Homework 7

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Problem 1. Fill in the blanks with either true (✓) or false (×)

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 2. 1. Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

Solution. Let $g(n) = 1$ for any integer $n \in N$.

$$f(x) = \begin{cases} \frac{1}{n}, n \neq 2^k (k \in N) \\ n, n = 2^k \end{cases}$$

And the statement that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$ is true because $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ both don't exist. □

2. Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is monotonically increasing if it satisfies the property '$x \leq y \implies h(x) \leq h(y)$'.
Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

Solution. for any integer $k \in N$:

$$f(x) = \begin{cases} 2^n, n = 2^{2k+1} \\ 2^{2^{2k+1}}, 2^{2k+1} < n < 2^{2k+3} \end{cases}$$

$$g(x) = \begin{cases} 2^n, n = 2^{2k} \\ 2^{2^{2k}}, 2^{2k} < n < 2^{2k+2} \end{cases}$$

And the statement that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$ is true because $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ and $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ both don't exist.

When $n = 2^{2k+1}$, $\lim_{k\to\infty} \frac{f(n)}{g(n)} = \frac{2^{2^{2k+1}}}{2^{2^{2k}}} = 2^{2^{2k}}$. When $n = 2^{2k+2}$, $\lim_{k\to\infty} \frac{g(n)}{f(n)} = \frac{2^{2^{2k+2}}}{2^{2^{2k+1}}} = 2^{2^{2k+1}}$.
So, $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$. $\square$

(Please give the detailed proof that your functions satisfy the requirements.)

Problem 3. Prove that

(a) $\left(1 + \frac{1}{n}\right)^n \leq e$ for all $n \geq 1$.

Proof. Since $1 + x \leqslant e^x$, $\left(1 + \frac{1}{n}\right)^n \leqslant (e^{\frac{1}{n}})^n = e$. $\square$

(b) $\left(1 + \frac{1}{n}\right)^{n+1} \geq e$ for all $n \geq 1$.

Proof. $\left(1 + \frac{1}{n}\right)^{n+1} = (\frac{1}{\frac{n}{n+1}})^{n+1} = (\frac{1}{1-\frac{1}{n+1}})^{n+1}$. Since $1 + x \leqslant e^x$,$1 - \frac{1}{n+1} \leqslant e^{-\frac{1}{n+1}}$, so $\frac{1}{1-\frac{1}{n+1}} \geqslant e^{\frac{1}{n+1}}$, so $(\frac{1}{1-\frac{1}{n+1}})^{n+1} \geqslant (e^{\frac{1}{n+1}})^{n+1} = e$. As a result, $\left(1 + \frac{1}{n}\right)^{n+1} \geq e$. $\square$

(c) Using (a) and (b), conclude that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.

Proof. From problem (a), we know that $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \leqslant e$.
From problem (b), we know that $\left(1 + \frac{1}{n}\right)^n \geq \frac{e}{1+\frac{1}{n}}$. So, $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \geqslant \lim_{n\to\infty} \frac{e}{1+\frac{1}{n}} = e$. As a result, $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. $\square$

Problem 4. Prove Bernoulli's inequality: for each natural number n and for every real $x \geq -1$, we have $(1 + x)^n \geq 1 + nx$.

Proof. We will prove it by induction. The statement $(1 + x)^n \geq 1 + nx$ is true when $n = 1$. Hypothesis that it's true for any natural number $n \geqslant 1$ that $(1+x)^n \geq 1+nx$. So, $(1+x)^{n+1} \geq (1+nx)(1+x) = 1+(n+1)x+nx^2 \geq 1+(n+1)x$. So, the statement is true for any natrual number n. $\square$

Problem 5. Prove that for $n = 1, 2, \ldots,$ we have

$$2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1.$$

Proof. We will prove it by induction. The statement is true when $n = 1$. Suppose that it holds true for any natrual number $n \geq 1$ that $2\sqrt{n+1} - 2 < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$. So, $2\sqrt{n+1} - 2 + \frac{1}{\sqrt{n+1}} < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}}$.
So, we should prove $2\sqrt{n} - 1 + \frac{1}{\sqrt{n+1}} \geq 2\sqrt{n+1} - 1$. The both sides times $\sqrt{n+1}$ and square, we can get $4n(n+1) \leq (2n+1)^2 \Rightarrow 4n^2 + 4n \leq 4n^2 + 4n + 1$. And that's true.
Additionally, we should prove $2\sqrt{n+2} - 2 \leq 2\sqrt{n+1} - 2 + \frac{1}{\sqrt{n+1}}$. The both sides times $\sqrt{n+1}$ and square, we can get $4(n+2)(n+1) \leq (2n+3)^2 \Rightarrow 4n^2 + 12n + 8 \leq 4n^2 + 12n + 9$. And that's true.
In conclusion, the statement is proved. □

Problem 6.

a) Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.

Proof. $\binom{2m}{m} = \frac{2m(2m-1)(2m-2)...2\times 1}{m(m-1)(m-2)...2\times 1}$. The numerator is the product of all the numbers that between m and $2m$. Since prime is a number only has divisors of 1 and itself, the numerator should divide by all the possible factors that the composite numbers between m and $2m$ hold, which is the denominator of $\binom{2m}{m}$. So, the statement has proved. □

b) Using a), prove the estimate $\pi(x) = O(\frac{x}{\ln x})$, where $\pi(x)$ denote the number of primes not exceeding the number x.

Proof. Suppose $x = 2^k$. Since the product of all primes p with $\frac{x}{2} < p < x$ is at most $\binom{x}{\frac{x}{2}}$, the largest number of primes between $\frac{x}{2}$ and x is $\log_{\frac{x}{2}}\binom{x}{\frac{x}{2}}$. We know from the estimate example that $\binom{n}{k} \leq (\frac{en}{k})^k$, so $\log_{\frac{x}{2}}\binom{x}{\frac{x}{2}} \leq \log_{\frac{x}{2}}(2e)^{\frac{x}{2}} = \frac{x}{2}\log_{\frac{x}{2}} 2e = \frac{x}{2}\frac{\ln 2e}{\ln\frac{x}{2}}$. So, $\pi(x) \leq \log_{\frac{x}{2}}\binom{x}{\frac{x}{2}} + \log_{\frac{x}{4}}\binom{\frac{x}{2}}{\frac{x}{4}} + \ldots + \log_2\binom{4}{2} + 1 \leq 2\ln 2e(\frac{\frac{x}{2}}{\ln\frac{x}{2}} + \frac{\frac{x}{4}}{\ln\frac{x}{4}} + \ldots\frac{\frac{x}{2^{k/2}}}{\ln\frac{x}{2^{k/2}}}) \leq 2\ln 2e(\frac{x}{\ln x} + \frac{\frac{x}{2}}{\ln x} + \frac{\frac{x}{4}}{\ln x} + \ldots + \frac{\frac{x}{2^{k/2-1}}}{\ln x}) \leq 2\ln 2e\frac{2x}{\ln x} = O(\frac{x}{\ln x})$.
So, the statement is proved. □