Homework 6

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Problem 1. Prove that any natural number $n \in \mathbb{N}$ can be written as a sum of mutually distinct Fibonacci numbers.

Solution. We can prove it by mathematical induction. For Fibonacci numbers, $f_n = f_{n-1} + f_{n-2}$, and $f_n \geqslant f_{n-1}$ for any integer $n \geqslant 1$. For any natural number $n \in \mathbb{N}$, let f_k be the largest Fibonacci number satisfying $f_k \leqslant n$. And for the remainded value $n - f_k$, we can know it is smaller than f_k from the definition that $f_k = f_{k-1} + f_{k-2}$ and f_k is an increasing sequence. Repeat this process. Because the minimized Fibonacci number that > 0 is $f_1 = 1$, this recurrence must have an end and must result in a solution. So, any natural number $n \in \mathbb{N}$ can be written as a sum of mutually distinct Fibonacci numbers. □

Problem 2. Express the n^{th} term of the sequences given by the following recurrence relations

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.
2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

Solution.

1. The characteristic polynomial is: $x^2 + 2x - 3 = 0$. The solution is: $x_1 = 1, x_2 = -3$. Since $x_1 \neq x_2$, the form of $a_n = c_1 + c_2(-3)^n$.
   With the condition that $a_0 = 2, a_1 = 3$, we can get the equation of c_1, c_2: $c_1 + c_2 = 2, c_1 - 3c_2 = 3$. So, $c_1 = 9/4, c_2 = -1/4$.
   As a result, $a_n = \frac{9}{4} - \frac{1}{4}(-3)^n$.
2. The homogeneous characteristic polynomial is $x = 2$. So the homogeneous solution is $c2^n$.
   We suppose that the special solution is $a_n = c'$. With the condition $a_{n+1} = 2a_n + 3$, we can get $c' = 2c' + 3$. So, $c' = -3$.
   As a result, the form of a_n is $a_n = c \times 2^n - 3$. Because $a_0 = 1$, we can get $c = 4$.
   So, $a_n = 2^{n+2} - 3$.

$\square$

Problem 3. Solve the recurrence relation $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$ and find $\lim_{n\to\infty} a_n$.

Solution. $a_{n+2} = \sqrt{a_{n+1}a_n} \Rightarrow a_{n+2}^2 = a_{n+1}a_n \Rightarrow 2\log_2 a_{n+2} = \log_2 a_{n+1} + \log_2 a_n$. Let $b_n = \log_2 a_n$, so the recurrence becomes $2b_{n+2} = b_{n+1} + b_n$. And $b_0 = 1, b_1 = 3$.
So, the characteristic polynomial of b_n is: $2x^2 - x - 1 = 0$. And the solution is $x_1 = 1, x_2 = -\frac{1}{2}$. So, the form of $b_n = c_1 + c_2(-\frac{1}{2})^n$.
Because $b_0 = 1, b_1 = 3$, we can get $c_1 = \frac{7}{3}, c_2 = -\frac{4}{3}$. So, $b_n = \frac{7}{3} - \frac{4}{3}(-\frac{1}{2})^n$.
So, $a_n = 2^{\frac{7}{3}-\frac{4}{3}(-\frac{1}{2})^n}$. And $\lim_{n\to\infty} a_n = 2^{\lim_{n\to\infty}\frac{7}{3}-\frac{4}{3}(-\frac{1}{2})^n} = 2^{\frac{7}{3}}$. $\square$

Problem 4. Show that for any $n \geq 1$, the number $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$ is an integer.

Solution. Suppose that there is a recurrence sequence a_n, and the expression of $a_n = \frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$. Regard this recurrence relation: $h_n = 2h_{n-1} + h_{n-2}$. The characteristic polynomial is $x^2 - 2x - 1 = 0$. And the answer is $x_1 = 1+\sqrt{2}, x_2 = 1-\sqrt{2}$. So, the form of $a_n = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n$. If $h_0 = 1, h_1 = 1$, we can get $c_1 = c_2 = \frac{1}{2}$. So, $a_n = \frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$. According to the discussion above, $\frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$ is the n^{th} expression of the recurrence relation $h_n = 2h_{n-1} + h_{n-2}$. And because $h_0 = 1, h_1 = 1$, h_n is an incremental sequence of integers. So, the statement has proved. $\square$