STONY BROOK UNIVERSITY

DEPARTMENT OF PHYSICS AND ASTRONOMY

Graduate Placement Exam Part 1, August 25, 2009

General Instructions: This exam is for incoming graduate students who wish to demonstrate mastery in one or more areas of the graduate core curriculum, in order to skip one or more of the first-year courses. Do two of the three problems in either or both areas.

Each solution should typically take on the order of 45 minutes.

Use one exam book for each problem, and label it carefully with the problem topic and number and your name. Make sure to do every part of the problems you choose.

You may use a one page help sheet, a calculator, and with the proctor's approval, a foreign language dictionary. No other materials may be used.

Electromagnetism 1

A cylindrical shell having conductivity σ, radius R, thickness $\Delta << R$, and magnetic permitivity μ, extends to infinity in the z direction. Inside and outside the shell is free space. When $t = 0$, the shell supports a surface current density

$$\vec{\mathbf{K}}(0, \phi) = \hat{\mathbf{z}}\, K_0 \sin 2\phi$$

Find $\vec{\mathbf{K}}(t, \phi)$ for $t > 0$.

Electromagnetism 2

A relativistic charged particle of mass M, charge q and velocity v passes an electron of mass m and charge e at a fixed impact parameter b. Assume the electron to be harmonically bound classically in an atom with frequency ω, and assume the incident particle continues undeflected.

1. (8 points) Assume that the electron does not move appreciably during the time of interaction. How much energy is transferred to the electron?

2. (8 points) For what range of the impact parameter b is this energy transfer valid?

3. (4 points) If the density per unit volume of bound electrons in matter is n what is the energy loss per unit length suffered by the charge q passing through this matter?

Electromagnetism 3

A line charge with linear charge density λ is set parallel to and a distance R away from the axis of a conducting and grounded cylinder of radius $b < R$. Take the origin at the center of the cylinder, the z axis along the cylinder, that the x axis in the direction of the line charge.

1. (12 points) Find the potential at any point in polar coordinates.

2. (5 points) Find the induced surface charge density on the cylinder and plot it qualitatively in units of λ/b for $R/b = 2, 4$.

3. (3 points) What is the force per length on the line charge?

The test paper given to the students had the explicit forms of vector operations from the back page of Jackson.

EM1

In view of the initial current distribution we seek the potentials

$$
\begin{aligned}
\text{in}: &\qquad \Psi_i = C\, r^2 \cos(2\phi) \\
\text{out}: &\qquad \Psi_o = A\, \frac{\cos(2\phi)}{r^2}
\end{aligned}
\tag{1}
$$

which solve Ampere's law with $\vec{H} = -\vec{\nabla}\Psi$ outside/inside. The corresponding magnetic fields are

$$
\begin{aligned}
\vec{H}_i = &\qquad -2Cr\cos(2\phi)\hat{r} + 2Cr\sin(2\phi)\,\hat{\phi} \\
\vec{H}_o = &\qquad +\frac{2A}{r^3}\cos(2\phi)\hat{r} + \frac{2A}{r^3}\sin(2\phi)\,\hat{\phi}
\end{aligned}
\tag{2}
$$

To make the magnetic normal continuous $\partial_r \Psi_i = \partial_r \Psi_o$ at $r = R$. So $A = -CR^4$. Ohm's law implies $E_z = K_z/(\sigma\Delta)$. Ampere's boundary condition implies $H_{o\phi} - H_{i\phi} = 4\pi\, K_z$. The radial component of Faraday's law then require

$$
\frac{1}{R\Delta\sigma}\frac{\partial}{\partial\phi}(H_{o\phi} - H_{i\phi}) = -\frac{\mu}{c}\frac{\partial H_{or}}{\partial t}
\tag{3}
$$

Substituting (2) and using $C = -A/R^4$ yield

$$
\frac{1}{4\pi\, R\Delta\sigma}\left(\frac{4A}{R^3} - 4CR\right) = \frac{1}{4\pi\, R\Delta\sigma}\left(\frac{8A}{R^3}\right) = -\frac{2\mu}{R^3}\frac{dA}{dt}
\tag{4}
$$

It follows that the decay constant is $\tau = \pi\mu\sigma\Delta R/c$. Given the initial current density, we have

$$
\vec{\mathbf{K}}(t,\phi) = \vec{\mathbf{K}}(0,\phi)\, e^{-t/\tau}
\tag{5}
$$

EM2

1. The Transverse electric field associated with the moving relativistic particle is

$$
E_\perp = \frac{\gamma q b}{(b^2 + \gamma^2 v^2 t^2)^{3/2}}
\tag{6}
$$

It acts like a pulse of strength $\gamma q/b^2$ for about a time $\Delta t \approx b/\gamma v$. This pulse causes a change in the electron transverse momentum of order $\Delta p_\perp \approx eE_\perp \Delta t \approx qe/bv$. Thus a change in the kinetic energy

$$\Delta K(b) \approx \frac{(\Delta p_\perp)^2}{2m} \approx \frac{q^2 e^2}{mv^2} \frac{1}{b^2} \tag{7}$$

2. The minimum impact parameter $bmin$ for which this approximation is valid is $bmin \approx v_\perp \, \Delta t \approx qe/\gamma mv^2$, for otherwise the transverse kick exceeds the impact parameter. The maximum impact parameter $bmax$ is set by the passing time not to exceed $1/\omega$, i.e. $bmax/\gamma v \approx 1/\omega$, for otherwise the classical electron is not considered bound. Thus

$$bmin \approx \frac{qe}{\gamma mv^2} \leq b \leq bmax \approx \frac{\gamma v}{\omega} \tag{8}$$

3. The total energy loss in the Coulomb encounter is

$$\frac{\Delta E}{\Delta x} = \int_{bmin}^{bmax} d\vec{q}_\perp \Delta K(b) = \frac{2\pi n q^2 e^2}{mv^2} \, ln \left(\frac{bmax}{bmin} \right) \tag{9}$$

EM3

1. We will use the image method because of cylindrical symmetry to solve this problem. The potential is

$$\phi(\vec{r}) = -2\lambda \left(\, ln\rho + \, ln\rho' \right) + \phi_0 \tag{10}$$

with ρ, ρ' the distances from the line and its image to an arbitrary point r in the plane xy. The image charge distance from the center of the cylinder is in harmonic proportions with the line charge, i.e. $RR' = b^2$ and the charge is opposite $\lambda' = -\lambda$. Also $\rho'/\rho = b/R$ for any point on the cylinder. By requiring the cylinder to be grounded

$$\phi_0 = 2\lambda \, ln(R/b) \tag{11}$$

or

$$\phi(\vec{r}) = 2\lambda \, ln \left(\frac{\rho' R}{\rho b} \right) \tag{12}$$

2. The induced surface charge density is $4\pi\sigma = -\partial_r \phi$ at $r = b$

$$\sigma = -\frac{\lambda}{2\pi}\frac{\partial}{\partial r}\left(ln\left(\frac{\rho' R}{\rho b}\right)\right)_{r=b} \tag{13}$$

For $R/b = 2, 4$,

$$\begin{aligned}\sigma_2 &= -\frac{\lambda}{2\pi b}\left(\frac{3}{5 - 4\,cos(\theta)}\right) \\ \sigma_4 &= -\frac{\lambda}{2\pi b}\left(\frac{15}{17 - 8\,cos(\theta)}\right)\end{aligned} \tag{14}$$

The force per length between the wire and the cylinder is attractive

$$\vec{F} = \lambda\vec{E}(r = b) = -\frac{2\lambda^2 R}{R^2 - b^2}\,\hat{x} \tag{15}$$

Classical Mechanics Placement. Aug. 25, 2009

Problem 1

A meteorite with the initial velocity v_∞ approaches an atmosphere-free planet of mass M and radius R.

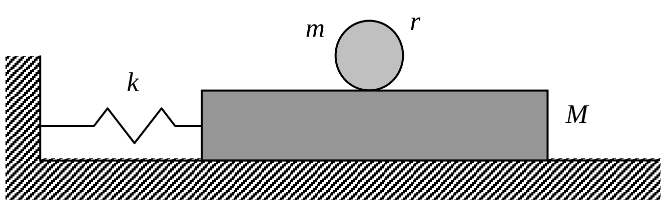
(a) Find the condition on the impact parameter b for the meteorite to hit planet's surface.

(b) If the meteorite barely avoids the collision, what is its scattering angle?

Problem 2.

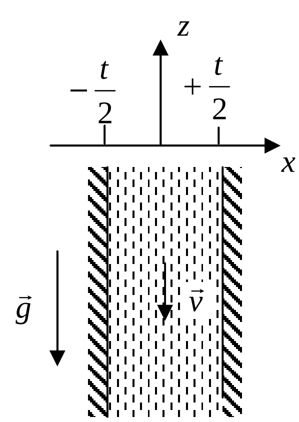
A round uniform cylinder of radius r and mass m may roll, without slipping, on a horizontal surface of a block of mass M. The block, in turn, may move, without friction, on an immobile horizontal surface, being connected to it with a spring (see Figure below).

(a) Find the equations of motion of the system (within the plane of the picture).
(b) Find and interpret the integral(s) of motion.
(c) Find the frequency of small (linear) oscillations of the system near the equilibrium.
(d) Find the distribution coefficients and sketch the oscillation mode.



Problem 3.

An incompressible fluid with density ρ and viscosity η flows down, under its own weight, through a vertical gap of thickness t between two wide, parallel plates. Find the stationary velocity distribution and fluid discharge (per unit width).



Solution 1

(a) A massive, spherically-symmetric sphere produces a gravity field similar to that of a particle of the same mass, located in the sphere's center. (This fact may be readily proven by the gravitational analog of the Gauss theorem.)

This is why we may relate the planet's radius R and the largest value $b_{\max}$ of the impact parameter using equation $U_{\text{ef}}(R) = E = (m/2)v_\infty^2$, where

$$U_{ef}(R) = U(R) + \frac{L_z^{\ 2}}{2mR^2} = -\frac{GMm}{R} + \frac{mv_\infty^2 b_{\max}^2}{2R^2},$$

since $L_z = mv_\infty b_{\max}$. From this equation, we readily get

$$b < b_{\max} = R\left(1 + \frac{2GM}{Rv_\infty^2}\right)^{1/2}.$$

(b) Using Eq. (4.44) of the lecture notes, we get

$$\tan\frac{|\theta|}{2} = \frac{GM}{v_\infty^2 b_{\max}} = \frac{(GM/v_\infty^2 R)}{\left[1 + (GM/v_\infty^2 R)\right]^{1/2}}.$$

Solution 2

(a) A good choice of generalized coordinates are the horizontal positions of the centers of mass (say, X for the block and x for the cylinder) measured in the lab frame. In these coordinates,

$$L = \frac{M}{2}\dot{X}^2 + \frac{m}{2}\dot{x}^2 + \frac{I}{2}\omega^2 - \frac{k}{2}X^2,$$

where $I = mr^2/2$ is the cylinder's moment of inertia, and ω its angular velocity. From kinematics (no-slipping condition), $r\omega = \dot{X} - \dot{x}$. Using this relation to exclude ω from L, we get

$$L = \left(\frac{M}{2} + \frac{m}{4}\right)\dot{X}^2 + \frac{3m}{4}\dot{x}^2 - \frac{m}{2}\dot{X}\dot{x} - \frac{k}{2}X^2.$$

This gives the following Lagrangian equations:

$$\left(M + \frac{m}{2}\right)\ddot{X} - \frac{m}{2}\ddot{x} + kX = 0,$$
$$\frac{3}{2}m\ddot{x} - \frac{m}{2}\ddot{X} = 0. \quad (1)$$

(b) The last equation immediately gives the first integral of motion,

$$\frac{3}{2}\dot{x} - \frac{m}{2}\dot{X} = const.$$

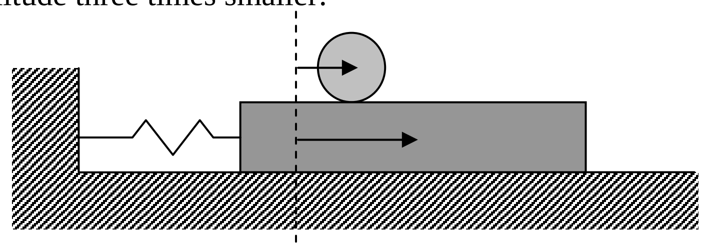
The second integral is given by conservation of energy

$$E = \left(\frac{M}{2} + \frac{m}{4}\right)\dot{X}^2 + \frac{3m}{4}\dot{x}^2 - \frac{m}{2}\dot{X}\dot{x} + \frac{k}{2}X^2.$$

(c) Solving Eqs. (1) for either X or x, we get a standard differential equation of linear oscillations with frequency

$$\omega = \left(\frac{k}{M + m/3}\right)^{1/2}.$$

(d) As immediately visible from the second of Eqs. (1), at these oscillations $x(t) = X(t)/3$ (for any mass ratio), i.e. the cylinder's center oscillates in phase with the block, but with an amplitude three times smaller:



The cylinder turn angle is $\varphi = (X - x)/r = 2x/r = (2/3)X/r$.

Solution 3

With the coordinate choice shown in Fig., fluid velocity has only one component: $\vec{v} = v_z(x)\vec{n}_z,$ for which the Navier-Stokes equation gives

$$-g + \frac{\eta}{\rho}\frac{d^2 v_z}{dx^2} = 0.$$

(Note that in this case there is no pressure gradient along the gap.) Solving this equation with boundary conditions $v_z(\pm t/2) = 0$, we get

$$v_z = -\frac{\rho g}{2\eta}\left(\frac{t^2}{4} - x^2\right), \quad \left|v_z\right|_{\max} = \frac{\rho g t^2}{8\eta}.$$

(Note that velocity scales as the reciprocal kinetic viscosity $\nu = \eta/\rho$ rather than the dynamic viscosity η, which gives a very clear physical meaning of ν.) From here, the discharge per unit width is

$$q = \rho \int_{-t/2}^{+t/2} \left|v_z\right| dx = \frac{\rho^2 g t^3}{12\eta}.$$