Essential

CALCULUS

Skills Practice Workbook

with Full Solutions

$$\frac{d}{dx}\tan(5x)$$

$$\int \sqrt{1 - x^2}\, dx$$

Chris McMullen, Ph.D.

Essential Calculus Skills Practice Workbook with Full Solutions

Improve Your Math Fluency

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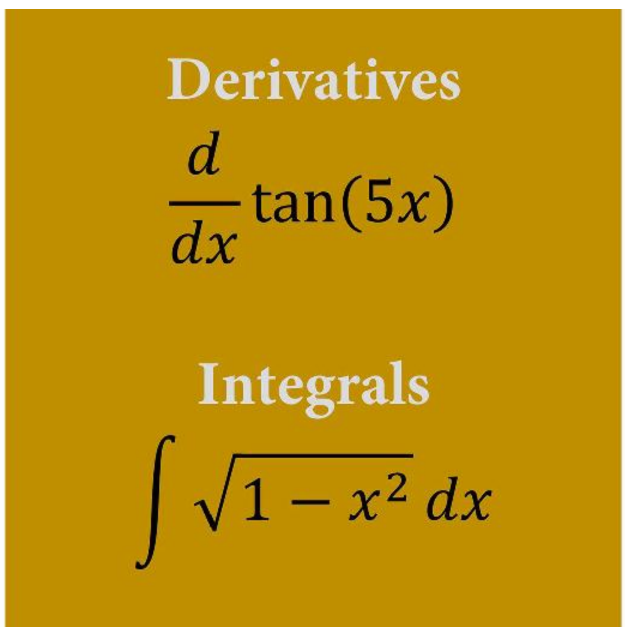

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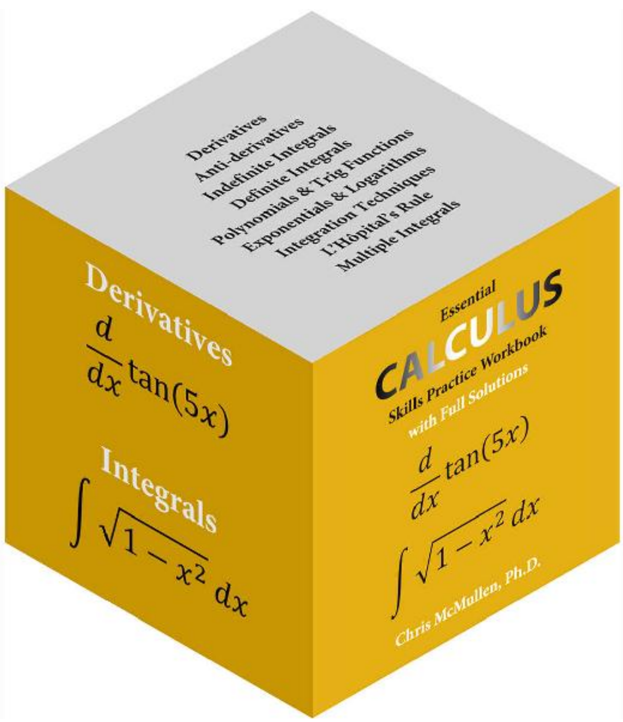
Zishka Publishing

Version 2.07.18A

Textbooks > Math > Calculus
Study Guides > Workbooks> Math
Education > Math > Calculus

Paperback ISBN: 978-1-941691-24-3



Contents

Introduction

This workbook is designed to help practice essential calculus techniques, especially the art of finding derivatives and performing integrals. Each chapter focuses on one main topic such as how to apply the chain rule or how to perform an integral with a trigonometric substitution.

Every chapter begins with a concise explanation of the main concept, followed by a few examples. The examples are fully solved step-by-step with explanations, and should serve as a valuable guide for solving the practice problems. The solution to every practice exercise is tabulated at the back of the book.

A variety of essential calculus skills are covered in this workbook. The first chapter starts out simple with derivatives of polynomials, and the difficulty of the lessons progresses as the book continues. Students will learn:

- how to find derivatives and anti-derivatives of polynomials
- how to find derivatives and anti-derivatives of trigonometric functions
- how to find derivatives and anti-derivatives of logarithms and exponentials
- how to perform definite integrals
- how to perform multiple integrals
- a variety of integration techniques

May you (or your students) find this workbook useful and become more fluent with these essential calculus skills.

1 DERIVATIVES OF POLYNOMIALS

Given a polynomial term of the form **ax^b** (where **a** is a constant coefficient and **b** is a constant exponent), to take a derivative with respect to the variable **x**, first multiply the coefficient **a** by the exponent **b**, and then reduce the exponent by **1** according to the following formula:

$$\frac{d}{dx}(ax^b) = bax^{b-1}$$

Note that a few special exponents can make a polynomial look somewhat different:

- If you see a variable in a denominator, you may bring it to the numerator by negating its exponent according to the following rule:

$$\frac{1}{x^n} = x^{-n}$$

- If you see a squareroot, rewrite it using an exponent of ½:

$$\sqrt{x} = x^{1/2}$$

- If no coefficient or exponent is observed, the number **1** is implied:

$$x = 1x^1$$

$$\frac{1}{x} = \frac{1}{x^1} = x^{-1}$$

It may help to recall the following rules of algebra regarding exponents:

$$x^m x^n = x^{m+n} \quad , \quad \frac{x^m}{x^n} = x^{m-n}$$

$$x^0 = 1 \quad , \quad (x^m)^n = x^{mn}$$

$$(ax)^m = a^m x^m \quad , \quad (a^m a^n)^p = a^{mp} x^{np}$$

$$\sqrt{ax} = (ax)^{1/2} = a^{1/2} x^{1/2}$$

If a polynomial includes multiple terms, find the derivative of each term individually and then add the results together:

$$\frac{d}{dx}(y_1 + y_2 + \cdots + y_n) = \frac{dy_1}{dx} + \frac{dy_2}{dx} + \cdots + \frac{dy_n}{dx}$$

For example, if $\mathbf{y_1 = 4x^2}$, $\mathbf{y_2 = -3x}$, and $\mathbf{y_3 = 6}$, this means:

$$\frac{d}{dx}(4x^2 - 3x + 6) = \frac{d}{dx}(4x^2) + \frac{d}{dx}(-3x) + \frac{d}{dx}(6)$$

Note that the derivative of a constant equals zero:

$$\frac{d}{dx}c = 0$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}(5x^3)$$

Compare $5x^3$ with the general form ax^b to see that the coefficient is $a = 5$ and the exponent is $b = 3$. Plug $a = 5$ and $b = 3$ into the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$.

$$\frac{d}{dx}(5x^3) = (3)(5)x^{3-1} = 15x^2$$

Example: Perform the following derivative with respect to **t**.

$$\frac{d}{dt}(t^4)$$

Note that this derivative is with respect to t (instead of the usual x). Simply replace x with t in the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$ to get the formula $\frac{d}{dt}(at^b) = bat^{b-1}$. Compare t^4 with the general form at^b to see that the coefficient is $a = 1$ (since the number 1 is implied when no coefficient is present: $1t^4 = t^4$) and the exponent is $b = 4$. Plug $a = 1$ and $b = 4$ into the formula $\frac{d}{dt}(at^b) = bat^{b-1}$.

$$\frac{d}{dt}(t^4) = (4)(1)t^{4-1} = 4t^3$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}\left(6x^{2/3}\right)$$

Compare $6x^{2/3}$ with the general form ax^b to see that the coefficient is $a = 6$ and the exponent is $b = 2/3$. Plug $a = 6$ and $b = 2/3$ into the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$.

$$\frac{d}{dx}\left(6x^{2/3}\right) = \left(\frac{2}{3}\right)(6)x^{2/3-1} = 4x^{-1/3}$$

Note that $\left(\frac{2}{3}\right)(6) = \frac{2}{3}\frac{6}{1} = \frac{12}{3} = 4$ and that $\frac{2}{3} - 1 = \frac{2}{3} - \frac{3}{3} = \frac{2-3}{3} = -\frac{1}{3}$ (recall that the way to subtract fractions is to make a common denominator).

Since $x^{-1/3} = \frac{1}{x^{1/3}}$, the answer may alternatively be expressed as:

$$4x^{-1/3} = \frac{4}{x^{1/3}}$$

Example: Perform the following derivative with respect to **u**.

$$\frac{d}{du}\left(\frac{7}{u}\right)$$

Note that this derivative is with respect to u (instead of the usual x). Simply replace x with u in the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$ to get the formula $\frac{d}{du}(au^b) = bau^{b-1}$. Rewrite $\frac{7}{u}$ as $7u^{-1}$ using the fact that $u^{-1} = \frac{1}{u}$. Compare $7u^{-1}$ with the general form au^b to see that the coefficient is $a = 7$ and the exponent is $b = -1$. Plug $a = 7$ and $b = -1$ into the formula $\frac{d}{du}(au^b) = bau^{b-1}$.

$$\frac{d}{du}\left(\frac{7}{u}\right) = \frac{d}{du}(7u^{-1}) = (-1)(7)u^{-1-1} = -7u^{-2}$$

$$= -\frac{7}{u^2}$$

Note that $(-1)(7) = -7$, that $-1 - 1 = -2$, and that $u^{-2} = \frac{1}{u^2}$.

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}\left(\sqrt{3x}\right)$$

Rewrite $\sqrt{3x}$ as $(3x)^{1/2}$ using the fact that $\sqrt{y} = y^{1/2}$. Then rewrite $(3x)^{1/2}$ as $3^{1/2}x^{1/2}$ using the rule $(cx)^n = c^n x^n$. Compare $3^{1/2}x^{1/2}$ with the general form ax^b to see that the coefficient is $a = 3^{1/2}$ and the exponent is $b = 1/2$. Plug $a = 3^{1/2}$ and $b = 1/2$ into the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$.

$$\frac{d}{dx}\left(\sqrt{3x}\right) = \frac{d}{dx}\left(3^{1/2}x^{1/2}\right) = \left(\frac{1}{2}\right)\left(3^{1/2}\right)x^{1/2-1}$$

$$= \frac{3^{1/2}}{2}x^{-1/2}$$

Note that $\left(\frac{1}{2}\right)\left(3^{1/2}\right) = \frac{3^{1/2}}{2}$ and that $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (recall that the way to subtract fractions is to make a common denominator). Since $x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$ and $3^{1/2} = \sqrt{3}$, the answer may alternatively be expressed as:

$$\frac{3^{1/2}}{2}x^{-1/2} = \frac{\sqrt{3}}{2\sqrt{x}} = \frac{\sqrt{3}}{2\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{3}\sqrt{x}}{2x} = \frac{\sqrt{3x}}{2x}$$

We multiplied by $\frac{\sqrt{x}}{\sqrt{x}}$ in order to rationalize the denominator. Note that $\sqrt{x}\sqrt{x} = \sqrt{x^2} = x$ and that $\sqrt{3}\sqrt{x} = \sqrt{3x}$ according to the rule $\sqrt{p}\sqrt{x} = \sqrt{px}$.

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}(6x^3 - 12x)$$

Since this expression has two terms ($6x^3$ and $-12x$), we may find the derivative of each term individually and then add the results together according to the following formula, where $y_1 = 6x^3$ and $y_2 = -12x$:

$$\frac{d}{dx}(y_1 + y_2) = \frac{dy_1}{dx} + \frac{dy_2}{dx}$$

Find the two individual derivatives:

- Compare $6x^3$ with the general form ax^b to see that the coefficient is $a = 6$ and the exponent is $b = 3$. Plug $a = 6$ and $b = 3$ into the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$.

$$\frac{dy_1}{dx} = \frac{d}{dx}(6x^3) = (3)(6)x^{3-1} = 18x^2$$

- Compare $-12x$ with the general form ax^b to see that the coefficient is $a = -12$ and the exponent is $b = 1$ (recall that an exponent of 1 is implied when no exponent is present: $x^1 = x$). Plug $a = -12$ and $b = 1$ into the formula $\frac{d}{dx}(ax^b) = bax^{b-1}$. Recall from algebra that $x^0 = 1$.

$$\frac{dy_2}{dx} = \frac{d}{dx}(-12x) = (1)(-12)x^{1-1} = -12x^0$$
$$= -12$$

Now substitute $\frac{dy_1}{dx} = \frac{d}{dx}(6x^3) = 18x^2$ and $\frac{dy_2}{dx} = \frac{d}{dx}(-12x) = -12$ into the equation $\frac{d}{dx}(y_1 + y_2) = \frac{dy_1}{dx} + \frac{dy_2}{dx}$:

$$\frac{d}{dx}(y_1 + y_2) = \frac{dy_1}{dx} + \frac{dy_2}{dx} = 18x^2 + (-12)$$
$$= 18x^2 - 12$$

Our final answer is:

$$\frac{d}{dx}(6x^3 - 12x) = 18x^2 - 12$$

Chapter 1 Exercises

Directions: Perform each derivative with respect to the indicated variable.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $\frac{d}{dx}(8x^4) =$

❷ $\frac{d}{dx}(5x^{-2}) =$

❸ $\frac{d}{dt}\left(\frac{1}{t}\right) =$

❹ $\frac{d}{dx}\left(8x^{7/4}\right) =$

❺ $\frac{d}{dx}\left(\frac{x^{3/5}}{6}\right) =$

❻ $\frac{d}{du}(u) =$

❼ $\frac{d}{dt}(\sqrt{2t}) =$

❽ $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) =$

Part B

❾ $\dfrac{d}{dx}(5x^3 + 4x^2 - 3x + 2) =$

❿ $\dfrac{d}{du}(1 - u) =$

⓫ $\dfrac{d}{dx}\left(4x^{3/2} + 12x^{1/2}\right) =$

⓬ $\dfrac{d}{dt}\left(\sqrt{t} - \dfrac{1}{t}\right) =$

Chapter 1 Solutions

Part A

❶ $\frac{d}{dx}(8x^4) = (4)(8)x^{4-1} = \boxed{32x^3}$

❷ $\frac{d}{dx}(5x^{-2}) = (-2)(5)x^{-2-1} = -10x^{-3}$

$= \boxed{-\frac{10}{x^3}}$

Note: Both $-10x^{-3}$ and $-\frac{10}{x^3}$ are correct answers.

❸ $\frac{d}{dt}\left(\frac{1}{t}\right) = \frac{d}{dt}(t^{-1}) = \frac{d}{dt}(1t^{-1}) = (-1)(1)t^{-1-1}$

$= -t^{-2} = \boxed{-\frac{1}{t^2}}$

Note: Both $-t^{-2}$ and $-\frac{1}{t^2}$ are correct answers.

❹ $\frac{d}{dx}\left(8x^{7/4}\right) = \left(\frac{7}{4}\right)(8)x^{7/4-1} = \frac{56}{4}x^{3/4}$

$= \boxed{14x^{3/4}}$

Note: $\frac{7}{4} - 1 = \frac{7}{4} - \frac{4}{4} = \frac{7-4}{4} = \frac{3}{4}$ (subtract fractions with a common denominator).

❺ $\frac{d}{dx}\left(\frac{x^{3/5}}{6}\right) = \left(\frac{3}{5}\right)\left(\frac{1}{6}\right)x^{3/5-1} = \frac{3}{30}x^{-2/5}$

$= \frac{x^{-2/5}}{10} = \boxed{\frac{1}{10x^{2/5}}}$

Notes:

- $\frac{3}{5} - 1 = \frac{3}{5} - \frac{5}{5} = \frac{3-5}{5} = \frac{-2}{5}$ (subtract fractions with a common denominator).
- Both $\frac{x^{-2/5}}{10}$ and $\frac{1}{10x^{2/5}}$ are correct answers.

❻ $\frac{d}{du}(u) = \frac{d}{du}(1u^1) = (1)(1)u^{1-1} = 1u^0 = \boxed{1}$

Notes: $1u^1 = u$ and $u^0 = 1$.

❼ $$\frac{d}{dt}\left(\sqrt{2t}\right) = \frac{d}{dt}(2t)^{1/2} = \frac{d}{dt}\left(2^{1/2}t^{1/2}\right)$$

$$= \left(\frac{1}{2}\right)\left(2^{1/2}\right)t^{1/2-1} = \frac{2^{1/2}}{2}t^{-1/2}$$

$$= 2^{1/2-1}t^{-1/2} = 2^{-1/2}t^{-1/2} = \frac{1}{2^{1/2}t^{1/2}} = \frac{1}{(2t)^{1/2}}$$

$$= \frac{1}{\sqrt{2t}} = \frac{1}{\sqrt{2t}}\frac{\sqrt{2t}}{\sqrt{2t}} = \boxed{\frac{\sqrt{2t}}{2t}}$$

Notes:

- $(2t)^{1/2} = 2^{1/2}t^{1/2}$ because $(cx)^n = c^n x^n$.
- $\left(\frac{1}{2}\right)\left(2^{1/2}\right) = \frac{2^{1/2}}{2} = 2^{1/2}2^{-1} = 2^{1/2-1} = 2^{-1/2}$ because $x^m x^n = x^{m+n}$.
- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).

- $2^{-1/2}t^{-1/2}$, $\frac{1}{2^{1/2}t^{1/2}}$, $\frac{1}{\sqrt{2t}}$, and $\frac{\sqrt{2t}}{2t}$ are all correct answers. However, only the answer $\frac{\sqrt{2t}}{2t}$ has a rational denominator. We multiplied $\frac{1}{\sqrt{2t}}$ by $\frac{\sqrt{2t}}{\sqrt{2t}}$ in order to rationalize the denominator. Note that $\sqrt{2t}\sqrt{2t} = 2t$ because $\sqrt{x}\sqrt{x} = x$.

❽ $$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{1}{x^{1/2}}\right) = \frac{d}{dx}\left(x^{-1/2}\right)$$

$$= \frac{d}{dx}\left(1x^{-1/2}\right) = \left(-\frac{1}{2}\right)(1)x^{-1/2-1}$$

$$= \left(-\frac{1}{2}\right)(1)x^{-3/2} = -\frac{x^{-3/2}}{2} = -\frac{1}{2x^{3/2}} = -\frac{1}{2x\sqrt{x}}$$

$$= -\frac{1}{2x\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \boxed{-\frac{\sqrt{x}}{2x^2}}$$

Notes:

- $\sqrt{x} = x^{1/2}$
- $\frac{1}{x^{1/2}} = x^{-1/2}$ because $x^{-n} = \frac{1}{x^n}$.
- $x^{3/2} = x^1 x^{1/2} = x\sqrt{x}$ because $x^{m+n} = x^m x^n$.
- $\sqrt{x}\sqrt{x} = x$.
- $-\frac{x^{-3/2}}{2}$, $-\frac{1}{2x^{3/2}}$, $-\frac{1}{2x\sqrt{x}}$, and $-\frac{\sqrt{x}}{2x^2}$ are all correct answers. However, only the answer $-\frac{\sqrt{x}}{2x^2}$ has a rational denominator. We multiplied $-\frac{1}{2x\sqrt{x}}$ by $\frac{\sqrt{x}}{\sqrt{x}}$ in order to rationalize the denominator.

Part B

❾ $\frac{d}{dx}(5x^3 + 4x^2 - 3x + 2)$

$$= \frac{d}{dx}(5x^3) + \frac{d}{dx}(4x^2) - \frac{d}{dx}(3x) + \frac{d}{dx}(2)$$

$$= (3)(5)x^{3-1} + (2)(4)x^{2-1} - (1)(3)x^{1-1} + 0$$

$$= \boxed{15x^2 + 8x - 3}$$

Notes:

- $\frac{d}{dx}(3x) = \frac{d}{dx}(3x^1) = (1)(3)x^{1-1} = 3x^0 = 3$ because $x^1 = x$ and $x^0 = 1$.
- $\frac{d}{dx}(2) = 0$ because the derivative of a constant is zero.

❿ $\frac{d}{du}(1 - u) = \frac{d}{du}(1) - \frac{d}{du}(u) = 0 - 1 = \boxed{-1}$

Notes:

- $\frac{d}{dx}(1) = 0$ because the derivative of a constant is zero.
- $\frac{d}{du}(u) = \frac{d}{du}(1u^1) = (1)(1)u^{1-1} = 1u^0 = 1$ because $1u^1 = u$ and $u^0 = 1$.

⓫ $\frac{d}{dx}\left(4x^{3/2} + 12x^{1/2}\right)$

$$= \left(\frac{3}{2}\right)(4)x^{3/2-1} + \left(\frac{1}{2}\right)(12)x^{1/2-1}$$

$$= \frac{12}{2}x^{1/2} + \frac{12}{2}x^{-1/2}$$

$$= 6x^{1/2} + 6x^{-1/2} = 6x^{1/2} + \frac{6}{x^{1/2}} = 6\sqrt{x} + \frac{6}{\sqrt{x}}$$

$$= 6\sqrt{x} + \frac{6}{\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \boxed{6\sqrt{x} + \frac{6\sqrt{x}}{x}}$$

Notes:

- $\frac{3}{2} - 1 = \frac{3}{2} - \frac{2}{2} = \frac{3-2}{2} = \frac{1}{2}$ (subtract fractions with a common denominator).
- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $x^{-1/2} = \frac{1}{x^{1/2}}$ because $x^{-n} = \frac{1}{x^n}$.
- $x^{1/2} = \sqrt{x}$.

- $6x^{1/2} + 6x^{-1/2}$, $6x^{1/2} + \frac{6}{x^{1/2}}$, $6\sqrt{x} + \frac{6}{\sqrt{x}}$, and $6\sqrt{x} + \frac{6\sqrt{x}}{x}$ are all correct answers. However, only the answer $6\sqrt{x} + \frac{6\sqrt{x}}{x}$ has a rational denominator. We multiplied $\frac{6}{\sqrt{x}}$ by $\frac{\sqrt{x}}{\sqrt{x}}$ in order to rationalize the denominator.

⓬ $$\frac{d}{dt}\left(\sqrt{t} - \frac{1}{t}\right) = \frac{d}{dt}\left(t^{1/2} - t^{-1}\right)$$

$$= \frac{d}{dt}\left(1t^{1/2} - 1t^{-1}\right)$$

$$= \left(\frac{1}{2}\right)(1)t^{1/2-1} - (-1)(1)t^{-1-1} = \frac{t^{-1/2}}{2} - \left(-t^{-2}\right)$$

$$= \frac{t^{-1/2}}{2} + t^{-2} = \frac{1}{2t^{1/2}} + t^{-2}$$

$$= \frac{1}{2\sqrt{t}} + \frac{1}{t^2} = \frac{1}{2\sqrt{t}}\frac{\sqrt{t}}{\sqrt{t}} + \frac{1}{t^2} = \boxed{\frac{\sqrt{t}}{2t} + \frac{1}{t^2}}$$

Notes:

- $\sqrt{t} = t^{1/2}$ and $\frac{1}{t} = t^{-1}$.
- $-(-t^{-2}) = t^{-2}$ (two negatives make a positive).
- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $-1 - 1 = -2$.

- $\frac{t^{-1/2}}{2} + t^{-2}$, $\frac{1}{2t^{1/2}} + t^{-2}$, $\frac{1}{2\sqrt{t}} + \frac{1}{t^2}$, and $\frac{\sqrt{t}}{2t} + \frac{1}{t^2}$ are all correct answers. However, only the answer $\frac{\sqrt{t}}{2t} + \frac{1}{t^2}$ has a rational denominator. We multiplied $\frac{1}{2\sqrt{t}}$ by $\frac{\sqrt{t}}{\sqrt{t}}$ in order to rationalize the denominator.

2 THE CHAIN RULE, PRODUCT RULE, AND QUOTIENT RULE

If **f(u)** is a function of one variable **u**, and if **u(x)** is itself a function of a second variable **x**, then the chain rule may be applied in order to find a derivative of the function **f(u(x))** with respect to the second variable **x**:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx}$$

Note the following:

- **f(u)** is called the outside function.
- **u(x)** is called the inside function.
- The chain rule basically states that **df/dx** equals the product of the derivatives of the outside function (**df/du**) and the inside function (**du/dx**).
- The chain rule is often helpful when taking derivatives of functions like these:

$$\frac{d}{dx}(3x-2)^{20} \quad , \quad \frac{d}{dx}\sqrt{2x^2+1} \quad , \quad \frac{d}{dx}\frac{1}{x^2-4x+5} \quad , \quad \frac{d}{dx}\sin(4x^3+3)$$

The product rule allows you to take a derivative of two different functions, **f(x)** and **g(x)**, of the same argument (**x**), when the functions are multiplied together:

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

The quotient rule allows you to take a derivative of two different functions, **f(x)** and **g(x)**, of the same argument (**x**), when the functions are being divided:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

The product rule is useful when taking derivatives like these:

$$\frac{d}{dx}(2x+3)^8(x^2-1)^5 \quad , \quad \frac{d}{dx}(5x^2-2)\sqrt{8x+3} \quad , \quad \frac{d}{dx}x^3\sin(2x)$$

The quotient rule is useful when taking derivatives like these:

$$\frac{d}{dx}\frac{x^2+4}{x-3} \quad , \quad \frac{d}{dx}\frac{\sqrt{x}}{3x-4} \quad , \quad \frac{d}{dx}\frac{\tan(4x)}{x}$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}(2x^2 - 5)^5$$

It would be tedious to multiply $(2x^2 - 5)$ by itself multiple times. The chain rule lets you avoid multiplying $(2x^2 - 5)^5$ out. We will make the following definition in order to apply the chain rule:

$$u = 2x^2 - 5$$

Now the problem looks like this:

$$f(u) = u^5 \quad , \quad u(x) = 2x^2 - 5 \quad , \quad \frac{df}{dx} = ?$$

Apply the chain rule:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^5)\right]\left[\frac{d}{dx}(2x^2 - 5)\right]$$

$$= (5u^4)(4x) = 20xu^4$$

Substitute $u = 2x^2 - 5$ into the previous expression to complete the solution:

$$\frac{df}{dx} = 20x(2x^2 - 5)^4$$

Example: Perform the following derivative with respect to **t**.

$$\frac{d}{dt}\left(\sqrt{t^3-8}\right)$$

We will make the following definition in order to apply the chain rule:

$$u = t^3 - 8$$

Now the problem looks like this:

$$f(u) = \sqrt{u} = u^{1/2} \quad , \quad u(t) = t^3 - 8 \quad , \quad \frac{df}{dt} = ?$$

Apply the chain rule:

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = \left[\frac{d}{du}\left(u^{1/2}\right)\right]\left[\frac{d}{dt}(t^3-8)\right]$$

$$= \left(\frac{1}{2}u^{-1/2}\right)(3t^2) = \frac{3t^2u^{-1/2}}{2} = \frac{3t^2}{2u^{1/2}}$$

$$\frac{df}{dt} = \frac{3t^2}{2\sqrt{u}}$$

Recall that $z^{-n} = \frac{1}{z^n}$ such that $u^{-1/2} = \frac{1}{u^{1/2}}$. Also recall that $u^{1/2} = \sqrt{u}$. Substitute $u = t^3 - 8$ into the previous expression to complete the solution:

$$\frac{df}{dt} = \frac{3t^2}{2\sqrt{u}} = \frac{3t^2}{2\sqrt{t^3-8}}$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}(3x^4 - 7)(2x^3 - 6)$$

We will make the following definitions in order to apply the product rule:

$$f(x) = 3x^4 - 7 \quad , \quad g(x) = 2x^3 - 6$$

Now the problem looks like this:

$$f(x) = 3x^4 - 7 \quad , \quad g(x) = 2x^3 - 6$$

$$\frac{d}{dx}(fg) = ?$$

Apply the product rule:

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= (2x^3 - 6)\left[\frac{d}{dx}(3x^4 - 7)\right]$$

$$+ (3x^4 - 7)\left[\frac{d}{dx}(2x^3 - 6)\right]$$

$$= (2x^3 - 6)(12x^3) + (3x^4 - 7)(6x^2)$$

$$= 24x^6 - 72x^3 + 18x^6 - 42x^2$$

$$= 42x^6 - 72x^3 - 42x^2$$

Alternatively, this example could be solved without applying the product rule, by multiplying $(3x^4 - 7)(2x^3 - 6)$ before taking the derivative:

$$\frac{d}{dx}(3x^4 - 7)(2x^3 - 6)$$

$$= \frac{d}{dx}(6x^7 - 18x^4 - 14x^3 + 42)$$

$$= 42x^6 - 72x^3 - 42x^2$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}\frac{x^4 + 3x^2 - 6}{x^2 + 1}$$

We will make the following definitions in order to apply the quotient rule:

$$f(x) = x^4 + 3x^2 - 6 \quad , \quad g(x) = x^2 + 1$$

Now the problem looks like this:

$$f(x) = x^4 + 3x^2 - 6 \quad , \quad g(x) = x^2 + 1$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

Apply the quotient rule:

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{(x^2 + 1)\left[\frac{d}{dx}(x^4 + 3x^2 - 6)\right] - (x^4 + 3x^2 - 6)\left[\frac{d}{dx}(x^2 + 1)\right]}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(4x^3 + 6x) - (x^4 + 3x^2 - 6)(2x)}{x^4 + 2x^2 + 1}$$

$$= \frac{4x^5 + 6x^3 + 4x^3 + 6x - 2x^5 - 6x^3 + 12x}{x^4 + 2x^2 + 1}$$

$$= \frac{2x^5 + 4x^3 + 18x}{x^4 + 2x^2 + 1}$$

Chapter 2 Exercises

Directions: Perform each derivative with respect to the indicated variable.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $\frac{d}{dx}(x^3 - 3x^2 + 4x - 5)^8 =$

❷ $\frac{d}{dt}\frac{1}{\sqrt{5t^2 - 3t + 6}} =$

❸ $\frac{d}{dx}(4x^2 - 6)\sqrt{x} =$

❹ $\frac{d}{dx}\frac{3 - 2x^2}{4 - 3x^2} =$

Part B

❺ $$\frac{d}{dx}\left(\frac{1}{x^3-4x}\right)=$$

❻ $$\frac{d}{dx}(4+x)^9(2-x)^5=$$

❼ $$\frac{d}{dt}\left(\sqrt{t^2+9}\right)=$$

❽ $$\frac{d}{dx}\frac{x^4}{\sqrt{x^2+4}}=$$

Part C

❾ $\frac{d}{dx}\left(2x^{5/2}-8x^{3/2}\right)^6=$

❿ $\frac{d}{dx}\frac{x^2+3x-4}{2x+5}=$

⓫ $\frac{d}{dx}\sqrt{2+\sqrt{x}}=$

⓬ $\frac{d}{dt}(4t^2-9)(t^4+8t^2-3)^9=$

Chapter 2 Solutions

Part A

❶ $\frac{d}{dx}(x^3 - 3x^2 + 4x - 5)^8 = ?$

Apply the chain rule: $u = x^3 - 3x^2 + 4x - 5$

$$f = u^8 \quad , \qquad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^8)\right]\left[\frac{d}{dx}(x^3 - 3x^2 + 4x - 5)\right]$$

$$= (8u^7)(3x^2 - 6x + 4)$$

$$= \boxed{8(x^3 - 3x^2 + 4x - 5)^7(3x^2 - 6x + 4)}$$

❷ $\dfrac{d}{dt}\dfrac{1}{\sqrt{5t^2-3t+6}} = ?$

Apply the chain rule: $u = 5t^2 - 3t + 6$

$$f = \frac{1}{\sqrt{u}} = \frac{1}{u^{1/2}} = u^{-1/2} \quad , \qquad \frac{df}{dt} = ?$$

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = \left[\frac{d}{du}\left(u^{-1/2}\right)\right]\left[\frac{d}{dt}\left(5t^2 - 3t + 6\right)\right]$$

$$= \left(-\frac{1}{2}u^{-3/2}\right)(10t - 3)$$

$$= -\frac{10t-3}{2u^{3/2}} = \boxed{-\frac{10t-3}{2(5t^2-3t+6)^{3/2}}}$$

Notes: $-\frac{1}{2} - 1 = -\frac{1}{2} - \frac{2}{2} = \frac{-1-2}{2} = -\frac{3}{2}$ and $u^{-3/2} = \frac{1}{u^{3/2}}$.

❸ $\frac{d}{dx}(4x^2-6)\sqrt{x} = ?$

Apply the product rule: $f(x) = 4x^2 - 6$

$$g(x) = \sqrt{x} = x^{\frac{1}{2}} \quad , \quad \frac{d}{dx}(fg) = ?$$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= x^{1/2}\left[\frac{d}{dx}(4x^2-6)\right] + (4x^2-6)\left(\frac{d}{dx}x^{1/2}\right)$$

$$= x^{1/2}(8x) + (4x^2-6)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= 8x^{1/2}x + 2x^2x^{-1/2} - 3x^{-1/2}$$

$$= 8x^{3/2} + 2x^{3/2} - 3x^{-1/2} = 10x^{3/2} - 3x^{-1/2}$$

$$= 10x^{3/2} - \frac{3}{x^{1/2}} = 10x^{3/2} - \frac{3}{\sqrt{x}}$$

$$= 10x^{3/2} - \frac{3\sqrt{x}}{x} = \boxed{10x\sqrt{x} - \frac{3\sqrt{x}}{x}}$$

Notes:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $x^{1/2}x = x^{1/2}x^1 = x^{3/2}$ because $x^1 = x$ and $x^m x^n = x^{m+n}$.
- $x^2 x^{-1/2} = x^{3/2}$ because $x^p x^{-q} = x^{p-q}$.
- Going from $(4x^2 - 6)\left(\frac{1}{2}x^{-1/2}\right)$ to $2x^2x^{-1/2} - 3x^{-1/2}$, distribute according to $(r + s)t = rt + st$ with $r = 4x^2$, $s = -6$, and $t = \frac{1}{2}x^{-1/2}$.
- $10x^{3/2} - 3x^{-1/2}$, $10x^{3/2} - \frac{3}{x^{1/2}}$, $10x^{3/2} - \frac{3}{\sqrt{x}}$, $10x^{3/2} - \frac{3\sqrt{x}}{x}$, and $10x\sqrt{x} - \frac{3\sqrt{x}}{x}$ are all correct answers. We multiplied $\frac{3}{\sqrt{x}}$ by $\frac{\sqrt{x}}{\sqrt{x}}$ in order to rationalize the denominator. In the last step, we used $x^{3/2} = x^1x^{1/2} = xx^{1/2} = x\sqrt{x}$.

❹ $\frac{d}{dx}\frac{3-2x^2}{4-3x^2} = ?$

Apply the quotient rule: $f(x) = 3 - 2x^2$

$$g(x) = 4 - 3x^2 \quad , \quad \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{(4-3x^2)\left[\frac{d}{dx}(3-2x^2)\right] - (3-2x^2)\left[\frac{d}{dx}(4-3x^2)\right]}{(4-3x^2)^2}$$

$$= \frac{(4-3x^2)(-4x) - (3-2x^2)(-6x)}{16 - 12x^2 - 12x^2 + 9x^4}$$

$$= \frac{-16x + 12x^3 + 18x - 12x^3}{16 - 24x^2 + 9x^4}$$

$$= \boxed{\frac{2x}{16 - 24x^2 + 9x^4}}$$

Notes: $-(3)(-6x) = 18x$ and $-(-2x^2)(-6x) = -12x^3$.

Part B

❺ $\frac{d}{dx}\left(\frac{1}{x^3 - 4x}\right) = ?$

Apply the chain rule: $\quad u = x^3 - 4x$

$$f = \frac{1}{u} = u^{-1} \quad , \qquad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(u^{-1})\right]\left[\frac{d}{dx}(x^3 - 4x)\right]$$

$$= (-u^{-2})(3x^2 - 4) = -\frac{(3x^2 - 4)}{u^2}$$

$$= \frac{-(3x^2 - 4)}{(x^3 - 4x)^2} = \frac{-3x^2 - (-4)}{(x^3 - 4x)^2} = \boxed{\frac{-3x^2 + 4}{(x^3 - 4x)^2}}$$

Note that $\frac{d}{du}(u^{-1}) = -u^{-2}$ because $-1 - 1 = -2$.

❻ $\dfrac{d}{dx}(4+x)^9(2-x)^5 = ?$

Apply the product rule: $f(x) = (4+x)^9$

$$g(x) = (2-x)^5 \quad , \quad \frac{d}{dx}(fg) = ?$$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= (2-x)^5\left[\frac{d}{dx}(4+x)^9\right] + (4+x)^9\left[\frac{d}{dx}(2-x)^5\right]$$

Now apply the chain rule: $u = 4+x$

$$f = u^9 \quad , \quad v = 2-x \quad , \quad g = v^5$$

$$\frac{d}{dx}(fg) = (2-x)^5\left[\frac{df}{du}\frac{du}{dx}\right] + (4+x)^9\left[\frac{dg}{dv}\frac{dv}{dx}\right]$$

$$= (2-x)^5\left[\frac{d}{du}(u^9)\frac{d}{dx}(4+x)\right]$$

$$+ (4+x)^9\left[\frac{d}{dv}(v^5)\frac{d}{dx}(2-x)\right]$$

$$= (2-x)^5[(9u^8)(1)] + (4+x)^9[(5v^4)(-1)]$$

$$= 9u^8(2-x)^5 - 5v^4(4+x)^9$$

$$= \boxed{9(4+x)^8(2-x)^5 - 5(4+x)^9(2-x)^4}$$

Although $9(4+x)^8(2-x)^5 - 5(4+x)^9(2-x)^4$ is a correct answer, if you want to be fancy, you could factor out $(4+x)^8(2-x)^4$ as follows:

$$9(4+x)^8(2-x)^5 - 5(4+x)^9(2-x)^4$$
$$= (4+x)^8(2-x)^4[9(2-x) - 5(4+x)]$$
$$= (4+x)^8(2-x)^4(18 - 9x - 20 - 5x)$$
$$= (4+x)^8(2-x)^4(-2 - 14x)$$
$$= (4+x)^8(2-x)^4(-1)(2 + 14x)$$
$$= \boxed{-(4+x)^8(2-x)^4(2+14x)}$$

Note:

- $-5(4+x) = -5(4) - 5(x) = -20 - 5x$ (the minus sign gets distributed).
- $(-2 - 14x) = (-1)(2 + 14x) = -(2 + 14x)$, where again a minus sign gets distributed.

❼ $\frac{d}{dt}\left(\sqrt{t^2+9}\right) = ?$

Apply the chain rule: $u = t^2 + 9$

$$f = \sqrt{u} = u^{1/2} \quad , \qquad \frac{df}{dt} = ?$$

$$\frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = \left[\frac{d}{du}\left(u^{1/2}\right)\right]\left[\frac{d}{dt}(t^2+9)\right]$$

$$= \left(\frac{1}{2}u^{-1/2}\right)(2t) = \frac{2t}{2u^{1/2}} = \frac{t}{\sqrt{u}} = \frac{t}{\sqrt{t^2+9}}$$

$$= \frac{t}{\sqrt{t^2+9}}\frac{\sqrt{t^2+9}}{\sqrt{t^2+9}} = \boxed{\frac{t\sqrt{t^2+9}}{t^2+9}}$$

Notes:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $\frac{t}{\sqrt{t^2+9}}$ and $\frac{t\sqrt{t^2+9}}{t^2+9}$ are both correct, but $\frac{t\sqrt{t^2+9}}{t^2+9}$ has a rational denominator.

❽ $\frac{d}{dx}\frac{x^4}{\sqrt{x^2+4}} = ?$

Apply the quotient rule: $f(x) = x^4$

$$g(x) = \sqrt{x^2+4} \quad , \quad \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\,\frac{df}{dx} - f\,\frac{dg}{dx}}{g^2}$$

$$= \frac{\sqrt{x^2+4}\left[\frac{d}{dx}x^4\right] - x^4\left[\frac{d}{dx}\sqrt{x^2+4}\right]}{\left(\sqrt{x^2+4}\right)^2}$$

$$= \frac{\sqrt{x^2+4}(4x^3) - x^4\left(\frac{d}{dx}\sqrt{x^2+4}\right)}{x^2+4}$$

Now apply the chain rule: $\quad u = x^2 + 4$

$$g = \sqrt{u} = u^{1/2}$$

$$\frac{4x^3\sqrt{x^2+4} - x^4\left(\frac{dg}{du}\frac{du}{dx}\right)}{x^2+4}$$

$$= \frac{4x^3\sqrt{x^2+4} - x^4\left[\frac{d}{du}\left(u^{1/2}\right)\frac{d}{dx}\left(x^2+4\right)\right]}{x^2+4}$$

$$= \frac{4x^3\sqrt{x^2+4} - x^4\left[\left(\frac{1}{2}u^{-1/2}\right)(2x)\right]}{x^2+4}$$

$$= \frac{4x^3\sqrt{x^2+4} - x^5u^{-1/2}}{x^2+4}$$

$$= \frac{4x^3\sqrt{x^2+4} - x^5(x^2+4)^{-1/2}}{x^2+4}$$

$$= 4x^3(x^2+4)^{-1/2} - x^5(x^2+4)^{-3/2}$$

$$= \frac{4x^3}{(x^2+4)^{1/2}} - \frac{x^5}{(x^2+4)^{3/2}} = \frac{4x^3(x^2+4) - x^5}{(x^2+4)^{3/2}}$$

$$= \boxed{\frac{3x^5 + 16x^3}{(x^2+4)^{3/2}}}$$

Notes:

- In the second to last line, we distributed: $\frac{A+B}{C} = \frac{A}{C} + \frac{B}{C} = AC^{-1} + BC^{-1}$.
- $\sqrt{x^2+4}(x^2+4)^{-1} = (x^2+4)^{1/2}(x^2+4)^{-1} = (x^2+4)^{-1/2}$ according to $x^m x^n = x^{m+n}$. Similarly, $(x^2+4)^{-1/2}(x^2+4)^{-1} = (x^2+4)^{-3/2}$.
- $\frac{4x^3}{(x^2+4)^{1/2}} = \frac{4x^3}{(x^2+4)^{1/2}}\frac{(x^2+4)}{(x^2+4)} = \frac{4x^3(x^2+4)}{(x^2+4)^{3/2}} = \frac{4x^5+16x^3}{(x^2+4)^{3/2}}$.

Part C

❾ $\dfrac{d}{dx}\left(2x^{5/2}-8x^{3/2}\right)^6=?$

Apply the chain rule: $u=2x^{5/2}-8x^{3/2}$

$$f=u^6\quad,\qquad \frac{df}{dx}=?$$

$$\frac{df}{dx}=\frac{df}{du}\frac{du}{dx}=\left[\frac{d}{du}(u^6)\right]\left[\frac{d}{dx}\left(2x^{5/2}-8x^{3/2}\right)\right]$$

$$=(6u^5)\left[\left(\frac{5}{2}\right)(2)x^{3/2}-\left(\frac{3}{2}\right)(8)x^{1/2}\right]$$

$$=(6u^5)\left(5x^{3/2}-12x^{1/2}\right)$$

$$=\boxed{6\left(2x^{5/2}-8x^{3/2}\right)^5\left(5x^{3/2}-12x^{1/2}\right)}$$

⑩ $\dfrac{d}{dx}\dfrac{x^2+3x-4}{2x+5} = ?$

Apply the quotient rule: $f(x) = x^2+3x-4$

$$g(x) = 2x+5 \quad , \quad \frac{d}{dx}\left(\frac{f}{g}\right) = ?$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2}$$

$$= \frac{(2x+5)\left[\frac{d}{dx}(x^2+3x-4)\right] - (x^2+3x-4)\left[\frac{d}{dx}(2x+5)\right]}{(2x+5)^2}$$

$$= \frac{(2x+5)(2x+3) - (x^2+3x-4)(2)}{4x^2+10x+10x+25}$$

$$= \frac{4x^2+6x+10x+15-2x^2-6x+8}{4x^2+20x+25}$$

$$= \boxed{\frac{2x^2+10x+23}{4x^2+20x+25}}$$

Note: Distribute the minus sign in $-(x^2+3x-4)(2)$ to get $(-x^2-3x+4)(2)$.

⓫ $\dfrac{d}{dx}\sqrt{2+\sqrt{x}} = ?$

Apply the chain rule:

$$u = 2+\sqrt{x} = 2 + x^{1/2} \quad , \quad f = \sqrt{u} = u^{1/2}$$

$$\frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}\left(u^{1/2}\right)\right]\left[\frac{d}{dx}\left(2+x^{1/2}\right)\right]$$

$$= \left(\frac{1}{2}u^{-1/2}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{4}u^{-1/2}x^{-1/2}$$

$$= \frac{1}{4u^{1/2}x^{1/2}} = \frac{1}{4\sqrt{u}\sqrt{x}} = \frac{1}{4\sqrt{x}\sqrt{u}} = \frac{1}{4\sqrt{x}\sqrt{2+\sqrt{x}}}$$

$$= \frac{1}{4\sqrt{x\left(2+\sqrt{x}\right)}} = \frac{1}{4\sqrt{2x+x\sqrt{x}}}$$

$$= \frac{1}{4\sqrt{2x+x\sqrt{x}}}\frac{\sqrt{2x+x\sqrt{x}}}{\sqrt{2x+x\sqrt{x}}} = \frac{\sqrt{2x+x\sqrt{x}}}{4\left(2x+x\sqrt{x}\right)}$$

$$= \frac{\sqrt{2x+x\sqrt{x}}}{4\left(2x+x\sqrt{x}\right)}\frac{\left(2x-x\sqrt{x}\right)}{\left(2x-x\sqrt{x}\right)}$$

$$= \frac{\left(2x - x\sqrt{x}\right)\sqrt{2x + x\sqrt{x}}}{4\left(4x^2 - x^2\sqrt{x}\sqrt{x}\right)}$$

$$= \boxed{\frac{\left(2x - x\sqrt{x}\right)\sqrt{2x + x\sqrt{x}}}{4(4x^2 - x^3)}} \quad \left(\begin{array}{c}\text{any of the last 8 steps} \\ \text{is a correct answer}\end{array}\right)$$

⓬ $\dfrac{d}{dt}(4t^2 - 9)(t^4 + 8t^2 - 3)^9 = ?$

Apply the product rule: $f(t) = 4t^2 - 9$

$$g(t) = (t^4 + 8t^2 - 3)^9 \quad , \quad \frac{d}{dt}(fg) = ?$$

$$\frac{d}{dt}(fg) = g\frac{df}{dt} + f\frac{dg}{dt}$$

$$= (t^4 + 8t^2 - 3)^9\left[\frac{d}{dt}(4t^2 - 9)\right]$$

$$+ (4t^2 - 9)\left[\frac{d}{dt}(t^4 + 8t^2 - 3)^9\right]$$

$$= (t^4 + 8t^2 - 3)^9(8t)$$

$$+ (4t^2 - 9)\left[\frac{d}{dt}(t^4 + 8t^2 - 3)^9\right]$$

Now apply the chain rule: $u = t^4 + 8t^2 - 3$

$$g = u^9$$

$$\frac{d}{dt}(fg) = (8t)(t^4 + 8t^2 - 3)^9 + (4t^2 - 9)\left[\frac{dg}{du}\frac{du}{dt}\right]$$

$$= (8t)(t^4 + 8t^2 - 3)^9$$

$$+ (4t^2 - 9)\left[\frac{d}{du}(u^9)\frac{d}{dt}(t^4 + 8t^2 - 3)\right]$$

$$= (8t)(t^4 + 8t^2 - 3)^9$$

$$+ (4t^2 - 9)[(9u^8)(4t^3 + 16t)]$$

$$= (8t)(t^4 + 8t^2 - 3)^9 + 9u^8(4t^3 + 16t)(4t^2 - 9)$$

$$= \boxed{(8t)(t^4 + 8t^2 - 3)^9 + 9(t^4 + 8t^2 - 3)^8(4t^3 + 16t)(4t^2 - 9)}$$

Although $(8t)(t^4 + 8t^2 - 3)^9 + 9(t^4 + 8t^2 - 3)^8(4t^3 + 16t)(4t^2 - 9)$ is a correct answer, if you want to be fancy, you could factor out $(t^4 + 8t^2 - 3)^8$ as follows:

$$
\begin{aligned}
&(8t)(t^4+8t^2-3)^9 \\
&\quad +9(t^4+8t^2-3)^8(4t^3+16t)(4t^2-9) \\
&= (t^4+8t^2-3)^8[(8t)(t^4+8t^2-3) \\
&\quad +9(4t^3+16t)(4t^2-9)] \\
&= (t^4+8t^2-3)^8[8t^5+64t^3-24t \\
&\quad +9(16t^5-36t^3+64t^3-144t)] \\
&= (t^4+8t^2-3)^8[8t^5+64t^3-24t \\
&\quad +9(16t^5+28t^3-144t)] \\
&= (t^4+8t^2-3)^8(8t^5+64t^3-24t+144t^5 \\
&\quad +252t^3-1296t) \\
&= \boxed{(t^4+8t^2-3)^8(152t^5+316t^3-1320t)}
\end{aligned}
$$

3 DERIVATIVES OF TRIG FUNCTIONS

The derivatives of the basic trig functions are:

$$\frac{d}{d\theta}\sin\theta = \cos\theta \quad , \quad \frac{d}{d\theta}\cos\theta = -\sin\theta \quad , \quad \frac{d}{d\theta}\tan\theta = \sec^2\theta$$

$$\frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta \quad , \quad \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta \quad , \quad \frac{d}{d\theta}\cot\theta = -\csc^2\theta$$

The derivatives of the inverse trig functions are:

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \text{ where } |x| < 1 \quad , \quad \frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}} \text{ where } |x| < 1$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad , \quad \frac{d}{dx}\cot^{-1}(x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \text{ where } |x| > 1 \quad , \quad \frac{d}{dx}\csc^{-1}(x) = \frac{-1}{|x|\sqrt{x^2-1}} \text{ where } |x| > 1$$

Recall the following trigonometric relations and identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad , \quad \csc\theta = \frac{1}{\sin\theta} \quad , \quad \sec\theta = \frac{1}{\cos\theta} \quad , \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \quad , \quad \tan^2\theta + 1 = \sec^2\theta \quad , \quad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad , \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

Example: Perform the following derivative with respect to theta.

$$\frac{d}{d\theta} 4 \sin 3\theta$$

We will make the following definition in order to apply the chain rule:

$$u = 3\theta$$

Now the problem looks like this:

$$f(u) = 4 \sin u \quad , \quad u(\theta) = 3\theta \quad , \quad \frac{df}{d\theta} = ?$$

Apply the chain rule:

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(4 \sin u)\right]\left[\frac{d}{d\theta}(3\theta)\right]$$

$$= (4 \cos u)(3) = 12 \cos u = 12 \cos 3\theta$$

Example: Perform the following derivative with respect to theta.

$$\frac{d}{d\theta} 2\cos^3\theta$$

We will make the following definition in order to apply the chain rule:

$$u = \cos\theta$$

Now the problem looks like this:

$$f(u) = 2u^3 \quad , \quad u(\theta) = \cos\theta \quad , \quad \frac{df}{d\theta} = ?$$

Apply the chain rule:

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(2u^3)\right]\left[\frac{d}{d\theta}(\cos\theta)\right]$$

$$= (6u^2)(-\sin\theta) = -6u^2\sin\theta$$

$$= -6\cos^2\theta\sin\theta$$

Example: Perform the following derivative with respect to theta.

$$\frac{d}{d\theta}\sec\theta\tan\theta$$

We will make the following definitions in order to apply the product rule:

$$f(\theta) = \sec\theta \quad , \quad g(\theta) = \tan\theta$$

Apply the product rule:

$$\frac{d}{d\theta}(fg) = g\frac{df}{d\theta} + f\frac{dg}{d\theta}$$

$$= \tan\theta\left(\frac{d}{d\theta}\sec\theta\right) + \sec\theta\left(\frac{d}{d\theta}\tan\theta\right)$$

$$= \tan\theta\,(\sec\theta\tan\theta) + \sec\theta\,(\sec^2\theta)$$

$$= \tan^2\theta\sec\theta + \sec^3\theta$$

Example: Perform the following derivative with respect to theta.

$$\frac{d}{d\theta}\frac{1+\sin\theta}{\cos\theta}$$

We will make the following definitions in order to apply the quotient rule:

$$f(\theta) = 1 + \sin\theta \quad , \quad g(\theta) = \cos\theta$$

Apply the quotient rule:

$$\frac{d}{d\theta}\left(\frac{f}{g}\right) = \frac{g\frac{df}{d\theta} - f\frac{dg}{d\theta}}{g^2}$$

$$= \frac{\cos\theta\left[\frac{d}{d\theta}(1+\sin\theta)\right] - (1+\sin\theta)\left(\frac{d}{d\theta}\cos\theta\right)}{\cos^2\theta}$$

$$= \frac{\cos\theta\,(0+\cos\theta) - (1+\sin\theta)(-\sin\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta - (-\sin\theta - \sin^2\theta)}{\cos^2\theta}$$

$$= \frac{\cos^2\theta + \sin\theta + \sin^2\theta}{\cos^2\theta} = \frac{1+\sin\theta}{\cos^2\theta}$$

$$(\text{because } \sin^2\theta + \cos^2\theta = 1)$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}4\tan^{-1}(x^3)$$

We will make the following definition in order to apply the chain rule:

$$u = x^3$$

Now the problem looks like this:

$$f(u) = 4\tan^{-1}u \quad , \quad u(x) = x^3 \quad , \quad \frac{df}{dx} = ?$$

Apply the chain rule:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(4\tan^{-1}u)\right]\left[\frac{d}{dx}(x^3)\right]$$

$$= \left(\frac{4}{1+u^2}\right)(3x^2) = \frac{12x^2}{1+u^2}$$

$$= \frac{12x^2}{1+(x^3)^2} = \frac{12x^2}{1+x^6}$$

Note that $(x^3)^2 = x^{(3)(2)} = x^6$ according to $(x^m)^n = x^{mn}$.

Example: Perform the following derivative with respect to **x** (where **x** isn't equal to **–1** or **+1**).

$$\frac{d}{dx}\sin^{-1}x\cos^{-1}x$$

We will make the following definitions in order to apply the product rule:

$$f(x) = \sin^{-1}x \quad , \quad g(x) = \cos^{-1}x$$

Apply the product rule:

$$\frac{d}{d\theta}(fg) = g\frac{df}{d\theta} + f\frac{dg}{d\theta}$$

$$= \cos^{-1}x\left(\frac{d}{dx}\sin^{-1}x\right)$$

$$+ \sin^{-1}x\left(\frac{d}{dx}\cos^{-1}x\right)$$

$$= \cos^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1}x\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$= \frac{\cos^{-1}x}{\sqrt{1-x^2}} - \frac{\sin^{-1}x}{\sqrt{1-x^2}} = \frac{\cos^{-1}x - \sin^{-1}x}{\sqrt{1-x^2}}$$

Chapter 3 Exercises

Directions: Perform each derivative with respect to the indicated variable.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\frac{d}{d\theta} 7 \tan 5\theta =$$

❷ $$\frac{d}{d\theta} 3 \sin^4 \theta =$$

❸ $$\frac{d}{d\theta} \csc\theta \sec\theta =$$

❹ $$\frac{d}{d\theta} \frac{\sin\theta + \cos\theta}{\sin\theta} =$$

Part B

❺ $\dfrac{d}{d\theta}\cos(\theta^2 - 2\theta + 4) =$

❻ $\dfrac{d}{d\theta} 2\cot\sqrt{\theta} =$

❼ $\dfrac{d}{d\theta}(\theta\sin\theta) =$

❽ $\dfrac{d}{d\theta}\sqrt{1 + \sin\theta} =$

Part C

❾ $\dfrac{d}{dx} 3\cot^{-1}(4x) =$

❿ $\dfrac{d}{dx} x\csc^{-1} x =$

⓫ $\dfrac{d}{dx} (\sin^{-1} x + \cos^{-1} x) =$

⓬ $\dfrac{d}{dx} \dfrac{\sec^{-1} x}{x} =$

Chapter 3 Solutions

Part A

❶ $\dfrac{d}{d\theta} 7\tan 5\theta$

Apply the chain rule:

$u = 5\theta \quad , \quad f = 7\tan u \quad , \quad \dfrac{df}{d\theta} = ?$

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(7\tan u)\right]\left[\frac{d}{d\theta}(5\theta)\right]$$

$$= (7\sec^2 u)(5) = 35\sec^2 u$$

$$= \boxed{35\sec^2(5\theta)}$$

Note: As usual, the constant coefficient simply comes out: $\frac{d}{du}(7\tan u) = 7\frac{d}{du}(\tan u)$.

❷ $\dfrac{d}{d\theta} 3\sin^4\theta$

Apply the chain rule:

$$u = \sin\theta \quad , \quad f = 3u^4 \quad , \quad \frac{df}{d\theta} = ?$$

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(3u^4)\right]\left[\frac{d}{d\theta}(\sin\theta)\right]$$

$$= (12u^3)(\cos\theta) = \boxed{12\sin^3\theta\cos\theta}$$

Note: It is instructive to compare this solution to the previous solution.

❸ $\dfrac{d}{d\theta}\csc\theta\sec\theta$

Apply the product rule:

$$f(\theta) = \csc\theta \quad , \quad g(\theta) = \sec\theta \quad , \quad \frac{d}{d\theta}(fg) = ?$$

$$\frac{d}{d\theta}(fg) = g\frac{df}{d\theta} + f\frac{dg}{d\theta}$$

$$= \sec\theta\left(\frac{d}{d\theta}\csc\theta\right) + \csc\theta\left(\frac{d}{d\theta}\sec\theta\right)$$

$$= \sec\theta\,(-\csc\theta\cot\theta) + \csc\theta\,(\sec\theta\tan\theta)$$

$$= -\sec\theta\csc\theta\cot\theta + \csc\theta\sec\theta\tan\theta$$

$$= \sec\theta\csc\theta\,(-\cot\theta + \tan\theta)$$

$$= \frac{1}{\cos\theta \sin\theta}\left(-\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{-1}{\cos\theta \sin\theta}\frac{\cos\theta}{\sin\theta} + \frac{1}{\cos\theta \sin\theta}\frac{\sin\theta}{\cos\theta}$$

$$= -\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} = \boxed{-\csc^2\theta + \sec^2\theta}$$

Notes:

- $\csc\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and $\cot\theta = \frac{\cos\theta}{\sin\theta}$.
- $\sec\theta \csc\theta\,(-\cot\theta + \tan\theta)$ and $-\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta}$, are also correct answers.

❹ $\frac{d}{d\theta}\frac{\sin\theta+\cos\theta}{\sin\theta}$

Apply the quotient rule:

$f(\theta)=\sin\theta+\cos\theta \quad , \quad g(\theta)$

$$=\sin\theta \quad , \quad \frac{d}{d\theta}\left(\frac{f}{g}\right)=?$$

$$\frac{d}{d\theta}\left(\frac{f}{g}\right)=\frac{g\frac{df}{d\theta}-f\frac{dg}{d\theta}}{g^2}$$

$$=\frac{\sin\theta\left[\frac{d}{d\theta}(\sin\theta+\cos\theta)\right]-(\sin\theta+\cos\theta)\left(\frac{d}{d\theta}\sin\theta\right)}{\sin^2\theta}$$

$$=\frac{\sin\theta\,(\cos\theta-\sin\theta)-(\sin\theta+\cos\theta)(\cos\theta)}{\sin^2\theta}$$

$$=\frac{\sin\theta\cos\theta-\sin^2\theta-\sin\theta\cos\theta-\cos^2\theta}{\sin^2\theta}$$

$$=\frac{-\sin^2\theta-\cos^2\theta}{\sin^2\theta}=-\frac{1}{\sin^2\theta}=\boxed{-\csc^2\theta}$$

Notes:

- $\frac{d}{d\theta}(\sin\theta+\cos\theta)=\frac{d}{d\theta}\sin\theta+\frac{d}{d\theta}\cos\theta=\cos\theta-\sin\theta.$
- $\sin^2\theta+\cos^2\theta=1$ and $\csc\theta=\frac{1}{\sin\theta}.$

Part B

❺ $\dfrac{d}{d\theta}\cos(\theta^2 - 2\theta + 4)$

Apply the chain rule:

$$u = \theta^2 - 2\theta + 4 \quad , \quad f = \cos u \quad , \quad \frac{df}{d\theta} = ?$$

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(\cos u)\right]\left[\frac{d}{d\theta}(\theta^2 - 2\theta + 4)\right]$$

$$= (-\sin u)(2\theta - 2)$$

$$= -(2\theta - 2)\sin(\theta^2 - 2\theta + 4)$$

$$= \boxed{-2(\theta - 1)\sin(\theta^2 - 2\theta + 4)}$$

Note that $2\theta - 2 = 2(\theta - 1)$. We factored out the 2.

❻ $\dfrac{d}{d\theta} 2\cot\sqrt{\theta}$

Apply the chain rule:

$$u = \sqrt{\theta} = \theta^{1/2} \quad , \quad f = 2\cot u \quad , \quad \frac{df}{d\theta} = ?$$

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(2\cot u)\right]\left[\frac{d}{d\theta}\left(\theta^{1/2}\right)\right]$$

$$= (-2\csc^2 u)\left(\frac{1}{2}\theta^{-1/2}\right)$$

$$= -2\csc^2 u\left(\frac{1}{2\theta^{1/2}}\right) = -\frac{2\csc^2 u}{2\theta^{1/2}} = -\frac{\csc^2 u}{\theta^{1/2}}$$

$$= -\frac{\csc^2\sqrt{\theta}}{\sqrt{\theta}} = -\left(\frac{\csc^2\sqrt{\theta}}{\sqrt{\theta}}\right)\frac{\sqrt{\theta}}{\sqrt{\theta}}$$

$$= \boxed{-\frac{\sqrt{\theta}\csc^2\sqrt{\theta}}{\theta}} = -\frac{\sqrt{\theta}}{\theta\sin^2\sqrt{\theta}}$$

Notes:

- $\theta^{-1/2} = \frac{1}{\theta^{1/2}}$ according to $x^{-m} = \frac{1}{x^m}$.
- $\sqrt{\theta}\sqrt{\theta} = \theta$.
- $-\frac{\csc^2\sqrt{\theta}}{\theta^{1/2}}$, $-\frac{\csc^2\sqrt{\theta}}{\sqrt{\theta}}$, and $-\frac{\sqrt{\theta}\csc^2\sqrt{\theta}}{\theta}$ are all correct answers. However, only the answers $-\frac{\sqrt{\theta}\csc^2\sqrt{\theta}}{\theta}$ and $-\frac{\sqrt{\theta}}{\theta\sin^2\sqrt{\theta}}$ have a rational denominator. We multiplied $-\frac{\csc^2\sqrt{\theta}}{\sqrt{\theta}}$ by $\frac{\sqrt{\theta}}{\sqrt{\theta}}$ in order to rationalize the denominator.
- $\csc\theta = \frac{1}{\sin\theta}$.
- Note that $\csc^2\sqrt{\theta}$ doesn't simplify to $\csc\theta$.

❼ $\dfrac{d}{d\theta}(\theta \sin\theta)$

Apply the product rule:

$f(\theta) = \theta \quad , \quad g(\theta) = \sin\theta \quad , \quad \dfrac{d}{d\theta}(fg) = ?$

$$\frac{d}{d\theta}(fg) = g\frac{df}{d\theta} + f\frac{dg}{d\theta}$$

$$= \sin\theta\left(\frac{d}{d\theta}\theta\right) + \theta\left(\frac{d}{d\theta}\sin\theta\right)$$

$$= (\sin\theta)(1) + \theta(\cos\theta)$$

$$= \boxed{\sin\theta + \theta\cos\theta}$$

Note that $\frac{d}{d\theta}\theta = 1$ just as $\frac{d}{dx}x = 1$. In more steps, $\frac{d}{dx}(1x^1) = (1)(1)x^{1-1} = x^0 = 1$.

❽ $\dfrac{d}{d\theta}\sqrt{1+\sin\theta}$

Apply the chain rule:

$$u = 1+\sin\theta \quad , \quad f = \sqrt{u} = u^{1/2} \quad , \quad \frac{df}{d\theta} = ?$$

$$\frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left[\frac{d}{du}\left(u^{1/2}\right)\right]\left[\frac{d}{d\theta}(1+\sin\theta)\right]$$

$$= \left(\frac{1}{2}u^{-1/2}\right)(0+\cos\theta)$$

$$= \left(\frac{1}{2u^{1/2}}\right)\cos\theta = \frac{\cos\theta}{2u^{1/2}} = \frac{\cos\theta}{2\sqrt{u}} = \frac{\cos\theta}{2\sqrt{1+\sin\theta}}$$

$$= \frac{\cos\theta}{2\sqrt{1+\sin\theta}}\frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}} = \boxed{\frac{\cos\theta\sqrt{1+\sin\theta}}{2(1+\sin\theta)}}$$

Part C

❾ $\frac{d}{dx} 3\cot^{-1}(4x)$

Apply the chain rule:

$$u = 4x \quad , \quad f = 3\cot^{-1} u \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(3\cot^{-1} u)\right]\left[\frac{d}{dx}(4x)\right]$$

$$= \left(\frac{-3}{1+u^2}\right)(4) = \frac{-12}{1+u^2}$$

$$= \frac{-12}{1+(4x)^2} = \boxed{\frac{-12}{1+16x^2}}$$

Note that $(4x)^2 = 4^2x^2 = 16x^2$ according to $(ax)^n = a^n x^n$.

⑩ $\frac{d}{dx} x \csc^{-1} x$

Apply the product rule:

$f(x) = x \quad , \quad g(x) = \csc^{-1} x \quad , \quad \frac{d}{dx}(fg) = ?$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= \csc^{-1} x \left(\frac{d}{dx} x\right) + x \left(\frac{d}{dx} \csc^{-1} x\right)$$

$$= (\csc^{-1} x)(1) + x\left(\frac{-1}{|x|\sqrt{x^2 - 1}}\right)$$

$$= \boxed{\csc^{-1} x - \frac{x}{|x|\sqrt{x^2 - 1}}} \text{ provided that } |x| > 1$$

⑪ $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x)$

$$= \frac{d}{dx}\sin^{-1} x + \frac{d}{dx}\cos^{-1} x$$

$$= \frac{1}{\sqrt{1 - x^2}} + \frac{-1}{\sqrt{1 - x^2}} = \boxed{0}$$

⓬ $\frac{d}{dx}\frac{\sec^{-1}x}{x}$

Apply the quotient rule:

$f(x) = \sec^{-1}x \quad , \quad g(x) = x \quad , \quad \frac{d}{dx}\left(\frac{f}{g}\right) = ?$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

$$= \frac{x\left[\frac{d}{dx}(\sec^{-1}x)\right] - (\sec^{-1}x)\left(\frac{d}{dx}x\right)}{x^2}$$

$$= \frac{x\left(\frac{1}{|x|\sqrt{x^2-1}}\right) - (\sec^{-1} x)(1)}{x^2}$$

$$= \frac{\frac{x}{|x|\sqrt{x^2-1}} - \sec^{-1} x}{x^2}$$

$$= \left(\frac{x}{|x|\sqrt{x^2-1}} - \sec^{-1} x\right)\left(\frac{1}{x^2}\right)$$

$$= \boxed{\frac{1}{x|x|\sqrt{x^2-1}} - \frac{\sec^{-1} x}{x^2}} \text{ where } |x| > 1$$

Note that $\frac{x}{x^2} = \frac{1}{x}$. Also, note the absolute values on $|x|$ (the absolute values in $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$ reflect that a graph of secant inverse has a positive slope for all possible values of both $x > 1$ and $x < 1$).

4 DERIVATIVES OF EXPONENTIALS

The derivative of the basic exponential function is:

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

Two simple cases correspond to setting **a** equal to **+1** or **–1**:

$$\frac{d}{dx}e^{x} = e^{x} \quad , \quad \frac{d}{dx}e^{-x} = -e^{-x}$$

Numerically, Euler's number is **e = 2.718281828**… Recall the following properties regarding exponentials:

$$e^{x+y} = e^{x}e^{y} \quad , \quad e^{x-y} = e^{x}e^{-y}$$

$$e^{-x} = \frac{1}{e^{x}} \quad , \quad (e^{x})^{a} = e^{ax} \quad , \quad e^{0} = 1$$

The hyperbolic functions are defined in terms of exponentials:

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

Note the following properties regarding hyperbolic functions:

$\cosh^2 x - \sinh^2 x = 1$

$\cosh(-x) = \cosh(x)$

$\sinh(-x) = -\sinh(x)$

sinh 0 = 0
cosh 0 = 1
tanh 0 = 0

The derivatives of the hyperbolic functions (**not** the ordinary trig functions) are:

$$\frac{d}{dx}\sinh x = \cosh x \quad , \quad \frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x}$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}3e^{x^2}$$

We will make the following definition in order to apply the chain rule:

$$u = x^2$$

Now the problem looks like this:

$$f(u) = 3e^u \quad , \quad u(x) = x^2 \quad , \quad \frac{df}{dx} = ?$$

Apply the chain rule:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(3e^u)\right]\left[\frac{d}{dx}(x^2)\right]$$

$$= (3e^u)(2x) = 6xe^u = 6xe^{x^2}$$

Example: Perform the following derivative with respect to **t**.

$$\frac{d}{dt} t \cosh t$$

We will make the following definitions in order to apply the product rule:

$$f(t) = t \quad , \quad g(t) = \cosh t$$

Apply the product rule:

$$\frac{d}{dt}(fg) = g\frac{df}{dt} + f\frac{dg}{dt}$$

$$= \cosh t \left(\frac{d}{dt} t\right) + t\left(\frac{d}{dt} \cosh t\right)$$

$$= (\cosh t)(1) + t(\sinh t) = \cosh t + t \sinh t$$

Note that the derivative of hyperbolic cosine, $\frac{d}{dx}(\cosh x) = \sinh x$, is positive, whereas the derivative of ordinary cosine, $\frac{d}{dx}\cos x = -\sin x$, is negative.

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}\frac{2+e^{-2x}}{e^{3x}}$$

By realizing that $\frac{1}{e^{3x}} = e^{-3x}$, we can rewrite this derivative as:

$$\frac{d}{dx}\frac{2+e^{-2x}}{e^{3x}} = \frac{d}{dx}\left[(2+e^{-2x})e^{-3x}\right]$$

We will make the following definitions in order to apply the product rule:

$$f(x) = 2 + e^{-2x} \quad , \quad g(x) = e^{-3x}$$

Apply the product rule:

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= (2+e^{-2x})\frac{d}{dx}e^{-3x} + e^{-3x}\frac{d}{dx}(2+e^{-2x})$$

$$= (2+e^{-2x})(-3e^{-3x}) + e^{-3x}(0-2e^{-2x})$$

$$= (2)(-3)e^{-3x} + e^{-2x}(-3)e^{-3x} + e^{-3x}(-2)e^{-2x}$$

$$= -6e^{-3x} - 3e^{-5x} - 2e^{-5x} = -6e^{-3x} - 5e^{-5x}$$

Note the following:

- $\frac{d}{dx}e^{-2x} = -2e^{-2x}$ and $\frac{d}{dx}e^{3x} = 3e^{3x}$ according to $\frac{d}{dx}e^{ax} = ae^{ax}$.
- A derivative of a constant is zero: $\frac{d}{dx}(2) = 0$.
- $e^{-2x}e^{-3x} = e^{-2x-3x} = e^{-5x}$ according to $e^{x}e^{y} = e^{x+y}$.
- $\frac{-6e^{3x}-5e^{x}}{e^{6x}}$, $-\frac{6e^{3x}+5e^{x}}{e^{6x}}$ (the minus sign distributes), and $-6e^{-3x} - 5e^{-5x}$ are all correct answers. It's just a matter of preference.

Chapter 4 Exercises

Directions: Perform each derivative with respect to the indicated variable.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\frac{d}{dx}\left(4e^{x^2} - 6e^{4x} + 9\right) =$$

❷ $$\frac{d}{dx} 4\cosh^3 x =$$

❸ $$\frac{d}{dt} \sinh t \cosh t =$$

❹ $$\frac{d}{dx} \sinh[\cosh(x)] =$$

Part B

❺ $$\frac{d}{dx}\tanh\sqrt{x} =$$

❻ $$\frac{d}{dt}(t^2e^t) =$$

❼ $$\frac{d}{dx}\sqrt{1+e^{-x}} =$$

❽ $$\frac{d}{dx}\frac{1-\sinh x}{\sinh x} =$$

Chapter 4 Solutions

Part A

❶ $$\frac{d}{dx}\left(4e^{x^2} - 6e^{4x} + 9\right)$$
$$= \frac{d}{dx}\left(4e^{x^2}\right) - \frac{d}{dx}\left(6e^{4x}\right) + \frac{d}{dx}9$$
$$= 8xe^{x^2} - 24e^{4x} + 0 = \boxed{8xe^{x^2} - 24e^{4x}}$$

Notes:

- $\frac{d}{dx}\left(4e^{x^2}\right) = \frac{d}{dx}(4e^u)$ where $u = x^2$. Apply the chain rule with $f = 4e^u$ and $u = x^2$:
  $\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(4e^u)\right]\left(\frac{d}{dx}x^2\right)$
  $= (4e^u)(2x) = 8xe^u = 8xe^{x^2}$.
  Therefore, $\frac{d}{dx}\left(4e^{x^2}\right) = 8xe^{x^2}$.
- $\frac{d}{dx}\left(6e^{4x}\right) = 6\frac{d}{dx}\left(e^{4x}\right) = 6\left(4e^{4x}\right) = 24e^{4x}$ according to $\frac{d}{dx}e^{ax} = ae^{ax}$.
- A derivative of a constant is zero: $\frac{d}{dx}(9) = 0$.

❷ $\frac{d}{dx} 4\cosh^3 x$

Apply the chain rule:

$$u = \cosh x \quad , \quad f = 4u^3 \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left[\frac{d}{du}(4u^3)\right]\left(\frac{d}{dx}\cosh x\right)$$

$$= (12u^2)(\sinh x) = \boxed{12\cosh^2 x \sinh x}$$

Note that the derivative of hyperbolic cosine, $\frac{d}{dx}(\cosh x) = \sinh x$, is positive, whereas the derivative of ordinary cosine, $\frac{d}{dx}\cos x = -\sin x$, is negative.

❸ $\dfrac{d}{dt}\sinh t \cosh t$

Apply the product rule:

$$f(t) = \sinh t \quad , \quad g(t) = \cosh t \quad , \quad \frac{d}{dt}(fg) = ?$$

$$\frac{d}{dt}(fg) = g\,\frac{df}{dt} + f\,\frac{dg}{dt}$$

$$= \cosh t \left(\frac{d}{dt}\sinh t\right) + \sinh t \left(\frac{d}{dt}\cosh t\right)$$

$$= \cosh t\,(\cosh t) + \sinh t\,(\sinh t)$$

$$= \boxed{\cosh^2 t + \sinh^2 t}$$

Notes:

- The answer doesn't reduce to 1. Compare the hyperbolic identity, $\cosh^2 x - \sinh^2 x = 1$, to the ordinary trig identity, $\cos^2 x + \sin^2 x = 1$: Note the minus sign in the hyperbolic identity.
- Note that the derivative of hyperbolic cosine, $\frac{d}{dx}(\cosh x) = \sinh x$, is positive, whereas the derivative of ordinary cosine, $\frac{d}{dx}\cos x = -\sin x$, is negative.

❹ $\frac{d}{dx}\sinh[\cosh(x)]$

This is not a product of hyperbolic functions. Rather, hyperbolic cosine is inside the argument of hyperbolic sine. This is a chain rule problem, not a product rule.

Apply the chain rule:

$$u = \cosh x \quad , \quad f = \sinh u \quad , \qquad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}\sinh u\right)\left(\frac{d}{dx}\cosh x\right)$$

$$= (\cosh u)(\sinh x) = \sinh x \cosh u$$

$$= \boxed{\sinh x \cosh[\cosh(x)]}$$

Note that $\cosh[\cosh(x)]$ has $\cosh(x)$ inside of its argument. This isn't multiplication. Also, note that the derivative of hyperbolic cosine, $\frac{d}{dx}(\cosh x) = \sinh x$, is positive, whereas the derivative of ordinary cosine, $\frac{d}{dx}\cos x = -\sin x$, is negative.

Part B

❺ $\dfrac{d}{dx}\tanh\sqrt{x}$

Apply the chain rule:

$$u = \sqrt{x} = x^{1/2} \quad , \quad f = \tanh u \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}\tanh u\right)\left(\frac{d}{dx}x^{1/2}\right)$$

$$= (\operatorname{sech}^2 u)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}\operatorname{sech}^2 u$$

$$= \frac{\operatorname{sech}^2 u}{2x^{1/2}} = \frac{\operatorname{sech}^2 u}{2\sqrt{x}} = \frac{\operatorname{sech}^2 u}{2\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}} = \frac{\sqrt{x}\operatorname{sech}^2 u}{2x}$$

$$= \frac{\sqrt{x}\operatorname{sech}^2\sqrt{x}}{2x} = \boxed{\frac{\sqrt{x}}{2x\cosh^2\sqrt{x}}}$$

Notes:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $\frac{1}{x^{1/2}} = x^{-1/2}$ because $x^{-n} = \frac{1}{x^n}$.
- $\sqrt{x}\sqrt{x} = x$.

- $\frac{1}{2}x^{-1/2}\operatorname{sech}^2\sqrt{x}$, $\frac{\operatorname{sech}^2\sqrt{x}}{2x^{1/2}}$, $\frac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$, $\frac{\sqrt{x}\operatorname{sech}^2\sqrt{x}}{2x}$, and $\frac{\sqrt{x}}{2x\cosh^2\sqrt{x}}$ are all correct answers. However, only the last two answers have a rational denominator. We multiplied $\frac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$ by $\frac{\sqrt{x}}{\sqrt{x}}$ in order to rationalize the denominator.
- $\operatorname{sech} x = \frac{1}{\cosh x}$.
- Note that $\cosh^2\sqrt{x}$ doesn't simplify to $\cosh x$.

❻ $\frac{d}{dt}(t^2e^t)$

Apply the product rule:

$$f(t) = t^2 \quad , \quad g(t) = e^t \quad , \quad \frac{d}{dt}(fg) = ?$$

$$\frac{d}{dt}(fg) = g\frac{df}{dt} + f\frac{dg}{dt} = e^t\left(\frac{d}{dt}t^2\right) + t^2\left(\frac{d}{dt}e^t\right)$$

$$= e^t(2t) + t^2(e^t) = 2te^t + t^2e^t$$

$$= e^t(2t + t^2) = \boxed{e^t(t^2 + 2t)}$$

Notes:

- $\frac{d}{dt}e^t = e^t$ according to $\frac{d}{dx}e^{ax} = ae^{ax}$ with $a = 1$ and x replaced by t.
- $2te^t + t^2e^t = e^t(2t + t^2)$. Factor out e^t.
- $2t + t^2 = t^2 + 2t$ since addition is commutative (order doesn't matter).

❼ $\frac{d}{dx}\sqrt{1 + e^{-x}}$

Apply the chain rule:

$$u = 1 + e^{-x} \quad , \quad f = \sqrt{u} = u^{1/2} \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}u^{1/2}\right)\left[\frac{d}{dx}(1 + e^{-x})\right]$$

$$= \left(\frac{1}{2}u^{-1/2}\right)(0 - e^{-x}) = -\frac{u^{-1/2}e^{-x}}{2}$$

$$= -\frac{e^{-x}}{2u^{1/2}} = -\frac{e^{-x}}{2\sqrt{u}} = -\frac{e^{-x}}{2\sqrt{1+e^{-x}}}$$

$$= -\frac{e^{-x}}{2\sqrt{1+e^{-x}}}\frac{\sqrt{1+e^{-x}}}{\sqrt{1+e^{-x}}}$$

$$= \boxed{-\frac{e^{-x}\sqrt{1+e^{-x}}}{2(1+e^{-x})}}$$

Notes:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $\frac{1}{u^{1/2}} = u^{-1/2}$ because $x^{-n} = \frac{1}{x^n}$.
- A derivative of a constant is zero: $\frac{d}{dx}(1) = 0$.
- $\frac{d}{dx}e^{-x} = -e^{-x}$ according to $\frac{d}{dx}e^{ax} = ae^{ax}$ with $a = -1$.
- $\sqrt{1+e^{-x}}\sqrt{1+e^{-x}} = 1 + e^{-x}$ because $\sqrt{u}\sqrt{u} = u$.

- $-\frac{e^{-x}}{2\sqrt{1+e^{-x}}}$ and $-\frac{e^{-x}\sqrt{1+e^{-x}}}{2(1+e^{-x})}$ are both correct answers. However, only the answer $-\frac{e^{-x}\sqrt{1+e^{-x}}}{2(1+e^{-x})}$ has a rational denominator. We multiplied $-\frac{e^{-x}}{2\sqrt{1+e^{-x}}}$ by $\frac{\sqrt{1+e^{-x}}}{\sqrt{1+e^{-x}}}$ in order to rationalize the denominator.

❽ $\frac{d}{dx}\frac{1-\sinh x}{\sinh x}$

Apply the quotient rule: $f(x) = 1-\sinh x$, $g(x) = \sinh x$, $\frac{d}{dx}\left(\frac{f}{g}\right) = ?$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\,\frac{df}{dx} - f\,\frac{dg}{dx}}{g^2}$$

$$= \frac{\sinh x\left[\frac{d}{dx}(1-\sinh x)\right] - (1-\sinh x)\left(\frac{d}{dx}\sinh x\right)}{\sinh^2 x}$$

$$= \frac{\sinh x\,(0-\cosh x)-(1-\sinh x)(\cosh x)}{\sinh^2 x}$$

Distribute: $-(a-b) = -a-(-b)$

$$= \frac{-\sinh x\cosh x-(1)\cosh x-(-\sinh x)\cosh x}{\sinh^2 x}$$

Note: $-(-\sinh x) = +\sinh x$

$$= \frac{-\sinh x\cosh x-\cosh x+\sinh x\cosh x}{\sinh^2 x}$$

$$= \boxed{\frac{-\cosh x}{\sinh^2 x}}$$

Note: The answer may alternatively be expressed as $-\coth x\,\operatorname{csch} x$ or $-\cosh x\,\operatorname{csch}^2 x$ because $\coth x = \frac{\cosh x}{\sinh x}$ and $\operatorname{csch} x = \frac{1}{\sinh x}$.

5 DERIVATIVES OF LOGARITHMS

The derivative of the natural logarithm function is:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Note that a constant multiplying the variable inside of the argument has no effect on the derivative of a logarithm. We will explore why in one of the examples.

$$\frac{d}{dx}\ln(ax) = \frac{1}{x} \quad (a\text{ vanishes})$$

Recall that a natural logarithm is a function that is used to determine the exponent in the equation $\mathbf{y = e^x}$. Specifically, the equation $\mathbf{\ln y = x}$ is exactly the same as the equation $\mathbf{y = e^x}$; the only difference is that $\mathbf{\ln y = x}$ lets you solve for the exponent. Recall the following properties regarding logarithms:

$$\ln(xy) = \ln x + \ln y$$
$$\ln(x/y) = \ln x - \ln y$$
$$\ln(1/x) = \ln(x^{-1}) = -\ln x$$
$$\ln(x^a) = a \ln x$$
$$\ln(e^x) = x$$
$$e^{\ln x} = x$$
$$\ln(e) = 1$$
$$\ln(1) = 0$$

The natural logarithm $\mathbf{\ln x}$ is a logarithm where the base equals Euler's number (**$e \approx 2.718281828$**). That is, $\mathbf{\ln x = \log_e x}$. A general logarithm of base **b** can be related to the natural logarithm using the change of base formula:

$$\log_b x = \frac{\ln x}{\ln b}$$

This allows you to take a derivative of a logarithm of any base (where **b** is the base):

$$\frac{d}{dx}\log_b x = \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}\ln(x) = \frac{1}{\ln b}\frac{1}{x} = \frac{1}{x \ln b}$$

A natural logarithm is also involved in the derivative of the power function:

$$\frac{d}{dx}b^x = b^x \ln b \quad , \quad \frac{d}{dx}b^{-x} = -b^{-x}\ln b$$

If you set **b = e** in the above formulas, you get the special cases

$$\frac{d}{dx}e^x = e^x \quad \text{and} \quad \frac{d}{dx}e^{-x} = -e^{-x}$$

which we explored in the previous chapter (note that **ln e = 1**).

Example: Show that the constant **a** has no effect on the following derivative.

$$\frac{d}{dx}\ln(ax)$$

Method 1: We will make the following definition in order to apply the chain rule:

$$f(u) = \ln u \quad , \quad u(x) = ax \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}\ln u\right)\left(\frac{d}{dx}ax\right)$$

$$= \left(\frac{1}{u}\right)(a) = \frac{a}{ax} = \frac{1}{x}$$

The constant a canceled out.

Method 2: Apply the identity $\ln(ax) = \ln(a) + \ln(x)$.

$$\frac{d}{dx}\ln(ax) = \frac{d}{dx}(\ln a + \ln x)$$

$$= \frac{d}{dx}\ln a + \frac{d}{dx}\ln x = 0 + \frac{1}{x} = \frac{1}{x}$$

The derivative of a constant equals zero, and $\ln a$ is a constant (since a is constant).

Example: Perform the following derivative with respect to **t**.

$$\frac{d}{dt} t \ln t$$

We will make the following definitions in order to apply the product rule:

$$f(t) = t \quad , \quad g(t) = \ln t$$

Apply the product rule:

$$\frac{d}{dt}(fg) = g\frac{df}{dt} + f\frac{dg}{dt} = \ln t\left(\frac{d}{dt}t\right) + t\left(\frac{d}{dt}\ln t\right)$$

$$= (\ln t)(1) + t\left(\frac{1}{t}\right) = \ln t + 1$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}\log_{10} x$$

Use the change of base formula with $b = 10$:

$\log_{10} x = \frac{\ln x}{\ln 10}$.

$$\frac{d}{dx}\log_{10} x = \frac{d}{dx}\left(\frac{\ln x}{\ln 10}\right) = \frac{1}{\ln 10}\frac{d}{dx}\ln x$$

$$= \frac{1}{\ln 10}\frac{1}{x} = \frac{1}{x\ln 10}$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d}{dx}2^x$$

Use the formula for the derivative of the power function with $b = 2$:

$$\frac{d}{dx}2^x = 2^x \ln 2$$

Chapter 5 Exercises

Directions: Perform each derivative with respect to the indicated variable.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $\dfrac{d}{dx} e^x \ln x =$

❷ $\dfrac{d}{dt} \dfrac{\ln t}{t} =$

❸ $\dfrac{d}{dx} \ln|\cos x| =$

❹ $\dfrac{d}{dx} \ln(\cosh x) =$

Part B

❺ $\dfrac{d}{dx}\sqrt{\ln x} =$

❻ $\dfrac{d}{dx}\ln\sqrt{x} =$

❼ $\dfrac{d}{dt}\log_2 t =$

❽ $\dfrac{d}{dx}\dfrac{2^x}{x^2} =$

Chapter 5 Solutions

Part A

❶ $\dfrac{d}{dx}e^x \ln x$

Apply the product rule:

$$f(x) = e^x \quad , \quad g(x) = \ln x \quad , \quad \frac{d}{dx}(fg) = ?$$

$$\frac{d}{dx}(fg) = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= e^x\left(\frac{d}{dx}\ln x\right) + \ln x\left(\frac{d}{dx}e^x\right)$$

$$= e^x\left(\frac{1}{x}\right) + \ln x\,(e^x)$$

$$= \frac{e^x}{x} + e^x \ln x = \boxed{e^x\left(\frac{1}{x} + \ln x\right)}$$

❷ $\frac{d}{dt} = \frac{\ln t}{t}$

Apply the quotient rule:

$$f(t) = \ln t \quad , \quad g(t) = t \quad , \quad \frac{d}{dt}\left(\frac{f}{g}\right) = ?$$

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \frac{g\,\frac{df}{dt} - f\,\frac{dg}{dt}}{g^2} = \frac{t\left(\frac{d}{dt}\ln t\right) - \ln t\left(\frac{d}{dt}t\right)}{t^2}$$

$$= \frac{t\left(\frac{1}{t}\right) - \ln t\,(1)}{t^2} = \boxed{\frac{1 - \ln t}{t^2}}$$

❸ $\frac{d}{dx}\ln|\cos x|$

Apply the chain rule:

$$u = \cos x \quad , \quad f = \ln u \quad , \qquad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}\ln u\right)\left(\frac{d}{dx}\cos x\right)$$

$$= \left(\frac{1}{u}\right)(-\sin x) = \frac{-\sin x}{u} = -\frac{\sin x}{\cos x}$$

$$= \boxed{-\tan x}$$

(Why put absolute values on $\ln|\cos x|$? The logarithm is only real when the argument is positive. The absolute values prevent cosine from being negative.)

❹ $\frac{d}{dx}\ln(\cosh x)$

Apply the chain rule:

$u = \cosh x \quad , \quad f = \ln u \quad , \qquad \frac{df}{dx} = ?$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}\ln u\right)\left(\frac{d}{dx}\cosh x\right)$$

$$= \left(\frac{1}{u}\right)(\sinh x) = \frac{\sinh x}{u} = \frac{\sinh x}{\cosh x}$$

$$= \boxed{\tanh x}$$

Note: The derivative of hyperbolic cosine, $\frac{d}{dx}(\cosh x) = \sinh x$, is positive, whereas the derivative of ordinary cosine, $\frac{d}{dx}\cos x = -\sin x$, is negative.

Part B

❺ $\frac{d}{dx}\sqrt{\ln x}$

Apply the chain rule:

$$u = \ln x \quad , \quad f = \sqrt{u} = u^{1/2} \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}u^{1/2}\right)\left(\frac{d}{dx}\ln x\right)$$

$$= \left(\frac{1}{2}u^{1/2-1}\right)\left(\frac{1}{x}\right) = \left(\frac{1}{2}u^{-1/2}\right)\left(\frac{1}{x}\right)$$

$$= \left(\frac{1}{2u^{1/2}}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{2xu^{1/2}} = \frac{1}{2x\sqrt{u}} = \frac{1}{2x\sqrt{\ln x}} = \frac{1}{2x\sqrt{\ln x}}\frac{\sqrt{\ln x}}{\sqrt{\ln x}}$$

$$= \boxed{\frac{\sqrt{\ln x}}{2x\ln x}}$$

Notes:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $\frac{1}{x^{1/2}} = x^{-1/2}$ because $x^{-n} = \frac{1}{x^n}$.

- $\sqrt{\ln x}\sqrt{\ln x} = \ln x$ because $\sqrt{u}\sqrt{u} = \left(\sqrt{u}\right)^2 = u$.
- $\frac{1}{2x\sqrt{\ln x}}$ and $\frac{\sqrt{\ln x}}{2x\ln x}$ are both correct answers. However, only the answer $\frac{\sqrt{\ln x}}{2x\ln x}$ has a rational denominator. We multiplied $\frac{1}{2x\sqrt{\ln x}}$ by $\frac{\sqrt{\ln x}}{\sqrt{\ln x}}$ in order to rationalize the denominator.

❻ $\frac{d}{dx}\ln\sqrt{x}$

Method 1: Apply the chain rule:

$$u = \sqrt{x} = x^{1/2} \quad , \quad f = \ln u \quad , \quad \frac{df}{dx} = ?$$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = \left(\frac{d}{du}\ln u\right)\left(\frac{d}{dx}x^{1/2}\right)$$

$$= \left(\frac{1}{u}\right)\left(\frac{1}{2}x^{1/2-1}\right) = \left(\frac{1}{x^{1/2}}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \left(\frac{1}{x^{1/2}}\right)\left(\frac{1}{2x^{1/2}}\right) = \frac{1}{2x^{1/2}x^{1/2}} = \frac{1}{2\sqrt{x}\sqrt{x}} = \boxed{\frac{1}{2x}}$$

Note that $\sqrt{x}\sqrt{x} = \left(\sqrt{x}\right)^2 = x$.

Method 2: Apply the identity $\ln x^a = a \ln x$ with $a = \frac{1}{2}$.

$$\frac{d}{dx}\ln\sqrt{x} = \frac{d}{dx}\ln x^{1/2} = \frac{d}{dx}\left(\frac{1}{2}\ln x\right) = \frac{1}{2}\frac{d}{dx}\ln x$$

$$= \frac{1}{2}\left(\frac{1}{x}\right) = \boxed{\frac{1}{2x}}$$

❼ $\dfrac{d}{dt}\log_2 t$

Use the change of base formula with $b = 2$: $\log_2 t = \frac{\ln t}{\ln 2}$.

$$\frac{d}{dt}\log_2 t = \frac{d}{dt}\left(\frac{\ln t}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dt}\ln t = \frac{1}{\ln 2}\frac{1}{t}$$

$$= \boxed{\frac{1}{t\ln 2}}$$

❽ $\frac{d}{dx}\frac{2^x}{x^2}$

Apply the quotient rule:

$f(x) = 2^x \quad , \quad g(x) = x^2 \quad , \quad \frac{d}{dx}\left(\frac{f}{g}\right) = ?$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{x^2\left(\frac{d}{dx}2^x\right) - 2^x\left(\frac{d}{dx}x^2\right)}{(x^2)^2}$$

Use the formula for the derivative of the power function with $b = 2$:

$$\frac{d}{dx}2^x = 2^x \ln 2$$

Substitute the above derivative into the previous equation.

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{x^2(2^x \ln 2) - 2^x(2x)}{(x^2)^2}$$

$$= \frac{x^2 2^x \ln 2 - 2x2^x}{x^4}$$

$$= \boxed{\frac{2^x(x^2 \ln 2 - 2x)}{x^4}} = \boxed{\frac{2^x(x \ln 2 - 2)}{x^3}}$$

Notes:

- $(x^2)^2 = x^4$ according to $(x^a)^b = x^{ab}$.
- $x^2 2^x \ln 2 - 2x2^x = 2^x(x^2 \ln 2 - 2x)$. Factor out 2^x.
- In the last step, we divided the numerator and denominator both by x.
- An alternative answer is $\frac{2^x \ln 2}{x^2} - \frac{2^{x+1}}{x^3}$ because $\frac{x^2}{x^4} = \frac{1}{x^2}$ and $\frac{2^x 2x}{x^4} = \frac{2^x 2}{x^3} = \frac{2^{x+1}}{x^3}$ (since $2^x 2 = 2^x 2^1 = 2^{x+1}$ according to $x^m x^n = x^{m+n}$).

6 SECOND DERIVATIVES

A second derivative simply means to take two consecutive derivatives, as illustrated by the following examples.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d^2}{dx^2}(2x^6)$$

First take a derivative of $\mathbf{y = 2x^6}$ with respect to **x**, and then take a second derivative:

$$\frac{dy}{dx} = \frac{d}{dx}2x^6 = 12x^5$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(12x^5) = 60x^4$$

Example: Perform the following derivative with respect to theta.

$$\frac{d^2}{d\theta^2}(\sin\theta)$$

First take a derivative of **y = sin(theta)** with respect to theta, and then take a second derivative:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\sin\theta = \cos\theta$$

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = \frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

Example: Perform the following derivative with respect to **x**.

$$\frac{d^2}{dx^2}\left(\frac{1}{x^2}\right)$$

First take a derivative of **y = 1/x^2** with respect to **x**, and then take a second derivative:

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}x^{-2} = -2x^{-3}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-2x^{-3}) = (-3)(-2)x^{-4} = 6x^{-4} = \frac{6}{x^4}$$

Chapter 6 Exercises

Directions: Perform each derivative with respect to the indicated variable.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\frac{d^2}{dx^2}(x^7 - 3x^5 + 5x^3 - 7x) =$$

❷ $$\frac{d^2}{d\theta^2}\cos 3\theta =$$

❸ $$\frac{d^2}{dx^2}\ln x =$$

❹ $$\frac{d^2}{dt^2}e^{-3t^2} =$$

Part B

❺ $$\frac{d^2}{d\theta^2}\sin(\theta^2) =$$

❻ $$\frac{d^2}{d\theta^2}\tan\theta =$$

❼ $$\frac{d^2}{dt^2}(t\ln t) =$$

❽ $$\frac{d^2}{dx^2}\sqrt{x} =$$

Part C

❾ $\dfrac{d^2}{dx^2}\dfrac{1}{x} =$

❿ $\dfrac{d^2}{d\theta^2}\sec\theta =$

⓫ $\dfrac{d^2}{dt^2}\tan^{-1} t =$

⓬ $\dfrac{d^2}{d\theta^2}(\theta \sin\theta) =$

Chapter 6 Solutions

Part A

❶ $$\frac{dy}{dx} = \frac{d}{dx}(x^7 - 3x^5 + 5x^3 - 7x)$$
$$= 7x^6 - 15x^4 + 15x^2 - 7$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(7x^6 - 15x^4 + 15x^2 - 7)$$
$$= \boxed{42x^5 - 60x^3 + 30x}$$

❷ $$\frac{dy}{d\theta} = \cos 3\theta$$

Apply the chain rule with $u = 3\theta$ and $y = \cos u$:
$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left(\frac{d}{du}\cos u\right)\left(\frac{d}{d\theta}3\theta\right) = (-\sin u)(3)$$
$$= -3\sin u = -3\sin 3\theta$$

Now take a second derivative:
$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = \frac{d}{d\theta}(-3\sin 3\theta)$$

Apply the chain rule with $u = 3\theta$ and $g = -3\sin u$:

$$\frac{d^2y}{d\theta^2} = \frac{dg}{d\theta} = \frac{dg}{du}\frac{du}{d\theta} = \left[\frac{d}{du}(-3\sin u)\right]\left(\frac{d}{d\theta}3\theta\right)$$
$$= (-3\cos u)(3) = -9\cos u = \boxed{-9\cos 3\theta}$$

❸ $\frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{x} \quad \rightarrow \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2} = \boxed{-\frac{1}{x^2}}$$

❹ $\frac{dy}{dt} = \frac{d}{dt}e^{-3t^2}$

Apply the chain rule with $u = -3t^2$ and $y = e^u$:

$$\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = \left(\frac{d}{du}e^u\right)\left[\frac{d}{dt}(-3t^2)\right] = (e^u)(-6t)$$
$$= -6te^u = -6te^{-3t^2}$$

Now take a second derivative:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(-6te^{-3t^2}\right)$$

Apply the product rule with $g(t) = -6t$ and $h(t) = e^{-3t^2}$.

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(gh) = h\frac{dg}{dt} + g\frac{dh}{dt}$$

$$= e^{-3t^2}\left[\frac{d}{dt}(-6t)\right] + (-6t)\left(\frac{d}{dt}e^{-3t^2}\right)$$

$$= e^{-3t^2}(-6) - 6t\left(-6te^{-3t^2}\right)$$

$$= -6e^{-3t^2} + 36t^2e^{-3t^2}$$

$$= (-6 + 36t^2)e^{-3t^2}$$

$$= (36t^2 - 6)e^{-3t^2} = 6(6t^2 - 1)e^{-3t^2}$$

$$= \boxed{\frac{6(6t^2 - 1)}{e^{3t^2}}}$$

Notes:

- We used $\frac{d}{dt}e^{-3t^2} = -6te^{-3t^2}$ from $\frac{dy}{dt}$ when finding $\frac{d^2y}{dt^2}$.
- $-6e^{-3t^2} + 36t^2e^{-3t^2} = (-6 + 36t^2)e^{-3t^2}$. (Factor out e^{-3t^2}.)
- Note that $-6 + 36t^2 = 36t^2 - 6$.
- Each of the last five answers is a correct answer for this problem.

Part B

❺ $\frac{dy}{d\theta} = \frac{d}{d\theta}\sin(\theta^2)$

Apply the chain rule with $u = \theta^2$ and $y = \sin u$:

$$\frac{dy}{d\theta} = \frac{dy}{du}\frac{du}{d\theta} = \left(\frac{d}{du}\sin u\right)\left(\frac{d}{d\theta}\theta^2\right) = (\cos u)(2\theta)$$

$$= 2\theta\cos u = 2\theta\cos(\theta^2)$$

Now take a second derivative:

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left[\frac{dy}{d\theta}\right] = \frac{d}{d\theta}[2\theta\cos(\theta^2)]$$

Apply the product rule with $g(\theta) = 2\theta$ and $h(\theta) = \cos(\theta^2)$.

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(gh) = h\frac{dg}{d\theta} + g\frac{dh}{d\theta}$$

$$= \cos(\theta^2)\left(\frac{d}{d\theta}2\theta\right) + 2\theta\left[\frac{d}{d\theta}\cos(\theta^2)\right]$$

$$= \cos(\theta^2)\,(2) + 2\theta\left[\frac{d}{d\theta}\cos(\theta^2)\right]$$

$$= 2\cos(\theta^2) + 2\theta\left[\frac{d}{d\theta}\cos(\theta^2)\right]$$

Apply the chain rule with $u = \theta^2$ and $k = \cos u$:

$$\frac{dk}{d\theta} = \frac{dk}{du}\frac{du}{d\theta} = \left(\frac{d}{du}\cos u\right)\left(\frac{d}{d\theta}\theta^2\right) = (-\sin u)(2\theta)$$

$$= -2\theta \sin u = -2\theta \sin(\theta^2)$$

Substitute $\frac{d}{d\theta}\cos(\theta^2) = -2\theta \sin(\theta^2)$ into the equation for the second derivative:

$$\frac{d^2y}{d\theta^2} = 2\cos(\theta^2) + 2\theta\left[\frac{d}{d\theta}\cos(\theta^2)\right]$$

$$= 2\cos(\theta^2) + 2\theta[-2\theta \sin(\theta^2)]$$

$$= \boxed{2\cos(\theta^2) - 4\theta^2 \sin(\theta^2)}$$

Note that the angle is squared, not the sine function. Compare $\sin(\theta^2)$, which means to square the angle first and then take the sine, with $\sin^2\theta$, which instead means to take the sine first and then square the result.

❻ $\frac{dy}{d\theta} = \frac{d}{d\theta}\tan\theta = \sec^2\theta \quad \rightarrow \quad \frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right)$

$$= \frac{d}{d\theta}\sec^2\theta$$

Apply the chain rule with $u = \sec\theta$ and $f = u^2$:

$$\frac{d^2y}{d\theta^2} = \frac{df}{d\theta} = \frac{df}{du}\frac{du}{d\theta} = \left(\frac{d}{du}u^2\right)\left(\frac{d}{d\theta}\sec\theta\right)$$

$$= (2u)(\sec\theta\tan\theta) = (2\sec\theta)(\sec\theta\tan\theta)$$

$$= \boxed{2\sec^2\theta\tan\theta}$$

Note that an alternate answer is $\frac{2\sin\theta}{\cos^3\theta}$ because $\sec^2\theta = \frac{1}{\cos^2\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

❼ $\frac{dy}{dt} = \frac{d}{dt}(t\ln t)$

Apply the product rule with $f(t) = t$ and $g(t) = \ln t$.

$$\frac{dy}{dt} = \frac{d}{dt}(fg) = g\frac{df}{dt} + f\frac{dg}{dt}$$

$$= \ln t\left(\frac{d}{dt}t\right) + t\left(\frac{d}{dt}\ln t\right)$$

$$= (\ln t)(1) + t\left(\frac{1}{t}\right)$$

$$= \ln t + 1 = 1 + \ln t$$

Now take a second derivative:

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}(1+\ln t) = \frac{d}{dt}1 + \frac{d}{dt}\ln t$$

$$= 0 + \frac{1}{t} = \boxed{\frac{1}{t}}$$

❽ $$\frac{dy}{dx} = \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2}\frac{d}{dx}x^{-\frac{1}{2}}$$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)x^{-\frac{3}{2}}$$

$$= -\frac{1}{4}x^{-3/2} = -\frac{1}{4x^{3/2}} = -\frac{1}{4x\sqrt{x}} = -\frac{1}{4x\sqrt{x}}\frac{\sqrt{x}}{\sqrt{x}}$$

$$= \boxed{-\frac{\sqrt{x}}{4x^2}}$$

Notes:

- $\frac{1}{2} - 1 = \frac{1}{2} - \frac{2}{2} = \frac{1-2}{2} = -\frac{1}{2}$ (subtract fractions with a common denominator).
- $-\frac{1}{2} - 1 = -\frac{1}{2} - \frac{2}{2} = \frac{-1-2}{2} = -\frac{3}{2}$.
- $\sqrt{x} = x^{1/2}$ and $\sqrt{x}\sqrt{x} = \left(\sqrt{x}\right)^2 = x$.
- $x^{-3/2} = \frac{1}{x^{3/2}}$ according to $x^{-n} = \frac{1}{x^n}$. Also, $x^{3/2} = x^1 x^{1/2} = x\sqrt{x}$.
- There are multiple correct answers, but $-\frac{\sqrt{x}}{4x^2}$ has a rational denominator.

Part C

❾ $$\frac{dy}{dx} = \frac{d}{dx}\frac{1}{x} = \frac{d}{dx}x^{-1} = -x^{-2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-x^{-2}) = (-2)(-1)x^{-3}$$

$$= 2x^{-3} = \boxed{\frac{2}{x^3}}$$

Notes:

- $-1 - 1 = -2$ and $-2 - 1 = -3$.
- $(-2)(-1) = 2$.
- $x^{-1} = \frac{1}{x}$ and $x^{-3} = \frac{1}{x^3}$.
- $2x^{-3}$ and $\frac{2}{x^3}$ are both correct answers.

⑩ $\frac{dy}{d\theta} = \frac{d}{d\theta}\sec\theta = \sec\theta\tan\theta \quad \rightarrow \quad \frac{d^2y}{d\theta^2}$

$$= \frac{d}{d\theta}\left(\frac{dy}{d\theta}\right) = \frac{d}{d\theta}(\sec\theta\tan\theta)$$

Apply the product rule with $f = \sec\theta$ and $g = \tan\theta$:

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}(fg) = g\frac{df}{d\theta} + f\frac{dg}{d\theta}$$

$$= \tan\theta\left(\frac{d}{d\theta}\sec\theta\right) + \sec\theta\left(\frac{d}{d\theta}\tan\theta\right)$$

$$= \tan\theta\,(\sec\theta\tan\theta) + \sec\theta\,(\sec^2\theta)$$

$$= \sec\theta\tan^2\theta + \sec^3\theta$$

$$= \boxed{\sec\theta\,(\tan^2\theta + \sec^2\theta)}$$

An alternate answer is $\frac{\sin^2\theta+1}{\cos^3\theta}$ because $\sec\theta = \frac{1}{\cos\theta}$ and $\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta}$.

⓫ $\frac{dy}{dt} = \frac{d}{dt}\tan^{-1} t = \frac{1}{1+t^2} \quad \to \quad \frac{d^2y}{dt^2}$

$$= \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{1}{1+t^2}\right)$$

The formula for the derivative of the inverse tangent was given in Chapter 3. Apply the chain rule with $u = 1 + t^2$ and $f = \frac{1}{u} = u^{-1}$.

$$\frac{d^2y}{dt^2} = \frac{df}{dt} = \frac{df}{du}\frac{du}{dt} = \left(\frac{d}{du}u^{-1}\right)\left[\frac{d}{dt}(1+t^2)\right]$$

$$= (-u^{-2})(0 + 2t)$$

$$= -2tu^{-2} = -\frac{2t}{u^2} = \boxed{-\frac{2t}{(1+t^2)^2}}$$

An alternate answer is $= -\frac{2t}{1+2t^2+t^4}$.

⓬ $\frac{dy}{d\theta} = \frac{d}{d\theta}(\theta \sin\theta)$

Apply the product rule with $f = \theta$ and $g = \sin\theta$:

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(fg) = g\frac{df}{d\theta} + f\frac{dg}{d\theta}$$

$$= \sin\theta\left(\frac{d}{d\theta}\theta\right) + \theta\left(\frac{d}{d\theta}\sin\theta\right)$$

$$= \sin\theta\,(1) + \theta\cos\theta$$

$$\frac{dy}{d\theta} = \sin\theta + \theta\cos\theta$$

Now take a second derivative:

$$\frac{d^2y}{d\theta^2} = \frac{d}{d\theta}\left[\frac{dy}{d\theta}\right] = \frac{d}{d\theta}(\sin\theta + \theta\cos\theta)$$

$$= \cos\theta + \frac{d}{d\theta}(\theta\cos\theta)$$

Apply the product rule with $f = \theta$ and $h = \cos\theta$:

$$\frac{d}{d\theta}(\theta\cos\theta) = \frac{d}{d\theta}(fh) = h\frac{df}{d\theta} + f\frac{dh}{d\theta}$$

$$= \cos\theta\left(\frac{d}{d\theta}\theta\right) + \theta\left(\frac{d}{d\theta}\cos\theta\right)$$

$$= \cos\theta\,(1) + \theta(-\sin\theta) = \cos\theta - \theta\sin\theta$$

Substitute $\frac{d}{d\theta}(\theta\cos\theta) = \cos\theta - \theta\sin\theta$ into the equation for $\frac{d^2y}{d\theta^2}$:

$$\frac{d^2y}{d\theta^2} = \cos\theta + \frac{d}{d\theta}(\theta\cos\theta)$$

$$= \cos\theta + (\cos\theta - \theta\sin\theta)$$

$$= \boxed{2\cos\theta - \theta\sin\theta}$$

7 EXTREME VALUES

To find the relative extrema for a function, **f(x)**, follow these steps (as illustrated by the example that follows):

1. Take the first derivative of the function, **df/dx**.
2. Set the first derivative equal to zero: **df/dx = 0**. (Why? The slope of the tangent line is zero, corresponding to **df/dx = 0**, at the relative extrema.)
3. Solve for the values of **x** that make the first derivative zero. Call these $\mathbf{x_c}$. (A few books also include points where the slope is vertical or undefined.)
4. Take the second derivative of the function, $\mathbf{d^2f/dx^2}$, by taking a derivative of the algebraic result from step 1.
5. Evaluate the second derivative at each value of $\mathbf{x_c}$ found in step 3.
6. For each value of $\mathbf{x_c}$ obtained in step 3, follow these steps to classify the type of relative extrema associated with that value of $\mathbf{x_c}$:

• If $\mathbf{d^2f/dx^2 > 0}$ at $\mathbf{x_c}$, classify that point as a relative (or local) minimum.

• If $\mathbf{d^2f/dx^2 < 0}$ at $\mathbf{x_c}$, classify that point as a relative (or local) maximum.

• If $\mathbf{d^2f/dx^2 = 0}$ at $\mathbf{x_c}$, examine the first derivative just before and just after $\mathbf{x_c}$ (as described on the next page).

1. Evaluate the function, **f(x)**, at each value of $\mathbf{x_c}$ obtained in step 3 corresponding to a relative minimum or maximum. These are the values of the function at each relative (or local) extremum.

2. To find the absolute extrema of the function over a specified interval, **(a,b)**, evaluate the function at the endpoints of the interval, **a** and **b**. Follow these steps:

- Compare the values of $\mathbf{f(x_c)}$ from step 7. Also compare **f(a)** and **f(b)**. The largest of these values is the absolute maximum.
- Compare the values of $\mathbf{f(x_c)}$ from step 7. Also compare **f(x)**. The smallest of these values is the absolute minimum.

If the second derivative equals zero at $\mathbf{x_c}$, follow these steps:

1. Determine the sign of the first derivative just before $\mathbf{x_c}$.
2. Determine the sign of the first derivative just after $\mathbf{x_c}$.
3. Interpret your answers to steps 1-2 as follows:

• The sign of the first derivative changes from positive to negative at a relative (or local) maximum.

• The sign of the first derivative changes from negative to positive at a relative (or local) maximum.

• If the first derivative doesn't change sign, this is a point of inflection (where the second derivative, relating to concavity, changes sign.)

Example: Find the absolute extrema for the function below over the interval **(–3,5)**.

f(x) = x³ – 12x + 4

Take a derivative of **f(x)** with respect to **x**:

$$\frac{df}{dx} = \frac{d}{dx}(x^3 - 12x + 4) = 3x^2 - 12$$

Set the first derivative equal to zero. Solve for **x**. Call these values **x_c**.

$$\frac{df}{dx} = 0 \quad \rightarrow \quad 3x_c^2 - 12 = 0$$

$$3x_c^2 = 12 \quad \rightarrow \quad x_c^2 = 4 \quad \rightarrow \quad x_c = \pm 2$$

Take a second derivative of **f(x)** with respect to **x**:

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(3x^2 - 12) = 6x$$

Evaluate the second derivative at **x_c = –2** and **x_c = 2** (which we found previously).

$$\left.\frac{d^2f}{dx^2}\right|_{x=-2} = 6(-2) = -12 \left(\begin{matrix}\text{relative}\\ \text{maximum}\end{matrix}\right)$$

$$\left.\frac{d^2f}{dx^2}\right|_{x=-2} = 6(2) = 12 \left(\begin{matrix}\text{relative}\\ \text{minimum}\end{matrix}\right)$$

There is a relative maximum at **x_c = –2** (where the second derivative is negative), and a relative minimum at **x_c = 2** (where the second derivative is positive). Evaluate the function at **x_c = –2**, **x_c = 2**, and the endpoints (**x = –3** and **x = 5**).

$$f(-2) = (-2)^3 - 12(-2) + 4 = -8 + 24 + 4 = 20$$
$$f(2) = 2^3 - 12(2) + 4 = 8 - 24 + 4 = -12$$
$$f(-3) = (-3)^3 - 12(-3) + 4 = -27 + 36 + 4 = 13$$
$$f(5) = 5^3 - 12(5) + 4 = 125 - 60 + 4 = 69$$

Over the interval **(–3,5)**, the absolute extrema are:

- **f(x)** has an absolute maximum value of **69** (when **x = 5**).
- **f(x)** has an absolute minimum value of **–12** (when **x = 2**).

Chapter 7 Exercises

Directions: Find the absolute extrema for each function over the specified interval.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

1. Find the absolute extrema for the function below over the interval **(2,4)**.

$$f(x) = 2x^4 - 8x^3$$

2. Find the absolute extrema for the function below over the interval **(½,10)**.

$$f(x) = \frac{4}{x^6} - \frac{3}{x^8}$$

Chapter 7 Solutions

❶ $f(x) = 2x^4 - 8x^3 \quad , \quad 2 \leq x \leq 4$

Take a derivative of $f(x)$ with respect to x:

$$\frac{df}{dx} = \frac{d}{dx}(2x^4 - 8x^3) = 8x^3 - 24x^2$$

Set the first derivative equal to zero. Solve for x. Call these values x_c.

$$\frac{df}{dx} = 0 \quad \rightarrow \quad 8x_c^3 - 24x_c^2 = 0$$

Factor out $8x_c^2$:

$$8x_c^2(x_c - 3) = 0$$

$$8x_c^2 = 0 \quad \text{or} \quad x_c - 3 = 0$$

$$x_c = 0 \quad \text{or} \quad x_c = 3$$

Of these, only $x_c = 3$ lies in the specified interval, $2 \leq x \leq 4$.

Take a second derivative of $f(x)$ with respect to x:

$$\frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(8x^3 - 24x^2) = 24x^2 - 48x$$

Evaluate the second derivative at $x_c = 3$ (which we found previously).

$$\left.\frac{d^2f}{dx^2}\right|_{x=3} = 24(3)^2 - 48(3) = 216 - 144$$

$$= 72 \begin{pmatrix}\text{relative}\\ \text{minimum}\end{pmatrix}$$

There is a relative minimum at $x_c = 3$ (where the second derivative is positive).

Evaluate the function at $x_c = 3$, and the endpoints ($x = 2$ and $x = 4$).

$$f(3) = 2(3)^4 - 8(3)^3 = 162 - 216 = -54$$

$$f(2) = 2(2)^4 - 8(2)^3 = 32 - 64 = -32$$

$$f(4) = 2(4)^4 - 8(4)^3 = 512 - 512 = 0$$

Over the interval $(2,4)$, the absolute extrema are:

- $f(x)$ has an absolute maximum value of $\boxed{0}$ (when $x = 4$).
- $f(x)$ has an absolute minimum value of $\boxed{-54}$ (when $x = 3$).

❷ $f(x) = \frac{4}{x^6} - \frac{3}{x^8} \quad , \quad \frac{1}{2} \le x \le 10$

Take a derivative of $f(x)$ with respect to x:

$$\frac{df}{dx} = \frac{d}{dx}\left(\frac{4}{x^6} - \frac{3}{x^8}\right) = \frac{d}{dx}(4x^{-6} - 3x^{-8})$$

$$= -24x^{-7} + 24x^{-9} = -\frac{24}{x^7} + \frac{24}{x^9}$$

Set the first derivative equal to zero. Solve for x. Call these values x_c.

$$\frac{df}{dx} = 0 \quad \to \quad -\frac{24}{x_c^7} + \frac{24}{x_c^9} = 0 \quad \to \quad \frac{24}{x_c^9} = \frac{24}{x_c^7}$$

$$\to \quad \frac{1}{x_c^9} = \frac{1}{x_c^7}$$

Multiply both sides of the equation by x_c^9:

$$1 = \frac{x_c^9}{x_c^7} \quad \to \quad 1 = x_c^2 \quad \to \quad \pm 1 = x_c$$

Note that $\frac{x_c^9}{x_c^7} = x_c^{9-7} = x_c^2$ because $\frac{x^m}{x^n} = x^{m-n}$. The reason that $x_c = -1$ and $x_c = 1$ both solve $x_c^2 = 1$ is that $(-1)^2 = 1$ and $1^2 = 1$. However, the problem specified the interval $\frac{1}{2} \leq x < 10$, and the only solution that satisfies this interval is $x_c = 1$. Take a second derivative of $f(x)$ with respect to x:

$$\frac{d^2 f}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}(-24x^{-7} + 24x^{-9})$$

$$= 168x^{-8} - 216x^{-10} = \frac{168}{x^8} - \frac{216}{x^{10}}$$

Evaluate the second derivative at $x_c = 1$ (which we found previously).

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = \frac{168}{1^8} - \frac{216}{1^{10}} = 168 - 216$$

$$= -48 \begin{pmatrix} \text{relative} \\ \text{maximum} \end{pmatrix}$$

There is a relative maximum at $x_c = 1$ (where the second derivative is negative). Evaluate the function at $x_c = 1$, and the endpoints ($x = \frac{1}{2}$ and $x = 10$).

$$f(1) = \frac{4}{1^6} - \frac{3}{1^8} = 4 - 3 = 1$$

$$f\left(\frac{1}{2}\right) = \frac{4}{\left(\frac{1}{2}\right)^6} - \frac{3}{\left(\frac{1}{2}\right)^8} = 4(2)^6 - 3(2)^8$$

$$= 256 - 768 = -512$$

To divide by a fraction, multiply by its reciprocal:

$\frac{4}{\left(\frac{1}{2}\right)^6} = 4\left(\frac{2}{1}\right)^6 = 4(2)^6 = 256.$

$$f(10) = \frac{4}{10^6} - \frac{3}{10^8} = 0.000004 - 0.00000003$$

$$= 0.00000397$$

Over the interval $\left(-\frac{1}{2}, 10\right)$, the absolute extrema are:

- $f(x)$ has an absolute maximum value of $\boxed{1}$ (when $x = 1$).
- $f(x)$ has an absolute minimum value of $\boxed{-512}$ (when $x = \frac{1}{2}$).

8 LIMITS AND L'HÔPITAL'S RULE

The following notation means, "What value does the function **f(x)** approach as the variable **x** approaches **c**?" The limit asks how the function behaves as **x** gets closer and closer to the value of **c**.

$$\lim_{x \to c} f(x)$$

A one-sided limit asks how the function behaves as **x** approaches **c** from a specified direction: A **+** sign means to let **x** approach **c** from the right (meaning to start with values of **x > c** and let **x** decrease towards **c**), whereas a **–** sign means to let **x** approach **c** from the left (meaning to start with values of **x < c** and let **x** increase towards **c**). The notation for one-sided limits is shown below.

$$\lim_{x \to c^+} f(x) \quad , \quad \lim_{x \to c^-} f(x)$$

When finding the limit of the ratio of two functions, if both functions approach zero (or if both grow to infinity), l'Hôpital's rule may help to evaluate the limit:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\left.\frac{df}{dx}\right|_{x=c}}{\left.\frac{dg}{dx}\right|_{x=c}} \qquad \begin{array}{c} \text{if } \lim_{x \to c} f(x) = 0 \text{ and } \lim_{x \to c} g(x) = 0 \\ \text{or} \\ \text{if } \lim_{x \to c} f(x) = \pm\infty \text{ and } \lim_{x \to c} g(x) = \pm\infty \end{array}$$

Example: Evaluate the following limit.

$$\lim_{x \to \infty} e^{-x}$$

Note that $e^{-x} = \frac{1}{e^x}$ according to the rule $x^{-n} = \frac{1}{x^n}$.

$$\lim_{x \to \infty} e^{-x} = \lim_{x \to \infty} \frac{1}{e^x}$$

Ask yourself, "What happens to $\frac{1}{e^x}$ as x grows very large?" Answer it in two steps:

- As x gets larger and larger, e^x gets much larger.
- As e^x gets much larger, $\frac{1}{e^x}$ gets much smaller.

The answer is zero: As x grows to infinity, $\frac{1}{e^x}$ approaches zero.

$$\lim_{x \to \infty} e^{-x} = 0$$

Example: Evaluate the following limit.

$$\lim_{x\to\infty} \frac{6x^3 - 8x^2}{2x^3 + 4x^2}$$

At first glance, it might seem necessary to apply l'Hôpital's rule, since $6x^3 - 8x^2$ and $2x^3 + 4x^2$ both grow indefinitely as x gets larger and larger. However, this problem can actually be solved by dividing both the numerator and denominator x^3.

$$\lim_{x\to\infty} \frac{6x^3 - 8x^2}{2x^3 + 4x^2} = \lim_{x\to\infty} \frac{\frac{6x^3}{x^3} - \frac{8x^2}{x^3}}{\frac{2x^3}{x^3} + \frac{4x^2}{x^3}} = \lim_{x\to\infty} \frac{6 - \frac{8}{x}}{2 + \frac{4}{x}}$$

Since $\frac{8}{x}$ and $\frac{4}{x}$ each approach zero as x grows to infinity, the limit simplifies to:

$$\lim_{x\to\infty} \frac{6 - \frac{8}{x}}{2 + \frac{4}{x}} = \frac{6 - 0}{2 + 0} = \frac{6}{2} = 3$$

Example: Evaluate the following limit.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

This problem requires l'Hôpital's rule because $\lim_{x \to 0} \sin x = 0$ and $\lim_{x \to 0} x = 0$. (Unlike the previous example, this problem can't be solved just by algebra.)

$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\left.\frac{d}{dx} \sin x\right|_{x=0}}{\left.\frac{d}{dx} x\right|_{x=0}} = \frac{\cos x|_{x=0}}{1} = \frac{\cos 0}{1} = \frac{1}{1}$$

$$= 1$$

Recall from trig that $\cos 0 = 1$. Note that $\frac{d}{dx} x = 1$ for any value of x.

Example: Evaluate the following limit.

$$\lim_{x \to 0} \frac{2x}{e^x - 1}$$

This problem requires l'Hôpital's rule because $\lim_{x \to 0} 2x = 0$ and $\lim_{x \to 0}(e^x - 1) = 1 - 1 = 0$.

$$\lim_{x \to 0} \frac{2x}{e^x - 1} = \frac{\left.\frac{d}{dx} 2x\right|_{x=0}}{\left.\frac{d}{dx}(e^x - 1)\right|_{x=0}} = \frac{2}{e^x|_{x=0}} = \frac{2}{e^0} = \frac{2}{1}$$

$$= 2$$

Recall that $e^0 = 1$. Note that $\frac{d}{dx} e^x = e^x$.

Chapter 8 Exercises

Directions: Evaluate each limit.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $\lim_{x \to 2} \frac{x^2 - 3x + 6}{x^2 + 3x - 2} =$

❷ $\lim_{x \to 0} \frac{\sqrt{144 - x^2}}{\sqrt{9 - x^2}} =$

❸ $\lim_{x \to 1} e^x \ln x =$

❹ $\lim_{x \to \pi} x \cos x =$

❺ $\lim_{x \to \infty} \frac{2x - 8}{6x - 4} =$

❻ $\lim_{x \to 4} \frac{x^2 - 16}{x - 4} =$

❼ $\lim_{x \to \infty} \frac{2x^2 - 3x}{4x^2 + 9x} =$

❽ $\lim_{x \to 1} \frac{\ln x}{x - 1} =$

Part B

❾ $\lim_{x \to 0} \frac{\tan x}{x} =$

❿ $\lim_{x \to \infty} \frac{2x^4 + 4x^2 + 6}{x^4 + 2x^2 + 3} =$

⓫ $\lim_{x \to \infty} \frac{x}{e^x} =$

⓬ $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} =$

⓭ $\lim_{x \to \infty} \frac{8x^5 - 3x^3}{2x^6 - 9x^2} =$

⓮ $\lim_{x \to 0} \frac{e^x - e^{-x}}{x} =$

⓯ $\lim_{x \to \frac{\pi}{2}} \frac{2x - \pi}{\cos x} =$

⓰ $\lim_{x \to \infty} \frac{\ln x}{x} =$

Chapter 8 Solutions

Part A

❶ $$\lim_{x\to 2}\frac{x^2-3x+6}{x^2+3x-2}=\frac{2^2-3(2)+6}{2^2+3(2)-2}=\frac{4-6+6}{4+6-2}$$
$$=\frac{4}{8}=\boxed{\frac{1}{2}}$$

Note that l'Hôpital's rule doesn't apply because neither the numerator nor the denominator approach 0 (or $\pm\infty$) in the specified limit.

❷ $$\lim_{x\to 0}\frac{\sqrt{144-x^2}}{\sqrt{9-x^2}}=\frac{\sqrt{144-0^2}}{\sqrt{9-0^2}}=\frac{\sqrt{144}}{\sqrt{9}}=\frac{12}{3}$$
$$=\boxed{4}$$

Note that l'Hôpital's rule doesn't apply because neither the numerator nor the denominator approach 0 (or $\pm\infty$) in the specified limit.

❸ $\lim_{x\to 1} e^x \ln x = e^1 \ln(1) = e(0) = \boxed{0}$

Note that $e^1 = e$ and $\ln(1) = 0$.

❹ $\lim_{x\to \pi} x\cos x = \pi\cos\pi = \pi(-1) = \boxed{-\pi}$

Note that $\cos\pi = \cos 180° = -1$.

❺ $$\lim_{x\to\infty}\frac{2x-8}{6x-4} = \lim_{x\to\infty}\frac{\frac{2x}{x}-\frac{8}{x}}{\frac{6x}{x}-\frac{4}{x}} = \lim_{x\to\infty}\frac{2-\frac{8}{x}}{6-\frac{4}{x}} = \frac{2-0}{6-0}$$

$$= \frac{2}{6} = \boxed{\frac{1}{3}}$$

Notes:

- We divided the numerator and denominator each by x in the first step.
- $\frac{8}{x}$ and $\frac{4}{x}$ each approach 0 as x grows larger and larger.
- Calculator check: If you plug in $x = 1000$, you will get $\frac{1992}{5996} \approx 0.33 \approx \frac{1}{3}$.

❻ $$\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x+4)(x-4)}{x-4}$$
$$= \lim_{x \to 4} (x+4) = 4 + 4 = \boxed{8}$$

Notes:

- $(x+4)(x-4) = x^2 + 4x - 4x - 16 = x^2 - 16$.
- We're exploring what happens to the function as x gets close to 4, but since x never quite reaches 4, we don't need to be worried about dividing by zero.
- Calculator check: If you plug in $x = 4.1$, you will get $\frac{0.81}{0.1} = 8.1 \approx 8$.

❼ $$\lim_{x\to\infty}\frac{2x^2-3x}{4x^2+9x}=\lim_{x\to\infty}\frac{\frac{2x^2}{x^2}-\frac{3x}{x^2}}{\frac{4x^2}{x^2}+\frac{9x}{x^2}}=\lim_{x\to\infty}\frac{2-\frac{3}{x}}{4+\frac{9}{x}}$$

$$=\frac{2-0}{4+0}=\frac{2}{4}=\boxed{\frac{1}{2}}$$

Notes:

- We divided the numerator and denominator each by x^2 in the first step.
- $\frac{3}{x}$ and $\frac{9}{x}$ each approach 0 as x grows larger and larger.
- Calculator check: If you plug in $x = 1000$, you will get $\frac{1{,}997{,}000}{4{,}009{,}000} \approx 0.5 = \frac{1}{2}$.

❽ $$\lim_{x\to 1}\frac{\ln x}{x-1} = \frac{\left.\frac{d}{dx}\ln x\right|_{x=1}}{\left.\frac{d}{dx}(x-1)\right|_{x=1}} = \frac{\left.\frac{1}{x}\right|_{x=1}}{1} = \frac{1/1}{1} = \frac{1}{1}$$

$$= \boxed{1}$$

Notes:

- We applied l'Hôpital's rule since $\lim_{x\to 1}\ln x = 0$ and $\lim_{x\to 1}(x-1) = 0$.
- $\frac{d}{dx}\ln(x) = \frac{1}{x}$ and $\frac{d}{dx}(x-1) = \frac{d}{dx}x - \frac{d}{dx}1 = 1 - 0 = 1$.

- Note that $\frac{d}{dx}x = 1$ for any value of x.
- Calculator check: If you plug in $x = 1.01$, you will get $\frac{0.009950331}{0.01} \approx 0.995 \approx 1$.

Part B

❾ $$\lim_{x\to 0}\frac{\tan x}{x}=\frac{\left.\frac{d}{dx}\tan x\right|_{x=0}}{\left.\frac{d}{dx}x\right|_{x=0}}=\frac{\left.\sec^2 x\right|_{x=0}}{1}$$

$$=\frac{\left.\frac{1}{\cos^2 x}\right|_{x=0}}{1}=\frac{\frac{1}{\cos^2 0}}{1}=\frac{1/1^2}{1}=\boxed{1}$$

Notes:

- We applied l'Hôpital's rule since $\lim_{x\to 0}\tan x=0$ and $\lim_{x\to 0}x=0$.
- $\frac{d}{dx}\tan x=\sec^2 x$ and $\frac{d}{dx}x=1$.
- $\sec x=\frac{1}{\cos x}$, $\cos 0=1$, and $\sec 0=1$.
- Calculator check: If you plug in $x=0.01$, you will get $\frac{0.01}{0.01}\approx 1$ (but first check that your calculator is in radians mode and not degrees mode).

⓾ $\lim_{x\to\infty} \frac{2x^4 + 4x^2 + 6}{x^4 + 2x^2 + 3} = \lim_{x\to\infty} \frac{\frac{2x^4}{x^4} + \frac{4x^2}{x^4} + \frac{6}{x^4}}{\frac{x^4}{x^4} + \frac{2x^2}{x^4} + \frac{3}{x^4}}$

$$= \lim_{x\to\infty} \frac{2 + \frac{4}{x^2} + \frac{6}{x^4}}{1 + \frac{2}{x^2} + \frac{3}{x^4}} = \frac{2 + 0 + 0}{1 + 0 + 0} = \boxed{2}$$

Notes:

- We divided the numerator and denominator each by x^4 in the first step.
- $\frac{4}{x^2}, \frac{6}{x^4}, \frac{2}{x^2}$, and $\frac{3}{x^4}$ each approach 0 as x grows larger and larger.
- Calculator check: If you plug in $x = 100$, you will get $\frac{200{,}040{,}006}{100{,}020{,}003} \approx 2$.

⓫ $\lim_{x\to\infty}\frac{x}{e^x} = \frac{\left.\frac{d}{dx}x\right|_{x\to\infty}}{\left.\frac{d}{dx}e^x\right|_{x\to\infty}} = \frac{1}{e^x|_{x\to\infty}} = \frac{1}{\lim_{x\to\infty} e^x} = \boxed{0}$

Notes:

- We applied l'Hôpital's rule since x and e^x both grow indefinitely as x gets larger and larger.
- $\frac{d}{dx}e^x = e^x$ and $\frac{d}{dx}x = 1$. Note that $\frac{d}{dx}x = 1$ for any value of x.
- As e^x grows larger and larger, $\frac{1}{e^x}$ gets closer and closer to zero.
- Calculator check: If you plug in $x = 20$, you will get $\frac{20}{485{,}165{,}195.4} \approx 0.000000041 \approx 0$.

⓬ $\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}=\lim\limits_{x\to 0}\dfrac{\sqrt{x+4}-2}{x}\dfrac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$

$$=\lim_{x\to 0}\frac{x+4-4}{x\left(\sqrt{x+4}+2\right)}$$

$$=\lim_{x\to 0}\frac{x}{x\left(\sqrt{x+4}+2\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}$$

$$=\frac{1}{\sqrt{0+4}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{2+2}=\boxed{\frac{1}{4}}$$

Notes:

- $\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)=\sqrt{x+4}\sqrt{x+4}+2\sqrt{x+4}-2\sqrt{x+4}-4$
  $=x+4-4=x$ for the same reason that $(a-b)(a+b)=a^2-b^2$.
- Although x approaches 0, since it never quite reaches 0 in the limit, we don't need to worry about dividing by zero when we cancel $\frac{x}{x}$.

- If instead you apply l'Hôpital's rule, you will get the same answer. Note that $\frac{d}{dx}\sqrt{x+4} = \frac{1}{2\sqrt{x+4}}$, $\frac{d}{dx}x = 1$, and $\lim\limits_{x\to 0}\frac{1}{2\sqrt{x+4}} = \frac{1}{2\sqrt{4}} = \frac{1}{2(2)} = \frac{1}{4}$.
- Calculator check: If you plug in $x = 0.01$, you will get $\frac{0.002498439}{0.01} \approx 0.25 = \frac{1}{4}$.

⓭ $$\lim_{x\to\infty}\frac{8x^5-3x^3}{2x^6-9x^2} = \lim_{x\to\infty}\frac{\frac{8x^5}{x^6}-\frac{3x^3}{x^6}}{\frac{2x^6}{x^6}-\frac{9x^2}{x^6}} = \lim_{x\to\infty}\frac{\frac{8}{x}-\frac{3}{x^3}}{2-\frac{9}{x^4}}$$

$$= \frac{0-0}{2-0} = \frac{0}{2} = \boxed{0}$$

Notes:

- We divided the numerator and denominator each by x^6 in the first step.
- $\frac{8}{x}$, $\frac{3}{x^3}$, and $\frac{9}{x^4}$ each approach 0 as x grows larger and larger.

- Conceptually, the denominator, which grows as x^6, dominates the numerator, which grows as x^5, causing the ratio to approach zero.
- Calculator check: If you plug in $x = 100$, you will get $\frac{7.9997\times10^{10}}{1.99999991\times10^{12}} \approx 0.04 \approx 0$.

⓮ $$\lim_{x\to0}\frac{e^x - e^{-x}}{x} = \frac{\frac{d}{dx}(e^x - e^{-x})\Big|_{x=0}}{\frac{d}{dx}x\Big|_{x=0}}$$

$$= \frac{(e^x + e^{-x})|_{x=0}}{1} = \frac{1+1}{1} = \frac{2}{1} = \boxed{2}$$

Notes:

- We applied l'Hôpital's rule since $\lim_{x\to 0}(e^x - e^{-x}) = 0$ and $\lim_{x\to 0} x = 0$.
- $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}e^{-x} = -e^{-x}$, and $\frac{d}{dx}x = 1$.
- $\frac{d}{dx}(e^x - e^{-x}) = \frac{d}{dx}e^x - \frac{d}{dx}e^{-x} = e^x - (-e^{-x}) = e^x + e^{-x}$.
- $e^0 = 1$.
- Calculator check: If you plug in $x = 0.01$, you will get $\frac{0.020000333}{0.01} \approx 2$.

⓯ $$\lim_{x\to\frac{\pi}{2}} \frac{2x-\pi}{\cos x} = \frac{\frac{d}{dx}(2x-\pi)\Big|_{x=\frac{\pi}{2}}}{\frac{d}{dx}\cos x\Big|_{x=\frac{\pi}{2}}} = \frac{2}{-\sin x|_{x=\frac{\pi}{2}}}$$

$$= \frac{2}{-\sin\frac{\pi}{2}} = \frac{2}{-1} = \boxed{-2}$$

Notes:

- We applied l'Hôpital's rule since $\lim_{x\to\frac{\pi}{2}}(2x - \pi) = 0$ and $\lim_{x\to\frac{\pi}{2}}\cos x = 0$.

- $\frac{d}{dx}\cos x = -\sin x$ and $\frac{d}{dx}(2x - \pi) = \frac{d}{dx}2x - \frac{d}{dx}\pi = 2 - 0 = 2$.
- $\sin\frac{\pi}{2} = \sin 90° = 1$ and $\cos\frac{\pi}{2} = \cos 90° = 0$.
- Calculator check: If you plug in $x = 1.6$, you will get $\frac{0.058407346}{-0.029199522} \approx -2$ (but first check that your calculator is in radians mode and not degrees mode).

⓰ $$\lim_{x\to\infty}\frac{\ln x}{x} = \frac{\frac{d}{dx}\ln x\Big|_{x\to\infty}}{\frac{d}{dx}x\Big|_{x\to\infty}} = \frac{\frac{1}{x}\Big|_{x\to\infty}}{1} = \frac{0}{1} = \boxed{0}$$

Notes:

- We applied l'Hôpital's rule since $\ln x$ and x both grow indefinitely as x gets larger and larger.
- $\frac{d}{dx}\ln x = \frac{1}{x}$ and $\frac{d}{dx}x = 1$. Note that $\frac{d}{dx}x = 1$ for any value of x.
- Calculator check: If you plug in $x = 1{,}000{,}000$, you will get $\frac{13.81551056}{1{,}000{,}000} \approx 0.000013816 \approx 0$.

9 INTEGRALS OF POLYNOMIALS

Given a polynomial term of the form **ax^b** (where **a** is a constant coefficient and **b** is a constant exponent), to perform an indefinite integral over the variable **x**, increase the exponent to **b + 1** and then divide by **b + 1** according to the following formulas, where **c** is a constant of integration:

$$\int ax^b \, dx = \frac{ax^{b+1}}{b+1} + c \quad (\text{if } b \neq -1)$$

$$\int ax^{-1} \, dx = \int \frac{a}{x} dx = a \ln x + c$$

Note that a few special exponents can make a polynomial look somewhat different:

- If you see a variable in a denominator, you may bring it to the numerator by negating its exponent according to the following rule:

$$\frac{1}{x^n} = x^{-n}$$

- If you see a squareroot, rewrite it using an exponent of ½:

$$\sqrt{x} = x^{1/2}$$

• If no coefficient or exponent is observed, the number **1** is implied:

$$x = 1x^1 \quad , \quad \frac{1}{x} = \frac{1}{x^1} = x^{-1}$$

It may help to recall the following rules of algebra regarding exponents:

$$x^m x^n = x^{m+n} \quad , \quad \frac{x^m}{x^n} = x^{m-n} \quad , \quad x^0 = 1 \quad , \quad (x^m)^n = x^{mn}$$

$$(ax)^m = a^m x^m \quad , \quad (a^m a^n)^p = a^{mp} x^{np} \quad , \quad \sqrt{ax} = (ax)^{1/2} = a^{1/2} x^{1/2}$$

If a polynomial includes multiple terms, integrate each term individually and then add the results together:

$$\int (y_1 + y_2 + \cdots + y_n)\, dx = \int y_1\, dx + \int y_2\, dx + \cdots + \int y_n\, dx$$

For example, if $\mathbf{y_1 = 3x^2}$, $\mathbf{y_2 = -2x}$, and $\mathbf{y_3 = 4}$, this means:

$$\int (3x^2 - 2x + 4)\, dx = \int 3x^2\, dx - \int 2x\, dx + \int 4\, dx$$

Tip: When you finish an indefinite integral, take a derivative to check your answer.

Example: Perform the following integral.

$$\int 12x^3\,dx$$

Compare $12x^3$ with the general form ax^b to see that the coefficient is $a = 12$ and the exponent is $b = 3$. Plug $a = 12$ and $b = 3$ into the formula $\frac{ax^{b+1}}{b+1} + c$.

$$\int 12x^3\,dx = \frac{12x^{3+1}}{3+1} + c = \frac{12x^4}{4} + c = 3x^4 + c$$

Check: Take a derivative of $3x^4 + c$ to check the answer.

$$\frac{d}{dx}(3x^4 + c) = 12x^3$$

Example: Perform the following integral.

$$\int \frac{6}{t^4}\,dt$$

Note that this derivative is over t (instead of the usual x). Simply replace x with t in the formula $\frac{ax^{b+1}}{b+1} + c$ to get the formula $\frac{at^{b+1}}{b+1} + c$. Rewrite $\frac{6}{t^4}$ as $6t^{-4}$ using the rule $t^{-n} = \frac{1}{t^n}$. Compare $6t^{-4}$ with the general form at^b to see that the coefficient is $a = 6$ and the exponent is $b = -4$. Plug $a = 6$ and $b = -4$ into the formula $\frac{at^{b+1}}{b+1} + c$.

$$\int \frac{6}{t^4}\,dt = \int 6t^{-4}\,dt = \frac{6t^{-4+1}}{-4+1} + c = \frac{6t^{-3}}{-3} + c$$

$$= -2t^{-3} + c$$

Check: Take a derivative of $2t^3 + c$ to check the answer.

$$\frac{d}{dt}\left(-2t^{-3} + c\right) = 6t^{-4} = \frac{6}{t^4}$$

Example: Perform the following integral.

$$\int \frac{2}{x} dx$$

Rewrite $\frac{2}{x}$ as $2x^{-1}$ using the rule $x^{-1} = \frac{1}{x}$. Compare $2x^{-1}$ with the general form ax^b to see that the coefficient is $a = 2$ and the exponent is $b = -1$. Note that $b = -1$ is a special case: This integral equals a natural logarithm.

$$\int \frac{2}{x} dx = \int 2x^{-1}\, dx = 2\ln x + c$$

Check: Take a derivative of $2\ln x + c$ to check the answer.

$$\frac{d}{dx}(2\ln x + c) = \frac{2}{x}$$

Example: Perform the following integral.

$$\int \sqrt{x}\,dx$$

Rewrite $\sqrt{x}$ as $x^{1/2}$ using the rule $x^{1/2} = \sqrt{x}$. Compare $x^{1/2}$ with the general form ax^b to see that the coefficient is $a = 1$ and the exponent is $b = 1/2$. Plug $a = 1$ and $b = 1/2$ into the formula $\frac{ax^{b+1}}{b+1} + c$.

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{1x^{1/2+1}}{1/2+1} + c = \frac{x^{3/2}}{3/2} + c$$

$$= \frac{2x^{3/2}}{3} + c$$

Note that $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$ (add fractions with a common denominator) and $\frac{1}{3/2} = 1 \div \frac{3}{2} = 1 \times \frac{2}{3} = \frac{2}{3}$ (to divide by a fraction, multiply by its reciprocal).

Check: Take a derivative of $\frac{2x^{3/2}}{3} + c$ to check the answer.

$$\frac{d}{dx}\left(\frac{2x^{3/2}}{3} + c\right) = \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)x^{1/2} = x^{1/2} = \sqrt{x}$$

Example: Perform the following integral.

$$\int (5u^4 + 9u^2)\, du$$

Since this expression has two terms ($5u^4$ and $9u^2$), we may integrate each term individually and then add the results together, where $y_1 = 5u^4$ and $y_2 = 9u^2$:

$$\int (y_1 + y_2)\, dx = \int y_1\, dx + \int y_2\, dx$$

Find the two individual anti-derivatives:

$$\int (5u^4 + 9u^2)\, du = \int 5u^4\, du + \int 9u^2\, du$$

$$= \frac{5u^{4+1}}{4+1} + \frac{9u^{2+1}}{2+1} + c = \frac{5u^5}{5} + \frac{9u^3}{3} + c$$

$$= u^5 + 3u^3 + c$$

Check: Take a derivative of $u^5 + 3u^3 + c$ to check the answer.

$$\frac{d}{du}(u^5 + 3u^3 + c) = 5u^4 + 9u^2$$

Chapter 9 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $\int 35x^6\,dx =$

❷ $\int \frac{63}{t^8}\,dt =$

❸ $\int 15x^{2/3}\,dx =$

❹ $\int 48u^{-5}\,du =$

Part B

❺ $\int t\,dt =$

❻ $\int \frac{dx}{\sqrt{x}} =$

❼ $\int 4\,du =$

❽ $\int \frac{8dx}{x^{1/3}} =$

Part C

❾ $$\int (x^2 - 3x + 4)\, dx =$$

❿ $$\int \left(\frac{1}{u^2} - \frac{4}{u}\right) du =$$

⓫ $$\int \left(10x^{3/2} + 6x^{1/2}\right) dx =$$

⓬ $$\int \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) dt =$$

Chapter 9 Solutions

Part A

❶ $$\int 35x^6\,dx = \frac{35x^{6+1}}{6+1} + c = \frac{35x^7}{7} + c$$

$$= \boxed{5x^7 + c}$$

Check your answer:

$$\frac{d}{dx}(5x^7 + c) = 35x^6$$

❷ $$\int \frac{63}{t^8}\,dt = \int 63t^{-8}\,dt = \frac{63t^{-8+1}}{-8+1} + c$$

$$= \frac{63t^{-7}}{-7} + c = -9t^{-7} + c = \boxed{-\frac{9}{t^7} + c}$$

Notes:

- $-8 + 1 = -7$ and $t^{-7} = \frac{1}{t^7}$.
- $-9t^{-7} + c$ and $-\frac{9}{t^7} + c$ are both correct answers.

Check your answer:

$$\frac{d}{dt}\left(-9t^{-7} + c\right) = 63t^{-8} = \frac{63}{t^8}$$

❸ $\int 15x^{2/3}\,dx = \dfrac{15x^{2/3+1}}{\frac{2}{3}+1} + c = \dfrac{15x^{5/3}}{5/3} + c$

$= \boxed{9x^{5/3} + c}$

Note that $\frac{2}{3} + 1 = \frac{2}{3} + \frac{3}{3} = \frac{2+3}{3} = \frac{5}{3}$ and $15 \div \frac{5}{3} = 15 \times \frac{3}{5} = \frac{45}{5} = 9$.

Check your answer:

$$\frac{d}{dx}\left(9x^{5/3} + c\right) = \left(\frac{5}{3}\right)9x^{2/3} = \frac{45}{3}x^{2/3} = 15x^{2/3}$$

❹ $$\int 48u^{-5}\,du = \frac{48u^{-5+1}}{-5+1} + c = \frac{48u^{-4}}{-4} + c$$

$$= -12u^{-4} + c = \boxed{-\frac{12}{u^4} + c}$$

Notes:

- $-5 + 1 = -4$ and $u^{-4} = \frac{1}{u^4}$.
- $-12u^{-4} + c$ and $-\frac{12}{u^4} + c$ are both correct answers.

Check your answer:

$$\frac{d}{du}(-12u^{-4} + c) = 48u^{-5}$$

Part B

❺ $\int t\,dt = \int 1t^1\,dt = \frac{1t^{1+1}}{1+1} + c = \boxed{\frac{t^2}{2} + c}$

Note that $1t^1 = t$. Check your answer:

$$\frac{d}{dt}\left(\frac{t^2}{2} + c\right) = \frac{2t}{2} = t$$

❻ $\int \frac{dx}{\sqrt{x}} = \int \frac{dx}{x^{1/2}} = \int x^{-1/2}\,dx = \frac{x^{-1/2+1}}{-\frac{1}{2}+1} + c$

$$= \frac{x^{1/2}}{1/2} + c = 2x^{1/2} + c = \boxed{2\sqrt{x} + c}$$

Notes:

- $\sqrt{x} = x^{1/2}$ and $x^{-1/2} = \frac{1}{x^{1/2}}$.
- $-\frac{1}{2} + 1 = -\frac{1}{2} + \frac{2}{2} = \frac{-1+2}{2} = \frac{1}{2}$ and $\frac{1}{1/2} = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2$.
- $2x^{1/2} + c$ and $2\sqrt{x} + c$ are both correct answers.

Check your answer:

$$\frac{d}{dx}(2x^{1/2}+c)=\left(\frac{1}{2}\right)2x^{-1/2}=x^{-1/2}=\frac{1}{x^{1/2}}=\frac{1}{\sqrt{x}}$$

❼ $$\int 4\,du=\int 4u^0\,du=\frac{4u^{0+1}}{0+1}+c=\frac{4u^1}{1}+c$$

$$=\boxed{4u+c}$$

Note that $u^0=1$ and $u^1=u$. Check your answer:

$$\frac{d}{du}(4u+c)=4$$

❽ $$\int\frac{8dx}{x^{1/3}}=\int 8x^{-1/3}\,dx=\frac{8x^{-1/3+1}}{-\frac{1}{3}+1}+c$$

$$=\frac{8x^{2/3}}{2/3}+c=\boxed{12x^{2/3}+c}$$

Notes:

- $x^{-1/3}=\frac{1}{x^{1/3}}$.
- $-\frac{1}{3}+1=-\frac{1}{3}+\frac{3}{3}=\frac{-1+3}{3}=\frac{2}{3}$ and $8\div\frac{2}{3}=8\times\frac{3}{2}=\frac{24}{2}=12$.

Check your answer:

$$\frac{d}{dx}\left(12x^{2/3} + c\right) = \left(\frac{2}{3}\right)12x^{2/3-1} = \frac{24}{3}x^{-1/3}$$

$$= 8x^{-1/3} = \frac{8}{x^{1/3}}$$

Part C

❾ $\int (x^2 - 3x + 4)\,dx$

$$= \int x^2\,dx - \int 3x\,dx + \int 4\,dx$$

$$= \boxed{\frac{x^3}{3} - \frac{3x^2}{2} + 4x + c}$$

Check your answer:

$$\frac{d}{dx}\left(\frac{x^3}{3} - \frac{3x^2}{2} + 4x + c\right) = \frac{3x^2}{3} - \frac{6x}{2} + 4$$

$$= x^2 - 3x + 4$$

❿ $\int \left(\frac{1}{u^2} - \frac{4}{u}\right) du = \int (u^{-2} - 4u^{-1})\,du$

$$= \int u^{-2}\,du - \int 4u^{-1}\,du$$

$$= \frac{u^{-2+1}}{-2+1} - 4\ln u + c = \frac{u^{-1}}{-1} - 4\ln u + c$$

$$= -u^{-1} - 4\ln u + c = \boxed{-\frac{1}{u} - 4\ln u + c}$$

Notes:

- $\frac{1}{u^2} = u^{-2}$ and $\frac{1}{u} = u^{-1}$.
- $-2 + 1 = -1$.
- Recall that a power of $b = -1$ is a special case: $\int \frac{du}{u} = \int u^{-1}\, du = \ln u + c$ (whereas $\int au^b\, du = \frac{au^{b+1}}{b+1} + c$ when $b \neq -1$).

Check your answer:

$$\frac{d}{du}(-u^{-1} - 4\ln u + c) = u^{-2} - \frac{4}{u} = \frac{1}{u^2} - \frac{4}{u}$$

⓫ $\int \left(10x^{3/2} + 6x^{1/2}\right) dx$

$$= \int 10x^{3/2}\, dx + \int 6x^{1/2}\, dx$$

$$= \frac{10x^{5/2}}{5/2} + \frac{6x^{3/2}}{3/2} + c$$

$$= \frac{20x^{5/2}}{5} + \frac{12x^{3/2}}{3} + c = \boxed{4x^{5/2} + 4x^{3/2} + c}$$

Notes:

- $\frac{3}{2}+1=\frac{3}{2}+\frac{2}{2}=\frac{3+2}{2}=\frac{5}{2}$ and $\frac{1}{2}+1=\frac{1}{1}+\frac{2}{2}=\frac{1+2}{2}=\frac{3}{2}$.
- $10\div\frac{5}{2}=10\times\frac{2}{5}=\frac{20}{5}=4$ and $6\div\frac{3}{2}=6\times\frac{2}{3}=\frac{12}{3}=4$.

Check your answer:

$$\frac{d}{dx}\left(4x^{5/2}+4x^{3/2}+c\right)$$

$$=\left(\frac{5}{2}\right)(4)x^{3/2}+\left(\frac{3}{2}\right)(4)x^{1/2}$$

$$=\frac{20}{2}x^{3/2}+\frac{12}{2}x^{1/2}$$

$$=10x^{3/2}+6x^{1/2}$$

⓬ $$\int\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)dt=\int\left(t^{1/2}+\frac{1}{t^{1/2}}\right)dt$$

$$=\int\left(t^{1/2}+t^{-1/2}\right)dt$$

$$=\int t^{1/2}\,dt+\int t^{-1/2}\,dt=\frac{t^{1/2+1}}{\frac{1}{2}+1}+\frac{t^{-1/2+1}}{-\frac{1}{2}+1}+c$$

$$=\frac{t^{3/2}}{3/2}+\frac{t^{1/2}}{1/2}+c=\boxed{\frac{2t^{3/2}}{3}+2t^{1/2}+c}$$

$$=\frac{2t\sqrt{t}}{3}+2\sqrt{t}+c$$

Notes:

- $\frac{1}{2}+1=\frac{1}{1}+\frac{2}{2}=\frac{1+2}{2}=\frac{3}{2}$ and $-\frac{1}{2}+1=-\frac{1}{1}+\frac{2}{2}=\frac{-1+2}{2}=\frac{1}{2}$.
- $1\div\frac{3}{2}=1\times\frac{2}{3}=\frac{2}{3}$ and $1\div\frac{1}{2}=1\times\frac{2}{1}=2$. Note that $t^{3/2}=t^1t^{1/2}=t\sqrt{t}$.

Check your answer:

$$\frac{d}{dt}\left(\frac{2t^{3/2}}{3}+2t^{1/2}+c\right)$$

$$=\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)t^{1/2}+\left(\frac{1}{2}\right)(2)t^{-1/2}$$

$$=t^{1/2}+t^{-1/2}=\sqrt{t}+\frac{1}{\sqrt{t}}$$

10 DEFINITE INTEGRALS

First, we will discuss indefinite integrals, and then we will explore definite integrals. An indefinite integral basically equals the antiderivative of a function, including a constant of integration. For example, if **f(x)** equals a derivative of **g(x)** with respect to **x**, then **g(x)** is the antiderivative of **f(x)**, apart from a constant of integration:

$$f(x) = \frac{d}{dx} g(x) \quad , \quad \int f(x)\, dx = g(x) + c$$

To help make this clear, we will rewrite the above equations using **g(x) = x^4**.

$$4x^3 = \frac{d}{dx}(x^4) \quad , \quad \int 4x^3\, dx = x^4 + c$$

In the above equations:

- **f(x) = $4x^3$** is the derivative of **g(x) = x^4** with respect to **x**.
- **g(x) = x^4** is the antiderivative of **f(x) = $4x^3$**. An indefinite integral is the opposite of a derivative.

A definite integral includes limits, specifying an interval, **(a,b)**, as shown below.

- $\int_{x=a}^{b} f(x)\, dx$ is an example of a definite integral.
- $\int f(x)\, dx$ is an example of an indefinite integral.

The fundamental theorem of calculus helps to evaluate a definite integral:

$$\int_{x=a}^{b} f(x)\,dx$$

1. To evaluate the definite integral above, first find the antiderivative, **g(x)**, of the function specified in the integrand, **f(x)**.
2. Next, evaluate the antiderivative at the upper and lower limits, **g(b)** and **g(a)**.
3. Subtract the answers from step 2, as illustrated in the examples that follow.

$$\int_{x=a}^{b} f(x)\,dx = g(b) - g(a)$$

Note: It isn't necessary to include the constant of integration in the antiderivative when performing a definite integral because it will cancel out in step 3.

Example: Perform the following integral.

$$\int_{x=1}^{2} 9x^2\,dx$$

First find the antiderivative of $9x^2$.

$$\int_{x=1}^{2} 9x^2\,dx = \left[\frac{9x^{2+1}}{2+1}\right]_{x=1}^{2} = \left[\frac{9x^3}{3}\right]_{x=1}^{2} = [3x^3]_{x=1}^{2}$$

The notation $[3x^3]_{x=1}^{2}$ means to evaluate the antiderivative, $3x^3$, at each limit and subtract:

$$[3x^3]_{x=1}^{2} = 3(2)^3 - 3(1)^3 = 3(8) - 3(1)$$
$$= 24 - 3 = 21$$

Check the antiderivative: Take a derivative of $3x^3$ to check the antiderivative.

$$\frac{d}{dx}(3x^3) = 9x^2$$

Example: Perform the following integral.

$$\int_{x=4}^{12} \frac{dx}{x^2}$$

First find the antiderivative of $\frac{1}{x^2}$, noting that $x^{-2} = \frac{1}{x^2}$.

$$\int_{x=4}^{12} \frac{dx}{x^2} = \int_{x=4}^{12} x^{-2}\,dx = \left[\frac{x^{-2+1}}{-2+1}\right]_{x=4}^{12} = \left[\frac{x^{-1}}{-1}\right]_{x=4}^{12}$$

$$= [-x^{-1}]_{x=4}^{12} = \left[-\frac{1}{x}\right]_{x=4}^{12}$$

The notation $\left[-\frac{1}{x}\right]_{x=4}^{12}$ means to evaluate the antiderivative, $-\frac{1}{x}$, at each limit and subtract:

$$\left[-\frac{1}{x}\right]_{x=4}^{12} = -\frac{1}{12} - \left(-\frac{1}{4}\right) = -\frac{1}{12} + \frac{1}{4} = -\frac{1}{12} + \frac{3}{12}$$

$$= \frac{-1+3}{12} = \frac{2}{12} = \frac{1}{6} \approx 0.167$$

Note that $-\left(-\frac{1}{4}\right) = \frac{1}{4}$. Two minus signs make a plus sign.

Check the antiderivative: Take a derivative of $-x^{-1}$ to check the antiderivative.

$$\frac{d}{dx}(-x^{-1}) = (-1)(-x^{-2}) = x^{-2} = \frac{1}{x^2}$$

Chapter 10 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\int_{x=1}^{2} 8x^3\,dx =$$

❷ $$\int_{x=3}^{6} \frac{dx}{x^3} =$$

❸ $$\int_{t=4}^{9} \frac{dt}{\sqrt{t}} =$$

❹ $$\int_{x=-2}^{2} (x^3 - 6x^2 + 4x - 8)\,dx =$$

Part B

❺ $$\int_{x=0}^{5} \frac{x^4}{5}\,dx =$$

❻ $$\int_{t=-3}^{6} (t^2 - 2)\,dt =$$

❼ $$\int_{x=1}^{4} \sqrt{x}\,dx =$$

❽ $$\int_{x=1}^{e} \frac{dx}{x} =$$

Chapter 10 Solutions

Part A

❶ $$\int_{x=1}^{2} 8x^3\,dx = \left[\frac{8x^{3+1}}{3+1}\right]_{x=1}^{2} = \left[\frac{8x^4}{4}\right]_{x=1}^{2}$$

$$= [2x^4]_{x=1}^{2} = 2(2)^4 - 2(1)^4$$

$$= 2(16) - 2(1) = 32 - 2 = \boxed{30}$$

Check your antiderivative (by taking a derivative):

$$\frac{d}{dx} 2x^4 = 8x^3$$

❷ $$\int_{x=3}^{6} \frac{dx}{x^3} = \int_{x=3}^{6} x^{-3}\,dx = \left[\frac{x^{-3+1}}{-3+1}\right]_{x=3}^{6}$$

$$= \left[\frac{x^{-2}}{-2}\right]_{x=3}^{6} = \left[-\frac{x^{-2}}{2}\right]_{x=3}^{6} = \left[-\frac{1}{2x^2}\right]_{x=3}^{6}$$

$$= -\frac{1}{2(6^2)} - \left[-\frac{1}{2(3^2)}\right] = -\frac{1}{72} + \frac{1}{18} = -\frac{1}{72} + \frac{4}{72}$$

$$= \frac{-1+4}{72} = \frac{3}{72} = \boxed{\frac{1}{24}} \approx \boxed{0.0417}$$

Check your antiderivative (by taking a derivative):

$$\frac{d}{dx}\left(-\frac{x^{-2}}{2}\right) = (-2)\left(-\frac{1}{2}\right)x^{-3} = x^{-3} = \frac{1}{x^3}$$

❸ $$\int_{t=4}^{9} \frac{dt}{\sqrt{t}} = \int_{t=4}^{9} \frac{dt}{t^{1/2}} = \int_{t=4}^{9} t^{-1/2}\, dt = \left[\frac{t^{-1/2+1}}{-\frac{1}{2}+1}\right]_{t=4}^{9}$$

$$= \left[\frac{t^{1/2}}{1/2}\right]_{t=4}^{9} = \left[2t^{1/2}\right]_{t=4}^{9}$$

$$= \left[2\sqrt{t}\right]_{t=4}^{9} = 2\sqrt{9} - 2\sqrt{4} = 2(3) - 2(2) =$$

$$6 - 4 = \boxed{2}$$

Check your antiderivative (by taking a derivative):

$$\frac{d}{dt}\left(2t^{1/2}\right) = \left(\frac{1}{2}\right)(2)t^{-1/2} = t^{-1/2} = \frac{1}{t^{1/2}} = \frac{1}{\sqrt{t}}$$

❹ $$\int_{x=-2}^{2} (x^3 - 6x^2 + 4x - 8)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{4x^2}{2} - 8x\right]_{x=-2}^{2}$$

$$= \left[\frac{x^4}{4} - 2x^3 + 2x^2 - 8x\right]_{x=-2}^{2}$$

$$= \frac{2^4}{4} - 2(2)^3 + 2(2)^2 - 8(2)$$

$$- \left[\frac{(-2)^4}{4} - 2(-2)^3 + 2(-2)^2 - 8(-2)\right]$$

$$= \frac{16}{4} - 2(8) + 2(4) - 16$$

$$- \left[\frac{16}{4} - 2(-8) + 2(4) + 16\right]$$

$$= 4 - 16 + 8 - 16 - (4 + 16 + 8 + 16)$$

$$= -20 - (44) = -20 - 44 = \boxed{-64}$$

Check your antiderivative (by taking a derivative):

$$\frac{d}{dx}\left(\frac{x^4}{4} - \frac{6x^3}{3} + \frac{4x^2}{2} - 8x\right)$$

$$= \frac{4x^3}{4} - \frac{(3)(6)x^2}{3} + \frac{(2)(4)x}{2} - 8$$

$$= x^3 - 6x^2 + 4x - 8$$

Part B

❺ $$\int_{x=0}^{5} \frac{x^4}{5}\,dx = \frac{1}{5}\int_{x=0}^{5} x^4\,dx = \frac{1}{5}\left[\frac{x^{4+1}}{4+1}\right]_{x=0}^{5}$$

$$= \frac{1}{5}\left[\frac{x^5}{5}\right]_{x=0}^{5} = \frac{1}{5}\left(\frac{5^5}{5} - \frac{0^5}{5}\right) = \frac{1}{5}\left(\frac{5^5}{5} - 0\right)$$

$$= \frac{1}{5}\left(\frac{5^5}{5}\right) = \frac{5^5}{5(5)} = \frac{5^5}{5^2} = 5^{5-2} = 5^3 = \boxed{125}$$

Check your antiderivative (by taking a derivative):

$$\frac{d}{dx}\left(\frac{1}{5}\frac{x^5}{5}\right) = \frac{5x^4}{25} = \frac{x^4}{5}$$

❻ $$\int_{t=-3}^{6} (t^2 - 2)\,dt = \left[\frac{t^3}{3} - 2t\right]_{t=-3}^{6}$$

$$= \frac{6^3}{3} - 2(6) - \left[\frac{(-3)^3}{3} - 2(-3)\right]$$

$$= \frac{216}{3} - 12 - \left(-\frac{27}{3} + 6\right) = 72 - 12 - (-9 + 6)$$

$$= 60 - (-3) = 60 + 3 = \boxed{63}$$

Check your antiderivative (by taking a derivative):

$$\frac{d}{dt}\left(\frac{t^3}{3} - 2t\right) = \frac{3t^2}{3} - 2 = t^2 - 2$$

❼ $$\int_{x=1}^{4} \sqrt{x}\,dx = \int_{x=1}^{4} x^{1/2}\,dx = \left[\frac{x^{1/2+1}}{\frac{1}{2}+1}\right]_{x=1}^{4}$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_{x=1}^{4} = \left[\frac{2x^{3/2}}{3}\right]_{x=1}^{4}$$

$$= \frac{2(4)^{3/2}}{3} - \frac{2(1)^{3/2}}{3} = \frac{2(8)}{3} - \frac{2(1)}{3} = \frac{16}{3} - \frac{2}{3}$$

$$= \frac{16-2}{3} = \boxed{\frac{14}{3}} \approx \boxed{4.667}$$

Note that $(4)^{3/2} = (4^3)^{1/2} = 64^{1/2} = \sqrt{64} = 8$, or use a calculator to verify that $4\text{^}(3/2) = 8$.

Check your antiderivative (by taking a derivative):

$$\frac{d}{dx}\left(\frac{2x^{3/2}}{3}\right) = \left(\frac{3}{2}\right)\left(\frac{2}{3}\right)x^{1/2} = x^{1/2} = \sqrt{x}$$

❽ $\displaystyle\int_{x=1}^{e} \frac{dx}{x} = [\ln x]_{x=1}^{e} = \ln e - \ln 1 = 1 - 0 = \boxed{1}$

Notes:

- Recall from Chapter 9 that $\int \frac{dx}{x} = \int x^{-1}\,dx = \ln x + c$ (but, as usual, the constant of integration doesn't matter in a definite integral because it would cancel out during the subtraction after plugging in the limits).
- Recall from Chapter 5 that $\ln e = 0$ and $\ln 1 = 0$.
- We first encountered Euler's number, e, in Chapter 4.

Check your antiderivative (it may help to review Chapter 5):

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

11 INTEGRALS OF TRIG FUNCTIONS

The integrals of the basic trig functions are:

$$\int \sin\theta \, d\theta = -\cos\theta + c$$

$$\int \cos\theta \, d\theta = \sin\theta + c$$

$$\int \tan\theta \, d\theta = \ln|\sec\theta| + c$$

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$\int \cot\theta \, d\theta = \ln|\sin\theta| + c$$

$$\int \csc\theta \, d\theta = -\ln|\csc\theta + \cot\theta| + c$$

Recall the following trigonometric relations and identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad , \quad \csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad , \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta \quad , \quad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad , \quad \cos 2\theta$$

$$= \cos^2\theta - \sin^2\theta$$

$$\pi \text{ radians} = 180°$$

Example: Perform the following integral.

$$\int_{\theta=0}^{\pi/2} \cos\theta \, d\theta$$

First find the antiderivative of $\cos\theta$.

$$\int_{\theta=0}^{\pi/2} \cos\theta \, d\theta = [\sin\theta]_{\theta=0}^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin 0$$

$$= 1 - 0 = 1$$

Recall from trig that:

- $\frac{\pi}{2} rad = \frac{\pi}{2}\frac{180°}{\pi} = 90°$ and $\sin\left(\frac{\pi}{2}\right) = \sin(90°) =$ 1.
- $\sin 0 = 0$.

Check the antiderivative: Take a derivative of $\sin\theta$ to check the antiderivative (it may help to review Chapter 3):

$$\frac{d}{d\theta}\sin\theta = \cos\theta$$

Chapter 11 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

❶ $$\int_{\theta=0}^{\pi} \sin\theta \, d\theta =$$

❷ $$\int_{\theta=0}^{\pi/3} \tan\theta \, d\theta =$$

❸ $$\int_{\theta=-\pi/6}^{\pi/6} \sec\theta \, d\theta =$$

❹ $$\int_{\theta=0}^{\pi/3} (\theta + \cos\theta) \, d\theta =$$

Chapter 11 Solutions

❶ $$\int_{\theta=0}^{\pi} \sin\theta \, d\theta = [-\cos\theta]_{\theta=0}^{\pi}$$
$$= -\cos\pi - (-\cos 0) = -(-1) + 1$$
$$= 1 + 1 = \boxed{2}$$

Notes:

- π rad $= 180°$ and $\cos(\pi) = \cos(180°) = -1$.
- $\cos 0 = 1$.
- $-\cos\pi = -(-1) = 1$ and $-(-\cos 0) = \cos 0 = 1$.

Check your antiderivative (it may help to review Chapter 3):

$$\frac{d}{d\theta}(-\cos\theta) = -\frac{d}{d\theta}\cos\theta = -(-\sin\theta) = \sin\theta$$

❷ $\displaystyle\int_{\theta=0}^{\pi/3} \tan\theta \, d\theta = [\ln|\sec\theta|]_{\theta=0}^{\frac{\pi}{3}}$

$$= \ln\left|\sec\frac{\pi}{3}\right| - \ln|\sec 0| = \ln 2 - \ln 1$$

$$\ln 2 - 0 = \boxed{\ln 2} \approx \boxed{0.693}$$

Notes:

- $\frac{\pi}{3}$ rad $= \frac{\pi}{3}\frac{180°}{\pi} = 60°$, $\cos\left(\frac{\pi}{3}\right) = \cos(60°) = \frac{1}{2}$, and $\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{1/2} = 2$.
- $\cos 0 = 1$ and $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$.
- $\ln 1 = 0$. Use a calculator to determine that $\ln 2 \approx 0.693147181$.

❸ $$\int_{\theta=-\frac{\pi}{6}}^{\pi/6} \sec\theta\, d\theta = [\ln|\sec\theta + \tan\theta|]_{\theta=-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \ln\left|\sec\frac{\pi}{6} + \tan\frac{\pi}{6}\right| - \ln\left|\sec\left(-\frac{\pi}{6}\right) + \tan\left(-\frac{\pi}{6}\right)\right|$$

$$= \ln\left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln\left|\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right|$$

$$= \ln\left|\frac{2+1}{\sqrt{3}}\right| - \ln\left|\frac{2-1}{\sqrt{3}}\right|$$

$$= \ln\left|\frac{3}{\sqrt{3}}\right| - \ln\left|\frac{1}{\sqrt{3}}\right|$$

$$= \ln\left|\frac{3}{\sqrt{3}} \div \frac{1}{\sqrt{3}}\right| = \ln\left|\frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{1}\right| = \boxed{\ln 3} \approx \boxed{1.099}$$

Notes:

- $\frac{\pi}{6}$ rad $= \frac{\pi}{6}\frac{180°}{\pi} = 30°$.
- $\cos\left(\frac{\pi}{6}\right) = \cos(30°) = \frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{\pi}{6}\right) = \cos(-30°) = \frac{\sqrt{3}}{2}$.
- $\sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$ and $\sec\left(-\frac{\pi}{6}\right) = \frac{1}{\cos\left(-\frac{\pi}{6}\right)} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$.
- $\tan\left(\frac{\pi}{6}\right) = \tan(30°) = \frac{1}{\sqrt{3}}$ and $\tan\left(-\frac{\pi}{6}\right) = \tan(-30°) = -\frac{1}{\sqrt{3}}$.

- Are you wondering if $\tan(30°) = \frac{\sqrt{3}}{3}$ instead of $\tan(30°) = \frac{1}{\sqrt{3}}$? You shouldn't be wondering this because both are the same. That's because $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$, as you can see by rationalizing the denominator: $\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

- Note that $\sec\left(-\frac{\pi}{6}\right)$ is positive whereas $\tan\left(-\frac{\pi}{6}\right)$ is negative. That's because secant is an even function (symmetric about the vertical axis) whereas tangent is an odd function (with an anti-symmetric graph).
- How does $\ln\left|\frac{3}{\sqrt{3}}\right| - \ln\left|\frac{1}{\sqrt{3}}\right|$ equal $\ln\left|\frac{3}{\sqrt{3}} \div \frac{1}{\sqrt{3}}\right|$? The way to see this is to apply the property of logarithms that $\ln\left(\frac{p}{q}\right) = \ln p - \ln q$, setting $p = \frac{3}{\sqrt{3}}$ and $q = \frac{1}{\sqrt{3}}$.
- $\frac{3}{\sqrt{3}} \div \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{1} = 3$. Recall that the way to divide two fractions is to multiply by the reciprocal of the second fraction. The reciprocal of $\frac{1}{\sqrt{3}}$ is $\frac{\sqrt{3}}{1}$.
- Use a calculator to determine that $\ln 3 \approx 1.098612289$.

❹ $$\int_{\theta=0}^{\pi/3} (\theta + \cos\theta)\, d\theta = \int_{\theta=0}^{\pi/3} \theta\, d\theta + \int_{\theta=0}^{\pi/3} \cos\theta\, d\theta$$

$$= \left[\frac{\theta^2}{2}\right]_{\theta=0}^{\frac{\pi}{3}} + [\sin\theta]_{\theta=0}^{\frac{\pi}{3}}$$

$$= \frac{\left(\frac{\pi}{3}\right)^2}{2} - \frac{0^2}{2} + \sin\left(\frac{\pi}{3}\right) - \sin 0$$

$$= \frac{\left(\frac{\pi^2}{9}\right)}{2} - 0 + \frac{\sqrt{3}}{2} - 0 = \boxed{\frac{\pi^2}{18} + \frac{\sqrt{3}}{2}}$$

$$\approx 0.548311356 + 0.866025404 \approx \boxed{1.414}$$

Notes:

- The given integral is like $\int (y_1 + y_2)\, dx = \int y_1\, dx + \int y_2\, dx$, with $y_1 = \theta$ and $y_2 = \cos\theta$.
- The first integral has the same form as $\int x\, dx = \frac{x^2}{2}$, with θ in place of x.
- $\frac{\pi}{3}$ rad $= \frac{\pi}{3}\frac{180°}{\pi} = 60°$ and $\sin\left(\frac{\pi}{3}\right) = \sin(60°) = \frac{\sqrt{3}}{2}$.

- $\sin 0 = 0$.
- $\left(\frac{\pi}{3}\right)^2 = \frac{\pi^2}{9}$ because $\left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2}$.
- $\frac{\left(\frac{\pi^2}{9}\right)}{2} = \frac{\pi^2}{9} \div 2 = \frac{\pi^2}{9} \div \frac{2}{1} = \frac{\pi^2}{9} \times \frac{1}{2} = \frac{\pi^2}{18}$. To divide by the fraction $\frac{2}{1}$, multiply by its reciprocal, which is $\frac{1}{2}$.
- We used a calculator to get the numerical values at the end of the solution.

Check your antiderivative (it may help to review Chapter 3):

$$\frac{d}{d\theta}\left(\frac{\theta^2}{2} + \sin\theta\right) = \frac{d}{d\theta}\frac{\theta^2}{2} + \frac{d}{d\theta}\sin\theta = \frac{2\theta}{2} + \cos\theta$$

$$= \theta + \cos\theta$$

12 INTEGRALS OF EXPONENTIALS AND LOGARITHMS

The integrals of the basic exponential, logarithm, and power functions are:

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int b^{x}\,dx = \frac{b^{x}}{\ln b} + c$$

The integrals of the hyperbolic functions (**<u>not</u>** the ordinary trig functions) are:

$$\int \sinh x \, dx = \cosh x + c$$

$$\int \cosh x \, dx = \sinh x + c$$

$$\int \tanh x \, dx = \ln|\cosh x| + c$$

Recall the following relations and identities:

$$e = 2.718281828 \ldots \quad , \quad e^{-x} = \frac{1}{e^x} \quad , \quad e^0 = 1 \quad , \quad \ln(e) = 1 \quad , \quad \ln(1) = 0$$

$$\ln(xy) = \ln x + \ln y \quad , \quad \ln\left(\frac{x}{y}\right) = \ln x - \ln y \quad , \quad \ln\left(\frac{1}{x}\right) = \ln(x^{-1}) = -\ln x$$

$$\ln(x^a) = a \ln x \quad , \quad \ln(e^x) = x \quad , \quad e^{\ln x} = x \quad , \quad \log_b x = \frac{\ln x}{\ln b}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad , \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad , \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1 \quad , \quad \cosh(-x) = \cosh x \quad , \quad \sinh(-x) = -\sinh(x)$$

$$\sinh 0 = 0 \quad , \quad \cosh 0 = 1 \quad , \quad \tanh 0 = 0$$

Example: Perform the following integral.

$$\int_{x=0}^{1} \sinh x \, dx$$

First find the antiderivative of $\sinh x$.

$$\int_{x=0}^{1} \sinh x \, dx = [\cosh x]_{x=0}^{1}$$

$$= \cosh(1) - \cosh 0$$

$$\approx 1.543 - 1 \approx 0.543$$

Note that $\cosh x = \frac{e^x + e^{-x}}{2}$. This is a hyperbolic function (**<u>not</u>** the cosine function from trig). We used a calculator to get the numerical values at the end of the solution.

Chapter 12 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

❶ $$\int_{x=0}^{\infty} e^{-x}\,dx =$$

❷ $$\int_{x=1}^{e} \ln x\,dx =$$

❸ $$\int_{t=1}^{2} \cosh t\,dt =$$

❹ $$\int_{x=3}^{4} 2^{x}\,dx =$$

Chapter 12 Solutions

❶ $$\int_{x=0}^{\infty} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_{x=0}^{\infty} = [-e^{-x}]_{x=0}^{\infty}$$

$$= \left[-\frac{1}{e^{x}}\right]_{x=0}^{\infty} = -\lim_{L\to\infty}\frac{1}{e^{L}} - \left(-\frac{1}{e^{0}}\right)$$

$$= -0 + \frac{1}{1} = \boxed{1}$$

This is an 'improper' integral: Imagine that the upper limit is $x = L$, such that the integral is $\int_{x=0}^{L} e^{-x}\,dx$. Integration gives $\left[-\frac{1}{e^{x}}\right]_{x=0}^{L} = -\frac{1}{e^{L}} - \left(-\frac{1}{e^{0}}\right)$. In the limit that L grows to ∞, the expression $\frac{1}{e^{L}}$ approaches zero.

❷ $$\int_{x=1}^{e} \ln x \, dx = [x \ln x - x]_{x=1}^{e}$$

$$= e \ln e - e - (1 \ln 1 - 1)$$

$$= e(1) - e - [1(0) - 1]$$

$$= e - e - (0 - 1) = e - e - (-1) = e - e + 1$$

$$= 0 + 1 = \boxed{1}$$

Notes:

- $\int \ln x \, dx = x \ln x - x + c$ (but as usual, we may ignore the constant of integration when performing a definite integral, since it would cancel out during the subtraction after plugging in the limits).
- $\ln e = 1$.
- $\ln 1 = 0$.

Check your antiderivative (it may help to review the product rule from Chapter 2 in addition to Chapter 5):

$$\frac{d}{dx}(x\ln x - x) = \frac{d}{dx}(x\ln x) - \frac{d}{dx}x$$

$$= \left(x\frac{d}{dx}\ln x + \ln x\frac{d}{dx}\right) - \frac{d}{dx}x$$

$$= x\left(\frac{1}{x}\right) + \ln x\,(1) - (1) = 1 + \ln x - 1 = \ln x$$

❸ $$\int_{t=1}^{2} \cosh t\,dt = [\sinh t]_{t=1}^{2} = \boxed{\sinh 2 - \sinh 1}$$

$$= \frac{e^2 - e^{-2}}{2} - \frac{e^1 - e^{-1}}{2}$$

$$\approx 3.626860408 - 1.175201194 \approx \boxed{2.452}$$

Notes:

- $\sinh x = \frac{e^x - e^{-x}}{2}$. This is a hyperbolic function (**<u>not</u>** the sine function from trig).
- We used a calculator to get the numerical values at the end of the solution.

❹ $$\int_{x=3}^{4} 2^x \, dx = \left[\frac{2^x}{\ln 2}\right]_{x=3}^{4} = \frac{2^4}{\ln 2} - \frac{2^3}{\ln 2} = \frac{2^4 - 2^3}{\ln 2}$$

$$= \frac{16 - 8}{\ln 2} = \boxed{\frac{8}{\ln 2}} \approx \boxed{11.542}$$

Note that we used a calculator to get the numerical value in the last step.

Check your antiderivative (it may help to review Chapter 5):

$$\frac{d}{dx}\left(\frac{2^x}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dx}(2^x) = \frac{1}{\ln 2}(2^x \ln 2) = 2^x$$

13 INTEGRATION BY POLYNOMIAL SUBSTITUTION

A common method for performing integration is to make a change of variables. If an integral is over one variable **x**, it can be transformed to an integral over a different variable **u** by making a substitution.

$$\int_{x=x_1}^{x_2} f(x)\,dx = \int_{u=u_1}^{u_2} g(u)\,du$$

To apply the method of substitution to an integral, follow these steps:
1. Try to think of a substitution that will make the integral simpler. Experience solving a variety of integration problems can help with this step.
2. Write down the equation for your substitution, which relates **u** to **x**. The two variables, **u** and **x**, should appear on separate sides of this equation (with one variable on the left and the other variable on the right).
3. Implicitly differentiate both sides of the equation from step 2: On one side of the equation, take a derivative with respect to **u** and multiply by **du**, and on the other side, take a derivative with respect to **x** and multiply by **dx**. This is illustrated in the examples.
4. In the equation from step 3, solve for **dx** in terms of **du**.
5. For a definite integral, you must also determine the new limits of integration. Plug the original lower limit, $\mathbf{x_1}$, into the equation from step 2 to determine the new lower limit, $\mathbf{u_1}$. Similarly, plug in the original upper limit, $\mathbf{x_2}$, into the equation from step 2 to determine the new upper limit, $\mathbf{u_2}$.

6. Make all of the following substitutions into the original integral:
• Replace **dx** with the equation from step 4.
• Replace **x** with the equation from step 2.
Replace the old limits, $\mathbf{x_1}$ and $\mathbf{x_2}$, with the new limits, $\mathbf{u_1}$ and $\mathbf{u_2}$.

7. Sometimes, the substitution that you try in steps 1-2 doesn't work out. When that happens, either try making a different substation or try another method of integration (such as those described in Chapters 14-16).

Example: Perform the following integral.

$$\int (3x+4)^5 \, dx$$

It would be tedious to multiply $(3x+4)$ by itself multiple times. The substitution below makes this integral much simpler:

$$u = 3x + 4$$

Implicitly differentiate the above equation:

- On the left, take a derivative with respect to u and multiply by du.
- On the right, take a derivative with respect to x and multiply by dx.

$$1\, du = 3\, dx$$

$$dx = \frac{du}{3}$$

Since this is an indefinite integral, there are no limits of integration to worry about. Substitute $dx = \frac{du}{3}$ and $u = 3x + 4$ into the original integral.

$$\int (3x+4)^5 \, dx = \int u^5 \frac{du}{3} = \int \frac{u^5}{3} du = \frac{u^6}{3(6)} + c$$

$$= \frac{u^6}{18} + c = \frac{(3x+4)^6}{18} + c$$

Example: Perform the following integral.

$$\int_{\theta=0}^{\pi/8} \cos(4\theta)\, d\theta$$

Make the following substitution:

$$u = 4\theta$$

Implicitly differentiate the above equation:

- On the left, take a derivative with respect to u and multiply by du.
- On the right, take a derivative with respect to θ and multiply by $d\theta$.

$$1\, du = 4\, d\theta$$

$$d\theta = \frac{du}{4}$$

Use $u = 4\theta$ to determine the new limits of integration: $u_1 = 4(0) = 0$ and $u_2 = 4\left(\frac{\pi}{8}\right) = \frac{\pi}{2}$. Substitute $d\theta = \frac{du}{4}$, $u = 4\theta$, $u_1 = 0$, and $u_2 = \frac{\pi}{2}$ into the original integral.

$$\int_{\theta=0}^{\pi/8} \cos(4\theta)\, d\theta = \int_{u=0}^{\pi/2} \cos(u) \frac{du}{4} = \frac{1}{4} [\sin u]_{u=0}^{\pi/2}$$

$$= \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{4}(1 - 0) = \frac{1}{4}$$

Note that $\frac{\pi}{2}$ rad $= 90°$, $\sin\frac{\pi}{2} = \sin 90° = 1$, and $\sin 0 = 0$.

Example: Perform the following integral.

$$\int_{x=1}^{2} x^2\sqrt{x^3+8}\,dx$$

This problem would be more difficult if the x^2 weren't present. How does x^2 help? Because the derivative of the inside function is $\frac{d}{dx}(x^3+8) = 3x^2$, which only differs from x^2 by a constant. Make the following substitution:

$$u = x^3 + 8$$

Implicitly differentiate the above equation:

- On the left, take a derivative with respect to u and multiply by du.
- On the right, take a derivative with respect to x and multiply by dx.

$$1\,du = 3x^2\,dx$$

$$dx = \frac{du}{3x^2}$$

Use $u = x^3 + 8$ to determine the new limits of integration:

$$u_1 = 1^3 + 8 = 1 + 8 = 9 \quad , \quad u_2 = 2^3 + 8$$
$$= 8 + 8 = 16$$

Substitute $dx = \frac{du}{3x^2}$, $u = x^3 + 8$, $u_1 = 9$, and $u_2 = 16$ into the original integral.

$$\int_{x=1}^{2} x^2\sqrt{x^3 + 8}\, dx = \int_{u=9}^{16} x^2\sqrt{u}\,\frac{du}{3x^2} = \int_{u=9}^{16} \frac{\sqrt{u}}{3}\, du$$

$$= \int_{u=9}^{16} \frac{u^{1/2}}{3}\, du = \frac{1}{3}\int_{u=9}^{16} u^{1/2}\, du$$

$$= \frac{1}{3}\left[\frac{u^{1+1/2}}{1+\frac{1}{2}}\right]_{u=9}^{16} = \frac{1}{3}\left[\frac{u^{3/2}}{\frac{3}{2}}\right]_{u=9}^{16} = \frac{1}{3}\left[\frac{2u^{3/2}}{3}\right]_{u=9}^{16}$$

$$= \frac{2}{9}\left[u^{3/2}\right]_{u=9}^{16} = \frac{2}{9}\left[u^1 u^{1/2}\right]_{u=9}^{16}$$

$$= \frac{2}{9}\left[u\sqrt{u}\right]_{u=9}^{16} = \frac{2}{9}\left(16\sqrt{16} - 9\sqrt{9}\right)$$

$$= \frac{2}{9}[16(4) - 9(3)] = \frac{2}{9}(64 - 27) = \frac{74}{9}$$

$$\approx 8.222$$

Note that $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$ (add fractions with a common denominator) and $\frac{1}{3/2} = 1 \div \frac{3}{2} = 1 \times \frac{2}{3} = \frac{2}{3}$ (to divide by a fraction, multiply by its reciprocal). Also note that $u^{3/2} = u^1 u^{1/2} = u\sqrt{u}$ because $u^m u^n = u^{m+n}$ and $u^{1/2} = \sqrt{u}$. Alternatively, you can use a calculator to check that 16^(3/2) = 64 and 9^(3/2) = 27.

Chapter 13 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $\int \frac{dx}{2x+3} =$

❷ $\int_{\theta=0}^{\pi/9} 6\sin(3\theta)\, d\theta =$

Part B

❸ $$\int \frac{dx}{\sqrt{x-1}} =$$

❹ $$\int_{t=0}^{\infty} t\, e^{-t^2}\, dt =$$

Part C

❺ $$\int \frac{8x^3}{x^4 - 2}\,dx =$$

❻ $$\int_{\theta=0}^{\frac{\sqrt{\pi}}{2}} \theta \cos(\theta^2)\,d\theta =$$

Part D

❼ $$\int \frac{12x^2}{(x^3+16)^2}\,dx =$$

❽ $$\int_{\theta=0}^{\pi^2/4} \frac{\sin\left(\sqrt{\theta}\right)}{\sqrt{\theta}}\,d\theta =$$

Part E

❾ $\int \sqrt{1-\sqrt{x}}\,dx =$

❿ $\int_{x=0}^{4} \sqrt{3x+4}\,dx =$

Chapter 13 Solutions

Part A

❶ $\int \frac{dx}{2x+3}$

$$u = 2x + 3 \quad , \quad du = 2\,dx \quad , \quad dx = \frac{du}{2}$$

$$\int \frac{dx}{2x+3} = \int \frac{1}{2x+3}\,dx = \int \frac{1}{u}\left(\frac{du}{2}\right) = \int \frac{du}{2u}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln u + c = \boxed{\frac{1}{2}\ln(2x+3) + c}$$

Recall from Chapter 9 that an exponent of $b = -1$ is a special case of integrals of the form $\int au^b\,du$: In this case, $\int \frac{du}{u} = \int u^{-1}\,du = \ln u + c$.

❷ $$\int_{\theta=0}^{\pi/9} 6\sin(3\theta)\,d\theta$$

$$u = 3\theta \quad , \quad du = 3\,d\theta \quad , \quad d\theta = \frac{du}{3}$$

$$u_1 = u(0) = 3(0) = 0 \quad , \quad u_2 = u\left(\frac{\pi}{9}\right) = 3\left(\frac{\pi}{9}\right)$$

$$= \frac{\pi}{3}$$

$$\int_{\theta=0}^{\pi/9} 6\sin(3\theta)\,d\theta = \int_{u=0}^{\pi/3} 6\sin u\,\frac{du}{3}$$

$$= \int_{u=0}^{\pi/3} 2\sin u\,du = 2[-\cos u]_{u=0}^{\frac{\pi}{3}}$$

$$= -2[\cos u]_{u=0}^{\frac{\pi}{3}}$$

$$= -2\left(\cos\frac{\pi}{3} - \cos 0\right) = -2\left(\frac{1}{2} - 1\right) = -2\left(-\frac{1}{2}\right)$$

$$= \boxed{1}$$

Notes:

- $\frac{\pi}{3}$ rad $= \frac{\pi}{3}\frac{180°}{\pi} = 60°$.
- $\cos\left(\frac{\pi}{3}\right) = \cos(60°) = \frac{1}{2}$.
- $\cos 0 = 1$.
- $\int \sin\theta \, d\theta = -\cos\theta + c$ (but as usual, we may ignore the constant of integration when performing a definite integral, since it would cancel out during the subtraction after plugging in the limits).

Part B

❸ $\int \frac{dx}{\sqrt{x-1}}$

$$u = x - 1 \quad , \quad du = dx \quad , \quad dx = du$$

$$\int \frac{dx}{\sqrt{x-1}} = \int \frac{du}{\sqrt{u}} = \int \frac{du}{u^{1/2}} = \int u^{-1/2}\, du$$

$$= \frac{u^{-1/2+1}}{-\frac{1}{2}+1} + c = \frac{u^{1/2}}{1/2} + c = 2u^{1/2} + c$$

$$= 2\sqrt{u} + c = \boxed{2\sqrt{x-1} + c}$$

Notes:

- $\frac{du}{dx} = \frac{d}{dx}(x-1) = \frac{d}{dx}x - \frac{d}{dx}1 = 1 - 0 = 1.$ Therefore, $du = 1\, dx = dx.$
- $\sqrt{u} = u^{1/2}$ and $u^{-1/2} = \frac{1}{u^{1/2}}.$
- $-\frac{1}{2} + 1 = -\frac{1}{2} + \frac{2}{2} = \frac{-1+2}{2} = \frac{1}{2}$ and $\frac{1}{1/2} = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = \frac{2}{1} = 2.$

❹ $\displaystyle\int_{t=0}^{\infty} t\, e^{-t^2}\, dt$

$$u = t^2 \quad , \quad du = 2t\, dt \quad , \quad dt = \frac{du}{2t}$$

$$u_1 = u(0) = 0^2 = 0 \quad , \quad u_2 = u(\infty) = (\infty)^2 = \infty$$

$$\int_{t=0}^{\infty} t\, e^{-t^2}\, dt = \int_{u=0}^{\infty} t\, e^{-u} \left(\frac{du}{2t}\right) = \int_{u=0}^{\infty} \frac{t\, e^{-u}}{2t}\, du$$

$$= \frac{1}{2} \int_{u=0}^{\infty} e^{-u}\, du = \frac{1}{2} [-e^{-u}]_{u=0}^{\infty}$$

$$= -\frac{1}{2} [e^{-u}]_{u=0}^{\infty} = -\frac{1}{2}\left(\lim_{L\to\infty} e^{-L} - e^{-0}\right)$$

$$= -\frac{1}{2}\left(0 - \frac{1}{e^0}\right) - \frac{1}{2}\left(0 - \frac{1}{1}\right) = -\frac{1}{2}(-1)$$

$$= \boxed{\frac{1}{2}}$$

This is an 'improper' integral: Imagine that the upper limit is L, such that the integral is $\frac{1}{2}\int_{u=0}^{L} e^{-u}\, du$. Integration gives $\frac{1}{2}[-e^{-u}]_{u=0}^{L} = \frac{1}{2}[-e^{-L} - (-e^{0})]$. In the limit that L grows to ∞, the expression $e^{-L} = \frac{1}{e^{L}}$ approaches zero, such that $\frac{1}{2}[-e^{-L} - (-e^{0})]$ approaches $\frac{1}{2}[-0 - (-1)] = \frac{1}{2}(-0 + 1) = \frac{1}{2}(1) = \frac{1}{2}$.

Part C

❺ $\int \frac{8x^3}{x^4 - 2} dx$

$$u = x^4 - 2 \quad , \quad du = 4x^3\,dx \quad , \quad dx = \frac{du}{4x^3}$$

$$\int \frac{8x^3}{x^4 - 2} dx = \int \frac{8x^3}{u} \frac{du}{4x^3} = \int \frac{2}{u} du = 2 \int \frac{du}{u}$$

$$= 2 \ln u + c = \boxed{2 \ln(x^4 - 2) + c}$$

Notes:

- $\frac{du}{dx} = \frac{d}{dx}(x^4 - 2) = \frac{d}{dx} x^4 - \frac{d}{dx} 2 = 4x^3 - 0 = 4x^3$. Therefore, $du = 4x^3\,dx$.
- Recall from Chapter 9 that an exponent of $b = -1$ is a special case of integrals of the form $\int au^b\,du$: In this case, $\int \frac{du}{u} = \int u^{-1}\,du = \ln u + c$.

❻ $$\int_{\theta=0}^{\frac{\sqrt{\pi}}{2}} \theta\cos(\theta^2)\,d\theta$$

$$u=\theta^2 \quad , \quad du = 2\theta\,d\theta \quad , \quad d\theta = \frac{du}{2\theta}$$

$$u_1 = u(0) = 0^2 = 0$$

$$u_2 = u\left(\frac{\sqrt{\pi}}{2}\right) = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$$

$$\int_{\theta=0}^{\frac{\sqrt{\pi}}{2}} \theta\cos(\theta^2)\,d\theta = \int_{u=0}^{\pi/4} \theta\cos u\,\frac{du}{2\theta}$$

$$= \frac{1}{2}\int_{u=0}^{\pi/4} \cos u\,du = \frac{1}{2}[\sin u]_{u=0}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sin\frac{\pi}{4} - \sin 0\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2} - 0\right) = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \boxed{\frac{\sqrt{2}}{4}}$$

Notes:

- $\frac{\pi}{4}$ rad $= \frac{\pi}{4}\frac{180°}{\pi} = 45°$.
- $\sin\left(\frac{\pi}{4}\right) = \sin(45°) = \frac{\sqrt{2}}{2}$.
- $\sin 0 = 0$.
- $\int \cos\theta \, d\theta = \sin\theta + c$ (but as usual, we may ignore the constant of integration when performing a definite integral, since it would cancel out during the subtraction after plugging in the limits).

Part D

❼ $\displaystyle\int \frac{12x^2}{(x^3+16)^2}\,dx$

$$u = x^3 + 16 \quad , \quad du = 3x^2\,dx \quad , \quad dx = \frac{du}{3x^2}$$

$$\int \frac{12x^2}{(x^3+16)^2}\,dx = \int \frac{12x^2}{u^2}\frac{du}{3x^2} = \int \frac{4}{u^2}\,du$$

$$= 4\int \frac{du}{u^2} = 4\int u^{-2}du = \frac{4u^{-2+1}}{-2+1} + c$$

$$= \frac{4u^{-1}}{-1} + c = -4u^{-1} + c = -\frac{4}{u} + c$$

$$= \boxed{-\frac{4}{x^3+16} + c}$$

Notes:

- $\frac{du}{dx} = \frac{d}{dx}(x^3+16) = \frac{d}{dx}x^3 + \frac{d}{dx}16 = 3x^2 + 0 = 3x^2$. Therefore, $du = 3x^2\,dx$.
- $-2+1 = -1$, $u^{-2} = \frac{1}{u^2}$, and $u^{-1} = \frac{1}{u}$.

❽ $\displaystyle\int_{\theta=0}^{\pi^2/4} \frac{\sin(\sqrt{\theta})}{\sqrt{\theta}}\,d\theta$

$$u = \sqrt{\theta} = \theta^{1/2} \quad , \quad du = \frac{1}{2}\theta^{-1/2}\,d\theta = \frac{d\theta}{2\theta^{1/2}}$$

$$= \frac{d\theta}{2\sqrt{\theta}} \quad , \quad d\theta = 2\sqrt{\theta}\,du$$

$$u_1 = u(0) = \sqrt{0} = 0 \quad , \quad u_2 = u\left(\frac{\pi^2}{4}\right) = \sqrt{\frac{\pi^2}{4}} = \frac{\pi}{2}$$

$$\int_{\theta=0}^{\pi^2/4} \frac{\sin(\sqrt{\theta})}{\sqrt{\theta}}\,d\theta = \int_{u=0}^{\pi/2} \frac{\sin u}{\sqrt{\theta}}\left(2\sqrt{\theta}\,du\right)$$

$$= 2\int_{u=0}^{\pi/2} \sin u\,du = 2[-\cos u]_{u=0}^{\frac{\pi}{2}}$$

$$= -2[\cos u]_{u=0}^{\frac{\pi}{2}} = -2\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= -2(0-1) = -2(-1) = \boxed{2}$$

Notes:

- $\frac{\pi}{2}$ rad $= \frac{\pi}{2}\frac{180°}{\pi} = 90°$.
- $\cos\left(\frac{\pi}{2}\right) = \cos(90°) = 0$.
- $\cos 0 = 1$.
- $\int \sin\theta\, d\theta = -\cos\theta + c$ (but as usual, we may ignore the constant of integration when performing a definite integral, since it would cancel out during the subtraction after plugging in the limits).

Part E

❾ $\int \sqrt{1-\sqrt{x}}\,dx$

$$u = 1-\sqrt{x} = 1-x^{1/2} \quad , \quad du = -\frac{1}{2}x^{-1/2}\,dx$$

$$= -\frac{dx}{2x^{1/2}} = -\frac{dx}{2\sqrt{x}} \quad , \quad dx = -2\sqrt{x}\,du$$

$$\int \sqrt{1-\sqrt{x}}\,dx = \int \sqrt{u}\left(-2\sqrt{x}\,du\right)$$

$$= -2\int \sqrt{u}\sqrt{x}\,du$$

Add $\sqrt{x}$ to both sides of the equation $u = 1-\sqrt{x}$ to get $u+\sqrt{x} = 1$. Subtract u from both sides to get $\sqrt{x} = 1-u$. Therefore, we may replace $\sqrt{x}$ with $1-u$.

$$-2\int \sqrt{u}\sqrt{x}\,du = -2\int \sqrt{u}(1-u)\,du$$

$$= -2\int u^{1/2}(1-u^1)\,du$$

$$= -2\int u^{1/2}\,du - 2\int u^{1/2}(-u^1)\,du$$

$$= -2\int u^{1/2}\,du + 2\int u^{1/2}u^1\,du$$

$$= -2\int u^{1/2}\,du + 2\int u^{3/2}\,du$$

$$= -\frac{2u^{1/2+1}}{\frac{1}{2}+1} + \frac{2u^{3/2+1}}{\frac{3}{2}+1} + c$$

$$= -\frac{2u^{3/2}}{3/2} + \frac{2u^{5/2}}{5/2} + c = -\frac{4u^{3/2}}{3} + \frac{4u^{5/2}}{5} + c$$

$$= \frac{4u^{5/2}}{5} - \frac{4u^{3/2}}{3} + c$$

$$= \boxed{\frac{4\left(1-\sqrt{x}\right)^{5/2}}{5} - \frac{4\left(1-\sqrt{x}\right)^{3/2}}{3} + c}$$

Notes:

- $\frac{du}{dx} = \frac{d}{dx}\left(1 - x^{1/2}\right) = \frac{d}{dx}1 - \frac{d}{dx}x^{1/2} = 0 - \frac{1}{2}x^{-1/2} = -\frac{1}{2}x^{-1/2} = -\frac{dx}{2x^{1/2}} = -\frac{dx}{2\sqrt{x}}$. Therefore, $du = -\frac{dx}{2\sqrt{x}}$, such that $dx = -2\sqrt{x}\,du$.
- $\sqrt{u}(1-u) = \sqrt{u} - u\sqrt{u} = u^{1/2} - uu^{1/2} = u^{1/2} - u^1u^{1/2} = u^{1/2} - u^{3/2}$ because $x^mx^n = x^{m+n}$.

- Note that $u = u^1$ and $\sqrt{u} = u^{1/2}$.
- $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$ and $\frac{3}{2} + 1 = \frac{3}{2} + \frac{2}{2} = \frac{3+2}{2} = \frac{5}{2}$.
- $\frac{2}{3/2} = 2 \div \frac{3}{2} = 2 \times \frac{2}{3} = \frac{4}{3}$ and $\frac{2}{5/2} = 2 \div \frac{5}{2} = 2 \times \frac{2}{5} = \frac{4}{5}$.
- Going from $-\frac{4u^{3/2}}{3} + \frac{4u^{5/2}}{5} + c$ to $\frac{4u^{5/2}}{5} - \frac{4u^{3/2}}{3} + c$, we applied the identity $-a + b = b - a$.

⓾ $$\int_{x=0}^{4} \sqrt{3x+4}\,dx$$

$$u = 3x+4 \quad , \quad du = 3\,dx \quad , \quad dx = \frac{du}{3}$$

$$u_1 = u(0) = 3(0)+4 = 4 \quad , \quad u_2 = u(4)$$
$$= 3(4)+4 = 12+4 = 16$$

$$\int_{x=0}^{4} \sqrt{3x+4}\,dx = \int_{u=4}^{16} \sqrt{u}\,\frac{du}{3} = \frac{1}{3}\int_{u=4}^{16} \sqrt{u}\,du$$

$$= \frac{1}{3}\int_{u=4}^{16} u^{1/2}\,du = \frac{1}{3}\left[\frac{u^{1/2+1}}{\frac{1}{2}+1}\right]_{u=4}^{u=16}$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_{u=4}^{u=16} = \frac{1}{3}\left[\frac{2u^{3/2}}{3}\right]_{u=4}^{u=16} = \frac{2}{9}\left[u^{3/2}\right]_{u=4}^{u=16}$$

$$= \frac{2}{9}\left(16^{3/2} - 4^{3/2}\right)$$

$$= \frac{2}{9}(64-8) = \frac{2}{9}(56) = \boxed{\frac{112}{9}} \approx \boxed{12.444}$$

Notes:

- $\frac{d}{dx}(3x+4) = \frac{d}{dx}3x + \frac{d}{dx}4 = 3 + 0 = 3$. Therefore, $du = 3\,dx$.
- $u^{1/2} = \sqrt{u}$.
- $\frac{1}{2} + 1 = \frac{1}{2} + \frac{2}{2} = \frac{1+2}{2} = \frac{3}{2}$ and $\frac{1}{3/2} = 1 \div \frac{3}{2} = 1 \times \frac{2}{3} = \frac{2}{3}$.
- Note that $(16)^{3/2} = \left(16^{1/2}\right)^3 = 4^3 = 64$ and $(4)^{3/2} = \left(4^{1/2}\right)^3 = 2^3 = 8$ because $x^{mn} = (x^m)^n = (x^n)^m$. Alternatively, use a calculator to verify that 16^(3/2) = 64 and 4^(3/2) = 8.

14 INTEGRATION BY TRIGONOMETRIC SUBSTITUTION

Trigonometric substitutions are often helpful for two types of integrals:

- integrals involving trigonometric functions, like those below.

$$\int \sin\theta \cos\theta \, d\theta \quad , \quad \int \sin^3\theta \;\, d\theta \quad , \quad \int \frac{\sin\theta}{\cos^2\theta} \, d\theta$$

- integrals where $\mathbf{x^2}$ is added or subtracted to a constant, like those below.

$$\int \sqrt{x^2+4}\, dx \quad , \quad \int \frac{dx}{9-x^2} \quad , \quad \int \frac{dx}{(x^2-1)^{3/2}}$$

It may help to recall the following trigonometric identities and relations:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad , \quad \csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \quad , \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \quad , \quad \tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta \quad , \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \quad , \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Notes:

For problems that involve expressions of the form $(a^2 - x^2)^p$:

- Make the substitutions $x = a \sin\theta$ and $dx = a\cos\theta\, d\theta$.
- Note that $a^2 - x^2 = a^2 - a^2 \sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta$. The reason is that $\sin^2\theta + \cos^2\theta = 1$, such that $\cos^2\theta = 1 - \sin^2\theta$.

For problems that involve expressions of the form $(x^2 + a^2)^p$:

- Make the substitutions $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$.
- Note that $x^2 + a^2 = a^2\tan^2\theta + a^2 = a^2(\tan^2\theta + 1) = a^2\sec^2\theta$. The reason is that $\tan^2\theta + 1 = \sec^2\theta$.

For problems that involve expressions of the form $(x^2 - a^2)^p$:

- Make the substitutions $x = a \sec\theta$ and $dx = a \sec\theta \tan\theta \, d\theta$.
- Note that $x^2 - a^2 = a^2 \sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2 \tan^2\theta$. The reason is that $\tan^2\theta + 1 = \sec^2\theta$, such that $\tan^2\theta = \sec^2\theta - 1$.

Note that the constant a is squared in $(a^2 - x^2)^p$, $(x^2 + a^2)^p$, and $(x^2 - a^2)^p$.

- Example: $\sqrt{x^2 + 4} = \sqrt{x^2 + 2^2} = (x^2 + 2^2)^{1/2}$. Here, $a = 2$ and $p = 1/2$.
- Example: $9 - x^2 = 3^2 - x^2$. Here, $a = 3$ and $p = 1$.

Example: Perform the following integral.

$$\int \sin^3\theta \cos\theta \, d\theta$$

Cosine is the derivative of sine, which makes the following substitution helpful:

$$u = \sin\theta$$

Implicitly differentiate the above equation:

- On the left, take a derivative with respect to u and multiply by du.
- On the right, take a derivative with respect to θ and multiply by $d\theta$.

$$1\, du = \cos\theta \ d\theta$$

Since this is an indefinite integral, there are no limits of integration to worry about. Substitute $du = \cos\theta \ d\theta$ and $u = \sin\theta$ into the original integral.

$$\int \sin^3\theta \cos\theta \, d\theta = \int u^3 \, du = \frac{u^4}{4} + c = \frac{\sin^4\theta}{4} + c$$

Example: Perform the following integral.

$$\int \sin^2\theta\, d\theta$$

The following trig identity is helpful for this integral:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Substitute the above expression into the integral:

$$\int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2} d\theta$$

$$= \int \frac{1}{2} d\theta - \int \frac{\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta\, d\theta$$

The first integral is trivial: $\int d\theta = \theta + c$. For the second integral, make the following substitution:

$$u = 2\theta$$

$$du = 2\, d\theta$$

$$d\theta = \frac{du}{2}$$

Since this is an indefinite integral, there are no limits of integration to worry about. Substitute $d\theta = \frac{du}{2}$ and $u = 2\theta$ into the second integral.

$$\frac{1}{2}\int d\theta - \frac{1}{2}\int \cos 2\theta \, d\theta = \frac{1}{2}\int d\theta - \frac{1}{2}\int \cos u \frac{du}{2}$$

$$= \frac{1}{2}\int d\theta - \frac{1}{4}\int \cos u \, du$$

$$= \frac{1}{2}\theta - \frac{1}{4}\sin u + c = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + c$$

Example: Perform the following integral.

$$\int_{\theta=0}^{\pi/2} \cos^3\theta \, d\theta$$

Apply the identity $\sin^2\theta + \cos^2\theta = 1$, after rewriting it as $\cos^2\theta = 1 - \sin^2\theta$.

$$\int_{\theta=0}^{\pi/2} \cos^3\theta \, d\theta = \int_{\theta=0}^{\pi/2} \cos^2\theta \cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} (1 - \sin^2\theta)\cos\theta \, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \cos\theta \, d\theta - \int_{\theta=0}^{\pi/2} \sin^2\theta \cos\theta \, d\theta$$

We distributed: $(1 - \sin^2\theta)\cos\theta = \cos\theta - \sin^2\theta\cos\theta$. The first integral is trivial. For the second integral, make the following substitution:

$$u = \sin\theta$$

Implicitly differentiate the above equation:

- On the left, take a derivative with respect to u and multiply by du.
- On the right, take a derivative with respect to θ and multiply by $d\theta$.

$$1\,du = \cos\theta\;\;d\theta$$

Use $u = \sin\theta$ to determine the new limits of integration:

$$u_1 = u(0) = \sin 0 = 0$$

$$u_2 = u\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = \sin 90° = 1$$

Substitute $du = \cos\theta\;\;d\theta$, $u = \sin\theta$, $u_1 = 0$, and $u_2 = 1$ into the original integral.

$$\int_{\theta=0}^{\pi/2} \cos\theta\,d\theta - \int_{\theta=0}^{\pi/2} \sin^2\theta\cos\theta\,d\theta$$

$$= [\sin\theta]_{\theta=0}^{\pi/2} - \int_{u=0}^{1} u^2\,du$$

$$= [\sin\theta]_{\theta=0}^{\pi/2} - \left[\frac{u^3}{3}\right]_{u=0}^{1} = [\sin\theta]_{\theta=0}^{\frac{\pi}{2}} - \frac{1}{3}[u^3]_{u=0}^{1}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin 0 - \frac{1}{3}(1^3 - 0^3)$$

$$= 1 - 0 - \frac{1}{3}(1 - 0) = 1 - \frac{1}{3}(1) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\approx 0.667$$

Note that $\frac{\pi}{2}$ rad $= \frac{\pi}{2}\frac{180°}{\pi} = 90°$, $\sin\left(\frac{\pi}{2}\right) = \sin 90° = 1$, and $\sin 0 = 0$.

Example: Perform the following integral.

$$\int_{x=0}^{3} \frac{dx}{\sqrt{x^2+9}}$$

Rewrite $x^2 + 9$ in the form $x^2 + a^2$. This requires that $a^2 = 9$, which means $a = 3$. Now the integral looks like this:

$$\int_{x=0}^{3} \frac{dx}{\sqrt{x^2+9}} = \int_{x=0}^{3} \frac{dx}{\sqrt{x^2+3^2}}$$

Since x^2 is added to a constant, according to the first page of this chapter, we should make the following substitutions:

$$x = a\tan\theta = 3\tan\theta$$

$$dx = a\sec^2\theta\, d\theta = 3\sec^2\theta\, d\theta$$

Solve for θ in $x = 3\tan\theta$:

$$\theta = \tan^{-1}\left(\frac{x}{3}\right)$$

Use the above equation to determine the new limits of integration:

$$\theta_1 = \theta(0) = \tan^{-1}\left(\frac{0}{3}\right) = \tan^{-1}(0) = 0$$

$$\theta_2 = \theta(3) = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Recall from trig that $\tan 0 = 0$ and $\tan\left(\frac{\pi}{4}\right) = \tan 45° = 1$. Substitute $dx = 3\sec^2\theta\ d\theta$, $x = 3\tan\theta$, $\theta_1 = 0$, and $\theta_2 = \frac{\pi}{4}$ into the integral.

$$\int_{x=0}^{3} \frac{dx}{\sqrt{x^2 + 3^2}} = \int_{\theta=0}^{\pi/4} \frac{3\sec^2\theta\ d\theta}{\sqrt{(3\tan\theta)^2 + 3^2}}$$

$$= \int_{\theta=0}^{\pi/4} \frac{3\sec^2\theta\ d\theta}{\sqrt{3^2\tan^2\theta + 3^2}}$$

$$= \int_{\theta=0}^{\pi/4} \frac{3\sec^2\theta\ d\theta}{\sqrt{3^2(1 + \tan^2\theta)}}$$

$$= \int_{\theta=0}^{\pi/4} \frac{3\sec^2\theta \; d\theta}{\sqrt{3^2\sec^2\theta}} = \int_{\theta=0}^{\pi/4} \frac{3\sec^2\theta \; d\theta}{3\sec\theta}$$

$$= \int_{\theta=0}^{\pi/4} \sec\theta \, d\theta = [\ln|\sec\theta + \tan\theta|]_{\theta=0}^{\frac{\pi}{4}}$$

$$= \ln\left|\sec\left(\frac{\pi}{4}\right) + \tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0) + \tan(0)|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0|$$

$$= \ln\left|\sqrt{2} + 1\right| - \ln(1) = \ln\left|\sqrt{2} + 1\right| - 0$$

$$= \ln\left|\sqrt{2} + 1\right| \approx 0.881$$

Note the following:

- $\sec\left(\frac{\pi}{4}\right) = \sec 45° = \frac{1}{\cos 45°} = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $\tan\left(\frac{\pi}{4}\right) = \tan 45° = 1$.
- $\frac{2}{\sqrt{2}} = \sqrt{2}$ because $\sqrt{2}\sqrt{2} = 2$.
- $\sec(0) = \frac{1}{\cos 0} = \frac{1}{1} = 1$, $\tan 0 = 0$, and $\ln(1) = 0$.
- $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + c$.

Chapter 14 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\int \frac{dx}{\sqrt{16 - x^2}} =$$

❷ $$\int_{\theta=0}^{\pi/3} \cos^4\theta \sin\theta \, d\theta =$$

Part B

❸ $$\int \frac{dx}{x^2 + 25} =$$

❹ $$\int_{\theta=0}^{\pi/4} \tan^3 \theta \, d\theta =$$

Part C

❺ $\int \sin^4 \theta \, d\theta =$

❻ $\displaystyle\int_{x=0}^{1} \frac{dx}{(x^2+1)^2} =$

Part D

❼ $$\int \frac{dx}{(9 - x^2)^{3/2}} =$$

❽ $$\int_{\theta=\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^3\theta}\, d\theta =$$

Chapter 14 Solutions

Part A

❶ $$\int \frac{dx}{\sqrt{16 - x^2}} = \int \frac{dx}{\sqrt{4^2 - x^2}}$$

$$x = a \sin\theta = 4 \sin\theta$$

$$dx = a \cos\theta \, d\theta = 4 \cos\theta \, d\theta$$

$$\int \frac{dx}{\sqrt{4^2 - x^2}} = \int \frac{4 \cos\theta \, d\theta}{\sqrt{4^2 - (4 \sin\theta)^2}}$$

$$= \int \frac{4 \cos\theta \, d\theta}{\sqrt{4^2 - 4^2 \sin^2\theta}} = \int \frac{4 \cos\theta \, d\theta}{\sqrt{4^2 (1 - \sin^2\theta)}}$$

$$= \int \frac{4 \cos\theta \, d\theta}{\sqrt{4^2 \cos^2\theta}} = \int \frac{4 \cos\theta \, d\theta}{4 \cos\theta} = \int d\theta = \theta + c$$

$$= \boxed{\sin^{-1}\left(\frac{x}{4}\right) + c}$$

Notes:

- $4^2 - 4^2\sin^2\theta = 4^2(1-\sin^2\theta) = 4^2\cos^2\theta$ because $\sin^2\theta + \cos^2\theta = 1$.
- In the last step, $\theta = \sin^{-1}\left(\frac{x}{4}\right)$ follows from $x = 4\sin\theta \rightarrow \frac{x}{4} = \sin\theta$.

❷ $$\int_{\theta=0}^{\pi/3} \cos^4\theta \sin\theta \, d\theta$$

$$u = \cos\theta \quad , \quad du = -\sin\theta \, d\theta$$

$$u_1 = u(0) = \cos 0 = 1$$

$$u_2 = u\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \cos(60°) = \frac{1}{2}$$

$$\int_{\theta=0}^{\pi/3} \cos^4\theta \sin\theta \, d\theta = \int_{u=1}^{1/2} u^4 \, (-du) = -\int_{u=1}^{\frac{1}{2}} u^4 \, du$$

$$= -\left[\frac{u^5}{5}\right]_{u=1}^{\frac{1}{2}} = -\frac{1}{5}\left[u^5\right]_{u=1}^{\frac{1}{2}}$$

$$= -\frac{1}{5}\left(\frac{1}{2^5} - 1^5\right) = -\frac{1}{5}\left(\frac{1}{32} - 1\right) = -\frac{1}{5}\left(\frac{1}{32} - \frac{32}{32}\right)$$

$$= -\frac{1}{5}\left(\frac{1-32}{32}\right)$$

$$= -\frac{1}{5}\left(-\frac{31}{32}\right) = \boxed{\frac{31}{160}} \approx \boxed{0.194}$$

Notes:

- $\frac{\pi}{3}\text{rad} = \frac{\pi}{3}\frac{180°}{\pi} = 60°.$
- $\left(\frac{1}{2}\right)^5 = \frac{1}{2^5} = \frac{1}{32}.$

Part B

❸ $\displaystyle\int \frac{dx}{x^2+25} = \int \frac{dx}{x^2+5^2}$

$$x = a\tan\theta = 5\tan\theta$$

$$dx = a\sec^2\theta\, d\theta = 5\sec^2\theta\, d\theta$$

$$\int \frac{dx}{x^2+5^2} = \int \frac{5\sec^2\theta\, d\theta}{(5\tan\theta)^2+5^2} = \int \frac{5\sec^2\theta\, d\theta}{5^2\tan^2\theta+5^2}$$

$$= \int \frac{5\sec^2\theta\, d\theta}{5^2(\tan^2\theta+1)}$$

$$= \int \frac{5\sec^2\theta\, d\theta}{5^2\sec^2\theta} = \int \frac{d\theta}{5} = \frac{1}{5}\int d\theta = \frac{\theta}{5} + c$$

$$= \boxed{\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c}$$

Notes:

- $5^2\tan^2\theta + 5^2 = 5^2(\tan^2\theta+1) = 5^2\sec^2\theta$ because $\tan^2\theta + 1 = \sec^2\theta$.
- In the last step, $\theta = \tan^{-1}\left(\frac{x}{5}\right)$ follows from $x = 5\tan\theta \to \frac{x}{5} = \tan\theta$.

❹ $$\int_{\theta=0}^{\pi/4} \tan^3\theta\, d\theta = \int_{\theta=0}^{\pi/4} \tan^2\theta \tan\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} (\sec^2\theta - 1)\tan\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \sec^2\theta \tan\theta\, d\theta - \int_{\theta=0}^{\pi/4} \tan\theta\, d\theta$$

Note that $\tan^2\theta = \sec^2\theta - 1$ follows from $\tan^2\theta + 1 = \sec^2\theta$. Distribute to see that $(\sec^2\theta - 1)\tan\theta = \sec^2\theta\tan\theta - \tan\theta$.

$$u = \tan\theta \quad , \quad du = \sec^2\theta\, d\theta$$

$$u_1 = u(0) = \tan 0 = 1$$

$$u_2 = u\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = \tan(45°) = 1$$

$$\int_{\theta=0}^{\pi/4} \sec^2\theta \tan\theta \, d\theta - \int_{\theta=0}^{\pi/4} \tan\theta \, d\theta$$

$$= \int_{u=0}^{1} u \, du - \int_{\theta=0}^{\pi/4} \tan\theta \, d\theta$$

$$= \left[\frac{u^2}{2}\right]_{u=0}^{1} - [\ln|\sec\theta|]_{\theta=0}^{\frac{\pi}{4}}$$

$$= \frac{1^2}{2} - \frac{0^2}{2} - \left(\ln\left|\sec\frac{\pi}{4}\right| - \ln|\sec 0|\right)$$

$$= \frac{1}{2} - 0 - \left(\ln\sqrt{2} - \ln 1\right) = \frac{1}{2} - \left(\ln\sqrt{2} - 0\right)$$

$$= \boxed{\frac{1}{2} - \ln\sqrt{2}} = \boxed{\frac{1}{2} - \ln 2^{1/2}} = \boxed{\frac{1}{2} - \frac{1}{2}\ln 2}$$

$$= \boxed{\frac{1}{2}(1 - \ln 2)} \approx \boxed{0.153}$$

Notes:

- On the last line, we applied the rule $\ln x^a = a \ln x$ to write $\ln 2^{1/2} = \frac{1}{2}\ln 2$.
- $\frac{\pi}{4}\text{rad} = \frac{\pi}{4}\frac{180°}{\pi} = 45°$.
- $\sec\frac{\pi}{4} = \sec 45° = \frac{1}{\cos 45°} = \frac{2}{\sqrt{2}} = \sqrt{2}$ and $\sec 0 = \frac{1}{\cos 0} = \frac{1}{1} = 1$. Note that $\frac{2}{\sqrt{2}} = \sqrt{2}$ because $\sqrt{2}\sqrt{2} = 2$.
- $\int \tan\theta \, d\theta = \ln|\sec\theta| + c$ (but as usual, we may ignore the constant of integration when performing a definite integral, since it would cancel out during the subtraction after plugging in the limits).
- Recall from Chapter 5 that $\ln 1 = 0$.

- We used a calculator to get the numerical value at the end of the solution.
- An alternative way to solve this problem is to write $\tan^3\theta = \frac{\sin^3\theta}{\cos^3\theta}$, rewrite $\sin^3\theta = \sin^2\theta\sin\theta = (1-\cos^2\theta)\sin\theta$, separate the integral into two integrals, and make the substitution $u = \sin\theta$ and $du = \cos\theta\, d\theta$.

Part C

❺ $\int \sin^4\theta\,d\theta = \int \sin^2\theta\sin^2\theta\,d\theta$

$$= \int (1-\cos^2\theta)\sin^2\theta\,d\theta$$

$$= \int (\sin^2\theta - \cos^2\theta\sin^2\theta)\,d\theta$$

$$= \int (\sin^2\theta - \sin^2\theta\cos^2\theta)\,d\theta$$

Note that $\sin^2\theta = 1-\cos^2\theta$ follows from $\sin^2\theta + \cos^2\theta = 1$. Distribute to see that $(1-\cos^2\theta)\sin^2\theta = \sin^2\theta - \cos^2\theta\sin^2\theta$.

$$\int (\sin^2\theta - \sin^2\theta\cos^2\theta)\,d\theta$$

$$= \int [\sin^2\theta - (\sin\theta\cos\theta)^2]\,d\theta$$

$$= \int \left[\sin^2\theta - \left(\frac{\sin 2\theta}{2}\right)^2\right] d\theta$$

$$= \int \left(\sin^2\theta - \frac{\sin^2 2\theta}{4}\right) d\theta$$

$$= \int \left(\frac{1-\cos 2\theta}{2} - \frac{1-\cos 4\theta}{8}\right) d\theta$$

$$= \int \left(\frac{4-4\cos 2\theta}{8} - \frac{1-\cos 4\theta}{8}\right) d\theta$$

$$= \frac{1}{8}\int [4 - 4\cos 2\theta - (1-\cos 4\theta)]\,d\theta$$

$$= \frac{1}{8}\int (4 - 4\cos 2\theta - 1 + \cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\int (3 - 4\cos 2\theta + \cos 4\theta)\,d\theta$$

$$= \frac{1}{8}\int 3\, d\theta - \frac{1}{8}\int 4\cos 2\theta\, d\theta$$
$$+ \frac{1}{8}\int \cos 4\theta\, d\theta$$

Notes:

- $\frac{\sin^2 2\theta}{4} = \sin^2\theta\cos^2\theta$ because $\sin 2\theta = 2\sin\theta\cos\theta \to \frac{\sin 2\theta}{2} = \sin\theta\cos\theta$.
- $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$ (double the angles on both sides).

$$u = 2\theta \quad , \quad du = 2\, d\theta d\theta$$
$$= \frac{du}{2} \quad \text{(second integral)}$$
$$v = 4\theta \quad , \quad dv = 4\, d\theta \quad , \quad d\theta$$
$$= \frac{dv}{4} \quad \text{(third integral)}$$

$$\frac{1}{8}\int 3\, d\theta - \frac{1}{8}\int 4\cos 2\theta\, d\theta + \frac{1}{8}\int \cos 4\theta\, d\theta$$
$$= \frac{3}{8}\int d\theta - \frac{4}{8}\int \cos u \frac{du}{2} + \frac{1}{8}\int \cos v \frac{dv}{4}$$

$$= \frac{3}{8}\int d\theta - \frac{1}{2}\int \cos u \frac{du}{2} + \frac{1}{8}\int \cos v \frac{dv}{4}$$

$$= \frac{3}{8}\int d\theta - \frac{1}{4}\int \cos u \, du + \frac{1}{32}\int \cos v \, dv$$

$$= \frac{3}{8}\theta - \frac{1}{4}\sin u + \frac{1}{32}\sin v + c$$

$$= \boxed{\frac{3}{8}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + c}$$

❻ $$\int_{x=0}^{1} \frac{dx}{(x^2+1)^2} = \int_{x=0}^{1} \frac{dx}{(x^2+1^2)^2}$$

$$x = a\tan\theta = 1\tan\theta = \tan\theta$$

$$dx = a\sec^2\theta \, d\theta = 1\sec^2\theta \, d\theta = \sec^2\theta \, d\theta$$

Since $x = \tan\theta$, it follows that $\theta = \tan^{-1}(x)$.

$$\theta_1 = \tan^{-1}(0) = 0$$

$$\theta_1 = \tan^{-1}(1) = 45° = \frac{\pi}{4}\,\text{rad}$$

$$\int_{x=0}^{1} \frac{dx}{(x^2+1)^2} = \int_{\theta=0}^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2}$$

$$= \int_{\theta=0}^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^2} = \int_{\theta=0}^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int_{\theta=0}^{\pi/4} \frac{d\theta}{\sec^2\theta} = \int_{\theta=0}^{\pi/4} \cos^2\theta\, d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{1+\cos 2\theta}{2} d\theta$$

$$= \int_{\theta=0}^{\pi/4} \frac{1}{2} d\theta + \int_{\theta=0}^{\pi/4} \frac{\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/4} d\theta + \frac{1}{2} \int_{\theta=0}^{\pi/4} \cos 2\theta \, d\theta$$

$$u = 2\theta \quad , \quad du = 2\, d\theta \quad , \quad d\theta = \frac{du}{2}$$

$$u_1 = 2(0) = 0 \quad , \quad u_2 = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\frac{1}{2}[\theta]_{\theta=0}^{\frac{\pi}{4}} + \frac{1}{2} \int_{u=0}^{\pi/2} \cos u \frac{du}{2}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) + \frac{1}{4} \int_{u=0}^{\pi/2} \cos u \, du$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}[\sin u]_{u=0}^{\pi/2} = \frac{\pi}{8} + \frac{1}{4}\left[\sin\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{\pi}{8} + \frac{1}{4}(1 - 0) = \boxed{\frac{\pi}{8} + \frac{1}{4}} = \frac{\pi}{8} + \frac{2}{8} = \boxed{\frac{\pi + 2}{8}} \approx \boxed{0.643}$$

Notes:

- $\tan^2\theta + 1 = \sec^2\theta$.
- $\sec\theta = \frac{1}{\cos\theta}$ such that $\frac{1}{\sec\theta} = \cos\theta$. Recall the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$.
- $\frac{\pi}{4}\,\text{rad} = \frac{\pi}{4}\frac{180°}{\pi} = 45°$ and $\frac{\pi}{2}\,\text{rad} = \frac{\pi}{2}\frac{180°}{\pi} = 90°$.
- $\sin\frac{\pi}{2} = \sin 90° = 1$ and $\sin 0 = 0$.
- Note carefully that the upper limit of the θ integrals is $\frac{\pi}{4}$, whereas the upper limit of the u integral is $\frac{\pi}{2}$ because $u = 2\theta$.
- We used a calculator to get the numerical value at the end of the solution.

Part D

❼ $$\int \frac{dx}{(9-x^2)^{3/2}} = \int \frac{dx}{(3^2-x^2)^{3/2}}$$

$$x = a\sin\theta = 3\sin\theta$$

$$dx = a\cos\theta\, d\theta = 3\cos\theta\, d\theta$$

$$\int \frac{dx}{(3^2-x^2)^{3/2}} = \int \frac{3\cos\theta\, d\theta}{[3^2-(3\sin\theta)^2]^{3/2}}$$

$$= 3\int \frac{\cos\theta\, d\theta}{(3^2-3^2\sin^2\theta)^{3/2}}$$

$$= 3\int \frac{\cos\theta\, d\theta}{[3^2(1-\sin^2\theta)]^{3/2}} = 3\int \frac{\cos\theta\, d\theta}{(3^2\cos^2\theta)^{3/2}}$$

$$= 3\int \frac{\cos\theta\, d\theta}{3^3\cos^3\theta} = \frac{3}{3^3}\int \frac{\cos\theta\, d\theta}{\cos^3\theta}$$

$$= \frac{1}{3^2}\int \frac{d\theta}{\cos^2\theta} = \frac{1}{9}\int \sec^2\theta\, d\theta = \frac{1}{9}\tan\theta + c$$

$$= \frac{\sin\theta}{9\cos\theta} + c = \frac{\sin\theta}{9\sqrt{1-\sin^2\theta}} + c$$

$$= \frac{\frac{x}{3}}{9\sqrt{1-\left(\frac{x}{3}\right)^2}} + c$$

$$= \frac{x}{27\sqrt{1-\frac{x^2}{9}}} + c \boxed{= \frac{x}{9\sqrt{9-x^2}} + c}$$

Notes:

- $3^2 - 3^2 \sin^2\theta = 3^2(1-\sin^2\theta) = 3^2\cos^2\theta$ because $\sin^2\theta + \cos^2\theta = 1$.
- $(3^2\cos^2\theta)^{3/2} = (3^2)^{3/2}(\cos^2\theta)^{3/2} = 3^3\cos^3\theta$ because $(x^m)^n = x^{mn}$.
- $\frac{3}{3^3} = \frac{3^1}{3^3} = 3^{1-3} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$ because $\frac{x^m}{x^n} = x^{m-n}$.
- $\frac{\cos\theta}{\cos^3\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$ because $\frac{x}{x^3} = \frac{1}{x^2}$ and $\frac{1}{\cos\theta} = \sec\theta$.

- $\int \sec^2\theta\, d\theta = \tan\theta$ because $\frac{d}{d\theta}\tan\theta = \sec^2\theta$ (as noted in Chapter 3).
- $\sin\theta = \frac{x}{3}$ since $x = 3\sin\theta$ and $\cos\theta = \sqrt{1-\sin^2\theta}$ since $\sin^2\theta + \cos^2\theta = 1$.
- $27\sqrt{1-\frac{x^2}{9}} = (9)(3)\sqrt{1-\frac{x^2}{9}} = 9\sqrt{9}\sqrt{1-\frac{x^2}{9}} = 9\sqrt{9\left(1-\frac{x^2}{9}\right)} = 9\sqrt{9-x^2}$.

❽ $$\int_{\theta=\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^3\theta}\, d\theta$$

$$u = \sin\theta \quad , \quad du = \cos\theta\, d\theta$$

$$u_1 = u\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \sin(30°) = \frac{1}{2}$$

$$u_2 = u\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = \sin(90°) = 1$$

$$\int_{\theta=\pi/6}^{\pi/2} \frac{\cos\theta}{\sin^3\theta}\, d\theta = \int_{u=1/2}^{1} \frac{du}{u^3} = \int_{u=1/2}^{1} u^{-3}\, du$$

$$= \left[\frac{u^{-3+1}}{-3+1}\right]_{u=\frac{1}{2}}^{1} = \left[\frac{u^{-2}}{-2}\right]_{u=\frac{1}{2}}^{1}$$

$$= -\frac{1}{2}[u^{-2}]_{u=\frac{1}{2}}^{1} = -\frac{1}{2}\left[\frac{1}{u^2}\right]_{u=\frac{1}{2}}^{1} = -\frac{1}{2}\left[\frac{1}{1^2} - \frac{1}{(1/2)^2}\right]$$

$$= -\frac{1}{2}(1-4) = \boxed{\frac{3}{2}} = \boxed{1.5}$$

Note that $\frac{\pi}{6}\text{rad} = \frac{\pi}{6}\frac{180°}{\pi} = 30°$, $\frac{\pi}{2}\text{rad} = \frac{\pi}{2}\frac{180°}{\pi} = 90°$, and $\frac{1}{(1/2)^2} = 1 \div \frac{1}{4} = 1 \times \frac{4}{1} = 4$.

15 INTEGRATION BY PARTS

Integration by parts involves using the following formula.

$$\int u\,dv = uv - \int v\,du$$

For a definite integral, remember to evaluate the term **uv** over the limits:

$$\int_i^f u\,dv = [uv]_i^f - \int_i^f v\,du$$

To apply the method of integration by parts, follow these steps:

1. Given an integral of the form $\int f(x)\, dx$, rearrange $f(x)\, dx$ in the form $u\, dv$, as illustrated by the examples that follow. Your goal is to choose u and dv in a way that makes the integral $\int v\, du$ easier to perform than $\int f(x)\, dx$.

- When you choose dv, make sure that you can find its antiderivative, v. Remember that dv includes dx.
- Focus on vdu. You want this to be simpler than udv.
- If the integrand is a product of two functions, like $\int x^2 \ln x\, dx$, first try either $u = x^2$ and $dv = \ln x\, dx$ or $u = \ln x$ and $dv = x^2\, dx$.
- Experience solving a variety of problems can help with this choice.

2. Find the antiderivative of dv. This is v.
3. Find the derivative of u. Multiply by dx. This is du.
4. Use the formula for integration by parts:

$$\int u\,dv = uv - \int v\,du$$

   For a definite integral, remember to evaluate the term uv over the limits:

$$\int_i^f u\,dv = [uv]_i^f - \int_i^f v\,du$$

5. If $\int v\,du$ doesn't seem any easier than $\int f(x)\,dx$ was, try starting over with a new choice for u and dv. It may take a little trial and error to get it right.

Example: Perform the following integral.

$$\int x \sin x \, dx$$

Identify u and dv from $x \sin x \, dx$:

- Neither a derivative nor an antiderivative of $\sin x$ would make this function any simpler than it already is.
- A derivative of x is simpler than an antiderivative of x. Going from u to du involves taking a derivative. Therefore, we should try $u = x$.
- Then $dv = \sin x \, dx$, such that $x \sin x \, dx = u \, dv$.

Use $u = x$ and $dv = \sin x \, dx$ to find du and v:

- Take a derivative of $u = x$ with respect to x and multiply by dx to find du:

$$du = \frac{du}{dx} dx = \left[\frac{d}{dx}(x)\right] dx = 1 \, dx = dx$$

- The antiderivative of $dv = \sin x \, dx$ equals v:

$$v = \int dv = \int \sin x \, dx = -\cos x$$

Plug $u = x$, $v = -\cos x$, $du = dx$, and $dv = \sin x \, dx$ into the equation for integration by parts:

$$\int u \, dv = uv - \int v \, du$$

$$\int x \sin x \, dx = x(-\cos x) - \int (-\cos x) \, dx$$

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + c$$

Check the answer: We can check the answer by taking a derivative. This involves the product rule with $f = x$ and $g = \cos x$.

$$\frac{d}{dx}(-x \cos x + \sin x + c)$$

$$= -\frac{d}{dx}(x \cos x) + \frac{d}{dx}\sin x + \frac{d}{dx}c$$

$$= -\frac{d}{dx}(fg) + \cos x + 0$$

$$= -\left(g\frac{df}{dx} + f\frac{dg}{dx}\right) + \cos x$$

$$= -\left(\cos x \frac{d}{dx} x + x \frac{d}{dx} \cos x\right) + \cos x$$

$$= -[\cos x\,(1) + x(-\sin x)] + \cos x$$

$$= -(\cos x - x \sin x) + \cos x$$

$$= -\cos x + x \sin x + \cos x = x \sin x$$

Distribute the minus sign to go from $-(\cos x - x \sin x)$ to $-\cos x + x \sin x$. The two minus signs effectively make a plus sign.

Example: Perform the following integral.

$$\int_{x=1}^{e} x^3 \ln x \, dx$$

Identify u and dv from $x^3 \ln x \, dx$:

- A derivative of $\ln x$ is simpler than an antiderivative of $\ln x$. Going from u to du involves taking a derivative. Therefore, we should try $u = \ln x$.
- Then $dv = x^3 dx$, such that $x^3 \ln x \, dx = u \, dv$.

Use $u = \ln x$ and $dv = x^3 dx$ to find du and v:

- Take a derivative of $u = \ln x$ with respect to x and multiply by dx to find du:

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} \ln x\right) dx = \frac{1}{x} \, dx = \frac{dx}{x}$$

- The antiderivative of $dv = x^3 dx$ equals v:

$$v = \int dv = \int x^3 \, dx = \frac{x^4}{4}$$

Plug $u = \ln x$, $v = \frac{x^4}{4}$, $du = \frac{dx}{x}$, and $dv = x^3 dx$ into the equation for integration by parts:

$$\int_i^f u\,dv = [uv]_i^f - \int_i^f v\,du$$

$$\int_{x=1}^{e} x^3 \ln x\,dx = \left[(\ln x)\left(\frac{x^4}{4}\right)\right]_{x=1}^{e} - \int_{x=1}^{e} \frac{x^4}{4}\frac{dx}{x}$$

$$= \frac{1}{4}[x^4 \ln x]_{x=1}^{e} - \frac{1}{4}\int_{x=1}^{e} x^3\,dx$$

$$= \frac{1}{4}(e^4 \ln e - 1^4 \ln 1) - \frac{1}{4}\left[\frac{x^4}{4}\right]_{x=1}^{e}$$

$$= \frac{1}{4}[e^4(1) - 1(0)] - \frac{1}{4}\left(\frac{e^4}{4} - \frac{1^4}{4}\right)$$

$$= \frac{1}{4}(e^4 - 0) - \frac{1}{4}\left(\frac{e^4}{4} - \frac{1}{4}\right) = \frac{e^4}{4} - \frac{e^4}{16} + \frac{1}{16}$$

$$= \frac{4e^4 - e^4 + 1}{16} = \frac{3e^4 + 1}{16} \approx 10.3$$

Note the following:

- $\ln 1 = 0$ and $\ln e = 0$.
- $-\frac{1}{4}\left(\frac{e^4}{4} - \frac{1}{4}\right) = -\frac{1}{4}\left(\frac{e^4}{4}\right) - \frac{1}{4}\left(-\frac{1}{4}\right) = -\frac{e^4}{16} + \frac{1}{16}$. The minus gets distributed.
- $\frac{e^4}{4} = \frac{4e^4}{16}$. This was used to make a common denominator.
- We used a calculator to get the numerical value at the end of the solution.

Chapter 15 Exercises

Directions: Perform the following integrals using integration by parts.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\int x\, e^x\, dx =$$

❷ $$\int_{x=0}^{\pi/2} x \cos x \, dx =$$

Part B

❸ $$\int \frac{\ln x}{x^2}\, dx =$$

❹ $$\int_{x=0}^{\pi/6} \sin x \tan x \, dx =$$

Part C

❺ $\int x^2 \cos x \, dx =$

❻ $\int e^x \sin x \, dx =$

Chapter 15 Solutions

Part A

❶ $\int x\, e^x\, dx$

$$u = x \quad , \quad dv = e^x dx$$

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} x\right) dx = 1\, dx = dx$$

$$v = \int dv = \int e^x\, dx = e^x$$

$$\int u\, dv = uv - \int v\, du$$

$$\int x\, e^x\, dx = xe^x - \int e^x\, dx$$

$$= xe^x - e^x + c = \boxed{e^x(x - 1) + c}$$

❷ $\displaystyle\int_{x=0}^{\pi/2} x\cos x\,dx$

$$u = x \quad , \quad dv = \cos x\,dx$$

$$du = \frac{du}{dx}dx = \left(\frac{d}{dx}x\right)dx = 1\,dx = dx$$

$$v = \int dv = \int \cos x\,dx = \sin x$$

$$\int_i^f u\,dv = [uv]_i^f - \int_i^f v\,du$$

$$\int_{x=0}^{\pi/2} x\cos x\,dx = [x\sin x]_{x=0}^{\pi/2} - \int_{x=0}^{\pi/2} \sin x\,dx$$

$$= \frac{\pi}{2}\sin\frac{\pi}{2} - 0\sin 0 - [-\cos x]_{x=0}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}(1) - 0 + [\cos x]_{x=0}^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0$$

$$= \frac{\pi}{2} + 0 - 1 = \boxed{\frac{\pi}{2} - 1} \approx \boxed{0.571}$$

Notes:

- $\int \sin x \, dx = -\cos x + c$.
- $\frac{\pi}{2}$ rad $= \frac{\pi}{2}\frac{180°}{\pi} = 90°$.
- $\sin\frac{\pi}{2} = \sin 90° = 1$ and $\cos\frac{\pi}{2} = \cos 90° = 0$.
- $-[-\cos x]_{x=0}^{\frac{\pi}{2}} = [\cos x]_{x=0}^{\frac{\pi}{2}}$. Two minus signs effectively make a plus sign.
- We used a calculator to get the numerical value at the end of the solution.

Part B

❸ $\int \frac{\ln x}{x^2}\,dx$

$$u = \ln x \quad , \quad dv = \frac{dx}{x^2}$$

$$du = \frac{du}{dx}dx = \left(\frac{d}{dx}\ln x\right)dx = \frac{1}{x}dx = \frac{dx}{x}$$

$$v = \int dv = \int \frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1}$$

$$= -x^{-1} = -\frac{1}{x}$$

$$\int u\,dv = uv - \int v\,du$$

$$\int \frac{\ln x}{x^2}dx = (\ln x)\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\frac{dx}{x}$$

$$= -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} + \int x^{-2}\,dx$$

$$= -\frac{\ln x}{x} + \frac{x^{-2+1}}{-2+1} + c = -\frac{\ln x}{x} + \frac{x^{-1}}{-1} + c$$

$$= -\frac{\ln x}{x} - x^{-1} + c = -\frac{\ln x}{x} + -\frac{1}{x} + c$$

$$= \boxed{-\frac{1}{x}(\ln x + 1) + c} \qquad \left(\text{factor out } -\frac{1}{x}\right)$$

❹ $\int\limits_{x=0}^{\pi/6} \sin x \tan x \, dx$

$$u = \tan x \quad , \quad dv = \sin x \, dx$$

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} \tan x\right) dx = \sec^2 x \;\; dx$$

$$v = \int dv = \int \sin x \, dx = -\cos x$$

$$\int\limits_{x=0}^{\pi/6} \sin x \tan x \, dx$$

$$= [(\tan x)(-\cos x)]_{x=0}^{\pi/6}$$

$$- \int\limits_{x=0}^{\pi/6} (-\cos x) \sec^2 x \, dx$$

$$= -[\cos x \tan x]_{x=0}^{\pi/6} + \int_{x=0}^{\pi/6} \cos x \sec^2 x \, dx$$

$$= -\left[\cos x \frac{\sin x}{\cos x}\right]_{x=0}^{\pi/6} + \int_{x=0}^{\pi/6} \cos x \frac{1}{\cos^2 x} dx$$

$$= -[\sin x]_{x=0}^{\pi/6} + \int_{x=0}^{\pi/6} \frac{dx}{\cos x}$$

$$= -\left(\sin \frac{\pi}{6} - \sin 0\right) + \int_{x=0}^{\pi/6} \sec x \, dx$$

$$= -\left(\frac{1}{2} - 0\right) + [\ln|\sec \theta + \tan \theta|]_{x=0}^{\frac{\pi}{6}}$$

$$= -\frac{1}{2} + \ln \left|\sec \frac{\pi}{6} + \tan \frac{\pi}{6}\right|$$
$$- \ln|\sec 0 + \tan 0|$$

$$= -\frac{1}{2} + \ln \left|\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right| - \ln(1 + 0)$$

$$= -\frac{1}{2} + \ln \frac{3}{\sqrt{3}} - \ln 1 = \boxed{-\frac{1}{2} + \ln \sqrt{3}}$$

$$\approx \boxed{0.0493}$$

Notes:

- $\frac{d}{dx}\tan x = \sec^2 x$.
- $\sec x = \frac{1}{\cos x}$ and $\tan x = \frac{\sin x}{\cos x}$.
- $\frac{\pi}{6}$ rad $= \frac{\pi}{6}\frac{180°}{\pi} = 30°$.
- $\sec\frac{\pi}{6} = \sec 30° = \frac{1}{\cos 30°} = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}$ and $\tan\frac{\pi}{6} = \tan 30° = \frac{1}{\sqrt{3}}$. (Note that $\frac{2}{\sqrt{3}}$ is the same as $\frac{2\sqrt{3}}{3}$ and that $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ if you rationalize the denominators.)
- $\ln 1 = 0$.
- We used a calculator to get the numerical value at the end of the solution.

Part C

❺ $$\int x^2 \cos x \, dx$$

$$u = x^2 \quad , \quad dv = \cos x \, dx$$

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} x^2\right) dx = 2x \, dx$$

$$v = \int dv = \int \cos x \, dx = \sin x$$

$$\int u \, dv = uv - \int v \, du$$

$$\int x^2 \cos x \, dx = x^2 \sin x - \int \sin x \, (2x \, dx)$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

Now integrate by parts a second time, with a new choice for u and dv:

$$u = x \quad , \quad dv = \sin x \, dx$$

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} x\right) dx = 1 \, dx = dx$$

$$v = \int dv = \int \sin x \, dx = -\cos x$$

$$x^2 \sin x - 2\int x \sin x \, dx$$

$$= x^2 \sin x - 2\left(uv - \int v \, du\right)$$

$$= x^2 \sin x - 2uv + 2\int v \, du$$

$$= x^2 \sin x - 2x(-\cos x) + 2\int (-\cos x) \, dx$$

$$= x^2 \sin x + 2x \cos x - 2\int \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$= \boxed{(x^2 - 2) \sin x + 2x \cos x + c}$$

❻ $\int e^x \sin x \, dx$

$$u = \sin x \quad , \quad dv = e^x dx$$

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} \sin x \right) dx = \cos x \ \ dx$$

$$v = \int dv = \int e^x \, dx = e^x$$

$$\int e^x \sin x \, dx = (\sin x) e^x - \int e^x \cos x \, dx$$

$$= e^x \sin x - \int e^x \cos x \, dx$$

Now integrate by parts a second time, with a new choice for u and dv:

$$u = \cos x \quad , \quad dv = e^x dx$$

$$du = \frac{du}{dx} dx = \left(\frac{d}{dx} \cos x \right) dx = - \sin x \ \ dx$$

$$v = \int dv = \int e^x \, dx = e^x$$

$$e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x$$

$$- \left[(\cos x) e^x - \int e^x (-\sin x) \, dx \right]$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

Set the original integral equal to the last equation above:

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

The 'trick' is to add $\int e^x \sin x \, dx$ to both sides (and add the constant of integration):

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + c$$

Divide both sides of the equation by 2.

$$\int e^x \sin x \, dx = \frac{e^x \sin x}{2} - \frac{e^x \cos x}{2} + c$$

$$= \boxed{\frac{e^x}{2} (\sin x - \cos x) + c}$$

Check your answer: Take a derivative, applying the product rule.

$$\frac{d}{dx}\left[\frac{e^x}{2}(\sin x - \cos x) + c\right] = \frac{d}{dx}(fg) + \frac{d}{dx}c$$

$$= \frac{d}{dx}(fg) + 0 = g\frac{df}{dx} + f\frac{dg}{dx}$$

$$= (\sin x - \cos x)\frac{d}{dx}\frac{e^x}{2} + \frac{e^x}{2}\frac{d}{dx}(\sin x - \cos x)$$

$$= (\sin x - \cos x)\left(\frac{e^x}{2}\right) + \frac{e^x}{2}(\cos x + \sin x)$$

$$= \frac{e^x}{2}(\sin x - \cos x + \cos x + \sin x)$$

$$= \frac{e^x}{2}(2\sin x) = e^x \sin x$$

16 MULTIPLE INTEGRALS

To perform a double or triple integral, follow these steps:

1. Note that you may reverse the order of the differential elements. Don't worry about whether you see **dxdy** or **dydx**. The order of the differential elements doesn't tell you which integral to do first.
2. Look at the limits of integration: If you see an integration variable (like **x**, **y**, or **z**) in the limits of integration, you must perform the integral with the variable limit first. You will see variable limits in three of the examples that follow.
3. When you integrate over one variable, treat the other independent variables as if they are constants. For example, when performing an integral over the variable **x**, treat **y** as a constant. Similarly, when performing an integral over the variable **y**, treat **x** as a constant. We will see this in the examples.
4. After you finish one integral, evaluate its antiderivative over the limits before you begin the next integral.

Example: Perform the following integral.

$$\int_{x=0}^{2} \int_{y=0}^{x^2} xy \, dx \, dy$$

We must integrate over y first because the integral over y has a variable limit (x^2). When we integrate over y, we treat the independent variable x as a constant. Pull x out of the y integral (but be careful not to pull x out of the x integral).

$$\int_{x=0}^{2} \int_{y=0}^{x^2} xy \, dx \, dy = \int_{x=0}^{2} x \left(\int_{y=0}^{x^2} y \, dy \right) dx$$

$$= \int_{x=0}^{2} x \left[\frac{y^2}{2} \right]_{y=0}^{x^2} dx = \frac{1}{2} \int_{x=0}^{2} x [y^2]_{y=0}^{x^2} \, dx$$

$$= \frac{1}{2} \int_{x=0}^{2} x[(x^2)^2 - 0^2]\, dx = \frac{1}{2} \int_{x=0}^{2} x x^4\, dx$$

$$= \frac{1}{2} \int_{x=0}^{2} x^5\, dx = \frac{1}{2} \left[\frac{x^6}{6}\right]_{x=0}^{2} = \frac{1}{12} [x^6]_{x=0}^{2}$$

$$= \frac{1}{12} (2^6 - 0^6) = \frac{1}{12} (64) = \frac{64}{12} = \frac{64 \div 4}{12 \div 4}$$

$$= \frac{16}{3} \approx 5.333$$

Example: Perform the following integral.

$$\int_{x=0}^{1/y} \int_{y=1}^{3} xy^2 \, dx \, dy$$

We must integrate over x first because the integral over x has a variable limit $(1/y)$. When we integrate over x, we treat the independent variable y as a constant. Pull y out of the x integral (but be careful not to pull y out of the y integral).

$$\int_{x=0}^{1/y} \int_{y=1}^{3} xy^2 \, dx \, dy = \int_{y=1}^{3} y^2 \left(\int_{x=0}^{1/y} x \, dx \right) dy$$

$$= \int_{y=1}^{3} y^2 \left[\frac{x^2}{2} \right]_{x=0}^{1/y} dy = \frac{1}{2} \int_{y=1}^{3} y^2 [x^2]_{x=0}^{1/y} \, dy$$

$$= \frac{1}{2} \int_{y=1}^{3} y^2 \left[\left(\frac{1}{y}\right)^2 - 0^2\right] dy = \frac{1}{2} \int_{y=1}^{3} \frac{y^2}{y^2} dy$$

$$= \frac{1}{2} \int_{y=1}^{3} dy = \frac{1}{2} [y]_{y=1}^{3} = \frac{1}{2} (3 - 1)$$

$$= \frac{2}{2} = 1$$

Example: Perform the following integral.

$$\int_{x=0}^{1/3} \int_{y=0}^{3} x^2 y^2 \, dx \, dy$$

Since all of the limits are constants, we may perform these integrals in any order.

$$\int_{x=0}^{1/3} \int_{y=0}^{3} x^2 y^2 \, dx \, dy = \int_{x=0}^{1/3} x^2 \left(\int_{y=0}^{3} y^2 \, dy \right) dx$$

$$= \int_{x=0}^{1/3} x^2 \left[\frac{y^3}{3} \right]_{y=0}^{3} dx$$

$$= \frac{1}{3} \int_{x=0}^{1/3} x^2 [y^3]_{y=0}^{3} \, dx = \frac{1}{3} \int_{x=0}^{1/3} x^2 (3^3 - 0^3) \, dx$$

$$= \frac{1}{3} \int_{x=0}^{1/3} x^2 (27) \, dx = 9 \int_{x=0}^{1/3} x^2 \, dx$$

$$= 9 \left[\frac{x^3}{3}\right]_{x=0}^{1/3} = 3[x^3]_{x=0}^{1/3} = 3\left[\left(\frac{1}{3}\right)^3 - 0^3\right] = 3\left(\frac{1}{27}\right)$$

$$= \frac{3}{27} = \frac{3 \div 3}{27 \div 3} = \frac{1}{9} \approx 0.111$$

Example: Perform the following integral.

$$\int_{x=0}^{1} \int_{y=0}^{x^2} \int_{z=0}^{y} xyz \, dx \, dy \, dz$$

We must integrate over y and z before x because the integrals over y and z have variable limits (x^2 and y). Furthermore, we must integrate over z before y because the limit in the z integration includes y. We will first integrate over z, then integrate over y, and lastly integrate over x.

When we integrate over z, we treat the independent variables x and y as constants. Pull x and y out of the z integral (but be careful not to pull y out of the y integral).

$$\int_{x=0}^{1} \int_{y=0}^{x^2} \int_{z=0}^{y} xyz \, dx \, dy \, dz$$

$$= \int_{x=0}^{1} x \int_{y=0}^{x^2} y \left(\int_{z=0}^{y} z \, dz \right) dy \, dx$$

$$= \int_{x=0}^{1} x \int_{y=0}^{x^2} y \left[\frac{z^2}{2} \right]_{z=0}^{y} dy \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x \int_{y=0}^{x^2} y[z^2]_{z=0}^{y} \, dy \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x \int_{y=0}^{x^2} y(y^2 - 0) \, dy \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x \int_{y=0}^{x^2} yy^2 \, dy \, dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x \int_{y=0}^{x^2} y^3 \, dy \, dx = \frac{1}{2} \int_{x=0}^{1} x \left(\int_{y=0}^{x^2} y^3 \, dy \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{1} x \left[\frac{y^4}{4} \right]_{y=0}^{x^2} dx$$

$$= \frac{1}{8} \int_{x=0}^{1} x [y^4]_{y=0}^{x^2} \, dx = \frac{1}{8} \int_{x=0}^{1} x[(x^2)^4 - 0^4] \, dx$$

$$= \frac{1}{8} \int_{x=0}^{1} x(x^8 - 0) \, dx = \frac{1}{8} \int_{x=0}^{1} x x^8 \, dx$$

$$= \frac{1}{8} \int_{x=0}^{1} x^9 \, dx = \frac{1}{8} \left[\frac{x^{10}}{10} \right]_{x=0}^{1} = \frac{1}{80} [x^{10}]_{x=0}^{1}$$

$$= \frac{1}{80} (1^{10} - 0^{10})$$

$$= \frac{1}{80} (1 - 0) = \frac{1}{80} = 0.0125$$

Note that $(x^2)^4 = x^8$ according to the rule $(x^m)^n = x^{mn}$.

Chapter 16 Exercises

Directions: Perform the following integrals.

Write your solutions down on a piece of paper. You can find the full solution to each problem at the end of this chapter.

Part A

❶ $$\int_{x=0}^{3}\int_{y=0}^{\sqrt{x}} xy\,dx\,dy =$$

❷ $$\int_{x=0}^{y^2}\int_{y=1}^{2} \frac{y^2}{\sqrt{x}}\,dx\,dy =$$

Part B

❸ $$\int_{x=1}^{4}\int_{y=4}^{9}\frac{dxdy}{\sqrt{xy}} =$$

❹ $$\int_{x=0}^{5}\int_{y=x}^{2x} x^2y\,dx\,dy =$$

Part C

❺ $$\int_{x=0}^{y}\int_{y=0}^{2}\int_{z=0}^{x} xy^2z^3\,dx\,dy\,dz =$$

❻ $$\int_{x=0}^{\sqrt{y}}\int_{y=0}^{\sqrt{z}}\int_{z=0}^{4} x^3y\,dx\,dy\,dz =$$

Chapter 16 Solutions

Part A

❶ $$\int_{x=0}^{3}\int_{y=0}^{\sqrt{x}} xy\,dx\,dy = \int_{x=0}^{3} x\left(\int_{y=0}^{\sqrt{x}} y\,dy\right)dx$$

$$= \int_{x=0}^{3} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{x}} dx = \frac{1}{2}\int_{x=0}^{3} x[y^2]_{y=0}^{\sqrt{x}}\,dx$$

$$= \frac{1}{2}\int_{x=0}^{3} x\left[\left(\sqrt{x}\right)^2 - 0^2\right]dx = \frac{1}{2}\int_{x=0}^{3} x(x-0)\,dx$$

$$= \frac{1}{2}\int_{x=0}^{3} x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{x=0}^{3} = \frac{1}{6}[x^3]_{x=0}^{3}$$

$$= \frac{1}{6}(3^3 - 0^3) = \frac{27}{6} = \boxed{\frac{9}{2}} = \boxed{4.5}$$

❷ $$\int_{x=0}^{y^2}\int_{y=1}^{2}\frac{y^2}{\sqrt{x}}\,dx\,dy = \int_{y=1}^{2} y^2\left(\int_{x=0}^{y^2}\frac{dx}{\sqrt{x}}\right)dy$$

$$= \int_{y=1}^{2} y^2\left(\int_{x=0}^{y^2}\frac{dx}{x^{1/2}}\right)dy$$

$$= \int_{y=1}^{2} y^2\left(\int_{x=0}^{y^2} x^{-1/2}\,dx\right)dy$$

$$= \int_{y=1}^{2} y^2 \left[\frac{x^{-1/2+1}}{-\frac{1}{2}+1} \right]_{x=0}^{y^2} dy$$

$$= \int_{y=1}^{2} y^2 \left[\frac{x^{1/2}}{1/2} \right]_{x=0}^{y^2} dy$$

$$= \int_{y=1}^{2} y^2 \left[2x^{1/2}\right]_{x=0}^{y^2} dy = \int_{y=1}^{2} y^2 \left[2\sqrt{x}\right]_{x=0}^{y^2} dy$$

$$= 2 \int_{y=1}^{2} y^2 \left[\sqrt{x}\right]_{x=0}^{y^2} dy$$

$$= 2 \int_{y=1}^{2} y^2 \left(\sqrt{y^2} - \sqrt{0}\right) dy = 2 \int_{y=1}^{2} y^2 y \, dy$$

$$= 2 \int_{y=1}^{2} y^3 \, dy = 2 \left[\frac{y^4}{4}\right]_{y=1}^{2}$$

$$= \frac{2}{4}[y^4]_{y=1}^{2} = \frac{1}{2}[y^4]_{y=1}^{2} = \frac{1}{2}[2^4 - 1^4]$$

$$= \frac{1}{2}(16 - 1) = \boxed{\frac{15}{2}} = \boxed{7.5}$$

Notes:

- $-\frac{1}{2} + 1 = -\frac{1}{2} + \frac{2}{2} = \frac{-1+2}{2} = \frac{1}{2}$ (add fractions with a common denominator).
- $\frac{1}{1/2} = 1 \div \frac{1}{2} = 1 \times \frac{2}{1} = 2$ (to divide by a fraction, multiply by its reciprocal).

Part B

❸ $$\int_{x=1}^{4}\int_{y=4}^{9}\frac{dx\,dy}{\sqrt{xy}}=\int_{x=1}^{4}\int_{y=4}^{9}\frac{dx\,dy}{\sqrt{x}\sqrt{y}}$$

$$=\int_{x=1}^{4}\frac{1}{\sqrt{x}}\left(\int_{y=4}^{9}\frac{dy}{\sqrt{y}}\right)dx$$

$$=\int_{x=1}^{4}\frac{1}{\sqrt{x}}\left(\int_{y=4}^{9}\frac{dy}{y^{1/2}}\right)dx$$

$$=\int_{x=1}^{4}\frac{1}{\sqrt{x}}\left(\int_{y=4}^{9}y^{-1/2}\,dy\right)dx$$

$$= \int_{x=1}^{4} \frac{1}{\sqrt{x}} \left[\frac{y^{-1/2+1}}{-\frac{1}{2}+1} \right]_{y=4}^{9} dx$$

$$= \int_{x=1}^{4} \frac{1}{\sqrt{x}} \left[\frac{y^{1/2}}{1/2} \right]_{y=4}^{9} dx$$

$$= \int_{x=1}^{4} \frac{1}{\sqrt{x}} \left[2y^{1/2}\right]_{y=4}^{9} dx = 2 \int_{x=1}^{4} \frac{1}{\sqrt{x}} \left[y^{1/2}\right]_{y=4}^{9} dx$$

$$= 2 \int_{x=1}^{4} \frac{1}{\sqrt{x}} \left[\sqrt{y}\right]_{y=4}^{9} dx$$

$$= 2 \int_{x=1}^{4} \frac{1}{\sqrt{x}} \left(\sqrt{9} - \sqrt{4}\right) dx = 2 \int_{x=1}^{4} \frac{1}{\sqrt{x}} (3 - 2)\, dx$$

$$= 2 \int_{x=1}^{4} \frac{1}{\sqrt{x}} (1)\, dx = 2 \int_{x=1}^{4} \frac{dx}{\sqrt{x}}$$

$$= 2 \int_{x=1}^{4} \frac{dx}{x^{1/2}} = 2 \int_{x=1}^{4} x^{-1/2}\, dx = 2 \left[\frac{x^{-1/2+1}}{-\frac{1}{2} + 1}\right]_{x=1}^{4}$$

$$= 2 \left[\frac{x^{1/2}}{1/2}\right]_{x=1}^{4} = 2\left[2x^{1/2}\right]_{x=1}^{4}$$

$$= 4\left[x^{1/2}\right]_{x=1}^{4} = 4\left[\sqrt{x}\right]_{x=1}^{4} = 4\left(\sqrt{4} - \sqrt{1}\right)$$

$$= 4(2 - 1) = 4(1) = \boxed{4}$$

❹ $$\int_{x=0}^{5}\int_{y=x}^{2x} x^2 y\,dx\,dy = \int_{x=0}^{5} x^2\left(\int_{y=x}^{2x} y\,dy\right)dx$$

$$= \int_{x=0}^{5} x^2\left[\frac{y^2}{2}\right]_{y=x}^{2x} dx$$

$$= \frac{1}{2}\int_{x=0}^{5} x^2[y^2]_{y=x}^{2x}\,dx = \frac{1}{2}\int_{x=0}^{5} x[(2x)^2 - x^2]\,dx$$

$$= \frac{1}{2}\int_{x=0}^{5} x(4x^2 - x^2)\,dx$$

$$= \frac{1}{2}\int_{x=0}^{5} x^2(3x^2)\,dx = \frac{3}{2}\int_{x=0}^{5} x^4\,dx = \frac{3}{2}\left[\frac{x^5}{5}\right]_{x=0}^{5}$$

$$= \frac{3}{10}[x^5]_{x=0}^{5} = \frac{3}{10}(5^5 - 0^5)$$

$$= \frac{3(3125)}{10} = \frac{3(625)}{2} = \boxed{\frac{1875}{2}} = \boxed{937.5}$$

Note that $(2x)^2 = 2^2x^2 = 4x^2$ because $x^m x^n = x^{m+n}$.

Part C

❺ $$\int_{x=0}^{y}\int_{y=0}^{2}\int_{z=0}^{x} xy^2z^3\,dx\,dy\,dz$$

$$= \int_{y=0}^{2} y^2 \int_{x=0}^{y} x\left(\int_{z=0}^{x} z^3\,dz\right)dx\,dy$$

$$= \int_{y=0}^{2} y^2 \int_{x=0}^{y} x\left[\frac{z^4}{4}\right]_{z=0}^{x} dx\,dy$$

$$= \frac{1}{4}\int_{y=0}^{2} y^2 \int_{x=0}^{y} x[z^4]_{z=0}^{x}\,dx\,dy$$

$$= \frac{1}{4} \int_{y=0}^{2} y^2 \int_{x=0}^{y} x(x^4 - 0^4)\, dx\, dy$$

$$= \frac{1}{4} \int_{y=0}^{2} y^2 \int_{x=0}^{y} x x^4\, dx\, dy$$

$$= \frac{1}{4} \int_{y=0}^{2} y^2 \left(\int_{x=0}^{y} x^5\, dx \right) dy$$

$$= \frac{1}{4} \int_{y=0}^{2} y^2 \left[\frac{x^6}{6}\right]_{x=0}^{y} dy = \frac{1}{24} \int_{y=0}^{2} y^2 [x^6]_{x=0}^{y} \, dy$$

$$= \frac{1}{24} \int_{y=0}^{2} y^2 (y^6 - 0^6) \, dy$$

$$= \frac{1}{24} \int_{y=0}^{2} y^2 y^6 \, dy = \frac{1}{24} \int_{y=0}^{2} y^8 \, dy = \frac{1}{24} \left[\frac{y^9}{9}\right]_{y=0}^{2}$$

$$= \frac{1}{216} [y^9]_{y=0}^{2} = \frac{2^9 - 0^9}{216} = \frac{512}{216}$$

$$= \boxed{\frac{64}{27}} \approx \boxed{2.370}$$

❻ $$\int_{x=0}^{\sqrt{y}} \int_{y=0}^{\sqrt{z}} \int_{z=0}^{4} x^3 y \, dx \, dy \, dz$$

$$= \int_{z=0}^{4} \int_{y=0}^{\sqrt{z}} y \left(\int_{x=0}^{\sqrt{y}} x^3 \, dx \right) dy \, dz$$

$$= \int_{z=0}^{4} \int_{y=0}^{\sqrt{z}} y \left[\frac{x^4}{4} \right]_{x=0}^{\sqrt{y}} dy \, dz$$

$$= \frac{1}{4} \int_{z=0}^{4} \int_{y=0}^{\sqrt{z}} y[x^4]_{x=0}^{\sqrt{y}} \, dy \, dz$$

$$= \frac{1}{4} \int_{z=0}^{4} \int_{y=0}^{\sqrt{z}} y\left[\left(\sqrt{y}\right)^4 - 0^4\right] dy \, dz$$

$$= \frac{1}{4} \int_{z=0}^{4} \int_{y=0}^{\sqrt{z}} y(y^2) \, dy \, dz$$

$$= \frac{1}{4} \int_{z=0}^{4} \left(\int_{y=0}^{\sqrt{z}} y^3 \, dy \right) dz = \frac{1}{4} \int_{z=0}^{4} \left[\frac{y^4}{4} \right]_{y=0}^{\sqrt{z}} dz$$

$$= \frac{1}{16} \int_{z=0}^{4} [y^4]_{y=0}^{\sqrt{z}} \, dz$$

$$= \frac{1}{16} \int_{z=0}^{4} \left[\left(\sqrt{z} \right)^4 - 0^4 \right] dz$$

$$= \frac{1}{16} \int_{z=0}^{4} z^2 \, dz = \frac{1}{16} \left[\frac{z^3}{3} \right]_{z=0}^{4} = \frac{1}{48} [z^3]_{z=0}^{4}$$

$$= \frac{1}{48} (4^3 - 0) = \frac{64}{48} = \boxed{\frac{4}{3}} \approx \boxed{1.333}$$

WAS THIS BOOK HELPFUL?

A great deal of effort and thought was put into this book, such as:

- Breaking down the solutions to help make the math easier to understand.
- Careful selection of examples and problems for their instructional value.
- Full solutions to every problem in the answer key, including helpful notes.
- Every answer was independently checked by an international math guru.
- Multiple stages of proofreading, editing, and formatting.
- Beta testers provided valuable feedback.
- Elaborate conversion to Kindle format.

If you appreciate the effort that went into making this book possible, there is a simple way that you could show it:

<u>Please take a moment to post an honest review.</u>

For example, you can review this book at Amazon.com or Barnes & Noble's website at BN.com.

Even a short review can be helpful and will be much appreciated. If you're not sure what to write, following are a few ideas, though it's best to describe what's important to you.

- How much did you learn from reading and using this workbook?
- Were the solutions at the back of the book helpful?
- Were you able to understand the solutions?
- Was it helpful to follow the examples while solving the problems?

- Would you recommend this book to others? If so, why?

Do you believe that you found a mistake? Please email the author, Chris McMullen, at mailto:greekphysics@yahoo.com to ask about it. One of two things will happen:

- You might discover that it wasn't a mistake after all and learn why.
- You might be right, in which case the author will be grateful and future readers will benefit from the correction. Everyone is human.

ABOUT THE AUTHOR

Dr. Chris McMullen has over 20 years of experience teaching university physics in California, Oklahoma, Pennsylvania, and Louisiana. Dr. McMullen is also an author of math and science books. Whether in the classroom or as a writer, Dr. McMullen loves sharing knowledge and the art of motivating and engaging students.

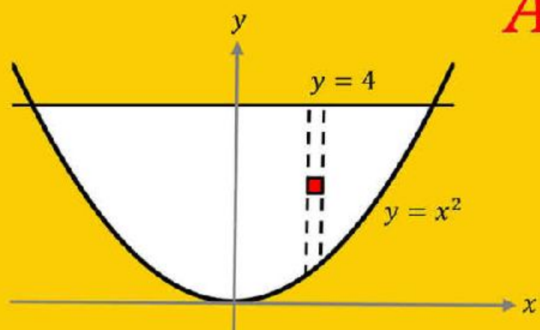
Chris McMullen earned his Ph.D. in phenomenological high-energy physics (particle physics) from Oklahoma State University in 2002. Originally from California, Dr. McMullen earned his Master's degree from California State University, Northridge, where his thesis was in the field of electron spin resonance.

As a physics teacher, Dr. McMullen observed that many students lack fluency in fundamental math skills. In an effort to help students of all ages and levels master basic math skills, he published a series of math workbooks on arithmetic, fractions, long division, algebra, trigonometry, and calculus entitled *Improve Your Math Fluency*. Dr. McMullen has also published a variety of science books, including introductions to basic astronomy and chemistry concepts in addition to physics workbooks.

www.amazon.com/author/chrismcmullen

50 CHALLENGING CALCULUS PROBLEMS

$$A = \int_{x=-2}^{2} (4 - x^2)\, dx$$



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50 CHALLENGING ALGEBRA PROBLEMS

$3x - 2y$

$9x^2 - 12xy + 4y^2$

$27x^3 - 54x^2y + 36xy^2 - 8y^3$

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Vol. 1

Essential Calculus-based

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Study Guide Workbook

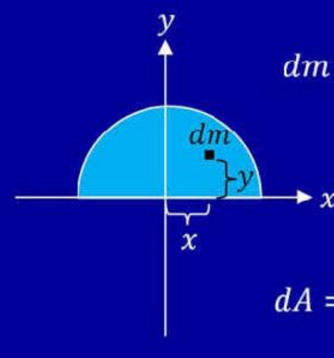
Volume 1: The Laws of Motion

$$y_{cm} = \frac{1}{m}\int y\,dm \qquad m = \int dm = \int \sigma\,dA = \frac{\sigma\pi R^2}{2}$$

$$y_{cm} = \frac{\sigma}{m}\int_{r=0}^{R}\int_{\theta=0}^{\pi}(r\sin\theta)\,r\,dr\,d\theta$$

$$dm = \sigma dA$$

$$y_{cm} = \frac{\sigma}{m}\int_{r=0}^{R} r^2[-\cos\pi - (-\cos 0)]\,dr$$



$$y_{cm} = \frac{2\sigma}{m}\int_{r=0}^{R} r^2\,dr = \frac{2\sigma R^3}{3m}$$

$$dA = rdrd\theta$$

$$y_{cm} = \frac{2\sigma R^3}{3}\frac{2}{\sigma\pi R^2} = \frac{4R}{3\pi}$$

$$y = r\sin\theta$$

Chris McMullen, Ph.D.

Includes Answers!

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Dr. McMullen has published a variety of science books, including:

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- Calculus-based physics textbook
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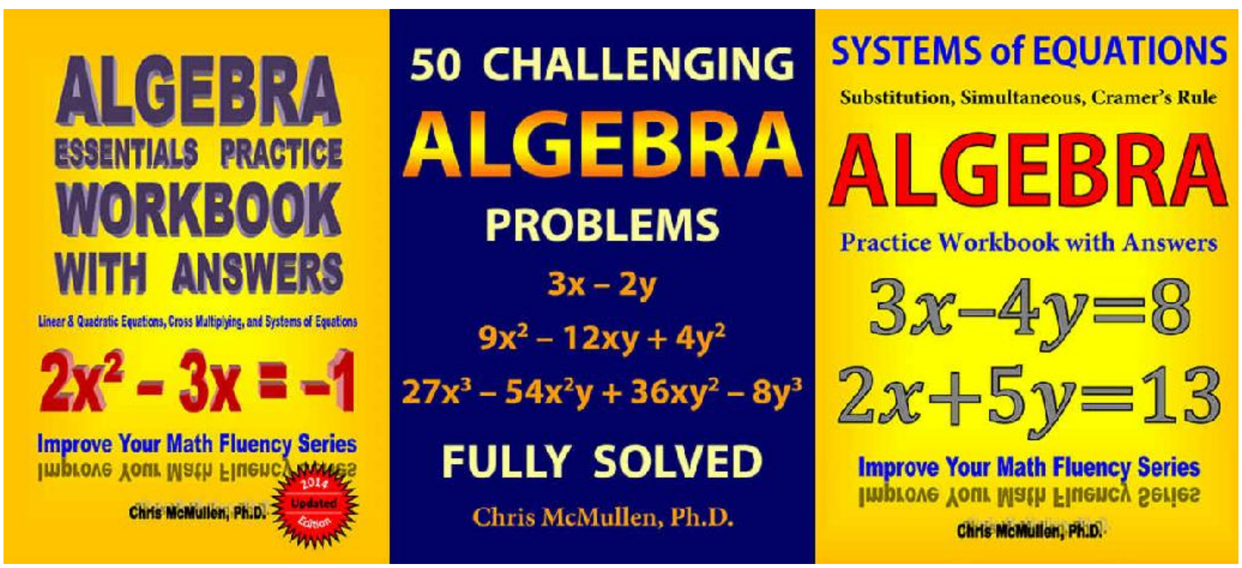

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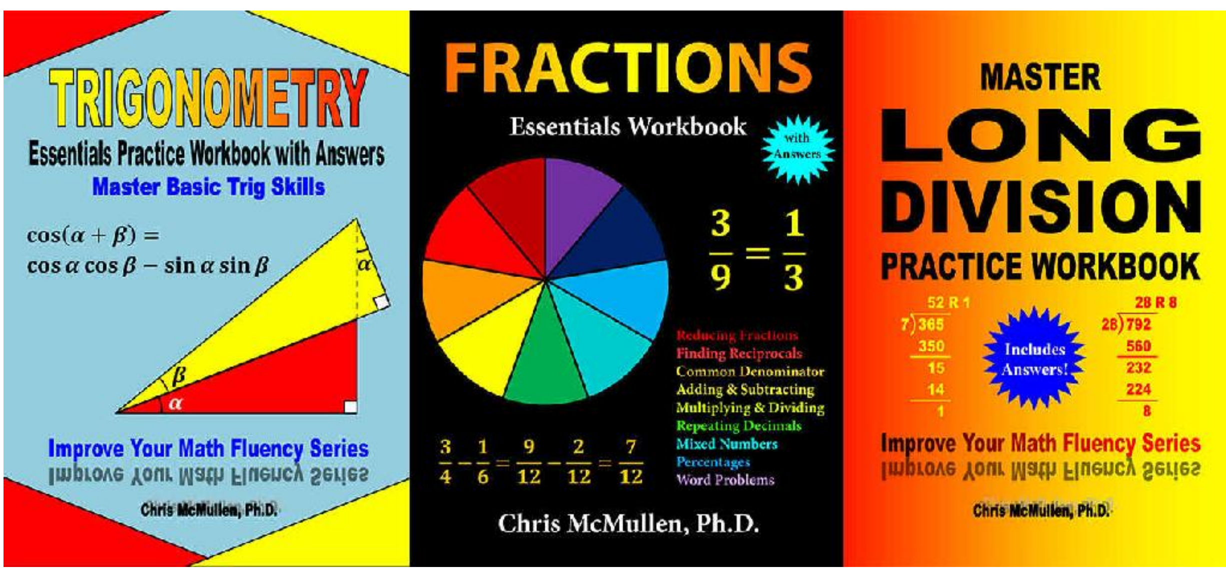
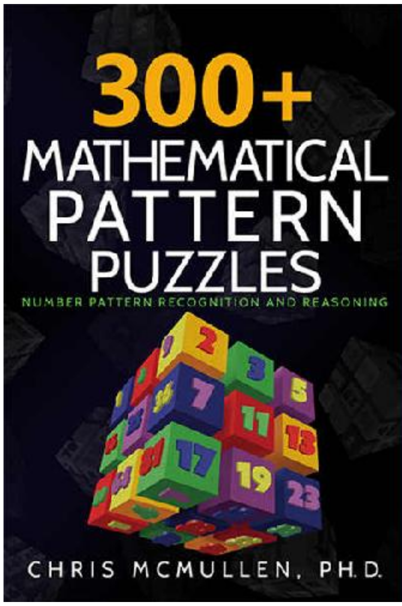


PUZZLES

The author of this book, Chris McMullen, enjoys solving puzzles. His favorite puzzle is Kakuro (kind of like a cross between crossword puzzles and Sudoku). He once taught a three-week summer course on puzzles. If you enjoy mathematical pattern puzzles, you might appreciate:

Number Pattern Recognition & Reasoning:

- Pattern recognition
- Visual discrimination
- Analytical skills
- Logic and reasoning
- Analogies
- Mathematics


300+
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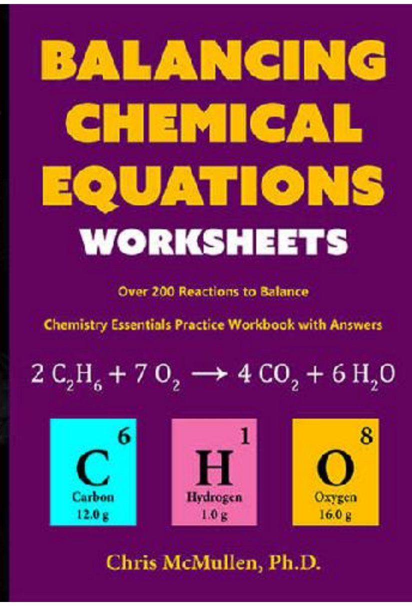
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O 8 Oxygen 16.0 g
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Word Scrambles with a Chemical Flavor
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2 C + N + 2 I + P → P I C N I C
Ti + C + Cr + P + Y → Cr Y P Ti C
2 C + U + 2 S + Es → S U C C Es S
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- Explore the world of hypercubes and hyperspheres.
- Imagine living in a two-dimensional world.
- Try to understand the fourth dimension by analogy.
- Several illustrations help to try to visualize a fourth dimension of space.
- Investigate hypercube patterns.
- What would it be like to be a four-dimensional being living in a four-dimensional world?
- Learn about the physics of a possible four-dimensional universe.