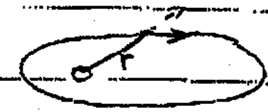
Solutions Fall '96 I. 1

Mechanics I

a.

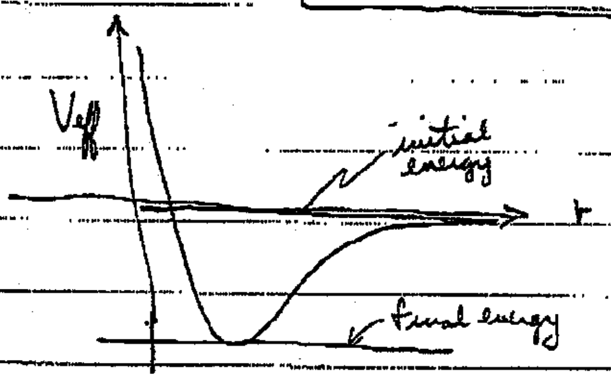


$J = m\omega r^2$ — In a frame that rotates with the planet, the centrifugal force is $m\omega^2 r$. The corresponding fictitious potential is $\frac{m\omega^2 r^2}{2}$

Thus, $V_{eff} = \frac{-GMm}{r} - \frac{m\omega^2 r^2}{2}$ ← zero is referenced to $r=0$

written in terms of J, we have

$$\boxed{V_{eff} = -\frac{GMm}{r} + \frac{J^2}{2mr^2}}$$ ← zero is referenced to $r=\infty$



b. Energy

Near perihelion $E = -\frac{GMm}{r_0} + \frac{1}{2}mv_0^2 \simeq 0$ (given in the problem)

Angular momentum $J_0 = mr_0v_0$ (initially)

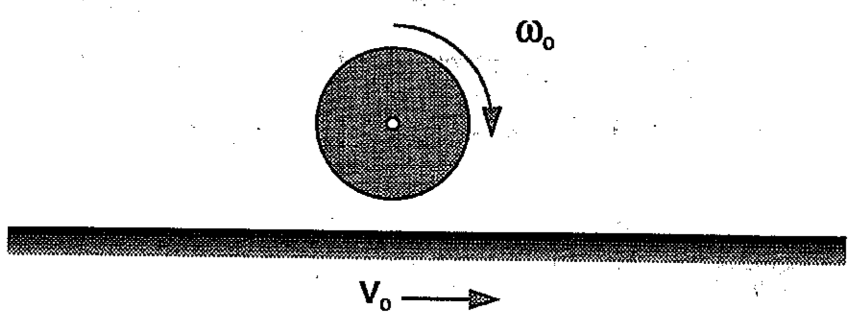
$J_{final} = mr_fv_f = mr_0v_0$

$v_0 = \sqrt{\frac{2GM}{r_0}}$ and for a final circular orbit $\frac{GMm}{r_f^2} = \frac{mv_f^2}{r_f}$

$\Rightarrow r_0v_0 = \sqrt{2GMr_0} = r_fv_f = \sqrt{GMr_f} \quad \Rightarrow \quad \boxed{r_f = 2r_0}$

Mechanics II

A long belt of a conveyer is moving horizontally with a constant speed v_o. A cylinder of mass M and radius R is rotating with an angular velocity ω_o and is quietly dropped on the belt. Find the relative distance the cylinder skips on the belt before it starts to rotate without slipping. The kinetic friction between the cylinder and the belt is μ_k.



<Solution>

The equations of motion for the cylinder with respect to the belt is

$Ma = F$ and $I\alpha = - RF$

where F is the frictional force and $I = MR^2/2$. The relationship between the friction and the normal force which is Mg is $F = \mu_k Mg$. Thus the linear and angular acceleration are given by

$a = \mu_k g$ and $R\alpha = - 2\,\mu_k g$

and thus $v = v_o - \mu_k gt$ and $R\omega = R\omega_o - 2\,\mu_k gt.$

When $v = R\omega$ there is no slipping. Thus

$$t = \frac{v_o + R\omega_o}{3\mu_k g}$$

During this time the relative distance moved by the cylinder is

$$x = \frac{(v_o + R\omega_o)^2}{18\mu_k g}$$

<End>.

Before the collision (lab frame) | After the collision (CM frame)

$h\nu \rightsquigarrow \quad \Longleftarrow$ Proton, γ | π proton (at rest)

$$E^2-(pc)^2 = \text{invariants} = E_{i,lab}^2 - (P_{i,lab}c)^2 = E_{f,cm}^2 - (P_{f,cm}c)^2$$

$$(h\nu + \gamma m_o c^2)^2 - (h\nu - \gamma\beta m_o c^2)^2 = (m_o + m_\pi)^2 c^4$$

$$\cancel{(h\nu)^2} + 2\gamma m_o c^2 h\nu + \gamma^2 (m_o c^2)^2 - \cancel{(h\nu)^2} + 2h\nu\gamma\beta m_o c^2 - \gamma^2\beta^2 (m_o c^2)^2 = m_o^2 c^4 + m_\pi^2 c^4 + 2m_\pi m_o c^4$$

$$2h\nu m_o c^2(\gamma + \gamma\beta) + \underbrace{(\gamma^2 - \gamma^2\beta^2)}_{1}\cancel{(m_o c^2)^2} = \cancel{(m_o c^2)^2} + (m_\pi c^2)^2 + 2m_\pi m_o c^4$$

$$h\nu\,\gamma(1+\beta) = m_\pi c^2 + \frac{(m_\pi c^2)^2}{2m_o c^2}$$

$$\boxed{\gamma(1+\beta) = \frac{m_\pi c^2\left(1 + \frac{m_\pi}{2m_o}\right)}{h\nu}}$$

for $h\nu \simeq 10^{-3}\,eV$ $\qquad m_\pi c^2 \simeq 140\ MeV$

$$\gamma(1+\beta) = \frac{140\times10^6}{10^{-3}}\left(1 + \frac{140}{2\times 940}\right) \simeq 1.4\times10^{11} \qquad \Rightarrow \beta \simeq 1$$

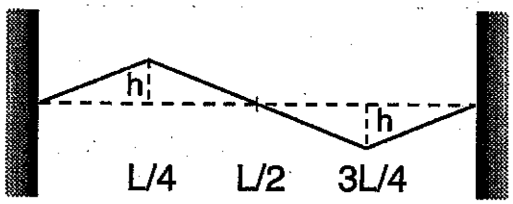
$$\boxed{\gamma_p = 0.7\times10^{11}}$$

$$\boxed{E_{proton} \simeq 0.7\times10^{20}\,eV}$$

I.4

Mechanics IV

A string of length L and mass M is under tension T with its two ends fixed on the wall. The two points, $L/4$ from both ends, are pulled and slightly displaced from equilibrium by distance h as shown in the diagram. Suddenly the string is released. Assume $h << L$.



(1) Find the normal modes of the string.
(2) Which normal modes have non-zero amplitudes?
(3) Find the displacement $u(x, t)$ indicating the vertical position of the string.

<Solution>
(1) The normal modes are

$$w_n(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\sqrt{\frac{TL}{M}}\frac{n\pi}{L}t\right)$$

(2) The displacement is obtained as the sum of Fourier transform:

$$u(x,t) = \sum_n A_n w_n(x,t)$$

where

$$A_n \equiv \int_0^L u(x,0)w_n(x,0) = \begin{cases} \dfrac{4\sqrt{2}h}{\sqrt{L}}\dfrac{1}{n} & \textit{for} \quad n = 2(2m+1) \\ 0 & \textit{otherwise} \end{cases}$$

and

$$h = \frac{L}{4}\tan\theta \cong \frac{L}{4}\theta \cong \frac{L}{4}\frac{F}{2T} = \frac{FL}{8T}$$

<End>

I.5

E&M. I

For nonrelativistic motion

$$E(\vec{r},t) = \frac{e}{cr}\left(n \times (n \times \dot{\beta})\right)_{ret} \quad \text{far from the accelerating charge}$$

$$c\dot{\beta}_1 = s/2\,(\sin\omega t)\omega^2 \; ; \; c\dot{\beta}_2 = -s/2(\sin\omega t)\omega^2 \quad (\underline{\underline{180^\circ}} \text{ out of phase})$$

$$E(\vec{r},t) \simeq \frac{e\,\omega^2}{c^2 r}\frac{s}{2}\sin\theta\left[\sin\omega t_{ret_1} - \sin\omega t_{ret_2}\right]$$

where one neglects the small difference in direction of the E field and the small difference in $1/r$ factor

$$t_{ret_1} = t + \frac{r}{c} - \frac{s}{c2}\cos\theta \quad ; \quad t_{ret_2} = t + \frac{r}{c} + \frac{s}{c2}\cos\theta$$

Thus, $$E(\vec{r},t) = \frac{\omega^2 e s}{2c^2 r}\sin\theta\left\{2\cos\left(\omega(t+r/c)\right)\sin\left(\frac{\omega s}{2c}\cos\theta\right)\right\}$$

but, $\frac{\omega s}{c} \ll 1$

$$\Rightarrow \quad E(\vec{r},t) = \frac{\omega^2 e s 2}{2c^2 r}\frac{\omega s}{2c}\sin\theta\,\overset{\cos\theta}{\wedge}\cos\left(\omega(t+r/c)\right)$$

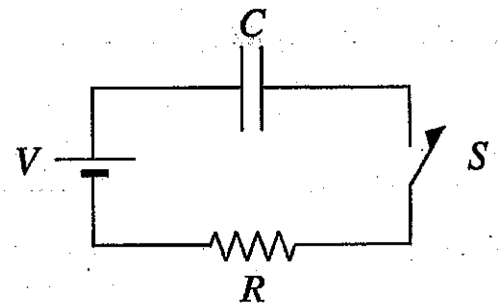
$$E_0 = \frac{s^2 e\,\omega^3}{2c^3 r}\sin\theta\cos\theta$$

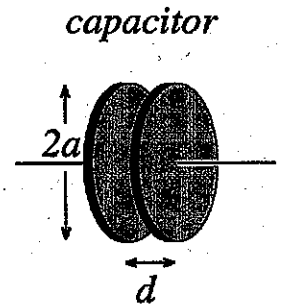
$$\boxed{S_{Poyn} \propto E_0^2 \propto \frac{s^4 e^2 \omega^6}{4c^6 r^2}\sin^2\theta\cos^2\theta}$$

I.6

Electromagnetism II

Consider the current flowing in a circuit containing a capacitor C made of two circular plates of radius a and distance d. At time $t = 0$, the switch is turned on. Find the displacement current density and the magnetic field inside the capacitor. Assume d<<a.





<Solution>

The electric field is given by

$$E(t) = \frac{V(t)}{d} = \frac{Q(t)}{C \cdot d}$$

where

$$V(t) = V\left(1 - \exp\left(-\frac{t}{RC}\right)\right).$$

The displacement current density is given by

$$J_d = \frac{1}{4\pi}\frac{\partial E(t)}{\partial t} = \frac{1}{4\pi C \cdot d}\frac{\partial Q(t)}{\partial t} = \frac{I(t)}{4\pi C \cdot d}$$

where

$$I(t) = \frac{V}{R}\exp\left(-\frac{t}{RC}\right).$$

The direction of the displacement current is in the direction of the current.

Integrating a Maxwell's equation over a circular area of radius r,

$$\int_{S(r)} curl\, \mathbf{B} \cdot da = \frac{1}{c}\int_{S(r)} \frac{\partial \mathbf{E}}{\partial t} \cdot da$$

Thus

$$2\pi r B(r) = \frac{4\pi I(r)}{c}\left(\frac{r}{a}\right)^2$$

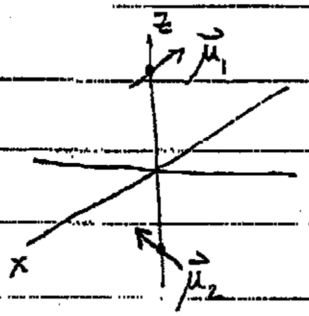
Thus the magnetic field in the capacitor is $B(r) = \dfrac{2rI}{ca^2}$.

<End>

I.7

E&M III

Call the line joining the ~~two~~ dipoles, the z axis



Break $\vec{\mu}_1$ and $\vec{\mu}_2$ up into their cartesian components

$|B_z| = \frac{2\mu}{r^3}$ and $|B_x| = \frac{\mu}{r^3}$; $|B_y| = \frac{\mu}{r^3}$

$$E = -\vec{\mu}_2 \cdot \vec{B}_1 = +\frac{\mu_{2x}\,\mu_{1x}}{r^3} + \frac{\mu_{2y}\,\mu_{1y}}{r^3} - \frac{2\mu_{2z}\,\mu_{1z}}{r^3}$$

$$E = \frac{\vec{\mu}_1 \cdot \vec{\mu}_2}{r^3} - \frac{3(\vec{\mu}_1 \cdot \vec{r})(\vec{\mu}_2 \cdot \vec{r})}{r^5}$$

$\uparrow$	$\downarrow$	$\rightarrow$	$\rightarrow$
$\uparrow$	$\uparrow$	$\rightarrow$	$\leftarrow$
$-\frac{2\mu^2}{r^3}$	$+\frac{2\mu^2}{r^3}$	$\frac{\mu^2}{r^3}$	$-\frac{\mu^2}{r^3}$
Min. Energy			

I.8

Electromagnetism IV

What is the charge distribution when the potential in space is given by Yukawa potential:

$$\phi(r) = \frac{q}{4\pi\varepsilon_o r}\exp\left(-\frac{r}{r_o}\right)$$

<Solution>

The electric field by the potential is

$$E_r(r) = -\frac{\partial}{\partial r}\left[\frac{q}{4\pi\varepsilon_o r}\exp\left(-\frac{r}{r_o}\right)\right] = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r^2}+\frac{1}{rr_o}\right)\exp\left(-\frac{r}{r_o}\right)$$

$$E_\theta = E_\phi = 0$$

Using Maxwell's equation:

$$\frac{1}{r^2}\frac{d}{dr}\left\{r^2 \cdot E_r(r)\right\} = \frac{\rho(r)}{\varepsilon_o}$$

we have

$$\rho(r) = -\frac{q}{4\pi r_o^2 r^2}\exp\left(-\frac{r}{r_o}\right) \qquad (r \neq 0)$$

For $r \to 0$,

$$\phi(r) = \frac{q}{4\pi\varepsilon_o r}\exp\left(-\frac{r}{r_o}\right) \quad \to \quad \frac{q}{4\pi\varepsilon_o r}$$

$$E_r(r) = \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r^2}+\frac{1}{rr_o}\right)\exp\left(-\frac{r}{r_o}\right) \quad \to \quad \frac{q}{4\pi\varepsilon_o r^2}$$

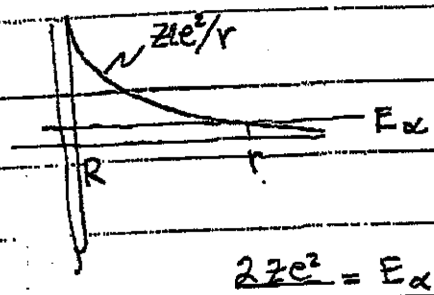
$\rho(r)$ can be expressed as $q\delta(r)$. Thus

$$\rho(r) = q\delta(r) - \frac{q}{4\pi r_o^2}\frac{1}{r}\exp\left(-\frac{r}{r_o}\right)$$

<End>

QM I

The tunneling prob. $T \sim e^{-2\int_{r_1}^{R}\sqrt{(V(r)-E)2m/\hbar^2}\,dr}$



$$\sqrt{\left(\frac{2Ze^2}{r}-E_\alpha\right)\frac{2m}{\hbar^2}} \to \frac{\sqrt{2mE_\alpha}}{\hbar}\sqrt{\frac{r_1}{r}-1}$$

The $-\int_{r_1}^{R}\sqrt{r_1/r-1}\,dr$ integral has its largest contribution for $r \ll r_1$

$$\frac{2Ze^2}{r_1}=E_\alpha$$

$$\simeq -\int_{r_1}^{R}\sqrt{r_1/r}\,dr = \sqrt{r_1}\left(\sqrt{r_1}-\sqrt{R}\right) \simeq r_1$$

Thus, the entire exponent $\simeq +2\frac{\sqrt{2mE_\alpha}}{\hbar}r_1 = 2\frac{\sqrt{2mE_\alpha}}{\hbar}\frac{2Ze^2}{E_\alpha}$

$$\boxed{T \sim e^{-Z4\sqrt{2m}\,e^2/\hbar E_\alpha^{1/2}}}$$

Done more correctly $Z \to Z-2$, $4 \to 2\pi$

The decay rate $\simeq \frac{R}{v_\alpha}T$

τ = The decay time $\simeq \frac{v_\alpha}{TR}$

$$\log_{10}\tau = -\underbrace{\log_{10}}T + \text{const} + \log_{10}E_\alpha$$

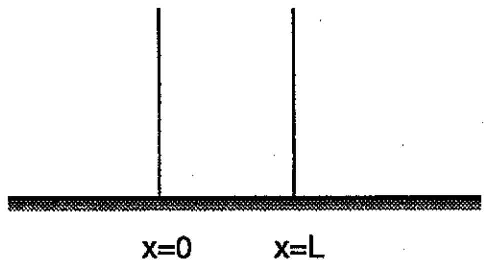
$$\boxed{\log_{10}\tau \propto \frac{1}{E_\alpha^{1/2}}}$$

Quantum Mechanics II

A particle of mass m with energy E is incident from the left in one dimensional onto a double potential wall given by

$$V(x) = A\{\delta(x) + \delta(x - L)\}$$

Find the condition that the particle is not reflected by the wall and fully transmitted.



<Solution>

Assuming that the wave function is

$$\psi(x,t) = \begin{cases} e^{i(kx-\omega t)} + R \cdot e^{i(-kx-\omega t)} & x < 0 \\ a \cdot e^{i(kx-\omega t)} + b \cdot e^{i(-kx-\omega t)} & 0 \le x \le L \\ T \cdot e^{i(kx-\omega t)} & L < x \end{cases}$$

where $\omega \equiv \dfrac{E}{\hbar}$ and $k \equiv \dfrac{\sqrt{2mE}}{\hbar}$.

The boundary conditions at $x = 0$ and $x = L$ are

$$\psi_+(0) = \psi_-(0) \quad \text{and} \quad \psi'_+(0) = \psi'_-(0) + k^2 \frac{A}{E}\psi(0)$$

$$\psi_+(L) = \psi_-(L) \quad \text{and} \quad \psi'_+(L) = \psi'_-(L) + k^2 \frac{A}{E}\psi(L)$$

These can be rewritten for no reflection,

$$\begin{aligned} 1 &= a + b \\ 1 &= a - b + (a+b)A' \\ a + b \cdot e^{-2ikL} &= 1 \\ a - b \cdot e^{-2ikL} &= 1 - A' \end{aligned}$$

where $A' \equiv -ik\dfrac{A}{E}$. This is satisfied by $kL = n\pi$ for a given value of A'. <End>

$$\text{Prob} = |\langle \Psi_f^* \, \Psi_i \rangle|^2$$

$$\langle \Psi_f^* | \Psi_i \rangle^* = \int \left[2 \left(\frac{2}{a_0}\right)^{3/2} \exp\left(\frac{-2r}{a_0}\right) Y_0^0 \right] \left[2 \cdot \left(\frac{1}{a_0}\right)^{3/2} \exp\left(\frac{-r}{a_0}\right) Y_0^0 \; d^3r \right.$$

$$= 4 \times 2^{3/2} \left(\frac{1}{a_0}\right)^3 \left(Y_0^0\right)^2 \int \exp\left(\frac{-2r}{a_0}\right) \exp\left(\frac{-r}{a_0}\right) d^3r \qquad ①$$

$$\cancel{= 2^{3/2} \left(\frac{}{a_0}\right)^3 \left(\frac{1}{4\pi}\right)} \int \exp\left(\frac{-3r}{a_0}\right) d^3r$$

Note
$$1 = \langle \Psi_i^* \, \Psi_i \rangle$$

$$= \int \left(2 \left(\frac{1}{a_0}\right)^{3/2} \exp\left(\frac{-r}{a_0}\right) Y_0^0 \right)^2 d^3r$$

$$= 4 \left(\frac{1}{a_0}\right)^3 \left(Y_0^0\right)^2 \int \exp\left(\frac{-2r}{a_0}\right) d^3r$$

$$\Rightarrow \quad 4 \left(\frac{1}{a_0}\right)^3 \left(Y_0^0\right)^2 = \frac{1}{\int \exp\left(\frac{-2r}{a_0}\right) d^3r} \qquad ②$$

Subs ② into ①

$$\Rightarrow \quad \text{Prob} = 2^{3/2} \frac{\int \exp\left(\frac{-3r}{a_0}\right) d^3r}{\int \exp\left(\frac{-2r}{a_0}\right) d^3r}$$

$$= 2^{3/2} \frac{\int_0^\infty r^2 \exp\left(\frac{-3r}{a_0}\right) dr}{\int_0^\infty r^2 \exp\left(\frac{-2r}{a_0}\right) dr}$$

by parts
Twice

$$\int_0^\infty r^2 e^{-\alpha r}\, dr = \left(\frac{2}{\alpha^3}\right)$$

$$\Rightarrow \frac{\langle \psi_f | \psi_i \rangle}{\cancel{\text{prob}}} = \frac{2^{3/2} \times 2\left(\frac{a_0}{3}\right)^3}{2\left(\frac{a_0}{2}\right)^3}$$

$$= 2^{3/2} \times \frac{8}{27} = \frac{16\sqrt{2}}{27}$$

$$= \frac{2^{9/2}}{27}$$

$$\text{prob} = 0.8381^2 = 0.70$$

Given the potential $V(x) = \beta x$ for $x \geq 0$
$V(x) = \infty$ for $x < 0$

Estimate ground-state energy of a particle

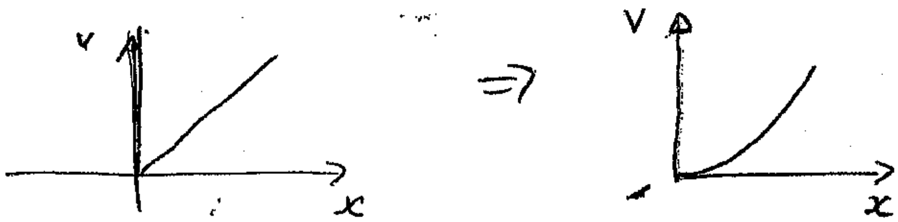
$$\left[-\frac{\hbar^2}{2m} \frac{d}{dx^2} + V \right] \psi(x) = E\,\psi(x)$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{d}{dx^2} \psi(x) + \beta(x)\,\psi(x) = E\,\psi(x)$$

non-linear differential eq^n.

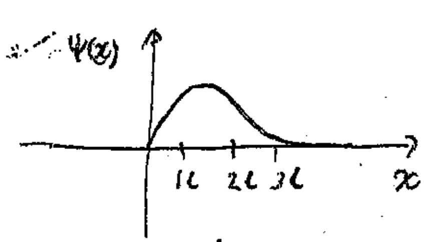
Two ~~Three~~ valid types of solution to this problem, (Other solutions also given credit eg Bohr-Sommerfeld.)

1) Approximate the linear potential as $\frac{1}{2}$ a SHO



ground state is taken to be the first state of the SHO that has a node at $x=0$.

This wavefunction has the following shape, $x>0$



and Energy $E_0 = \frac{3}{2}\hbar\omega$ (1)

where l is the typical length scale, $l = \sqrt{\frac{\hbar}{m\omega}}$

hence to make a link between the linear and SHO potentials we need to consider the region $0 < x < 3l$. matching the mean potentials in this region provides a link between β and w

$$\langle V_{SHO} \rangle \sim \langle V_{linear} \rangle$$

$$\Rightarrow \int_0^3 V_{SHO}(x)\,dx \sim \int_0^3 V_{linear}(x)\,dx$$

$$\Rightarrow \frac{1}{2} m w^2 \int_0^{3\ell} x^2\,dx \sim \beta \int_0^{3\ell} x\,dx$$

$$\Rightarrow \frac{1}{2} m w^2 \left.\frac{x^3}{3}\right|_0^{3\ell} \sim \beta \left.\frac{x^2}{2}\right|_0^{3\ell}$$

$$\Rightarrow \frac{1}{2} m w^2\, 9\ell^3 \sim \beta \frac{9}{2}\ell^2$$

$$\Rightarrow w^2 \ell \sim \frac{\beta}{m}$$

solve for w, substitute into ①

$$w^2 \sqrt{\frac{\hbar}{mw}} \sim \frac{\beta}{m}$$

$$\Rightarrow w^{3/2} \sim \frac{\beta}{\sqrt{\hbar}\sqrt{m}}$$

$$\Rightarrow w \sim \frac{\beta^{2/3}}{\hbar^{1/3} m^{1/3}}$$

into ①

$$\Rightarrow E_0 \sim \frac{3}{2} \hbar \frac{\beta^{2/3}}{\hbar^{1/3} m^{1/3}}$$

$$= \frac{3}{2} \frac{(\hbar\beta)^{2/3}}{m^{1/3}}$$

$\longrightarrow$ check units! ✓

2) An alternative technique is to use the variational method and minimise

$$E(a) = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle}$$

let $\psi_a(x) = x e^{-ax}$ and minimize wrt a

$$H = -\frac{\hbar^2}{2m}\frac{d}{dx^2} + \beta x$$

$$\Rightarrow E(a) = \frac{-\int_0^\infty x e^{-ax} \frac{\hbar^2}{2m} \frac{d^2}{dx^2}\left(x e^{-ax}\right) dx + \int_0^\infty x e^{-ax} \beta x^2 e^{-ax} dx}{\int_0^\infty x^2 e^{-2ax} dx}$$

$$\frac{d^2}{dx^2}\left(x e^{-ax}\right) = \frac{d\left(e^{-ax} + -ax e^{-ax}\right)}{dx}$$

$$= -a e^{-ax} - a e^{-ax} - ax \cdot -a e^{-ax}$$

$$= -2a e^{-ax} + a^2 x e^{-ax}$$

$$\Rightarrow E(a) = \frac{\left[+2a \frac{\hbar^2}{2m} \int_0^\infty x e^{-2ax} dx - \frac{\hbar^2}{2m} a^2 \int_0^\infty x^2 e^{-2ax} dx + \beta \int_0^\infty x^3 e^{-2ax} dx \right]}{\int_0^\infty x^2 e^{-2ax} dx} \qquad 1$$

Integrating by parts

$$\int_0^\infty x e^{-2ax} dx = \frac{1}{(2a)^2}$$

$$\int_0^\infty x^2 e^{-2ax} dx = \frac{2}{(2a)^3}$$

$$\int_0^\infty x^3 e^{-2ax}\,dx = \frac{6}{(2a)^4}$$

substituting into ①

$$E(a) = \frac{\dfrac{2a\hbar^2}{2m}\dfrac{1}{(2a)^2} - \dfrac{\hbar^2}{2m}a^2 \times \dfrac{2}{(2a)^3} + \beta\dfrac{6}{(2a)^4}}{\dfrac{2}{(2a)^3}}$$

$$= \frac{a\hbar^2}{m} \times \frac{2a}{2} - \frac{\hbar^2 a^2}{2m} + \frac{3\beta}{2a}$$

$$E(a) = \frac{3\beta}{2a} + \frac{\hbar^2 a^2}{2m} \qquad ②$$

minimise wrt a

$$\frac{dE(a)}{da} = -\frac{3\beta}{2a^2} + \frac{\hbar^2 a}{m} = 0$$

$$\Rightarrow \quad \frac{\hbar^2 a}{m} = \frac{3\beta}{2a^2}$$

$$\Rightarrow \quad a^3 = \frac{3}{2}\frac{m}{\hbar^2}\beta \qquad \text{substitute into ②}$$

$$a = \sqrt[3]{\frac{3}{2}\frac{m}{\hbar^2}\beta}$$

$$E_0 = \frac{3}{2}\frac{\beta}{\sqrt[3]{\frac{3}{2}\frac{m}{\hbar^2}\beta}} + \frac{\hbar^2}{2m}\left(\frac{3}{2}\frac{m}{\hbar^2}\beta\right)^{2/3}$$

$$= \left(\frac{3}{2}\right)^{2/3}\left(\frac{m}{\hbar^2}\right)^{-1/3}\beta^{2/3} + \left(\frac{3}{2}\right)^{2/3}\frac{1}{2}\left(\frac{m}{\hbar^2}\right)^{-1/3}\beta^{2/3}$$

$$= \left(3/2\right)^{5/3}\beta^{2/3}\left(\frac{\hbar^2}{m}\right)^{1/3} = \left(\frac{3}{2}\right)^{5/3}\frac{(\hbar\beta)^{2/3}}{m^{1/3}}$$

The answers are very similar, within a constant ~1.

Doctor's General Examination
Part I
Fall 1996
Section IV Statistical Mechanics

Problem I

(1) Single Particle partition function:

$$Z=\sum_{n=0}^{4} e^{-\beta n E_0}=1+e^{-\beta E_0}+e^{-2\beta E_0}+e^{-3\beta E_0}+e^{-4\beta E_0}$$

In general, $\sum_{n=0}^{N} x^n=\frac{x^{N+1}-1}{x-1}$. therefore, $Z=\frac{e^{-5\beta E_0}-1}{e^{-\beta E_0}-1}$

(2) Average energy:

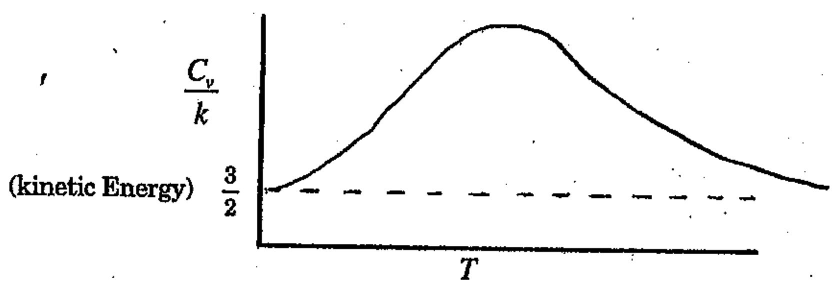
$$\langle E\rangle=\frac{\sum_{n=0}^{4} nE_0 e^{-n\beta E_0}}{\sum_{n=0}^{4} e^{-n\beta E_0}}=-\frac{\partial}{\partial\beta}\ln Z=-\frac{1}{Z}\frac{\partial Z}{\partial\beta}=E_0\left[\frac{1}{e^{\beta E_0}-1}-\frac{5}{e^{5\beta E_0}-1}\right]$$

$\langle E\rangle\to 0,\ T\to 0$ and $\langle E\rangle\to 2E_0,\ T\to\infty$

(3) Compute, sketch specific heat:

$$C_v=\frac{\partial\langle E\rangle}{\partial T}=k\left(\frac{E_0}{kT}\right)^2\left[\frac{e^{\beta E_0}}{\left(e^{\beta E_0}-1\right)^2}-\frac{25e^{5\beta E_0}}{\left(e^{5\beta E_0}-1\right)^2}\right]$$

$\frac{C_v}{k}\to 0,\ T\to 0$ and $\frac{C_v}{k}\to 0,\ T\to\infty$



Doctor's General Examination
Part I
Fall 1996
Section IV Statistical Mechanics

Problem II

$$N = \sum_{\varepsilon} \frac{1}{e^{-\mu\beta} e^{\beta\varepsilon} - 1} = \frac{1}{e^{-\mu\beta} - 1} + \sum_{\varepsilon \neq 0} \frac{1}{e^{-\mu\beta} e^{-\beta\varepsilon} - 1},$$

Where $\beta = \frac{1}{kT}$ and $\mu =$ chemical potential. For N large, many close lying states, so sum can be transformed to integral weighted by density in phase-space:

$$\rho(p)dp = \frac{V 4\pi p^2 dp}{h^3}, \text{ where: } V = \text{ spatial volume, } p = \text{ momentum}$$

Momentum density distribution can be written as energy distribution, since:

$$\varepsilon = \frac{p^2}{2m},\ dp = \frac{m d\varepsilon}{\sqrt{2m\varepsilon}} \text{ and } \rho(\varepsilon)d\varepsilon = \frac{V 4\pi 2m^2 \varepsilon d\varepsilon}{h^3 \sqrt{2m\varepsilon}} = \frac{V 2\pi (2m)^{\frac{3}{2}} \varepsilon^{\frac{1}{2}} d\varepsilon}{h^3}$$

$$\therefore \frac{N}{V} = \frac{2\pi (2m)^{\frac{3}{2}}}{h^3} \int_0^{\infty} \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{-\mu\beta} e^{\beta\varepsilon} - 1} + \frac{1}{V} \frac{1}{e^{-\mu\beta} - 1} = \frac{N_e}{V} + \frac{N_0}{V}$$

First term is number per volume in excited states (none ground state) Second term is number per volume in ground state. As temperature decreases, occupation number in excited states decrease and when less than total number N, number in ground state begins to increases indefinitely: this is onset of Bose Einstein condensation. Thus, the condition for condensation is:

$$\frac{N}{V} \geq \frac{N_e}{V}, \text{ or, } \frac{N}{V} \geq \frac{2\pi (2m)^{\frac{3}{2}}}{h^3} \int_0^{\infty} \frac{\varepsilon^{\frac{1}{2}} d\varepsilon}{e^{-\mu\beta} e^{\beta\varepsilon} - 1} = \frac{2\pi (2mkT)^{\frac{3}{2}}}{h^3} \int_0^{\infty} \frac{x^{\frac{1}{2}} dx}{e^{-\mu\beta} e^{x} - 1}$$

Where $x = e^{\beta\varepsilon}$. Note, last integral is largest when $\mu = 0$. Temperature when B-E condensation takes place is then:

$$T \leq \frac{h^3}{2mk} \left[\frac{N}{2\pi V \int_0^{\infty} \frac{x^{\frac{1}{2}} dx}{e^{x} - 1}} \right]^{\frac{2}{3}}$$

I. 19

Doctor's General Examination
Part I
Fall 1996
Section IV Statistical Mechanics

Problem III

Some interpreted the problem as stating that the internal energy, U, of the system does not change during the volume expansion, ie, $dU = 0$. In this case, the following reasoning is taken since the internal energy of a black-body system depends on temperature only, following the Stefan-Boltzmann law of black-body radiation:

$$\frac{U}{V} \propto T^4 = \frac{4\sigma T^4}{c}$$

where σ is Stefan constant, and c is speed of light. The T^4 dependence can be inferred readily by considering B-E distribution of photons, Planck's black body distribution, etc. Thus the proportionality of energy density to T^4 is all that is required here. Since the internal energy does not change,

$$U \propto V_{init} T_{init}^4 \text{ and } U \propto V_{fin} T_{fin}^4 \text{ and } \therefore T_{fin} = T_{init}\left(\frac{V_{init}}{V_{fin}}\right)^{\frac{1}{4}}$$

A subtle point here. The problem states that the system is insulated and no energy (presumably in the form of heat) leaks in or out of the system. Thus, from the first law of thermodynamics, ($dU = dQ - pdV$), we have $dU = -pdV$.

The radiation pressure in the volume is, $p = \frac{1}{3}\frac{U}{V}$. Thus,

$$pdV = \frac{1}{3}\frac{U}{V}dV = -dU \text{ from which we get,}$$

$$\frac{1}{3}\frac{dV}{V} = -\frac{dU}{U}, \text{ and } \frac{1}{3}\int_{V_{init}}^{V_{fin}}\frac{dV}{V} = -\int_{U_{init}}^{U_{fin}}\frac{dU}{U}. \text{ Resulting in,}$$

$$\frac{1}{3}\ln V\Big]_{V_{init}}^{V_{fin}} = -\ln U\Big]_{U_{init}}^{U_{fin}} = \ln U\Big]_{U_{fin}}^{U_{init}}, \text{ or, } \frac{V_{fin}}{V_{init}} = \left(\frac{U_{init}}{U_{fin}}\right)^3.$$

From Stefan-Boltzmann, $U_{init} = const.V_{init}T_{init}^4$ and $U_{fin} = const.V_{fin}T_{fin}^4$ Thus,

$$\frac{V_{fin}}{V_{init}} = \left(\frac{V_{init}T_{init}^4}{V_{fin}T_{fin}^4}\right)^3 \text{ and } \therefore T_{fin} = T_{in}\left(\frac{V_{init}}{V_{fin}}\right)^{\frac{1}{3}}$$

Doctor's General Examination
Part I
Fall 1996
Section IV Statistical Mechanics

Problem IV

Total energy of system is:

$$E=\frac{m}{2}\left(\dot{x}_1^2+\dot{x}_2^2\right)+\frac{k}{2}\left(x_1^2+x_2^2\right)+\frac{k}{2}\left(x_1-x_2\right)^2$$

The normal coordinates, by inspection, are:

$z_1=x_1+x_2$ and $z_2=x_1-x_2$

In terms of the normal coordinates,

$$E=\frac{m}{4}\left(\dot{z}_1^2+\dot{z}_2^2\right)+\frac{k}{4}\left(z_1^2+3z_2^2\right)$$

Equipartition theorem relates average of quadratic conjugate variables in energy to $\frac{1}{2}k_BT$. Here, we are interested in spatial coordinates only, so:

$\frac{k}{4}\langle z_1^2\rangle=\frac{1}{2}k_BT$ and $\frac{3k}{4}\langle z_2^2\rangle=\frac{1}{2}k_BT$,

or $\langle z_1^2\rangle=\langle x_1^2\rangle+\langle x_2^2\rangle+2\langle x_1x_2\rangle=2k_BT$ and $\langle z_2^2\rangle=\langle x_1^2\rangle+\langle x_2^2\rangle-2\langle x_1x_2\rangle=\frac{2}{3}k_BT$

from which we get: $2\langle x_1^2\rangle+2\langle x_2^2\rangle=\frac{8}{3}k_BT$

but, from symmetry, $\langle x_1^2\rangle=\langle x_2^2\rangle\equiv\langle x^2\rangle$

$$\therefore\langle x^2\rangle=\frac{2}{3}k_BT$$