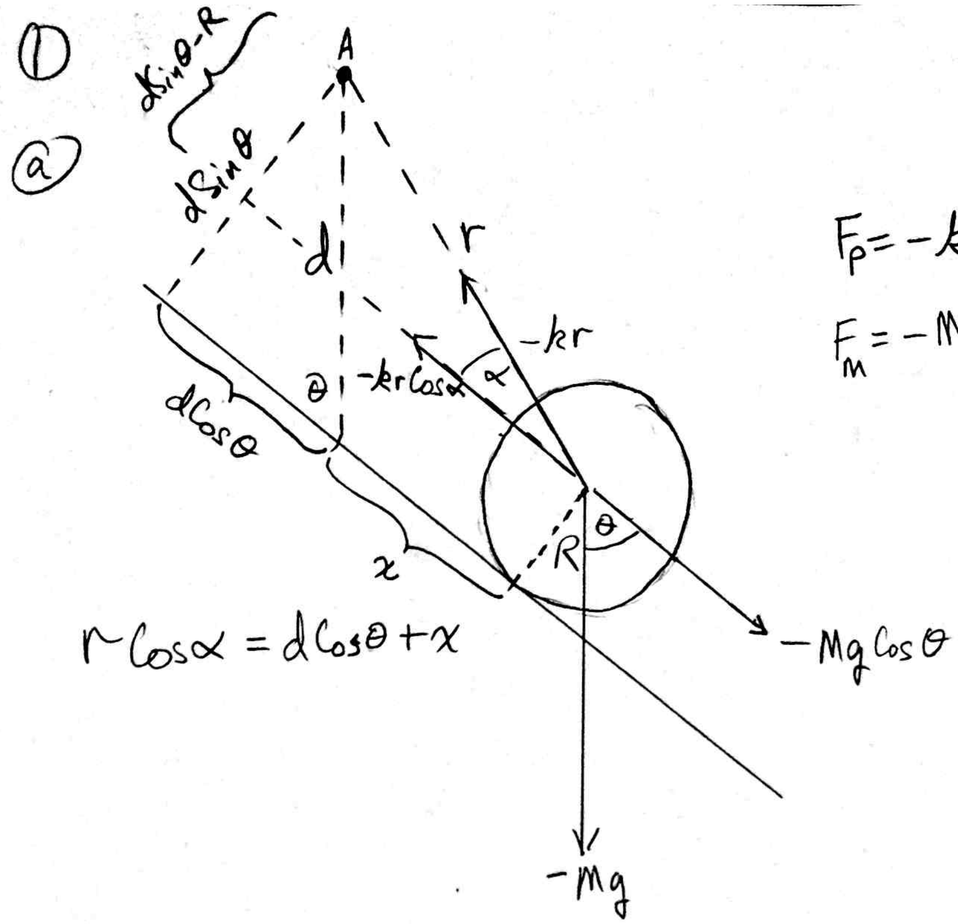
① ⓐ



$$r\cos\alpha = d\cos\theta + x$$

$$\left.\begin{aligned} F_p &= -kr\,\hat{r} \\ F_M &= -Mg\,\hat{z} \end{aligned}\right\} \; kr\cos\alpha = Mg\cos\theta, \quad \underbrace{kr\cos\alpha}_{d\cos\theta + x}$$

$$\Downarrow$$

Equlibrium position

$$\boxed{x_0 = \left(\frac{Mg}{k} - d\right)\cos\theta}$$

ⓑ ⓘ $F = -kr \Rightarrow V_p = -\int F\,dr = \frac{1}{2}kr^2 \quad V_m = +Mgx\cos\theta$

ⓘⓘ $r^2 = (x + d\cos\theta)^2 + (d\sin\theta - R)^2$

ⓘⓘⓘ $y = s = R\theta \Rightarrow \dot{y} = \dot{s} = R\dot{\theta}$

ⓘⓥ $I = \frac{1}{2}MR^2$

Define $y = x - x_0$
Then $y = x - \left(\frac{Mg}{k} - d\right)\cos\theta$
And $\dot{y} = \dot{x}\,;\; \ddot{y} = \ddot{x}$

ⓥ $T = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{x}}{R}\right)^2 + \frac{1}{2}M\dot{x} = \frac{3}{4}M\dot{x}^2$

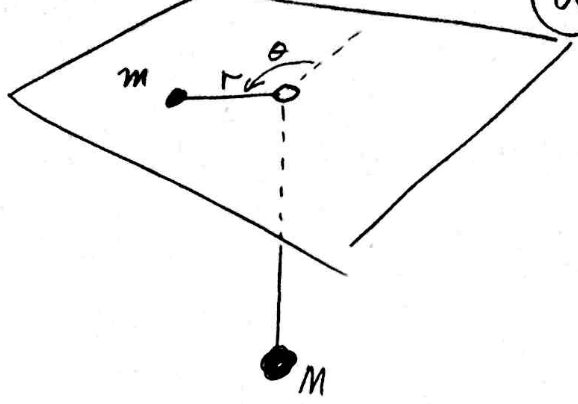
Thus $\mathcal{L} = T - V = \frac{3}{4}M\dot{x}^2 - \frac{1}{2}k\left[(x + d\cos\theta)^2 + (d\sin\theta - R)^2\right] + Mgx\cos\theta$

Using the ELEs $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = \frac{3}{2}M\ddot{x} = \frac{\partial\mathcal{L}}{\partial x} = -k(x + d\cos\theta) + Mg\cos\theta$

Hence $\ddot{x} + \frac{2k}{3M}\underbrace{\left(x - \left[\frac{Mg}{k} - d\right]\cos\theta\right)}_{y} = 0 \Rightarrow \ddot{y} + \frac{2k}{3M}y = 0$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{2k}{3M}}}$$

② ⓐ



① **Generalized Coordinates:**

$$\boxed{r, \theta}$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2$$

$$V = -Mg(l - r)$$

Lagrangian:

$$\boxed{\mathcal{L} = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2 + Mg(l-r)}$$

② **Conserved Quantity**

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2\ddot{\theta} = \frac{\partial \mathcal{L}}{\partial \theta} = 0 \Rightarrow \boxed{\begin{array}{l} mr^2\dot{\theta} = \text{constant} = L \\ \text{Angular momentum conserved} \end{array}}$$

$$\frac{1}{2}mr^2\dot{\theta}^2 = \frac{L^2}{2mr^2}$$

③ **Stable circular Orbit**

$$E_0 = T + V = \frac{1}{2}m\dot{r}^2 + \left[\frac{L^2}{2mr^2} - Mg(l-r)\right] = \frac{1}{2}m\dot{r}^2 + V_{eff}$$

A circular orbit requires

$$0 \equiv \frac{d}{dr}V_{eff}\Big|_{r_0} = \frac{L^2}{2m}\left[-\frac{2}{r^3}\right]_{r_0} + Mg \Rightarrow \frac{L^2}{mr_0^3} = Mg \Rightarrow \boxed{r_0 = \left(\frac{L^2}{mMg}\right)^{1/3}}$$

Test if the orbit is stable

$$0 < \frac{d^2}{dr^2}V_{eff}\Big|_{r_0} = \frac{L^2}{2m}\left[\frac{6}{r^4}\right]_{r_0} = \frac{3Mg}{r_0} = \boxed{\text{stable orbit}}$$

④ **Total Energy of circular Orbit**

$$E_0 = \frac{1}{2}m\dot{r}_0^2 + \left[\frac{L^2}{2mr_0^2} - Mg(l-r_0)\right] = 0 + \left[\frac{L^2 r_0}{2mr_0^3} - Mg(l-r_0)\right] = \boxed{+Mg\left(\frac{3r_0}{2} - l\right)}$$

② Continued

ⓑ Initially $r_1 = \frac{\ell}{2}$ and $\vec{v}_1 = \dot{r}_1\hat{r} + r_1\dot{\theta}_1\hat{\theta} = 0\hat{r} + \sqrt{\ell g}\,\hat{\theta}$

① Since $\dot{r}_1 = 0$, the conserved total E is

$$E = T + V = \tfrac{1}{2}m\dot{r}_1^2 + \frac{L^2}{2mr_1^2} - mg(\ell - r_1)$$

$$= 0 + \frac{L^2}{2mr_1^2} - mg(\ell - r_1)$$

Now $L_1 = mr_1^2\dot{\theta}_1 = mr_1\sqrt{\ell g} = m\left(\frac{\ell}{2}\right)\sqrt{\ell g}$

Hence $$\boxed{E = \frac{\left(m\frac{\ell}{2}\sqrt{\ell g}\right)^2}{2m\left(\frac{\ell}{2}\right)^2} - mg\left(\ell - \frac{\ell}{2}\right) = mg\frac{\ell}{2} - mg\frac{\ell}{2} = 0}$$

② The effective potential is $V_{eff}(r) = \frac{1}{2m}\frac{L^2}{r^2} - mg(\ell - r)$

Clearly $\boxed{V_{eff}(0) = +\infty}$

Also $\boxed{V_{eff}(r = r_1 = \tfrac{\ell}{2}) = E(\dot{r}_1, r_1) = 0}$ (from above)

Now $V_{eff}(r)$ has a minimum for a circular orbit $r = r_c$

From part ⓐ the radius of a circular orbit is

$$r_c^3 = \frac{L_1^2}{mMg} = \frac{\left[m\left(\frac{\ell}{2}\right)\sqrt{\ell g}\right]^2}{m^2 g} = \frac{\ell^3}{4} \Rightarrow r_c = 2^{-2/3}\ell$$

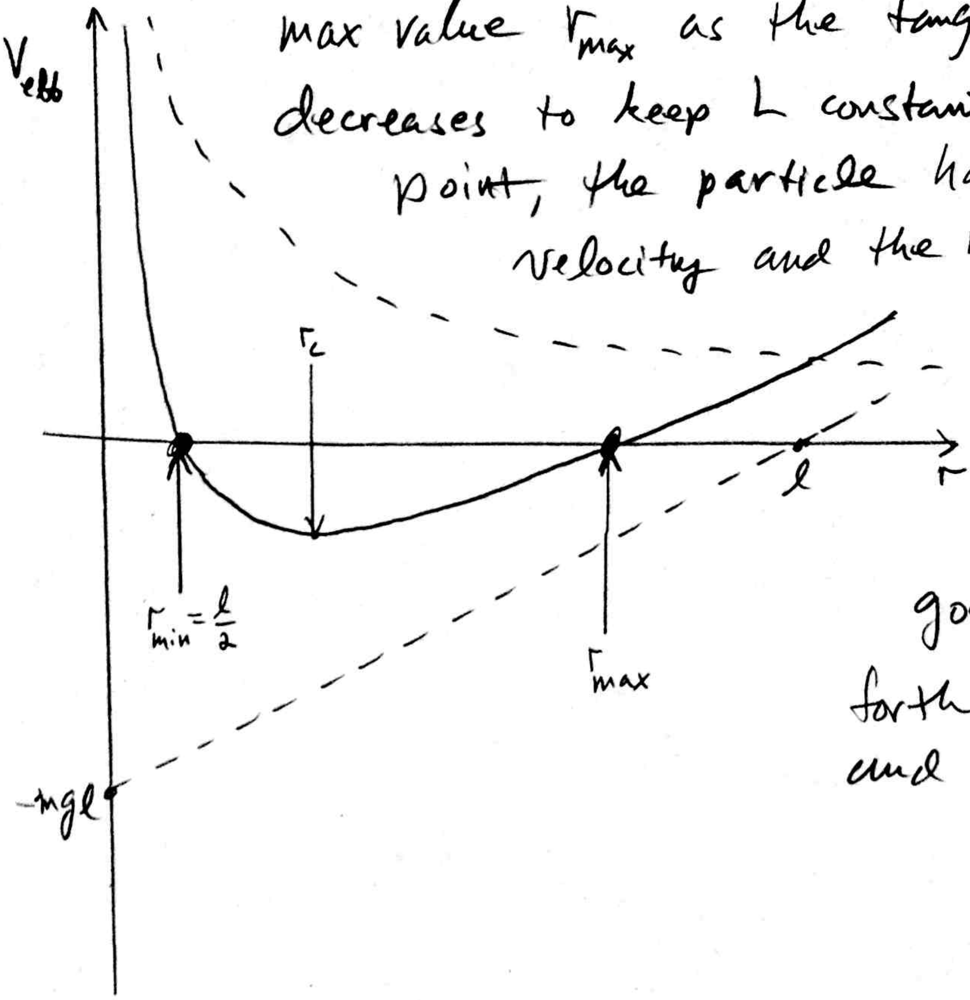
Hence

$$\boxed{\begin{aligned} V_{eff}(r_c) &= \frac{L_1^2}{2mr_c^2} - mg(\ell - r_c) = \frac{1}{2m}\frac{L_1^2}{r_c^3}r_c - mg(\ell - r_c) \\ &= \frac{1}{2m}(m^2 g)r_c - mg(\ell - r_c) = \\ &= mg(2^{-1}2^{-2/3})\ell - mg\ell + mg(2^{-2/3})\ell \\ &= mg\ell\left(\frac{3}{2^{5/3}} - 1\right) < 0 \end{aligned}}$$

(2) Continued

(b)

(2) Hence the radius of the orbit increases to the max value r_{max} as the tangential v decreases to keep L constant. At some point, the particle has no radial velocity and the radius starts decreasing back to $r_{min} = \frac{l}{2}$. The motion then goes back and forth between r_{min} and r_{max}



(c) To find r_{max}, we recognise that $V_{eff}(r_{max}) = 0$

Hence $V_{eff} = \frac{L_1^2}{2 m r_{max}^2} - mg(l - r_{max}) = 0$

Substituting L_1

$$= \frac{\left[m\left(\frac{l}{2}\right)\sqrt{lg}\right]^2}{2 m r_{max}^2} - mg(l - r_{max}) = \frac{mg}{8}\frac{l^3}{r_{max}^2} - mgl + mgr_{max}$$

Or

$$8 r_{max}^3 - 8 l r_{max}^2 + l^3 = 0 \quad \left(\text{let } x = \frac{r}{l}\right)$$

which is

$$8x^3 - 8x^2 + 1 = 0$$

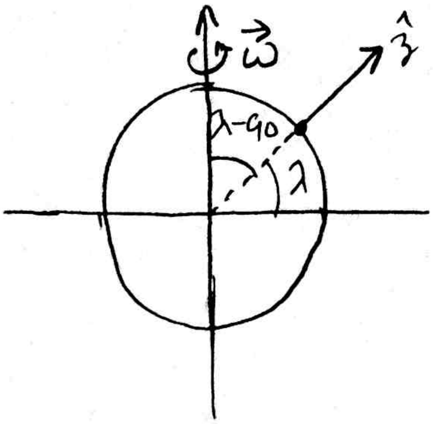
we know that one solution must be $r_{min} = \frac{l}{2} \Rightarrow x = \frac{1}{2}$

Hence

$$\left(x - \tfrac{1}{2}\right)(8x^2 - 4x - 2) = 0 \Rightarrow x = \frac{+4 \pm \sqrt{16 - 4(8)(2)}}{2(8)} = \frac{1 \pm \sqrt{5}}{4}$$

Choos + root $\Rightarrow \boxed{r_{max} = \left(\frac{1 + \sqrt{5}}{4}\right) l}$

③



ⓐ The velocity in an inertial frame, expressed in a rotating one

$$\dot{\vec{r}}_{init} = \dot{\vec{r}}_{rot} + \vec{\omega}\times\vec{r}_{rot}$$

the acceleration in the inertial frame is

$$\ddot{\vec{r}}_{init} = \ddot{\vec{r}}_{rot} + \dot{\vec{\omega}}\times\dot{\vec{r}}_{rot} + \vec{\omega}\times\dot{\vec{r}}_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{rot})$$

① $$= \ddot{\vec{r}}_{rot} + 2\vec{\omega}\times\dot{\vec{r}}_{rot} + \cancel{\vec{\omega}\times(\vec{\omega}\times\vec{r}_{rot})} \rightarrow \text{2nd Order in } \vec{\omega}$$

the effective accel in the rotating frame is

② $$\ddot{\vec{r}}_{rot} = -g\hat{z}$$

Hence the velocity in the rotating frame is

$$\int_{v_0\hat{z}}^{\dot{\vec{r}}_{rot}} d\dot{r}_{rot} = \dot{\vec{r}}_{rot} - v_0\hat{z} = \int_0^t -g\hat{z}\,dt = -g\hat{z}t \Rightarrow \dot{\vec{r}}_{rot} = (v_0 - gt)\hat{z}$$

Now

$$|\vec{\omega}\times\dot{\vec{r}}_{rot}| = \omega\dot{r}_{rot}\sin(\lambda-90) = \omega\dot{r}_{rot}(-\cos\lambda)$$

③ $$= -\omega(v_0 - gt)\cos\lambda$$

④ $$\hat{\omega}\times\hat{\dot{r}}_{rot} = \hat{\omega}\times(-\hat{z}) = -\hat{y}$$

③

ⓐ continued

Hence, putting everything together ①,②,③,④

$$\ddot{\vec{r}}_{init} = \ddot{\vec{r}}_{rot} + 2\vec{\omega}\times\dot{\vec{r}}_{rot}$$

$$= -g\hat{z} - 2\omega(v_0 - gt)(\cos\lambda)\hat{y}$$

the effective force on the projectile is then

$$\vec{F}_{eff} = -mg\hat{z} - 2m\omega(\cos\lambda)(v_0 - gt)\hat{y}$$

ⓑ To find out where it lands, we 1st find the time of flight then twice integrate $\ddot{\vec{r}}_{init}$

up+down

$$\int_{v_0}^{0} dv_z = \int_0^t -g\,dt \Rightarrow -v_0 = -gt \Rightarrow t_{flight} = 2\frac{v_0}{g}$$

Integrating $\ddot{\vec{r}}_{init}$ to get $\dot{\vec{r}}_{init}$

$$\int_{v_0}^{v} d\vec{v} = \int_0^t \left[-g\hat{z} - 2\omega(v_0 - gt)(\cos\lambda)\hat{y}\right]dt$$

$$= -gt\hat{z} - 2\omega\left(v_0 t - \frac{1}{2}gt^2\right)(\cos\lambda)\hat{y} = \vec{v} - \vec{v}_0$$

Integrating $\dot{\vec{r}}_{init}$ to get $\vec{r}_{init}$

$$\int_0^{\vec{r}} d\vec{r} = \int_0^{2\frac{v_0}{g}} dt\left[(v_0 - gt)\hat{z} - 2\omega\left(v_0 t - \frac{1}{2}gt^2\right)(\cos\lambda)\hat{y}\right]$$

$$\vec{r} = \left(v_0 t - \frac{1}{2}gt^2\right)\hat{z}\Big|_0^{2\frac{v_0}{g}} - 2\omega\left(v_0\frac{1}{2}t^2 - \frac{1}{6}gt^3\right)\cos\lambda\,\hat{y}\Big|_0^{2\frac{v_0}{g}}$$

$$= 0\hat{z} - 2\omega\left(v_0 2\frac{v_0^2}{g^2} - \frac{4}{3}\frac{v_0^3}{g^2}\right)\cos\lambda\,\hat{y} = \boxed{-\frac{4}{3}\omega\frac{v_0^3}{g^2}(\cos\lambda)\hat{y}}$$

③ ⓑ continued

Since angular momentum is conserved, the projectile's eastward angular velocity must be less than that of the earth's surface (since $r > R_{earth}$). Hence the projectile lands to the west of where it started. Thus the sign of $\vec{r}$ at impact is correct.

(4) (a) As the particle moves along a great-circle, the moment of inertia increases, since it increases its distance $R \sin\theta$ from the center of rotation. Hence by conservation of angular momentum, the angular velocity must slow down. After a time T, the angle of rotation will have been retarded.

(b) Next page

(4) (b) To determine the angle by which the sphere would be retarded, consider 1st the case when $\vec{v}$ of the particle is zero

If $\vec{v}=\vec{0}$ then $I=\frac{2}{5}MR^2$ which leads to

$$\frac{d\omega}{dt}=\frac{d}{dt}\left(\frac{L}{I}\right)=L\frac{d}{dt}\left(\frac{1}{I}\right)$$

Hence

$$\int_0^{\alpha_1}\omega dt=\int_0^{\alpha_1}\frac{d\phi}{dt}dt=\int_0^{\alpha_1}d\phi=\int_0^T\frac{L}{I}dt$$

$$\alpha_1-0=\frac{L}{I}(T-0)=\frac{I\omega}{I}T\Rightarrow\alpha_1=\omega T$$

Now consider the case when $\vec{v}\neq\vec{0}$. Then

$$I=\frac{2}{5}MR^2+m[R\sin\theta]^2$$

Following in the same steps as above

$$\int_0^{\alpha_2}d\phi=\int\frac{L}{I}dt=L\left[\int_0^{\pi/2}\frac{\frac{v}{R}d\theta}{\frac{2}{5}MR^2+m[R\sin\theta]^2}+\right.$$
$$\left.+\int_{\pi/2}^{\pi}\frac{\frac{v}{R}d\theta}{\frac{2}{5}MR^2+m[R\sin\theta]^2}\right]$$

$$\boxed{\begin{array}{l}\text{Define:}\\ a^2=\frac{2}{5}MR^2\\ b^2=mR^2\end{array}}$$

so that

$$\alpha_2=2\frac{v}{R}L\left[\frac{1}{a\sqrt{a^2+b^2}}\tan^{-1}\left(\frac{\sqrt{a^2+b^2}\tan\theta}{a}\right)\right|_0^{\pi/2}=$$

$$=2\frac{v}{R}L\frac{1}{a\sqrt{a^2+b^2}}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]=2\frac{v}{R}L\frac{\pi/2}{a\sqrt{a^2+b^2}}$$

④ Continued

Since angular momentum is conserved, $L = I_{sphere}\,\omega$

Hence (and simplifying a bit)

$$\alpha_2 = \underbrace{\frac{v}{R}\pi}_{T} \frac{I_{sphere}\,\omega}{\sqrt{\frac{2}{5}MR^2}} \frac{1}{\sqrt{\frac{2}{5}MR^2 + mR^2}}$$

$$= \omega T \frac{\frac{2}{5}MR^2}{\sqrt{\frac{2}{5}MR^2}} \frac{1}{\sqrt{\frac{2}{5}MR^2 + mR^2}}$$

$$= \omega T \sqrt{\frac{\frac{2}{5}MR^2}{\frac{2}{5}MR^2 + mR^2}} = \omega T \sqrt{\frac{2M}{2M + 5m}}$$

Thus the angle by which the sphere would be retarded is

$$\Delta\alpha = \alpha_1 - \alpha_2 = \omega T - \omega T \sqrt{\frac{2M}{2M+5m}}$$

$$= \omega T \left(1 - \sqrt{\frac{2M}{2M+5m}}\right)$$

(5) Choose units so that $m = k = d = 1$

	L	M	T
m	0	1	0
k	3	1	-2
d	1	0	0

	m	k	d
L	0	0	1
M	1	0	0
T	1/2	-1/2	3/2

$f = -\frac{1}{r^2}$

$\Rightarrow v = \frac{1}{\sqrt{2}}$

$\overline{OP} = 2 = r$

(a) $f = -\frac{1}{r^2} \Rightarrow V = -\frac{1}{r}$

$$E = T + V = \frac{1}{2}v^2 + V = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{2} = \boxed{-\frac{1}{4}} \qquad (1)$$

(b) The angular momentum points in a direction perpendicular to $\vec{r}$ and $\vec{v}$ with magnitude

$$L = rv\sin\alpha = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\sqrt{\frac{3}{2}}} \qquad (2)$$

(c) To find the max, min distances from the origin Apply conservation of E and L, using (1) & (2)

$$E = T + V = \underbrace{\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2}_{T} \underbrace{- \frac{1}{r}}_{V} = -\frac{1}{4} \qquad (3)$$

$$L = r^2\dot{\theta} = \sqrt{\frac{3}{2}} \qquad (4)$$

Inserting (4) in (3) we get

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\left(\frac{3}{2}\frac{1}{r^4}\right) - \frac{1}{r} = -\frac{1}{4}$$

or

$$\frac{1}{2}\dot{r}^2 + \frac{3}{4}\frac{1}{r^2} - \frac{1}{r} = -\frac{1}{4} \qquad (5)$$

The max, min r correspond to $\dot{r} = 0$, hence

$$\frac{3}{4}\frac{1}{r^2} - \frac{1}{r} = -\frac{1}{4} \Rightarrow r^2 - 4r + 3 = 0$$

$$\Rightarrow (r-3)(r-1) = 0$$

$$\Rightarrow \boxed{r_{min} = 1 \;;\; r_{max} = 3}$$

5

d From 5 in part c

$$\frac{dr}{dt} = \dot{r} = \pm\sqrt{-\frac{3}{2}\frac{1}{r^2} + \frac{2}{r} - \frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{1}{r}\sqrt{-r^2+4r-3}$$

Hence

$$dt = dr \frac{\sqrt{2}\, r}{\sqrt{-r^2+4r-3}}$$

to obtain the period, we integrate between the extremum and multiply by 2 (as we have a half period)

$$\tau = 2\int_1^3 dr \frac{\sqrt{2}\, r}{\sqrt{-r^2+4r-3}}$$

Lets use the hint $x \equiv r-2 \Rightarrow r = x+2$; $dr = dx$

$r^2 = x^2+4x+4$

$-r^2+4r-3 = 1-x$

$$\tau = 2\sqrt{2}\int_{-1}^{1} dx \frac{(x+2)}{\sqrt{1-x^2}}$$

the integral over $\frac{x}{\sqrt{1-x^2}}$ is zero by symmetry

$$\tau = 4\sqrt{2}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = 4\sqrt{2}\,\sin^{-1}x\Big|_{-1}^{1} = 4\sqrt{2}\,\pi$$

Restoring the constants, our answers are

a $E = -\frac{1}{4}\left(\frac{k}{d}\right)$

b $L = \sqrt{\frac{3}{2}}\left(\sqrt{kmd}\right)$

c $r_{min} = d$; $r_{max} = 3d$

d $\tau = 4\sqrt{2}\,\pi\left(\sqrt{\frac{md^3}{k}}\right)$