Homework 8

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Problem 1. Which of the following statements about graph G and H are true?

1. G and H are isomorphic if and only if for every map $f : V(G) \to V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.

2. G and H are isomorphic if and only if there exists a bijection $f : E(G) \to E(H)$.

3. If there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.

4. If G and H are isomorphic, then there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.

5. If G and H are isomorphic, then there exists a bijection $f : E(G) \to E(H)$.

6. G and H are isomorphic if and only if there exists a map $f : V(G) \to V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.

7. Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \ldots, n\}$.

8. Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.

Solution. Statement 4,5,6 and 7 are true. □

Problem 2. Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \to V'$. Now if f satisfies:

i) It is a bijective function;

ii) $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;

Then we say that graph G and G' are isomorphic to each other. We use $G \cong G'$ to stand for the isomorphism relation.

Consider the following questions:

1. $G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \to V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n-vertices.

2. $G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called complete bipartite graphs), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.

Solution.

1. Proof. Since G has n vertices and function g is a map from V to V', $Ran(g) \leq n$. Since G' contains n vertices, g is a bijective function. So, G and G' are isomorphic, which means the statement is true. □

2. The simplest graph G' is a graph contains two vertices and an edge that connects the two vertices.

□

Problem 3. How many graphs on the vertex set $\{1, 2, \ldots, 2n\}$ are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set {{1,2},{3,4},..., {2n-1,2n}}?

Solution. Suppose the number of graphs isomorphic to graph with edge set {{1,2},{3,4},..., {2n-1,2n}} is $T(2n)$. By induction, $T(2n) = (2n-1)T(2n-2) = (2n-1)(2n-3)T(2n-4) = \ldots = (2n-1)!!$. □

Problem 4. Construct an example of a sequence of length n in which each term is some of the numbers $1, 2, \ldots, n-1$ and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.

Solution. We can know that $n > 3$. We reorder the sequence so that the value is increasing. The first element is 1 and the last two elements are $n-1, n-1$. The other number can be randomly chosen to satisfies the condition that "has an even number of odd terms".
To prove it not a graph score, we first delete the last element $n-1$ and change the value of remainded number. So, the first element becomes 0 and the last element becomes $n-2$. When we delete $n-2$, the action cannot continue, so the sequence isn't a graph score. □

Problem 5. Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

Proof. We will prove it by contradiction. Hypothesis that G at most has 4 vertices of degree 6 and at most has 5 vertices of degree 5. Since G has 9 vertices and the degree of each vertex is 5 or 6, G has exactly 4 vertices of 6 and 5 vertices of 5.
The total degree is $4 \times 6 + 5 \times 5 = 49$. According to handshake lemma, the answer is wrong. So the hypothesis is wrong. The original statement is true. □

Problem 6. Given a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers (where $n \geq 1$):

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$.

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$.

Prove that (i) and (ii) are equivalent.

Proof. $(i) \leftarrow (ii)$:
The number of edges of a tree with n vertices is $n-1$. According to handshake lemma, $\sum_{i=1}^{n} d_i = 2(n-1) = 2n-2$. The statement is proved.
$(i) \leftarrow (ii)$:
According to handshake lemma, the number of edges is $n-1$. For there are n vertices with the sum of all degree is $2n-2$, there must be a vertex with degree 1. Regard it as v_n. It's a leaf obviously. We remove v_n from the score, and there remains $n-1$ vertices and the sum of degree becomes $2n-2-2 = 2(n-1)-2$. So, by induction, everytime we remove one leaf from the graph, and the graph finally becomes empty, which satisfies the definition of the tree. The statement is proved. □

Problem 7. Let N_k denote the number of spanning trees of K_n in which the vertex n has degree k, $k = 1, 2, \ldots, n-1$ (recall that we assume $V(K_n) = \{1, 2, \ldots, n\}$).

i) Prove that $(n-1-k)N_k = k(n-1)N_{k+1}$.

ii) Using i), derive $N_k = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

iii) Prove Cayley's formula from ii).

Solution.

1. From N_k to N_{k+1}, we use this method to construct a new spanning tree: add a new node to v_n and delete the edge that the new part attached to originally. So, $(n-1-k)N_k$ is the ways to construct N_{k+1}. But there are some repetition: for the tree N_{k+1}, every node v_i except v_n can attach to the other k nodes v_n directly attaches to except the node in v_i's branch. So, there are $k(n-1)$ repetitions. And in conclusion, $N_{k+1} = \frac{(n-1-k)}{k(n-1)}N_k$. So, the statement is true.

2. $N_k = \frac{(n-k)}{(k-1)(n-1)}N_{k-1} = \frac{(n-k)(n-k+1)}{(k-1)(k-2)(n-1)^2}N_{k-2} = \ldots = \frac{(n-k)(n-k+1)...(n-2)}{(k-1)(k-2)...1(n-1)^{k-1}}N_1 = \binom{n-2}{k-1}\frac{N_1}{(n-1)^{k-1}}$. Since $N_{n-1} = 1$, we know from the equation above that $N_{n-1} = \frac{N_1}{(n-1)^{n-2}} = 1$. So, $N_1 = (n-1)^{n-2}$. As a result, $N_k = \binom{n-2}{k-1}\frac{N_1}{(n-1)^{k-1}} = \binom{n-2}{k-1}(n-1)^{n-1-k}$.

3. We assume $C(n)$ as the number of all different spanning trees with n nodes. $C(n) = N_1 + N_2 + \ldots + Nn - 1 = \binom{n-2}{0}(n-1)^{n-2} + \binom{n-2}{1}(n-1)^{n-3} + \ldots \binom{n-2}{n-2}(n-1)^0 = (n-1+1)^{n-2} = n^{n-2}$. So, Cayley's formula is proved.

□