17/20

a) because lithium has 3 electrons and 7 nucleons, the spins from the nucleus always form an even angular momentum sum with the electrons

$l_{nucleus} = 1/2, 3/2, 5/2, or\ 7/2$ I don't know which

$l_{electrons} = 1/2$ or $3/2$

and the spherical tensors available to these to add together are $\rightarrow$ (odd half integer) $\otimes$ (odd half integer)

$= |odd - odd| + \ldots + |odd + odd|$

↑ range of available final states

and all odd ± odd = even, ∴

since $\{l_{final}\}$ = even we have a boson here,

2/2

OK

b) $E(a) = \frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{\int_{-\infty}^{\infty} d^3x\, \Psi^*_{(x)} H \Psi_{(x)}}{\int_{-\infty}^{\infty} |\Psi_{(x)}|^2 d^3x}$

letting $\Psi_{(x)} = C \cdot e^{-r^2/2a^2}$

we get $\langle\Psi|\Psi\rangle = C^2 \int_{-\infty}^{\infty} d^3x\, e^{-r^2/a^2} = C^2 4\pi \int_{-\infty}^{\infty} dr\, e^{-r^2/a^2} \cdot r^2$

now the fundamental gaussian integral is $\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$

then $\left(-\frac{d}{da}\right)^n \int_{-\infty}^{\infty} dx\, e^{-ax^2} = \int_{-\infty}^{\infty} dx\, x^{2n} e^{-ax^2} = \left(-\frac{d}{da}\right)^n \sqrt{\frac{\pi}{a}}$

so $|C|^2 \int_{-\infty}^{\infty} dx\, x^2 e^{-x^2/a^2} = |C|^2 - 1 \cdot \frac{d}{d(1/a^2)} \sqrt{\frac{\pi}{(1/a^2)}}$

$= |C|^2 - 1 \cdot \frac{d}{dx} x^{-1/2} \cdot \sqrt{\pi}$

$= |C|^2 \frac{1}{2} \cdot x^{-3/2} \sqrt{\pi} = \boxed{\frac{\sqrt{\pi}}{2} a^3 \cdot |C|^2}$

$x = \frac{1}{a^2}$

normalization

" 1 iff

$\boxed{C = \sqrt{2} \frac{1}{\pi^{3/4}} \frac{1}{\sqrt{a^3}}}$

Then the numerator is

$\int_{-\infty}^{\infty} d^3\vec{r}\; e^{-r^2/2a} \cdot C^* \left(\frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 r^2 + \frac{Ng}{2} |C|^2 e^{-r^2/a^2} \right) \cdot e^{-r^2/2a} \cdot C$

$E = |C|^2 \cdot 4\pi \cdot \int_{-\infty}^{\infty} r^2 dr \left\{ e^{-r^2/2a} \frac{p^2}{2m} e^{-r^2/2a} + \frac{1}{2} m\omega^2 r^2 e^{-r^2/a^2} \ldots e^{-2r^2/a^2} \right\}$

$\partial_r^2 + \frac{2}{r}\partial_r$

$\partial_r e^{-r^2/2a^2} = -\frac{2r}{2a^2} e^{-r^2/2a^2}$

$E(a) = |C|^2 \cdot 4\pi \cdot (\text{3 terms})$

① $\int_{-\infty}^{\infty} r^2 dr\, e^{-r^2/2a} \cdot \frac{-\hbar^2}{2m} \vec{\nabla}^2 e^{-r^2/2a} = \frac{-\hbar^2}{2m} \int_{-\infty} dr\, r^2 e^{-r^2/2a^2} \left(\partial_r^2 + \frac{2}{r}\partial_r\right) e^{-r^2/2a^2}$

$= \frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} r^2 dr\, e^{-r^2/2a^2} \cdot \left(\partial_r + \frac{2}{r}\right) \left(\frac{-2r}{2a^2} e^{-r^2/2a^2} \right)$

$= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} r^2 dr\, e^{-r^2/2a^2} \cdot \left[\frac{-2}{a^2} e^{-r^2/2a^2} - \frac{1}{a^2} \left(e^{-r^2/2a^2} - \frac{r^2}{a^2} e^{-r^2/2a^2} \right) \right]$

$= \frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} dr \left(-\frac{3r^2}{a^2} e^{-r^2/a^2} + \frac{r^4}{a^4} e^{-r^2/a^2} \right)$

$\downarrow$ $\frac{-\hbar^2 \cdot -3}{2m a^2} \cdot \frac{\sqrt{\pi}}{2} a^3$

$\downarrow$ $\frac{-\hbar^2}{2m} \cdot \frac{1}{a^4} \cdot -1 \frac{d}{dx} \left(\frac{1}{2} x^{-3/2} \sqrt{\pi} \right)$

$= \frac{-\hbar^2}{2ma^4} \cdot \sqrt{\pi} \cdot \frac{3}{4} \left(\frac{1}{a^2}\right)^{-5/2}$

$= \frac{-\hbar^2}{2ma^4} \sqrt{\pi} \frac{3}{4} a^5$

① $= \frac{3\hbar^2 \sqrt{\pi}\, a}{4m} - \frac{3\hbar^2 \sqrt{\pi}}{8m} a = \frac{3\hbar^2 \sqrt{\pi}}{8m} a$

$4\pi |C|^2$ factor makes this

① $= A/a^2 \to 0$

$\times\; 4\pi |C|^2 = a 3\hbar^2 \pi \sqrt{\pi} \cdot \frac{2}{2m \sqrt{\pi}\, a \ldots}$

$\boxed{\text{kinetic} = \frac{3\hbar^2 \pi}{\ldots^2}}$

PE

$$② = \int_{-\infty}^{\infty} r^2 dr \cdot \frac{1}{2} m\omega^2 r^2 e^{-r^2/a^2} = \frac{1}{2} m\omega^2 \int_{-\infty}^{\infty} dr\, r^4 e^{-r^2/a^2}$$

$$\sqrt{\pi}\,\frac{3}{2}\left(\frac{1}{a^2}\right)^{-5/2} = \frac{3}{2}\sqrt{\pi}\, a^5$$

$$② = \frac{1}{2} m\omega^2 \cdot \frac{3}{2}\sqrt{\pi}\, a^5$$

Interaction energy

$$③ = \int_{-\infty}^{\infty} \frac{N_g}{2} |c|^2 e^{-2r^2/a^2} r^2 dr$$

$$= \frac{N_g |c|^2}{2} \int_{-\infty}^{\infty} dr\, r^2 e^{-\frac{2}{a^2} r^2}$$

$$2^{-3/2} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$$

$$\frac{\sqrt{\pi}}{2}\left(\frac{2}{a^2}\right)^{-3/2} = \frac{a^3\sqrt{\pi}}{2\cdot 2\sqrt{2}} = \frac{a^3}{4}\sqrt{\frac{\pi}{2}}$$

$$③ = \frac{N_g |c|^2}{2} \frac{a^3}{4}\sqrt{\frac{\pi}{2}}$$

$$\therefore\ E = KE + \left(\frac{1}{2} m\omega^2 \frac{3}{2}\sqrt{\pi}\, a^5 + N_g \frac{|c|^2 a^3 \sqrt{\pi/2}}{8}\right) 4\pi |c|^2$$

$$|c|^2 = \frac{2}{\sqrt{\pi}} \frac{1}{a^3}$$

$$KE = \frac{3\hbar^2\pi}{2ma^2}$$

$$\therefore\ E = \frac{4\pi\cdot 2}{\sqrt{\pi}} \frac{1}{a^3}\left(\frac{3\sqrt{\pi}}{4} m\omega^2 a^5 + \frac{2}{\sqrt{\pi}} \frac{1}{a^3} \frac{N_g}{8} a^3 \frac{\sqrt{\pi}}{\sqrt{2}}\right) + KE$$

$$= 8\cdot\sqrt{\pi}\left(\frac{3}{4}\sqrt{\pi}\, m\omega^2 a^2 + \frac{\sqrt{2}}{8} N_g \frac{1}{a^3}\right) + \frac{3\hbar^2\pi}{2ma^2}$$

Very minor problems

$A = \frac{3\hbar^2\pi}{2m}$, $B = 6\pi m\omega^2$, $C = \sqrt{2\pi}\, N_g$

~~8/10~~ 7/7

$$\boxed{E_{(a)} = \underbrace{6\pi m\omega^2 a^2}_{\uparrow PE} + \underbrace{\sqrt{2\pi}\, N_g / a^3}_{\uparrow \text{interaction}} + \underbrace{\frac{3\hbar^2\pi}{2m} / a^2}_{\uparrow KE}}$$

c) $\lim \quad \frac{\sqrt{2\pi}\, N_g}{a^3} \gg \frac{3\hbar^2\pi}{2m} \frac{1}{a^2}$

$$\downarrow$$

$$\frac{N_g}{\hbar^2} \gg a$$

small $\hbar$

we ignore KE

5/5

$$\frac{d E(a)}{da} = 0 = 12\pi m\omega^2 a - \frac{3\sqrt{2\pi}\, N_g}{a^4} - \underbrace{\frac{3\hbar^2\pi}{ma^3}}_{\to 0}$$

$$\therefore\ 12\pi m\omega^2 a = \frac{3\sqrt{2\pi}\, N_g}{a^4} \quad \therefore$$

$$a^5 = \frac{\sqrt{2}}{4\sqrt{\pi}} \frac{N_g}{m\omega^2} \rightarrow a = \left(\frac{\sqrt{2}}{4\sqrt{\pi}} \frac{N_g}{m\omega^2}\right)^{1/5}$$

So $\boxed{a_{min} \propto N^{1/5}}$

and $H_{min} = 6\pi m\omega^2 a^2 + \sqrt{2\pi}\, Ng/a^3$

$$\propto \underbrace{N^{2/5}} + N^{-3/5}$$

$$\langle H \rangle_{min} \propto N^{2/5}$$

d) for $a_{min} = \left(\frac{\sqrt{2}}{4\sqrt{\pi}} \frac{Ng}{m\omega^2}\right)^{1/5}$

$$N = a_{min}^{5} \frac{4\sqrt{\pi}\, m\omega^2}{\sqrt{2}\, g}$$

We maximize the number of particles as in part c) as the Number of particles whose interaction energy matches the Potential energy (KE = 0), else the particles just escape.

$\therefore$ KE + PE = $-$Int E $\quad \frac{3\hbar^2\pi}{2ma^2} + 6\pi m\omega^2 a^2 = -\frac{\sqrt{2\pi}\, Ng}{a^3}$

mass $\propto$ 7 geV/c^2

then $N = \frac{-a_{min}^3}{g\sqrt{2\pi}} \cdot a_{min}^2\, 6\pi m\omega^2 - \frac{3\hbar^2\pi}{2m} \frac{a_{min}}{\sqrt{2\pi}} \frac{1}{g}$

$$a_{min} = \left(\frac{\sqrt{2}}{4\sqrt{\pi}} \frac{Ng}{m\omega^2}\right)^{1/5}$$

$$N = -\frac{\sqrt{2}\, Ng}{4\sqrt{\pi}\, m\omega^2} \frac{6\pi m\omega^2}{g\sqrt{2\pi}} - \frac{3\hbar^2\pi}{2m\sqrt{2\pi}} \frac{1}{g} \left(\frac{\sqrt{2}}{4\sqrt{\pi}} \frac{Ng}{m\omega^2}\right)^{1/5}$$

$$N^{4/5}\left(1 + \frac{6}{4}\right) = -\frac{3\hbar^2\pi}{g\, 2m\sqrt{2\pi}} \left(\frac{\sqrt{2}\, g}{4\sqrt{\pi}\, m\omega^2}\right)^{1/5}$$

$3/2$

$$N^4 \left(5/2\right)^5 = \left(-\frac{3\hbar^2\pi}{g\, 2m\sqrt{2\pi}}\right)^5 \cdot \frac{\sqrt{2}\, g}{4\sqrt{\pi}\, m\omega^2}$$

$$N^4 \cdot 97.656 = -\left(\frac{3\pi}{2\sqrt{2\pi}}\right)^5 \cdot \frac{\hbar^{10}}{g^5 m^5} \frac{\sqrt{2}\, g}{4\sqrt{\pi}\, m\omega^2}$$

23.5

$$= -0.048 \cdot \frac{\hbar^{10}}{g^4 m^6 \omega^2}$$

$=$

$g = \frac{4\pi\hbar^2}{m} \cdot -1.5\,\text{nm}$

d) $\longrightarrow$

d) We solve for the Number which yields the critical BEC density → that is, What number$/a^3$ yields us a BEC.
→ this happens when the H_{int} cancels the $H_{KE} + H_{PE}$ terms.

good thoughts

$\therefore\ H(N/a^3)\ \text{critical} = 0$

3/6

$$\therefore -\sqrt{2\pi}\, g\left(\frac{N}{a^3}\right) = 6\pi\omega^2 m a^2 + \frac{3\hbar^2\pi}{2m}\frac{1}{a^2}$$

then
$$N/a^3 = -\frac{6\sqrt{\pi}\,\omega^2 m a^2}{\sqrt{2}\,g} - \frac{3\hbar^2\sqrt{\pi}}{2\sqrt{2}\,g m a^2}$$

$m \simeq 7\,\text{GeV}/c^2$

$$g = \frac{4\pi(\hbar c)^2(-1.5\,\text{nm})}{mc^2} = \frac{4\pi(197\,\text{eV nm})^2}{7\,\text{GeV}}\cdot(-1.5\,\text{nm}) = -0.105\,\text{eV}(\text{nm})^3$$

$\omega = 2\pi \times 145\,\text{Hz}$

$$N/a^3 = -\frac{6\sqrt{\pi}}{\sqrt{2}}\cdot\frac{(2\pi)^2\, 145/\text{s}^2\; 7\,\text{GeV}/c^2\, a^2}{-0.105\,\text{eV}(\text{nm})^3}$$

$$-\left(\frac{3\sqrt{\pi}}{2\sqrt{2}}\frac{1}{a^2}\frac{\hbar c\cdot c}{mc^2}\frac{1}{g}\right) = \frac{3\sqrt{\pi}}{2\sqrt{2}}\frac{1}{a^2}\frac{197\,\text{eV nm}\cdot c}{7\,\text{GeV}\cdot -.105\,\text{eV}(\text{nm})^3}$$

$$= \underbrace{\cancel{0.0046\,\frac{a^2/m^2}{(\text{nm})^3}}}_{\text{negligible}} + \frac{0.000151\,m^2/a^2}{(\text{nm})^3}$$

$$N/a^3 \propto \frac{0.00015}{(\text{nm})^3}\left(\frac{m^2}{a^2}\right)$$

$N \propto a \qquad N = 0.00015\, a\, \frac{m^2}{(\text{nm})^3}$

$= 1.5\times10^{14}\cdot a/\text{nm}$

So the exact number depends on the Trap size → maybe I left out an additional constraint on this problem?