QM II Final HW 7-9 (old solutions written too)

QM II Final Notes

Time Dependent Perturbation theory

- Interaction Picture $\quad (H = H_0 + V(t))$

$$|\alpha, t\rangle_I = e^{iH_0 t/\hbar} |\alpha, t\rangle_S$$

(5.5.13) $$|\alpha, t\rangle_I = \sum_n C_n(t) |n\rangle$$

? picture kets (actually doesn't matter since all observables don't care about phase).

$$C_n(t) = C_n^{(0)} + C_n^{(1)} + C_n^{(2)} + \dots$$

Starting state

(5.7.17) $$C_n^{(0)} = \delta_{ni} \quad \text{independent of time}$$

$$C_n^{(1)} = \frac{-i}{\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t')\, dt' = \frac{-i}{\hbar}\int_{t_0}^{t} \langle n | V_I(t') | i\rangle\, dt$$

$$C_n^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, e^{i\omega_{nm}t'} V_{nm}(t')\, e^{i\omega_{mi}t''} V_{mi}(t''$$

$\vdots$

$$\omega_{nm} = \frac{E_n}{\hbar} - \frac{E_m}{\hbar}$$

$$V_{nm}(t) = \langle n | V(t) | m\rangle \quad \text{(schrodinger picture kets)}$$

$$\langle \text{expectation} \rangle = {}_I\langle \text{state} | \text{expect}_I | \text{state}\rangle_I = \langle \text{State} | \text{expt} | \text{state}$$

where $(\text{expect}_I) = e^{iH_0 t/\hbar} (\text{expect}) e^{-iH_0 t/\hbar}$ Schro pict

$\therefore$ & $|\alpha \text{ state}\rangle_I = \text{expansion or } e^{iH_0 t/\hbar} |\alpha, \text{stat}$

$$P_{i\to n}(t) \cong |C_n^{(1)}(t)|^2 = |{}_I\langle n | i\rangle_I|^2$$

Spin Hamiltonians

$$(J_1 + J_2)^2 = J_1^2 + J_2^2 + 2 J_1 \cdot J_2$$

$$\therefore \quad J_1 \cdot J_2 = \frac{1}{2}\left[\underbrace{(J_1 + J_2)}_{J_{tot}}^2 - J_1^2 - J_2^2 \right]$$

$$J_1^2 = j_1(j_1+1)\hbar^2 \xrightarrow{\text{if}} j_1 = \frac{1}{2} \rightarrow \frac{3}{4}\hbar^2$$

Fermi's Golden Rule

deltafunction ~~often~~

$$\omega_{i \to n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \cdot \rho(E_n \approx E_i) \quad = \text{transition rate}$$

density of states $= \dfrac{d\Omega}{dE}$ & Ω = multiplicity of stat or phase space volum

Rabi - Oscillations

$$H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} + \gamma \begin{pmatrix} 0 & e^{i\omega t} \\ e^{-i\omega t} & 0 \end{pmatrix}$$

$$P_{1\to 2}(t) = \frac{\gamma^2}{\gamma^2 + \Delta^2} \sin^2\left(\frac{\sqrt{\gamma^2 + \Delta^2}}{2\hbar} t \right)$$

$$\Delta = \frac{\hbar}{2}(\omega - \omega_{21}) \quad , \quad E_{21} = \hbar\omega_{21} = E_2 - E_1$$

EM/Harmonic interaction

$$V(t) = \frac{-e}{mc} \vec{A} \cdot \vec{P} \quad \text{minimal coupling}$$

$$\overline{\omega_{i\to f}} = \frac{2\pi}{\hbar}\left(\frac{e\omega A_0}{c}\right)^2 |\langle i | \hat{\varepsilon} \cdot \vec{x} | f \rangle|^2 \qquad \hbar m \frac{d}{dt} \to \frac{m\omega\hbar}{\hbar} \to m$$

$\pm$ for absorbtion / emission

Cross-Section: $\sigma_{i \to f} = 4\pi^2 \alpha \hbar\omega |X_{if}|^2 \delta(E_f - E_i - \hbar\omega)$

TRK Sum Rule: $K_0 = 4\pi^2\alpha \langle i | [H_0, \hat{\varepsilon}\cdot\vec{x}] \hat{\varepsilon} \cdot \vec{x} | i \rangle$

Variational Principle

$$E(\alpha) = \frac{\int d\vec{x}\, \phi^*(\alpha,\vec{x})\ H\ \phi(\alpha,\vec{x})}{\int d\vec{x}\ |\phi(\alpha,\vec{x})|^2}$$

$$\left.\frac{\partial E(\alpha)}{\partial \alpha}\right|_{\alpha^*} = 0$$ fixes α parameters + minimizes E.

$$E(\alpha_*) \geq E$$

Many Particle Systems

Nature favors that systems of indistinguishable particles fall into symmetric (Bosons) and antisymmetric (fermions) states.

2 Particles: $H_{N=2} = H_1 + H_2$ (free, no interaction) $\rightarrow \Psi(x_1,x_2) = \phi_\alpha(x_1)\phi_\beta(x_2)$

(no spin) Bosons: $\underset{sym}{\Psi_B} = \frac{1}{\sqrt{2}}\left(\phi_\alpha(x_1)\phi_\beta(x_2) + \phi_\alpha(x_2)\phi_\beta(x_1)\right)$

fermions: $\underset{anti\text{-}sym}{\Psi_F} = \frac{1}{\sqrt{2}}\left(\phi_\alpha(x_1)\phi_\beta(x_2) - \phi_\alpha(x_2)\phi_\beta(x_1)\right)$

etc: $\rightarrow$ Slater Determinant generates Ψ_F (N particles) antisymmetric wavefunctions

$$\Psi_F(x_1, x_2, \ldots x_n) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_{\alpha_1}(x_1) & \phi_{\alpha_2}(x_1) & \cdots \\ \vdots & \vdots & \ddots \\ \phi_{\alpha_1}(x_N) & \phi_{\alpha_2}(x_N) & \cdots \end{vmatrix}$$

Young Tableaux

$\square$ = Primitive $\rightarrow$ N values

$\square \times \square \rightarrow \begin{smallmatrix}\square\\\square\end{smallmatrix}$ or $\square\square$

antisymm — increase down

symmetric — at least increase right

$$D = \frac{\prod N_i}{\prod \text{hooks}}$$

Dimensionality

$$D = \frac{N(N+1)(N-1)}{3 \cdot 1 \cdot 1}$$

Scattering Processes

$$\Psi^+_{(\vec{x})} = \phi_k(\vec{x}) + \frac{1}{E - H_0 + i\varepsilon} V \Psi^+_{(\vec{x})} = \phi_k(\vec{x}) + \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right)\int d\vec{x}' \frac{e^{ik|\vec{x}-\vec{x}'|} V(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$G^+(\vec{x}-\vec{x}') = \frac{-e^{ik|\vec{x}-\vec{x}'|}}{4\pi|\vec{x}-\vec{x}'|}$$

local Approximation

$$\Psi^+_{(\vec{x})} = \frac{1}{(2\pi)^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k},\vec{k}')\right]$$

$$f(\vec{k},\vec{k}') = \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right)\int d\vec{x}' e^{-i\vec{k}'\cdot\vec{x}'} V(\vec{x}') \Psi(\vec{x}') \cdot (2\pi)^3$$

$$\frac{d\sigma}{d\Omega} = |f(\vec{k},\vec{k}')|^2 \qquad \sigma_{tot} = \int \frac{d\sigma}{d\Omega} d\Omega$$

Born Approximation $\quad \Psi^{(0)+} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$ $\quad 0^{th}$ order in $f(\vec{k},\vec{k}')$

$$f_B(\vec{k},\vec{k}') = \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right)\int d^3x' e^{i\vec{q}\cdot\vec{x}'} V(\vec{x}') \quad + \quad \vec{q} = \vec{k}-\vec{k}'$$

Fourier transform of interaction w.r.t. momentum transfer

Transfer Matrix : $\quad T|\phi\rangle = V|\phi\rangle + VG^+V|\Psi^+\rangle = V|\Psi^+\rangle$

$\hookrightarrow |\Psi^+\rangle = |\phi\rangle + G^+V|\Psi$

(Examples of T matrix)?

$$T = \frac{1}{1-VG^+} V$$

$$f(\vec{k},\vec{k}') = \left(\frac{2m}{\hbar^2}\right)\left(\frac{-1}{4\pi}\right)(2\pi)^3 \langle \vec{k}'|T|\vec{k}\rangle$$

Optical Theorem : $\quad \sigma_{tot} = \frac{4\pi}{k} \mathrm{Im}\{f(\vec{k},\vec{k}'=\vec{k})\}$

Partial Wave analysis

for low Energy scattering $f_\ell \propto k^{2\ell}$ $\therefore$ $\ell = 0$ dominate

$$f(k\hat{z}, \vec{k}') = \sum_\ell (2\ell+1) f_\ell(E) P_\ell(\cos\theta)$$

$$f_\ell(E) = \frac{-\pi}{k} T_\ell(E) \rightarrow \langle E\ell m | T | E \ell' m' \rangle = T_\ell(E) \delta_{\ell\ell'} \delta_{m\ldots}$$

Smatrix : $S_\ell(E) = 1 - 2i\pi T_\ell(E)$ $\quad \langle E\ell m | S | E \ell' m' \rangle = S_\ell(E) \delta_{\ell\ell'} \delta_{m'm}$

$$|S_\ell| = 1 \rightarrow S_\ell(E) = e^{i2\delta_\ell(E)}$$

$$\sigma_{tot} = \sum_\ell (2\ell+1) \frac{4\pi}{k} \operatorname{Im}\{f_\ell\}$$

$$f_\ell = -\frac{\pi}{k} T_\ell, \quad S_\ell = e^{2i\delta_\ell} = 1 - 2i\pi T_\ell$$

$$T_\ell = \frac{1 - e^{2i\delta_\ell}}{2i\pi}, \quad f_\ell = \frac{-\pi}{k} \frac{1}{2i\pi} \left(1 - e^{2i\delta_\ell}\right)$$

$$\boxed{f_\ell = \frac{e^{i\delta_\ell} \sin(\delta_\ell)}{k}}$$

$$\sigma_{TOT} = \frac{4\pi}{k^2} \sum_\ell (2\ell+1) \sin^2(\delta_\ell)$$

obtain phase shift from phase shifted wave function at boundary of potential

General free solution

$$\psi^+(\vec{x}) = \sum_\ell (2\ell+1) \frac{P_\ell(\cos\theta)}{2ikr} (-1)^\ell \left[\frac{e^{2i\delta\ell}}{2} e^{ikr} - \frac{e^{-ikr}}{2} \right]$$

Bound states correspond to poles in the S-Matrix.

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random QM scratch notes

1) Solve Comp problems
2) review material
3) write review sheets

Trig

* $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$
* $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

$\begin{matrix}\sin^2\\ \cos^2\end{matrix}(x) = \frac{1 \mp \cos(2x)}{2}$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\tan(2x) = 2\tan(x)/(1-\tan^2(x))$

$\sin^2(x) + \cos^2(x) = 1$ $\quad \Big\downarrow 1/\cos^2(x)$

$\tan^2(x) + 1 = \sec^2(x)$ $\quad \Big\downarrow 1/\sin^2(x)$

$1 + \cot^2(x) = \csc^2(x)$

$$\left.\begin{aligned}\sin(a)\sin(b) &= \tfrac{1}{2}(\cos(a-b) - \cos(a+b))\\ \cos(a)\cos(b) &= \tfrac{1}{2}(\cos(a-b) + \cos(a+b))\end{aligned}\right\} *$$

$$\left.\begin{aligned}\sin(a)\cos(b) &= \tfrac{1}{2}(\sin(a+b) + \sin(a-b))\\ \cos(a)\sin(b) &= \tfrac{1}{2}(\sin(a+b) - \sin(a-b))\end{aligned}\right\} \text{simply flip } a-b \to b-a = -(a-b)$$

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Scattering theory Basics

$$e^{a+b} = e^{a} e^{b} e^{-[a,b]/2}$$
$$e^{a} e^{b} = e^{b} e^{a} e^{[a,b]}$$

$$\frac{d}{dt} A = \frac{1}{i\hbar} [A, H]$$

$$\sigma^2 = (\Delta A)^2 = \langle \Psi | (\underset{\substack{\text{difference} \\ \text{value} \downarrow}}{A} - \underset{\text{mean} \downarrow}{\langle A \rangle})^2 | \Psi \rangle \quad , \quad \langle A \rangle = \langle \Psi | A | \Psi \rangle$$

Minimum uncertainty relation $(\Delta A)^2 (\Delta B)^2 \geq -\frac{1}{4} \langle \Psi | [\Delta A, \Delta B] | \Psi \rangle$

$$\Delta A \, \Delta B = \sigma_A \sigma_B \geq \frac{i}{2} \langle [A, B] \rangle$$

Translation operator $\quad \vec{p} = \frac{\hbar}{i} \frac{\partial}{\partial \vec{x}} \quad \vec{\Pi} = \vec{p} - \frac{q\vec{A}}{c}$

$$\hat{T}(\vec{x}) \simeq 1 - \frac{i \hat{\vec{x}} \cdot \hat{\vec{p}}}{\hbar} = e^{-\frac{i \vec{x} \cdot \vec{p}}{\hbar}}$$

$$\Psi(\vec{r} - \vec{x}) = \hat{T}(\vec{x}) \, \Psi(\vec{r}) = e^{-\frac{i \vec{x} \cdot \vec{p}}{\hbar}} \Psi(\vec{r}) = e^{-\vec{x} \cdot \vec{\nabla}} \cdot \Psi(r) \simeq \Psi(r) - \vec{x} \cdot \vec{\nabla} \Psi$$

Operators acting on other operators

$$T_x H_0 T_x^{\dagger} = e^{-\frac{i \vec{x} \cdot \vec{p}}{\hbar}} H_0 \, e^{+\frac{i \vec{x} \cdot \vec{p}}{\hbar}} = H_0 (\vec{r} - \vec{x})$$

$$T_a \Psi(x) = \Psi(x - a)$$

Rotation operator

$$R(\hat{r}, \theta) = e^{-\frac{i \theta \hat{r} \cdot \vec{J}}{\hbar}}$$ rotates by angle θ about $\hat{r}$ axis

Time evolution operator

$$U(t) = e^{-\frac{i t H}{\hbar}}$$

z component of angular momentum

$$L_z = -i\hbar \frac{\partial}{\partial z}$$

$[p_i, x_j] = \frac{\hbar}{i} \delta_{ij} \rightarrow [\vec{\nabla}, x] = 1 \quad , [\vec{x}, \vec{y}] = 0, \ [p_x, p_y] = 0,$

$[L_i, L_j] = i\hbar \, \varepsilon_{ijk} L_k \quad , [p, x^n] = n \frac{\hbar}{i} x^{n-1} = n x^{n-1} [p, x]$

$[E, t] = i\hbar \quad , [L^2, L_i] = 0$

$[a, a^{\dagger}] = 1 \ , [N, a^{\dagger}] = a^{\dagger}$

$J_{\pm} = J_x \pm i J_y \quad [J_z, J_{\pm}] = \pm \hbar J_{\pm} \quad [J_+, J_-] = 2\hbar J_z \quad [N, a] = -a$