DOCTORAL GENERAL EXAMINATION

PART 1

Solutions

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Updated June 13, 2002 based on corrections from Dave Pritchard and Gianpaolo Carosi.

Group I

1. **Dipole antenna**

Choose the dipole along $\hat{z}$ with the midpoint of the antenna at the origin. The vector potential is

$$\vec{A}(\vec{x},t) = \int_V \frac{\vec{J}(\vec{x}',t')}{rc} d^3x'.$$

For the propagating wave, we find $\vec{A}$ far from the antenna, $|\vec{x}| >> l$. The integral over the cross section of the antenna is just $I(t')$ and t' is just the retarded time $t' = t - r/c$. Then

$$\vec{A} = \frac{lI_o \cos\omega(t - r/c)}{rc}\hat{z}.$$

2. **Parking Orbit**
Solution 1
The statement of the problem is not very good. I should have written: "A spacecraft of mass m is approaching Planet X (of mass M) at radial velocity $\dot{r} \sim v$ and impact parameter $b << a$. At this time, the spacecraft is a distance a from Planet X. Give the condition on b for a bound orbit in terms of M, m, a and/or v and physical constants."

The student is to assume the radial velocity is v and the angular momentum is mbv. The solution is as follows:

The total energy is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}ma^2\dot{\theta}^2 - \frac{GMm}{a}$$

with $E < 0$ for a bound orbit. The angular momentum is $l = mbv = mr^2\dot{\theta}$, so the condition for a bound orbit is

$$\begin{aligned} 0 &> \frac{1}{2}mv^2 + \frac{l^2}{2ma^2} - \frac{GMm}{a} \\ a\sqrt{\frac{2GM}{av^2} - 1} &< b. \end{aligned}$$

Some students may have assumed an angular momentum or velocity. Given the nature of my error, that is fine; they should receive full credit.

Apologies to Dave P., who pointed out the error, which I did not understand.

Solution 2

In order to get into a stable orbit at radius r,

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r} < 0$$

and you are given $\dot{r} = v$ when $r = a$. Then orbit condition is

$$\frac{1}{2}m(v^2 + a^2\dot{\theta}^2) < \frac{GMm}{a} \tag{1}$$

In order to solve, we must eliminate $\dot{\theta}$ in the Eq. 1. To do this, find a relation between a, b, v and $\dot{\theta}$ which is shown in the figure below.

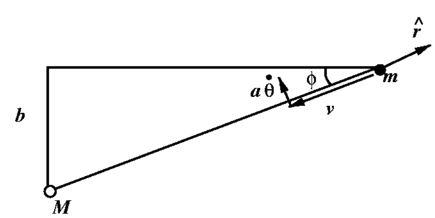


Figure 1:

$$\begin{aligned} \tan\phi &= \frac{b}{\sqrt{a^2 - b^2}} = \frac{a\dot{\theta}}{v} \\ a\dot{\theta} &= \frac{vb}{\sqrt{a^2 - b^2}} \end{aligned}$$

and eliminating $\dot{\theta}$ from Eq. 1 gives

$$\begin{aligned} v^2 + \frac{v^2 b^2}{\sqrt{a^2 - b^2}} &< \frac{2GM}{a} \\ b &< a\sqrt{1 - \frac{av^2}{2GM}} \end{aligned}$$

for a stable orbit.

The first solution relies on making the assumption that $l = mbv = mr^2\dot{\theta}$, which is not exactly true, but full credit was given for solution 1. Solution 2 is exact. Note in both cases $M >> m$ has been assumed, so the reduced mass is $\mu = m$. This was not stated in the problem explicitely, but it is okay to assume a planet is much heavier than a spacecraft.

3. **Interacting bosons**

The unperturbed solution is $\psi_j^{(o)}(x_1, x_2) = (2/L)\sin(\pi x_1/L)\sin(\pi x_2/L)$. The effect on the energy is then given by first order time independent perturbation theory as

$$\begin{aligned} E_j^{(1)} &= <\phi_j^o|V|\phi_j^o> \\ &= -LV_o\frac{4}{L^2}\int_0^L dx \sin^4\frac{\pi x}{L} \\ &= -\frac{4V_o}{\pi}\int_0^\pi dy \sin^4 y \\ &= -\frac{3}{2}V_o. \end{aligned}$$

The δ function requires $x_1 = x_2$ in the first line.

4. **Solar corona**

The steady state heat flows means the temperature is independent of time and we are told $T \propto r^n$, so we just need to find n, so

$$\begin{aligned} \vec{\nabla}T &= \frac{\partial T}{\partial r}\hat{r} = \alpha n r^{n-1}\hat{r} \\ k\vec{\nabla}T &= \alpha' n r^{5n/2+n-1}\hat{r} = \alpha' n r^{7n/2-1}\hat{r} \\ \vec{\nabla}\cdot k\vec{\nabla}T &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 k\vec{\nabla}T) = n\alpha' r^{7n/2}(11n/2 - 1) = 0 \end{aligned}$$

where α, α' are proportionality constants. Then $T \propto r^{2/11}$.

Group II

1. **Ring in a magnetic field**
   The flux through the ring is $\Phi = B(t)\pi a^2$, so the EMF from the changing flux is
$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} = -\frac{1}{c}\pi a^2 \frac{\partial B}{\partial t}.$$
   This makes an azimuthal electric field
$$\vec{E} = -\frac{\mathcal{E}}{2\pi a}\hat{\phi}$$
   which acts on the charge in the ring to make a torque
$$\vec{N} = -Q\vec{E}a = \dot{\vec{L}}$$
   and integrating gives
$$\vec{L} = \frac{Qa^2}{2c}\int_0^\infty \frac{\partial \vec{B}(t)}{\partial t} dt = \frac{Qa^2 B_o}{2c}\vec{z}.$$
   The results does not depend how the field changes with time, only the total change in the field as shown by the integral.

2. **van der Waals gas**

   From second law,
$$\begin{aligned} dE &= TdS - PdV \\ &= T\left(\left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV\right) - PdV \\ \rightarrow \left(\frac{\partial E}{\partial V}\right)_T &= \left(T\left(\frac{\partial S}{\partial V}\right)_T - P\right). \end{aligned}$$
   Next, use the Free energy, $dF = dE - TdS - SdT = -SdT - PdV$ to get the Maxwell relation
$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$
   which gives
$$\left(\frac{\partial E}{\partial V}\right)_T = \left(T\left(\frac{\partial P}{\partial T}\right)_V - P\right) = \frac{a}{V^2}$$

3. **Barely bound particle**
   The solution must satisfy the time independent Schroedinger equation on both sides of the delta function

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V = E\psi.$$

The solution on each side of the *delta* function is

$$\psi_L = A_L e^{k_L x} \to k_L^2 = -\frac{2mE}{\hbar^2}$$
$$\psi_R = A_R e^{k_R x} \to k_R^2 = -\frac{2m(E - V_o)}{\hbar^2}.$$

The wave functions must go to zero at $x = \pm\infty$, so $k_L > 0$ and $k_R < 0$. The wave functions must match at $x = 0$

$$\psi_L(0) = \psi_R(0) = A_L = A_R.$$

Next, integrate the TISE from over $-\epsilon, \epsilon$ which gives

$$\begin{aligned}
-\frac{\hbar^2}{2m}\left(\frac{\partial\psi_R}{\partial x} - \frac{\partial\psi_L}{\partial x}\right)_{x=0} &= W\psi(0) = -\frac{\hbar^2}{2m}(-|k_R| - |k_L|)A_L \\
\to \sqrt{2m(V_o - E)} + \sqrt{2mE)} &= 2mW/\hbar \\
\to W &= \hbar\sqrt{\frac{V_o}{2m}}
\end{aligned}$$

for the "barely bound" case, $E = 0$.

4. **Bessel function**

   The key for this problem is to Taylor expand and equate terms of t^n:

$$\begin{aligned}
e^{xt/2} &= \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{xt}{2}\right)^r \\
e^{-x/2t} &= \sum_{s=0}^{\infty}\frac{1}{s!}\left(-\frac{x}{2t}\right)^s \\
g(x,t) &= \Sigma_r\Sigma s\left(\frac{1}{r!}\frac{x^r t^r}{2^r}\right)(-1)^s\left(\frac{1}{s!}\frac{x^s}{2^s t^s}\right) \\
&= \sum_{r,s}\frac{1}{r!}\frac{1}{s!}(-1)^s\frac{x^{r+s}}{2^{r+s}}t^{r-s}.
\end{aligned}$$

Now substitute $n = r - s$ and get

$$g(x,t) = \sum_{s=0}^{\infty} \sum_{n=-s}^{\infty} \frac{1}{(n+s)!} \frac{1}{s!} (-1)^s \left(\frac{x}{2}\right)^{n+2s} t^n .$$

From the definition of the generating function then

$$J_n(x) = \sum_{s=0}^{\infty} \frac{1}{(n+s)!} \frac{1}{s!} (-1)^s \left(\frac{x}{2}\right)^{n+2s}$$

Group III

1. **Oscillating disk**

Define quantities as shown in the figure below.

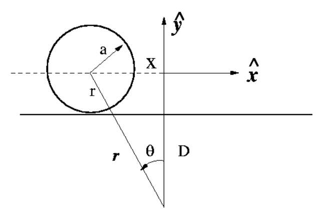


Figure 2:

Then

$$\begin{aligned}\vec{F} &= -kr^{-\alpha}\hat{r} \\ &= -kr^{-\alpha}\left(\frac{D}{r}\hat{y} + \frac{x}{r}\hat{x}\right).\end{aligned}$$

we are told $|x/D| << 1$, so $r \sim D$ and

$$\vec{F} = -kD^{-\alpha}\left(\hat{y} + \frac{x}{D}\hat{x}\right).$$

Now due to the constraint of the plane, the normal force will always cancel the $\hat{y}$ component of the force, so we ignore it for the rest of the problem. Since the disk rolls without slipping, there will be a torque around the point of contact between the plane andthe disk:

$$\vec{\tau} = a\left(-kD^{-(\alpha+1)}x\right)\hat{y} \times \hat{x} = akD^{-(\alpha+1)}x\hat{z}$$

and the no-slip constraint is $x = a\theta$. The moment of interia around the center of the disk is $I_c = (1/2)ma^2$ and, using the parallel axis theorem, this gives a moment of $I = I_c + ma^2 = (3/2)ma^2$ around the point of contact. Then

$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F} = a^2kD^{-(\alpha+1)}\theta\hat{z} \\ &= -I\ddot{\theta}\hat{z}\end{aligned}$$

which gives

$$\ddot{\theta} + \frac{2kD^{-(\alpha+1)}}{3m}\theta = 0$$

so the frequency of small oscillations is

$$\omega = \sqrt{\frac{2kD^{-(\alpha+1)}}{3m}}$$

and oscillations may occur for any value of α.

2. **Vibrations of a monatomic crystal**
   The Boltzmann probability for state r is $p_r = \exp(-\hbar\omega(r+1/2)/kT)$. Since $\hbar\omega >> 1$, only the $r = 0, 1$ states are important. Then, $p_o = \exp(\hbar\omega/2kT)$ and $p_1 = 1 - p_o$. Take $x = \hbar\omega/2kT$ and find the entropy

$$S = -k\sum_{i=0,1} p_i \ln p_i = -3Nk\left[(1-e^{-x})\ln(1-e^{-x}) + e^{-x}\ln e^{-x}\right].$$

   As $T \to 0$, $x \to \infty$ and the first term goes to zero and

$$S \to 3Nkxe^{-x}$$

3. **Radially confined quantum system**
   The eigenstates of the system are $\phi_n = \exp(ik\phi)/\sqrt{2\pi}$ and confined to a circle gives $\phi_n(0) = 1/\sqrt{2\pi} = \phi_n(2\pi) = \exp(2ik\pi)/\sqrt{2\pi} \to k = n/2\pi$, so $\phi_n = \exp(in\phi)/\sqrt{2\pi}$. The angular momentum operator is $L_z = \hbar/i\partial/\partial\phi$ and

$$L_z\phi_n = n\hbar\phi_n.$$

   Normalizing the wave function gives

$$1 = \int_0^{2\pi} \psi^*\psi d\phi = A^2(2\pi + 2\pi) \to A = 1/\sqrt{6\pi}$$

   and writing ψ as $\psi = \left(e^{2i\phi} + e^{i\phi} + e^{-i\phi}\right)/\sqrt{6\pi}$ gives

$$\begin{aligned} P(L_z = -1) &= \frac{1}{3} \\ P(L_z = 1) &= \frac{1}{3} \\ P(L_z = 2) &= \frac{1}{3} \end{aligned}$$

4. **Novel spaceship**
   The pressure $P = U(T)/3 = \sigma T^4$, where U is the energy per unit volume. Then, $F = \sigma T^4 A \rightarrow a = \sigma T^4 A/m$

Group IV

1. **Adiabatic oscillator**
At time $t = 0$. the mass has total energy $E_o(t) = (1/2)mv^2$. The viral theorem says for an adiabatic change, $\overline{T} = \overline{V} = \overline{E}/2$ where $\overline{T}$ and $\overline{V}$ refer to averages over one oscillation period. The Hamiltonian for the system is
$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2$$
where $p = m\dot{x}$. Since the change is adiabatic
$$\frac{dE}{dt} = \frac{\partial H}{\partial t} = \frac{1}{2}x^2\frac{\partial k}{\partial t} = -\frac{V}{k}\alpha$$
Then averaging over one period gives
$$\frac{d\overline{E}}{dt} == -\frac{\overline{V}\alpha}{k} = -\frac{mv^2}{4k_o}\alpha$$

2. **Black hole entropy**
From relativity, $dE = c^2dM = TdS = \hbar c^3(dS/k/8\pi MG)$. Then
$$d(S/k) = \frac{4\pi G}{\hbar c}2MdM \rightarrow S = \frac{4\pi k M^2 G}{\hbar c}.$$

3. **Cartesian resonator cavity**
The frequency will be lowest for the modes which depend on x and y since they are the longest dimensions, so choose
$$E_z = E_o \sin k_x x \sin k_y y \sin \omega t$$
and substitute into the wave equation to relate the k's and ω
$$(-k_x^2 - k_y^2)E_z = \frac{1}{c^2}(-\omega^2)E_z \rightarrow \omega^2 = c^2(k_x^2 + k_y^2).$$
The boundary conditions require $E_z(x = 0) = E_z(y = 0) = 0$, which are obviously satisfied. Then
$$\begin{aligned} E_z(x = a) &= 0 \rightarrow k_x = n\pi/a \\ E_z(y = a) &= 0 \rightarrow k_y = m\pi/a \end{aligned}$$

where m and n are integers. The lowest frequency is $\omega = \sqrt{2}c\pi/a$. Then use Faraday's law to get

$$\vec{B}(\vec{x},t) = E_o\hat{x}\left[\frac{k_y}{c\omega}\sin k_x x \cos k_y y \cos\omega t\right] + E_o\hat{y}\left[-\frac{k_y}{c\omega}\cos k_x x \sin k_y y \cos\omega t\right].$$

4. **Diffraction limited lens**

The effect of the lens is to focus the light on the focal plane a distance f from the lens. Then the lens can be thought of as an aperture of diameter D and the focal plane is effectively in the far field, but only f away. The radius of the visible disk occurs at the first zero of the Bessel function, which gives

$$\frac{2\pi(D/2)q}{\lambda f} = 3.8 \to q = 1.2\frac{f\lambda}{D}.$$

We asked for an ESTIMATE, so

$$\frac{2\pi(D/2)q}{\lambda f} \sim 1 \to q \sim 0.2\frac{f\lambda}{D}$$

is perfectly fine.

Group V

1. **Off center cylinder**
   Let y be the distance above the horizontal. The potential energy of the cylinder is just mgy and of the mass point is $mgy + mga\cos\theta/2$ and $y = a\theta\sin\alpha$, so the total potential energy is

$$V(\theta) = mg\left(2a\theta\sin\alpha + a\cos\theta/2\right).$$

   The cylinder will roll freely when there are no stable points:

$$\begin{aligned} \frac{dV}{d\theta} &= mg\left(2\sin\alpha - \frac{1}{2}\sin\theta\right) = 0 \rightarrow \\ 4\sin\alpha &= \sin\theta \end{aligned}$$

   so when $\sin\theta > 1/4$, the cylinder begins to roll.

   Alternatively, one may take torques around the contact point.

2. **Non-ideal gas**

   The entropy is

$$S = S(T,V) \rightarrow dS = \left(\frac{\partial S}{\partial T}\right)_V dT + \left(\frac{\partial S}{\partial V}\right)_T dV.$$

   From first law,

$$dE = dQ - dW \rightarrow dQ = dE + dW$$

   then the heat capacity at constant volume is

$$C_v = \frac{dQ}{dT}|_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V = 2bV^{2/3}T.$$

   Use the Maxwell relation

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V = \frac{4}{3}bV^{-1/3}T \rightarrow dS = 2bV^{2/3}dT + \frac{4}{3}V^{-1/3}bTdV.$$

Since

$$\frac{\partial}{\partial V}\left(2bV^{2/3}\right) = \frac{4}{3}bV^{-1/3} = \frac{\partial}{\partial T}\left(\frac{4}{3}bV^{-1/3}T\right)$$

the entropy has the form

$$S = 2bV^{2/3}T + f(V).$$

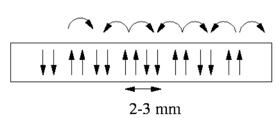
To find f

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{4}{3}bV^{-1/3}T + f' = \frac{4}{3}bV^{-1/3}$$

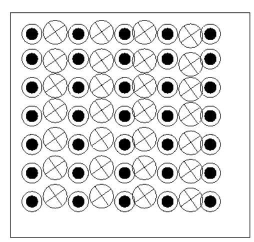
so $f(V) = S_o$, a constant.

3. **Quantum mechanical translation**
For a small, $T(a)\psi(x) = \psi(x) + (\partial\psi/\partial x)a = \psi(x) + (p_x\psi/i\hbar)a = (1 + p_x a/i\hbar)\psi$. If we want to translate by a large amount $A = Na$, in the limit $N \to \infty$, $T(A) \to \exp(-ip_xA/\hbar)$.

4. **Refrigerator magnets**
Sliding the magnets back and forth under various orientations, one find the pattern are repeating stripes pointing either up or down as shown below.



Side view



Top view

To estimate the magnetization, use the force on a dipole $\vec{m}$,

$$\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}).$$

At the surface of the magnet, $B = 4\pi\vec{M}$. The force drops away over $d = 1mm$ and, over the entire surface of the magnets, is about 1 N.

The magnets have an area $A = 24cm^2$. The magnetic material is about $t = 1mm$ thick. Then

$$F = (MAt)\frac{4\pi M}{d} \rightarrow \frac{F}{A} = 4\pi M^2 \frac{t}{d}$$

where the last factor is about 1. Solving gives $M \sim 13G$, which gives $B = 4\pi M \sim 160G$.

Note: The magnets handed out were 1×2.5 cm^2, not 24 cm^2 as in the solution. To estimate the force necessary to get them apart, recall lifting a 1 kg book is 10 N and a penny is 10^{-2} N. One would guess the force necessary to get the magnetic apart is somewhere in between, nearer the lower limit, say 0.1 N. This gives $M = 13Oe$ (or Gauss). The rest of the solution is as stated.