i

6)

Old Problems

test 1

1.1.a) Write the explicit form of $|+\rangle_n$ using rotation matrices with $\hat{n}$ a unit vector of angles β (polar) and γ (azimuthal)

page 13

$|+\rangle_n$ is just a rotated vector, analogous to euler angles

$$|+\rangle_n = D_z(\gamma)\, D_y(\beta)\, |+\rangle_z$$

$$= \left(\mathbb{1}\cos\left(\tfrac{\gamma}{2}\right) - i\sin\left(\tfrac{\gamma}{2}\right)\sigma_z\right)\left(\mathbb{1}\cos\left(\tfrac{\beta}{2}\right) - i\sin\left(\tfrac{\beta}{2}\right)\sigma_y\right)|+\rangle$$

$$= \begin{pmatrix} c-is & 0 \\ 0 & c+is \end{pmatrix}_{\frac{\gamma}{2}} \begin{pmatrix} c & -s \\ s & c \end{pmatrix}_{\frac{\beta}{2}} |+\rangle_z \qquad |+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_z$$

$$= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} c \\ s \end{pmatrix}_{\frac{\beta}{2}}$$

$$= \boxed{\begin{pmatrix} e^{-i\frac{\gamma}{2}} \cos\left(\frac{\beta}{2}\right) \\ e^{i\frac{\gamma}{2}} \sin\left(\frac{\beta}{2}\right) \end{pmatrix} = |+\rangle_n} \qquad \text{vector}$$

HW 3.6.b) & 3.2.b)

b) Work out $\vec{S}\cdot\hat{n}\left[-i\sigma_2 |+\rangle_n^*\right]$

$$\vec{S}\cdot\hat{n} = \frac{\hbar}{2}\vec{\sigma}\cdot\hat{n} = \frac{\hbar}{2}\begin{pmatrix} \cos\beta & \sin\beta\, e^{-i\gamma} \\ \sin\beta\, e^{+i\gamma} & -\cos\beta \end{pmatrix}$$

and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

and $|+\rangle_n^* = \begin{pmatrix} e^{i\frac{\gamma}{2}} \cos\left(\frac{\beta}{2}\right) \\ e^{-i\frac{\gamma}{2}} \sin\left(\frac{\beta}{2}\right) \end{pmatrix}$

1)

1.1.a) ctd. So $\vec{S}\cdot\hat{n}\left[-i\sigma_2 |+\rangle_n^*\right]$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\gamma} \\ \sin\beta e^{i\gamma} & -\cos\beta\end{pmatrix}\cdot\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}e^{i\frac{\gamma}{2}}\cos(\frac{\beta}{2}) \\ e^{-i\frac{\gamma}{2}}\sin(\frac{\beta}{2})\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\beta & \sin\beta e^{-i\gamma} \\ \sin\beta e^{i\gamma} & -\cos\beta\end{pmatrix}\begin{pmatrix}-e^{-i\frac{\gamma}{2}}\sin(\frac{\beta}{2}) \\ e^{i\frac{\gamma}{2}}\cos(\frac{\beta}{2})\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}-\cos\beta\sin\frac{\beta}{2}e^{-i\frac{\gamma}{2}} + \sin\beta\cos\frac{\beta}{2}e^{-i\gamma/2} \\ -\sin\beta\sin\frac{\beta}{2}e^{+i\frac{\gamma}{2}} - \cos\beta\cos\frac{\beta}{2}e^{i\gamma/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}e^{-i\frac{\gamma}{2}}\left(-\sin\frac{\beta}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)+\cos\frac{\beta}{2}\left(2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right)\right) \\ -e^{i\frac{\gamma}{2}}\left(\sin\frac{\beta}{2}\left(2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\right)+\cos\frac{\beta}{2}\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)\right)\end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix}e^{-i\frac{\gamma}{2}}\sin\frac{\beta}{2} \\ -e^{i\frac{\gamma}{2}}\cos\frac{\beta}{2}\end{pmatrix} = \frac{\hbar}{2}\cdot|-\rangle_n$$

So

c) What does $-i\sigma_2\vec{K}$ stand for and why?

it is the parity operator on $|\pm\rangle_n$ because it flips about the K axis which is our rotation angle.

81

?

1.2 Consider a particle in a finite well

$$V(x) = -V_0\,\Theta(a - |\vec{x}|)$$ repeat of 2.4

1.3. Let T_{ij} be a 2-rank tensor. Show that

a) $\sum_i T_{ii}$ is a scalar

What is $T'_{ii} = R_{ij} R_{ik} T_{jk} = \delta_{jk} T_{jk} = T_{jj}$

so $T'_{ii} = T_{jj} = T_{ii}$ so $T' = T$ in all possible frames and must be a scalar.

b) $\sum_{jk} \varepsilon_{ijk} T_{jk}$ is a vector

What is $X_i = \sum_{jk} \varepsilon_{ijk} T_{jk}$

$$\underbrace{A\, a_j b_k = T_{jk}}_{\text{tensor decomposition}} \quad \therefore \quad X_i = A \sum_{jk} \varepsilon_{ijk} a_j b_k = \underbrace{A(\vec{a}\times\vec{b})_i}_{\text{Certainly a vector!}}$$

1.4 We recall that $|j_1 j_2 j m\rangle$ refers to the recoupling basis and $|j_1 m_1 j_2 m_2\rangle$ to the free basis of two particles of angular momentum

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

Calculate the overlap

$$|\langle j, j, j_{tot} = 2j-1, m = 2j-1 | j, m_1 = j, j, m_2 = j-1\rangle|$$

We must find the composition of $|j_{tot}, m, j_1, j_2\rangle$

$$|j_{tot}, m, j_1, j_2\rangle = |2j-1, 2j-1, j, j\rangle$$

Using

A $$J^2 = J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}$$

B $$\& \quad m = m_1 + m_2 \qquad \& \quad |m_i| \le j_i$$

C $$\& \quad |j_1 - j_2| \le j \le |j_1 + j_2|$$

Wherein

$$J^2|j, m\rangle = j(j+1)|j, m\rangle$$

$$J_z|j, m\rangle = m|j, m\rangle$$

$$J_\pm|j, m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

5
1101

1.9. (cont.) So we start by defining an ensemble of $|\text{left}\rangle = |j, m, j_1, j_2\rangle$ in terms of $|j_1, m_1, j_2, m_2\rangle$ which satisfy the selection rules.

- $$|j_{tot} = 2j-1, m = 2j-1, j, j\rangle = a \cdot |\overset{m_1}{j}, j-1, \overset{m_2}{j}, j\rangle + b \cdot |j, j, j, j-1\rangle$$

where $m = 2j-1 = m_1 + m_2$ where $|m_1| + |m_2| \leq j$ is very restrictive. & $a^2+b^2=1$.

explicitly writing out the tensor decomposition.

$$|\text{left}\rangle = a\,|\overset{1}{j}, j-1\rangle \otimes |\overset{2}{j}, j\rangle + b\,|\overset{1}{j}, j\rangle \otimes |\overset{2}{j}, j-1\rangle$$

- Now we can act J^2 on both sides to find.

$$(2j-1)(2j-2)\,|\text{left}\rangle = \left(J_1^2 + J_2^2 + 2J_{1z}J_{2z} + J_{1+}J_{2-} + J_{1-}J_{2+}\right) \cdot \left(a|j, j-1\rangle|j, j\rangle + b|j, j\rangle|j, j-1\rangle\right)$$

$$= 2j(j-1)\,|\text{right}\rangle + 2 \cdot j \cdot (j-1)\,|\text{right}\rangle$$

$$+ a\left[\underbrace{\sqrt{j(j+1) - (j-1)(j)}}_{j^2-j^2+j+j\;=2j}\sqrt{j(j+1) - (j)(j-1)}\,|j, j\rangle|j, j-1\rangle\right]$$

$$+ b \cdot 0 + a \cdot 0$$

$$+ b \cdot \left[\sqrt{j(j+1) - j(j-1)}\sqrt{j(j+1) - (j-1)j}\,|j, j-1\rangle|j, j\rangle\right]$$

$$= 4j(j-1)\,|\text{right}\rangle + 4j^2 \cdot \left(a|j, j\rangle|j, j-1\rangle + b|j, j-1\rangle|j, j\rangle\right)$$

acting $\langle \text{left}|J^2|\text{left}\rangle$ and using orthonormality we get

6.
(II)

1.4. (td.) $(2\cancel{j})^2 - 3\cdot 2j + 2 = 4\cancel{j^2} - 4j + 4j^2 \cdot 2ab$

$$\frac{2-2j}{2\cdot 4j^2} = a\cdot b \qquad a\cdot b = \frac{1-j}{4j^2} \quad \to b = \frac{1-j}{4j^2 a}$$

$$a^2+b^2=1, \quad \therefore a = \pm\sqrt{1-b^2}$$

$$a^2 = 1 - \left(\frac{1-j}{4j^2 a}\right)^2$$

$$a^4 - a^2 + \frac{1-2j+j^2}{16j^4} = 0$$

$$a^2 = \frac{-(-1) \pm \sqrt{1 - \frac{1-2j+j^2}{4j^4}}}{2}$$

So $$\boxed{a = \pm\frac{1}{\sqrt{2}}\cdot\sqrt{1 \pm \sqrt{1 - \frac{1-2j+j^2}{4j^4}}} \quad + \quad b = \frac{1-j}{4j^2 a}}$$

overall sign doesn't matter!

So our overlap is exactly equal to $|b|$ $\boxed{\text{QED}}$

$b\pm = a\mp$ option!

1

(2)

test 2

2.1. a) What is the most general decomposition of a tensor of rank 2 in terms of irreducible tensors?

$$\text{Cart: } T_{ij} = \underbrace{\tfrac{1}{2}\left(T_{ij} + T_{ji} - \tfrac{2}{3}\delta_{ij}T\right)}_{\text{sym traceless}} + \underbrace{\tfrac{1}{2}\left(T_{ij} - T_{ji}\right)}_{\text{anti sym}} + \underbrace{\tfrac{1}{3}\delta_{ij}T}_{\text{trace}}$$

$$\text{Sph: } T_q^{k} = \underbrace{\sum_{1,2}}_{} \underbrace{\langle k_1 q_1 k_2 q_2 | k k_1 k_2 q\rangle}_{\text{Clebsh / Wigner Eckart Coefficient}} \underbrace{T_{q_1}^{k_1} T_{q_2}^{k_2}}_{\text{irriducible tensors}} : \; -k_i \le q_i \le k_i$$

$$\downarrow \; \ell \quad \downarrow \; m \quad \downarrow \; \ell$$

where $k = k_1 \otimes k_2 = |k_1 - k_2| \oplus \cdots \oplus |k_1 + k_2|$ tensor ranks

$+ \; q = q_1 + q_2$

b) Use the result in a) to evaluate $T_{ij} = P_i P_j$ in the ground state of the hydrogen atom.

$$\langle 0 | p_i p_j | 0\rangle = \left[\overset{?}{\langle 0 | A T_{ij} | 0\rangle} = \langle 0 | B T_q^{k} \delta_{kj} \delta^{q}_{i} | 0\rangle\right.$$

$$= \langle 0 | B T_q^{k} | 0\rangle \delta_{kj} \delta^{i}_{q} = \left. B' \delta_{ij} \cdot Y_0^{0*} Y_0^{0}\right]^{?} = C\delta_{ij}$$

$$\text{or } \delta^{ij}\langle 0 | p_i p_j | 0\rangle = \langle 0 | \hat{p}^2 | 0\rangle = 3\cdot C$$

$$= \int d^3x\, d^3y \,\langle 0 | x\rangle \langle x | \hat{p}^2 | y\rangle \langle y | 0\rangle$$

$$= \int d^3\vec{x}\, d^3\vec{y}\, \frac{1}{\sqrt{\pi a_0^3}} e^{-x/a_0} \cdot -\hbar^2 \delta^3(\vec{x} - \vec{y}) \cdot \vec{\nabla}^2\left(\frac{1}{\sqrt{\pi a_0^3}} e^{-y/a_0}\right)$$

$$\vec{\nabla}_y^2 e^{-y/a_0} = \frac{1}{y^2}\frac{\partial}{\partial y}\left(y^2 \frac{\partial}{\partial y} e^{-y/a_0}\right) = \frac{1}{y^2}\frac{\partial}{\partial y}\left(-\frac{y^2}{a_0} e^{-y/a_0}\right) = +\frac{1}{a_0^2} e^{-y/a_0} - \frac{2}{y a_0} e^{-y/a_0}$$

$$= \frac{-\hbar^2}{\pi a_0^3}\int d^3\vec{x}\, d^3\vec{y}\, e^{-x/a_0} \cdot \left(\frac{1}{a_0^2} - \frac{2}{y a_0}\right) e^{-y/a_0}$$

$$\int_0^\infty x^n e^{-x/a} dx = n!\, a^{n+1}$$

$$= \frac{-\hbar^2}{\pi a_0^5}\int d^3\vec{x}\, e^{-2x/a_0} \cdot \left(1 - \frac{2a_0}{x}\right) = \frac{4\pi \cdot -\hbar^2}{\pi a_0^5}\int dx\left(x^2 - 2a_0 x\right) e^{-2x/a_0}$$

$$= \frac{-4\hbar^2}{a_0^5}\cdot\left(2! \cdot \left(\frac{a_0}{2}\right)^3 - 2a_0 \cdot 1 \cdot \left(\frac{a_0}{2}\right)^2\right) = 3C$$

$\therefore \; C = \frac{\hbar^2}{a_0^2}\cdot\frac{1}{3}$ [illegible] $+\hbar^2$ [illegible] QED

2
13)

2.2 The Hamiltonian in tigh-binding approximation reads

page 42.

$$H|n\rangle = E_0|n\rangle - \Delta|n+1\rangle - \Delta|n-1\rangle$$

for a chain of length Na, i.e. $|N+1\rangle = |1\rangle$

a) Identify the discrete symmetry of the chain T.

$\hat{T}_a = e^{\frac{-iap}{\hbar}}$ where $\boxed{\hat{T}_a|n\rangle = |n+1\rangle = \lambda_a|n\rangle}$ a discrete translation by a

since $T_a^N|1\rangle = 1|1\rangle$

$1 = (\lambda_a)^{2N}$ $\lambda_a = e^{i\theta_a} = e^{\frac{-ika}{\hbar}}$ | discrete translational symmetries necessarily yield bloch state-solutions.

b) Use this symmetry to find the spectrum of H.

$T_a|n\rangle = \lambda_a|n\rangle$, $|n\rangle = \sum_i C_i|i\rangle$ tight binding leads

$T_a|n\rangle = \lambda_a|n\rangle = T_a\sum_i C_i|i\rangle = \sum_i C_i|i+1\rangle$

$= \lambda_a\sum_i C_i|i\rangle = \sum_i C_{i-1}|i\rangle$

$$\therefore \boxed{\lambda_a = \frac{C_{i-1}}{C_i}}$$

since $|\lambda_a|^2 = 1$ then $\frac{\lambda_a}{\lambda_a} = \lambda_a\lambda_a^*$ $\therefore \lambda_a^* = \frac{1}{\lambda_a}$

so $C_i = \lambda_a^* C_{i-1} = (\lambda_a^*)^i$

sequence
(set $C_0 = 1$)

3.
(14)

2.2.b (td.) then with $C_i = (\lambda_a^*)^i$ we get

$$|n\rangle = \sum_i^N (\lambda_a^*)^i |i\rangle$$

$$H|n\rangle = E_n|n\rangle = E_0|n\rangle - \Delta|n+1\rangle - \Delta|n-1\rangle$$

$$\therefore H = \sum_i^N \left[E_0 |i\rangle\langle i| - \Delta\left(|i\rangle\langle i+1| + |i\rangle\langle i-1| \right)\right]$$

$$H|n\rangle = H\sum_i^N (\lambda_a^*)^i |i\rangle = \sum_i \left\{ E_0 (\lambda_a^*)^i |i\rangle - \Delta\left((\lambda_a^*)^{i+1} + (\lambda_a^*)^{i-1} \right) |i\rangle \right\}$$

$$\therefore E_n = E_0 - \Delta\left(\lambda_a^{*1} + \lambda_a^{*-1}\right)$$

$$\lambda_a = e^{-\frac{ipa}{\hbar}}$$

$$\therefore \boxed{E_n = E_0 - 2\Delta \cos\left(\frac{pa}{\hbar}\right)}$$

QED

4
15)

2.3 a) Show that the matrices

HW 1.1 $(S_i)_{jk} = -i\hbar\, \varepsilon_{ijk}$ matrix row column $i,j,k \in 1,2,3$ or x,y,z

Obey the commutation relations of angular momentum.

$$S_1 = S_x = -i\hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \rightarrow \varepsilon_{ijk} \qquad \varepsilon_{123} = +1$$

row column

$$S_2 = S_y = -i\hbar \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$S_3 = S_z = -i\hbar \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can try to evaluate the commutation relation directly or we can use ε_{ijk} notations.

$\rightarrow [S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$ is our goal to match J comm. rel.

$[S_i, S_j] = S_i S_j - S_j S_i$ $\quad + (S_i S_j)_{mn} = (S_i)_{ml}(S_j)_{ln}$

$+ (S_j S_i)_{mn} = (S_j)_{ml}(S_i)_{ln}$

So $[S_i, S_j]_{mn} = (S_i)_{ml}(S_j)_{ln} - (S_j)_{ml}(S_i)_{ln}$

$= -\hbar^2 (\varepsilon_{iml}\varepsilon_{jln} - \varepsilon_{jml}\varepsilon_{iln})$

$= -\hbar^2 (\varepsilon_{lim}\varepsilon_{lnj} - \varepsilon_{ljm}\varepsilon_{lni})$

5

(16)

2.3 a) Ctd. $[S_i, S_j]_{mn} = -\hbar^2\left(\delta_{in}\delta_{mj} - \cancel{\delta_{ij}\delta_{mn}} - \left(\delta_{jn}\delta_{mi} - \cancel{\delta_{ji}\delta_{mn}}\right)\right)$

$$= -\hbar^2\left(\delta_{in}\delta_{mj} - \delta_{jn}\delta_{mi}\right)$$

$$= \hbar^2\left(\delta_{jn}\delta_{im} - \delta_{in}\delta_{jm}\right)$$

$$= \hbar^2 \cdot \varepsilon_{kji}\,\varepsilon_{knm}$$ double negative

$$= \hbar^2\,\varepsilon_{ijk}\,\varepsilon_{mnk}$$

$$= i\hbar\,\varepsilon_{ijk}\left(-i\hbar\,\varepsilon_{kmn}\right)$$

$\therefore \boxed{[S_i, S_j] = i\hbar\,\varepsilon_{ijk}\,S_k}$ $\boxed{\text{QED}}$

b) Give an interpretation of $\vec{S}$.

$\vec{S}$ is therefore able to represent the spin of a spin 1 particle (since the dimension of the matrix should be $D = 2s+1$ and $D = 3$, $\therefore s = 1$ spin 1) $\boxed{\text{QED}}$

6
(17)

2.4. Consider a particle in the finite well

$$V(\vec{x}) = -V_0\, \Theta(a - |\vec{x}|) \quad \text{(radial step well)}$$

a) Write down a transcendental equation for the $l=0$ state.

(Page 54) 3D Schrodinger equation (TISE)

(TISE) $$H\Psi(\vec{x}) = \left[\frac{-\hbar^2}{2m}\left(\partial_r^{\,2} + \frac{2}{r}\partial_r\right) + \frac{L^2}{2mr^2} + V(r)\right]\Psi(\vec{x}) = E\Psi$$

Our potential is spherically symmetric, so $\Psi(\vec{x}) \propto Y_l^m R_n(r)$ and we set $l=0$ so (TISE) becomes

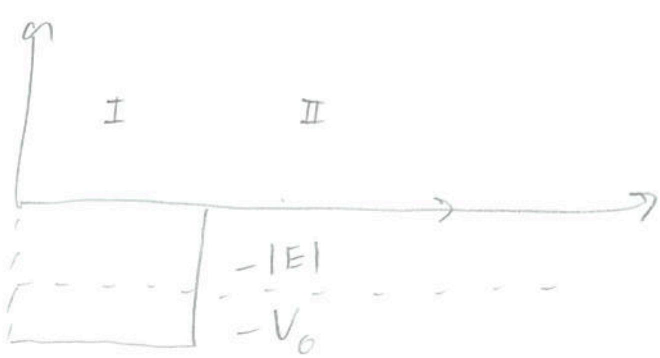
$$H R_n(r) = \frac{-\hbar^2}{2m}\left(\partial_r^{\,2} + \frac{2}{r}\partial_r\right) - V_0\,\Theta(a - |\vec{x}|)\Big) R_n(r) = E R_n(r)$$

switching $R(r) = \frac{U(r)}{r}$ we get

$$-\frac{\hbar^2}{2m}\frac{U''(r)}{r} - \frac{V_0}{r}\Theta(a-r)\,U(r) = \frac{E\,U(r)}{r}$$

or $$U'' = \frac{2m}{\hbar^2}\Big(|E| - V_0\,\Theta(a-r)\Big)\,U(r)$$

1D problem



7
(8)

2.4.(a) (cl'd.) In region I we have

$$u'' - \frac{2m(|E|-V_0)}{\hbar^2} u = 0 \qquad K_1^2 = +\frac{2m(V_0-|E|)}{\hbar^2}$$

$$u'' + K_1^2 u = 0 \qquad \therefore \quad u_I = A\sin(K_1 r) + B\cos(K_1 r)$$

$$u_I(0) = 0 \quad \therefore B = 0$$

In region II we have

$$u'' - \frac{2m|E|}{\hbar^2} u = 0$$

$u(\infty) = 0$

$$u'' - k^2 u = 0 \qquad u_{II} = Ce^{-kr} + \cancel{D}e^{+kr}$$

then $u_I(a) = u_{II}(a)$ so $A\sin(K_1 a) = Ce^{-ka}$

$u_I''(a) = u_{II}'(a)$ so $K_1 A\cos(K_1 a) = -kCe^{-ka}$

$$\frac{1}{K_1}\tan(K_1 a) = -\frac{1}{k}$$

$$\boxed{k = -K_1 \operatorname{Cotan}(K_1 a) \quad \text{with } k = \sqrt{\frac{2m|E|}{\hbar^2}} \;\&\; K_1 = \sqrt{\frac{2m(V_0-|E|)}{\hbar^2}}}$$

QED

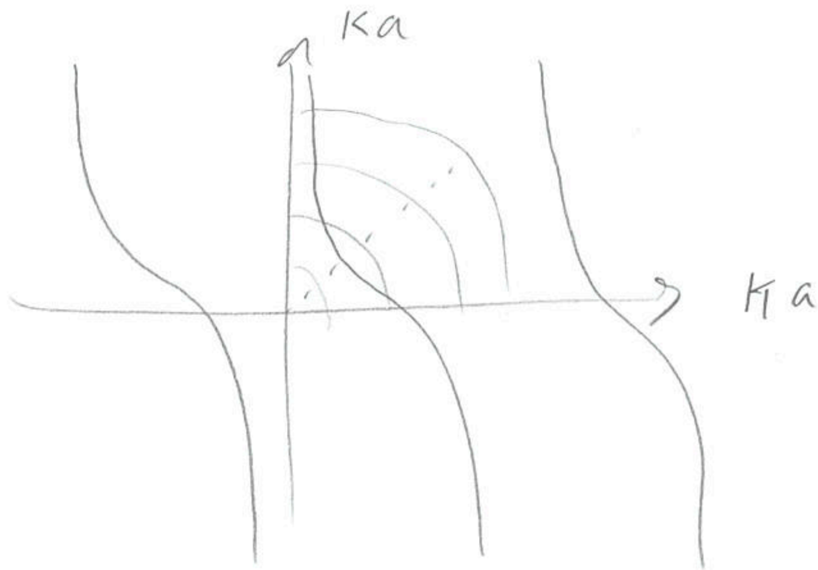
8

(9)

2.4.b) For a fixed range a, what is the minimal value of V_0 for which there is a bound state?

$$a^2 K^2 - a^2 k^2 = a^2 \frac{2mV_0}{\hbar^2}$$ ovals of radius $a\sqrt{\frac{2mV_0}{\hbar^2}}$

$$Ka = ka \cot(ka)$$



When $k \cot(ka) = \pm\sqrt{k^2 + \frac{2mV_0}{\hbar^2}}$

$$\underbrace{\sqrt{\frac{2m(|E|-V_0)}{\hbar^2}}}_{k} \underbrace{\cot(ka)}_{\sim \frac{1}{ka} - \frac{1}{3}ka} = \pm\sqrt{\frac{2m|E|}{\hbar^2}}$$

$$1 - \frac{1}{3}\frac{2m(|E|-V_0)}{\hbar^2} = \pm a\sqrt{\frac{2m|E|}{\hbar^2}}$$

QED $\boxed{V_0 = \left(\left(\pm a\sqrt{\frac{2m|E|}{\hbar^2}} - 1\right) \cdot \frac{-3\hbar^2}{2m} - |E|\right) \cdot -1}$

85

3/31/2015

191

Midterm Revisions

$V = -\vec{d}\cdot\vec{E}$ $\qquad \vec{d} = p$

$\vec{d} = d\vec{s}$ & $\vec{E} = -\vec{\nabla}\Phi$

1) electric dipole moment proportional to electron spin.

$$H = \frac{\vec{p}^{\,2}}{2m} + (-\vec{p}\cdot\vec{E}) \quad , \quad \vec{p} = d\cdot\vec{s}, \quad \vec{E} = -\vec{\nabla}\Phi(r)$$

$$= \frac{\vec{p}^{\,2}}{2m} + d\cdot\Phi'(r)\,\vec{s}\cdot\hat{r}$$

Under rotations $\quad H^R = D^\dagger(R)\, H\, D(R)$

$$D^\dagger(\vec{s}\cdot\hat{r})D = (D^\dagger\vec{s}D)\cdot(D^\dagger\hat{r}D)$$

and for vector operators $\quad D^\dagger V_i D = R_{ij}V_j$

& $R^T R = 1$

$\therefore$ Whole H is rotationally invariant

Under parity $\qquad \Pi|\vec{x}\rangle = |-\vec{x}\rangle \qquad \Pi^2 = 1$

$$\left.\begin{aligned}\Pi\,\hat{r}\,\Pi &= -\hat{r}\\ \Pi\,\vec{s}\,\Pi &= +\vec{s}\end{aligned}\right\}$$ parity odd interaction

time reversal $$\left.\begin{aligned}\Theta^{-1}\vec{s}\,\Theta &= -\vec{s}\\ \Theta^{-1}\hat{r}\,\Theta &= +\hat{r}\end{aligned}\right\}$$ time odd interaction

but p^2 is even for both

$\therefore [\Pi, H] \neq 0$

$[\Theta, H] \neq 0$

Assuming the potential is linear in z

$$H = \frac{P_x^2}{2m} + \frac{P_\perp^2}{2m} + d\cdot E S_z \qquad \text{so } \Psi \to \Psi_{n,P_\perp,m} = \sqrt{\frac{2}{L}}\cos\left(\frac{n\pi x}{L}\right)\frac{e^{i\vec{P}_\perp\cdot\vec{x}}}{2\pi}$$

$$\text{\& } E_{nP_\perp m} = \frac{\hbar^2\left(\frac{n\pi}{L}\right)^2}{2m} + \frac{P_\perp^2}{2m} \pm d\cdot E\, m\frac{\hbar}{2}$$

2. State of $|j, m=j\rangle$ then rotate $R_y(\varepsilon)$

$$|jj\rangle_\varepsilon = D(R_y(\varepsilon))\,|jj\rangle = e^{-i\frac{\varepsilon J_y}{\hbar}}|jj\rangle = \left[\mathbb{1} - i\frac{\varepsilon J_y}{\hbar} + \frac{\left(\frac{-i\varepsilon J_y}{\hbar}\right)^2}{2}\right]|jj\rangle$$

What is $|\langle jj|jj\rangle_\varepsilon|^2 \simeq |\langle jj|[\quad]|jj\rangle|^2$

(A) $$\simeq \left|\langle jj|\,\mathbb{1} - \frac{i\varepsilon J_y}{\hbar} + \left(\frac{-i\varepsilon}{\hbar}\right)^2\frac{1}{2}J_y{}^2\,|jj\rangle\right|^2$$

$$J_y = \frac{J_+ - J_-}{2i}$$

$$J_x = \frac{J_+ + J_-}{2}$$

$$\simeq \left|\langle jj|\,\mathbb{1} - \cancel{\frac{i\varepsilon J_y}{\hbar}} + \left(\frac{-i\varepsilon}{\hbar}\right)^2\frac{1}{2\cdot 4}\left(\underset{0}{\cancel{J_+^2}} + \underset{0}{\cancel{J_-^2}} - J_+J_- - \underset{0 \text{ can't go higher}}{J_-J_+}\right)|jj\rangle\right.$$

raises & lowers $\therefore 0$

$$\simeq \left|\langle jj|\,\mathbb{1} - \frac{\varepsilon^2}{2\hbar^2}\cdot\frac{1}{4}\cdot J_+J_-\,|jj\rangle\right|^2$$

$$\simeq \mathbb{1} - \frac{\varepsilon^2}{4\hbar^2}\langle jj|J_+J_-|jj\rangle^2$$

one way

$$J_\pm|jm\rangle = \sqrt{j(j-1) - m(m\pm 1)}\,|j, m\pm 1\rangle$$

two way

$$2J_y{}^2 = J_y{}^2 + J_x{}^2 = J_\perp{}^2$$

or from (A) we get $\simeq \mathbb{1} - \frac{\varepsilon^2}{2\hbar^2}\cdot 2\langle jj|J_y{}^2|jj\rangle$

$$\simeq \mathbb{1} - \frac{\varepsilon^2}{2\hbar^2}\cdot\langle jj|J_\perp{}^2|jj\rangle$$

$$J^2 - J_z{}^2$$

$$\simeq 1 - \frac{\varepsilon^2}{2\hbar^2}\left(j(\cancel{j}+1) - \cancel{j^2}\right)\hbar^2 = 1 - \frac{\varepsilon^2}{2}j$$

87

(21)

3. $$H = \left[\frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2} m\omega^2 (x^2+y^2) \right] + \left[\delta m \omega^2 xy \right]$$

$$E_{n_x n_y} = \hbar\omega (n_x + n_y + 1) \qquad n_{x,y} \in \mathbb{N}$$

$$E_{00} = \hbar\omega \qquad |00\rangle$$
$$E_{10} = E_{01} = 2\hbar\omega \qquad |01\rangle \;\&\; |10\rangle$$

G.S. $\overset{1}{E}_{00} = \langle 00|V|00\rangle = 0$ by parity

1.E.S $\overset{1}{E}_1$ = diagonalized. $\langle 01|V|01\rangle$ etc.

$$V_{nm} = \begin{pmatrix} \langle 01|V|01\rangle & \langle 01|V|10\rangle \\ \langle 10|V|01\rangle & \langle 10|V|10\rangle \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

$$\Delta = \langle 10|V|01\rangle = \delta m\omega^2 |\langle 0|x|1\rangle|^2$$

$$x = \frac{a^\dagger + a}{\sqrt{2}} \cdot x_0 \qquad \& \;\langle 1|x|0\rangle = \frac{x_0}{\sqrt{2}} \langle 1|a^\dagger + a|0\rangle$$
$$= \frac{x_0}{\sqrt{2}}$$

$$\Delta = \frac{\delta m\omega^2 x_0^2}{2} = \delta\hbar\omega \underbrace{\left(\frac{m\omega^2 x_0^2}{2\hbar\omega}\right)}_{=1 \text{ by SHO Normalization}}$$

$$V_{2\times 2} = \begin{pmatrix} 0 & \delta\hbar\omega \\ \delta\hbar\omega & 0 \end{pmatrix} \rightarrow \pm\,\delta\hbar\omega \text{ excitation energies.}$$