Qualifying Examination - Part I

Time: 9:30-11:30 a.m. **Saturday, February 12, 2005**

The examination is to be written on the single sheets that are provided; no more than one question is to be answered on a single sheet. Number each question. You are to answer all questions in Part I; however, if you do omit any questions, ***cross out those numbers on your title page***. When you are finished, collect the answer sheets in order and place them together (with the title page on top) back in the envelope.

Place your code letter (from your title page) on the back of each sheet of paper.

Part I counts one-third (1/3) of the final grade.
Part II is in this same room at 1:00 p.m.

PHYSICAL CONSTANTS

Planck's constant	$h = 6.63 \times 10^{-34}$ J•s $= 4.14 \times 10^{-15}$ eV•s
Vacuum speed of light	$c = 3.00 \times 10^{8}$ m/sec
	$\hbar c = 197$ MeV•fm $= 1.97 \times 10^{-5}$ eV•cm
Electron charge	$e = 1.60 \times 10^{-19}$ C $= 4.80 \times 10^{-10}$ esu
Boltzmann constant	$k = 1.38 \times 10^{-23}$ J/K $= 8.62 \times 10^{-5}$ eV/K
Gas constant	$R = 8.31$ J/(mol•K)
Gravitational constant	$G = 6.67 \times 10^{-11}$ N•m^2/kg^2
Permeability of free space	$\mu_0 = 4\pi \times 10^{-7}$ H/m
Permittivity of free space	$\epsilon_0 = 8.85 \times 10^{-12}$ F/m
Electron mass	$m_e = 9.11 \times 10^{-31}$ kg $= 0.511$ MeV/c^2
Proton mass	$m_p = 1.67 \times 10^{-27}$ kg $= 938$ MeV/c^2
Bohr radius of hydrogen	$a_B = 5.3 \times 10^{-11}$ m
Ionization energy of hydrogen	13.6 eV
Avogadro's number	6.02×10^{23} / mole

Conversion Factors

1 eV $= 1.6 \times 10^{-19}$ J $= 1.6 \times 10^{-12}$ erg

1 m $= 10^{10}$ Å$= 10^{15}$ fm $= 6.25 \times 10^{-4}$ miles

1 atm $= 1.01 \times 10^{5}$ N/m^2 $= 760$ Torr

1 cal $= 4.186$ J

Divergence and curl in spherical coordinates

$$\nabla \cdot E = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\varphi}{\partial\varphi}$$

$$\nabla \times E = r\frac{1}{r sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\varphi) - \frac{\partial E_\theta}{\partial\varphi}\right] + \theta\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\varphi} - \frac{\partial(rE_\varphi)}{\partial r}\right] + \varphi\frac{1}{r}\left[\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta}\right]$$

where r, θ, φ are the unit vectors associated with the spherical coordinates r, θ, φ.

Useful integrals

$$\int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty xe^{-\alpha x^2}\,dx = \frac{1}{2\alpha}$$

$$\int_0^\infty x^2e^{-\alpha x^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}}$$

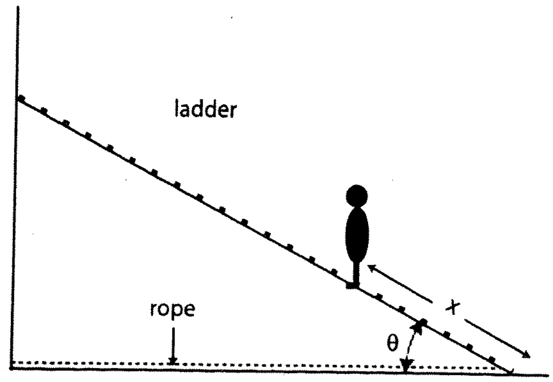
$$\int_0^x \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$\Sigma_{n=0}^{\infty} x^n = \frac{1}{(1-x)};\ -\Sigma_{n=1}^{\infty}(-)^n\frac{x^n}{n} = \ell n\ (1+x)$$

Spring 2005 Qualifier **7-Minute Questions**

1. A person of mass 50 kg is climbing up a 5 m long, 20 kg ladder shown on the right. Assume that the ladder rests on a frictionless wall and floor, and that the base of the ladder is held in place by a rope attached to the wall. The diameter of the rope is 3 mm and the breaking point stress of the material is $10^8\,\mathrm{N/m^2}$. The ladder makes an angle $\theta = 30°$ with respect to the floor.

   How far can the person climb up the ladder before the rope breaks (i.e., find the maximum x).



2. Suppose the current density in a copper wire is $1000\,\mathrm{A/cm^2}$. Calculate the average drift velocity of the electrons assuming there is one conduction electron per copper atom.

   Here is some information about copper: density $8.9\,\mathrm{g/cm^3}$; atomic number 29; atomic weight 63.5; resistivity $1.7 \times 10^{-8}\,\Omega{\cdot}\mathrm{m}$.

3. Two speakers driven in phase are playing a steady tone. You are 4 m from one speaker and 4.5 m from the other. Find the frequencies (the two lowest ones) at which the sound intensity you hear is minimized. Take the speed of sound to be 330 m/s.

4. 1.5 moles of water are vaporized at a temperature 100°C and a pressure of 1 atm. Find the change in the internal energy of the water in this process.

   The molar volumes of water and water vapor are $V_{\mathrm{water}} = 18.8\,\mathrm{cm^3/mol}$ and $V_{\mathrm{vapor}} = 3.01 \times 10^4\,\mathrm{cm^3/mol}$, respectively. The latent heat of vaporization is $4.06 \times 10^4\,\mathrm{J/mol}$.

5. A particle of mass m is in a stable circular orbit with a potential energy

$$V(r) = -cr^{-\lambda},$$

   where c and λ are positive constants.

   (a) Find the radial force that acts on the particle.

   (b) Derive a relation between the angular momentum of the particle and the radius, r_0 of the circular orbit.

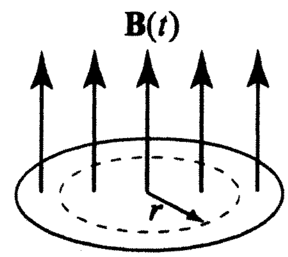
6. A wave group travels along the x-axis in a transparent but dispersive medium. The wave numbers are centered at k_0 with width $\Delta k << k_0$.

   (a) What, roughly, is the minimum spatial size of the group?

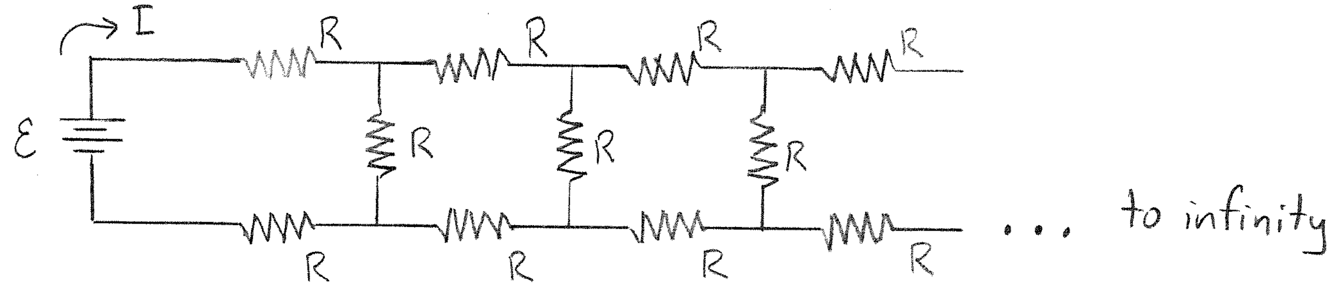
   (b) For light propagating in glass with "normal" dispersion, the index of refraction increases with frequency. Do the individual Fourier components of the group move faster or slower than the group envelope? Explain

   (c) For gravity waves in deep water, the dispersion relation is $\omega = \sqrt{gk}$. What is the ratio of phase to group velocity in this case?

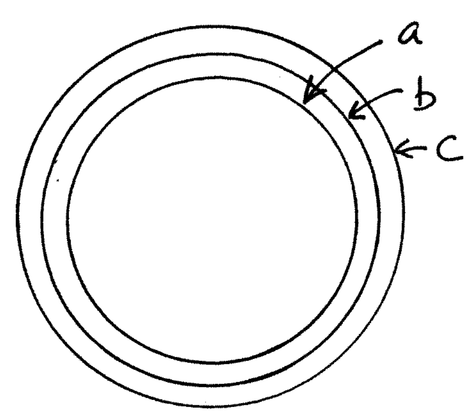
7. A uniform magnetic field, $B(t) = \hat{z} A \cos \omega t$, pointing upward in the drawing, fills the circular region of the figure. Find the magnitude and direction of the induced electric field as a function of time and the radial coordinate r. A and ω are both constants.



8. The circuit shown below consists of a network of identical resistors. Find the current I in terms of the EMF $\mathcal{E}$ and the resistance R.



9. Three concentric spheres of radii a, b and c are all perfect blackbodies. The inner sphere is held at temperature T_a while the outer sphere is held at temperature T_c. Find the equilibrium temperature of the middle sphere for the situation in which c is only slightly larger than b, and b is only slightly larger than a.



10. For each of the functions listed below, indicate whether ψ is a solution to the time-independent Schrodinger equation for a free particle ($V = 0$) in one dimension. Justify your answers. Assume that $A \neq B$ and $a \neq b$.

   (a) $\psi = Ae^{iax}$.

   (b) $\psi = A[e^{iax} + e^{-ibx}]$.

   (c) $\psi = Ae^{iax} + Be^{-iax}$.

   (d) $\psi = Ae^{iax} + Be^{ibx}$.

Qualifying Examination - Part II

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Part II counts two-thirds (2/3) of the final grade.

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$$+ \varphi\frac{1}{r}\left[\frac{\partial(rE_\theta)}{\partial r} - \frac{\partial E_r}{\partial\theta}\right]$$

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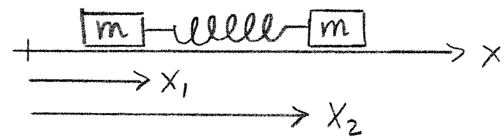
$$\int_0^x \sin^2 x\,dx = \frac{x}{2} - \frac{1}{4}\sin 2x$$

$$\Sigma_{n=0}^{\infty} x^n = \frac{1}{(1-x)}; \quad -\Sigma_{n=1}^{\infty} (-)^n \frac{x^n}{n} = \ell n\,(1+x)$$

11. Two equal masses m are free to slide along the x-axis and are connected by a massless spring of constant k and relaxed length L. At time $t = 0$ the left mass is suddenly given a velocity v_0.

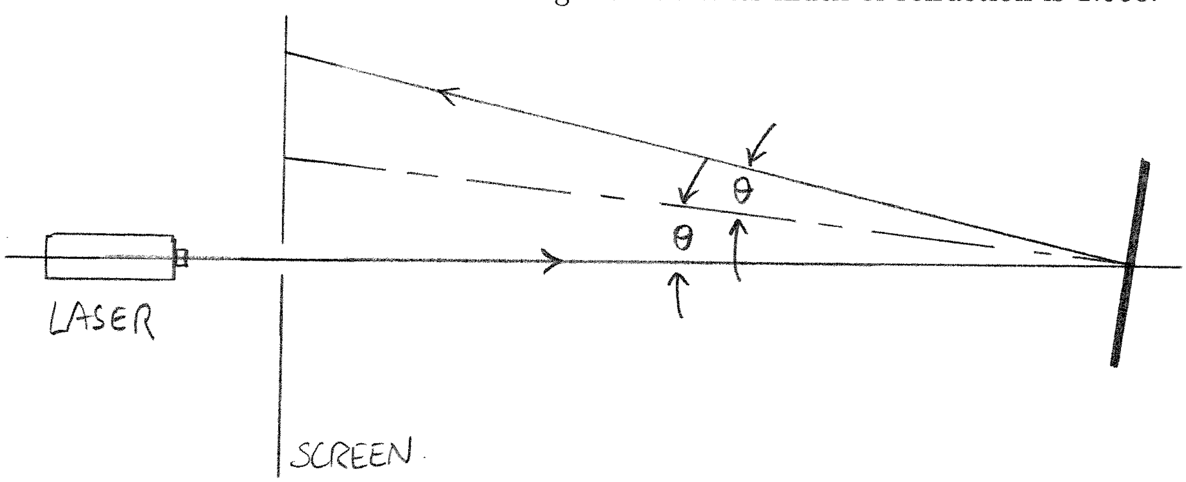


(a) Write down the Lagrangian of the system in terms of the coordinates x_1 and x_2.

(b) Identify two constants of the motion.

(c) Find the equations of motion and solve them for $t > 0$.

12. Laser light of wavelength 630 nm is reflected from a thin plane-parallel glass slab onto a screen where interference maxima and minima of the reflected spot are observed as the slab is tilted with respect to the incident laser beam. An interference maximum (not necessarily the first maximum) is seen when the slab is tilted 3.00° from normal incidence. The tilt angle is increased to the next order of constructive interference, which is found to occur at 4.50°. Calculate the thickness of the glass slab if its index of refraction is 1.500.



13. Spin $\frac{1}{2}$ particles of mass m are confined in a 3-dimensional infinite square well potential (width a in each dimension).

(a) Write down a general formula for the normalized wavefunctions and the corresponding energy eigenvalues for a single particle in the well.

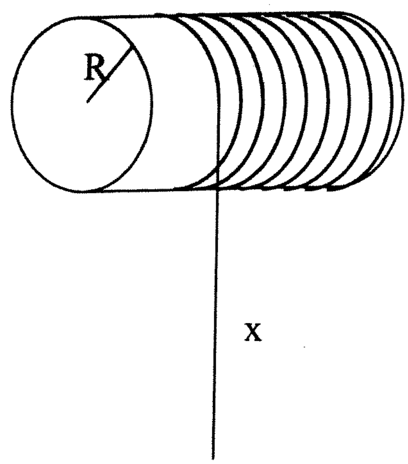
(b) Give the degeneracy of each of the three lowest energy levels.

(c) Write down the wave function (or wave functions) for the ground state of two non-interacting particles in the well. What is the degeneracy of this state?

(d) Repeat (c) for the first excited state of the two-particle system.

14. A homogeneous sphere of radius R, made of material with dielectric constant κ, is placed in an otherwise uniform electric field $\vec{E} = E_0 \hat{z}$. Find an expression for the electric field inside the sphere.

15. In the drawing shown at the right, a cable of length L and total mass M is wound onto a cylinder of radius R. The moment of inertia of the cylinder - not including the cable - is I. Let x represent the length of cable that is unspooled and exposed to the gravitational force. The spool is initially at rest, with a length x_0 unspooled.



    (a) Determine the initial acceleration ($\ddot{x}$ or $\ddot{\theta}$ if you prefer) of the system.

    (b) Find the equation of motion.

    (c) Find x as a function of time (assuming that the cable is not yet completely unspooled).

16. A uniform thin insulating disk of mass M and radius b rotates counter-clockwise about the z-axis with angular velocity ω. A total charge $+Q$ is uniformly distributed on the surface.

    (a) What is the magnitude and direction of the magnetic moment $\vec{\mu}$?

    (b) Now suppose that a weak, uniform magnetic field, B_0, is applied along the y-axis. Determine the magnitude and the direction of the resulting precession, $\vec{\Omega}$.

17. Extremely relativistic particles have a momentum p such that $pc >> mc^2$ (where m is the rest mass of the particles). Show that an ideal gas of such particles has a mean energy of $3kT$ per particle, in contrast to $\frac{3}{2}kT$ for nonrelativistic particles.

18. Positrons of total relativistic energy E collide with electrons at rest. Find the minimum energy at which the reaction $e^+ \, e^- \to \mu^+ \, \mu^-$ can occur. The muon mass is $m_\mu = 105\,\mathrm{MeV/c}$.

19. Assume that we produce radioactive ${}^{13}_{7}\mathrm{N}$, whose half-life is 10 min, by bombarding a 0.2 mm thick foil of isotopically pure ${}^{10}_{5}\mathrm{B}$ (boron-10) with a $2\,\mu\mathrm{A}$ beam of alpha particles. The density of boron is $3.3\,\mathrm{g/cm^3}$

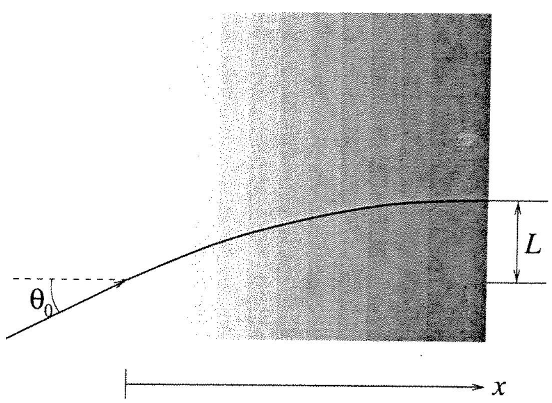
   (a) Write the above reaction in the form X(x,y)Y.

   (b) Find the production rate of ${}^{13}\mathrm{N}$ if the reaction cross section is $400\,\mathrm{mb}$, where $1\,\mathrm{b} = 10^{-28}\,\mathrm{m}^2$.

20. A basic principle of statistical physics states that the most probable macroscopic configuration of a system in equilibrium is the one that is observed. Use this principle to show that two systems in thermodynamic equilibrium with each other, and isolated from everything else, will have the same temperature ($\frac{1}{T} \equiv \frac{\partial S}{\partial E}$).

21. Consider a medium whose index of refraction is given by

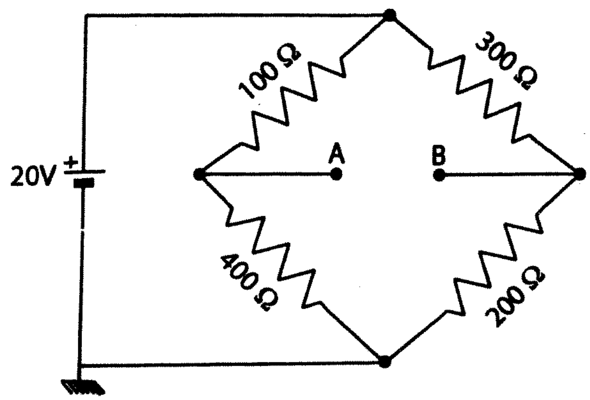
$$n(x) = \begin{cases} 1 & \text{for } x < 0 \\ e^{ax} & \text{for } x > 0 \end{cases}$$



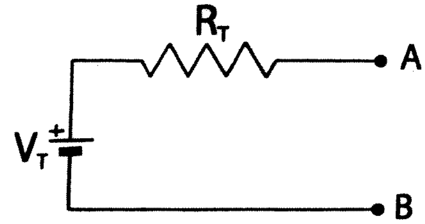
Light is incident on this medium at an angle θ_0. Recalling Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$ determine the distance L which this light will travel in the vertical direction while traveling from $x = 0$ to $x = \infty$ horizontally. You may find one of the following integrals to be useful:

$$\int_1^\infty \frac{b\,dz}{z\sqrt{z^2 - b^2}} = \sin^{-1} b \qquad\qquad \int \frac{dz}{\sqrt{a^2 - z^2}} = \sin^{-1}\frac{z}{a}$$

22. The bridge circuit shown below is composed of four resistors and an ideal voltage source.



Find the Thevenin equivalent circuit. (For those who don't know the terminology, the problem is to find appropriate values for V_T and R_T so that the simple circuit at the right is equivalent to the original circuit.)



23. Consider an atom with nuclear spin I in a level with angular momentum J. The hyperfine interaction couples $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$ to give a total angular momentum operator $\hat{\mathbf{F}} = \hat{\mathbf{I}} + \hat{\mathbf{J}}$.

(a) What are the maximum and minimum possible values of $\langle \hat{\mathbf{F}}^2 \rangle$ for $I = \frac{3}{2}$ and $J = \frac{1}{2}$?

(b) The Hamiltonian describing the Zeeman shifts of the hyperfine levels due to an external magnetic field $\vec{B}$ is

$$H = -g_I' \mu_B \hat{\mathbf{I}} \cdot \vec{B} + g_J \mu_B \hat{\mathbf{J}} \cdot \vec{B}$$

where g_I' and g_J are the nuclear and electronic g-factors and μ_B is the Bohr magneton. With m_F the projection of F along the direction of the applied magnetic field, show that the Zeeman shifts are given by $\Delta E = g_F \mu_B B m_F$ and find an expression for g_F.

24. Consider the problem of electron spin precession in a constant magnetic field $\vec{B} = B\hat{z}$. Find the expectation value of the x-component of the spin as a function of time, assuming that the wave function at time $t=0$ is

$$|\psi(t=0)\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + |\downarrow\rangle \right).$$

Here $|\uparrow\rangle$ and $|\downarrow\rangle$ are eigenstates of σ_z, $\sigma_z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$. The Hamiltonian of the system is $H = -\vec{\mu}\cdot\vec{B} = \mu_B B \sigma_z$, and the Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

25. (a) Draw energy level diagrams for n-type and for p-type semiconductors. The diagrams should show the band edges of the valence band (VB) and the conduction (CB) and the energy level of the dopants (D for donor, A for acceptor). Finally, show the Fermi level E_F, assuming that the temperature is low (essentially zero). Separate diagrams should be given for n-type and p-type.

(b) Draw a similar diagram (band edges, dopant levels, Fermi energy) for a junction between a p-type and n-type semiconductor. Assume zero external bias and low temperature. Which way is charge transferred across the junction when the junction is formed?

(c) Describe how the Fermi level moves as the temperature is increased.