Homework 2

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Problem 1. Let$(X, \preceq_1)$, $(Y, \preceq_2)$ be (partially) ordered sets. We say that they are isomorphic if there exists a bijection $f : X \to Y$ such that for every $x, y \in X$, we have $x \preceq_1 y$ if and only if $f(x) \preceq_2 f(y)$.

1. Draw Hasse diagrams for all nonisomorphic 3-element posets.
2. Prove that any two n-element linearly ordered sets are isomorphic.
3. Prove that $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$ are not isomorphic. (where $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational numbers, $\leq$ is the usual 'less or equal to' between numbers).

Solution. 1. the picture is below.

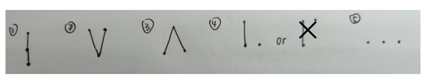

Figure 1: 3-element nonisomorphic posets

2. Proof. Supposing that A and B are two $n-$element linearly ordered sets, we define a bijection function f by recurrence such that the minimal element a_0 in $A \leftrightarrow$ the minimal element b_0 in B, and it's the same with sets $A' = A \setminus \{a_0\}$ and $B' = B \setminus \{b_0\}$ until reaching to the maximal element. Because of the definition of linearly ordered set, every time there is just a minimal element in each recurrency, so the recurrency is right.
   Thus, we have the statement :for every $a_i, a_j \in A$, we have $a_i \preceq_1 a_j$ if and noly if $f(a_i) = b_i, f(a_j) = b_j$, which satisfies $b_i \preceq_2 b_j$. □

3. Proof. Because $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$ don't have the same equivalence, there isn't any bijection function between $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$. So, $(\mathbb{N}, \leq)$ and $(\mathbb{Q}, \leq)$ are not isomorphic. □

Problem 2. Prove or disprove: If a partially ordered set $(X, \preceq)$ has a single minimal element, then it is a smallest element as well.

Proof. The statement is true. We will prove it by induction.
Basis. When the partially ordered set $(X, \preceq)$ only has one element, the only element is the minimal element as well as the smallest element.
Hypothesis. For any partially ordered set $(X, \preceq)$ with n elements $n \geqslant 1$, if $(X, \preceq)$ have a single minimal element, then it is a smallest element as well.
Induction. For a partially ordered set $(X, \preceq)$ with $n + 1$ elements, we first pick out an element x_0, and consider the left n elements set $<x_1, x_2, ..., x_n>$. According to our hypothesis, the $n-$element set has a single minimal element which is also the smallest element. Without generity, we assume it x_1. Then, we would put the element x_0 back.
case a. $x_0 \preceq x_1$. In this case, x_0 becomes the minimal element. Because x_1 is the minimal element and smallest element of $n-$element set, $x_1 \preceq x_2, x_3, ..., x_n$ and $x_0 \preceq x_1$. Thus $x_0 \preceq x_1, x_2, ..., x_n$, which means x_0 is the smallest element too.
case b. x_0 is in the second floor of the $n-$element, which means $x_1 \preceq x_0$ directly. In this case, x_0 is the only minimal element. And for our hypothesis, $x_1 \preceq x_2, x_3, ..., x_n$ and $x_1 \preceq x_0$, so x_1 is the smallest element too.
case c. x_0 is in the higher floor of the $n-$element. In this case, x_1 is the minimal element. We assume that x_0 is in the k^{th} floor so there is a connection to $(k-1)^{th}$ floor, and $(k-1)^{th}$ floor has a connection to $(k-2)^{th}$ floor, and .... Finally this chain down to x_1, which means $x_1 \preceq x_0$. So $x_1 \preceq x_0, x_2, x_3, ..., x_n$, x_1 is the smallest element too. $\square$

Problem 3. Let $(X, \preceq)$ and $(X', \preceq')$ be partially ordered sets. A mapping $f : X \to X'$ is called an embedding of $(X, \preceq)$ into $(X', \preceq')$ if the following conditions hold:

- f is an injective mapping;
- $f(x) \preceq' f(y)$ if and only if $x \preceq y$.

Now consider the following problem

a) Describe an embedding of the set $\{1, 2\} \times \mathbb{N}$ with the lexicographic ordering into the ordered set $(\mathbb{Q}, \leq)$.

b) Solve the analog of a) with the set $\mathbb{N} \times \mathbb{N}$ (ordered lexicographically) instead of $\{1, 2\} \times \mathbb{N}$.

Solution. a) The set $\{1,2\} \times \mathbb{N}$ has the element (x, y) satisfying $x \in \{1,2\}$ and $y \in \mathbb{N}$. We define a function f with features below: Firstly, f maps the element (x, y) in to a decimal, and if $x = 1$, then the single digit is 1, else the single digit is 2. Then, the number of 1 behind the decimal point is y. For example, $(1,1)$ maps to 1.1, $(1,2)$ maps to 1.11, $(2,1)$ maps to 2.1, $(2,3)$ maps to 2.111.

b) We assume an element $(x, y) \in \mathbb{N} \times \mathbb{N}$. And the function f can maps (x, y) to a decimal $x.111...111$, the number of 1 behind decimal point is y.

Problem 4. Prove the following strengthening of the Erdös-Szekeres Lemma: Let κ, ℓ be natural numbers. Then every sequence of real numbers of length $\kappa\ell + 1$ contains an nondecreasing subsequence of length $\kappa + 1$ or a decreasing subsequence of length $\ell + 1$.

Proof. Let $\kappa\ell = n^2$. Assume the $(n^2 + 1)$-element sequence of real numbers is $(x_1, x_2, x_3, ..., x_{n^2+1})$ and the set $I = \{1, 2, ..., n^2 + 1\}$. We define the relation $\preccurlyeq$ on I such that $i \preccurlyeq j$ if and only if $(i \leqslant j) \wedge (x_i \leqslant x_j)$. And $(I, \preccurlyeq)$ is partially ordered set for the definition. According to the deduction of Mirsky's Theorem, for any partially ordered set $P = (S, \preccurlyeq)$, we have $\alpha(P) \times \omega(P) \geqslant |S|$. So, we have two cases:
case a. $\omega(I, \preccurlyeq) > n$: there is a nondecreasing subsequence that $x_{i1} \leqslant x_{i2} \leqslant x_{i3} \leqslant ... \leqslant x_{in} \leqslant ... \leqslant x_{im}$.
case b. $\alpha(I, \preccurlyeq) > n$: For the index must can be compared, so, if $i_1 < i_2 < ... < i_m, m > n$, there is an increasing subsequence of $x_{i1} > x_{i2} > x_{i3} > ... > x_{im}$.
Above all, we should prove that $\kappa \leqslant n$ or $\ell \leqslant n$. If they both $> n$, then $\kappa\ell > n^2$, which is contrast to our premiss. So, $\kappa \leqslant n$ or $\ell \leqslant n$.
In a nutshell, this statement is proved. □