人工智能导论第三次作业

1

$$\sum_{i=1}^{n} \log p(\boldsymbol{x} \mid \mu, \boldsymbol{\Sigma}) = -\frac{n}{2} \log |2\pi \boldsymbol{\Sigma}| - \frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{x}_i - \mu)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}_i - \mu)$$

(1) 令 $\nabla_\mu J(\mu, \boldsymbol{\Sigma}) = 0$ ，即:

$$\begin{aligned}
\frac{\partial \sum_{i=1}^{n} \log p(\boldsymbol{x} \mid \mu, \boldsymbol{\Sigma})}{\partial \mu} &= 0 \\
\frac{\partial \left[-\frac{n}{2} \log |2\pi \boldsymbol{\Sigma}| - \frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{x}_i - \mu)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}_i - \mu) \right]}{\partial \mu} &= 0 \\
\frac{\partial \left[\sum_{i=1}^{n} (\boldsymbol{x}_i - \mu)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}_i - \mu) \right]}{\partial \mu} &= 0 \\
\boldsymbol{\Sigma}^{-1} \sum_{i=1}^{n} (\boldsymbol{x}_i - \mu) &= 0 \\
\mu &= \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i
\end{aligned}$$

故 $\hat{\mu}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i$ 。另一方面，令 $\nabla_\Sigma J(\mu, \boldsymbol{\Sigma}) = 0$ ，即:

$$\begin{aligned}
\frac{\partial \sum_{i=1}^{n} \log p(\boldsymbol{x} \mid \mu, \boldsymbol{\Sigma})}{\partial \boldsymbol{\Sigma}} &= 0 \\
\frac{\partial \left[-\frac{n}{2} \log |2\pi \boldsymbol{\Sigma}| - \frac{1}{2} \sum_{i=1}^{n} (\boldsymbol{x}_i - \mu)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x}_i - \mu) \right]}{\partial \boldsymbol{\Sigma}} &= 0 \\
-\frac{n}{2} \boldsymbol{\Sigma}^{-1} + \frac{1}{2} \boldsymbol{\Sigma}^{-1} \left(\sum_{i=1}^{n} (\boldsymbol{x}_i - \mu)^T (\boldsymbol{x}_i - \mu) \right) \boldsymbol{\Sigma}^{-1} &= 0 \\
n \boldsymbol{\Sigma} &= \sum_{i=1}^{n} (\boldsymbol{x}_i - \mu) (\boldsymbol{x}_i - \mu)^T \\
\boldsymbol{\Sigma} &= \frac{1}{n} \sum_{i=1}^{n} (\boldsymbol{x}_i - \mu) (\boldsymbol{x}_i - \mu)^T
\end{aligned}$$

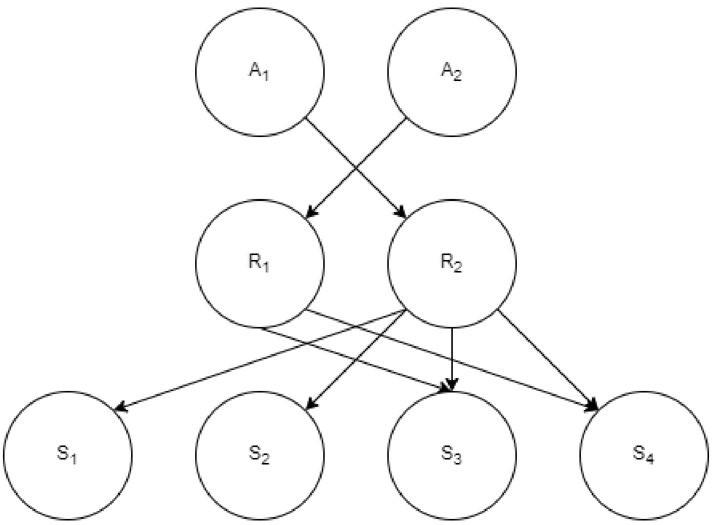
故 $\hat{\Sigma}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} (\boldsymbol{x}_i - \hat{\mu}_{\text{MLE}}) (\boldsymbol{x}_i - \hat{\mu}_{\text{MLE}})^T$ 。

(2) 证明:

$$
\begin{aligned}
\mathrm{E}\left[\hat{\boldsymbol{\Sigma}_{\mathrm{MLE}}}\right] &= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\boldsymbol{x}_i-\hat{\mu}_{\mathrm{MLE}}\right)\left(\boldsymbol{x}_i-\hat{\mu}_{\mathrm{MLE}}\right)^T\right] \\
&= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^2-2x_i\hat{\mu}_{\mathrm{MLE}}+\hat{\mu}_{\mathrm{MLE}}^2\right)\right] \\
&= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^2\right)-\hat{\mu}_{\mathrm{MLE}}^2\right] \\
&= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^2\right)-\mu^2+\mu^2-\hat{\mu}_{\mathrm{MLE}}^2\right] \\
&= \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(x_i^2\right)-\mu^2\right]-\mathrm{E}\left(\hat{\mu}_{\mathrm{MLE}}^2-\mu^2\right) \\
&= \mathrm{E}\left[\frac{\sum_{i=1}^{n}\left(x_i^2-\mu^2\right)}{n}\right]-\left[\mathrm{E}\left(\hat{\mu}_{\mathrm{MLE}}^2\right)-\mu^2\right] \\
&= \boldsymbol{\Sigma}-\left[\mathrm{E}\left(\hat{\mu}_{\mathrm{MLE}}^2\right)-\mathrm{E}^2\left(\hat{\mu}_{\mathrm{MLE}}\right)\right] \\
&= \boldsymbol{\Sigma}-\mathrm{Var}\left(\hat{\mu}_{\mathrm{MLE}}\right) \\
&= \boldsymbol{\Sigma}-\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) \\
&= \boldsymbol{\Sigma}-\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\left(x_i\right) \\
&= \boldsymbol{\Sigma}-\frac{1}{n}\boldsymbol{\Sigma} \\
&= \frac{n-1}{n}\boldsymbol{\Sigma}
\end{aligned}
$$

2

(1)



$$
\begin{aligned}
&p(S_1,S_2,S_3,S_4,R_1,R_2,A_1,A_2) \\
=&p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)p(R_1|A_2)p(R_2|A_1)p(A_1)p(A_2)
\end{aligned}
$$

(2) R_1, R_2

(3) 需要 $2+2+4+4+2+2+1+1=18$ 个参数。

取消独立性假设：$2^8-1=255$ 个参数。

(4) 若有吸烟经历，则 R_1 的概率发生变化；若已知 $S_3=1$ ，则 R_1, R_2 的概率都发生变化。

(5)

$$
\begin{aligned}
&p(R_2) \\
=& \sum_{A_1,A_2,S_1,S_2,S_3,S_4,R_1} p(A_1,A_2,S_1,S_2,S_3,S_4,R_1,R_2) \\
=& \sum_{A_1,A_2,S_1,S_2,S_3,S_4,R_1} p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)p(R_1|A_2)p(R_2|A_1)p(A_1)p(A_2) \\
=& \sum_{A_1,S_1,S_2,S_3,S_4,R_1} p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)p(R_2|A_1)p(A_1)\sum_{A_2} p(R_1|A_2)p(A_2) \\
=& \sum_{A_1,S_1,S_2,S_3,S_4,R_1} p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)p(R_2|A_1)p(A_1)\tau(R_1) \\
=& \sum_{S_1,S_2,S_3,S_4,R_1} p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)\tau(R_1)\sum_{A_1} p(R_2|A_1)p(A_1) \\
=& \sum_{S_1,S_2,S_3,S_4,R_1} p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)\tau(R_1)\tau(R_2)
\end{aligned}
$$

$$
\begin{aligned}
&p(R_2|A_1,A_2,S_1,S_2,S_3,S_4) \\
=&\frac{p(A_1,A_2,S_1,S_2,S_3,S_4,R_2)}{p(A_1,A_2,S_1,S_2,S_3,S_4)} \\
=&\frac{p(A_1)p(A_2)p(S_1|R_2)p(S_2|R_2)p(S_3|R_1,R_2)p(S_4|R_1,R_2)}{}
\end{aligned}
$$