Solutions

Physics PhD Qualifying Examination
Part I – Tuesday, August 22, 2006

Name:____
(please print)
Identification Number:________

STUDENT: Designate the problem numbers that you are handing in for grading in the appropriate left hand boxes below. Initial the right hand box.
PROCTOR: Check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on each preprinted answer sheet.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least three passed problems from problems 1-5 (Mechanics) and three problems from problems 6-10 (Electricity and Magnetism). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[I-1] **[10]**

A particle moves in a medium under the influence of a retarding force equal to $-mk(v^3 + a^2 v)$, where k and a are constants. *There are no other forces present.* The particle is initially at the origin $(x = 0)$ and is given an initial velocity v_o. Treating the problem classically (not relativistically),

(a) What is the distance the particle travels before coming to a stop? (Your answer should be given in terms of v_0, k, and a.)

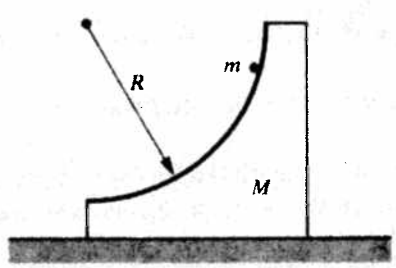
(b) What is the absolute maximum possible distance it can travel for any initial velocity?

[I-2] **[10]**

A particle of mass m slides down a smooth circular wedge of mass M as shown below, starting from an arbitrarily specified position on the circular surface. The wedge rests on a smooth horizontal table.

(a) Defining suitable generalized coordinates, find the equations of motion of m and M.

(b) Find the force exerted by the circular wedge on m as a function of position.



[I-3] **[10]**

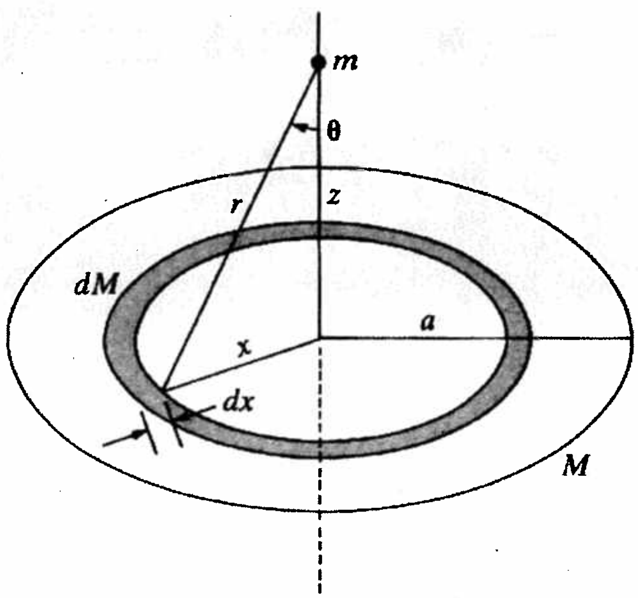
Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{6} & 3/2 & 1/\sqrt{12} \\ 1/\sqrt{2} & 1/\sqrt{12} & 5/2 \end{pmatrix}.$$

(a) Determine the eigenvalues. (One of the eigenvalues is 2.)
(b) Determine the eigenvectors.

[I-4] **[6, 4]**

Consider a thin uniform disk of mass M and radius a. Find
(a) the gravitational potential $\Phi(z)$ and
(b) the gravitational force $f(z)$ on a mass m located along the axis of the disk.



[I-5] [10]

A racer attempting to break the land speed record speeds by two markers spaced 100m apart on the ground in a time of 0.4ms as measured by an observer on the ground.
(a) How far apart do the two markers appear to the racer?
(b) What elapsed time does the racer measure?
(c) What speeds do the racer and ground observer measure?

[I-6] **[5, 5]**

The *linear* charge density on a ring of radius a is given by

$$\rho_l(\vartheta) = \frac{q}{a}[\cos(\vartheta) - \sin(2\vartheta)] \ ,$$

where ϑ is the polar angle in the plane of the ring.

(a) Find the monopole and the dipole moments of the system.

(b) Calculate the potential at an arbitrary point in space, accurate to terms in r^{-3}.

[I-7] **[10]**

Consider a parallel-plate capacitor immersed in seawater and driven by an alternating voltage $V(t)=V_0\cos(2\pi ft)$. Sea water at frequency $f = 4\times10^8$ Hz has a permittivity $\varepsilon=81\varepsilon_0$, a permeability $\mu=\mu_0$, and a resistivity $\rho=0.23$ Ωm. $\varepsilon_0=8.85\text{x}10^{-12}$ As/Vm and $\mu_0=4\pi\text{x}10^{-7}$Vs/Am.

What is the ratio of the amplitudes of the conduction current to the displacement current in sea water at $f = 4\times10^8$ Hz?

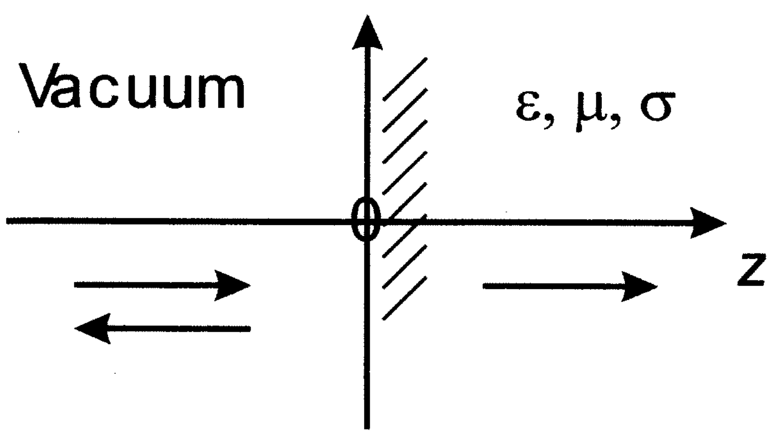
[I-8] [10]

An alternating current $I = I_o \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a.
(a) In what *direction* does the induced electric field point (radial, circumferential, or longitudinal)?
(b) Assuming the field goes to zero as the distance from the wire $r \to \infty$, find $\boldsymbol{E}(r,t)$.

[I-9] [10]

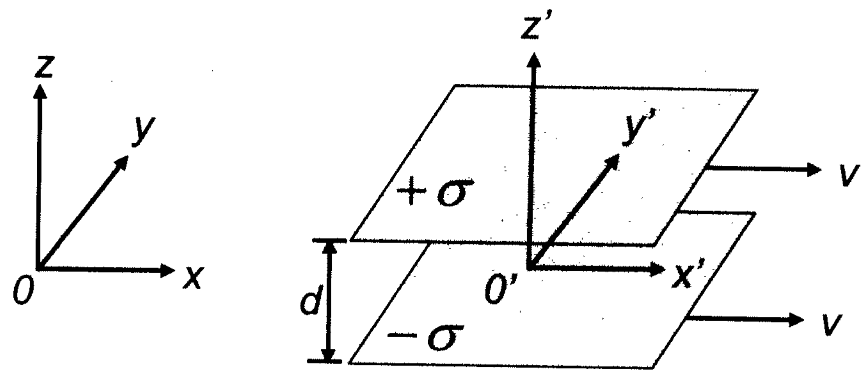
A homogenous, uncharged medium (dielectric constant ε, magnetic permeability μ and electric conductivity σ) is limited to the right-hand half space ($z > 0$). There is vacuum on the left-hand side ($z < 0$). A planar electromagnetic wave approaches the medium from $z<0$, propagating parallel to the z-axis. At $z = 0$ it meets the interface to the medium. At the interface, the wave is partially being reflected and partially penetrates the medium.

Calculate the electromagnetic wave in the medium. How far does the wave penetrate the medium (characteristic length)?



[I-10] **[10]**

Two large (non-conducting) parallel plates separated by a distance d and oriented as shown in the figure below, move together along the x-axis with a velocity v. The upper and lower plates have uniform charge densities $+\sigma$ and $-\sigma$ respectively in the rest frame of the plates. Find the magnitude and direction of the electric and magnetic fields between the plates. (You may neglect edge effects).



I-1

$$m\frac{dv}{dt} = -mk(v^3 + a^2 v)$$

use : $\frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\cdot\frac{dv}{dx}$

$$mv\frac{dv}{dx} = -mk(v^3 + a^2 v)$$

$$\int\frac{v\,dv}{v^3 + a^2 v} = -\int k\,dx$$

$$\int\frac{dv}{a^2 + v^2} = -kx + C$$

$$\frac{1}{a}\tan^{-1}\left(\frac{v}{a}\right) = -kx + C$$

$$\left.\begin{array}{l} x(0) = x_0 = \phi \\ v(0) = v_0 \end{array}\right\} \Rightarrow \frac{1}{a}\tan^{-1}\left(\frac{v_0}{a}\right) = -k\cancel{x_0} + C$$

$$C = \frac{1}{a}\tan^{-1}\left(\frac{v_0}{a}\right)$$

$$kx = \frac{1}{a}\tan^{-1}\left(\frac{v_0}{a}\right) - \frac{1}{a}\tan^{-1}\left(\frac{v}{a}\right)$$

$$x(v) = \frac{1}{ka}\left\{\tan^{-1}\left(\frac{v_0}{a}\right) - \tan^{-1}\left(\frac{v}{a}\right)\right\}$$

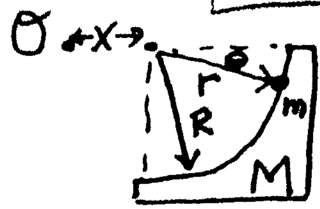
x_{max} : occurs when $v = 0$ (comes to a stop)

a) $$\boxed{x_{max} = \frac{1}{ka}\tan^{-1}\left(v_0/a\right)}$$

maximum possible distance: $v_0 \to \infty$

b) $$\lim_{v_0\to\infty} x_{max} = \frac{1}{ka}\lim_{v_0\to\infty}\tan^{-1}\left(\frac{v_0}{a}\right) = \frac{1}{ka}\frac{\pi}{2} = \boxed{\frac{\pi}{2ka}}$$

FALL 2006 : $\boxed{\text{I}-2}$



$x_M = X \qquad x_m = X + r\cos\theta$

$y_M = 0 \qquad y_m = -r\sin\theta$

Lagrangian: $L = \left(\frac{M+m}{2}\right)\dot{X}^2 + \frac{m}{2}\left\{\begin{array}{l}\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{X}\dot{r}\,\cdot \\ -2\dot{X}r\dot{\theta}\sin\theta\end{array}\right.$

$+ mgr\sin\theta$

constraint: $f(x,\theta,r) = r - R = 0$

$$\left.\begin{array}{l}\Rightarrow \ddot{X} = aR(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ \ddot{\theta} = (\ddot{X}\sin\theta + g\cos\theta)/R \\ \text{where } a = m/(M+m)\end{array}\right\} \text{(a) eqs. mot}$$

(b) force on m (of M)

$\lambda = m\ddot{X}\cos\theta + mR\dot{\theta}^2 - mg\sin\theta$

$\Rightarrow \lambda = \left[\frac{a-1}{1-a\sin^2\theta}\right](R\dot{\theta}^2 + g\sin\theta)$

Cons. energy $\Rightarrow H = \left(\frac{M+m}{2}\right)\dot{X}^2 + \frac{m}{2}(R^2\dot{\theta}^2 - 2\dot{X}R\dot{\theta}\sin\theta)$

$-mg\sin\theta$

$= -mgR\sin\theta_0 \quad [\theta_0 = \text{orig. posi}$

integ. $\ddot{x}$ eq $\Rightarrow \dot{\theta}^2 = \frac{2g(\sin\theta - \sin\theta_0)}{R(1-a\sin^2\theta)}$

$\Rightarrow \lambda = -\frac{mMg(3\sin\theta - a\sin^3\theta - 2\sin\theta_0)}{(M+m)(1-a\sin^2\theta)^2} = \text{fo on}$

Solution $\boxed{\text{I-3}}$

a) Eigen values

$$\begin{vmatrix} 2-\lambda & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{3}{2}-\lambda & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{12}} & \frac{5}{2}-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(\tfrac{3}{2}-\lambda)(\tfrac{5}{2}-\lambda) - \tfrac{1}{12} - \tfrac{1}{12} - (2-\lambda)\tfrac{1}{12} - (\tfrac{5}{2}-\lambda)\tfrac{1}{6} - (\tfrac{3}{2}-\lambda)\tfrac{1}{2} = 0$$

$$(3 - 2\lambda - \tfrac{3}{2}\lambda + \lambda^2)(\tfrac{5}{2}-\lambda) - \tfrac{1}{3} + \tfrac{\lambda}{12} - \tfrac{5}{12} + \tfrac{\lambda}{6} - \tfrac{3}{4} + \tfrac{\lambda}{2} = 0$$

$$\tfrac{15}{2} - 5\lambda - \tfrac{15}{4}\lambda + \tfrac{5}{2}\lambda^2 - 3\lambda + 2\lambda^2 + \tfrac{3}{2}\lambda^2 - \lambda^3 - \tfrac{3}{2} + \tfrac{3}{4}\lambda = 0$$

$$6 - 11\lambda + 6\lambda^2 - \lambda^3 = 0$$

$\underline{\underline{\lambda_1 = 2}}$ $\quad \lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda-2)(\lambda^2 - 4\lambda + 3)$

$$\begin{array}{l} \lambda^3 - 2\lambda^2 \\ \hline -4\lambda^2 + 11\lambda \\ -4\lambda^2 + 8\lambda \\ \hline 3\lambda - 6 \end{array}$$

$$\lambda_{2,3} = \frac{4 \pm \sqrt{16 - 4\cdot 3}}{2} = \frac{4 \pm 2}{2} \qquad \underline{\underline{\lambda_2 = 3}};\ \underline{\underline{\lambda_3 = 1}}$$

Eigen values are 1, 2, 3

b) Eigen vectors

$\lambda_1 = 1$:

$$\begin{array}{ccc} 1 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{2} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{12}} & \frac{3}{2} \\ \hline 1 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{2} \\ 0 & \sqrt{\frac{2}{3}} & \sqrt{2} \\ \hline 1 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \\ \hline 1 & 0 & \sqrt{2} \\ 0 & 1 & \sqrt{3} \\ 0 & 0 & 0 \\ \hline \end{array}$$

$\lambda_2 = 2$

$$\begin{array}{ccc} 0 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{2} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{12}} & \frac{1}{2} \\ \hline 1 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\sqrt{3} \\ -1 & -\sqrt{\frac{1}{2}} & \frac{1}{\sqrt{2}} \\ \hline 1 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & -\sqrt{3} \\ 0 & 1 & -\sqrt{3} \\ \hline 1 & 0 & \sqrt{2} \\ 0 & 1 & -\sqrt{3} \\ 0 & 0 & 0 \\ \hline \end{array}$$

$\lambda_3 = 3$

$$\begin{array}{ccc} -1 & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & -\frac{3}{2} & \frac{1}{\sqrt{12}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{12}} & -\frac{1}{2} \\ \hline 1 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -1 & -\sqrt{\frac{27}{2}} & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \hline 1 & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ 0 & -\sqrt{\frac{32}{3}} & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}$$

$$\vec{e}_1 = \frac{t}{\sqrt{6}} \begin{pmatrix} -\sqrt{2} \\ -\sqrt{3} \\ 1 \end{pmatrix} \qquad \vec{e}_2 = \frac{t}{\sqrt{6}} \begin{pmatrix} -\sqrt{2} \\ \sqrt{3} \\ 1 \end{pmatrix} \qquad \vec{e}_3 = \frac{t}{\sqrt{\frac{3}{2}}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 1 \end{pmatrix}$$

$$\mathcal{D} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix}$$

I-4) Gravity or Central Potential

(a) Gravitational potential is given by: $\phi = -G \cdot \frac{M}{r}$ ...... ①

$$d\phi = -G \frac{dM}{r} \quad \text{.......} ②$$

$$dM = \rho \, dA = \rho (2\pi x \, dx) \quad \text{......} ③$$

$$r = (x^2 + z^2)^{1/2} \quad \text{......} ④$$

Combine ②, ③ and ④, we have:

$$d\phi = -2\pi \rho G \frac{x \, dx}{\sqrt{x^2 + z^2}}$$

$$\phi(z) = -\pi \rho G \int_0^a \frac{2x \, dx}{\sqrt{x^2 + z^2}}$$

$$= -2\pi \rho G \sqrt{x^2 + z^2} \Big|_0^a$$

$$\therefore \phi(z) = -2\pi \rho G \left(\sqrt{a^2 + z^2} - z \right)$$

As $M \equiv \pi a^2 \rho$

$$\therefore \phi(z) = -2MG \left(\frac{\sqrt{a^2 + z^2} - z}{a^2} \right) \quad \text{......} ⑤$$

Gravity or Central Potential

(b) We find the force from:

$$\vec{F} = -\vec{\nabla} U = -m \vec{\nabla} \phi \quad \text{-------} \quad (6)$$

From symmetry, $\vec{F} \parallel \vec{z}$

$$\vec{F} = -m \frac{\partial}{\partial z} \phi(z)$$

$$\therefore F_z = \frac{2mMG}{a^2} \left(\frac{z}{\sqrt{a^2+z^2}} - 1 \right) \quad \text{------} \quad (7)$$

Solutions : 1. September 2006

I-5 Special relativity [redacted]

A racer attempting to break the land speed record rockets by two markers spaced 100m apart on the ground in a time of 0.4ms as measured by an observer on the ground. (10 pts)

- How far apart do the two markers appear to the racer?
- What elapsed does the racer measure?
- What speeds do the racer and ground observer measure?

The ground observer measures the speed to be

$$v = \frac{100\ \text{m}}{.4\ \mu\text{sec}} = \boxed{2.5\times 10^{8}\ \text{m/s}}$$

The length between the markers as measured by the racer is

$$\ell' = \ell\sqrt{1 - v^2/c^2}$$

$$= 100\ \text{m}\ \sqrt{1 - \left[\frac{2.5}{3}\right]^2} = \boxed{55.3\ \text{meters}}$$

The time measured in the racer's frame is given by

$$t' = \gamma\left(t - \frac{v}{c^2}x_1\right)$$

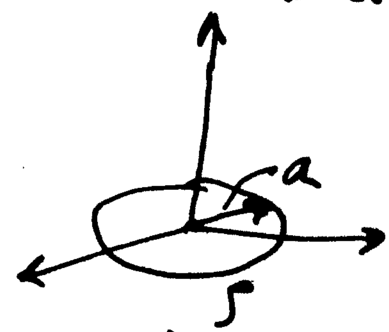
$$= \frac{\left(.4\ \mu\text{sec} - \dfrac{(2.5\times 10^{8}\ \text{m/s})(100\ \text{m})}{c^2}\right)}{\sqrt{1-(2.5/3)^2}}$$

$$= \boxed{22\ \mu\text{sec}}$$

The speed observed by the racer is

$$v = \frac{\ell'}{t'} = \frac{\ell}{t} = \boxed{2.5\times 10^{8}\ \text{m/s}}$$

(I-6.) Solution (Boundary value - Electrostatic



a) Monopole:

$$\rho_e = \frac{q}{a}(\cos\phi - \sin 2\phi)$$

$$Q = \int_0^{2\pi} a\,d\phi\,\frac{q}{a}(\cos\phi - \sin\phi\cos\phi)$$

$$Q = q\int_0^{2\pi}(\cos\phi - 2\sin\phi\cos\phi)\,d\phi = 0$$

($\cos\phi$ and $\sin 2\phi$ have period 2π) $\therefore Q = 0$

Dipole:

$$\vec{P} = \int_0^{2\pi} a\,d\phi\,(a\hat{e}_r)\frac{q}{a}(\cos\phi - \sin 2\phi)$$

$$\vec{P} = \int_0^{2\pi}\hat{e}_r(\cos\phi - \sin 2\phi)\,d\phi \cdot qa$$

now: $\hat{e}_r = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$ $\therefore$

$$P_x = qa\int_0^{2\pi} d\phi\,(\cos^2\phi - \sin 2\phi\cos\phi)$$

$$P_x = qa\int_0^{2\pi}\left[\left(\frac{1+\cos 2\phi}{2}\right) - 2\sin\phi\cos^2\phi\right]d\phi$$

$$P_x = qa\int_0^{2\pi}\frac{1}{2}\,d\phi = \pi q a$$

$$P_y = qa\int_0^{2\pi} d\phi\,(\cos\phi\sin\phi - 2\sin^2\phi\cos\phi) = 0$$

$\therefore$ dipole moment $\vec{P} = \pi q a\,\hat{x}$

(I-6) continued.

(b) Potential up to terms of order r^{-3}.
(made up of monopole and dipole contribu

$$\Phi^{(2)} = \frac{\vec{p}\cdot\hat{e}_r}{r^2} = \frac{1}{r^2}(\pi a q\,\hat{x}\cdot\hat{e}_r)$$

$$\Phi^{(2)} = \frac{\pi a q}{r^2}\sin\theta\cos\phi$$

quadrup
ter

actually : $\Phi = \underset{\overset{\shortparallel}{0}}{\Phi^{(0)}} + \Phi^{(2)} + \Phi^{(4)} + \cdots$

$$\left[\therefore \Phi = \frac{\pi a q}{r^2}\sin\theta\cos\phi\right]$$

Solutions : 1 September 2006

$\boxed{\text{I-7}}$ Maxwell's Equations (10 pts)

Consider a parallel-plate capacitor immersed in seawater and driven by an alternating voltage $V(t)=V_0\cos(2\pi ft)$.

Sea water at frequency $f=4\times10^8$ Hz has a permittivity $\varepsilon=81\varepsilon_0$, a permibility $\mu=\mu_0$, and a resistivity $\rho=0.23\ \Omega$m.

What is the ratio of the amplitudes of conduction current to displacement current in sea water at $f=4\times10^8$ Hz?

$\varepsilon_0=8.85\times10^{-12}$ As/Vm, $\mu_0=4p\times10^{-7}$Vs/Am

conduction current $j_c = \sigma E = \frac{1}{\rho}\frac{V}{d}$

displacement current $j_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E)$

$$= \frac{\partial}{\partial t}\left(\varepsilon \frac{V(t)}{d}\right)$$

$$= \frac{\varepsilon}{d}\frac{\partial}{\partial t}\left(V_0 \cos(2\pi f t)\right)$$

$$= \frac{\varepsilon V_0}{d}\left[-2\pi f \sin(2\pi f t)\right]$$

ratio of amplitudes :

$$\frac{j_c}{j_d} = \frac{V_0}{\rho d}\,\frac{d}{2\pi f \varepsilon V_0} = \frac{1}{2\pi f \varepsilon \rho}$$

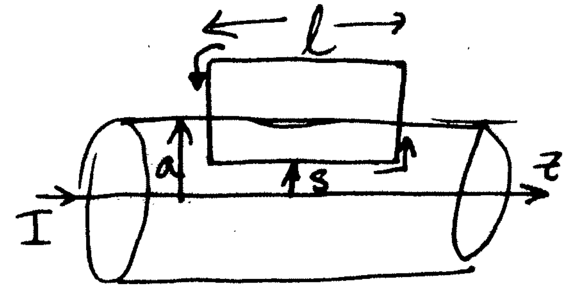
$$= \left[2\pi\,(4\times10^8)\,81\,(8.85\times10^{-12})\,(0.23)\right]^{-1}$$

$$\frac{j_c}{j_d} = 2.41$$

FALL 2006 : $\boxed{\text{I-8}}$

(a) longitudinal

(b) amperian loop:



$I = I_0 \cos($

outside: $\underline{B} = 0 \Rightarrow \underline{E} = 0$

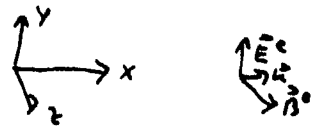
$$\oint \underline{E} \cdot d\underline{l} = E\,l = -\frac{d\phi}{dt} = -\frac{d}{dt}\int \underline{B}\cdot d\underline{a}$$

$$= -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'}\, l\, ds'$$

$$\Rightarrow \underline{E} = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right)\hat{z}$$

Solution I-9

Using SI units



Incoming wave

$(x<0)\quad E_y^e = a \exp\{j\omega(t - x/c)\}$

$$B_z^e = \frac{a}{c} \exp j\omega(t - x/c)$$

Reflected wave

$(x<0)\quad E_y^r = a' \exp j\omega(t + x/c)$

$$B_z^r = -a'/c \cdot \exp j\omega(t + x/c)$$

Penetrating wave

$(x>0)\quad E_y^t = a'' \exp j\omega(t - \tilde{n}x/c)$

$$B_z^t = \frac{\tilde{n}a''}{c} \exp j\omega(t - \tilde{n}x/c)$$

E_t continuous (no charges and currents in the interface) => H_t continuous

At t = 0:

$$\Rightarrow E_y^e + E_y^r = E_y^t \Rightarrow a + a' = a''$$

$$\frac{B_z^e}{1} + \frac{B_z^r}{1} = \frac{B_z^t}{\mu} \Rightarrow a - a' = \frac{\tilde{n}a''}{\mu}$$

$$\Rightarrow a = \frac{1}{2}a''\left(1 + \frac{\tilde{n}}{\mu}\right) \quad ; \quad a' = \frac{1}{2}a''\left(1 - \frac{\tilde{n}}{\mu}\right)$$

$$a'' = \frac{2a}{1 + \frac{\tilde{n}}{\mu}} \qquad a' = a\,\frac{1 - \frac{\tilde{n}}{\mu}}{1 + \frac{\tilde{n}}{\mu}} = \underline{a\,\frac{\mu - \tilde{n}}{\mu + \tilde{n}} = a'}$$

With $\tilde{n} = \sqrt{\varepsilon\mu\left(1 - \frac{j\sigma}{\varepsilon\varepsilon_0\omega}\right)}$ and $\tilde{n} = n - j\kappa$

Leads to penetration depth (Skin depth) (SI units)

$d = \frac{c}{\omega\kappa} = \sqrt{\frac{1}{\pi\mu\mu_0\sigma_0\nu}}$ (SI units) or $\delta_0 = \frac{c}{\sqrt{2\pi\sigma\omega}}$ (CGS units)

$\boxed{5.10}$

Let E&M fields be $\vec{E}'$ and $\vec{B}'$ in [illegible] frame S' (O, X', Y')

Let E&M fields be $\vec{E}$ and $\vec{B}$ in frame S (O, X, Y).

Let $\beta \equiv v/c$ and $\gamma \equiv \sqrt{1-\beta^2}$

We have:

$$\begin{cases} E_x = E_x' \\ E_y = \gamma\,(E_y' + \beta c B_z') \\ E_z = \gamma\,(E_z' - \beta c B_y') \end{cases}$$

$$\begin{cases} B_x = B_x' \\ B_y = \gamma\,(B_y' - \frac{\beta}{c} E_z') \\ B_z = \gamma\,(B_z' + \frac{\beta}{c} E_y') \end{cases}$$

In the rest frame S':

$$B_x' = B_y' = B_z' = 0$$

$$E_x' = E_y' = 0$$

$$E_z' = -\frac{\sigma}{\varepsilon_0}$$

[illegible]

Solutions

Physics PhD Qualifying Examination
Part II – Friday, August 25, 2006

Name:____
(please print)
Identification Number:________

STUDENT: insert a check mark in the left boxes to designate the problem numbers that you are handing in for grading.
PROCTOR: check off the right hand boxes corresponding to the problems received from each student. Initial in the right hand box.

	1	
	2	
	3	
	4	
	5	
	6	
	7	
	8	
	9	
	10	

Student's initials

problems handed in:

Proctor's initials

INSTRUCTIONS FOR SUBMITTING ANSWER SHEETS

1. DO NOT PUT YOUR NAME ON ANY ANSWER SHEET. EXAMS WILL BE COLLATED AND GRADED BY THE ID NUMBER ABOVE.
2. Use at least one separate preprinted answer sheet for each problem. Write on only one side of each answer sheet.
3. Write your **identification number** listed above, in the appropriate box on the preprinted sheets.
4. Write the **problem number** in the appropriate box of each preprinted answer sheet. If you use more than one page for an answer, then number the answer sheets with both problem number and page (e.g. Problem 9 – Page 1 of 3).
5. Staple together all the pages pertaining to a given problem. Use a paper clip to group together all eight problems that you are handing in.
6. Hand in a total of *eight* problems. A passing distribution will normally include at least four passed problems from problems 1-6 (Quantum Physics) and two problems from problems 7-10 (Thermodynamics and Statistical Mechanics). **DO NOT HAND IN MORE THAN EIGHT PROBLEMS.**
7. **YOU MUST SHOW ALL YOUR WORK.**

[II-1] **[10]**

A particle is in the ground state of a one-dimensional, infinitely-high potential well (hard-core box) of width L:

$$V_o(x) = \begin{cases} 0 & \text{if } 0 < x < L \\ \infty & \text{otherwise} \end{cases}.$$

At $t = 0$ we instantaneously expand the right wall to double the size of the box so that the new potential becomes

$$V(x) = \begin{cases} 0 & \text{if } 0 < x < 2L \\ \infty & \text{otherwise} \end{cases}.$$

(a) What is the probability that the particle is in the ground state of the new potential at time t?
(b) What state (in terms of the eigenstates of the new potential) is the particle most likely to be in at time t?

[II-2] **[10]**

A particle with charge e and mass m is confined to move on the circumference of a circle of radius r. Let ϕ be the angle around the circle. The only term in the Hamiltonian is the kinetic energy.

(a) Find the eigenfunctions of the system
(b) Find the eigenvalues
(c) An electric field $\mathbf{E}$ is imposed in the plane of the circle. Using time-independent perturbation theory, find the perturbed energy levels to orders in $O(|\mathbf{E}^2|)$.

[II-3] **[10]**

Consider the Pauli matrices σ_1, σ_2, and σ_3:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

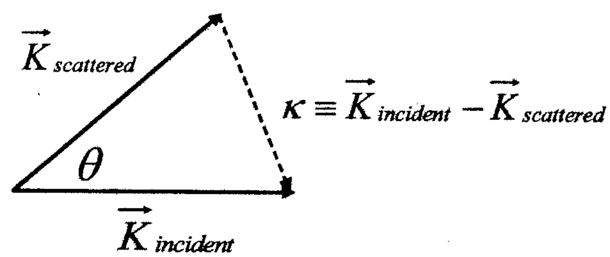
(a) Calculate the eigenvalues and eigenvectors of the above matrices σ_1, σ_2, and σ_3.

(b) Calculate the commutator $[\sigma_i, \sigma_j]$ for $i \neq j$ and for $i = j$.

[II-4] **[6, 4]**

(a) A particle of mass m is scattered by the Yukawa potential, $V(r) = V_o \dfrac{\exp(-\mu r)}{r}$, where μ is a constant. Calculate the scattering amplitude, $f(\theta)$, in the first Born approximation.

The two wave-vectors in the Born approximation.



(b) A particle of mass m is scattered by the following potential:

$V(r) = V_o$, if $r \le a$; and $V(r) = 0$, if $r > a$.

Calculate the scattering amplitude, $f(\theta)$, in the first Born approximation.

[II-5] **[10]**

The Hamiltonian for a three-level system is represented by

$$\boldsymbol{H} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix},$$

where a, b, and c are real numbers.

(a) If the system starts out in the state $|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, what is $|\mathcal{S}(t)\rangle$?

(b) If the system starts out in the state $|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, what is $|\mathcal{S}(t)\rangle$?

[II-6] **[10]**

A linear harmonic oscillator is acted upon by an electric field which is considered to be a perturbation and which depends as follows on time:

$$E(t) = \frac{A}{t^2 + \tau^2},$$

where A is some constant. The direction of the field is along the direction of motion of the unperturbed oscillator.

Assuming that when the field is switched on (that is, at $t = -\infty$) the oscillator is in its ground state, **evaluate to a first-order approximation** the probability that it is excited at the end of the action of the field (that is, at $t = +\infty$).

[II-7] [5, 5]

(a) It is easily verified that a rubber band heats up when it is stretched adiabatically. Given this fact, determine whether a rubber band will **contract** or **expand** when it is cooled at constant tension.
(b) The same amount of heat flows into two identical rubber bands, but the one is held at **constant tension** and the other at **constant length**. Which has the larger increase in temperature?

[II-8] [10]

The chemical potential of a *single-component* particle system is given by

$$\mu(T,P) = -k_B T \ln\left(a\frac{T^{5/2}}{P} \right),$$

where k_B is the Boltzman constant and a is a constant depending on material-specific and other fundamental constants of nature.
(a) Obtain the equation of state of the system.
(b) Obtain the internal energy of the system $E(T,V,N)$.
(c) Obtain the entropy of the system $S(E,V,N)$.
(T is the absolute temperature, P is the pressure, V is the volume, and N is the number of particles.)

[II-9] **[10]**

A small, just barely visible, dust particle has a mass of about 10^{-8} g. It falls onto a glass of ice-cold water where it is supported by surface tension, and moves freely in only two dimensions.

(a) What is the average translational energy of the dust particle?

(b) What is the root-mean-squared velocity of its Brownian motion?

$k_B = 1.38\times10^{-23}\,\mathrm{J/K}$

[II-10] **[6, 4]**

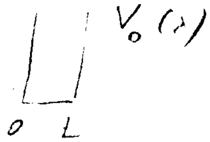
Consider an ideal Bose gas confined to a region of area A in *two dimensions.*

(a) Express the number of particles in the excited states, N_e, and the number of particles in the ground state, N_0, in terms of z, T, and A, and **show** that the system **does not** exhibit Bose-Einstein condensation unless $T \to 0\,\mathrm{K}$. Here, z is the fugacity of the gas and is related to the chemical potential, μ, through $z = \exp(\mu / k_B T)$.

(b) **Refine your argument to show** that, if the area A and the total number of particles N are held fixed and we require both N_e and N_0 to be of order N, we do achieve condensation when

$$T \sim \frac{h^2}{mk_B l^2}\frac{1}{\ln(N)}$$

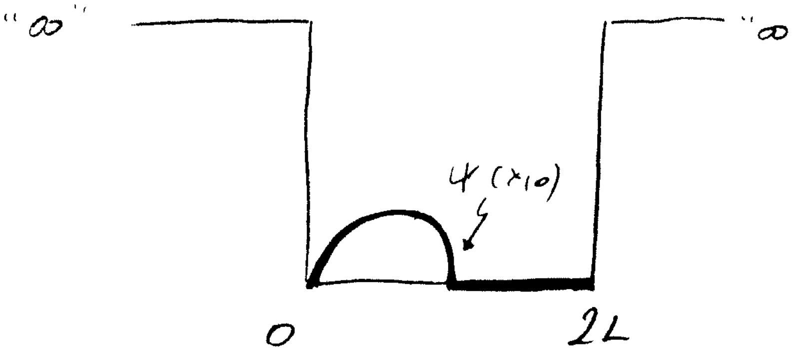
where l ($\sim\sqrt{A/N}$) is the mean inter-particle distance in the system. Of course, if both $A\to\infty$ and $N\to\infty$, keeping l fixed, then the desired T goes to 0.

[II-1] original box: 

$E = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 \quad \psi_n^0 = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$

i.e., $t=0$ $\quad \psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right) & 0<x<L \\ 0 & \text{otherwise} \end{cases}$

starting at $t=0$ $\quad V(x) = \begin{cases} 0 & \text{if } 0<x<2L \\ \infty & \text{otherwise} \end{cases}$



new potential:

$E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{2L}\right)^2 \quad \psi_n = \sqrt{\frac{2}{2L}}\sin\left(\frac{n\pi}{2L}x\right)$

$n = 1, 2, \ldots$

time-dependent Sch eq: $\quad \boxed{\hat{H}\psi(x,t) = i\hbar\frac{\partial\psi}{\partial t}}$

formal solution. $\quad |\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle \qquad \hat{H}|n\rangle = E|n\rangle$

$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}\overbrace{\sum_n |n\rangle\langle n|}^{\mathbb{1}}\psi(0)\rangle =$

$= \sum_n e^{-\frac{i}{\hbar}\hat{H}t}|n\rangle\langle n|\psi(0)\rangle = \sum_n e^{-\frac{i}{\hbar}E_n}|n\rangle\underbrace{\langle n|\psi(0)\rangle}_{a_n}$

$\Longrightarrow$

$$\psi(x,t) = \sum_{n=1}^{\infty}\underbrace{\int \psi_n^*(x')\psi(x',0)\,dx'}_{a_n}\cdot e^{-\frac{i}{\hbar}E_n t}\psi_n(x) =$$

$$= \sum_{n=1}^{\infty} a_n e^{-\frac{i}{\hbar}E_n t}\psi_n(x)$$

$P_n = \left|a_n e^{-\frac{i}{\hbar}E_n t}\right|^2 = |a_n|^2$

- 1

$$a_n = \int_0^{2L} \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi}{2L} x\right) \cdot \psi(x,0)\, dx$$

$$= \int_0^{L} \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi}{2L} x\right) \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L} x\right) dx$$

$$= \frac{\sqrt{2}}{L} \int_0^{L} \sin\left(\frac{n\pi}{2L} x\right) \sin\left(\frac{2\pi}{2L} x\right) dx = \frac{\sqrt{2}}{L} \int_0^{L} -\frac{1}{2} \left\{ \cos\left[\frac{(n+2)\pi}{2L} x\right] - \cos\left[\frac{(n-2)\pi}{2L} x\right] \right\} dx$$

$$= -\frac{\sqrt{2}}{2L} \left\{ \frac{2L}{(n+2)\pi} \sin\left[\frac{(n+2)\pi}{2L} x\right] \Big|_0^L - \frac{2L}{(n-2)\pi} \sin\left[\frac{(n-2)\pi}{2L} x\right] \Big|_0^L \right\} =$$

$$= -\sqrt{2} \left\{ \frac{1}{(n+2)\pi} \sin\left(\frac{n+2}{2}\pi\right) - \frac{1}{(n-2)\pi} \sin\left(\frac{(n-2)\pi}{2}\right) \right\}$$

a) $\underline{n = 1)}$ (GS)

$$a_1 = -\frac{\sqrt{2}}{\pi} \left\{ \frac{1}{3}(-1) - \frac{1}{(-1)}(-1) \right\} = \frac{\sqrt{2}}{\pi} \frac{4}{3}$$

$$p_1 = |a_1|^2 = \frac{16}{9} \cdot \frac{2}{\pi^2} \approx 0.36$$

$\underline{n = 2)}$

$$a_2 = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \qquad p_2 = |a_2|^2 = \frac{1}{2} = 0.50$$

$\underline{n \geq 3)}$

$a_n = 0$ for $n =$ even

$$a_n = \sqrt{2} \frac{4}{\pi} \frac{1}{n^2 - 4} (\pm 1)$$

$+1$: $n = 3, 7, 11, \ldots$

-1: $n = 5, 9, 13, \ldots$

$$p_n = |a_n|^2 = \frac{32}{\pi^2 [n^2 - 4]^2}$$

monotonically decaying with

$(p_3 = 0.1297)$

b) So p_n is max. for $n = 2$ $(p_1 \ldots \; p_2 = 0.50)$

Solution II-2:

a) The Eigenfunctions are

$$\psi_n(\phi) = \frac{1}{\sqrt{2\pi}} e^{in\phi}$$

b) The Eigenvalues are

$$E_n = \frac{\hbar^2 n^2}{2mr^2}$$

c) The perturbation is

$$V(\phi) = -e|E|r\cos\phi$$

If we assume the field is in the x-direction. The same result is obtained if we assume the perturbation is in the y-direction

$$V(\phi) = -e|E|r\sin\phi$$

In order to do perturbation theory, we need to find the matrix element of the perturbation between different Eigenstates. For first-order perturbation theory we need

$$\langle n|V|n\rangle = -e|E|r\int_0^{2\pi}\frac{d\phi}{2\pi}\cos\phi = 0$$

The Eigenvalues are unchanged to first-order in the field **E**.
To do second-order perturbation theory, we need off-diagonal matrix elements:

$$\langle n|V|m\rangle = -e|E|r\int_0^{2\pi}\frac{d\phi}{2\pi} e^{i\phi(m-n)}\cos\phi$$

$$= -\frac{1}{2}e|E|r\,\delta_{m,n\pm1}$$

If we recall that $\cos\phi = (e^{i\phi}+e^{-i\phi})/2$, then we see that $n-m$ can only equal ± 1 for the integral to be nonzero. In doing second-order perturbation theory for the state $|n\rangle$, the only permissible intermediate states are $m = n \pm 1$.

$$\delta E_n = \left\{\frac{\langle n|V|n+1\rangle^2}{E_n - E_{n+1}} + \frac{\langle n|V|n-1\rangle^2}{E_n - E_{n-1}}\right\}$$

$$= \left(\frac{e|E|r}{2}\right)^2\left(\frac{2mr^2}{\hbar^2}\right)\left[\frac{1}{n^2-(n+1)^2} + \frac{1}{n^2-(n-1)^2}\right]$$

$$= \frac{me^2r^4|E|^2}{\hbar^2}\,\frac{1}{4n^2-1}$$

This solution is valid for states $n > 0$.

For the ground state, with $n = 0$, the $n-1$ state does not exist, so answer for this case is

$$\delta E_0 = -\frac{me^2r^4|E|^2}{2\hbar^2}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

eigenvalues

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \qquad \lambda_{1,2} = \pm 1$$

eigenvectors

$\lambda = +1 \qquad -x + y = 0 \Rightarrow x = y \qquad x_1^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\qquad\qquad x - y = 0$

$\lambda = -1 \qquad x + y = 0 \Rightarrow = x - y \qquad x_1^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

for $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \lambda_{1,2} = \pm 1$

$\lambda = 1 \qquad x_2^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -1 \qquad x_2^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

for $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \lambda_{1,2} = \pm 1$

$\lambda = 1 \Rightarrow \qquad x_3^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1 \qquad x_3^{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$[\sigma_i, \sigma_j] = 2i\,\varepsilon_{ijk}\sigma_k$$

II-4 Born Approximation

(a) $V(r) = V_0 \, e^{-\mu r}/r$

$$f(\theta) \cong \frac{-2m}{\hbar^2} \int_0^\infty \frac{r^2 V(r) \sin \lambda r}{\lambda r} \cdot dr$$

$$f(\theta) \cong \frac{-2mV_0}{\hbar^2 \lambda} \int_0^\infty e^{-\mu r} \cdot \sin \lambda r \cdot dr$$

$$\therefore f(\theta) \cong \frac{-2mV_0}{\hbar^2} \frac{1}{\mu^2 + \lambda^2}$$

#

Born Approximation

(b) $V(r) = V_0 \,, \quad r \leq a$

$\qquad\qquad 0 \,, \quad r > a$

$$f(\theta) \cong -\frac{2m}{\hbar^2} \int_0^\infty \frac{r^2 V(r) \sin \kappa r}{\kappa r} \, dr$$

$$= -\frac{2mV_0}{\hbar^2 \kappa} \int_0^a r \cdot \sin \kappa r \cdot dr$$

$$= -\frac{2mV_0}{\hbar^2 \kappa} \left[\frac{\sin \kappa r}{\kappa^2} - \frac{r \cos \kappa r}{\kappa} \right]_0^a$$

$$f(\theta) = -\frac{2mV_0}{\hbar^2 \kappa} \left[\frac{\sin \kappa a}{\kappa^2} - \frac{a \cos \kappa a}{\kappa} \right]$$

FALL 2006 : $\boxed{\text{II}-5}$

Eigenvalues: $E = c,\ (a+b),\ (a-b)$

Eigenvectors: $|s_1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |s_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$|s_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(a) $|s(t)\rangle = e^{-iE_1 t/\hbar}\,|s_1\rangle = e^{-ict/\hbar}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

(b) $|s(0)\rangle = \frac{1}{\sqrt{2}}\left(|s_2\rangle + |s_3\rangle\right)$

$$|s(t)\rangle = e^{-iat/\hbar}\begin{pmatrix} \cos(bt/\hbar) \\ 0 \\ -i\sin(bt/\hbar) \end{pmatrix}$$

[II-6] Solution: (Perturbation theory-time depe

The field varies as $\mathcal{E}(t) \sim \frac{1}{t^2+\tau^2}$ and w
use this in $P = \int_{-\infty}^{\infty} e\,\mathcal{E}(t)dt = \text{const}$, l
to the following change of the field:

$$\mathcal{E}(t) = \frac{P}{e}\frac{\tau}{\pi}\frac{1}{\tau^2+t^2}$$

P is the total pulse, which is classically transferred to the oscillator by the electr field over the duration of the perturbati

$$P = \int_{-\infty}^{\infty} e\,\mathcal{E}(t)dt = eA\int_{-\infty}^{\infty}\frac{1}{t^2+\tau^2}dt = \text{constant}$$

The probability of a transition from the nth state (stationary state) of the discrete spectr to the kth. is equal to

$$w_{nk} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} V_{kn}\, e^{i(\omega_{kn}t)}dt\right|^2$$

where:

$$V_{kn} = \int_{-\infty}^{\infty}\psi_k^{(0)*}\hat{V}\,\psi_n^{(0)}dx$$

is the matrix element of the perturbation $\hat{V}$, $w_{kn} = \frac{1}{\hbar}[E_k^{(0)} - E_n^{(0)}]$

[II-6] solution- continued:

where $\psi_k^{(0)}$, $\psi_n^{(0)}$, $E_k^{(0)}$, $E_n^{(0)}$ are the wave functions and energy levels of the corresponding (unperturbed) stationary states. If we denote be e, μ and ω the charge, the mass, and the eigenfrequency of the oscillator, and by x its displacement from its equilibrium position we obtain under consideration of a uniform field for the perturbation operator,

$$\hat{V}(x,t) = -e x \mathcal{E}(t) \sim x$$

It is well known that in the matrix of the coordinate of the oscillator in the energy representation only the following matrix elements are different from zero,

$$x_{n,n+1} = x_{n+1,n} = \sqrt{[(n+1)\hbar/2\mu\omega]}.$$

Since we assumed that the oscillator was originally in its ground state ($n=0$), we are dealing with only the following non-vanishing matrix elements of the perturbation:

$$V_{01} = V_{10} = -P\frac{\tau}{\pi}\sqrt{\left(\frac{\hbar}{2\mu\omega}\right)}\frac{1}{\tau^2+t^2},$$

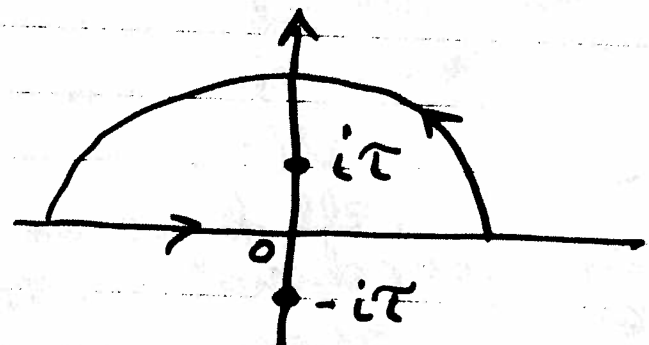
so we obtain for the probability of a

[II-6] continued.

transition to the first excited state

$$w_{01} = \frac{P^2\tau^2}{2\pi^2\mu\hbar\omega}\left|\int_{-\infty}^{\infty}\frac{e^{(i\omega t)}}{t^2+\tau^2}dt\right|^2.$$

Using the theory of residues one can evalu the above integral. Use the variable, in the complex plane and use the followi contour:



We close the circuit, which starts along the real axis corresponding to the integration in the above expressi by a semicircle of $R\to\infty$ in the upper half plane. the only singularity (a pol of first order) of the integrand insid the contour is at $t=+i\tau$.

$$\therefore \int_{-\infty}^{\infty}\frac{e^{i\omega t}}{t^2+\tau^2}dt = 2\pi i \,\text{Residue}\left(\frac{e^{i\omega t}}{t^2+\tau^2}\right|_{t=i\tau}$$

$$= 2\pi i\frac{e^{-\omega\tau}}{2i\tau} = \frac{\pi}{\tau}e^{-\omega\tau} \quad \text{and}$$

$$w_{01} = (P^2/2\mu\omega)\,e^{-2\omega\tau}.$$

[II-f] Solution (Thermo/Stat. Mech.)

(a) The rubber band heats up when stretched. Assume constant "n". So,

$$\left(\frac{\partial T}{\partial L}\right)_{s,n} > 0 \text{ or equivalently}$$

$$\left(\frac{\partial T}{\partial f}\right)_{s,n} > 0$$

with f = tension,

Note, these two derivatives have the same sign since

$$\left(\frac{\partial T}{\partial f}\right)_{s,n} = \left(\frac{\partial T}{\partial L}\right)_{s,n}\left(\frac{\partial L}{\partial f}\right)_{s,n}$$

and the second derivative is positive by stability.

To find the sign of $\left(\frac{\partial L}{\partial T}\right)_{f,n}$, we write

$$\left(\frac{\partial L}{\partial T}\right)_{f,n} = \left(\frac{\partial L}{\partial s}\right)_{f,n}\left(\frac{\partial s}{\partial T}\right)_{f,n} = -\underbrace{\left(\frac{\partial T}{\partial f}\right)_{s}}_{>0}\underbrace{\left(\frac{\partial s}{\partial T}\right)_{f}}_{>0}$$

$d(E-fL) = T\,ds - L\,df + \cdots$,

implies $\left(\frac{\partial L}{\partial s}\right)_{f,n} = -\left(\frac{\partial T}{\partial f}\right)_{s,n}$

Thus, $\left(\frac{\partial L}{\partial T}\right)_{f,n} < 0$

or, the rubber band stretches when cooled.

[II-7] continued.

(b) $\left(\frac{\partial S}{\partial T}\right)_f = \left(\frac{\partial S}{\partial T}\right)_L \left(\frac{\partial T}{\partial T}\right)_f + \left(\frac{\partial S}{\partial L}\right)_T \left(\frac{\partial L}{\partial T}\right)_f$

$$\left(\frac{\partial S}{\partial T}\right)_f - \left(\frac{\partial S}{\partial T}\right)_L = \left(\frac{\partial S}{\partial T}\right)_T \left[-\left(\frac{\partial f}{\partial T}\right)_L \left(\frac{\partial L}{\partial f}\right)_T\right]$$

note: $\left[d(E-TS) = -S\,dT + f\,dL + \cdots\right.$ implies $\left.\left(\frac{\partial f}{\partial T}\right)_L = -\left(\frac{\partial S}{\partial L}\right)_T\right]$

hence

$$\left(\frac{\partial S}{\partial T}\right)_f - \left(\frac{\partial S}{\partial T}\right)_L = \underbrace{\left(\frac{\partial S}{\partial L}\right)_T^2}_{>0} \underbrace{\left(\frac{\partial L}{\partial f}\right)_T}_{>0}$$ by stability

$\therefore\ C_f - C_L > 0$ and $\frac{1}{C_L} > \frac{1}{C_f}$.

$\therefore$ The constant length rubber band has the largest change in temperature.

$\boxed{\text{II - 8}}$ $\mu(T,P) = -k_B T \ln\left(a \frac{T^{5/2}}{P}\right)$

$G(T,P,N) = N\mu(T,P)$ Gibbs free energy

$S = -\left(\frac{\partial G}{\partial T}\right)_{P,N}$ $V = \left(\frac{\partial G}{\partial P}\right)_{T,N}$

Thus, $G(T,P,N) = -Nk_B T \ln\left(a \frac{T^{5/2}}{P}\right)$

a) $V = \left(\frac{\partial G}{\partial P}\right)_{T,N} = \frac{Nk_B T}{P}$ $\Rightarrow$ $\boxed{PV = Nk_B T}$

b) $G = E - TS + PV$
$E = G + TS - PV$

$S = -\left(\frac{\partial G}{\partial T}\right)_{P,N} = \frac{\partial}{\partial T}\left\{ Nk_B T \left(\ln(a) + \frac{5}{2}\ln(T) - \ln(P)\right)\right\}$

$= Nk_B \left\{\ln(a) + \frac{5}{2}\ln(T) - \ln(P)\right\} + \frac{5}{2}Nk_B$

$E = G + TS - PV = -Nk_B T\left\{\ln(a) + \frac{5}{2}\ln(T) - \ln(P)\right\}$
$+ Nk_B T\left\{\ln(a) + \frac{5}{2}\ln(T) - \ln(P)\right\} + \frac{5}{2}Nk_B T$
$- Nk_B T$ =

$= \frac{3}{2}Nk_B T$

$\Rightarrow$ $\boxed{E = \frac{3}{2}Nk_B T}$

c) Using the expression from (b) we can express S in terms of E, V, N:

$$S = Nk_B \ln(a) + \frac{5}{2} Nk_B \ln(T) - Nk_B \ln(p) + \frac{5}{2} Nk_B$$

$$= Nk_B \ln(a) + \frac{5}{2} Nk_B \ln\left(\frac{2}{3}\frac{E}{Nk_B}\right) - Nk_B \ln\left(\frac{Nk_B T}{V}\right) + \frac{5}{2} Nk_B$$

$$= Nk_B \ln(a) + \frac{5}{2} Nk_B \ln\left(\frac{2}{3}\frac{E}{Nk_B}\right) - Nk_B \ln\left(\frac{N}{V}\right) - Nk_B \ln(k_B)$$

$$- Nk_B \ln\left(\frac{2}{3}\frac{E}{Nk_B}\right) + \frac{5}{2} Nk_B$$

$$= Nk_B \ln(a) + \frac{3}{2} Nk_B \ln\left(\frac{2}{3}\frac{E}{Nk_B}\right) + Nk_B \ln\left(\frac{V}{N}\right)$$

$$+ \frac{5}{2} Nk_B - Nk_B \ln(k_B)$$

$$= Nk_B \ln(a) + \frac{3}{2} Nk_B \ln\left(\frac{2}{3}\frac{1}{k_B}\right) - Nk \ln(k_B)$$

$$+ \frac{3}{2} Nk_B \ln\left(\frac{E}{N}\right) + Nk_B \ln\left(\frac{V}{N}\right) + \frac{5}{2} Nk_B$$

$$= Nk_B \left\{\ln(a) + \frac{3}{2} \ln\left(\frac{2}{3}\right) - \frac{5}{2} \ln(k_B)\right\}$$

$$+ \frac{3}{2} Nk_B \ln\left(\frac{E}{N}\right) + Nk_B \ln\left(\frac{V}{N}\right) + \frac{5}{2} Nk_B$$

$$\rightarrow \boxed{S(E,V,N) = \text{Const.} + \frac{3}{2} Nk_B \ln\left(\frac{E}{N}\right) + Nk_B \ln\left(\frac{V}{N}\right) + \frac{5}{2} Nk_B}$$

Solution 1. September 2006

II-9 Classical Statistical Mechanics

A small, just barely visible, dust particle has a mass of about 10^{-8} g. It falls onto a glass of ice cold water where it is supported by surface tension, and moves freely in only two dimensions.

- What is the average translational energy of the dust particle?
- What is the root-mean-squared velocity of its Brownian motion?

average translational energy

$\rightarrow$ 2-dimensional movement, 2 degrees of freedom

$$E_{trans} = 2 \cdot \frac{1}{2} kT = kT = (1.38 \cdot 10^{-23} \cdot 273)$$
$$= 3.77 \cdot 10^{-21} J$$

root-mean-squared velocity in 2-dim.

$$\frac{1}{2} m v^2 = E_{trans} = kT$$

$$v^2 = \frac{2kT}{m}$$

$$v = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2 \cdot 3.77 \cdot 10^{-21}}{10^{-11}} \frac{m}{s}} = \sqrt{7.54 \cdot 10^{-10}}$$

$m = 10^{-8}$ g $= 10^{-11}$ kg

"icecold" $T = 273$ K

$$\boxed{k = 1.38 \cdot 10^{-23} J/K}$$

$v = 2.75 \times 10^{-5} \frac{m}{s}$

[II-10] Solution (Quantum-Stat. Mech.)

(a) It is straightforward to see that for a Bose gas in two dimensions

$$N_e = \int_0^\infty \frac{1}{z^{-1}e^{\beta\epsilon}-1}\,\frac{A\cdot 2\pi p\,dp}{h^2} =$$

$$= \frac{A\cdot 2\pi m k T}{h^2}\int_0^\infty \frac{dx}{z^{-1}e^{x}-1} = \frac{A}{\lambda^2}\,g_1(z)$$

while $N_0 = \frac{z}{1-z}$, Now since Bose-Einst. condensation requires that $z \to 1$, the critical temperature T_c, by the usual argument, is given by:

$$\left(\frac{N}{A}\right)\lambda_c^2 = g_1(1) = \infty$$

[for $g_1(z) = -\ln(1-z)$]. It follows now that $T_c = 0$.

(b) More accurately, the phenomenon of condensation requires both N_e and N_0 be of order N. This means that, while $z \simeq 1$, $(1-z)$ be of order N^{-1} and hence λ^2 be of order $(A \ln N/N)$. Since the ratio $(A/N) \sim l^2$, [II-10] Solution–Continued

the condition for condensation takes the form $(\lambda^2/\ell^2) = \bar{O}(\ln N)$.

It follows that

$$T \equiv \frac{h^2}{2m\pi k_B \lambda^2} \sim \frac{h^2}{m k_B \ell^2} \frac{1}{\ln N} .$$