Homework 3

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Problem 1. Prove the formula

1. $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$
2. $\sum_{k=0}^{n} \binom{m+k-1}{k} = \binom{n+m}{n}$

Proof. 1. $\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$
$= \binom{r+1}{r+1} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}$
$= \binom{r+2}{r+1} + \binom{r+2}{r} + \cdots + \binom{n}{r}$
$= \binom{n+1}{r+1}$

2. $\sum_{k=0}^{n} \binom{m+k-1}{k}$
$= \binom{m-1}{0} + \binom{m}{1} + \binom{m+1}{2} + \cdots + \binom{m+n-1}{n}$
$= \binom{m}{0} + \binom{m}{1} + \binom{m+1}{2} + \cdots + \binom{m+n-1}{n}$
$= \binom{m+1}{1} + \binom{m+1}{2} + \cdots + \binom{m+n-1}{n}$
$= \binom{m+n}{n}$

□

Problem 2. For natural numbers $m \leq n$ calculate (i.e. express by a simple formula not containing a sum) $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$.

Solution. By the meaning of combinatorial counting, this formula illustrates a situation where we pick k people to form a group from the total n people, and appoint m people in the group as leaders. Then, the possible number of this kind of appointments is the answer. We should note that different groups and different numbers mean different appointments. So, $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$ can be translated into: $\binom{n}{m} \sum_{k=0}^{n} \binom{n-m}{k}$. And $\binom{n}{m}$ means the possible appointments of the leaders, $\sum_{k=0}^{n} \binom{n-m}{k}$ means the possible situations for other group members. So,
$\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$
$= \binom{n}{m} \sum_{k=0}^{n} \binom{n-m}{k}$
$= \binom{n}{m}(1+1)^{n-m}$
$= \binom{n}{m} 2^{n-m}$

Problem 3. Calculate (i.e. express by a simple formula not containing a sum)

1. $\sum_{k=1}^{n}\binom{k}{m}\frac{1}{k}$

2. $\sum_{k=0}^{n}\binom{k}{m}k$

Solution. 1. $\sum_{k=1}^{n}\binom{k}{m}\frac{1}{k}$
$= \frac{k!}{(k-m)!m!} \times \frac{1}{k}$
$= \frac{(k-1)!}{(k-m)!(m-1)!} \times \frac{1}{m}$
$= \frac{1}{m}\sum_{k=1}^{n}\binom{k-1}{m-1}$
$= \frac{1}{m}\binom{n}{m}$

2. $\sum_{k=0}^{n}\binom{k}{m}k$
$= \sum_{k=0}^{n}\binom{k}{m}(k+1) - \sum_{k=0}^{n}\binom{k}{m}$
$= \frac{(k+1)!}{(k-m)!m!} - \binom{n+1}{m+1}$
$= (m+1)\frac{(k+1)!}{(k-m)!(m+1)!} - \binom{n+1}{m+1}$
$= (m+1)\sum_{k=0}^{n}\binom{k+1}{m+1} - \binom{n+1}{m+1}$
$= (m+1)\binom{n+2}{m+2} - \binom{n+1}{m+1}$

Problem 4. (a) Using Problem 1. for $r = 2$, calculate the sum $\sum_{i=2}^{n} i(i-1)$ and $\sum_{i=1}^{n} i^2$.

(b) Use (a) and Problem 1. for $r = 3$, calculate $\sum_{i=1}^{n} i^3$.

Solution. 1. $\sum_{i=2}^{n} i(i-1)$
$= 2 \times \frac{\sum_{i=2}^{n} i(i-1)}{2\times 1}$
$= 2 \times (\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r}),\ r = 2$
$= 2 \times \binom{n+1}{r+1}$, using Problem 1.. And $r = 2$.
$= 2\binom{n+1}{3}$
$\sum_{i=1}^{n} i^2$
$= \sum_{i=2}^{n} i(i-1) + 1^2 + \sum_{i=2}^{n} i$
$= 2\binom{n+1}{3} + 1 + \frac{(n+2)(n-1)}{2}$
$= 2\binom{n+1}{3} + \frac{n(n+1)}{2}$

2. $\sum_{i=1}^{n} i^3$
$= \sum_{i=1}^{n} i(i-1)(i-2) + 3i^2 - 2i$

$$= 6 \times \frac{\sum_{i=1}^{n} i(i-1)(i-2)}{3\times2\times1} + 6\binom{n+1}{3} + \frac{n(n+1)}{2}$$
$$= 6\binom{n+1}{4} + 6\binom{n+1}{3} + \frac{n(n+1)}{2}$$
$$= 6\binom{n+2}{4} + \frac{n(n+1)}{2}$$

Problem 5. How many functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are there that are monotone; that is, for $i < j$ we have $f(i) \leq f(j)$?

Solution. We can transfer this question into another question: for any integer $1 \leqslant i \leqslant n, x_i \geqslant 0, x_1 + x_2 + \cdots + x_n = n$. And x_i means the number of x that satisfies $f(x) = i$. So, we can easily know that the answer is $\binom{2n-1}{n-1}$.