1.a) $\vec{A} = Bx\hat{y}$ & $V(\vec{r}) = -ax$

3

Then because of the minimal coupling we see that our conjugate momentum is $\vec{\pi} = (\vec{P} - \frac{e}{c}\vec{A}) = \vec{P} - \vec{A}$

Then our Hamiltonian is that of an electric charge in a potential

$$H = \frac{\vec{\pi}^2}{2m} + V(r) = \frac{\vec{\pi}^2}{2m} - \alpha\hat{x} \qquad (\alpha = a)$$

$$= \frac{\hat{P}_x^2}{2m} + \frac{\hat{P}_z^2}{2m} + \frac{(\hat{P}_y - \hat{x}B)^2}{2m} - \alpha\hat{x}$$

Conserved quantities exist for $\partial_t A = \frac{1}{i\hbar}[A, H] \stackrel{?}{=} 0$

and we see that our P_y & $\cancel{P_z}$ momenta do in fact commute with the Hamiltonian and are therefore conserved

only

We can then solve the eigenvalue problem by separating $\Psi(\vec{r})$

$$\Psi = \phi(x)\phi(y) \quad \rightarrow \phi(y) = \frac{1}{\sqrt{2\pi}} e^{i\hbar \vec{k}_\perp \cdot \vec{r}_\perp}$$

for $\vec{k}_\perp = k_y\hat{y}$ vectors
$\vec{r}_\perp = y\hat{y}$

$\therefore H\Psi = E\Psi \Rightarrow$

$$\underbrace{\left(\frac{\hat{P}_x^2}{2m} + \frac{(\hbar k_y - \hat{x}B)^2}{2m} - \alpha\hat{x}\right)}_{\text{Shifted SHO in } X} \phi(x) = E\,\phi(x)$$

$\downarrow$

complete the square

$$\left(\frac{-\hbar^2\vec{\nabla}_x^2}{2m} + \frac{1}{2}m\omega^2(x - \bar{x})^2 + C\right)\phi(x) = E\,\phi(x)$$

$\omega = \frac{eB}{cm} = \boxed{\frac{B}{m} = \omega} \longrightarrow \boxed{X_0 = \frac{\hbar k_y}{m\omega}} = \frac{\hbar k_y}{B}$

$\therefore \frac{x^2 B^2}{2m} = \frac{1}{2} m\omega^2 x^2$ $\qquad \therefore \frac{\hbar^2 k_y^2}{2m} = \frac{1}{2} m\omega^2 X_0^2$ $\qquad \hbar = 1$, $X_0 = \boxed{k_y/B}$

$$\frac{(\hbar k_y - xB)^2}{2m} - \alpha x = \frac{1}{2} m\omega^2 (x - X_0)^2 - \alpha X$$

now we shift $X' = X - X_0 \therefore X = X' + X_0$

$$= \underbrace{\frac{1}{2} m\omega^2 X'^2 - \alpha X'}_{\text{Complete square}} - \alpha X_0$$

$X \to X' \therefore \vec{\nabla}_x \to \vec{\nabla}_{x'}$ freely

$$= \frac{1}{2} m\omega^2 \left(X'^2 - \frac{2\alpha}{m\omega^2} X'\right) - \alpha X_0$$

$$= \frac{1}{2} m\omega^2 \left(X' - \frac{\alpha}{m\omega^2}\right)^2 - \frac{\alpha^2}{m^2\omega^2} - \alpha X_0$$

$$\therefore \boxed{H = -\frac{\hbar^2 \vec{\nabla}_{x'}^2}{2m} + \frac{1}{2} m\omega^2 \left(X' - \frac{\alpha}{m\omega^2}\right)^2 - \frac{\alpha^2}{m^2\omega^2} - \alpha X_0 \quad \text{with } X_0 = k_y/B \ \& \ \omega = B/m}$$

This is simply a shifted SHO $\therefore$ $\boxed{E = \hbar\omega(n + 1/2) - \frac{\alpha^2}{m^2\omega^2} - \alpha X_0}$ $\quad \hbar = 1$

$-\alpha X_0$ part has $= -\frac{\alpha k_y}{B}$ — k_y conserved momentum in it, and k_y also contributes to the $X \to X' - X_0$ shift

If we swapped to $V = -ay$ then neither $\{P_y, H\}$ or $[\cancel{P_x}, H] = 0$ and so neither would be conserved, $\pi_x = p_x$

(this whole time z direction is unconstrained, so technically P_z is

b) $V(\vec{r}) = -\alpha x$, $|\psi(0)\rangle = \int_{-\infty}^{\infty} dk\, \psi(k) |k\rangle_0$

1

where $|k\rangle_0$ is the lowest energy eigenstate of momentum k,

& $\psi(k)$ is a normalized p space wavefunction

$\to$ try $\psi(k) = \frac{1}{\sqrt{2\pi}} e^{iky}$

$U(t,0) = e^{-\frac{iHt}{\hbar}}$

$\omega = B/m$

$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle$ is the time evolution $\quad \hbar = 1$

$$= \int_{-\infty}^{\infty} e^{-it\left(\frac{1}{2}\omega - \frac{\alpha^2}{m^2\omega^2} - \alpha k_y/B\right)} \cdot dk\, \psi(k) |k\rangle_0$$

$n = 0$ in lowest energy eigenstate.

$$\boxed{|\psi(t)\rangle = \underbrace{e^{-it\left(\frac{1}{2}\omega - \frac{\alpha^2}{m^2\omega^2}\right)}}_{\text{phase}} \cdot \int_{-\infty}^{\infty} e^{+it\frac{\alpha k}{B}} dk \frac{e^{ik\cdot y}}{\sqrt{2\pi}} |k\rangle_0}$$

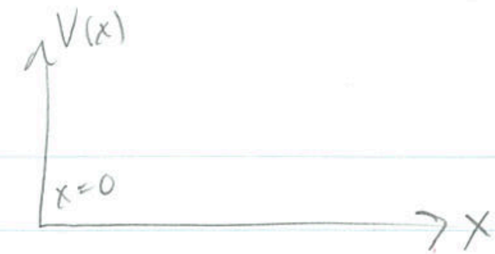
Then $\langle V \rangle = \langle \psi(t) | \frac{-i[y,H]}{m} | \psi(t) \rangle$

$$= \frac{1}{m} \iint_{-\infty}^{\infty} \underbrace{{}_0\langle k' | k | k \rangle_0}\, e^{\frac{it\alpha}{B}(k-k')} e^{iy(k-k')} dk\, dk'$$

$k \langle k' | k \rangle = k\, \delta(k - k')$

$$= \frac{1}{m} \int_{-\infty}^{\infty} e^0 e^0 \cdot k'\, dk' = \boxed{\frac{k'}{m} \cancel{=} \langle V \rangle}$$

$\therefore$ the particle moves at a constant velocity, as we

c) New potential → 

5 //

So our Hamiltonian now includes spin.

$$H = \frac{\Pi^2}{2m} - \vec{\mu}\cdot\vec{B} \quad + \text{ constrained to } x > 0.$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = B\hat{z} \qquad \text{from} \quad \vec{\nabla}\times\vec{A} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\vec{\mu} = \frac{e}{c} g \cdot \vec{S} = g\frac{\hbar}{2}\vec{\sigma}_{2\times 2} = \frac{g}{2}\vec{\sigma}$$

$$\underset{1}{\overset{\parallel}{}} \longrightarrow \hbar = c = e = 1$$

where now $-\frac{g}{2}\vec{\sigma}\cdot B\hat{z} = -\frac{g}{2}B\sigma_z$

$$H = \frac{P_x^2}{2m} + \frac{(P_y - Bx)^2}{2m} - \frac{g}{2}B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

↑ $|+?_z\rangle$ $|-?_z\rangle$ basis

$$H = SHO = \frac{P_x^2}{2m} + \frac{1}{2}m\omega^2(x-\bar{x})^2 - \frac{g}{2}B\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$B^2/m = m\omega^2 \therefore \boxed{\begin{array}{l} \omega = B/\omega \\ \bar{x} = k_y/B \end{array}}$$

→ k_y is conserved momentum

$$E_{n\pm} = \omega(n + \tfrac{1}{2}) \pm \frac{g}{2}B \quad \text{depending on spin state}$$

with $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ & $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and $\Psi_n(x) = H_n(x-\bar{x})\, e^{-\frac{(x-\bar{x})^2}{2}}$

and $\Psi_y(y) = \frac{1}{\sqrt{2\pi}} e^{ik_y \cdot y}$

✱ But our potential puts us in the half harmonic oscillator ∴ only odd n energies are allowed

lowest → $k_y = 0$

Since our 3 particles are fermions the lowest energy is when 2 are in the ground state of the Half SHO and one is in the first excited state with its spin anti-aligned

n=3 ↓

n=1 ↑↓

$$\text{Then} \quad E = E_a + E_b + E_c = E_{1+} + E_{1-} + E_{3-}$$

$$= 2\omega(1 + \tfrac{1}{2}) + \tfrac{g}{2}B - \tfrac{g}{2}B$$

$$+ 2\omega(3 + \tfrac{1}{2}) - \tfrac{g}{2}B$$

$$= 2\omega(10/2) - \tfrac{g}{2}B$$

$$\boxed{E_{gs} = \frac{13}{2}\omega - \frac{g}{2}B}$$

and our particles could be in any of the 6 combinations of a & b & c being in the n=1 ↑ or ↓ or n=3 ↓ states → ↑↓ is singlet but then we need to spatially anti-symmetrize w.r.t. state 3

Spin missing

$$\Psi_{+1/2}(x_1,x_2,x_3) = \frac{1}{\sqrt{6}}\Big(2\Psi_1(x_1)\Psi_1(x_2)\Psi_3(x_3) - \Psi_1(x_1)\Psi_3(x_2)\Psi_1(x_3) - \Psi_3(x_1)\Psi_1(x_2)\Psi_1(x_3)\Big)$$

$$? \frac{1}{\sqrt{2}} \cdot [\chi_+(1)\chi_-(2) - \chi_-(1)\chi_+(2)]\,\chi(3)$$