QM midterm notes Cameron

QM I

II projection of polarization measurements

$$P_\theta |\epsilon\rangle = |\theta\rangle\langle\theta|\epsilon\rangle = \underbrace{\langle\theta|\epsilon\rangle}_{\text{outcome}} \underbrace{|\theta\rangle}_{\text{2 vector unit}}$$

P_θ ← 2x2 matrix

Successive measurements add $|\theta_2\rangle\langle\theta_2| = P_{\theta_2}$ matrix in the

$$\frac{I_f}{I_{initial}} = \left| \langle \epsilon_{final}| P_{\theta_1} \cdot P_{\theta_2} \cdots |\epsilon_{initial}\rangle \right|^2$$

$$I_f = \left| \langle f|P|i\rangle \right|^2 I_i$$

Stern-Gerlach

$$S_z |\pm\rangle_z = \pm \frac{\hbar}{2} |\pm\rangle_z \quad , \quad |+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Pauli matrices ↓

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & - \\ i & 0 \end{pmatrix}$$

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle_z \mp i|-\rangle_z \right) \quad , \quad |\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle_z \pm |-\rangle_z \right)$$

$$S_{x,y,z} = \frac{\hbar}{2}\sigma_i \rightarrow \sigma_i^\dagger = \sigma_i \quad \sigma_i^2 = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1}_2 + i\epsilon_{ijk}\sigma_k$$

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3\mathbb{1}_2 \rightarrow \sigma_i\sigma^i = \delta_i^i \mathbb{1}_2$$

$$\vec{S}^2 = S_i S^i = \frac{\hbar^2}{4}\delta_i^i \mathbb{1}_2 = \frac{3\hbar^2}{4}\mathbb{1}_2 \rightarrow |\vec{S}| = \frac{\sqrt{3}\hbar}{2}$$

Mean of some measurement

$$\bar{A} = \langle\alpha|A|\alpha\rangle \quad , \quad \Delta A^2 = \langle\alpha|(A-\bar{A})^2|\alpha\rangle \quad , \quad P_{ob_A}(a) = |\langle a|\alpha\rangle|$$

basis can be chosen to minimize this uncertainty (gaussians...)

Uncertainty: if $[A,B] \neq 0$ then $\Delta A \Delta B \geq \frac{1}{2} \left| \langle [A,B] \rangle \right|$
if $[A,B] = 0$ then you can measure each precisely as they share eigenstate spectra.
proof with Cauchy-Schwartz inequality. (see 9/15 notes)

X-rep → (see 9/17 notes)

Translation operator $\Pi_a = e^{\frac{ia}{\hbar}\hat{p}}$ (X-rep), $\boxed{\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}}$

$\Pi_a^\dagger \Pi_a = \mathbb{1}$ unitary transformation group.

p-rep: $\mathbb{P}|p\rangle = p|p\rangle \rightarrow \langle x|\mathbb{P}|p\rangle = p\langle x|p\rangle$

$$\mathbb{P}\langle x|p\rangle = p\langle x|p\rangle$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}P(x) = p\,P(x)$$

$$\int \frac{\partial P(x)}{P(x)} = \int \frac{ip}{\hbar}\partial x \rightarrow \ln|P(x)| = \frac{ipx}{\hbar} + \mathbb{C}$$

$$P(x) = \mathbb{C}e^{ipx/\hbar}$$

↑ eigenvalue of $\mathbb{P}$ in x-rep

$\mathbb{C} = \frac{1}{\sqrt{2\pi}}$

Fourier → p is localized in p space but totally spread through x-space.

x-rep gaussian wavepacket

$$\alpha(x) = \frac{e^{-\frac{x^2}{2\sigma^2} + \frac{ipx}{\hbar}}}{(2\sigma\sqrt{\pi})^{1/2}}, \quad \bar{x} = \langle\alpha|\hat{x}|\alpha\rangle = 0$$

$\bar{p} = \langle\alpha|\hat{p}|\alpha\rangle = p$

$\Delta x^2 = \sigma^2/2$

$\Delta p^2 = \hbar^2/2\sigma^2$

$\Delta x \Delta p = \hbar/2$ saturates.

III Time evolution: $|\alpha\rangle \to |\alpha(t)\rangle$ time as a label

$$|\alpha(t)\rangle = U_{(t,0)}|\alpha(0)\rangle$$

$$U^{\dagger}_{(t,0)} U_{(t,0)} = \mathbb{1} \qquad U_{(t,0)} = e^{\frac{-itH}{\hbar}} \qquad \begin{pmatrix} x \text{ vs } p \\ t \text{ vs } E \end{pmatrix}$$

$$\therefore\ i\hbar \frac{d}{dt} e^{\frac{-itH}{\hbar}} = H\,U(t,0)$$

plug in $\alpha(0)$ states.

Schrodinger equation

$$i\hbar \frac{d}{dt}|\alpha(t)\rangle = H|\alpha(t)\rangle$$ Basis free schrodinger equation.

$$i\hbar \frac{d}{dt}\alpha(t,x) = \int dx' \langle x|H|x'\rangle\, \alpha(t,x')$$ x-specific S.E.

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$$ is energy operator often.

$$i\hbar \frac{d}{dt}\alpha(t,x) = \left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\alpha(t,x)$$

x-rep S.E.

We get "stationary states" when $|\alpha(t)\rangle = |E\rangle$, s.t. $H|E\rangle = E|E\rangle$, and $E(t) = e^{-\frac{itE}{\hbar}}|E(0)\rangle$

For $[A,H]=0$ you get degeneracy w.r.t. "a"

$$H|E,a\rangle = E|E,a\rangle$$
$$A|E,a\rangle = a|E,\underbrace{a}\rangle$$ double index.

non-stationary $\to$ $|\alpha(t)\rangle = \sum_E \underbrace{e^{\frac{-itE}{\hbar}} C_{E\alpha}(0)}_{C_{E\alpha}(t)} |E\rangle$

matrix elements

$(t \cdot \Delta E \geq \hbar)$ time heisenberg uncertainty relation.

Heisenberg business → $A_H = U^\dagger(t,0)\, A_S\, U(t,0)$ → operator is what moves.

iff $A_S(t) = A_S(0)$ then $\frac{d}{dt}A_H = \frac{1}{i\hbar}\left[A_H, H\right]$ since $U(t,0) = e^{\frac{-itH}{\hbar}}$

$\therefore\ i\hbar \frac{dA_H}{dt} = [A_H, H]$ is the heisenberg equation of motion.

$$\left[\hat{x}_i, F(\hat{\vec{p}})\right] = i\hbar \frac{\partial F}{\partial \hat{p}_i}, \quad \left[\hat{p}_i, G(\hat{\vec{x}})\right] = -i\hbar \frac{\partial G}{\partial \hat{x}_i}$$

$$\hookrightarrow \frac{dx_i}{dt} = \underbrace{\frac{1}{i\hbar}[x_i, H]}_{\text{heisenberg eqn}} = \frac{1}{i\hbar}\frac{1}{2m}\, i\hbar \frac{d}{dp_i}\left(\sum_{j=1} p_j^2\right) = \frac{p_i}{m} = \frac{p_i(0)}{m}$$

$$\left[x_i(t) = x_i(0) + \frac{p_i(0)}{m}t\right]$$

so $[x_i(t), x_i(0)] = \left[\frac{p_i(0)}{m}t, x_i(0)\right]$

$$\boxed{[x_i(t), x_i(0)] = \frac{-i\hbar t}{m}}$$

TDSE (earlier) + TISE → $\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\,\psi(x)$

- probability current conservation: $\vec{\nabla} \ln(\psi)\Big|_{\text{Boundary}}$ = Continuous (when no infinities exist) + ψ (point at ∞ or $V=\infty$) $= 0$

free particle ~ $\psi(x) = \frac{e^{\pm ikx}}{\sqrt{2\pi}}$ general solution

trapped ~ $\psi''(x) + \omega^2\psi(x) = 0 \rightarrow \psi(x) = \sin\omega x$ or $\cos\omega x$

Quantization comes from boundary conditions for trapped particles
+ probability current conservation (or zero ψ at infinite boundaries)
+ normalization helps too.

ctd. delta function potentials must be integrated across boundary.

$$\int_{-0}^{+0} dx \left[-\frac{\hbar^2}{2m}\phi''(x) + g\cdot\delta(x)\,\phi(x) = E\,\phi(x) \right]$$

$$-\frac{\hbar^2}{2m}\left[\phi'(x)\right]\Big|_{-0}^{+0} + g\int_{-0}^{+0}\delta(x)\,\phi(x)\,dx = E\int_{-0}^{+0}\phi(x)\,dx$$

$$-\frac{\hbar^2}{2m}\left[\overset{\text{derivative}}{\underset{\text{right}}{\phi'(0_+)} - \underset{\text{left}}{\phi'(0_-)}}\right] + g\cdot\phi(0) = E\,\phi(0)\cdot\cancelto{0}{(0_- - 0_+)}$$

choose either right or left side to evaluate at.

* $$\boxed{g\,\phi(x_0)\Big|_{x_0 = \text{on boundary}} = \frac{\hbar^2}{2m}\left[\phi'(x_0)\right]\Big|_{x_0 = \text{across boundary right-left.}}}$$

- delta function derivative discontinuity condition.

yields bound state energy additional term.

$$E_+ = -\frac{g^2 m}{2\hbar}$$

- Wave functions of form $\phi(x) \propto e^{-c|x|}$

$\hbar \propto x \cdot p$

Energy

S.H.O. : $H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \frac{\hbar\omega}{2}\left[\frac{p^2}{\hbar^2/x_0^2} + \frac{x^2}{x_0^2}\right]$ $\quad x_0^2 = \frac{\hbar}{m\omega}$

change variables dimensionless. $\begin{cases} p \to \frac{p}{\hbar/x_0} \text{ at end} \\ x \to x/x_0 \text{ at end} \end{cases}$

So $H = \frac{\hbar\omega}{2}\left[p^2 + x^2\right] \to [x,p] = i$ (originally $i\hbar$)

define $\hat{a} = \frac{\hat{x}+i\hat{p}}{\sqrt{2}}$, $\hat{a}^+ = \frac{\hat{x}-i\hat{p}}{\sqrt{2}}$ $\quad [a,a^+] = 1$, $[a^+,a] = -1$

s.t. $\hat{x} = \frac{\hat{a}+\hat{a}^+}{\sqrt{2}}$, $\hat{p} = \frac{\hat{a}-\hat{a}^+}{i\sqrt{2}} = i\left(\frac{a^+-a}{\sqrt{2}}\right)$

$H = \hbar\omega\left[a^+a + \frac{1}{2}\right]$ $\quad a^+a = N$, $N|n\rangle = n|n\rangle$

$[a^+a, a^+] = a^+$

* $\langle x|0\rangle = \frac{1}{\sqrt{2\sqrt{\pi}}}\frac{e^{-\frac{x^2}{2x_0^2}}}{\sqrt{x_0}}$

$E_n = \hbar\omega(n-\frac{1}{2})$ $n \in \mathbb{N}$

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a|0\rangle = 0 \to a^+|0\rangle = |1\rangle$

$\langle 0|a^+ = 0$, $\langle 0|a = \langle 1|$

BCH rule

$e^A e^B = e^B e^A e^{[A,B]}$ $\quad [A,B] \in \mathbb{C}$

$e^{A+B+\frac{1}{2}[A,B]} = e^A e^B$

$\therefore e^{\alpha a} e^{\alpha a^+} = e^{\alpha a^+} e^{\alpha a} e^{|\alpha|^2 [a,a^+]}$ (with $[a,a^+] = 1$)

$e^{\alpha a + \alpha a^+} = e^{\alpha a} e^{\alpha a^+} e^{-\frac{\alpha^2}{2}[a,a^+]}$ (with $[a,a^+] = 1$)

* $\langle x|n\rangle = \langle x|\frac{a^{+n}}{\sqrt{n!}}|0\rangle$

or $H_n(x) = 2x H_{n-1} - H'_{n-1}$
with $H_0 = 1$, $H_1 = 2x$, etc.

$\phi_n(x) = \frac{H_n(x)\, e^{-x^2/2}}{\sqrt{2^n n!}\,\pi^{1/4}}$

lowering

Coherent states: $a|\lambda\rangle = \lambda|\lambda\rangle$

$[a, a^{+n}] = n(a^{+})^{n-1}$, $\langle\lambda|\lambda\rangle = 1$

SHO GS.

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} e^{\lambda a^{+}} |0\rangle$$

BCH → $e^{A} e^{B} = e^{B} e^{A} e^{[A,B]}$

$e^{A+B} = e^{A} e^{B} e^{-[A,B]/2}$
$= e^{B} e^{A} e^{[A,B]/2}$

$$\langle\lambda'|\lambda\rangle = e^{-\frac{1}{2}(|\lambda'|^2 + |\lambda|^2 - 2\lambda'^{*}\lambda)}$$
$$\neq \delta(\lambda' - \lambda)$$

$$\mathbb{1}_{\lambda} = \frac{1}{\pi}\int d\lambda\, |\lambda\rangle\langle\lambda|$$

$$\langle n|(\sum_{\lambda} |\lambda\rangle)\langle\lambda|m\rangle = \pi\,\delta_{n,m} = \pi(\mathbb{1}_{\lambda})_{n,m}$$

Stationary phase approximation

$$|\lambda\rangle = e^{-\frac{|\lambda|^2}{2}} \sum \frac{\lambda^{n} a^{+n}}{\sqrt{n!}\sqrt{n!}} |0\rangle = e^{-\frac{|\lambda|^2}{2}} \sum \frac{\lambda^{n}}{\sqrt{n!}} |n\rangle$$

$$= \sum e^{n[\ln(\lambda) - \frac{1}{2}\cdot\frac{1}{n}\cdot\ln(n!)]} |n\rangle$$

λ is imaginary, ∴ we expect that for large n this exponent varies like sine & cosine rapidly, so the dominant contributions come from minima of this expression.

$$\frac{\delta}{\delta n}\left[n \cdot \ln|\lambda| - \frac{1}{2}\frac{1}{n} n\ln(n)\right] = 0$$

$$\ln|\lambda| - \frac{1}{2}\ln(\bar{n}) - \frac{1}{2} = 0$$

negligible

so $|\lambda\rangle \simeq e^{-\frac{|\lambda|^2}{2}} \frac{\lambda^{\bar{n}}}{\sqrt{\bar{n}!}} |\bar{n}\rangle$ is dominant term in the summation

[illegible]

Coherent States in time

$$|\lambda(t)\rangle = e^{-\frac{iHt}{\hbar}} |\lambda\rangle \;,\; H = \hbar\omega(a^\dagger a + \tfrac{1}{2})$$

$$= U(t)|\lambda\rangle = e^{-|\lambda|^2/2}\, U(t)\, e^{\lambda a^\dagger}|0\rangle$$

$$U(t)|0\rangle = e^{-\frac{i\omega t}{2}}|0\rangle$$

$$U(t)\, e^{\lambda a^\dagger} = \underbrace{U(t)\, e^{\lambda a^\dagger} U^\dagger(t)}\cdot U(t)$$

$$e^{\lambda a^\dagger(t)} \cdot U(t) \quad \& \quad a^\dagger(t) = e^{i\omega t} a^\dagger$$

$$\text{So } |\lambda(t)\rangle = e^{-i\omega t/2}\, e^{-|\lambda|^2/2}\, e^{\lambda e^{i\omega t} a^\dagger}|0\rangle$$

$$|\lambda(t)\rangle = e^{-i\omega t/2}\,|\lambda e^{i\omega t}\rangle$$ which is still a coherent state

Since $a|\lambda\rangle = \lambda|\lambda\rangle$ (definition of coherent state)

then $\langle\lambda| a^\dagger = \langle\lambda|\lambda^*$

$$\therefore \langle\lambda|\hat{x}|\lambda\rangle = \langle\lambda|\frac{a + a^\dagger}{\sqrt{2}}|\lambda\rangle$$

$$= \langle\lambda|\frac{\lambda^*}{\sqrt{2}}|\lambda\rangle + \langle\lambda|\frac{\lambda}{\sqrt{2}}|\lambda\rangle$$

$$\bar{x} = \frac{\lambda^* + \lambda}{\sqrt{2}}$$

$$\therefore \langle\lambda|\hat{x}^2|\lambda\rangle = \langle\lambda|\frac{a^2 + a^{\dagger 2} + aa^\dagger + a^\dagger a}{2}|\lambda\rangle$$

$$= \frac{1}{2}\langle\lambda|a^2 + a^{\dagger 2} + 2a^\dagger a + \underbrace{[a, a^\dagger]}_{1}|\lambda\rangle$$

$$= \frac{1}{2}\langle\lambda|(a^\dagger + a)^2 + 1|\lambda\rangle$$

$$\overline{x^2} = \frac{1}{2}\left((\lambda + \lambda^*)^2 + 1\right) = \bar{x}^2 + 1$$

$$\Delta x = \overline{x^2} - \bar{x}^2 = \frac{1}{2} \rightarrow \frac{\hbar}{2} \rightarrow$$ Saturates H.U.P.

Math: $\int_0^\infty x^{2n} e^{-x^2/a^2}\,dx = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$

$\int_0^\infty x^{2n+1} e^{-x^2/a^2}\,dx = \frac{n!}{2}\,a^{2n+2}$

$\text{\&} \quad \int_0^\infty x^n e^{-\frac{x}{a}}\,dx = n!\,a^{n+1}$

$[AB, C] = A[B,C] + [A,C]B$

derivative relation $= \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\,dx = (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{d\alpha^n} e^{-\alpha x^2}\,dx = \sqrt{\pi}\,(-1)^n \frac{d^n}{d\alpha^n}(\alpha^{-\frac{1}{2}}$

$n=0 \to \sqrt{\frac{\pi}{a}}$ = gaussian from $-\infty$ to ∞