Homework 5

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Problem 1.

1. Determine the coefficient of x^{50} in $(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$
2. Determine the coefficient of x^3 in $(2 + x)^{\frac{3}{2}}/(1 - x)$
3. Determine the coefficient of x^4 in $(2 + 3x)^5 \sqrt{1 - x}$

Solution.

1. We can simplify this formula:
   $(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$
   $= x^{42}(1 + x + x^2 + \ldots)^6$
   $= x^{42}\frac{1}{(1-x)^6}$
   $= x^{42}(\binom{5}{5} + \binom{6}{5}x + \binom{7}{5}x^2 + \ldots + \binom{5+k}{5}x^k + \ldots)$
   So, the coefficient of x^50 is $\binom{5+8}{5} = 1287$.
2. We can simplify this formula:
   $(2 + x)^{\frac{3}{2}}/(1 - x)$
   $= (2^{\frac{3}{2}} + \frac{3}{2}2^{\frac{1}{2}}x + \frac{3}{8}2^{\frac{-1}{2}}x^2 + \frac{-1}{16}2^{\frac{-3}{2}}x^3 + \ldots)(1 + x + x^2 + x^3 + \ldots)$
   So, the coefficient of x^3 is $235\sqrt{2}/64$.
3. We can simplify this formula:
   $(2 + 3x)^5 \sqrt{1 - x}$
   $= (32 + 240x + 720x^2 + 1080x^3 + 810x^4 + \ldots)(1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \ldots)$
   So, the coefficient of x^4 is $810 - 540 - 90 - 15 - \frac{5}{4} = 163.75$.

$\square$

Problem 2. Find generating functions for the following sequences (express them in a closed form, without infinite series!):

1. $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots$.
2. $1, 0, 1, 0, 1, 0, \cdots$.
3. $1, 2, 1, 4, 1, 8 \cdots$

Solution.

1. $(1,1,1,1,1,1,\cdots) \Rightarrow a(x) = \frac{1}{1-x}$.
   $(6,6,6,6,6,\cdots) \Rightarrow a(x) = \frac{6}{1-x}$.
   $(6,-6,6,-6,6,\cdots) \Rightarrow a(x) = \frac{6}{1+x}$.
   $(-6,6,-6,6,-6,\cdots) \Rightarrow a(x) = \frac{-6}{1+x}$
   $(0,0,0,0,-6,6,-6,6,-6,\cdots) \Rightarrow a(x) = \frac{-6x^4}{1+x}$.

2. $(1,1,1,1,1,1,\cdots) \Rightarrow a(x) = \frac{1}{1-x}$.
   $(1,-1,1,-1,1,-1,\cdots) \Rightarrow a(x) = \frac{1}{1+x}$.
   $(2,0,2,0,2,0,\cdots) \Rightarrow a(x) = \frac{2}{1-x^2}$.
   $(1,0,1,0,1,0,\cdots) \Rightarrow a(x) = \frac{1}{1-x^2}$.

3. $(2,2,2,2,2,2,\cdots) \Rightarrow a(x) = \frac{2}{1-x}$.
   $(2,4,8,16,32,64,\cdots) \Rightarrow a(x) = \frac{2}{1-2x}$.
   $(2,0,4,0,8,0,\cdots) \Rightarrow a(x) = \frac{2}{1-2x^2}$.
   $(0,2,0,4,0,8,0,\cdots) \Rightarrow a(x) = \frac{2x}{1-2x^2}$.
   $(1,0,1,0,1,0,\cdots) \Rightarrow a(x) = \frac{1}{1-x^2}$.
   $(1,2,1,4,1,8\cdots) \Rightarrow a(x) = \frac{-2x^3-2x^2+2x+1}{2x^4-3x^2+1}$.

□

Problem 3. Let a_n be the number of ordered triples $\langle i,j,k\rangle$ of integer numbers such that $i \geq 0, j \geq 1, k \geq 1$, and $i+3j+3k=n$. Find the generating function of the sequence $(a_0,a_1,a_2,\ldots)$ and calculate a formula for a_n.

Solution. We can know from the condition that: Firstly,$a_0 = a_1 = \ldots = a_5 = 0$. Secondly, every three numbers are the same since a_6. Finally, with the step of 3, $a_6, a_9, a_{12}, \ldots$ are $1, 1+2, 1+2+3, 1+2+3+4, \ldots$.
$(1,1,1,1,1,1,\cdots) \Rightarrow a(x) = \frac{1}{1-x}$
$(0,2,2,2,2,2,\cdots) \Rightarrow a(x) = \frac{2x}{1-x}$
$(0,0,3,3,3,3,\cdots) \Rightarrow a(x) = \frac{3x^2}{1-x}$
$\cdots$
$(1,1+2,1+2+3,1+2+3+4,\cdots) \Rightarrow a(x) = \frac{\frac{d\frac{x}{1-x}}{dx}}{1-x} = \frac{1}{(1-x)^3}$.
$(1,0,0,1+2,0,0,1+2+3,0,0,1+2+3+4,\cdots) \Rightarrow a(x) = \frac{1}{(1-x^3)^3}$.
$(0,1,0,0,1+2,0,0,1+2+3,0,0,1+2+3+4,\cdots) \Rightarrow a(x) = \frac{x}{(1-x^3)^3}$.
$(0,0,1,0,0,1+2,0,0,1+2+3,0,0,1+2+3+4,\cdots) \Rightarrow a(x) = \frac{x^2}{(1-x^3)^3}$.
$(1,1,1,1+2,1+2,1+2,1+2+3,1+2+3,1+2+3,1+2+3+4,1+2+3+4,1+2+3+4,\cdots) \Rightarrow a(x) = \frac{1+x+x^2}{(1-x^3)^3}$.
$(0,0,0,0,0,0,1,1,1,1+2,1+2,1+2,1+2+3,1+2+3,1+2+3,1+2+3+$

$4, 1+2+3+4, 1+2+3+4, \cdots) \Rightarrow a(x) = \frac{(1+x+x^2)x^6}{(1-x^3)^3}$.
So, $a(x) = \frac{(1+x+x^2)x^6}{(1-x^3)^3}$. $\square$

Problem 4. If $a(x)$ is the generating function of a sequence $(a_0, a_1, a_2, \ldots)$, please find the generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$.

Solution. $a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$
$xa(x) = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots$
$x^2 a(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \ldots$
$\ldots$
$(1 + x + x^2 + x^3 + \ldots)a(x) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots$
$\frac{a(x)}{1-x} = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \ldots$ $\square$