# Using Esscher Transforms to Value Initial Offerings for Options in Crypto-Markets

William H. Long

*Charlotte, North Carolina*

**Abstract**

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*Keywords:* None, Blank, Complicated

## 1. Introduction

Beginning last year, blockchains and cryptocurrencies became of great interest for the financial world. For those who are unaware, a blockchain is a way of storing data electronically without requiring a central trusted server. Several applications of this technology, such as smart contracts, could be revolutionary. In order to use blockchains to their full potential, users would need to have access to a stable store of value, such as a cryptocurrency. However, as the last year or so has show, the current widely used cryptocurrency are unusably volatilite. The use of derivative contracts would be one way to combat this volatility. The cryptocurrency derivatives market is still relatively new, and there is a lot of work to be done. This paper will discuss a potential model to value options on cryptocurrencies without the use of existing contract values to calibrate it.

## 2. Methods

While cryptocurrencies are on the cutting edge of finance, the problem of valuing derivatives on risky assets is a well studied one. As long as the returns of the risky asset have a similar distribution to a stock, the same modes which are used to value options on the stock market can be used to value options on other kinds of assets, such as cryptocurrencies. This matter is explored fully in the results section, but it is reasonable to assume that the daily returns for cryptocurrencies follow a similar distribution to the daily returns of the S&P 500, so it should be fine to adapt well established models for option valuation of stocks. The models discussed in this paper, the variance gamma (VG) and normal-inverse Gaussian (NIG), are both relatively easy to work with and calibrate yet still fairly accurate. In fact, using both models to value cryptocurrency options is virtually the same as valuing stock options. The issue comes from calibrating these models. When calibrating these models to value stock options, most practitioners use current market data from various options markets. However, since there is are no options on cryptocurrencies currently, doing the same for cryptocurrency options would be clearly impossible. However, by finding the distribution of the daily returns and changing the measure to be risk neutral, it is possible to calibrate these models for cryptocurrency markets.

*2.1. Overview of the Models*

Under both the variance gamma and normal-inverse Gaussian models, the current value of a call option which gives the holder the right, but not the obligation, to buy a risky asset S for price K at time T is:

$$C_0 = E_0^Q[\max[S_T - K, 0]] \tag{1}$$

$$S_t = S_0 \exp[(r-q)t + R_t] \tag{2}$$

In fact, the only difference between the two models is how they model the stochastic process $R_t$. As implied by the names, the VG model assumes that $R_t$ is a variance gamma process and the NIG model assumes that $R_t$ is a normal-inverse Gaussian process. [1] It also follows that the daily returns of $S_t$, $X_t = \frac{S_t - S_{t-1}}{S_{t-1}}$, follow either a VG or NIG distribution for each respective model. For the VG distribution, the probability density function, moment-generating function, and characteristic function follow [3]:

$$f_{VG}(x) = \frac{2\exp(\theta(x-c)/\sigma^2)}{\sigma\sqrt{2\pi}\nu^{1/\nu}\Gamma(1/\nu)} \left(\frac{\mid x-c \mid}{\sqrt{2\sigma^2/\nu+\theta^2}}\right)^{1/\nu-1/2} K_{1/\nu-1/2}(\frac{\mid x-c \mid \sqrt{2\sigma^2/\nu+\theta^2}}{\sigma^2}) \tag{3}$$

$$M_{VG}(u,t) = e^{cut}(1 - u\nu\theta - 1/2\sigma^2\nu u^2)^{-t/\nu} \tag{4}$$

$$\Phi_{VG}(z,t) = e^{izct}(1 - iz\nu\theta + 1/2\sigma^2\nu z^2)^{-t/\nu} \tag{5}$$

Note, $K_i()$ is a modified Bessel function of the third kind and $\Gamma()$ is the gamma function. Also, note that $\sigma, \nu > 0$. For the NIG distribution, the probability density function, moment-generating function, and characteristic function follow [2]:

$$f_{NIG}(x) = \frac{\alpha\delta K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\pi\sqrt{\delta^2 + (x-\mu)^2}} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \tag{6}$$

$$M_{NIG}(u,t) = e^{t(u\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+u)^2}))} \tag{7}$$

$$\Phi_{NIG}(z,t) = e^{t(iz\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+iz)^2}))} \tag{8}$$

While there are many known ways to solve eq (1), this paper will use the method outlined in Lewis (2001) [1]. As detailed in the paper, the present value of a call option is:

$$C_0 = S_0 - \frac{\sqrt{S_0 K} e^{-rT/2}}{\pi} \int_0^\infty \mathbf{Re}[e^{izk}\phi(z - i/2)] \frac{dz}{z^2 + 1/4} \tag{9}$$

Note that $k = \frac{\mathrm{S_0}}{\mathrm{K}} + \mathrm{rT}$, $\mathbf{Re}[z]$ is the real part of $z$, and $\phi(z)$ is the modified risk neutral characteristic function. According to Lewis, $\phi(z)$ for VG and NIG are as follows:

$$\phi_{VG}(z) = e^{iz\omega t}(1 - iz\nu\theta + 1/2\sigma^2\nu z^2)^{-t/\nu} \tag{10}$$

$$\omega_{VG} = \frac{1}{\nu}\ln(1 - \nu\theta - 1/2\sigma^2\nu) \tag{11}$$

$$\phi_{NIG}(z) = e^{t(iz\omega+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+iz)^2}))} \tag{12}$$

$$\omega_{NIG} = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}) \tag{13}$$

Note that this method requires additional constraints on the parameters. For the VG model, let $\alpha = \left(\frac{2}{\nu\sigma^2} + \frac{\theta^2}{\sigma^4}\right)^{0.5}$ and $\beta = \frac{\theta}{\sigma^2}$. So, for both models, the constraints are $\alpha + \beta \geq 0$ and $\beta - \alpha \leq -1$. Also, $\omega$ is defined such that $1 = \phi(-i)$

*2.2. Finding the Real Measure*

As is discussed in the results section, both the daily returns for the S&P 500 and Ethereum (ETH) are approximately log-normally distributed with slight left hand skew. Finding the VG distribution is fairly simple, and discussed at great length in Eugene Seneta's paper [Source]. It is fairly simple to use the same method to find the NIG distribution. First, a rough estimate for the parameters is found using the method of moments, and then that estimate is used as starting the starting parameters for the maximum likelihood estimation (MLE). The four standard moments of random variable $X$ are:

$$\begin{aligned} \mathrm{E}[X] &= \bar{X} \\ \mathrm{Var}[X] &= E[(X-\bar{X})^2] \\ \mathrm{Skew}[X] &= \frac{E[(X-\bar{X})^3]}{E[(X-\bar{X})^2]^{1.5}} \\ \mathrm{Kurt}[X] &= \frac{E[(X-\bar{X})^4]}{E[(X-\bar{X})^2]^2} \end{aligned} \tag{14}$$

For the VG distribution, it can be shown that the moments are approximately [4]:

$$\begin{aligned} \mathrm{E}[X] &= c+\theta \\ \mathrm{Var}[X] &= \sigma^2 \\ \mathrm{Skew}[X] &= 3\theta\nu/\sigma \\ \mathrm{Kurt}[X] &= 3(1+\nu) \end{aligned} \tag{15}$$

It can also be shown that the NIG has moments [2]:

$$\begin{aligned} \mathrm{E}[X] &= \mu+\delta\frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{0.5}} \\ \mathrm{Var}[X] &= \delta^2\alpha^{-1}\frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{1.5}} \\ \mathrm{Skew}[X] &= 3\alpha^{-1/4}\frac{\beta/\alpha}{(1-(\beta/\alpha)^2)^{0.25}} \\ \mathrm{Kurt}[X] &= 3\alpha^{-1/2}\frac{1+4(\beta/\alpha)^2}{(1-(\beta/\alpha)^2)^{0.5}}+3 \end{aligned} \tag{16}$$

It's fairly simple to solve both (15) and (16), and they both give us good estimates for the initial parameters for MLE. While there are a few ways to preform MLE, this paper minimizes this objective function using the L-BFGS-B algorithm:

$$\hat{l}(\theta; x) = -\frac{1}{n}\sum_{i=1}^{n} \ln f(x_i|\theta) \tag{17}$$

*2.3. Finding the Risk Neutral Measure using Esscher Transforms*

Now, the method described in the previous section will return the real probability measure. However, both the models in this paper require the risk neutral probability measure. This change of measure can be preformed using an Esscher transform. The Esscher transform of the probability measure $f(x)$ is defined as [5]:

$$f(x; h) = \frac{e^{hx}}{M(h, 1)} \tag{18}$$

Note that an Esscher transform is also a probability measure. According to Gerber and Shiu's paper, the Esscher parameter $h$ which yields the risk neutral measure is going to be the solution to this equation:

$$e^r = \frac{M(1+h, 1)}{M(h, 1)} \tag{19}$$

Note that $r$ is equal to the risk neutral rate of return.

The Esscher transform is rather simple in the NIG case. Here, equation (19) simplifies to:

$$\begin{aligned} e^r &= \frac{e^{(h+1)\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+h+1)^2})}}{e^{h\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+h)^2})}} \Rightarrow \\ r &= \mu + \delta(\sqrt{\alpha^2 - (\beta+h)^2} - \sqrt{\alpha^2 - (\beta+h+1)^2}) \end{aligned} \tag{20}$$

Solving for $h$ yields:

$$\beta + h = \hat{\beta} = -\frac{1}{2}\frac{\delta^4 + \delta^2 r^2 + r\sqrt{-\delta^2(\delta^2 + r^2 - 4\delta^2\alpha^2)}}{\delta^2(\delta^2 + r^2)} \tag{21}$$

Note that $\mu$ is likely going to be zero. If this is not the case, it is simple to replace r with $\hat{r} = r - \mu$ in equation (21). So, the risk neutral parameters for the NIG model will be $(r, \alpha, \hat{\beta}, \delta)$, which can be compared to the real parameters of the asset's distribution $(\mu, \alpha, \beta, \delta)$. Note that $\alpha$ and $\delta$ are the same, and $\hat{\beta}$ doesn't depend on $\beta$.

The Essher transform for the VG distribution is more complex. It is necessary to rewrite equation (4) in terms of $\alpha$ and $\beta$. So, the new MGF is:

$$M_{VG}(u,1) = e^{uc}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{1/\nu} \tag{22}$$

Plugging this into equation (19) yields:

$$\begin{aligned} e^r &= e^c\left(\frac{\alpha^2-(\beta+h)^2}{\alpha^2-(\beta+h+1)^2}\right)^{1/\nu} \Rightarrow \\ e^{\nu(r-c)} &= \frac{\alpha^2-(\beta+h)^2}{\alpha^2-(\beta+h+1)^2} \end{aligned} \tag{23}$$

Solving for $h$ yields:

$$\beta + h = \hat{\beta} = \frac{\sqrt{-2\alpha^2 e^{\nu(r-c)} + \alpha^2 e^{2\nu(r-c)} + \alpha^2 + e^{\nu(r-c)}} - e^{\nu(r-c)}}{e^{\nu(r-c)} - 1} \tag{24}$$

It is possible to find the risk neutral $\hat{\sigma}, \hat{\theta}$ by solving this system of equations:

$$\begin{aligned} \alpha &= \left(\frac{2}{\nu\hat{\sigma}^2} + \frac{\hat{\theta}^2}{\hat{\sigma}^4}\right)^{0.5} \\ \hat{\beta} &= \frac{\hat{\theta}}{\hat{\sigma}^2} \end{aligned} \tag{25}$$

So, the risk neutral parameters for the VG model will be $(r, \hat{\sigma}, \hat{\theta}, \nu)$.

## 3. Results

The usefulness of this model will first be demonstrated to value call options on the S&P 500. Since there is a liquid and well established market for these call options, it should be simple to test the accuracy of the model. This paper will consider the daily returns of the S&P 500 form March 31, 2015 to

March 31, 2016. Using the method from section 3, it can be found that the S&P 500 (at the date March 31st, 2016) has a NIG distribution with real parameters [6]:

| Parameter | Value | Standard Error | t-value |
|---|---|---|---|
| $\mu$ | -0.000696455 | 0.0010990625325 | -0.633681297764 |
| $\alpha$ | 103.575 | 0.0393202430401 | 2634.12898754 |
| $\beta$ | 8.06284 | 13.2290883486 | 0.609478010566 |
| $\delta$ | 0.00982864 | 0.000939948055802 | 10.4565827425 |

The values and statistical significance for $\mu, \alpha,$ and $\delta$ are as expected. However, the low t-value for $\beta$ is rather unexpected. According to the t-value, the null hypothesis that $\beta = 0$ can not be rejected. However, since $\hat{\beta}$ doesn't depend of $\beta$, and only depends on $\alpha, \delta,$ and r, thus $\hat{\beta}$ should be statistically significant.

The real parameters of the VG distribution are as follows:

| Parameter | Value | Standard Error | t-value |
|---|---|---|---|
| $c$ | -0.000575336 | 0.00043189302959 | -1.33212648749 |
| $\sigma$ | 0.00978294 | 0.000679266290001 | 14.4022174224 |
| $\theta$ | 0.000646343 | 0.00074226572304 | 0.870770833129 |
| $\nu$ | 0.827458 | 0.26259385857 | 3.15109452669 |

The values and statistical significance for these parameters is as expected. While the null hypothesis that $\theta = 0$ can not be rejected, this is normal when dealing with stocks [Source].

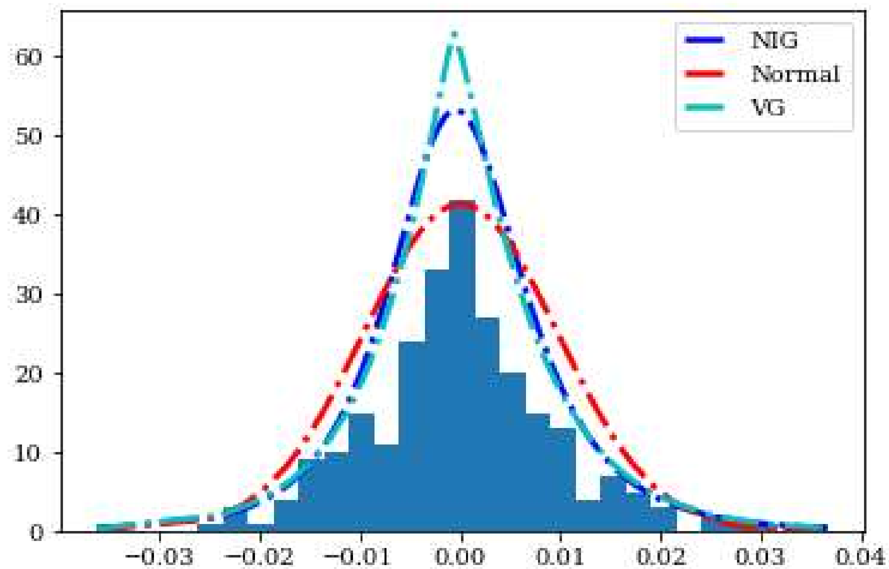
A simple normal regression was also preformed to establish a baseline. The parameters follow:

| Parameter | Value |
|---|---|
| Mean | 7.10149e-05 |
| Variance | 9.33308e-05 |

The p-values for the $\chi^2$ test and Kolmogorov Smirnov test for each of the distributions follow:

| Test | NIG P-value | VG P-value | Normal P-value |
|---|---|---|---|
| $\chi^2$ | 0.64899550751902157 | 0.9071711674806936 | 0 |
| Kolmogorov Smirnov | 0.9994952213295103 | 0.99949522132951041 | 0 |

According to these p-values, there's no evidence that the fitted NIG distribution or fitted VG distribution are different from the empirical S&P 500 distribution. However, it's clear that the normal distribution isn't a good fit. This is clear from the following chart:




Assume that the risk neutral rate of return is 5% annualized. Also, a market based calibration method is needed to compare results. This paper uses a simple SLSQP to minimize the RMSE of the predicted values and the market values. The risk neutral parameters for the NIG model are as follows:

| Parameter | r | $\alpha$ | $\hat{\beta}$ | $\delta$ |
|---|---|---|---|---|
| Model | 0.05 | 103.574591989 | -102.148583683 | 0.00982864461916 |
| Market | 0.05 | 109.73739 | 13.098993 | 0.8201415 |

The VG risk neutral parameters:

| Parameter | r | $\hat{\sigma}$ | $\hat{\theta}$ | $\nu$ |
|---|---|---|---|---|
| Model | 0.05 | 0.0098339883967 | -0.0016947834992 | 0.82745807048 |
| Market | 0.05 | 6.407e-02 | -5.86e-02 | 0 |

To assess how accurate these parameters are, the predicted call values are compared to real world market data from March 31, 2016 [7]. The RMSE of the predicted call values from all four methods is shown:

| Method | NIG Model | NIG Market | VG Model | VG Market |
|---|---|---|---|---|
| RMSE | 10.477134882 | 7.51262205 | 9.516228684 | 7.394267814 |

While it is clear that the Esscher transform calibration methods do produce different parameters from the market based calibration methods, and that the RMSEs are lower in the market based calibration methods, both of these facts are fine. It is important to note that that the range for call option values in the data set is around 400 dollars, so a difference of 2 or 3 dollars isn't that significant. It is helpful to look at the price and error charts and for the 4 calibration methods. (See Fig 1-8) It's clear to see that both of the Esscher transform calibration methods do produce similar price and error curves, and both have a large dip when pricing options that are right out of the money. This price dip is relatively common when using the Lewis method [Source]. While the prices could be more accurate, both the Esscher transform calibration and market based calibration yield similar results. Thus, it is possible to use Esscher transform calibration to price options without using market data. This is useful when such market data doesn't exist.

## 4. Application

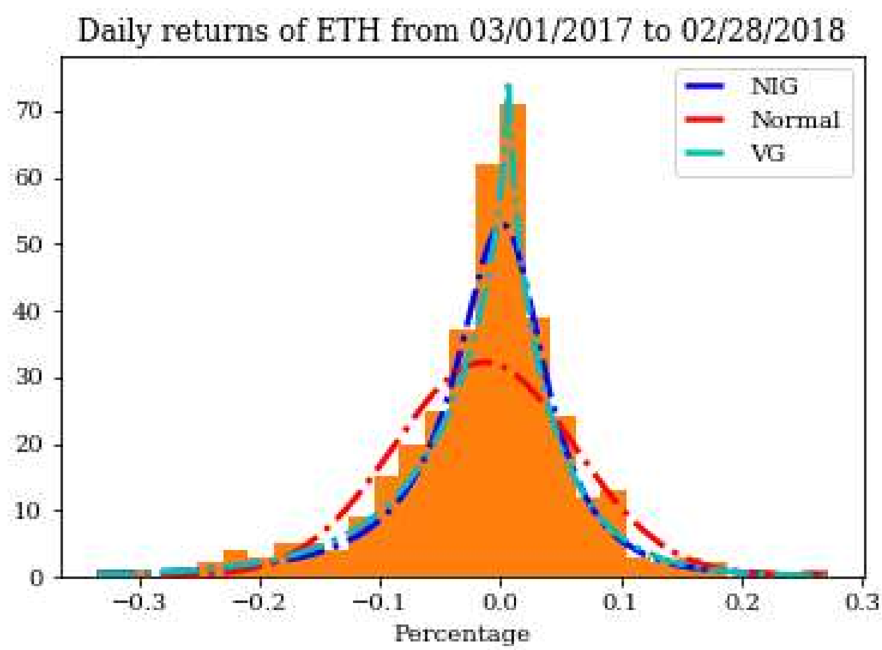
Now that there is a easy method to calibrate models such as the VG and NIG using only asset data, it is possible to price options on assets, such as cryptocurrencies, which do not have a well established options market. This paper will focus on Ethereum (ETH), mainly due to it's wide adoption and smart contract integration. The first step is finding the real measure. Using the daily returns from 03-01-2017 to 02-28-2018 [8], the real measures are:

| NIG | Value | Standard Error | t-value |
|---|---|---|---|
| $\mu$ | 5.027556e-03 | 0.002856104 | 1.760284541 |
| $\alpha$ | 9.17731507 | 3.318683916 | 2.765347740 |
| $\beta$ | -3.31953720 | 0.544025145 | -6.101808401 |
| $\delta$ | 0.04730191 | 0.011049360 | 4.280963950 |

| VG | Value | Standard Error | t-value |
|---|---|---|---|
| $c$ | 0.00689844 | 0.001327792 | 5.195418339 |
| $\sigma$ | 0.0724991 | 0.004206633 | 17.23447319 |
| $\theta$ | -0.02014797 | 0.003959816 | -5.088107642 |
| $\nu$ | 1.25351619 | 0.193007060 | 6.494664988 |

It's important to note that all the parameters are statistical significant. This may be a fluke, or an important fact about ETH. For study is needed. Using the same tests as before, it can be shown that both the NIG and VG distributions are good fits:

| Test | NIG P-value | VG P-value |
|---|---|---|
| $\chi^2$ | 0.555085211152320 | 0.831239354578552 |
| Kolmogorov Smirnov | 0.999026025987761 | 0.999863852746524 |




Making the same assumptions as before, it is possible to find the risk neutral parameters:

| Parameter | r | $\alpha$ | $\hat{\beta}$ | $\delta$ |
|---|---|---|---|---|
| Model | 0.05 | 9.1773151 | -6.805610 | 0.0473019 |

| Parameter | r | $\hat{\sigma}$ | $\hat{\theta}$ | $\nu$ |
|---|---|---|---|---|
| Model | 0.05 | 0.0784743 | -0.047366 | 0.82745807048 |

Note that the spot price on 02-28-2018 was around 867 USD. The price curves for the call options can be seen in Fig 9,10.

## 5. Conclusion

The goal of this project was to show that Esscher transforms can be used to approximate option values without the need for market data on those options. To this end, the project was successful. While the Esscher transform based calibration method didn't outperform market based methods, it didn't under-perform either. Since this method requires less data and computational power, there are quite a few scenarios where this method would be useful, such as pricing the initial call options on ETH.

## 6. Bibliography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## 7. Charts

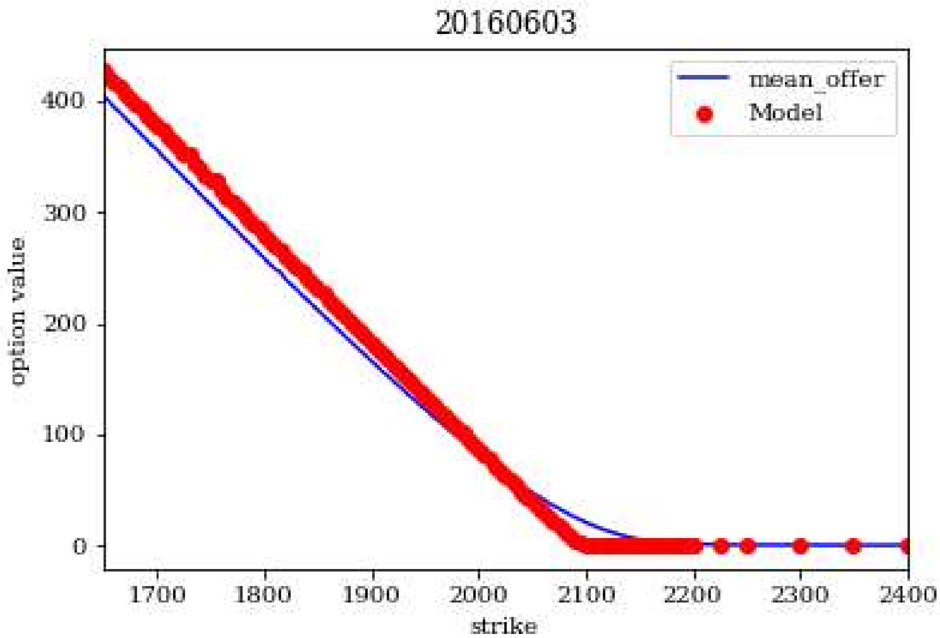


Figure 1: NIG Model

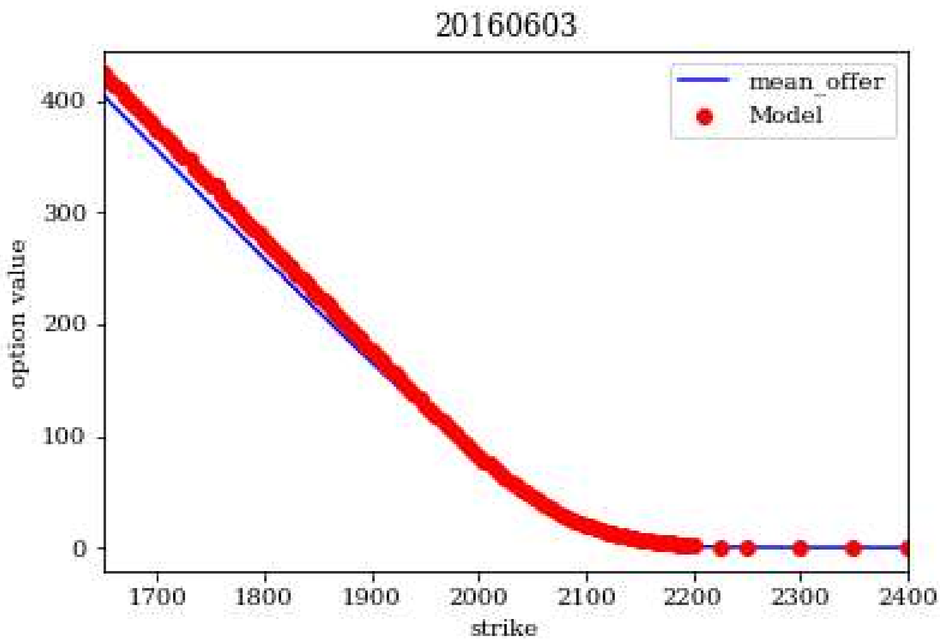


Figure 2: NIG Market

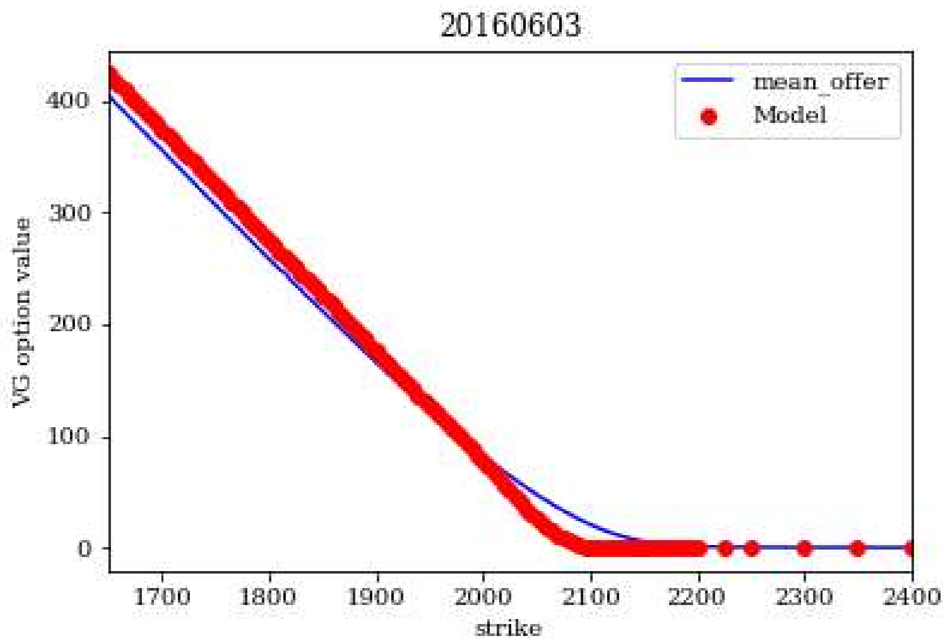


Figure 3: VG Model

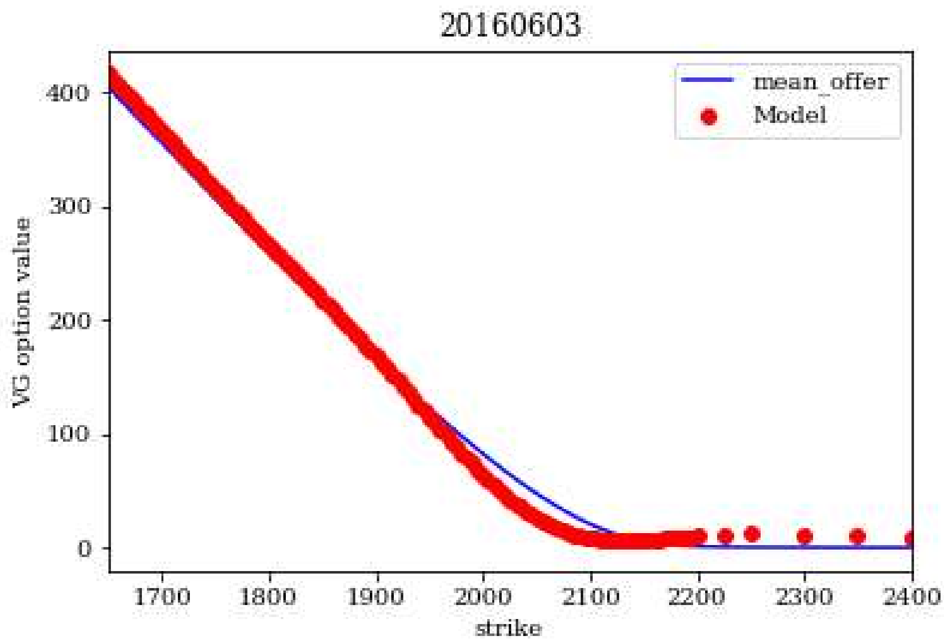


Figure 4: VG Market

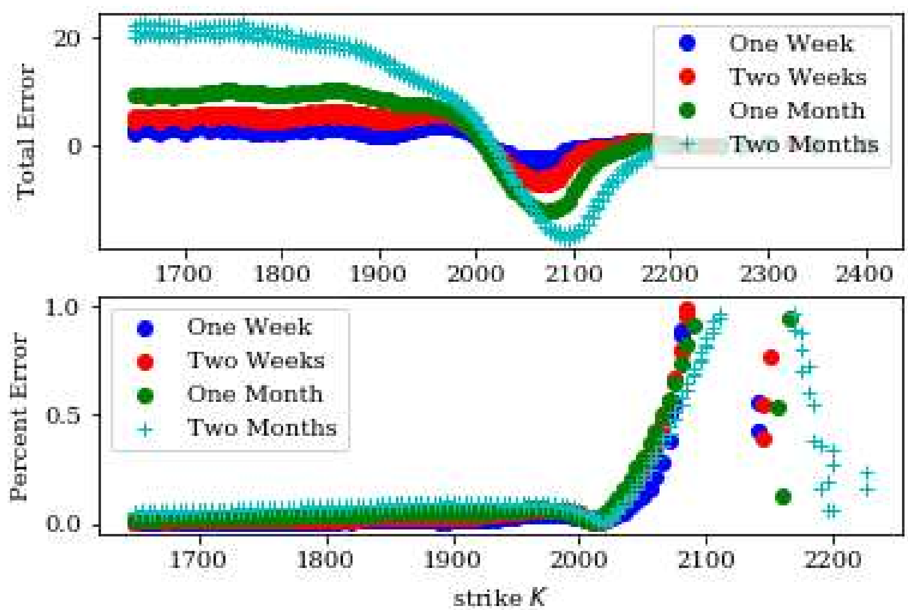


Figure 5: NIG Model Error

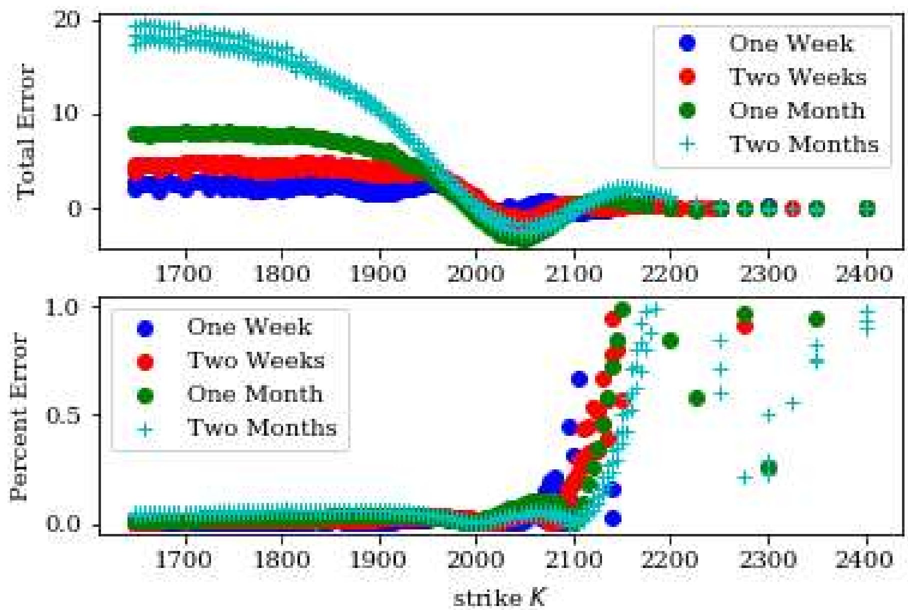


Figure 6: NIG Market Error

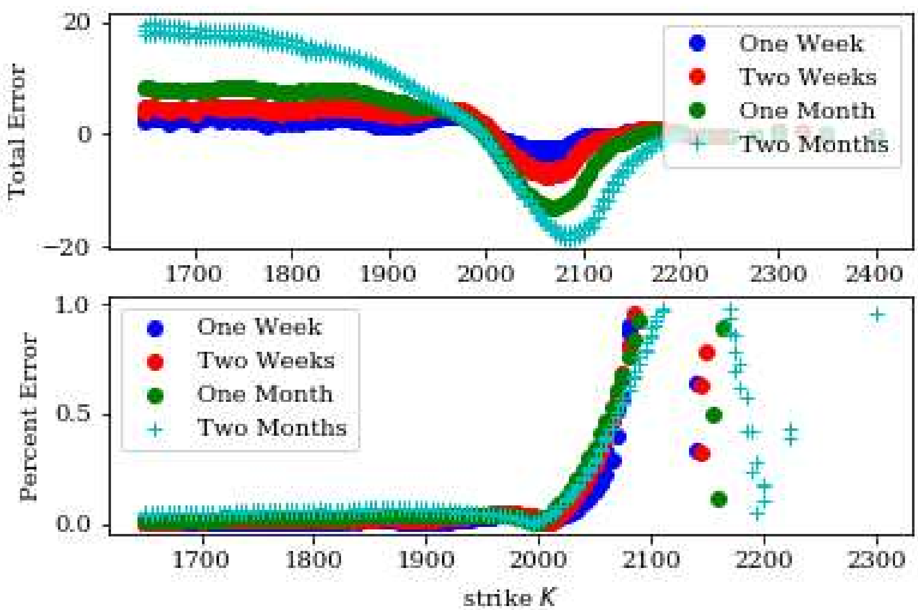


Figure 7: VG Model Error

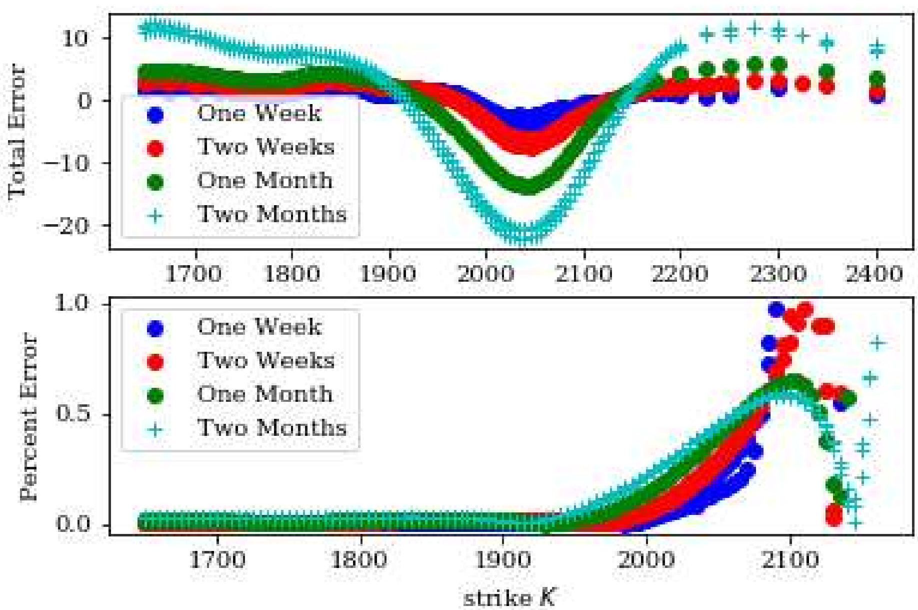


Figure 8: VG Market Error

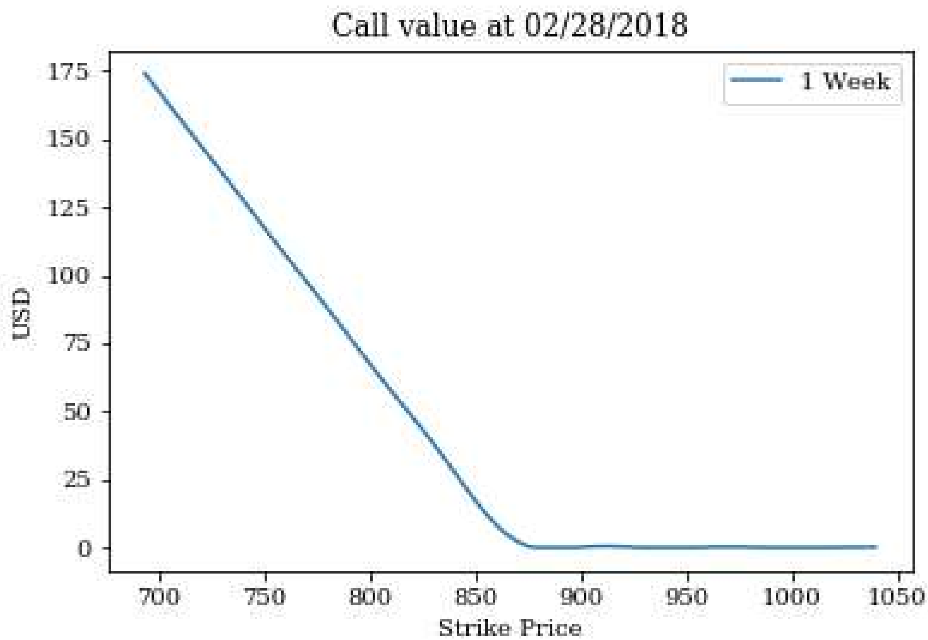


Figure 9: NIG value

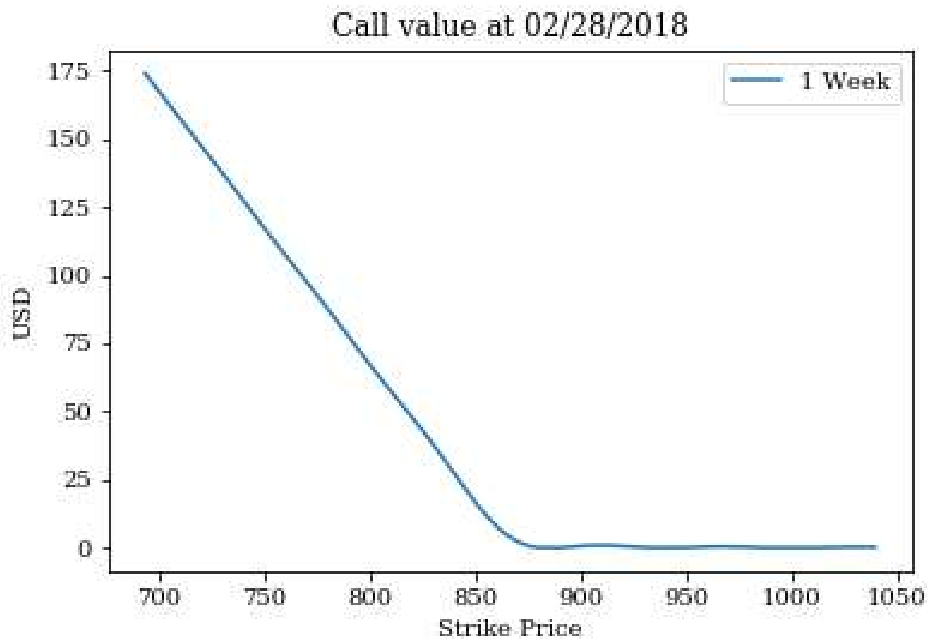


Figure 10: VG value