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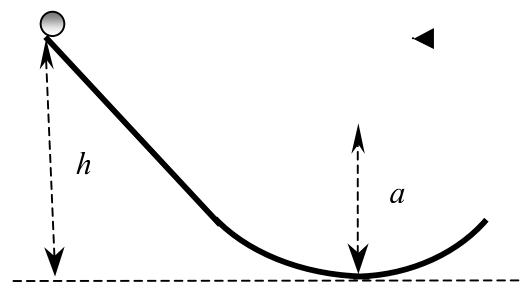
Department of Physics

Classical Mechanics Qualifying Exam Solutions

Problem 1.

A cylinder of a non-uniform radial density with mass *M,* length *l* and radius *R* rolls without slipping from rest down a ramp and onto a circular loop of radius *a*. The cylinder is initially at a height *h* above the bottom of the loop. At the bottom of the loop, the normal force on the cylinder is twice its weight.



a) Expressing the rotational inertia of the non-uniform cylinder in the general form ($I=\beta MR^2$), express the β in terms of *h* and *a*.

b) Find numerical value of β if the radial density profile for the cylinder is given by $\rho(r) = \rho_2 r^2$;

c) If for the cylinder of the same total mass *M* the radial density profile is given by $\rho_n(r) = \rho_n r^n$, where $n \in 0,1,2,3...$, describe qualitatively how do you expect the value of β to change with increasing *n*. Explain your reasoning.

Solution:

a) Centripetal acceleration as the ball rolls around the circular loop at the bottom of the track is $a_c = v^2/a$, and could be expressed from free body diagram equation:

$N\text{-}W = Mv^2/a$, where *N* is the normal force and the *W* is the weight. We are given that $N=2Mg$, so

$$2Mg - Mg - Mv^2/a, \text{ i. e. } v^2 = ga$$

Relating the angular and translational velocities by $v=a\omega$, we next use the expression for the total kinetic energy of rolling object (no slipping)

$$K = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2$$

And apply energy conservation for the ball between its initial position at rest and its position at the bottom of the loop:

$$Mgh = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2$$

Substituting for v and for ω (from above) and using $I = \beta MR^2$ expression we find

$$h = \tfrac{1}{2}a + \tfrac{1}{2}\beta a$$

Rearranging: $\beta = 2h/a - 1$

b) The moment of inertial about the rotational axis of the cylinder is given by $I = \int r^2 dm$. For the quadratic density profile $dm = \rho_2 r^2 l 2\pi r dr$

$$I = \int_0^R r^2 \rho_2 r^2 l 2\pi r dr = 2\pi\rho_2 l \frac{R^6}{6} = \frac{2}{3}MR^2 \text{, with } M = \int \rho(r)dV = \int_0^R 2\pi l\rho_2 r^3 dr = \frac{\pi l R^4 \rho_2}{2} \text{, so}$$

$$\beta = \frac{2}{3}$$

c) For an arbitrary value of n:

$$I = 2\pi l\rho_n \int_0^R r^{n+3} dr = 2\pi l\rho_n \frac{R^{n+4}}{n+4} \text{, with } M = 2\pi l\rho_n \int_0^R r^{n+1} dr = \frac{2\pi l\rho_n R^{n+2}}{n+2} \text{, so}$$

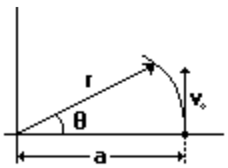
$$I = \frac{n+2}{n+4}MR^2, \qquad \beta = \frac{n+2}{n+4}, \qquad \text{e.g. } n = 0, \beta = \frac{1}{2}; \quad n \to \infty, \beta \to 1$$

Problem 2.

A particle of unit mass is projected with a velocity v_0 at right angles to the radius vector at a distance a from the origin of a center of attractive force, given by

$$f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

For initial velocity value given by $v_0^2 = \dfrac{9k}{2a^2}$, find the polar equation of the resulting orbit.



Solution: Calculating the potential energy

$$-\frac{dv}{dr} = f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

Thus, $V = -k\left(\dfrac{2}{r^2} + \dfrac{a^2}{4r^4}\right)$

The total energy is …

$$E = T_o + V_o = \frac{1}{2}v_o^2 - k\left(\frac{2}{a^2} + \frac{1}{4a^2}\right) = \frac{1}{2}\left(\frac{9k}{2a^2}\right) - \frac{9k}{4a^2} = 0$$

Its angular momentum is …

$$l^2 = a^2 v_o^2 = \frac{9k}{2} = constant = r^4\dot{\theta}^2$$

Its KE is …

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2 = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{l^2}{r^4}$$

The energy equation of the orbit is …

$$T + V = 0 = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{l^2}{r^4} - k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

$$= \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{9k}{4r^4} - k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

or $\quad \left(\dfrac{dr}{d\theta}\right)^2 = \dfrac{1}{9}\left(a^2 - r^2\right)$

Letting $\quad r = a\cos\phi \quad$ then $\quad \dfrac{dr}{d\theta} = -a\sin\phi\dfrac{d\phi}{d\theta}$

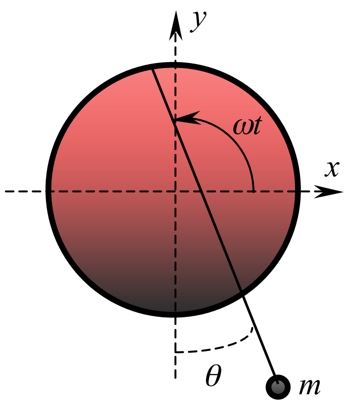
So $\quad \left(\dfrac{d\phi}{d\theta}\right)^2 = \dfrac{1}{9} \qquad \therefore \phi = \dfrac{1}{3}\theta$

Thus $\quad r = a\cos\dfrac{1}{3}\theta \qquad \left(r = a @ \theta = 0^{\circ}\right)$

Problem 3.

A simple pendulum of length b and mass m is suspended from a point on the circumference of a thin massless disc of radius a that rotates with a constant angular velocity ω about its central axis. Using Lagrangian formalism, find

a) the equation of motion of the mass m;
b) the solution for the equation of motion for small oscillations.



Solution:

a) Coordinates:

$$x = a\cos\omega t + b\sin\theta$$
$$y = a\sin\omega t - b\cos\theta$$

$$\dot{x} = -a\omega\sin\omega t + b\dot{\theta}\cos\theta$$
$$\dot{y} = a\omega\cos\omega t + b\dot{\theta}\sin\theta$$

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$
$$= \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin\left(\theta - \omega t\right)\right] - mg\left(a\sin\omega t - b\cos\theta\right)$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega\left(\dot{\theta} - \omega\right)\cos\left(\theta - \omega t\right)$$
$$\frac{\partial L}{\partial\theta} = mb\dot{\theta}a\omega\cos\left(\theta - \omega t\right) - mgb\sin\theta$$

The equation of motion $\quad \dfrac{\partial L}{\partial\theta} - \dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\theta}} = 0$ is

$$\ddot{\theta} - \frac{\omega^2 a}{b}\cos(\theta - \omega t) + \frac{g}{b}\sin\theta = 0$$

(Note – the equation reduces to equation of simple pendulum if $\omega \to 0$.)

b) For small θ the equation of motion reduces to that of a constant driving force harmonic oscillator :

$$\ddot{\theta} + \frac{g}{b}\theta = \frac{\omega^2 a}{b}\cos(\omega t)$$

The general solution for this equation consists of two parts, a complementary function *u(t)* and a particular solution *v(t)*. Complementary solution comes from the simple harmonic oscillator equation: $\ddot{\theta} + \frac{g}{b}\theta = 0$, and could be immediately written as $u(t) = A\cos\left(\sqrt{\frac{g}{b}}t - \varphi\right)$.

Given the form of the driving force, the particular solution could be defined as $v(t) = B\cos(\omega t)$.

Taking the derivatives and plugging back in the equation of motion, one finds the expression for B as $B = \frac{\omega^2 a}{g - \omega^2 b}$, and the general solution in form:

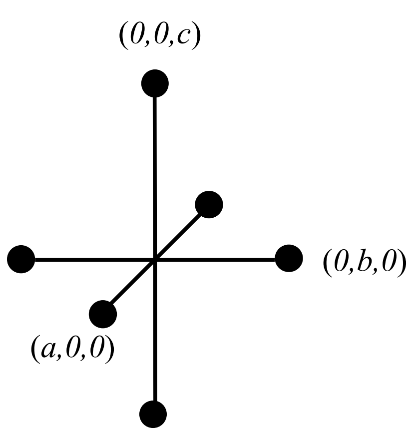
$$\theta(t) = u(t) + v(t) = A\cos\left(\sqrt{\frac{g}{b}}t - \phi\right) + \frac{\omega^2 a}{g - \omega^2 b}\cos(\omega t)$$

Problem 4.

A rigid body consists of six particles, each of mass *m*, fixed to the ends of three light rods of length *2a*, *2b*, and *2c* respectively, the rods being held mutually perpendicular to one another at their midpoints.

a) Write down the inertia tensor for the system in the coordinate axes defined by the rods;
b) Find angular momentum and the kinetic energy of the system when it is rotating with an angular velocity ω about an axis passing through the origin and the point *(a,b,c)*.

<u>Solution:</u>



a)

$I_{xy} = \sum_i m_i x_i y_i = 0$ since either x_i or y_i is zero for all six particles. Similarly, all the other products of inertia are zero. Therefore the coordinate axes are principle axes.

$$I_{xx} = \sum_i m_i \left(y_i^2 + z_i^2\right) = m\left[0 + 0 + b^2 + (-b)^2 + c^2 + (-c)^2\right]$$

$$I_{xx} = 2m\left(b^2 + c^2\right)$$

$$I_{yy} = 2m\left(a^2 + c^2\right)$$

$$I_{zz} = 2m\left(a^2 + b^2\right)$$

$$\overleftrightarrow{I} = 2m\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

b)
$$\vec{\omega} = \frac{\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\left(a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3\right) = \frac{\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Using $\vec{L} = \overleftrightarrow{I}\vec{\omega}$,

$$\vec{L} = \frac{2m\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{L} = \frac{2m\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right) \end{bmatrix}$$

Using $T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

$$T = \frac{1}{2}\frac{2m\omega^2}{\left(a^2+b^2+c^2\right)}\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right) \end{bmatrix}$$

$$T = \frac{m\omega^2}{a^2+b^2+c^2}\left[a^2\left(b^2+c^2\right)+b^2\left(a^2+c^2\right)+c^2\left(a^2+b^2\right)\right]$$

$$T = \frac{2m\omega^2}{a^2+b^2+c^2}\left(a^2b^2+a^2c^2+b^2c^2\right)$$

Problem 5.

The force of a charged particle in an inertial reference frame in electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$, where q is the particle charge and $\vec{v}$ is the velocity of the particle in the inertial system.

a) Prove that the transformation from a fixed frame to a rotating frame is given by $\ddot{\vec{r}} = \ddot{\vec{r}}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}'\right)$

b) Find the differential equation of motion referred to a non-inertial coordinate system rotating with angular velocity $\vec{\omega} = -\left(\frac{q}{2m}\right)\vec{B}$, for small $\vec{B}$ (neglect B^2 and higher order terms).

Solution:

a) Let's consider a coordinate system $(\vec{i}\,', \vec{j}\,', \vec{k}\,')$, rotating about the axis defined by the unit vector $\vec{n}$ with respect to the $(\vec{i}, \vec{j}, \vec{k})$ system with angular velocity $\vec{\omega} = \vec{n}\omega$. Position of any point P in space can be expressed in two systems (in case of common origin) as

$$\vec{r} = \vec{i}x + \vec{j}y + \vec{k}z = \vec{r}' = \vec{i}\,'x' + \vec{j}\,'y' + \vec{k}\,'z'$$

The velocity then can be written

$$\vec{v} = \vec{i}\frac{dx}{dt} + \ldots = \vec{i}\,'\frac{dx'}{dt} + \ldots + x'\frac{d\vec{i}\,'}{dt} + \ldots = \vec{v}' + x'\frac{d\vec{i}\,'}{dt} + \ldots = \vec{v}' + \vec{\omega} \times \vec{r}'$$

This finding is generally true for any vector, e.g. for derivative of the velocity vectors:

$$\left(\frac{d\vec{v}}{dt}\right)_{fixed} = \left(\frac{d\vec{v}}{dt}\right)_{rotating} + \vec{\omega} \times \vec{v}$$

$$\left(\frac{d\vec{v}}{dt}\right)_{fixed} = \left(\frac{d(\vec{v}' + \vec{\omega} \times \vec{r}')}{dt}\right)_{rotating} + \vec{\omega} \times (\vec{v}' + \vec{\omega} \times \vec{r}') =$$

$$\left(\frac{d\vec{v}'}{dt}\right)_{rotating} + \left(\frac{d(\vec{\omega} \times \vec{r}')}{dt}\right)_{rotating} + \vec{\omega} \times \vec{v}' + \vec{\omega} \times \vec{\omega} \times \vec{r}' =$$

$$\left(\frac{d\vec{v}'}{dt}\right)_{rotating} + \left(\frac{d\vec{\omega}}{dt}\right)_{rotating} \times \vec{r}' + \vec{\omega} \times \left(\frac{d\vec{r}'}{dt}\right)_{rotating} + \vec{\omega} \times \vec{v}' + \vec{\omega} \times \vec{\omega} \times \vec{r}'$$

Changing notations, one arrives at the transformation equations from a fixed to a rotating frame:

$$\vec{v} = \vec{v}' + \vec{\omega} \times \vec{r}' \qquad \ddot{\vec{r}} = \ddot{\vec{r}}' + \dot{\vec{\omega}} \times \vec{r}' + 2\vec{\omega} \times \vec{v}' + \vec{\omega} \times \left(\vec{\omega} \times \vec{r}'\right)$$

b) The equation of motion:

$$m\ddot{\vec{r}} = q\vec{E} + q\left(\vec{v} \times \vec{B}\right)$$

Transforming from a fixed frame to a moving rotating frame:

$\vec{\omega} = -\frac{q}{2m}\vec{B}$ so $\dot{\vec{\omega}} = 0$

$$m\ddot{\vec{r}}' - q\left(\vec{B} \times \vec{v}'\right) - \frac{q}{2}\vec{B} \times \left(\vec{\omega} \times \vec{r}'\right) = q\vec{E} + q\left[\left(\vec{v}' + \vec{\omega} \times \vec{r}'\right) \times \vec{B}\right]$$

$$m\ddot{\vec{r}}' + q\left(\vec{v}' \times \vec{B}\right) + \frac{q}{2}\left(\vec{\omega} \times \vec{r}'\right) \times \vec{B} = q\vec{E} + q\left(\vec{v}' \times \vec{B}\right) + q\left(\vec{\omega} \times \vec{r}'\right) \times \vec{B}$$

$$m\ddot{\vec{r}}' = q\vec{E} + \frac{q}{2}\left(\vec{\omega} \times \vec{r}'\right) \times \vec{B}$$

$$\left|\frac{q}{2}\left(\vec{\omega} \times \vec{r}'\right) \times \vec{B}\right| = \frac{q}{2}\left(\frac{qB}{2m}\right)\left(r'\right)\left(\sin\theta\right)\left(B\right) \propto B^2$$

Neglecting terms in B^2, $m\ddot{\vec{r}}' = q\vec{E}$ (Larmor's theorem)