DRAFT

January 2012 – Ph.D. Qualifying Exam

E&M

Question 1

A hollow metal sphere of radius R and mass M floats on an insulating dielectric liquid of density ρ and relative dielectric constant ε_r. When the metal sphere has no charge on it, it floats on the dielectric liquid as shown in Figure 1(a); i.e., the bottom of the sphere is $R/2$ below the surface of the dielectric liquid. Find the magnitude of the charge Q to which the sphere must be charged in order for it to be half submerged as shown in Figure 1(b). Express your answer in terms of ρ, R, ε_r, the vacuum permittivity ε_0, the acceleration due to gravity g, and other numerical factors.

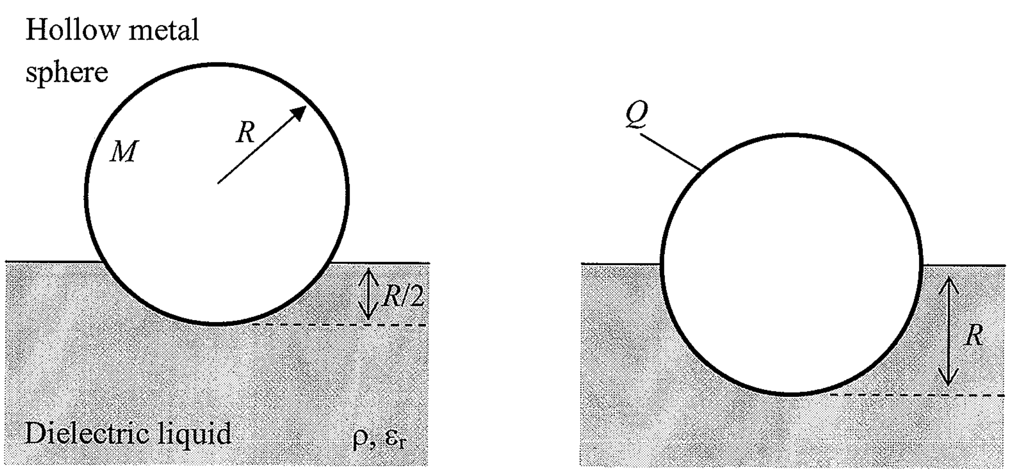
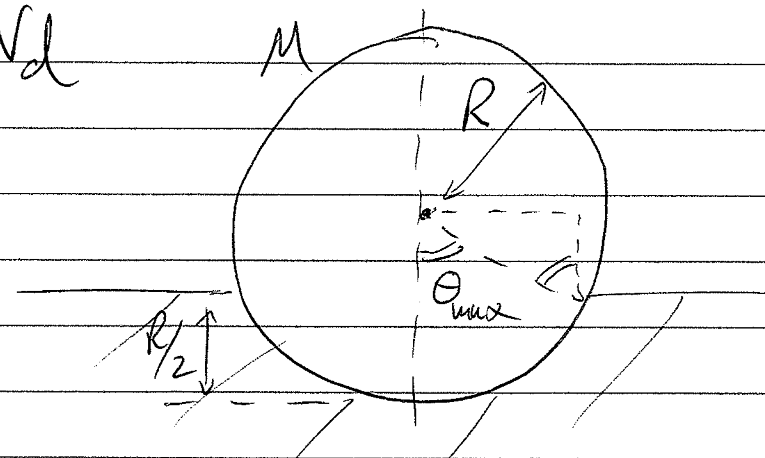


Figure 1(a) Figure 1(b)

1

First, determine the mass M in terms of ρ and R using displaced liquid volume V_d;

Bouyancy $\Rightarrow M = \rho V_d$



$$\theta_{max} = \cos^{-1}\left(\frac{R}{2R}\right) = \pi/3$$

$$d^3r = r\,dr\,d\phi\,dz$$
$$= 2\pi r\,dr\,dz$$

$$r = R\sin\theta \Rightarrow dr = R\cos\theta\,d\theta$$
$$z = R\cos\theta \Rightarrow dz = +R\sin\theta\,d\theta$$

$$\Rightarrow V_d = \int_0^{\pi/3} \pi (R\sin\theta)^2 (+R\sin\theta\,d\theta) = +\pi R^3 \int_0^{\pi/3} d\theta \sin^3\theta$$

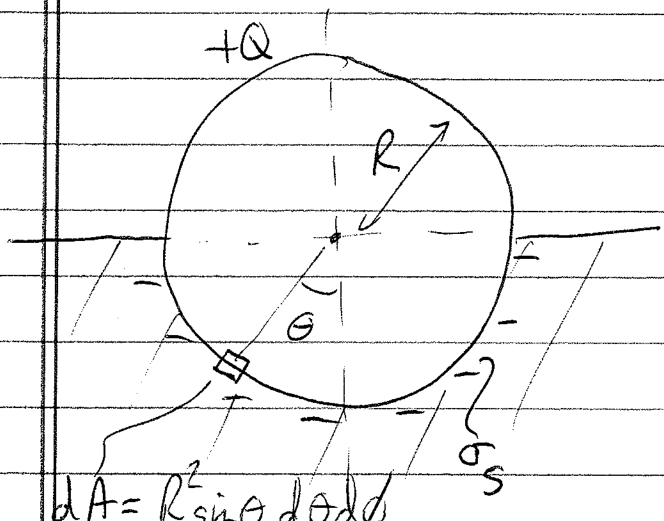
$$= +\pi R^3 \left\{ \left[-\cos\theta\right]_0^{\pi/3} + \left[\frac{1}{3}\cos^3\theta\right]_0^{\pi/3} \right\}$$

$$= +\pi R^3 \left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3}\right) = \frac{5}{24}\pi R^3$$

$$\therefore M = \frac{5}{24}\pi \rho R^3 \quad \ldots (A)$$

When the sphere is charged, there is an additional electric force between the charge Q and the induced surface charge in the dielectric liquid;

(1)



$dA = R^2 \sin\theta \, d\theta \, d\phi$

$dq = \sigma_s R^2 \sin\theta \, d\theta \, d\phi$

$$\Rightarrow F_e = \frac{Q}{4\pi\epsilon_0}\int_0^{\pi/2} \frac{2\pi\sigma_s \cancel{R^2} \sin\theta\cos\theta \, d\theta}{\cancel{R^2}}$$

... downward component

$$\Rightarrow F_e = \frac{\sigma_s Q}{2\epsilon_0}\int_0^{\pi/2} d\theta \cdot \frac{1}{2}\sin 2\theta = \frac{\sigma_s Q}{4\epsilon_0}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$$

$$= \frac{\sigma_s Q}{4\epsilon_0}\left(\frac{1}{2} - -\frac{1}{2}\right) = \frac{\sigma_s Q}{4\epsilon_0}$$

All that remains is to determine σ_s since we know that

$$F_e + Mg = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\rho g$$

... half sphere submerged

$$\Rightarrow \frac{\sigma_s Q}{4\epsilon_0} = \frac{11}{24}\pi R^3 \rho g$$

... using (A)

(1)

Now, $D = \epsilon_0 \epsilon_r E = \epsilon_0 E + P$, where $P = \sigma_s$ and $\epsilon_0 E = \sigma = \frac{Q}{4\pi R^2}$, hence

$$\epsilon_r \sigma = \sigma + \sigma_s \Rightarrow \sigma_s = \frac{Q(\epsilon_r - 1)}{4\pi R^2 \epsilon_r}$$

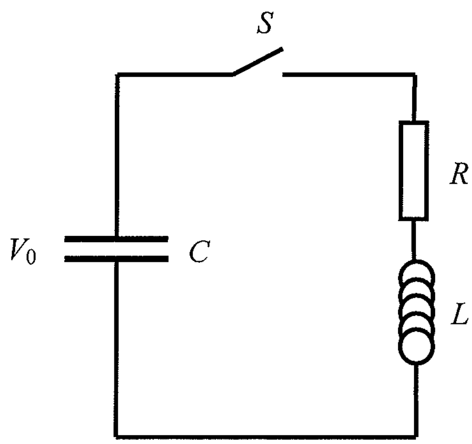
Thus,

$$\frac{Q^2(\epsilon_r - 1)}{4\pi R^2 \epsilon_0 \epsilon_r} = \frac{11}{6}\pi R^3 \rho g$$

$$\therefore \; Q = \sqrt{\frac{22\pi^2 R^5 \epsilon_0 \epsilon_r \rho g}{3(\epsilon_r - 1)}}$$

Question 2

Consider the circuit shown below, in which for times $t < 0$ the capacitor of capacitance C is charged to a voltage V_0. At $t = 0$, the switch S is closed, allowing the capacitor to discharge through a resistor R and an inductor L placed in series.



a) Using Kirchoff's voltage law, write down the second-order differential equation describing the evolution of the charge q on the capacitor for times $t > 0$.

b) For times $t > 0$, solve the differential equation obtained in part (a) subject to the boundary conditions $q(t = 0) = q_0$ and $\left.\dfrac{dq}{dt}\right|_{t=0} = 0$.

c) Verify that the solution obtained in part (b) satisfies the two boundary conditions.

d) Show that the current in the circuit is maximum at a time t given by the relation

$$\tanh(\Omega t) = \frac{2\Omega}{\alpha},$$

where $\Omega = \sqrt{\omega^2 + \dfrac{\alpha^2}{4}}$ and $\alpha = \dfrac{R}{L}$ with $\omega = \dfrac{1}{\sqrt{LC}}$.

(2)

a) KVL : Voltage drops = Voltage increase (Emf sources)

$$\Rightarrow L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = \frac{q}{C}$$

$$\therefore L\frac{d^2q}{dt^2} + R\frac{dq}{dt} - \frac{q}{C} = 0$$

b) Since this is a "transient" event from $t=0$, need to use Laplace transforms: $(p \leftrightarrow t)$

$$\frac{d^2q}{dt^2} = p^2Q - p\underbrace{q(t=0)}_{=q_0} - \cancel{\left.\frac{dq}{dt}\right|_{t=0}}^{=0} = p^2Q - pq_0$$

$$\frac{dq}{dt} = pQ - q(t=0) = pQ - q_0$$

So, transformed diff. eq^n reads

$$Q\left(Lp^2 + Rp - \frac{1}{C}\right) = Lq_0p + Rq_0$$

$$\Rightarrow Q = \frac{q_0\left(p + R/L\right)}{p^2 + \frac{R}{L}p - \frac{1}{LC}} = \frac{q_0(p+\alpha)}{p^2 + \alpha p + \omega^2}$$

$$\ldots \alpha = \frac{R}{L} \text{ and } \omega^2 = \frac{1}{LC}$$

②

b) contd.

Completing the square in the denominator gives

$$Q = \frac{q_0\left(p+\frac{\alpha}{2}\right) + q_0\left(\frac{\alpha}{2}\right)}{\left(p+\frac{\alpha}{2}\right)^2 - \Omega^2} \quad \ldots \Omega = \sqrt{\frac{\alpha^2}{4}+\omega^2}$$

$$= q_0\left[\underbrace{\frac{p+\frac{\alpha}{2}}{\left(p+\frac{\alpha}{2}\right)^2-\Omega^2}}_{\mathcal{L}\left[e^{-\alpha t/2}\cosh(\Omega t)\right]} + \frac{\alpha}{2\Omega}\left(\underbrace{\frac{\Omega}{\left(p+\frac{\alpha}{2}\right)^2-\Omega^2}}_{\mathcal{L}\left[e^{-\alpha t/2}\sinh(\Omega t)\right]}\right)\right]$$

So, the inverse Laplace transform gives

$$q(t) = q_0\left[\cosh(\Omega t) + \frac{\alpha}{2\Omega}\sinh(\Omega t)\right]e^{-\alpha t/2}$$

c) At $t=0$, we have

$$q(t=0) = q_0 \quad \ldots \begin{array}{l} \cosh(\Omega t) = 1 \\ \sinh(\Omega t) = 0 \\ \exp\left[-\alpha t/2\right] = 1. \end{array}$$

(2)

c) contd.

$$\frac{dq}{dt} = q_0 \frac{d}{dt}\left\{ \frac{1}{2}\left(e^{(\Omega-\frac{\alpha}{2})t} + e^{-(\Omega+\frac{\alpha}{2})t}\right) + \frac{\alpha}{4\Omega}\left(e^{(\Omega-\frac{\alpha}{2})t} - e^{-(\Omega+\frac{\alpha}{2})t}\right)\right\}$$

$$= \frac{q_0}{2}\frac{d}{dt}\left\{\left(1+\frac{\alpha}{2\Omega}\right)e^{(\Omega-\frac{\alpha}{2})t} + \left(1-\frac{\alpha}{2\Omega}\right)e^{-(\Omega+\frac{\alpha}{2})t}\right\}$$

$$= \frac{q_0}{2}\left[\underbrace{\left(1+\frac{\alpha}{2\Omega}\right)\left(\Omega-\frac{\alpha}{2}\right)}_{\frac{1}{\Omega}\left(\Omega^2-\frac{\alpha^2}{4}\right)}e^{(\Omega-\frac{\alpha}{2})t} - \underbrace{\left(1-\frac{\alpha}{2\Omega}\right)\left(\Omega+\frac{\alpha}{2}\right)}_{\frac{1}{\Omega}\left(\Omega^2-\frac{\alpha^2}{4}\right)}e^{-(\Omega+\frac{\alpha}{2})t}\right]$$

$$\Rightarrow \frac{dq}{dt} = \frac{q_0}{2\Omega}\left(\Omega^2-\frac{\alpha^2}{4}\right)\sinh(\Omega t).e^{-\alpha t/2}$$

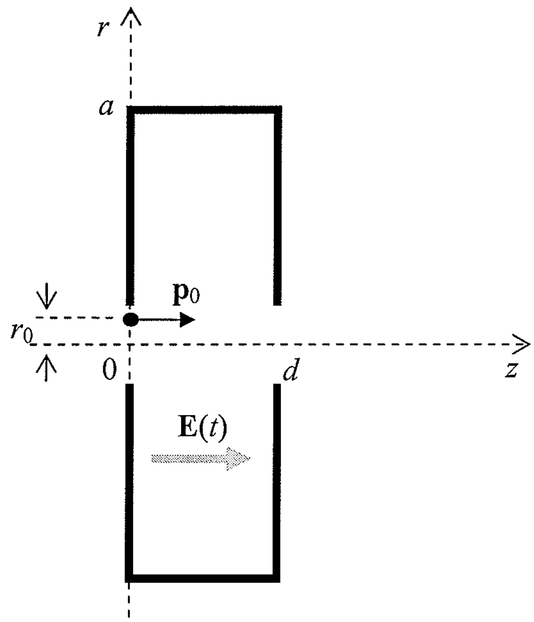
$$\therefore \left.\frac{dq}{dt}\right|_{t=0} = 0 \qquad \ldots \sinh(\Omega t) = 0.$$

d) Max. current at $\frac{d^2q}{dt^2} = 0$

$$\Rightarrow 0 = \frac{q_0}{2\Omega}\left(\Omega^2-\frac{\alpha^2}{4}\right)\left[\Omega\cosh(\Omega t) - \frac{\alpha}{2}\sinh(\Omega t)\right]e^{-\alpha t/2}$$

$$\therefore \text{When } \tanh(\Omega t) = \frac{2\Omega}{\alpha}$$

Question 3



A cylindrical 'pill-box' resonator of radius a and length d is driven at its fundamental TM_{010} mode for which the oscillating electric component of the RF field may be written as

$$\mathbf{E}(t) = \hat{\mathbf{z}} E_0 J_0\left(\frac{2.405r}{a}\right)\sin(\omega t + \phi)$$

where $J_0(x)$ is the Bessel function of zero order whose first zero is at $x = 2.405$.

a) What is the form of the magnetic component of the oscillating RF field in the cavity?
b) Verify that the average value of the Poynting vector (i.e., $\langle \mathbf{S} \rangle_{av.}$) is zero.
c) What is the stored energy of the oscillating TM_{010} mode?
d) Consider the propagation of a particle of mass m and charge q incident at $(r,\phi,z) = (r_0,0,0)$, where $r_0 << a$, and with momentum $\mathbf{p}_0 = (p_r,p_\phi,p_z) = (0,0,p_0)$ on the RF cavity at $t = 0$ (see diagram above). For the case when $\frac{p_0}{md} >> \omega$ (i.e., the time-of-flight through the RF cavity is much less than the RF period), determine $\mathbf{p}$ at $z = d$ to ***first order*** in E_0.

3)

a) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\Rightarrow \begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & 0 & E \end{vmatrix} = -\hat{\phi}\frac{\partial E}{\partial r} = -\frac{\partial \vec{B}}{\partial t}$$

... as E is only $f(r)$.

$$\therefore \frac{\partial \vec{B}}{\partial t} = \hat{\phi} E_0 \sin(\omega t + \phi) \cdot \left(\frac{2.405}{a}\right) J_1\left(\frac{2.405 r}{a}\right)$$

... as $\frac{\partial J_0(x)}{\partial x} = J_1(x)$.

$$\Rightarrow \vec{B} = -\hat{\phi}\left(\frac{2.405 E_0}{a\omega}\right) \cdot J_1\left(\frac{2.405 r}{a}\right) \cdot \cos(\omega t + \phi)$$

b) To find $\langle \vec{S} \rangle_{av}$, need to evaluate $\frac{1}{T}\int_0^T dt\, (\vec{E} \times \vec{H})$; that is, the average over one period of oscillation $\left(T = \frac{2\pi}{\omega}\right)$:

$$\vec{E} \times \vec{H} = \frac{1}{\mu}\begin{vmatrix} \hat{r} & \hat{\phi} & \hat{z} \\ 0 & 0 & E_z \\ 0 & B_\phi & 0 \end{vmatrix} = -\frac{1}{\mu_0}\hat{r} E_z B_\phi$$

$$\Rightarrow \langle \vec{S} \rangle_{av} = \frac{2.405 E_0^2}{a\omega\mu_0 T} J_0\left(\frac{2.405 r}{a}\right) \cdot J_1\left(\frac{2.405}{a}\right)$$

$$\times \int_0^T dt\, \sin(\omega t + \phi) \cos(\omega t + \phi)$$

(3)

b) contd.

As $\sin 2\theta = 2\sin\theta\cos\theta$, the integral becomes

$$\frac{1}{2}\int_0^T dt \sin\left[2(\omega t+\phi)\right] = \frac{1}{2}\left[-\frac{\cos\left[2(\omega t+\phi)\right]}{2\omega}\right]_0^T$$

$$= \frac{1}{4\omega}\left(\cancel{\cos[2\phi] - \cos[2(2\pi+\phi)]}\right)$$

$$= 0$$

$$\therefore \langle \vec{S} \rangle_{av.} = 0$$

c) Stored energy density $u = \frac{1}{2}(\vec{E}.\vec{D} + \vec{B}.\vec{H})$ with equal electric and magnetic contributions. So, since $\epsilon_r = 1$, we have

$$\text{Stored Energy} = \epsilon_0 \int_0^d dz \int_0^{2\pi} d\phi \int_0^a r\,dr.\, E^2$$

$$= 2\pi d \epsilon_0 E_0^2 \left(\frac{1}{2}\right) \int_0^a r\,dr\, J_0^2\left(\frac{2.405 r}{a}\right)$$

time average

Now, $\int_0^1 x\,dx\, J_0^2(\alpha x) = \frac{1}{2} J_1^2(\alpha)$ so $x = \frac{r}{a}$ and $\alpha = 2.405$;

$$r\,dr = a^2 x\,dx$$

③

c) contd.

Hence,

$$\text{Stored energy} = \frac{1}{2}\pi a^2 d \underline{\epsilon_0 E_0^2} \cdot J_1(2.405)$$

d) We are interested in $\Delta\vec{p}_{RF} = q\int_0^{t'}(\vec{E}+\vec{v}\times\vec{B})\,dt$ with $t' = \frac{d}{v} = \frac{md}{p_0} \ll \frac{1}{\omega}$.

For $r_0 \ll a$, $J_0\left(\frac{2.405 r_0}{a}\right) \simeq 1$ and $J_1\left(\frac{2.405 r_0}{a}\right) \simeq \frac{1}{2}\left(\frac{2.405 r_0}{a}\right)$;

$$\Delta p_z = qE_0\int_0^{t'} dt\,\sin(\omega t+\phi) = -\frac{qE_0}{\omega}\Big[\cos(\omega t+\phi)\Big]_0^{t'}$$

$$= -\frac{qE_0}{\omega}\Big[\cancel{\underbrace{\cos\omega t'}_{=1}\cos\phi} - \underbrace{\sin\omega t'}_{=\omega t'}\sin\phi \cancel{- \cos\phi}\Big]$$

$$\Rightarrow \Delta p_z = \frac{qmdE_0}{p_0}\sin\phi$$

$$\Delta p_\phi = 0$$

$$\Delta p_r = +\frac{qp_0}{m}\left(\frac{2.405E_0}{a\omega}\right)\frac{1}{2}\left(\frac{2.405 r_0}{a}\right)\int_0^{t'} dt\,\cos(\omega t+\phi)$$

... as $\vec{p}\times\vec{B} = -p_0 B_\phi$

$$\Rightarrow \Delta p_r = \frac{qp_0 r_0 E_0}{2m\omega}\left(\frac{2.405}{a}\right)^2 \cdot \left[\frac{1}{\omega}\sin(\omega t+\phi)\right]_0^{t'}$$

3

d) contd.

$$\Rightarrow \Delta p_r = \frac{q p_0 r_0 E_0}{2 m \omega^2}\left(\frac{2.405}{a}\right)^2 \Big[\underbrace{\sin(\omega t' + \phi) - \sin\phi}_{\underbrace{\sin \omega t'}_{=\omega t'} \cos\phi \, + \, \cancel{\underbrace{\cos\omega t'}_{=1} \sin\phi - \sin\phi}} \Big]$$

$$\therefore \Delta p_r = \frac{q d r_0 E_0}{2\omega}\left(\frac{2.405}{a}\right)^2 \cos\phi$$

So, after the RF cavity

$$\vec{p} = \left(\frac{q d r_0 E_0}{2\omega}\left(\frac{2.405}{a}\right)^2 \cos\phi,\ 0,\ p_0 + \frac{q m d E_0}{p_0}\sin\phi \right)$$

Question 4

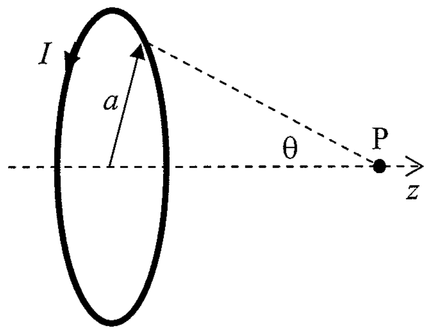


Figure 4(a)

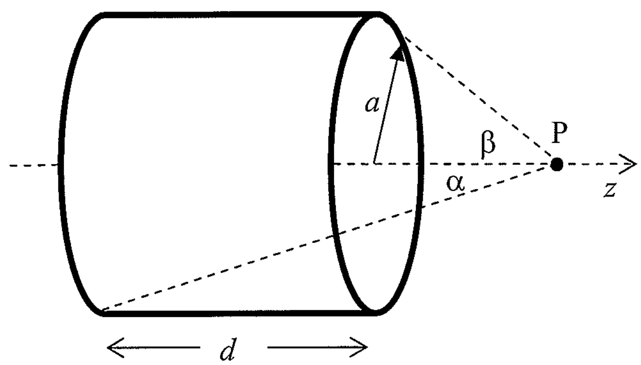


Figure 4(b)

a) Show that for a single wire loop of radius a carrying current I the axial magnetic field at point P in Figure 4(a) may be written as

$$\mathbf{B}(\theta) = \hat{\mathbf{z}}\frac{\mu_0 I}{2a}\sin^3\theta \quad ,$$

where θ is the angle subtended from the axis at point P to the circumference of the loop and μ_0 is the permeability of vacuum.

b) Use the result of part (a) to determine the axial magnetic field at point P for a solenoid of length d and radius a carrying current I in n turns (Figure 4(b)). Express your answer in terms of the angles α and β that the front and back coils of the solenoid subtend with its axis at point P.

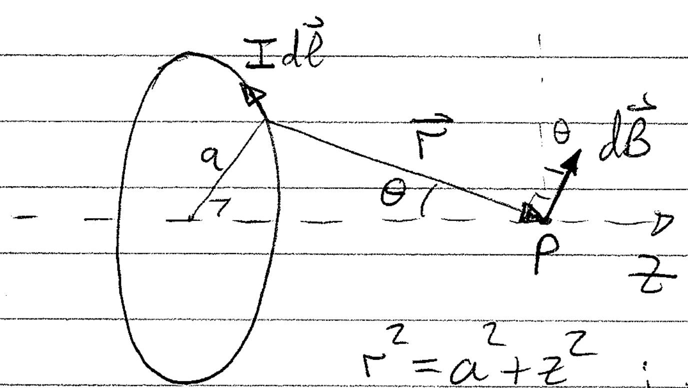
c) Verify that your answer in part (b) reduces to the expected result for an infinitely long solenoid (i.e., $d >> a$).

d) Show that for small distances $z << d$ from the center of the narrow ($d >> a$) finite solenoid that the axial dependence of the magnetic field strength is parabolic in z and of the form

$$B(z) = B_0\left[1 - \frac{2a^2}{d^2}\left(1 + \frac{12z^2}{d^2}\right)\right].$$

4

a)



$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \vec{r}}{r^3}$$

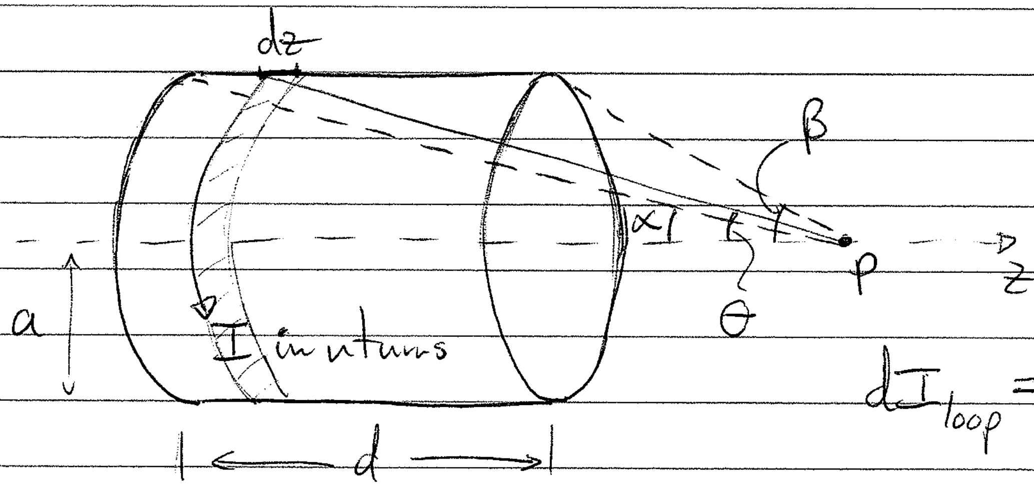
$$r^2 = a^2 + z^2 \; ; \; \sin\theta = \frac{a}{r}$$

Since $d\vec{l}$ and $\vec{r}$ are perpendicular we have

$$dB_z = \frac{\mu_0 I}{4\pi} \cdot \frac{a\,d\phi}{r^2} \sin\theta \ldots \quad dl = a\,d\phi \text{ around loop.}$$

$$\therefore B_z = \frac{\mu_0 I}{2} \cdot \frac{a \sin\theta}{r^2} \Rightarrow \vec{B}(\theta) = \hat{z} \frac{\mu_0 I}{2a} \sin^3\theta$$

b)



$$dI_{loop} = I\left(\frac{n}{d}\right) dz$$

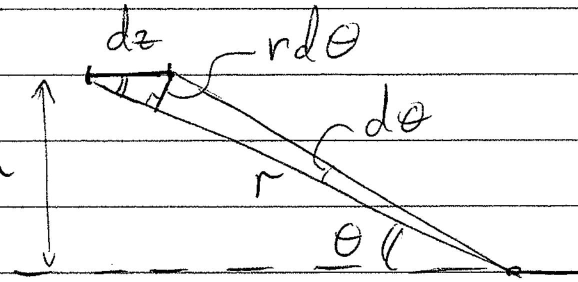
Field at P due to loop dz is given by

$$dB_{loop}(z) = \frac{\mu_0 I}{2a} \sin^3\theta \left(\frac{n}{d}\right) dz$$

(4)

b) contd.

Now, the relation between z and θ can be readily found;



$$\sin\theta = \frac{a}{r} = \frac{r\,d\theta}{dz}$$

$$\Rightarrow dB_{loop}(z) = \frac{\mu_0 I n}{2ad} \sin^3\theta \, \frac{r\,d\theta}{\sin\theta}$$

$$= \frac{\mu_0 I n}{2d} \sin\theta \, d\theta$$

So, total field at P for solenoid is

$$B(z) = \frac{\mu_0 I n}{2d}\Big[-\cos\theta\Big]_{\alpha}^{\beta} = \underline{\underline{\frac{\mu_0 I n}{2d}(\cos\alpha - \cos\beta)}}$$

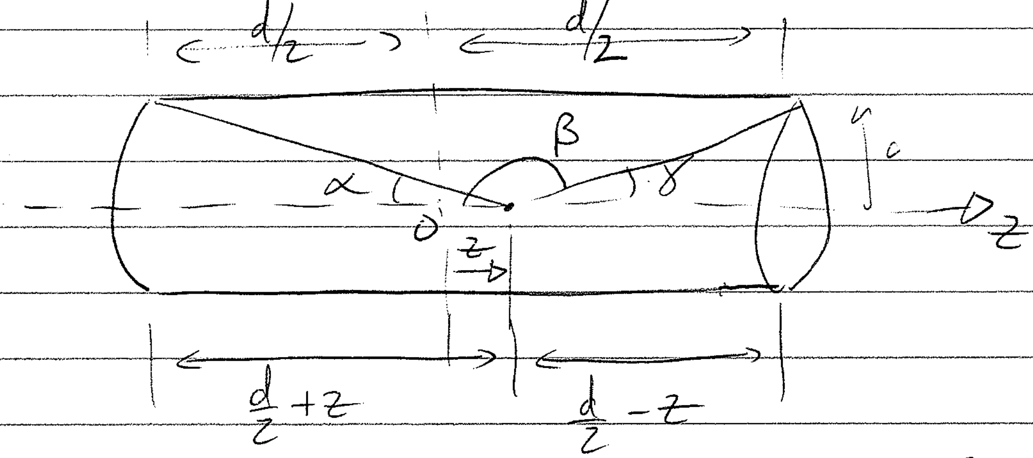
c) Magnetic field for infinitely long solenoid (inside) has $\alpha = 0$ and $\beta = \pi$, hence

$$B = \mu\left(\frac{n}{d}\right) I \qquad \text{... as expected}$$

turns/length

(4)

d) For a "narrow" solenoid ($d \gg a$), we have



$$\cos\alpha = \frac{\frac{d}{2}+z}{\sqrt{\left(\frac{d}{2}+z\right)^2+a^2}} \simeq 1-\frac{\alpha^2}{2} = 1-\frac{a^2}{2\left(\frac{d}{2}+z\right)^2}$$

$$\cos\gamma = -\cos\beta \simeq 1-\frac{a^2}{2\left(\frac{d}{2}-z\right)^2}$$

$$\therefore B = \frac{\mu_0 I n}{2d}\left[2-\frac{a^2}{2}\left(\frac{4}{d^2\left(1+\frac{2z}{d}\right)^2}+\frac{4}{d^2\left(1-\frac{2z}{d}\right)^2}\right)\right]$$

$$= \frac{\mu_0 I n}{d}\left[1-\frac{a^2}{d^2}\left(1-\cancel{\frac{4z}{d}}+\frac{24z^2}{2d^2}+1+\cancel{\frac{4z}{d}}+\frac{24z^2}{2d^2}+\dots\right)\right]$$

$$= \frac{\mu_0 I n}{d}\left[1-\frac{2a^2}{d^2}\left(1+\frac{12z^2}{d^2}\right)\right]$$

$$\therefore B(z) = B_0\left[1-\frac{2a^2}{d^2}\left(1+\frac{12z^2}{d^2}\right)\right]$$

Question 5

A material can be anisotropic in either or both its refractive index and absorption. These optical properties are described by a permittivity tensor, $\underline{\underline{\varepsilon}}$, for the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ directions (i.e., a 3×3 matrix). The wave equation in a non-conducting, non-magnetic medium then reads

$$\nabla^2 \mathbf{E} - \mu_0 \underline{\underline{\varepsilon}} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 .$$

a) For a electromagnetic wave with a polarization unit vector $\hat{\mathbf{e}}$ and amplitude $\mathbf{E}_0$ described by, $\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\hat{\mathbf{e}}\mathbf{E}_0 \exp[i(\mathbf{k}.\mathbf{r} - \omega t)] + c.c.$, show that the refractive index n experienced by the wave in the medium is given by the expression

$$n^2 = \frac{1}{\varepsilon_0}\left[\hat{\mathbf{e}}^* . \left(\underline{\underline{\varepsilon}}.\hat{\mathbf{e}}\right)\right] .$$

b) For $\hat{\mathbf{e}} = (\sin\theta, 0, \cos\theta)$ in the x-z plane, determine the refractive index experienced by the wave in a non-absorbing crystalline medium described by $\underline{\underline{\varepsilon}} = \varepsilon_0 \begin{pmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & (n_o + \Delta n)^2 \end{pmatrix}$,

where n_o and $n_e = n_o + \Delta n$ are the ordinary and extra-ordinary refractive indexes of the unaxial crystal respectively.

c) What is the walk-off angle ϕ between the Poynting vector $\mathbf{S}$ and the wave vector $\mathbf{k}$ of the wave in the anisotropic medium if its magnetic field amplitude is given by $\mathbf{H} = (0, H_0, 0) = \hat{\mathbf{y}} H_0$?

d) What is the angle between $\mathbf{E}$ and $\mathbf{D}$ in the uniaxial crystal?

e) Show that the angle θ for which the magnitude of ϕ is a maximum is given by the relation

$$\cos\theta = \sqrt{\frac{1}{\alpha + 1}}, \text{ where } \alpha = \left(\frac{n_o + \Delta n}{n_o}\right)^2 .$$

(5)

a) For $\vec{E} = \frac{1}{2}\hat{e}E_0 \exp[i(\vec{k}.\vec{r} - \omega t)] + cc.$

$$\nabla^2\vec{E} = -k^2\vec{E}$$

$$\frac{\partial^2\vec{E}}{\partial t^2} = -\omega^2\vec{E}$$

$$\Rightarrow \nabla^2\vec{E} - \mu_0\underline{\underline{\varepsilon}}\frac{\partial^2\vec{E}}{\partial t^2} = 0 = -k^2\vec{E} + \mu_0\omega^2\underline{\underline{\varepsilon}}\vec{E}$$

$$\therefore \; +\hat{e}k^2 = \mu_0\omega^2\,\underline{\underline{\varepsilon}}.\hat{e}$$

$$\Rightarrow k^2 = \mu_0\omega^2\left[\hat{e}^*.(\underline{\underline{\varepsilon}}.\hat{e})\right]$$

Since, $k^2 = \frac{n^2\omega^2}{c^2} = \mu_0\epsilon_0 n^2\omega^2$, we get

$$n^2 = \frac{1}{\epsilon_0}\left[\hat{e}^*.(\underline{\underline{\varepsilon}}.\hat{e})\right]$$

b) $$\frac{1}{\epsilon_0}\left[\hat{e}^*.(\underline{\underline{\varepsilon}}.\hat{e})\right] = (\sin\theta, 0, \cos\theta)\begin{pmatrix} n_0^2 & 0 & 0 \\ 0 & n_0^2 & 0 \\ 0 & 0 & (n_0+\Delta n)^2 \end{pmatrix}\begin{pmatrix} \sin\theta \\ 0 \\ \cos\theta \end{pmatrix}$$

$$= (\sin\theta, 0, \cos\theta)\begin{pmatrix} n_0^2\sin\theta \\ 0 \\ (n_0+\Delta n)^2\cos^2 \end{pmatrix}$$

(5)

b) contd.

$$\Rightarrow n^2 = n_0^2 \sin^2\theta + (n_0+\Delta n)^2 \cos^2\theta$$

$$= n_0^2 + (2n_0+\Delta n)\Delta n \cos^2\theta$$

c) $$\vec{S} = \vec{E} \times \vec{H} = E_0 H_0 \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{vmatrix}$$

$$= E_0 H_0 \left(-\cos\theta, 0, \sin\theta\right)$$

And $\vec{k}$ can be found from $\vec{\nabla} \times \vec{H} = \underbrace{\cancel{\vec{J}}}_{=0} + \frac{\partial \vec{D}}{\partial t} = \underline{\underline{\varepsilon}} \frac{\partial \vec{E}}{\partial t}$

$$\Rightarrow i\vec{k} \times \vec{H} = -i\omega\, \underline{\underline{\varepsilon}} \cdot \vec{E} \propto -i\omega \epsilon_0 E_0 \begin{pmatrix} n_0^2 \sin\theta \\ 0 \\ (n_0+\Delta n)^2 \cos\theta \end{pmatrix}$$

So, since $\vec{H} = (0, H_0, 0)$, $\vec{k}$ is in the x-z plane and perpendicular to $(n_0^2 \sin\theta, 0, (n_0+\Delta n)^2 \cos\theta)$

$$\Rightarrow \vec{k} \propto \left(-(n_0+\Delta n)^2 \cos\theta, 0, n_0^2 \sin\theta\right)$$

(5)

c) contd.

The angle ϕ between $\vec{S}$ and $\vec{k}$ is now readily found

$$\cos\phi = \frac{\vec{S}\cdot\vec{k}}{|\vec{S}|\,|\vec{k}|} = \frac{(n_0+\Delta n)^2\cos^2\theta + n_0^2\sin^2\theta}{\sqrt{(n_0+\Delta n)^4\cos^2\theta + n_0^4\sin^2\theta}}$$

$$\Rightarrow \cos\phi = \frac{\alpha\cos^2\theta + \sin^2\theta}{\sqrt{\alpha^2\cos^2\theta + \sin^2\theta}} \; ; \; \alpha = \left(\frac{n_0+\Delta n}{n_0}\right)^2$$

d) $\vec{E}$ is $\parallel$ to $\hat{e} = (\sin\theta, 0, \cos\theta)$ and $\vec{D} = \underline{\underline{\varepsilon}}\cdot\vec{E}$ which is given by

$$\underline{\underline{\varepsilon}}\cdot\vec{E} \propto \left(n_0^2\sin\theta,\, 0,\, (n_0+\Delta n)^2\cos\theta\right)$$

$$\therefore \cos\Phi = \frac{\vec{E}\cdot\vec{D}}{|\vec{E}|\cdot|\vec{D}|} = \frac{n_0^2\sin^2\theta + (n_0+\Delta n)^2\cos^3\theta}{\sqrt{n_0^4\sin^2\theta + (n_0+\Delta n)^4\cos^2\theta}}$$

... same as (c) ; $\therefore \Phi = \phi$.

(5)

e) As ϕ exists between 0° and 90°, it is sufficient to evaluate

$$\frac{d}{d\theta}(\cos\phi) = 0 = \frac{d}{d\theta}\left[\frac{\alpha\cos^2\theta + \sin^2\theta}{\sqrt{\alpha^2\cos^2\theta + \sin^2\theta}}\right] = \frac{d}{d\theta}\left[\frac{(\alpha-1)\cos^2\theta + 1}{\sqrt{(\alpha^2-1)\cos^2\theta + 1}}\right]$$

$$\ldots \alpha = \left(\frac{n_0 + \Delta n}{n_0}\right)^2$$

$$\Rightarrow 0 = \frac{d\cos^2\theta}{d\theta} \cdot \frac{d}{dx}\left[\frac{(\alpha-1)x + 1}{\sqrt{(\alpha^2-1)x+1}}\right]$$

$$= -2\cos\theta\sin\theta \cdot \left\{\frac{(\alpha-1)\sqrt{(\alpha^2-1)x+1} - \frac{1}{2}(\alpha^2-1)\left[(\alpha^2-1)x+1\right]^{-\frac{1}{2}}\left[(\alpha-1)x+1\right]}{(\alpha^2-1)x+1}\right\}$$

minima clearly

$$\Rightarrow (\alpha-1)\left[(\alpha^2-1)x+1\right] = \frac{1}{2}\left[(\alpha-1)x+1\right](\alpha^2-1)$$

$$\Rightarrow 2(\alpha-1) - (\alpha^2-1) = -(\alpha-1)(\alpha^2-1)x$$

$$\therefore x = \frac{\alpha^2-2\alpha+1}{(\alpha-1)(\alpha^2-1)} = \frac{\cancel{(\alpha-1)^2}}{\cancel{(\alpha-1)}\cancel{(\alpha-1)}(\alpha+1)}$$

$$\Rightarrow \quad \theta = \cos^{-1}\sqrt{\frac{n_0^2}{(n_0+\Delta n)^2 + n_0^2}} = \cos^{-1}\sqrt{\frac{1}{\alpha+1}}$$

... angle θ for which walk-off angle ϕ is maximum.