University of Illinois at Chicago

Department of Physics

Electricity and Magnetism

Qualifying Examination with Solutions

January 2015

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

Various equations, standard integrals, etc. are provided on the last page of the exam.

1. Boundary value

The electrostatic potential on a spherical surface of radius R is given by

$$\Phi(R,\theta,\phi) = V_0 + V_1 \sin\theta\cos\phi + V_2\cos 2\theta.$$

where θ and ϕ are the polar and azimuthal angles in the spherical coordinate system with the origin at the center of the sphere. There are no charges outside of this spherical surface.

(a) Find the potential $\Phi(r,\theta,\phi)$ outside of the spherical surface as a function of the distance r from the center and the spherical angles θ and ϕ.

Solution: Note that the boundary condition is a linear combination of $Y_{00} =$ const, $Y_{1,\pm1} \sim \sin\theta\cos\phi$ and $Y_{20} \sim 3\cos^2\theta - 1$. To see this for the last term one needs to write

$$\cos 2\theta = 2\cos^2\theta - 1 = \frac{2}{3}(3\cos^2\theta - 1) - \frac{1}{3} \sim Y_{20} + \text{const} \tag{1}$$

Thus we can seek solution as a linear combination

$$\Phi(r,\theta,\phi) = \frac{AR}{r} + \frac{BR^2\sin\theta\cos\phi}{r^2} + \frac{CR^3(3\cos^2\theta - 1)}{r^3} \tag{2}$$

Comparing to the boundary condition and using Eq. (1) we find the coefficients:

$$A = V_0 - \frac{1}{3}V_2; \quad B = V_1; \quad C = \frac{2V_2}{3} \tag{3}$$

$$\Phi(r,\theta,\phi) = \left(V_0 - \frac{1}{3}V_2\right)\frac{R}{r} + \frac{V_1R^2\sin\theta\cos\phi}{r^2} + \frac{2V_2R^3(3\cos^2\theta - 1)}{3r^3} \tag{4}$$

(b) Find the total charge Q inside or on the spherical surface.

Solution: By Gauss law only the Y_{00} term contributes and the total charge is given by

$$Q = 4\pi\varepsilon_0\left(V_0 - \frac{1}{3}V_2\right)R$$

(c) Now consider a different problem: a thin sheet lying in xy plane at $z = 0$ and carrying surface charge density $\sigma = \sigma_0\sin(kx)$. There are no other charges.

Find the potential and the electric field at all points $z > 0$.

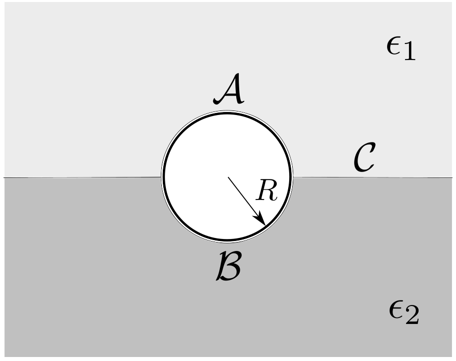
Solution:

Substitute $\Phi = Z(z)\sin(kx)$ into Laplace equation. Find $Z'' - k^2 Z = 0$. Conclude that $Z = A\exp(-k|z|)$ to avoid diverging Φ at infinity.

To find A, calculate $\boldsymbol{E}$: $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi = Ake^{-kz}(\hat{\boldsymbol{z}}\sin(kx) - \hat{\boldsymbol{x}}\cos(kx))$ and use Gauss law: $\sigma/\epsilon_0 = [E_n] = 2Ak\sin(kx)$. Thus $A = \sigma_0/(2\epsilon_0 k)$.

Thus $\Phi = \dfrac{\sigma_0}{2\epsilon_0 k}e^{-kz}\sin(kz)$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi = \dfrac{\sigma_0}{2\epsilon_0}e^{-kz}(\hat{\boldsymbol{z}}\sin(kx) - \hat{\boldsymbol{x}}\cos(kx))$.

2. Conducting sphere surrounded by dielectric



The center of a conducting sphere of radius R is located on the flat boundary between two dielectrics each filling half of the whole space outside the sphere. The dielectric permittivities are ε_1 and ε_2. The conducting sphere is held at potential V. Consider the space outside of the conducting sphere.

(a) Show that the potential $\Phi = VR/r$ satisfies the required boundary conditions on the plane $\mathcal{C}$ separating dielectrics as well as on the sphere.

Solution:

$\boldsymbol{D} = \varepsilon_{1,2}\boldsymbol{E}$ and $\boldsymbol{E} = -\boldsymbol{\nabla}\Phi = VR\,\hat{\boldsymbol{r}}/r^2$.

On the plane: $(D_1)_n = (D_2)_n = 0$ and $(E_1)_t = (E_2)_t = VR/r^2$.

On the sphere: $\Phi|_{r=R} = V$.

(b) Find the free charge density σ on the surface of the conducting sphere and the total amount of free charge Q on it.

Solution: $\sigma = D_r = \varepsilon_{1,2}V/R$.

Total charge $Q = 2\pi R^2(\sigma_1 + \sigma_2) = 2\pi(\varepsilon_1 + \varepsilon_2)VR$.

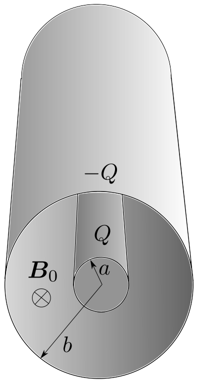
(c) Find the bound charge densities σ_b on the spherical boundaries $\mathcal{A}$ and $\mathcal{B}$ of the dielectrics.

Solution: $\sigma_b|_{\mathcal{A},\mathcal{B}} = -P_r = (\varepsilon_0 E - D)_r = (\varepsilon_0 - \varepsilon_{1,2})E_r = -(\varepsilon_{1,2} - \varepsilon_0)V/R$.

(d) Find the bound charge density σ_b on the flat boundary $\mathcal{C}$ between the dielectrics.

Solution: ($[\ldots]$ denotes discontinuity) $\sigma_b|_{\mathcal{C}} = -[P_n] = [\varepsilon_0 E_n] = 0$.

3. Cylindrical capacitor in a magnetic field



A capacitor is made out of two concentric conducting cylindrical surfaces of radii a and $b > a$. The charge on the inner conductor is $Q > 0$ and on the outer $-Q$. Uniform external magnetic field is applied in the direction of the axis of the cylinders. The magnetic flux density $\boldsymbol{B}$ is increasing with time from 0 to $\boldsymbol{B}_0$ along the axis of the cylinder. The time dependence of its magnitude is given: $B(t)$.
The magnetic field created by the currents on the cylinders is negligible.

(a) Find the magnitude of the torque experienced by the capacitor at time t and describe its direction relative to the direction of the magnetic field $\boldsymbol{B}_0$ (same or opposite).

Solution: The electromotive force per unit area acting on the outer cylinder is given by $-(Q/A)E_{\mathrm{emf}}(b)$, where A is the area and $E_{\mathrm{emf}}(b)$ is given by the Faraday's law:

$$E_{\mathrm{emf}}(b)\,2\pi b = -\frac{dB}{dt}\,\pi b^2. \tag{5}$$

The torque per unit area is $-(Q/A)E_{\mathrm{emf}}(b)b$ and the total torque on the outer cylinder is given by

$$\tau_b = -A(Q/A)E_{\mathrm{emf}}(b)b = QE_{\mathrm{emf}}(b)b = \frac{Qb^2}{2}\frac{dB}{dt}. \tag{6}$$

The torque on the inner cylinder is obtained by replacing $Q \to -Q$ and $b \to a$, and the total torque on the whole capacitor is given by

$$\tau = \tau_b + \tau_a = \frac{Q}{2}(b^2 - a^2)\frac{dB}{dt}. \tag{7}$$

The direction can be determined using Lenz's law: The magnetic field created by the Faraday currents is opposite $\boldsymbol{B}_0$, thus the current in the outer cylinder is anti-clockwise and so is the E_{emf}. Since the charge on the outer cylinder is negative, the force on the charge is clockwise and thus the torque is into the page – the same as $\boldsymbol{B}_0$.

(b) Find the magnitude of the total angular momentum that the capacitor receives as a result by the time the magnitude of the magnetic field reaches B_0 and compare with the magnitude of the total angular momentum of the electromagnetic field using the fact that the field carries the momentum density $\boldsymbol{\Pi} = \boldsymbol{S}/c^2 = \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}$. Compare the directions of these angular momenta.

Solution:

Angular momentum from torque (along $\boldsymbol{B}_0$):

$$L = \int \tau dt = \frac{Q}{2}(b^2 - a^2)B_0 \tag{8}$$

The angular momentum density is given by $\boldsymbol{r} \times \boldsymbol{\Pi}$ integrated over the volume of the capacitor. Electric field $\boldsymbol{E}$ is along the radial direction an the magnitude is determined by the Gauss law:

$$E = \frac{Q}{2\pi\varepsilon_0 \rho l} \tag{9}$$

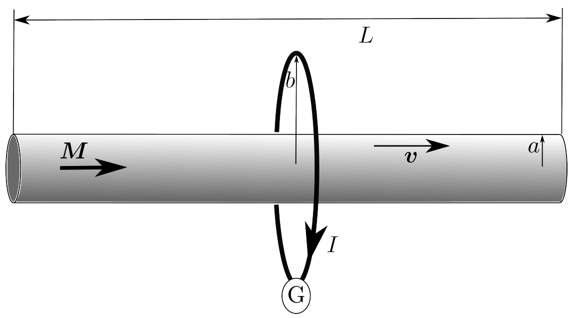
The angular momentum density $\rho\,\varepsilon_0 E B_0$ integrated over the interior of the capacitor

$$L = 2\pi l \int_a^b \rho d\rho\, \rho\varepsilon_0 E B_0 = Q B_0 \varepsilon_0 \int_a^b \rho d\rho = \frac{Q}{2}(b^2 - a^2)B_0 \tag{10}$$

This magnitude of the angular momentum is the same as the angular momentum from torque.

The direction is opposite: the $\boldsymbol{\Pi} \sim \boldsymbol{E} \times \boldsymbol{B}$ is anti-clockwise, thus $\boldsymbol{r} \times \boldsymbol{\Pi}$ is out of page (opposite $\boldsymbol{B}_0$).

4. Permanent magnet and a ring



A uniform permanent magnet in the shape of a long cylinder of radius a and length L carries magnetization M directed along its axis. The magnet is moving with constant velocity v along its axis. A ring of radius b of conducting wire is placed around the cylinder ($b > a$) in the plane perpendicular to its axis. A galvanometer with internal resistance R measures the current I through the wire of the ring.

The magnetic field created by the current in the wire, the thickness and the resistance of the wire are negligible. At time $t = 0$ the center of mass of the cylinder and the center of the conducting ring coincide.

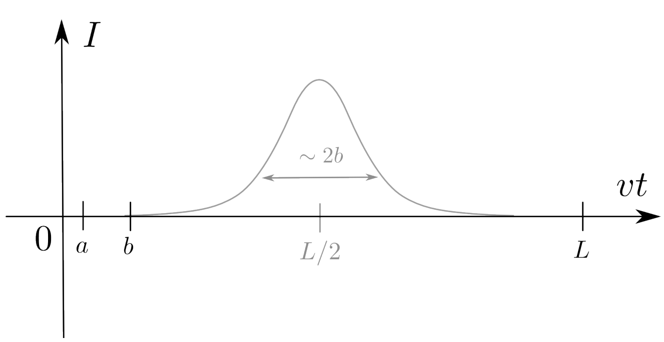
(a) At $t = 0$, find magnetic flux density B in the plane of the ring (inside and outside of the cylinder) assuming $L \gg a$.

> **Solution:** Far from the ends of the cylinder we need solution to obey boundary conditions on the surface of the cylinder. Such a solution is $\boldsymbol{H} = 0$ inside and outside and thus $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ is zero outside and $\boldsymbol{B} = \mu_0 \boldsymbol{M}$ inside.

(b) Find the total amount Q of the charge that will flow through the galvanometer between the time $t = 0$ and $t = \infty$.

> **Solution:** At time $t = 0$ the flux through the ring is $\Phi(0) = B\pi a^2 = \mu_0 M \pi a^2$ and at time $t = \infty$ it is $\Phi(\infty) = 0$. The electromotive force around the ring is $V = -d\Phi/dt$ and the current $I = V/R = -(d\Phi/dt)/R$. Thus the total charge is
>
> $$Q = \int_0^\infty I dt = -\frac{1}{R}(\Phi(0) - \Phi(\infty)) = \frac{\mu_0 M \pi a^2}{R} \tag{11}$$

(c) Sketch on the graph below (I vs vt) the time dependence of the current measured by the galvanometer. At what time is the current maximal?



Solution: Comment: In the limit $a \ll b$ the flux is proportional to the solid angle subtended by the ring from the point of view of the end of the cylinder. The derivative of it $d\Phi/dt \sim (x^2 + b^2)^{-3/2}$ where x is the distance of the end of the cylinder from the center of the ring.

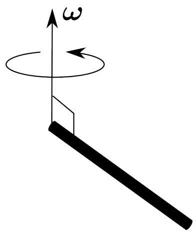
(d) By considering the characteristic time during which the current is significant, estimate the maximum value of the current I_{max}.

Solution: The magnetic field acting on the wire changes significantly when the end of the magnet is within a distance of order the ring radius b. The magnet covers this distance over time of order $\Delta t \sim 2b/v$. Since the total charge is $Q \sim I_{\text{max}} \Delta t$,

$$I_{\text{max}} \sim \frac{Q}{\Delta t} \sim \frac{\mu_0 M \pi a^2 v}{2bR} \tag{12}$$

(This is the exact result in the limit $a \ll b$.)

5. Radiation from a rotating rod



An infinitezimally thin uniformly charged rod of length L is rotated around the axis perpendicular to it going through its end with the angular frequency $\omega \ll c/L$. The total charge on the rod is Q.

(a) Find the electric dipole moment of the rotating rod.

Solution:

$$p = \int_0^L \frac{Q}{L} r dr = \frac{QL}{2} \tag{13}$$

$$\boldsymbol{p} = \frac{QL}{2}(\hat{\boldsymbol{x}} \cos\omega t + \hat{\boldsymbol{y}} \sin\omega t) \tag{14}$$

(b) Find the averaged electric dipole energy radiated by the rod per unit of time.

Solution:

$$\frac{d\mathcal{E}}{dt} = \frac{c}{12\pi\epsilon_0}\left(\frac{\omega}{c}\right)^4 |\boldsymbol{p}_\omega|^2 \tag{15}$$

where $\boldsymbol{p}_\omega = p(\hat{\boldsymbol{x}} + i\hat{\boldsymbol{y}})$, i.e., $|\boldsymbol{p}_\omega|^2 = 2p^2 = (QL)^2/2$ and

$$\frac{d\mathcal{E}}{dt} = \frac{\omega^4 Q^2 L^2}{24\pi\varepsilon_0 c^3} \tag{16}$$

(c) Find the magnetic dipole moment and the magnetic dipole energy radiated by the rod per unit of time.

Solution: The local velocity of the charge on the rod is $v = \omega r$ and it is perpendicular to the radius vector $\boldsymbol{r}$: $\boldsymbol{r} \times \boldsymbol{v} = \omega r^2 \hat{\boldsymbol{z}}$ and

$$\boldsymbol{m} = \frac{1}{2}\int dr \frac{Q}{L}\omega r^2 \hat{\boldsymbol{z}} = \frac{QL^2\omega}{6}\hat{\boldsymbol{z}}. \tag{17}$$

Since $\boldsymbol{m}$ is time-independent, the magnetic dipole radiation is absent.

Equations

$$\nabla\cdot\boldsymbol{D}=\rho;\quad \nabla\times\boldsymbol{E}=-d\boldsymbol{B}/dt;\quad \nabla\times\boldsymbol{H}=\boldsymbol{J}+d\boldsymbol{D}/dt;\quad \nabla\cdot\boldsymbol{B}=0;$$

$$\boldsymbol{D}=\varepsilon_0\boldsymbol{E}+\boldsymbol{P};\quad \boldsymbol{H}=\boldsymbol{B}/\mu_0-\boldsymbol{M};$$

$$\boldsymbol{E}=-\nabla\Phi-\partial\boldsymbol{A}/\partial t;\quad \boldsymbol{B}=\nabla\times\boldsymbol{A};$$

$$U=\int d^3\boldsymbol{r}\,\rho\,\Phi$$

$$\boldsymbol{p}=\int d^3\boldsymbol{r}\,\rho\boldsymbol{r};\quad \boldsymbol{m}=\frac{1}{2}\int d^3\boldsymbol{r}\,\boldsymbol{r}\times\boldsymbol{J};$$

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}=\sum_l\frac{r_<^l}{r_>^{l+1}}P_l(\cos\gamma)=\sum_{lm}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\Omega_{\boldsymbol{x}})Y^*_{lm}(\Omega_{\boldsymbol{y}})$$

$$\begin{array}{lll} Y_{00}\sim 1; & Y_{10}\sim\cos\theta; & Y_{20}\sim 3\cos^2\theta-1; \\ & Y_{11}\sim\sin\theta\,e^{i\phi}; & Y_{21}\sim\cos\theta\sin\theta\,e^{i\phi}; \\ & & Y_{22}\sim\sin^2\theta\,e^{2i\phi}. \end{array}$$

$$\frac{d\mathcal{E}}{dt}=\frac{c}{12\pi\varepsilon_0}k^4|\boldsymbol{p}_\omega|^2\qquad\text{for}\quad \boldsymbol{p}(t)=\mathrm{Re}\,\left(\boldsymbol{p}_\omega e^{-i\omega t}\right)$$

$$\frac{d\mathcal{E}}{dt}=\frac{\mu_0 c}{12\pi}k^4|\boldsymbol{m}_\omega|^2\qquad\text{for}\quad \boldsymbol{m}(t)=\mathrm{Re}\,\left(\boldsymbol{m}_\omega e^{-i\omega t}\right)$$