Homework 4

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Problem 1. Count the number of linear extensions for the following partial ordering:

X is a disjoint union of sets $X_1, X_2, \ldots, X_k$ of sizes $r_1, r_2, \ldots, r_k$, respectively. Each X_i is linearly ordered by $\preceq$, and no two elements from the different X are comparable.

Solution. There are $(r_1+r_2+\ldots+r_k)!$ ways of permutations if it's unrestricted. But actually for every case, the permutation of r_i elements in X_i is fixed, which means there is only 1 way of permutation contrast to $r_i!$ permutations with unrestriction. So, the final number of linear extensions is $\binom{(r_1+r_2+\ldots+r_k)!}{r_1,r_2,\ldots,r_k}$. □

Problem 2. Given a set X with $|X| = n$, determine the number of ordered set pairs $\langle A, B\rangle$ where $A \subseteq B \subseteq X$.

Solution. We can divide X into three parts:W, Y, Z, and let $W = A, Y = B\backslash A, Z = X\backslash B$. So, $A = W, B = W \cup Y, X = W \cup Y \cup Z$, which satisfies $A \subseteq B \subseteq X$. So, the number of ordered pairs is equal to the number of ways to put n elements into three parts W, Y, Z, and that is 3^n. □

Problem 3. There are n married couples attending a dance. How many ways are there to form n pairs for dancing if no wife should dance with their husband.

Solution. We can transfer this question to another question: How many ways are there to permutate n integers so that every integer cannot stand up its location? Because of inclusion-exclusion principle, the answer is $n! - \sum_{k=1}^{n}(-1)^{k-1}\frac{n!}{k!}$. □

Problem 4. Count the permutations with exactly k fixed points. (Remark: π is a permutation of the set {1,2,..., n}. Call an index i with $\pi(i) = i$ a fixed point of the permutation π.)

Solution. To choose k locations from total n locations, there are $\binom{n}{k}$ ways. And for these k points, there is only a fixed way to permutate. For the remainded $n-k$ locations, it's a same question of problem 3. So, there are $(n-k)! - \sum_{i=1}^{n-k}(-1)^{i-1}\frac{(n-k)!}{i!}$ to permutate the remainded $n-k$ elements. In conclusion, the total answer is $\binom{n}{k}((n-k)! - \sum_{i=1}^{n-k}(-1)^{i-1}\frac{(n-k)!}{i!})$. □

Problem 5. What is wrong with the following inductive "proof" that $D(n) = (n-1)!$ for all $n \geq 2$? Can you find a false step in it? For $n = 2$, the formula holds, so assume $n \geq 3$. Let π be a permutation of $\{1, 2, ..., n-1\}$ with no fixed point. We want to extend it to a permutation π' of $\{1, 2, ..., n\}$ with no fixed point. We choose a number $i \in \{1, 2, ..., n-1\}$, and we define $\pi'(n) = \pi(i), \pi'(i) = n$, and $\pi'(j) = \pi(j)$ for $j \neq i$, n. This defines a permutation of $\{1, 2, ..., n\}$, and it is easy to check that it has no fixed point. For each of the $D(n-1) = (n-2)!$ possible choices of π, the index i can be chosen in $n-1$ ways. Therefore, $D(n) = (n-2)! \cdot (n-1) = (n-1)!$.

Solution. The wrong step is that "We want to extend it to a permutation π' of $\{1, 2, ..., n\}$ with no fixed point." Actually, the permutation of $\{1, 2, ..., n\}$ without fixed point can be extended in both permutation of $\{1, 2, ..., n-1\}$ with no fixed point and with fixed point. For example, when $n = 3$, there are 2 permutations with no fixed point that's $\{3, 1, 2\}, \{2, 3, 1\}$. According to the wrong induction, when $n = 4$, there are 6 permutations with no fixed point arise from the former 2 permutations. But actually, there is a permutation of $\{2, 1, 4, 3\}$ extended from the permutation of $\{2, 1, 3\}$, which holds one fixed point when $n = 3$. But this induction proof doesn't take this condition to consideration, so it's wrong. □

Problem 6. How many ways are there to seat n married couples at a round table with $2n$ chairs in such a way that the couples never sit next to each other?

Solution. Let S_{2n} be the number of ways to permunate the n couples randomly. It's easy to know that $|S_{2n} \models (2n)!$. For $i = 1, 2, \ldots, n$, let $A_i = \{\pi \in S_{2n} | \pi(i) = \text{the } i-th \text{ couple sit together.}\}$. So, the answer $D(2n) = |S_{2n} \vdash |A_1 \cup A_2 \cup \ldots \cup A_n|$. And $|A_i \models 2 \times (2n-1)!, |A_i \cap A_j \models 2^2 \times (2n-2)! \ldots |A_{i+1} \cap A_{i+2} \cap A_{i+3} \cap \ldots \cap A_{i+k} \models 2^k \times (2n-k)!$. According to inclusion-exclusion principle, $|A_1 \cup A_2 \cup \ldots \cup A_n \models \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}2^k(2n-k)!$. So, the answer is $(2n)! - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}2^k(2n-k)!$. □