terms of m , k , and r_o .

I-5 [10]

As viewed from the laboratory frame of reference, two particles of rest mass m are emitted in the same direction. These particles (particle A and particle B) have momenta of $5mc$ and $10mc$, respectively. ($c = \text{the speed of light}$)

- What is the velocity of particle A, as viewed from the laboratory frame of reference?
- What is the velocity of particle B, as viewed from the laboratory frame of reference?
- What is the velocity of particle B, as viewed from the reference frame of particle A?
- What is the velocity of particle A, as viewed from the reference frame of particle B?

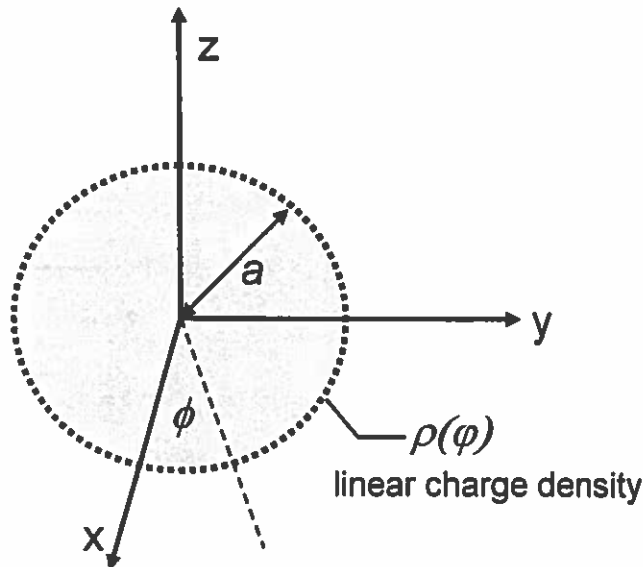
I-6 [10]

The linear charge density ρ on a ring of radius a is given by:

$$\rho(\phi) = \left(\frac{q}{a}\right) [\cos(\phi) - \sin(2\phi)],$$

where ϕ is the polar angle in the plane of the ring, i.e. the xy plane (see the illustration below).

- Find the mono-pole and the dipole moments of the system;
- Explain briefly the physical significance of the derived results.
- Calculate the potential at an arbitrary point in space, accurate to term in r^{-3} .

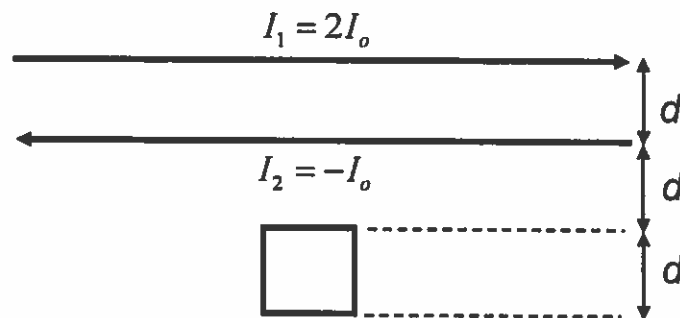


I-7 [5,3,2]

- (a) Show that it is possible for electromagnetic waves to be propagated in a hollow metal pipe of rectangular cross section with perfectly conducting walls.
- (b) What are the phase and group velocities for this system?
- (c) Show that there is a cutoff frequency below which no waves are propagated in this rectangular pipe.

I-8 [10]

Two infinite parallel wires separated by a distance d carry currents $I_1 = 2I_o$ and $I_2 = -I_o$, with current increasing at the rate dI_o/dt . (Note that the two currents flow in opposite directions.) A square loop of wire of length d on a side lies in the plane of the wires at a distance d from one of the parallel wires, as illustrated in the figure below.



- (a) Find the emf induced in the square loop;
- (b) Is the induced current clockwise or counterclockwise? Justify your answer.

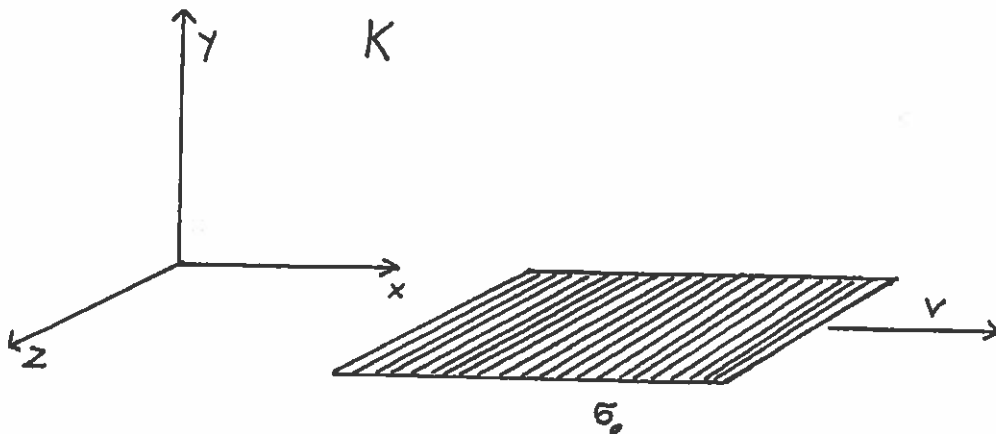
I-9 [10]

A non-relativistic positron of charge e and velocity v_1 ($v_1 \ll c$) impinges head-on on a *fixed* nucleus of charge Ze . The positron, which is coming from far away, is decelerated until it comes to rest and then is accelerated again in the opposite direction until it reaches a terminal velocity of v_2 . [Hint: Note that $\int \frac{dx}{x^3 \sqrt{x(x-a)}} = 2\sqrt{x(x-a)} * \frac{3a^2+4ax+8x^2}{15a^3x^3}$.]

- Taking radiation loss into account (but assuming it is small) find v_2 as a function of v_1 and Z .
- What is the angular distribution ($\frac{dP}{d\Omega}$) of the radiation?
- What is the polarization of the radiation?

I-10 [4,4,2]

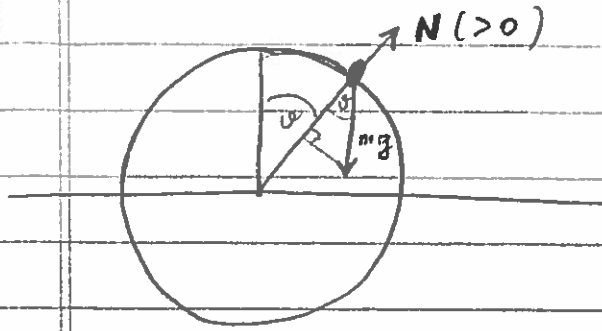
In reference frame K , an infinite, uniformly charged thin sheet in the x - z plane is moving to the $+\hat{x}$ direction with velocity v . The uniform surface charge density of the sheet is σ_0 , *if measured at rest*.



- What is the electric field vector in reference frame K ?
- What is the magnetic field vector in reference frame K ?
- What is the force on a point charge q_0 , that is moving at a distance y above the charged sheet with a velocity u to the $+\hat{x}$ direction? (Both the velocity u and the force are measured in reference frame K , while the charge q_0 is measured at rest.)

Solutions / Part-I

1-1



radial direction: $m \frac{v^2}{R} = mg \cos \theta - N$

conservation of energy: $mgR = mgR \cos \theta + \frac{1}{2} m v^2$

$$v^2 = 2gR(1 - \cos \theta)$$

$$N = mg \cos \theta - m \frac{v^2}{R} = mg \cos \theta - \frac{m}{R} 2gR(1 - \cos \theta) =$$
$$= mg(3 \cos \theta - 2) > 0$$

(N being the normal force has to be positive. $N=0$ precisely at the point where the particle 'takes off')

Thus, $3 \cos \theta - 2 > 0$

$$\cos \theta_c = \frac{2}{3}$$

$$\Rightarrow \theta_c = \cos^{-1}\left(\frac{2}{3}\right)$$

Q/2

(I-2)

By symmetry, the hinge will fall vertically.
The center-of-mass of the beams:

$$(i) \quad X_1 = \frac{1}{2} l \cos \theta, \quad Y_1 = \frac{1}{2} l \sin \theta \\ X_2 = -\frac{1}{2} l \cos \theta, \quad Y_2 = \frac{1}{2} l \sin \theta$$

(ii) Their velocity components:

$$\dot{X}_1 = -\frac{1}{2} l \dot{\theta} \sin \theta, \quad \dot{Y}_1 = \frac{1}{2} l \dot{\theta} \cos \theta \\ \dot{X}_2 = \frac{1}{2} l \dot{\theta} \sin \theta, \quad \dot{Y}_2 = \frac{1}{2} l \dot{\theta} \cos \theta$$

(iii) Each beam's moment-of-inertia: $ml^2/12$

(iv) The Lagrangian of the system:

$$L \equiv T - V \\ = \left(\frac{1}{4} ml^2 \dot{\theta}^2 + \frac{1}{12} ml^2 \dot{\theta}^2 \right) - mgl \sin \theta$$

$$\therefore L = \frac{1}{3} ml^2 \dot{\theta}^2 - mgl \sin \theta$$

(v) The Lagrange's Equ.:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

$$\therefore \ddot{\theta} + \frac{3g}{2l} \cos \theta = 0 \quad \#$$

I-2

No. ②/2

(b) Use $\ddot{\theta} = \frac{1}{2} \frac{d}{d\theta}(\dot{\theta}^2)$; $\dot{\theta} = 0$, when $\theta = 45^\circ$

also $\ddot{\theta} = -\frac{3}{2} \left(\frac{g}{l} \right) \cos \theta$

$$\therefore \boxed{\frac{1}{2} \frac{d}{d\theta}(\dot{\theta}^2) = -\frac{3}{2} \left(\frac{g}{l} \right) \cos \theta} \#$$

perform integration over θ

$$\frac{1}{2} [\dot{\theta}^2 - \dot{\theta}^2(\theta=45^\circ)] = -\frac{3}{2} \left(\frac{g}{l} \right) [\sin \theta - \sin 45^\circ]$$

\parallel \parallel
 0 $\frac{\sqrt{2}}{2}$

$$\therefore \underline{\dot{\theta}^2 = \frac{3}{2} \left(\frac{g}{l} \right) (\sqrt{2} - 2 \sin \theta), \theta \leq 45^\circ} \#$$

When the hinge hits the ground, $\theta = 0$

$$\dot{\theta}^2 = \frac{3}{2} \left(\frac{g}{l} \right) \sqrt{2}$$

$$\underline{\dot{\theta} = \left(\frac{3}{\sqrt{2}} \left(\frac{g}{l} \right) \right)^{1/2}} \#$$

$$(c) t = \int_{45^\circ}^{0^\circ} \frac{d\theta}{\dot{\theta}} = \int_{45^\circ}^{0^\circ} \frac{1}{\sqrt{\frac{3}{2} \left(\frac{g}{l} \right)}} \frac{1}{\sqrt{2} - 2 \sin \theta} d\theta \#$$

(I-3) Solution:

Let the vertical displacements from the equilibrium positions be X_1, X_2 . For the motion of the center of mass C , $F = ma$ yields

$$\frac{M}{2}(\ddot{X}_1 + \ddot{X}_2) = -k(X_1 + X_2) - mg \quad (1)$$

and the torque conditions for small X_1, X_2 gives

$$\frac{I_0}{L}(\ddot{X}_2 - \ddot{X}_1) = -\frac{L}{2}k(X_2 - X_1) \quad (2)$$

and $I_0 =$ moment of inertia about C which is $(ML^2/12)$.

The gravity term, which merely determines the unextended spring lengths can be transformed away. It does not affect the modes. Now, from (1) and (2) there are obviously two modes:

- (a) a symmetric one, $X_2 = X_1 + X_2$ with frequency $\omega_1^2 = 2k/M$ and
- (b) an antisymmetric one, $X_2 = X_1 - X_2$, with $\omega_2^2 = 6k/M$.

I-4 $U(r) = k \ln(r)$

$$L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\varphi}^2 - U(r)$$

$$\frac{\partial L}{\partial r} = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}}$$

$$\frac{\partial L}{\partial \varphi} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}}$$

(1) $m r \ddot{\varphi} - \frac{\partial U}{\partial r} = m \ddot{r}$

$$\varphi = \frac{d}{dt} (m r^2 \dot{\varphi})$$

$$l = m r^2 \dot{\varphi} = \text{const.}$$

$$U(r) = k \ln(r)$$

$\left(F(r) = - \frac{\partial U}{\partial r} = - \frac{k}{r} \right) \equiv - m g(r)$ (radial force)

Circular motion: $r = r_0 = \text{const.} \Rightarrow \dot{r} = 0, \ddot{r} = 0$

$$m r \ddot{\varphi} - \frac{k}{r} = 0 \Rightarrow \boxed{\ddot{\varphi} = \frac{k}{m r^2}}$$

From Eq. (1) (in general)

$$l = m r^2 \dot{\varphi} \\ l^2 = m^2 r^4 \dot{\varphi}^2 = m^2 r^4 \frac{k}{m r^2} = m r^2 k$$

$$\frac{l^2}{m r^3} - m g(r) = m \ddot{r}$$

$$= m r_0^2 k$$

\Rightarrow

$$\boxed{\ddot{r} = \frac{l^2}{m^2 r^3} - g(r)}$$

$$\ddot{r} = \frac{l^2}{m^2 r^3} - g(r) \quad , \text{ where } g(r) \equiv \frac{k}{m r}$$

Must expand about $r = r_0$

$$r = r_0 + x \quad (x \ll r_0)$$

$$\ddot{x} = \frac{l^2}{m^2 r_0^3} - \frac{3l^2}{m^2 r_0^4} x - g(r_0) - g'(r_0) x$$

$$\ddot{x} = \underbrace{\left[\frac{l^2}{m^2 r_0^3} - g(r_0) \right]}_{=0 \text{ for circ orbit @ } r_0} - \left[\frac{3l^2}{m^2 r_0^4} + g'(r_0) \right] x$$

$$\ddot{x} = - \left[\frac{3l^2}{m^2 r_0^4} + g'(r_0) \right] x$$

Frequency of small oscillations: see next page

$$\omega_0^2 \equiv \frac{3l^2}{m^2 r_0^4} - \frac{k}{m r_0^2} = \frac{3(m r_0^2 k)}{m^2 r_0^4} - \frac{k}{m r_0^2} =$$

$$= 3 \frac{k}{m r_0^2} - \frac{k}{m r_0^2} = \frac{2k}{m r_0^2}$$

$$\omega_0 = \sqrt{\frac{2k}{m r_0^2}} = \boxed{\sqrt{\frac{2k}{m r_0}}} \quad \checkmark$$

(General note)

For circular orbit: ($r = r_0$)

$$\frac{l^2}{m^2 r_0^3} - g(r_0) = 0$$

$$\frac{l^2}{m^2 r_0^4} = \frac{g(r_0)}{r_0}$$

Thus, eq. of motion for small oscillation:

$$\ddot{x} = - \left[3 \frac{g(r_0)}{r_0} + g'(r_0) \right] x$$

$$g(r) = \frac{k}{m} \frac{1}{r}$$

$$g'(r) = -\frac{k}{m} \frac{1}{r^2}$$

$$\omega_0^2 = 3 \frac{k}{m} \frac{1}{r_0^2} - \frac{k}{m} \frac{1}{r_0^2} = \boxed{\frac{2k}{m} \frac{1}{r_0^2}}$$

Alternative Solution:
(I-4)

Alternatively, using the effective
radial potential, $U_{\text{eff}}(r)$:

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\phi}^2 + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \left(\frac{l^2}{m^2 r^4} \right) + U(r)$$

$$= \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{l^2}{m r^2} + U(r) = \frac{1}{2} m \dot{r}^2 + U_{\text{eff}}(r)$$

where $U_{\text{eff}}(r) = \frac{l^2}{2mr^2} + U(r)$

effective (radial) equation of motion: $\left(\frac{\partial U}{\partial r} = m g(r) \right)$

$$m \ddot{r} = - \frac{\partial U_{\text{eff}}}{\partial r} = \frac{l^2}{mr^3} - \frac{\partial U}{\partial r}$$

$$m \ddot{r} = \frac{l^2}{mr^3} - m g(r)$$

$$\ddot{r} = \frac{l^2}{m^2 r^3} - g(r)$$

(same as obtained earlier)

freq. of small oscillations (about stable circular orbit):

$$m \omega^2 = \left. \frac{\partial^2 U_{\text{eff}}}{\partial r^2} \right|_{r_0} = \frac{3l^2}{mr_0^4} + m \left. \frac{dg}{dr} \right|_{r_0} =$$

$$= \frac{3l^2}{m\tau_0^4} + m \frac{k}{m} \left(-\frac{1}{\tau_0^2} \right) =$$

$$= \frac{3m\tau_0^2 k}{m\tau_0^4} - \frac{k}{\tau_0^2} = \frac{2k}{\tau_0^2}$$

$$\omega^2 = \frac{2k}{m\tau_0^2}$$

$$\boxed{\omega = \sqrt{\frac{2k}{m}} \frac{1}{\tau_0}}$$

$$|I + 5|$$

solution:

Solving for the velocities of particles A and B in the laboratory frame,

$$m\gamma v = p$$

$$mv \sqrt{\frac{1}{1 - \frac{v^2}{c^2}}} = p, \quad m^2 v^2 = p^2 \left(1 - \frac{v^2}{c^2}\right)$$

$$v^2 \left(m^2 + \frac{p^2}{c^2}\right) = p^2 \rightarrow v = \sqrt{\frac{p^2}{m^2 + p^2/c^2}}$$

$$v_A = \sqrt{\frac{(5mc)^2}{m^2 + (5mc)^2/c^2}} = c \sqrt{\frac{25}{26}}$$

$$v_B = \sqrt{\frac{(10mc)^2}{m^2 + (10mc)^2/c^2}} = c \sqrt{\frac{100}{101}}$$

Transforming the velocity of particle B v_B to the reference frame of particle A,

$$v'_B = (v_B - v_A)/(1 - v_A v_B/c^2) = c \left(\sqrt{\frac{100}{101}} - \sqrt{\frac{25}{26}} \right) / \left(1 - \sqrt{\frac{100}{101}} * \frac{25}{26} \right) \approx 0.595c$$

alternately, the velocity of particle A as measured from the reference frame of B $\rightarrow v'_A \approx -0.595c$

(1-6)

No. 10/3

(a) Along the ring, $dl = (a d\phi)$ For monopole, $Q = \int \rho_e(\phi) \cdot dl$

$$Q = \int_0^{2\pi} \left(\frac{Q}{a}\right) (\cos\phi - \sin 2\phi) (a d\phi) \quad \#$$

 $\cos\phi$, $\sin 2\phi$ have period of 2π and π , respectively.So, integral is zero, $Q = 0 \quad \#$ For dipole, $\vec{p} = \int \rho_e(\phi) (a \hat{e}_r) \cdot dl$

$$\vec{p} = \int_0^{2\pi} (a d\phi) (a \hat{e}_r) \left(\frac{Q}{a}\right) (\cos\phi - \sin 2\phi)$$

$$\vec{p} = (Qa) \int_0^{2\pi} d\phi (\hat{e}_r) (\cos\phi - \sin 2\phi)$$

Use: $\hat{e}_r = \hat{x} \cos\phi + \hat{y} \sin\phi$

$$\therefore p_x = (Qa) \int_0^{2\pi} d\phi (\cos^2\phi - \cos\phi \sin 2\phi)$$

$$p_y = (Qa) \int_0^{2\pi} d\phi (\cos\phi \sin\phi - 2 \sin\phi \sin 2\phi)$$

$$\therefore \begin{cases} p_x = \pi(Qa) \\ p_y = 0 \end{cases} \quad \#$$

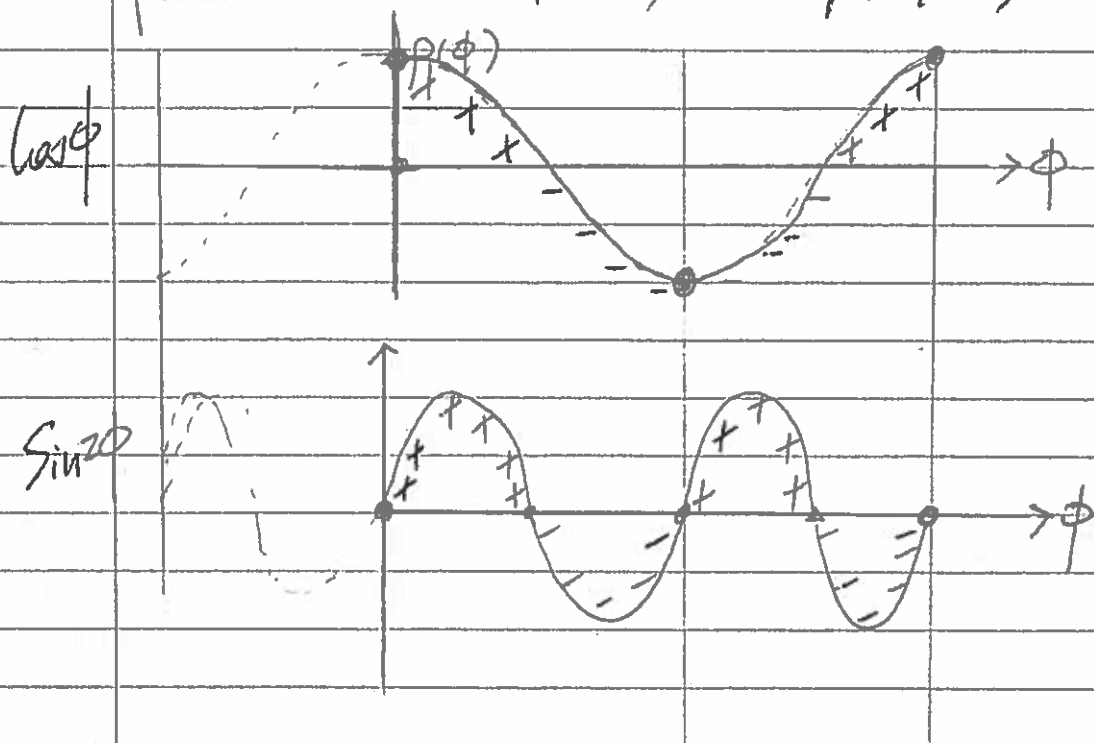
1-6

No. 2/3

(b) $\rho(\phi) \sim (\cos\phi - \sin 2\phi)$

for $\cos\phi$ term, dipole direction points

from $-x$ to $+x$, $\therefore p_x \neq 0$, By symmetry, $p_y = 0$ #



for $\sin 2\phi$ term p_x terms cancel, $p_x = 0$
 p_y terms cancel, $p_y = 0$ #

(1-6)

No.
③/3

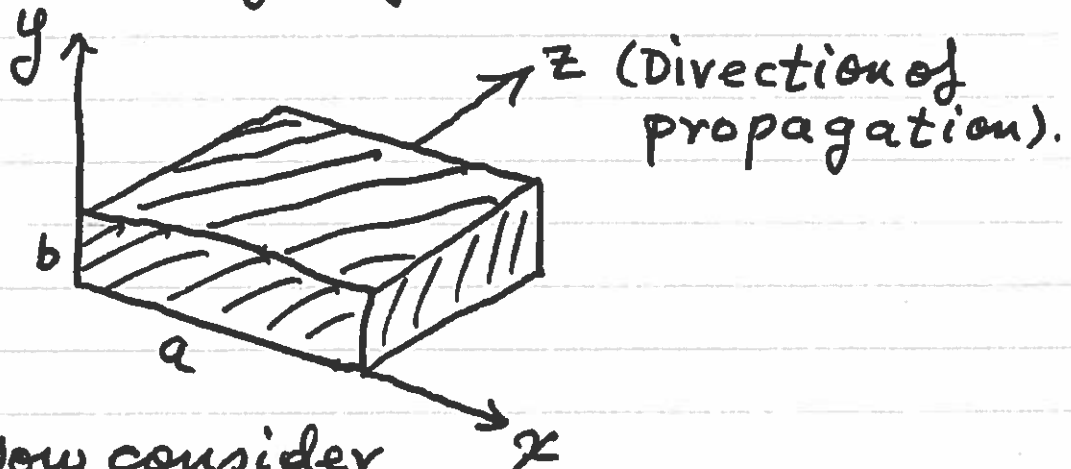
$$(c) \phi^{(2)} = \frac{\vec{p} \cdot \hat{e}_1}{r^2} = \frac{1}{r^2} (\pi q a) \hat{x} \cdot \hat{e}_r$$

$$\therefore \phi^{(2)} = \frac{\pi(qa)}{r^2} \sin\theta \cos\phi \quad \#$$

Ph.D. Q.E August 2014

[I-7] Solution

The cavity is of the form:



Now consider a solution of a Maxwell equation inside the above cavity of the form (transverse Electric (TE mode))

$$E_z = 0, \quad E_x = E_1(x, y) e^{i(kz - \omega t)}$$

$$E_y = E_2(x, y) e^{i(kz - \omega t)}$$

From the wave equation

$$\nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

one obtains the equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) E_i - \left(k^2 - \frac{\omega^2}{c^2} \right) E_i = 0 \quad (1)$$

The solution of this equation satisfying

2.

the boundary conditions that the tangential component of E and the normal component of H vanish on the boundaries is:

$$E_1 = E_{01} \cos\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$E_2 = E_{02} \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right)$$

with: $\frac{n}{a} E_{01} + \frac{m}{b} E_{02} = 0$ in order to guaranty $\vec{\nabla} \cdot \vec{E} = 0$

Now equation (1) becomes

$$\frac{\omega^2}{c^2} = k^2 + \pi^2 \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right)$$

(here n, m are integers, and $a > b$).

Since $k^2 \geq 0$ for transmission, there is a minimum frequency given by

$$\omega_0 = c\pi/a.$$

The phase velocity is given by

$$v_p = \frac{\omega}{k} = c \left\{ 1 + \left(\frac{\pi^2}{k^2} \right) \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \right\}^{1/2}$$

3.

while the group velocity is given by

$$v_g = \frac{d\omega}{dk} = c \left\{ 1 + \left(\frac{\pi^2}{k^2} \right) \left(\frac{n^2}{a^2} + \frac{m^2}{b^2} \right) \right\}^{-\frac{1}{2}}$$

Similar results are obtained for transverse magnetic modes; (TM) i.e. $H_z = 0$. however the cutoff frequency is higher.

One notices also that $v_p \cdot v_g = c^2$.

1-8

No.

①/2

The B-field produced by an infinite, straight wire:

$$B = \left(\frac{\mu_0}{2\pi} \right) \frac{I}{r}$$

The flux due to wire-1:

$$\phi_1 = \int_{2d}^{3d} \left(\frac{\mu_0}{2\pi} \right) \frac{2I_0}{r} (dr) = \frac{\mu_0}{2\pi} (2I_0 d) \ln\left(\frac{3}{2}\right) \quad \#$$

The flux due to wire-2:

$$\phi_2 = \int_d^{2d} \left(\frac{\mu_0}{2\pi} \right) \frac{I_0}{r} (dr) = \frac{\mu_0}{2\pi} (I_0 d) \ln 2 \quad \#$$

Note = ϕ_1 is directed into the page
 ϕ_2 " " " out-of the page.

$$|\phi_1| > |\phi_2|$$

The total flux:

$$\phi = \phi_1 - \phi_2 = \frac{\mu_0}{2\pi} (I_0 d) \left[2 \ln \frac{3}{2} - \ln 2 \right] \quad \#$$

The total flux is directed into the page

1-8

No. Q/2

The emf induced in the square loop due to (dI_0/dt) :

$$\mathcal{E} = - \frac{d\phi}{dt} = \underbrace{\left(\frac{\mu_0}{2\pi}\right)}_{\text{out-of-the-page}} (I_0 d) \left[\ln \frac{9}{8}\right] \frac{dI_0}{dt} \#$$

The induced field will be directed "out-of" the page so as to "oppose" the change in B-flux.

The induced current is "counter-clockwise" wise.

solution:

As the radiation loss of the positron is much smaller than its kinetic energy, it can be considered as a small perturbation. We therefore first neglect the effect of radiation. By the conservation of energy, when the distance between the positron and the fixed nucleus is r and its velocity is v we have,

$$\frac{1}{2}mv^2 + \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r} = \frac{1}{2}mv_1^2.$$

When $v = 0$, r reaches its minimum r_0 . Thus,

$$\frac{1}{4\pi\epsilon_0} \frac{Ze^2}{r_0} = \frac{1}{2}mv_1^2 \quad \text{or} \quad r_0 = \frac{Ze^2}{2\pi\epsilon_0 mv_1^2} \quad \text{with} \quad v^2 = v_1^2(1 - \frac{r_0}{r})$$

Differentiating the last equation we have,

$$2\dot{r}\ddot{r} = \frac{v_1^2 r_0}{r^2} \dot{r} \quad \rightarrow \quad \ddot{r} = \frac{v_1^2 r_0}{2r^2}$$

from Larmor's formula, the rate of radiation loss is given by

$$P = \frac{2q^2}{3c^3} \ddot{r}^2, \quad \text{with} \quad P = \frac{dW}{dt} = \frac{dW}{dr} \dot{r} = \frac{dW}{dr} v$$

Solving for dW yields, $dW = \frac{1}{v} \frac{2e^2}{3c^3} \ddot{r}^2 dr = \left(\frac{1}{v_1 \sqrt{1 - r_0/r}} \right) \left(\frac{2e^2}{3c^3} \right) \left(\frac{v_1^2 r_0}{2r^2} \right)^2 dr$

$$\Delta W = 2 \int_{r_0}^{\infty} dW = 2 \frac{e^2 v_1^3 r_0^2}{3c^3} \int_{r_0}^{\infty} \frac{dr}{r^3 \sqrt{r(r - r_0)}} = -\frac{8}{45} \frac{v_1^3}{Zc^3} mv_1^2$$

As $\frac{1}{2}mv_2^2 = \frac{1}{2}mv_1^2 - \Delta W$, we have

$$v_2^2 = v_1^2 \left(1 - \frac{16}{45} \frac{v_1^3}{c^3 Z} \right)$$

$$\text{Hence, } v_2 \approx v_1 \left(1 - \frac{8}{45} \frac{v_1^3}{c^3 Z} \right)$$

as $v_1 \ll c$.

Because $v_1 \ll c$, the radiation is dipole in nature so that the angular distribution of its radiated power is given by $\frac{dP}{d\Omega} \propto \sin^2 \theta$, with θ being the angle between the directions of the radiation and the particle velocity.

The radiation is plane polarized with the electric field vector in the plane containing the direction of the radiation and the acceleration (which is the same as that of the velocity in this case).

I-10

σ_0 : surface charge density in rest frame

("summary" of solution) In K , the sheet is moving with velocity v ,

$$\sigma = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q_0}{\sqrt{1-v^2/c^2} \Delta V_0} = \frac{1}{\sqrt{1-v^2/c^2}} \sigma_0 = \gamma \sigma_0$$

(since the charge is invariant, while $\Delta x = \sqrt{1-v^2/c^2} \Delta x_0$
"Lorentz contracted", $\Delta V = \Delta x \Delta y \Delta z$)

(a) electric field: (e.g. using Gauss' Law)

$$\vec{E} = \frac{\sigma}{2\epsilon_0} (\pm \hat{y}) = \frac{\gamma \sigma_0}{2\epsilon_0} (\pm \hat{y}) \quad \text{for } y \gtrless 0$$

$$(E_x = E_z = 0)$$

(b) magnetic field: (e.g. using Ampere's Law)

$$\vec{B} = \frac{\mu_0 \sigma v}{2} (\pm \hat{z}) = \frac{\mu_0 \gamma \sigma_0 v}{2} (\pm \hat{z}) \quad \text{for } y \gtrless 0$$

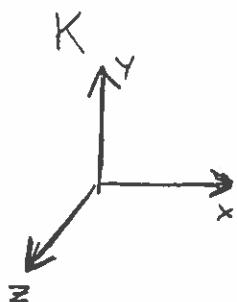
$$(B_x = B_y = 0)$$

(c) force on particle with q_0 , moving to $+\hat{x}$ with velocity u
above plane in frame K : $q = q_0$, $\vec{F} = q\vec{E} + q\vec{u} \times \vec{B}$

$$F_x = F_z = 0$$

$$F_y = q_0 E_y - q_0 u_x B_z = q_0 \frac{\gamma \sigma_0}{2\epsilon_0} - q_0 u \frac{\mu_0 \gamma \sigma_0 v}{2} =$$

$$= \frac{q_0 \sigma_0}{2\epsilon_0} \gamma \left(1 - \epsilon_0 \mu_0 u v\right) = \frac{q_0 \sigma_0}{2\epsilon_0} \gamma \left(1 - \frac{uv}{c^2}\right)$$



(Some further detail of alternative solution)

EM 1/10/10



$$2EA = \frac{Q}{\epsilon_0}$$

In reference frame K an infinite very thin uniformly charged plane is moving to the +x direction with velocity v . The uniform charge density of the plane is σ_0 if measured at rest.

4pts a) What is the electric field \vec{E} in K? (magnitude & direction)

4pts b) What is the magnetic field \vec{B} in K?

2pts c) What is the force on a charge measured in K if its velocity is $\vec{u} = u \hat{x}$ (in K)

solution (I)

$$\therefore EA + EA = \frac{Q}{\epsilon_0} \quad E = \frac{1}{\epsilon_0} \frac{Q}{2A} = \frac{1}{2\epsilon_0} \sigma_0$$

$$\begin{array}{l} \text{CGS} \\ EA + EA = 4\pi Q \\ E = \frac{4\pi}{2} \sigma \end{array}$$

a) $E_y = \frac{\sigma}{2\epsilon_0}$ (from Gauss's Law) $E = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \sigma_0$

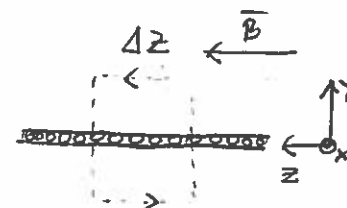
($E_x = E_z = 0$)

$E_y = \frac{\sigma_0}{2\epsilon_0}$

b) $B_z = \frac{\mu_0 \sigma v}{2}$ (from Ampere's Law:)

($B_x = B_y = 0$)

$$I = \sigma \Delta z \frac{\Delta x}{\Delta t} = \sigma \Delta z v$$



$$2B \Delta z = \mu_0 \sigma \Delta z v$$

$$B = \frac{\mu_0 \sigma v}{2}$$

$B_z = \frac{\mu_0 \sigma v}{2}$

CGS.

$$B = \frac{4\pi}{2c} \sigma v$$

c)

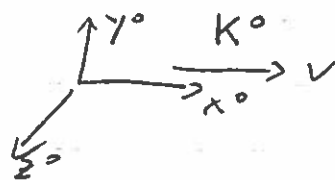
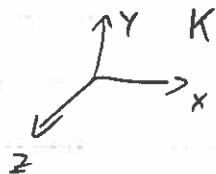
$$\begin{aligned}
 F_y &= q E_y - q u_x B_z = q E_y - q u B_z = \\
 &= q \frac{\partial \phi_0}{2 \epsilon_0} - q \frac{\mu_0 \partial \phi_0 v}{2} u = \\
 &= \frac{q \phi_0}{2 \epsilon_0} \gamma \left(1 - \epsilon_0 \mu_0 u v \right) = \frac{q \phi_0}{2 \epsilon_0} \gamma \left(1 - \frac{u v}{c^2} \right)
 \end{aligned}$$

solution (II)

$$E_y^0 = \frac{\phi_0}{2 \epsilon_0}$$

$$B_z^0 = B_x^0 = B_y^0 = 0$$

$$(E_x^0 = E_z^0 = 0)$$



$$E_y = \gamma (E_y^0 + v B_z^0) = \gamma \frac{\phi_0}{2 \epsilon_0}$$

$$B_z = \gamma \left(B_z^0 + \frac{v}{c^2} E_y^0 \right) = \gamma \frac{v}{c^2} \frac{\phi_0}{2 \epsilon_0} = \gamma \frac{\mu_0 \phi_0 v}{2}$$

Then, $F_y = \dots$ same as before.

Also, force may be directly obtained from F_z^0 : $\hat{z} \perp \vec{v}$

$$F_z^0 = \frac{q \phi_0}{2 \epsilon_0}$$

$$F_z = \frac{F_z^0}{\gamma \left(1 - \frac{u v}{c^2} \right)}$$

u is particle velocity in K

$$F_z = \gamma \left(1 - \frac{u v}{c^2} \right) F_z^0 = \gamma \left(1 - \frac{u v}{c^2} \right) \frac{q \phi_0}{2 \epsilon_0}$$

Physics PhD Qualifying Examination
Part II – Friday, August 20, 2014

Name: _____
(please print)

Identification Number: _____

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.

PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all problems that you are handing in.
6. A passing distribution for the individual components will normally include at least four passed problems (from problems 1-6) for Quantum Physics and two problems (from problems 7-10) for Thermodynamics and Statistical Mechanics.
7. **YOU MUST SHOW ALL YOUR WORK.**

II-1 [10]

The wavefunction for a particle of mass M in a one-dimensional potential $V(x)$ is given by the expression

$$\begin{aligned}\psi(x, t) &= \alpha x e^{-\beta x} e^{\frac{i\gamma t}{\hbar}}, & \text{for } x > 0 \\ \psi(x, t) &= 0, & \text{for } x < 0,\end{aligned}$$

where α , β , and γ are all positive constants.

- (a) Is the particle bound? Explain.
- (b) What is the probability density $\rho(E)$ for a measurement of the total energy E of the particle?
- (c) Find the potential $V(x)$ in terms of the given quantities.

II-2 [10]

Consider a particle of mass m in a one-dimensional box with infinite high potential wall at $x=0$ and $x=L$.

- (a) Find the eigenvalues E_n and eigenfunctions $\varphi_n(x)$ of the particle in the box.
- (b) Calculate the first-order correction to $E_2^{(0)}$ due to the following perturbation:

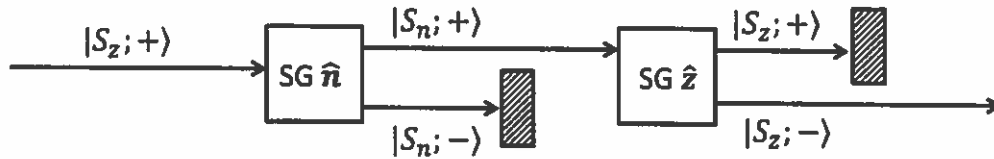
$$H' = 10^{-3} E_1 \frac{x^2}{L^2},$$

$$\text{where } E_1 = \frac{\hbar^2 \pi^2}{2mL^2}.$$

II-3 [10]

A beam of identically prepared spin $\frac{1}{2}$ atoms with $S_z = +\hbar/2$ orientation and with *unit intensity* goes through a series of Stern-Gerlach-type (SG) measurements (selective filtering) as follows:

- The first measurement accepts $S_n = +\hbar/2$ atoms (and rejects $S_n = -\hbar/2$ ones), where $S_n = \pm\hbar/2$ are the eigenvalues of the spin operator $S \cdot \hat{n}$ along the direction $\hat{n} = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, \cos\vartheta)$; (ϑ and φ are the polar and azimuthal angles, respectively).
- The second measurement accepts $S_z = -\hbar/2$ atoms (and rejects $S_z = +\hbar/2$ ones).



What is the intensity of the final $S_z = -\hbar/2$ beam? (You must express your answer in terms of ϑ and φ .)

II-4 [6,4]

Evaluate the differential scattering cross-section in a repulsive field, $V(r) = A/r^2$, in the **Born approximation** and according to **classical mechanics**. Determine the limit of applicability of the formulae obtained.

Hint: (a) Make an argument that the Born approximation is valid at all scattering angles where the classical result holds only for not too small angles.

(b) For the classical-mechanics case of scattering in a central force field you may want to use the integral of the form:

$$\psi = \int_{r_m}^{\infty} \frac{dr}{r^2 \sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{A}{r^2}}}$$

here ψ is the angle between the direction of the incoming asymptote and the periapsis (closest approach) direction. Hence, the scattering angle is given by $\Theta = \pi - 2\psi$, and $l = mv_0 s = s\sqrt{2mE}$ with s being the impact parameter and v_0 is the incident speed of the particle, m is the mass and E is the energy.

II-5 [10]

A beam of electrons is to be fired over a distance of $10^4 km$. If the size of the initial wavepacket is $1 mm$, estimate the size upon arrival, assuming a non-relativistic average kinetic energy, $K = 13.6 eV$.

II-6 [10]

Consider a charged one-dimensional harmonic oscillator with mass m , frequency ω_o , and charge q . Initially the oscillator is in its unperturbed ground state when there is no electric field present. At $t = 0$ a weak spatially uniform electric field $E = E_o e^{-\gamma t} \cos(\omega t)$ is imposed (the field is parallel to the direction of motion of the oscillator) with $\gamma < \omega_o$. Using *time-dependent perturbation theory*, find the transition probabilities to all excited states for $t = \infty$. For fixed ω_o and γ , what value of ω maximizes these transition probabilities? You may find the number representation of the harmonic oscillator with the annihilation and creation operators useful

$$a = \sqrt{\frac{m\omega_o}{2\hbar}} \left(x + \frac{i}{m\omega_o} p \right), \quad a^+ = \sqrt{\frac{m\omega_o}{2\hbar}} \left(x - \frac{i}{m\omega_o} p \right).$$

II-7 [10]

The Helmholtz free energy of a gas is given by $F(T, V) = -\frac{a}{3}T^4V$, where a is a positive constant. The gas is initially at temperature T and has volume V . Then the gas undergoes (Gay-Lussac — Joule) “*free expansion*” from V to $81V$. (In this process the gas suddenly and adiabatically expands into vacuum, i.e., no work is done.)

Your answers must be expressed in terms of the initial temperature T , the initial volume V , and the constant a , but may not necessarily involve all of them.

- (a) Obtain the final temperature T_2 of the gas.
- (b) Calculate the total entropy change ΔS of this gas during the above free expansion.

II-8 [10]

The equation of state of a simple ferromagnetic material is given by the implicit expression

$$m = \tanh\left(\frac{Jm + B}{kT}\right),$$

where $m = m(T, B)$ is the dimensionless magnetization (order parameter), B is the external magnetic field, T is the temperature, k is the Boltzmann constant, and J is a material-specific constant.

- (a) What is the *critical temperature* T_c below which the system exhibits spontaneous magnetization? (We refer to spontaneous magnetization when $m \neq 0$ at $B = 0$.)
- (b) Show that in the region just below T_c , the spontaneous magnetization behaves as

$$m(T, 0) \approx \text{const.} |T - T_c|^b,$$

and determine the value of the *critical exponent* b .

II-9 [6,4]

Consider a system composed of a very large number N of distinguishable atoms at rest and mutually noninteracting, each of which has only two (nondegenerate) energy levels: 0, $\varepsilon > 0$. Let E/N be the mean energy per atom in the limit $N \rightarrow \infty$.

- (a) What is the maximum possible value of E/N if the system is not necessarily in thermodynamic equilibrium? What is the maximum attainable value of E/N if the system is in equilibrium (at a positive temperature)?
- (b) For thermodynamic equilibrium compute the entropy per atom S/N as a function of E/N .

II-10 [10]

Consider a *three-dimensional* extreme-relativistic ($\varepsilon = cp$) free electron gas confined to a volume $V = L^3$. The number of electrons is N .

Obtain P_0 , the pressure of the system at $T = 0$. You must express your answer in terms of the electron (number) density N/V .

II = 1

Solutions

Part - II

solution:

(a) The particle is in a bound state because the wavefunction $\psi(x, t)$ satisfies
 $\lim_{x \rightarrow -\infty} \psi(x, t) = 0$, and $\lim_{x \rightarrow +\infty} \psi(x, t) = \lim_{x \rightarrow +\infty} \alpha x e^{-\beta x} e^{i\gamma t/\hbar} = 0$,

(b) $\hat{E}\Psi(x, t) = i\hbar \frac{\partial}{\partial t} \Psi(x, t) = i\hbar \left(\frac{i\gamma}{\hbar} \alpha x e^{-\beta x} e^{\frac{i\gamma t}{\hbar}} \right) = -\gamma \Psi(x, t)$

Hence, $\Psi(x, t)$ is an eigenfunction of the total energy with eigenvalue $-\gamma$, so

$$\rho(E) = \begin{cases} 1 & \text{for } E = -\gamma \\ 0 & \text{for } E \neq -\gamma \end{cases}$$

(c) $\left(-\frac{\hbar^2}{2m} \nabla^2 + V(x) \right) \Psi(x, t) = -\gamma \Psi(x, t)$

$$\begin{aligned} V(x) &= -\gamma + \frac{\hbar^2 \nabla^2 \Psi(x, t)}{2m \Psi(x, t)} = -\gamma + \frac{\frac{\hbar^2}{2m} \nabla (e^{-\beta x} (\alpha - \beta \alpha x))}{\alpha x e^{-\beta x}} \\ &= -\gamma + \frac{\frac{\hbar^2}{2m} (-\beta e^{-\beta x} (\alpha - \beta \alpha x) - \alpha \beta e^{-\beta x})}{\alpha x e^{-\beta x}} = -\gamma + \frac{\frac{\hbar^2}{2m} (-\beta \alpha (1 - \beta x) - \alpha \beta)}{\alpha x} \\ &= -\gamma + \frac{\hbar^2}{2m} \left(-\frac{\beta}{x} + \beta^2 - \frac{\beta}{x} \right) \rightarrow \boxed{V(x) = -\gamma + \frac{\hbar^2}{2m} \left(\beta^2 - \frac{2\beta}{x} \right)} \end{aligned}$$

(II-2)

No.

(a) Eigenvalues: $E_n = n^2 E_1$, $E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$ #

Eigen functions: $\phi_n = \sqrt{\frac{2}{L}} \sin(n\pi \frac{x}{L})$, $n=1, 2, 3, \dots$ #

(b) 1st order perturbation to $E_2^{(0)}$

$$E_2^{(1)} = \langle \phi_2 | H' | \phi_2 \rangle, H' = 10^{-3} E_1 \frac{x^2}{L^2} \#$$

$$E_2^{(1)} = \langle \phi_2 | 10^{-3} E_1 \frac{x^2}{L^2} | \phi_2 \rangle \#$$

$$E_2^{(1)} = (10^{-3} E_1 \frac{2}{L}) \int_0^L \sin^2 \frac{2\pi x}{L} \left(\frac{x}{L}\right)^2 dx$$
$$\left(\frac{2\pi x}{L}\right)^2 d\left(\frac{2\pi x}{L}\right) \left(\frac{1}{2\pi}\right)^2 \left(\frac{L}{2\pi}\right)$$

Let $Y \equiv \left(\frac{2\pi x}{L}\right)$, $Y: 0 \rightarrow 2\pi$

$$\therefore E_2^{(1)} = (10^{-3} E_1) \left(\frac{2}{(2\pi)^3}\right) \int_0^{2\pi} \sin^2 Y \cdot Y^2 \cdot dY \#$$

II-3

$$\begin{pmatrix} |+\rangle \equiv |S_{zi}+\rangle \\ |-\rangle \equiv |S_{zi}-\rangle \end{pmatrix}$$

$$|S_{ni}+\rangle = |\vec{S} \cdot \vec{n}_i+\rangle = \cos\left(\frac{\theta}{2}\right) |+\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\varphi} |-\rangle$$

$$\left(|S_{ni}-\rangle = |\vec{S} \cdot \vec{n}_i-\rangle = \sin\left(\frac{\theta}{2}\right) |+\rangle - \cos\left(\frac{\theta}{2}\right) e^{i\varphi} |-\rangle \right)$$

$$P_1 = |\langle S_{ni}+ | + \rangle|^2 = |\langle + | S_{ni}+ \rangle|^2 = \cos^2\left(\frac{\theta}{2}\right)$$

$$P_2 = |\langle - | S_{ni}+ \rangle|^2 = \sin^2\left(\frac{\theta}{2}\right)$$

$$I_{\text{final}} = 1 \cdot P_1 \cdot P_2 = \cos^2\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\theta}{2}\right) = \frac{1}{4} [2 \cos\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta}{2}\right)]$$

$$= \frac{1}{4} \sin^2(\theta)$$

[II-4] Solution

The scattering amplitude in the Born approximation is given by the equation

$$f_{\text{Born}}(\theta) = -\frac{\mu}{2\pi\hbar^2} \int e^{i(\vec{q} \cdot \vec{r})} V(\vec{r}) d\tau =$$

$$= -\frac{\pi\mu A}{\hbar^2 q} \quad \text{with}$$

$$\vec{q} = \vec{k}' - \vec{k}, \quad \text{and} \quad q = 2k \sin \frac{1}{2}\theta$$

Hence,

$$d\sigma_{\text{Born}} = |f(\theta)|^2 d\Omega = \frac{\pi^2 \mu^2 A^2}{2\hbar^4 E} \cot^2 \frac{1}{2}\theta d\theta.$$

In classical mechanics we have the following connection between the angle of scattering and the impact parameter b :

$$\int_{r_0}^{\infty} \frac{\mu v b dr}{r^2 [\sqrt{2\mu(E - V)} - (\mu v b/r)^2]} = \frac{\pi - \theta}{2},$$

where r_0 is the zero of the expression under the square root sign. Upon integrating we obtain

(note: v is the incident speed of the particle)

(II-4) cont.
$$f = \frac{A}{E} \frac{1}{\vartheta} \frac{(\pi - \vartheta)^2}{2\pi - \vartheta}$$

and thus

$$d\sigma = -2\pi f \frac{df}{d\vartheta} d\vartheta = \frac{2\pi^3 A}{E} \frac{\pi - \vartheta}{\vartheta^2 (2\pi - \vartheta)^2} d\vartheta.$$

If $\frac{8\mu A}{\hbar^2} \ll 1$

we can apply the Born approximation for all angles.

In the opposite limiting case, where

$\frac{8\mu A}{\hbar^2} \gg 1$ the classical result holds for not too small angles,

$$\vartheta \gtrsim \frac{\hbar^2}{8\mu A}$$

while for smaller angles,

$$\vartheta \lesssim \frac{\hbar^2}{8\mu A}$$

the Born approximation is valid.

II-5

solution:

Given a kinetic energy of $K = 13.6 \text{ eV}$,

$$v = \sqrt{2 * \frac{13.6 \text{ eV}}{.511 \text{ MeV}/c^2}} = 7.3 * 10^{-3} c$$

and the time spent in flight $t = \frac{10^4 [\text{km}]}{v} = \frac{10^7 [\text{m}]}{(7.3 * 10^{-3}) * 3 * 10^8 [\frac{\text{m}}{\text{s}}]} \approx 4.6 [\text{s}]$

For a non-relativistic velocity, we can approximate the spread of the wavepacket as a function of time,

$$\frac{w(t)}{w(0)} = \sqrt{1 + \frac{\beta^2 t^2}{2\alpha^2}} = \sqrt{1 + \frac{\hbar^2 t^2}{2m^2 \alpha^2}} = \sqrt{1 + \frac{2\hbar^2 t^2}{m^2 w^4(0)}}$$

yielding a width upon arrival of

$$w(t) = 10^{-3} [\text{m}] * \sqrt{1 + \frac{2(4.13 * 10^{-15} [\text{eV} * \text{s}])^2 (4.6 [\text{s}])^2}{10^{-12} [\text{m}^4] * \frac{.511^2 [\text{MeV}^2]}{c^4}}} \approx 7.5 * 10^{-1} [\text{m}]$$

II-6

Time-dependent perturbation

$$H = H_0 + V(t)$$

$$V(t) = -q E_0 x e^{-\gamma t} \cos(\omega t) \quad (t > 0)$$

(changed) harmonic oscillator: $H_0 = \frac{p^2}{2m} + \frac{m\omega_0^2}{2} x^2$

$$\left. \begin{aligned} a &= \sqrt{\frac{m\omega_0}{2\hbar}} \left(x + \frac{i}{m\omega_0} p \right) \\ a^+ &= \sqrt{\frac{m\omega_0}{2\hbar}} \left(x - \frac{i}{m\omega_0} p \right) \end{aligned} \right\} \Rightarrow x = \sqrt{\frac{\hbar}{2m\omega_0}} (a + a^+)$$

$$H_0 |n\rangle = \hbar\omega_0 (n + \frac{1}{2}) |n\rangle$$

$$n = 0, 1, 2, \dots$$

$$E_n = \hbar\omega_0 (n + \frac{1}{2})$$

$$\omega_{ek} \equiv \frac{1}{\hbar} (E_e - E_k) = \omega_0 (e - k)$$

$t = 0$: oscillator is in G.S. $|0\rangle$

$$\text{Note: } a^+ |0\rangle = |1\rangle$$

$$a |0\rangle = 0$$

Transition probabilities for $t = \infty$:

$$\begin{aligned} W(0 \rightarrow e) &= \frac{1}{\hbar^2} \left| \int_0^\infty \langle e | V(t) | 0 \rangle e^{i\omega_{e0} t} dt \right|^2 \\ &= \frac{q^2 E_0^2}{\hbar^2} |\langle e | x | 0 \rangle|^2 \left| \int_0^\infty e^{i\omega_{e0} t} e^{-\gamma t} \cos(\omega t) dt \right|^2 \\ &= \frac{q^2 E_0^2}{\hbar^2} \frac{\hbar}{2m\omega_0} |\langle e | 1 \rangle|^2 \left| \int_0^\infty e^{i\omega_{e0} t} e^{-\gamma t} \frac{e^{i\omega t} + e^{-i\omega t}}{2i} dt \right|^2 \\ &= \delta_{e1} \frac{q^2 E_0^2}{2\hbar m\omega_0} \left| \int_0^\infty \left\{ e^{[i(\omega_0 + \omega) - \gamma]t} + e^{[i(\omega_0 - \omega) - \gamma]t} \right\} dt \right|^2 = \end{aligned}$$

1) Kronecker-delta

II-6
(Cont'd)

$$= \delta_{el} \frac{q^2 E_0^2}{8 \hbar m \omega_0} \left| \frac{1}{\gamma - i(\omega_0 + \omega)} + \frac{1}{\gamma - i(\omega_0 - \omega)} \right|^2$$

$$= \delta_{el} \frac{q^2 E_0^2}{8 \hbar m \omega_0} \left| \frac{\gamma - i(\omega_0 - \omega) + \gamma - i(\omega_0 + \omega)}{[\gamma - i(\omega_0 + \omega)][\gamma - i(\omega_0 - \omega)]} \right|^2 =$$

$$= \delta_{el} \frac{q^2 E_0^2}{8 \hbar m \omega_0} \left| \frac{2(\gamma - i\omega_0)}{\gamma^2 - i\gamma(\omega_0 - \omega) - i\gamma(\omega_0 + \omega) - (\omega_0 - \omega)(\omega_0 + \omega)} \right|^2$$

$$= \delta_{el} \frac{q^2 E_0^2}{8 \hbar m \omega_0} \left| \frac{2(\gamma - i\omega_0)}{\gamma^2 - 2i\gamma\omega_0 + (\omega^2 - \omega_0^2)} \right|^2$$

$$= \delta_{el} \frac{q^2 E_0^2}{8 \hbar m \omega_0} 4 \frac{\gamma^2 + \omega_0^2}{[\gamma^2 + \omega^2 - \omega_0^2]^2 + 4\gamma^2\omega_0^2}$$

for $\gamma < \omega_0$, $W(0 \rightarrow 1)$ is maximum
if $\omega = \sqrt{\omega_0^2 - \gamma^2}$

transition is only possible to the first excited state $11\rangle$, and the prob. is maximum if

$$\omega = \sqrt{\omega_0^2 - \gamma^2}$$

II-7

$$F(T, V) = -\frac{a}{3} T^4 V$$

(a) $F = U - TS$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = \frac{4}{3} a T^3 V \quad \text{entropy}$$

$$U = F + TS = -\frac{a}{3} T^4 V + \frac{4}{3} a T^4 V = a T^4 V \quad \text{internal energy}$$

Joule "free-expansion" $U(T, V) = \text{const.}$

i.e. $U(T_1, V) = U(T_2, V_2)$
initial final

$$V_2 = 81V$$

$$a T^4 V = a T_2^4 V_2$$

$$T^4 V = T_2^4 81V$$

$$\frac{T^4}{81} = \frac{T_2^4}{1} \Rightarrow$$

$$T_2 = \frac{T}{3}$$

(b) $\Delta S = S_2 - S = S(T_2, V_2) - S(T, V)$
final initial

$$= \frac{4}{3} a T_2^3 V_2 - \frac{4}{3} a T^3 V = \frac{4}{3} a \left(\frac{1}{3}\right)^3 (81V) - \frac{4}{3} a T^3 V$$

$$= \frac{4}{3} a \{3 - 1\} T^3 V = \frac{4}{3} a 2 T^3 V = \frac{8}{3} a T^3 V$$

8

$$m = \tanh\left(\frac{Jm+B}{kT}\right)$$

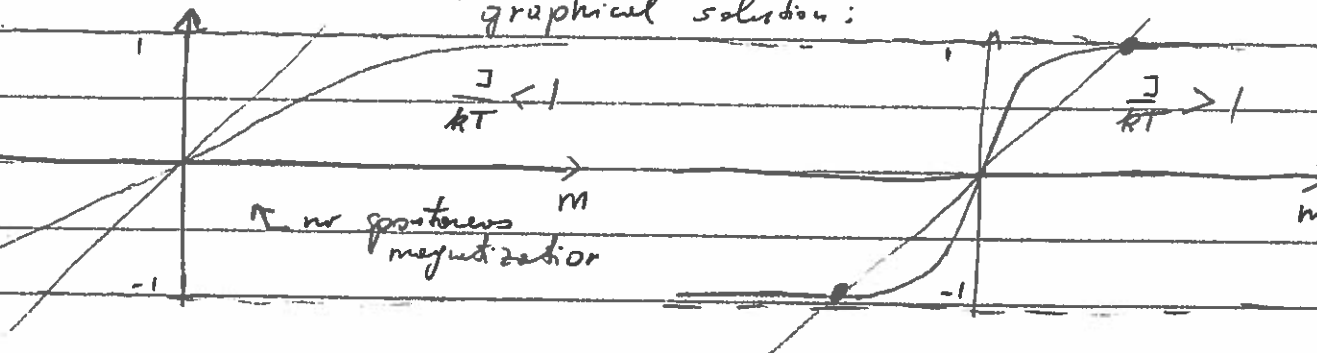
spontaneous magnetization: $m \neq 0$ when $B=0$

a)

$$m = \tanh\left(\frac{J}{kT}m\right)$$

$$\tanh(x) = x - \frac{1}{3}x^3 \pm$$

"graphical" solution:



For $\frac{J}{kT} > 1$ ($T < \frac{J}{k}$) there are 2 non-trivial solutions corresponding to spontaneous magnetization. Thus, the critical temperature: $\frac{J}{kT_c} = 1$

$$T_c = \frac{J}{k}$$

b) for $T \leq T_c$: ($B=0$)

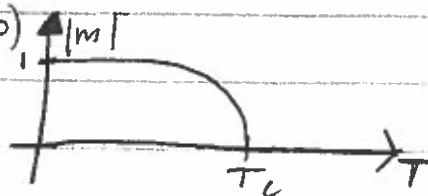
$$m \simeq \frac{J}{kT}m - \frac{1}{3}\left(\frac{J}{kT}\right)^3 m^3 \quad (m \neq 0)$$

$$1 = \frac{T_c}{T} - \frac{1}{3}\left(\frac{T_c}{T}\right)^3 m^2$$

$$m^2 = 3\frac{T}{T_c^3}\left(\frac{T_c}{T} - 1\right) = 3\frac{T}{T_c^3}\frac{T_c - T}{T} \simeq 3\frac{T_c - T}{T_c} \quad \text{for } \frac{T_c - T}{T_c} \ll 1$$

$$m = \pm \sqrt{3\frac{T_c - T}{T_c}} \sim \pm (T_c - T)^{1/2} \quad (B=0)$$

$$\beta = 1/2$$



[II-9] Solution:

(a) There is nothing to prevent giving each atom its larger energy ϵ ; hence $\eta = \frac{E}{N\epsilon}$ has a maximum of 1 with

$\frac{E}{N} = \epsilon$. Clearly, the system would not be in thermal

equilibrium. Now, to compute the problem in equilibrium, we need to determine the partition function, Z . For distinguishable non-interacting particles, the partition function factors, so for identical energy spectra

$$Z = Z_1^N = (1 + e^{-\epsilon/\tau})^N \quad (1)$$

The free energy would be

$$F = -\tau \ln Z = -N\tau \ln Z_1 = -N\tau \ln(1 + e^{-\epsilon/\tau}) \quad (2)$$

The energy becomes

$$\begin{aligned} E &= F + \tau S = F - \tau \frac{\partial F}{\partial \tau} = \\ &= -N\tau \ln Z_1 + N\tau^2 \frac{\partial}{\partial \tau} \ln Z_1 + N\tau \ln Z_1 \quad (3) \\ &= N\tau^2 \frac{\partial}{\partial \tau} \ln Z_1 \end{aligned}$$

2.

[II-9] continued.

$$\text{or: } \frac{E}{N} = \tau^2 \frac{\partial}{\partial \tau} \ln Z_1 = \tau^2 \frac{\partial}{\partial \tau} \ln(1 + e^{-\epsilon/\tau}) \quad (4.)$$

$$= \frac{\epsilon e^{-\epsilon/\tau}}{1 + e^{-\epsilon/\tau}} = \epsilon \frac{x}{1+x} = \epsilon f(x)$$

where: $x = e^{-\epsilon/\tau}$. Now $\epsilon > 0, \tau > 0$

thus, x cannot be greater than one. On the other hand $\left(\frac{x}{1+x}\right)$ is a monotonic

function which goes to $\frac{1}{2}$ when x goes to ∞ ; hence, $\max\left\{\frac{E}{N}\right\} = f(1) = \frac{1}{2}$ at $\tau \rightarrow \infty$.

(b.) The entropy may be found from eqs, (2-4).

$$S = \frac{E - F}{\tau} = \frac{E}{\tau} - \frac{F}{\tau} = \frac{N}{\tau} \frac{\epsilon x}{1+x} + N \ln(1+x) \quad (5.)$$

now the entropy per particle: $s = S/N$ is given by

$$s = \frac{\epsilon}{\tau} \frac{x}{1+x} + \ln(1+x) \quad (6.)$$

letting: $\eta = \frac{x}{1+x}$, $x = \frac{\eta}{1-\eta}$, $1+x = \frac{1}{1-\eta}$

$$s = \frac{\epsilon}{\tau} \eta - \ln(1-\eta) \quad (7.)$$

[II-9] continued.

$$\begin{aligned} S &= \eta [\ln(1-\eta) - \ln \eta] - \ln(1-\eta) = \\ &= -[\eta \ln \eta + (1-\eta) \ln(1-\eta)] \end{aligned}$$

we can check that

$$S \rightarrow \begin{cases} 0 & \text{as } T \rightarrow 0, \eta \rightarrow 0 \\ \ln 2 & \text{as } T \rightarrow \infty, \eta \rightarrow \frac{1}{2} \end{cases} \quad (8)$$

as it should.

II-10

$$\varepsilon(p) = cp$$

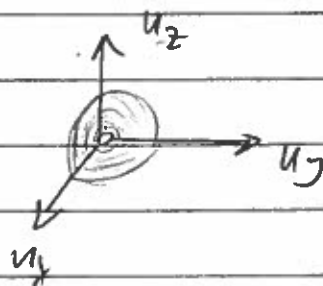
$$V = L^3$$

$$(s = 1/2)$$

$$\varepsilon = cp = c\hbar k = c\hbar \frac{\pi}{L} (n_x^2 + n_y^2 + n_z^2)^{1/2}$$

$$\left(\begin{aligned} k &= |\vec{k}| = (k_x^2 + k_y^2 + k_z^2)^{1/2} \\ k_x &= \frac{\pi}{L} n_x \quad n_x = 1, 2, \dots \\ &\text{etc.} \end{aligned} \right)$$

$$\varepsilon = \frac{c\hbar}{2L} \underbrace{(n_x^2 + n_y^2 + n_z^2)^{1/2}}_{\equiv n} = \frac{c\hbar}{2L} n$$



$$n_x, n_y, n_z > 0$$

$$N(\varepsilon) = \frac{1}{8} \frac{4\pi}{3} n^3$$

$$= \frac{1}{8} \frac{4\pi}{3} \left(\frac{2L}{c\hbar} \right)^3 \varepsilon^3 = \frac{1}{8} \frac{4\pi}{3} \frac{8L^3}{c^3 \hbar^3} \varepsilon^3 = \frac{4\pi}{3} \frac{V}{c^3 \hbar^3} \varepsilon^3$$

$$g(\varepsilon) = \underbrace{(2s+1)}_{\substack{\uparrow \\ \text{spin}}} \frac{\partial N}{\partial \varepsilon} = 2 \cdot \frac{4\pi}{3} \frac{V}{c^3 \hbar^3} 3 \varepsilon^2 = \frac{8\pi V}{c^3 \hbar^3} \varepsilon^2$$

$T=0$

$$N = \int_0^{\varepsilon_F} g(\varepsilon) d\varepsilon = \frac{8\pi V}{c^3 \hbar^3} \frac{\varepsilon_F^3}{3} = \frac{8\pi V}{3c^3 \hbar^3} \varepsilon_F^3$$

$$U_0 = \int_0^{\varepsilon_F} \varepsilon g(\varepsilon) d\varepsilon = \frac{8\pi V}{c^3 \hbar^3} \frac{\varepsilon_F^4}{4} = \frac{2\pi V}{c^3 \hbar^3} \varepsilon_F^4$$

eliminate ϵ_F : $\epsilon_F = \left(\frac{3c^3 h^3}{8\pi} \right)^{1/3} \left(\frac{N}{V} \right)^{1/3}$

$$\epsilon_F = \left(\frac{3}{8\pi} \right)^{1/3} c h \left(\frac{N}{V} \right)^{1/3}$$

$$U_0 = \frac{2\pi V}{c^3 h^3} \epsilon_F^4 = \frac{2\pi V}{c^3 h^3} \left(\frac{3}{8\pi} \right)^{4/3} c^4 h^4 \left(\frac{N}{V} \right)^{4/3}$$

$$= 2\pi V \left(\frac{3}{8\pi} \right)^{4/3} c h \left(\frac{N}{V} \right)^{4/3} = 2\pi h c \left(\frac{3}{8\pi} \right)^{4/3} \frac{N^{4/3}}{V^{1/3}}$$

- Recall: $PV = \frac{U}{3}$ for 3 dim extreme rel gas for any temperature

Thus, $P_0 = \frac{U_0}{3V}$

$$P_0 = \frac{U_0}{3V} = \frac{2\pi h c}{3} \left(\frac{3}{8\pi} \right)^{4/3} \left(\frac{N}{V} \right)^{4/3} = \frac{h c}{4} \left(\frac{3}{8\pi} \right)^{1/3} \left(\frac{N}{V} \right)^{4/3}$$

- or, Alternatively:

$$P = - \left(\frac{\partial F}{\partial V} \right)_{T,N} \quad \text{where } F = U - TS$$

Thus, at $T=0$

$$P_0 = - \left(\frac{\partial F_0}{\partial V} \right)_N = - \left(\frac{\partial U_0}{\partial V} \right)_N = \frac{2\pi}{3} h c \left(\frac{3}{8\pi} \right)^{1/3} \left(\frac{N}{V} \right)^{4/3} = \frac{h c}{4} \left(\frac{3}{8\pi} \right)^{1/3} \left(\frac{N}{V} \right)^{4/3}$$