Math Note

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Contents

My goal is to rewrite all undergraduate mathematics in my own language.
And, for review, to draw diagrams illustrating the relationships between mathematical objects.

Patch Note:

$\sim$ 2025/9/28 - Drafted the initial framework of the paper, and Transcribed previous works.
2025/09/29 - 1. Completed proof of Ring of Fractions.
2. Transcribed Integral, Ratio, and Root Test.
3. Transcribed Tube Lemma, Lindelöf and Countably Compact product Compact.
4. Transcribed Coproduct with Continuous, open, closed map.
2025/09/30 - 1. Proved Every open set in $\mathbb{R}^n$ is countable union of closed cubes, disjoint of interiors remains.
2. Transcribed Group action.
2025/10/01 - 1. Transcribed One-point Compactification.
2. Transcribed Definitions of subbasis, Borel set.
2025/10/02 - 1. Proved Euclidean Domain
2. Proving Existence of Nowhere-differentiable function.
2025/10/03 - 1. Drafted definition and propositions of Quotient Space.
2025/10/04 - 1. Studying Quotient Map.
2025/10/05 - 1. Studied Basic Properties of the Quotient Map, and Drew quotient map diagram.
2025/10/06 - 1. Drafted basic functions in a Metric space.
2025/10/07 - 1. Proved basic properties of Completely regular space.
2025/10/08 - 1. Proved Compact Hausdorff Space is Normal.
2. Proved Equivalent Conditions of Completely Regular Space.
3. Proving the Urysohn Metrization Theorem.
4. Understanding relations and characteristic of Domains.
2025/10/10 - 1. Transcribed basic statements of Polynomial Ring.
2025/10/11 - 1. Proved Gauss's Lemma in Polymonial Ring, and Transcribed.
2. Proving R U.F.D. iff $R[x]$ U.F.D.
2025/10/13 - 1. Understanding U.F.D and irreducible.
2. Proved R U.F.D. iff $R[x]$ U.F.D.
3. Drew the Diagram of Domains.
2025/10/14 - 1. Drew the Relations of Poloynomial Ring Diagram.
2. Study Quadratic Integer Ring.
3. Study Nilpotent in Ring, and Nilradical Ring.
4. Arranged Connected Space.
5. Proved S^n is One-point Compactification of $\mathbb{R}^n$.
6. Proving S^n is Connected via One-point Compactification.
2025/10/15 - 1. Completed Irreducibility Criteria.
2025/10/16 - 1. Observing Cyclotomic Polynomial.
2025/10/17 - 1. Studying Quadratic Field and Quadratic Integer Ring.
2025/10/18 - 1. Observed Nilradical Ideal.
2025/10/27 - 1. Drew Diagram in Connectedness.
2. Analyzed Topologist's Since curve.

Goal:

1. Brouwer fixed-point theorem.
2. Abel-Ruffini theorem.
3. Stokes's Theorem.
4. Three-Body Problem has no Analytic general solution.

2025/11/6 - 1. Completed Basic thoery of fields - Centre around Kronecker's Theorem.
2025/11/7 - 1. Studying Algebraic Extension.
2025/11/14 - 1. Proving $B^n/S^{n-1} \cong S^n$.
2. Revised proof of basic field theory.
2025/11/18 - 1. Revising the Basic Notation and Proving the Basic Properties of Vector Spaces.
2. Proved Extension field is the Vector Space.
2025/11/23 - 1. Wrote basic Definition of multivariable Derivative for PINN.
2. Studying Typical Quotient Space: Cylinder, Torus, Möbius band, Klein bottle, Projective Space.
2025/11/24 - 1. Stydied Typical Quotient Space: Cylinder, Torus, Möbius band.
2. Described Definition of Topological Manifold.
2025/11/27 - 1. Studying Basic Theory of Partial Differential Equation.
2025/12/8 - 1. Studied Definition of Manifold.
2. Understood Definition of Connected Sum.
3. Studied Connected Sum with Typical Space(Torus, P^n, Klein Bottle).
4. Studied Method to find Basis of Finitely Generated Field.

Chapter 1

Set Theory

Chapter 2

Group Theory

2.1 Isomorphism Theorems

Definition 2.1.0.1. Let G be a group, and $A \subset G$. Define *Subgroup generated by* A:

$$\langle A \rangle \overset{\text{def}}{=} \bigcap_{A \subseteq H \leq G} H$$

Lemma 2.1.0.1. Let G be a group, and $A \subset G$.

$$\langle A \rangle = \{ a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid n \in \mathbb{N}, \alpha_i \in \mathbb{Z}, a_i \in A \}$$

Theorem 2.1.0.1. *The First Isomorphism Theorem*
Let $\varphi : G \to H$ be a Group-Homomorphism. Then,

$$G/\ker\varphi \cong \varphi[G]$$

$$\begin{tikzcd}
G \arrow[r, "\overset{\text{homomorphism}}{\varphi}"] \arrow[dr, "\pi"', "\substack{\text{homomorphism}\\\text{surjection}}"'] & \varphi[G] \\
 & G/\ker\varphi \arrow[u, "\substack{\phi\\\text{isomorphism}}"']
\end{tikzcd}$$

Proof. Let $\pi : G \to G/\ker\varphi : x \mapsto x + \ker\varphi$. Then, the map $\phi : G/\ker\varphi \to \varphi[G] : a + \ker\varphi \mapsto \varphi(a)$ is isomorphism.
Well-defined and Injective:

$$a + \ker\varphi = b + \ker\varphi \iff a - b \in \ker\varphi \iff \varphi(a-b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. $\square$

Theorem 2.1.0.2. *The Second Isomorphism Theorem*
Let G be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

Proof. HK be a subgroup of G, being

$$HN = \bigcup_{h \in H} hN \overset{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.
Meanwhile, $H \cap N$ be a Normal Subgroup of H: for any $h \in H, n \in H \cap N$, $hnh^{-1} \in N$ because N is normal, and

$hnh^{-1} \in H$ since h, n contained in H. Thus, $hnh^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of H.
Now, Define a Map:

$$\varphi : H \to HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker \varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

> **Theorem 2.1.0.3.** *The Third Isomorphism Theorem*
> Let G be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and
>
> $$(G/H)/(K/H) \cong (G/K)$$

Proof. First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gkg^{-1})H \in K/H$$

since $gkg^{-1} \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \to G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \overset{H \leq K}{\Longrightarrow} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.
Clearly, for any $g_1H, g_2 \in G/H$,

$$\varphi(g_1Hg_2H) = \phi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.
4. Kernel.

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

□

> **Theorem 2.1.0.4.** *The Forth Isomorphism Theorem*
> Let G be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between
>
> $$D \overset{\text{def}}{=} \{H \leq G \mid N \leq H\},\ C \overset{\text{def}}{=} \{\overline{H} \leq G/N\}$$

Proof. Let $\pi : G \to G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \to C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.
To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.
To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear. □

2.2 Group Action

In this section, we follow that the notation of [Dummit and Foote, 2004, Abstract Algebra].

Definition 2.2.0.1. Let $(G, *)$ be a Group, and A be a non-empty set.
Define *Group Action* of a group G on a set A:

$$\alpha : G \times A \to A : (g, a) \mapsto g \cdot a$$

satisfies

1. For all $a \in A$, $1_G \cdot a = a$.
2. For all $g_1, g_2 \in G$, $a \in A$, $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$

In this, we said to be 'G acts on a set A'. Meanwhile, For each $g \in G$, Define a map

$$\sigma_g : A \to A : a \mapsto g \cdot a$$

Then, the *permutation representation*

$$\varphi : G \to S_A : g \mapsto \sigma_g$$

be a Homomorphism. Clearly, for each $g \in G$, $a \in A$,

$$\alpha(g, a) = g \cdot a = \sigma_g(a) = \varphi(g)(a)$$

Thus, there is one-to-one correspondence between group action and permutation representation.
For each $a \in A$, the *stabilizer* of a in G:

$$G_a \overset{\text{def}}{=} \{g \in G \mid g \cdot a = a\}$$

The *kernel of action*:

$$\ker \alpha \overset{\text{def}}{=} \{g \in G \mid g \cdot a = a,\ \ \forall a \in A\} = \bigcap_{a \in A} G_a$$

$G_a \leq G$ and $\ker \alpha \trianglelefteq G$.
If the kernel of action be trivial, the action is called *faithful*.

Definition 2.2.0.2. Let $\alpha : G \times A \to A$ be a Group Action. Define a relation on A:

$$a \sim b \iff a = g \cdot b \text{ for some } g \in G$$

Then, this relation be equivalence relation. Denote the equivalence relation, called *orbit*:

$$\mathcal{C}_a \overset{\text{def}}{=} \{b \mid b = g \cdot a \text{ for some } g \in G\} = \{g \cdot a \mid g \in G\}$$

And, the action is called *transitive* if there is only one orbit.

Lemma 2.2.0.1. For each $a \in A$,

$$|\mathcal{C}_a| = |G : G_a|$$

Proof. Since the map

$$\varphi_a : \mathcal{C}_a \to \{gG_a \mid g \in G\} : g \cdot a \mapsto gG_a$$

is well-defined, bijection.

□

Theorem 2.2.0.1. Let G be a Group, let $H \leq G$ and $A = \{gH \mid g \in G\}$, G acts by left multiplication on the set A.

$$\pi_H : G \to S_A : g \mapsto \sigma_g$$

be a permutation representation afforded by this action. Then

1. G acts transitively on A.
2. $G_{1H} = \{g \in G \mid gH = H\} = H$.
3. The kernel of the action $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$, this is the largest normal subgroup of G contained in H.

Proof. Let $aH, bH \in A$ be given. Then, for $g = ba^{-1}$, $g \cdot aH = (ga)H = bH$. Thus, $A = \mathcal{C}_a$ for any $a \in G$.
2 is clear, being $gH = H \iff g \in H$.
Now,

$$\begin{aligned}
\ker \pi_H &= \{g \in G \mid gxH = xH, \ \forall x \in G\} \\
&= \{g \in G \mid (x^{-1}gx)H = H, \ \forall x \in G\} \\
&= \{g \in G \mid x^{-1}gx \in H, \ \forall x \in G\} \\
&= \{g \in G \mid g \in xHx^{-1}, \ \forall x \in G\} = \bigcap_{x \in G} xHx^{-1}
\end{aligned}$$

And the second assertion given by:
Let N is a normal subgroup of G contained in H, then for any $x \in G$, $N = xNx^{-1} = xHx^{-1}$. Thus,

$$N \leq \bigcap_{x \in G} xHx^{-1}$$

□

Corollary 2.2.0.1. If G is a finite group of order n, p is the smallest prime dividing $|G|$.
Then, any subgroup of index p is normal.

Proof. Let $|G| = p^r p_1^{r_1} \cdots p_n^{r_n}$ be a prime decomposition, $H \leq G$ with $|G : H| = p$.
Let $K = \ker \pi_H \leq H$, $k = |H : K|$. Then, $|G : K| = |G : H||H : K| = pk$. By the First-Isomorphism Theorem,

$$G / \ker \pi_H \cong \pi_H[G] \leq S_A$$

and Since H has p left cosets, $A \cong \mathbb{Z}_p$, thus G/K is isomorphic to some subgroup of S_p.
Now, Lagrange's Theorem gives that $|G/K| = pk$ divides $|S_p| = p!$. This implies $k \mid (p-1)!$.
$|G : K| = pk$ implies $|G| = pk \cdot |K|$. Since p is the minimal prime that divides $|G|$, thus every prime divisor of k is greater than or equal to p. This implies must be $k = 1$. Thus $H = K \trianglelefteq G$. □

Definition 2.2.0.3. Let a Group action as:

$$\alpha : G \times G \to G : (g, a) \mapsto gag^{-1}$$

Now, the orbit drived from this action $[a] = \{b \in G \mid \exists g \in G \text{ s.t. } b = gag^{-1}\}$ is called be *Conjugacy Class*.
More generally,

$$\alpha : G \times \mathcal{P}(G) \to \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$$

Lemma 2.2.0.2. Let $\alpha : G \times \mathcal{P}(G) \to \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$ be a Group action acting as Conjugate.
Then, $G_S = N_G(S)$ and $|\mathcal{C}_S| = |G : N_G(S)|$, for any $S \subseteq G$.
In particular, if S is singleton, $S = \{g_i\}$, then $|\mathcal{C}_{\{g_i\}}| = |G : N_G(g_i)| = |G : C_G(g_i)|$.

Proof.

$$G_S = \{g \in G \mid gSg^{-1} = S\} = N_G(S)$$

Thus, for any $S \in \mathcal{P}(G)$,

$$|\mathcal{C}_S| = |G : N_G(S)|$$

□

2.2.1 Lagrange's Theorem

2.3 Generating subset of a Group

2.4 Commutator Subgroup

Chapter 3

Finite Group Theory

3.1 The Class Equation

Theorem 3.1.0.1. *The Class Equation*
Let G be a finite group, and
$g_1, \dots, g_r$ be representatives of the distinct conjugacy classes of G not contained in the center $Z(G)$ of G.
Then,

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

3.2 Cauchy's Theorem

Lemma 3.2.0.1. *Cauchy's Theorem*
Let G be a finite group, and p be a prime dividing $|G|$. Then, G has order p element.

Proof. Define a set:

$$S \overset{\text{def}}{=} \{(x_1, x_2, \dots, x_p) \mid x_i \in G,\ \ x_1 x_2 \cdots x_p = 1\}$$

Then, S has exactly $|G|^{p-1}$ elements because there are $|G|$ possible choices for each of the first $p-1$ elements in G.
Once $x_1, \cdots, x_{p-1}$ are chosen, then x_p is uniquely determined by the uniqueness of inverses.
Then, let $\sigma = (1, 2, \dots, p)$ be a permutation. Then, for any $\alpha \in S$, $\sigma^n(\alpha) \in S$ for all $n \in \mathbb{Z}$, being $ab = 1 \iff ba = 1$.
More precisely, let $n \in \mathbb{Z}$ be given, $\alpha = (x_1, \cdots, x_n)$. Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \dots, x_p, x_1, x_2, \dots x_n)$$

By $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$, $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$. Thus $\sigma^n(\alpha) \in S$. Now, define a relation on S as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on S. Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is silngeton if and only if } \alpha = (x, \dots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,
Suppose that $\alpha = (x_1, \dots, x_n)$ has different cordinate elements, let $x_i \neq x_j$, for some $i < j$. Then clearly

$$(x_1, \dots, x_i, \dots, x_p) \neq \sigma^{i-j}(x_1, \dots, x_i, \dots, x_j, \dots, x_p) = (\dots, \underbrace{x_j}_{\text{i'th element}}, \dots)$$

Meanwhile, if $[\alpha]$ has elements more than 1, $[\alpha]$ has exactly number of p elements. Because suppose that $\alpha = (x_1, \dots, x_p)$ has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \cdots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist $1 \leq i < j < p$ such that $\sigma^i(\alpha) = \sigma^j(\alpha)$, that is, $\sigma^{j-i}(\alpha) = \alpha$.
Now, $j - i \mid p$, this is contradiction with p is prime. Therefore, every equivalent class has order 1 or p. Consequently,

$$|G|^{p-1} = k + pd$$

where k is a number of classes of size 1, and d is a number of classes of size p. And $(1, 1, \ldots, 1) \in S$, k is at least 1.
Since p divides $|G|^{p-1} = k + pd$, thus k must be bigger than 1, thus there exists elements such that $x^p = 1$. $\square$

3.3 Sylow's Theorem

Theorem 3.3.0.1. *Sylow's Theorem*
Let G be a group of order $p^{\alpha}m$, where p is a prime such that $p \nmid m$.
A group of order p^r, $(r \geq 1)$ is called a p-group, Subgroups of G which are p-groups are called p-subgroup.
In particular, subgroups of order p^{α} is called *Sylow p-subgroup* of G. And, define a collection

$$\mathtt{Syl}_p(G) \overset{\text{def}}{=} \{P \leq G \mid |P| = p^{\alpha}\}, \ \ n_p(G) \overset{\text{def}}{=} \mathtt{Card}(\mathtt{Syl}_p(G))$$

The First Sylow Theorem
There exists a Sylow p-subgroup of G. i.e., $\mathtt{Syl}_p(G) \neq \emptyset$.

The Second Sylow Theorem
If $P \in \mathtt{Syl}_p(G)$ and $Q \leq G$ be a p-subgroup. Then, there exists $g \in G$ such that $Q \leq gPg^{-1}$.

The Third Sylow Theorem
$n_p \equiv 1(\text{mod} p)$, $n_p = |G : N_G(P)|$ for any $P \in \mathtt{Syl}_p(G)$, and $n_p \mid m$.

Before prove above statments, we show that:

Lemma 3.3.0.1. Let $P \in \mathtt{Syl}_p(G)$. If Q is p-subgroup of G, then $Q \cap N_G(P) = Q \cap P$.

Proof. Put $H = Q \cap N_G(P)$. Since $P \leq G$, for any $p \in P$, $pPp^{-1} = P$, thus $p \in N_G(P)$. i.e., $P \leq N_G(P)$.
Thus, Enough to Show that $H \leq Q \cap P$. Since $H \leq N_G(P)$,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus, $PH \leq G$. And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem, $H \leq P$ and $P \cap H \leq P$ must have order of powers of p, so PH be a p-group. Clearly, $P \leq PH$ and P is the largest p-group of G, thus, $PH = P$. This means, $H \leq P$. ☐

Proof. The First Theorem: The existence of Sylow p-subgroup. Proof by Induction:
If $|G| = 1$, there is nothing to prove.
Assume inductively the existence of Sylow p-subgroups for all groups of order less than $|G|$.
In case of $p||Z(G)|$, then by Cauchy's Theorem, $Z(G)$ has a subgroup N which has order of p.
Clearly N is Normal, and $G/N = |G|/|N| = p^{a-1}m$. By assumption, G/N has a subgroup P' of order $p^{\alpha-1}$.
By The Forth Isomorphism Theorem, Let $P \leq G$ be a subgroup such that $P/N = P'$.
Then, $|P| = |P/N| \cdot |N| = p^{\alpha}$, Thus P be a Sylow p-subgroup of G.
In case of $p \nmid |Z(G)|$.
Let $g_1, \ldots, g_r$ be represectatives of the distinct conjugacy classes of G, not contained in $Z(G)$. Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|$$

Since p divides $|G|$, if for all $i = 1, 2, \ldots, r$, $p \mid |G : C_G(g_i)|$ then $p \mid |Z(G)|$, this is contradiction.
Thus, for some j, $p \nmid |G : C_G(g_j)|$. Put $H = C_G(g_j) < G$. Then, $|H|$ has a factor of p^{α}, by $p \nmid |G : C_G(g_j)|$. Now,

$$|H| = p^{\alpha}m' \quad (m' < m)$$

By assumption, H has a Sylow p-group, order of p^{α}.
Consequently, the existence of Sylow p-subgroup was shown.
The Second Theorem: Relation of p-subgroups.
The First Theorem gives existence of Sylow p-subgroups. Let $P \in \mathtt{Syl}_p(G)$. Denote that:

$$S \overset{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \ldots, P_r\}$$

Let $Q \leq G$ be an any p-subgroup of G. And, Q acts by conjucation on S. i.e.,

$$\alpha : Q \times S \to S : (q, P_i) \mapsto qP_iq^{-1}$$

Write S as a disjoint union of orbits under this action by Q:

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_s$$

where $r = |\mathcal{O}_1| + \cdots + |\mathcal{O}_s|$. Rearrange a set S as: $P_i \in \mathcal{O}_i,\ 1 \leq i \leq s$. Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \overset{\text{Thm}}{=} |Q : N_Q(P_i)| \overset{\text{def}}{=} |Q : N_G(P_i) \cap Q| \overset{\text{lemma}}{=} |Q : P_i \cap Q|$$

for each $1 \leq i \leq s$. Since Q was arbitrary, Let $Q = P_1$, so that $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$. And, for each $i \geq 2$, $P_i \cap P_1 < P_1$,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since $P_1 \in \mathrm{Syl}_p(G)$, that is $|P_1| = p^\alpha$, $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$ where $1 \leq k < \alpha$.
This means for each $2 \leq i \leq s$, p divides $|\mathcal{O}_i|$. Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \cdots + |\mathcal{O}_s|) \equiv 1 (\mathrm{mod} p)$$

Now, Proof by Contradiction: Let $Q \leq G$ be a p-subgroup. Suppose that for any $1 \leq i \leq r$, $Q \nleq P_i$.
Then, $P_i \cap Q < Q$ for all i, this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any i, p divides $|\mathcal{O}_i|$, this is Contradiction. This proved Relation of p-subgroups.
Finally, The Third Theorem:
Since Second Theorem, this gives that $S = \mathrm{Syl}_p(G)$, thus $n_p(G) = r$. That is, $n_p \equiv 1 (\mathrm{mod} p)$.
Since all Sylow p-subgroups are Conjugate, for any $P \in \mathrm{Syl}_p(G)$,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. $\square$

3.4 More Theorems

> **Theorem 3.4.0.1.** n *Factorial Theroem*
> If G is simple and there is a subgroup H with $|G : H| = n$, then $|G| \mid n!$.

Proof. Let G act on $A = \{gH \mid g \in G\}$ by left multiplication. ($|A| = n$.).
Let $\varphi : G \to S_n$ be a homomorphism afforded above action. Then, $G \overset{G \text{ simp.}}{\cong} G/\ker\varphi \cong \varphi[G] \leq S_n$ □

3.5 Simple groups

3.6 Cyclic Group

3.7 Symmetric Group

3.8 Dihedral Group

Chapter 4

Ring Theory

4.1 Addition and Multiplication in $\mathbb{Z}$

4.1.1 $\mathbb{Z}_n^\times$

Theorem 4.1.1.1. For any integer $n > 1$, $(\mathbb{Z}_n^\times, \times_n)$ is a group where $\begin{cases} \mathbb{Z}_n^\times \overset{\text{def}}{=} \{k \in \mathbb{Z}_n \mid \gcd(k,n) = 1\} \\ a \times_n b \overset{\text{def}}{=} ab \bmod n \end{cases}$

Proof. Let $a \in \mathbb{Z}_n^\times$ be given. Then, the Bezout's identity gives $ax + ny = 1$ for some $x, y \in \mathbb{Z}$.
Now, $ax = 1 - ny \equiv 1 \bmod n$. □

Theorem 4.1.1.2. *Fermat's Little Theorem*
If $a \in \mathbb{Z}$ and p is prime not dividing a, then $a^{p-1} \equiv 1 \bmod p$.

Proof. Let $a \in \{1, 2, \ldots, p-1\}$. Put $k = |a|$. Then, $\{1, a^1, a^2, \ldots, a^{k-1}\}$ forms a subgroup of $\mathbb{Z}_p$.
By Lagrange's Theorem, $k \mid p - 1$. Now, $a^{p-1} = a^{km} = (a^k)^m = 1 \bmod p$. □

Theorem 4.1.1.3. $a \in \mathbb{Z}_n$ is a zero divisor *if and only if* $\gcd(a,n) \neq 1$.

Proof. Suppose that $\gcd(a,n) = 1$. If $ax + n\mathbb{Z}$ is zero for some $x \in \mathbb{Z}_n$, then $ax = nk$ for some $k \in \mathbb{Z}$.
Thus, $n \mid ax$ and $\gcd(a,n) = 1$, this implies $n \mid x$.
Conversely, Suppose that $d = \gcd(a,n) \neq 1$. Then,

$$a \cdot \frac{n}{d} = \frac{a}{d} \cdot n \in n\mathbb{Z}$$

But, $a \notin n\mathbb{Z}$ and $\frac{n}{d} \notin n\mathbb{Z}$, thus a is zero divisor. □

Corollary 4.1.1.1. The subset $G_n \overset{\text{def}}{=} \{a \in \mathbb{Z}_n \mid a \text{ is zero divisor}\}$ is a group under the multiplcation modulo n.

Theorem 4.1.1.4. *Euler's Theorem*
If $\gcd(a,n) = 1$, $a^{\varphi(n)} \equiv 1 \bmod n$.

Proof. Let $a \in \mathbb{Z}$ with $\gcd(a,n) = 1$. Then, there exists $0 < b \in a + n\mathbb{Z}$ such that $b < n$ and $\gcd(b,n) = 1$.
Then, $a^{\varphi(n)} \equiv b^{\varphi(n)} \bmod n$. Since $b \in \mathbb{Z}_n^\times$ and order of $\mathbb{Z}_n^\times$ is $\varphi(n)$, thus $b^{\varphi(n)} \equiv 1 \bmod n$. □

4.2 Ideal

Definition 4.2.0.1. Let R be a Ring. A subset $I \subseteq R$ is called ***ideal*** of R if:

1. $I \subseteq R$ is a subgroup of R.
2. I is closed under the multiplication.
3. For any $r \in R$, $rI \subseteq I$ and $Ir \subseteq I$. (In other word, for any $r \in R, a \in I$, $ra \in I$ and $ar \in I$.)

Theorem 4.2.0.1. Let R be a Ring. Then, TFAE:

1. $I \subset R$ is an Ideal of R.
2. The additive Quotient Group $R/I \overset{\text{def}}{=} \{r + I \mid r \in R\}$ be a Ring under the operation:

$$(r + I) \times (s + I) = (rs) + I$$

Proof. Observation:

$$r_1 + I = r_2 + I \iff r_1 - r_2 \in I \iff \exists a \in I \text{ s.t. } r_1 = r_2 + a$$

Now, for well-definedness, want to show that the equality

$$\begin{aligned}&(r + I) \times (s + I) = (rs) + I \\ \overset{(*)}{=} &[(r + \alpha) + I] \times [(s + \beta) + I] = (r + \alpha)(s + \beta) + I = (rs + r\beta + \alpha s + \alpha\beta) + I\end{aligned}$$

$(*)$ holds for any $r, s \in R$, $\alpha, \beta \in I$.
If I is Ideal, then $r\beta, \alpha s, \alpha\beta \in I$. Thus closed under the addition gives $(*)$.
Conversely, if this operation is well-defined, then for any $r, s \in R$, $\alpha, \beta \in I$, $(*)$ holds.
Substituting zero to each r, s, α, β gives I is ideal. $\square$

4.2.1 Properties of Ideal in Ring with identity

Definition 4.2.1.1. Let R be a Ring with identity, and $A \subseteq R$. Define *Ideal generated by* A as:

$$(A) \stackrel{\text{def}}{=} \bigcap_{\substack{I \text{ ideal} \\ A \subseteq I}} I$$

And,

$$RA \stackrel{\text{def}}{=} \{r_1 a_1 + \cdots + r_n a_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}$$
$$AR \stackrel{\text{def}}{=} \{a_1 r_1 + \cdots + a_n r_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\}$$
$$RAR \stackrel{\text{def}}{=} \{r_1 a_1 r_1' + \cdots + r_n a_n r_n' \mid n \in \mathbb{N}, r_i, r_i' \in R, a_i \in A\}$$

Lemma 4.2.1.1. Let R be a Ring with identity, and $A \subseteq R$. Then, $(A) = RAR$.

Proof. Since RAR is ideal which contains A, $(A) \subseteq RAR$.
And, conversely, if $\sum_{i=1}^{n} r_i a_i r_i' \in RAR$, then $\sum_{i=1}^{n} r_i a_i r_i' \in (A)$ because each $r_i a_i r_i'$ are contained in (A), being (A) is ideal containing A and ideal is closed under the addition. □

Theorem 4.2.1.1. Let I be an ideal of Ring R with identity.

$$I = R \textit{ if and only if } I \text{ contains a unit.}$$

Proof. Right direction is clear by $1 \in R = I$.
Denote $u \in I$ be a unit with $vu = 1$, and Let $r \in R$ be given. Then,

$$r = r1 = rvu \in I$$

□

Definition 4.2.1.2. An Ideal M of R is ***Maximal ideal*** if: There is no Ideal I such that $M \subsetneq I \subsetneq R$.

Theorem 4.2.1.2. Let R be a Ring with identity.
Then, every proper ideal $I \subsetneq R$ is contained in a maximal ideal.

Proof. □

Lemma 4.2.1.2. Let R be a commutative Ring with identity, M, P are proper ideals of R.

1. M is Maximal Ideal *if and only if* R/M is a field.
2. P is Prime Ideal *if and only if* R/M is an integral domain.

Summary: M maximal $\iff$ R/M field $\implies$ R/M integral domain $\iff$ M prime.

Proof.

$$\begin{aligned} M \text{ is maximal} &\iff \text{There is no ideal } I \text{ such that } M \subsetneq I \subsetneq R \\ &\iff \text{There are Ideals of } R/M \text{ only } 0 \text{ and } R/M \\ &\iff R/M \text{ is field} \end{aligned}$$

$$\begin{aligned} P \text{ is Prime Ideal} &\iff \text{If } ab \in P\text{, then } a \in P \text{ or } b \in P \\ &\iff \text{If } ab + P = P\text{, then } a + P = P \text{ or } b + P = P \\ &\iff \text{If } \overline{a}\overline{b} = \overline{0}\text{, then } \overline{a} = \overline{0} \text{ or } \overline{b} = \overline{0} \\ &\iff R/P \text{ is integral domain} \end{aligned}$$

$\square$

4.3 Ring of Fractions

Theorem 4.3.0.1. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases}\text{no zero, no zero divisors} \\ \text{closed under multiplication}\end{cases}$.
Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q, and every element of D becomes unit in Q. More precisely, $Q = \{rd^{-1} \mid r \in R,\ d \in D\}$.
2. Q is the smallest Ring contaning R with identity such that every element of D becomes unit in Q.

Proof. Let $\mathcal{F} \overset{\text{def}}{=} \{(r,d) \mid r \in R,\ d \in D\}$ and the relation $\sim$ on $\mathcal{F}$ by $(r_1,d_1) \sim (r_2,d_2) \iff r_1 d_2 = r_2 d_1$.
Then, $\sim$ is equivalent relation: reflexive and symmetirc are clear, and Suppose that $(r_1,d_1) \sim (r_2,d_2)$ and $(r_2,d_2) \sim (r_3,d_3)$.

$$r_2 d_3 = r_3 d_2 \implies r_2 d_1 d_3 = r_3 d_1 d_2 \implies r_1 d_2 d_3 = r_3 d_1 d_2 \implies d_2(r_1 d_3 - r_3 d_1) \implies r_1 d_3 = r_3 d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \overset{\text{def}}{=} [(r,d)] = \{(a,b) \mid (a,b) \sim (r,d)\},\ \ Q \overset{\text{def}}{=} \left\{\frac{r}{d} \,\middle|\, r \in R,\ d \in D\right\}$$

And define operations $+, \times$ on Q:

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 d_2 + r_2 d_1}{d_1 d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \overset{\text{def}}{=} \frac{r_1 r_2}{d_1 d_2}$$

Well-Definedness: If $\dfrac{r_1}{d_1} = \dfrac{r_1'}{d_1'}$ and $\dfrac{r_2}{d_2} = \dfrac{r_2'}{d_2'}$,

$$\frac{r_1 d_2 + r_2 d_1}{d_1 d_2} = \frac{r_1 d_2 d_1' d_2' + r_2 d_1 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1') d_2 d_2' + (r_2 d_2') d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1) d_2 d_2' + (r_2' d_2) d_1 d_1'}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_2' + r_2' d_1') d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' d_2' + r_2' d_1'}{d_1' d_2'}$$

$$\frac{r_1 r_2}{d_1 d_2} = \frac{r_1 r_2 d_1' d_2'}{d_1 d_2 d_1' d_2'} = \frac{(r_1 d_1')(r_2 d_2')}{d_1 d_2 d_1' d_2'} = \frac{(r_1' d_1)(r_2' d_2)}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2' d_1 d_2}{d_1 d_2 d_1' d_2'} = \frac{r_1' r_2'}{d_1' d_2'}$$

Now, $(Q,+,\times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \overset{\text{def}}{=} \dfrac{0}{d},\ 1_Q \overset{\text{def}}{=} \dfrac{d}{d}$. Then,

1. $(R,+,\times)$ closed under the operations since D is closed under the multiplication.
2. $(R,+)$ has a zero: $\dfrac{r_1}{d_1} + 0_Q = \dfrac{r_1}{d_1} + \dfrac{0}{d} = \dfrac{r_1 d + 0 d_1}{d_1 d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.
3. $(R,+)$ has an inverse: $\dfrac{r_1}{d_1} + \dfrac{-r_1}{d_1} = \dfrac{r_1 d_1 + (-r_1) d_1}{d_1 d_1} = \dfrac{[(r_1) + (-r_1)] d_1}{d_1 d_1} = \dfrac{0 d_1}{d_1 d_1} = \dfrac{0}{d_1 d_1} = 0_Q$.
4. $(R,+,\times)$ satisfies distributive law:

   4-1. The left law:

$$\begin{aligned}\frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3}\right) &= \frac{r_1}{d_1} \times \frac{r_2 d_3 + r_3 d_2}{d_2 d_3} = \frac{r_1 r_2 d_3 + r_1 r_3 d_2}{d_1 d_2 d_3} = \frac{r_1 r_2 d_1 d_3 + r_1 r_3 d_1 d_2}{d_1 d_2 d_1 d_3} = \frac{r_1 r_2}{d_1 d_2} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

   4-2. The right law:

$$\begin{aligned}\left(\frac{r_1}{d_1} + \frac{r_2}{d_2}\right) \times \frac{r_3}{d_3} &= \frac{r_1 d_2 + r_2 d_1}{d_1 d_2} \times \frac{r_3}{d_3} = \frac{r_1 r_3 d_2 + r_2 r_3 d_1}{d_1 d_2 d_3} = \frac{r_1 r_3 d_2 d_3 + r_2 r_3 d_1 d_3}{d_1 d_3 d_2 d_3} = \frac{r_1 r_3}{d_1 d_3} + \frac{r_2 r_3}{d_2 d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3}\end{aligned}$$

5. $(R,\times)$ has an identity: $\dfrac{r_1}{d_1} \times 1_Q = \dfrac{r_1}{d_1} \times \dfrac{d}{d} = \dfrac{r_1 d}{d_1 d} = \dfrac{r_1}{d_1}$.

6. Elements of D become unit in Q: Define $\iota : R \to Q : r \mapsto \dfrac{rp}{p}$ where $p \in D$ is any fixed element in D.
   Then, ι is Ring-Monomorphsim because:

   6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \dfrac{r_1 p}{p} = \dfrac{r_2 f}{f} \iff (r_1 - r_2)pp = 0 \iff r_1 = r_2$

   6-2. For any $d \in D$, $\iota(d)$ is a unit of Q: Put $(\iota(d))^{-1} \overset{\text{def}}{=} \dfrac{p}{dp}$, then

$$\iota(d) \times (\iota(d))^{-1} = \frac{dp}{p} \times \frac{p}{dp} = \frac{dpp}{dpp} = 1_Q$$

   That is, ι is embedding from R into Q such that $\iota[D]$ becomes units of Q except zero.
   Moreover, if $D = R \setminus \{0\}$, then Q is field.

7. Q is the *smallest* ring containing R with identity such that every element of D becomes units in Q.
   Let S be an any commutative ring with identity,
   and assume that $\varphi : R \to S$ is a Ring-Monomorphism such that for any $d \in D$, $\varphi(d)$ is unit in S.
   Define $\phi : Q \to S : \dfrac{r}{d} \mapsto \varphi(r)\varphi(d)^{-1}$. Then, this ϕ is well-defined and injective:

$$\phi\left(\frac{r_1}{d_1}\right) = \phi\left(\frac{r_2}{d_2}\right) \iff \varphi(r_1)\varphi(d_1)^{-1} = \varphi(r_2)\varphi(d_2)^{-1} \iff \varphi(r_1)\varphi(d_2) = \varphi(r_2)\varphi(d_1)$$
$$\overset{\text{homomor.}}{\iff} \varphi(r_1 d_2) = \varphi(r_2 d_1) \overset{\text{one-to-one}}{\iff} r_1 d_2 = r_2 d_1 \iff \frac{r_1}{d_1} = \frac{r_2}{d_2}$$

   That is, if a commutative ring S with identity contains a copy of R such that the denominator set D of R becomes unit in S, then S contains ring of fractions Q of R. Thus $S = Q$ is the smallest ring that satisfies these conditions.

$\square$

4.4 Commutative Ring with identity

Theorem 4.4.0.1. Finite integral doamin is field.

Proof. Let R be a finite integral domain, and non-zero $a \in R$ be given.
Then, the map from R into R $x \mapsto ax$ is injective: because $ax = ay \implies a(x-y) = 0$ and R has no zero divisor.
R is finite, so this map is surjective. Thus, there exists $b \in R$ such that $ab = 1$, a is unit. □

Lemma 4.4.0.1. Let R be a Commutative Ring, $a, b \in R$ with $b \neq 0$.

$$a = bx \text{ for some } x \in R \overset{\text{def}}{\iff} b \mid a \iff a \in (b) \iff (a) \subseteq (b)$$

Lemma 4.4.0.2. Let a, b be non-zero elements in a Commutative Ring R.
If $(a, b) = (d)$, then d is the greatest common divisor of a and b.

Theorem 4.4.0.2. Let R be an integral domain. If $(d) = (d')$, then $d' = ud$ for some unit $u \in R$.
Particular, d and d' are greatest common divisor of a and b, then $(d) = (d')$, thus $d' = ud$ for some unit $u \in R$.

Proof. If either d or d' is zero, then there is nothing to prove. Thus, Suppose that neither d nor d' is non-zero.
Since $(d) \subseteq (d')$ and $(d) \supseteq (d')$, $d' = dx$ for some $x \in R$ and $d = d'y$ for some $y \in R$.
Combining above, then $d' = dx = (d'y)x = d'(yx)$, this implies $d'(1 - yx) = 0$.
Since d' is non-zero and d' chosen in the integral domain, $1 - xy = 0$.
Now, both x and y are unit, we obtain the result. Second assertion is clear by the First. □

4.4.1 Euclidean Domain

Definition 4.4.1.1. An integral domain R is called ***Euclidean Domain*** if: there exists a norm N such that:

for any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ with $a = qb + r$ with $r = 0$ or $N(r) < N(b)$.

This definition allows us the ***Euclidean Algorithm*** on an integral domain R: for any $a, b \in R$ with $b \neq 0$,

$$\begin{aligned}
a &= q_0 b + r_0 \\
b &= q_1 r_0 + r_1 \\
r_0 &= q_2 r_1 + r_2 \\
r_1 &= q_3 r_2 + r_3 \\
&\vdots \\
r_k &= q_{k+2} r_{k+1} + r_{k+2} \\
&\vdots \\
r_{n-2} &= q_n r_{n-1} + r_n \\
r_{n-1} &= q_{n+1} r_n
\end{aligned}$$

This process gives a chain:

$$N(r_n) < N(r_{n-1}) < \cdots < N(r_2) < N(r_1) < N(r_0)$$

and this process terminates in finite iteration, since well-ordering principle.

Theorem 4.4.1.1. Let I be an ideal of a Euclidean Domain R. Then, I is principal ideal.

Proof. If I is zero ideal, there is nothing to prove. Let I be a non-zero ideal.
Since the set $\{N(a) \mid a \in I \setminus \{0\}\}$ has a minimum by Well-Ordering Principle,
choose $d \in I$ such that $N(d) \leq N(a),\ \forall a \in I \setminus \{0\}$. Clearly, $(d) \subseteq I$. Let $a \in I$. Then, there is $q, r \in R$ such that

$$a = qd + r \text{ with } r = 0 \text{ or } N(r) < N(d)$$

Since $r = a - qd \in I$ by $a, d \in I$, thus closed under the multiplication gives $r \in I$.
But, by minimality of d, r must be 0. Now, $a = qd + r = qd \in (d)$. □

Theorem 4.4.1.2. *Euclidean Algorithm*
Let R be a Euclidean Domain, $a, b \in R$ be non-zero.
Denote $d = r_n$ where r_n is the last nonzero remainder in the Euclidean Algorithm for a and b.
Then, d is the greatest common integer of a and b. And, $(d) = (a, b)$. That is, there exist $x, y \in R$ such that

$$d = ax + by$$

Proof. Note that: (a, b) is principal in Euclidean Domain.
Moreover, (a, b) is the smallest ideal contaning (a) and (b). That is,
If $(a) \subseteq (x)$ and $(b) \subseteq (x)$, then $(a, b) \subseteq (x)$. Now, Enough to Show:

1. $(a), (b) \subseteq (d)$. (It follows that $(a, b) \subseteq (d)$)
2. $(d) \subseteq (a, b)$. (That is, $(d) = (a, b)$)

$$\begin{aligned} a &= q_0 b + r_0 \\ b &= q_1 r_0 + r_1 \\ r_0 &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ &\vdots \\ r_{n-2} &= q_n r_{n-1} + r_n \\ r_{n-1} &= q_{n+1} r_n + 0 \end{aligned}$$

Since $(a), (b) \subseteq (d)$ if and only if $d \mid a, b$, show that d divides a, b.
In the last equation, $r_{n-1} = q_{n+1} r_n = q_{n+1} d$. Thus, $d \mid r_{n-1}$.
Clearly, $r_n \mid r_n$, thus $d \mid r_{n-2}$. Repeat this to finite times, then we obtain: $\forall 1 \leq i \leq n,\ d \mid r_i$. As result, $d \mid a$ and $d \mid b$. This proved 1.
For to show that 2., we will prove $d \in (a, b)$.
The first equation gives directly $r_0 \in (a, b)$.
That is, $(r_0) \subseteq (a, b)$, thus $r_1 = b - q_1 r_0 \in (a, b)$.
Inductively, $r_n = d \in (a, b)$, theorem completed. □

4.4.2 Principal Ideal Domain

Definition 4.4.2.1. An integral domain R is called ***Principal Ideal Domain*** if: every ideal of R is principal.

Theorem 4.4.2.1. Let R be a Principal Ideal Domain, and $a, b \in R$ be non-zero.
Let d be a generator for the principal ideal (a, b). Then,

d is the greatest common divisor of a and b, and unique up to multiplcation of unit of R.

Theorem 4.4.2.2. Every non-zero Prime Ideal in a Principal Domain is Maximal Ideal.

Proof. Let (p) be a non-zero Prime Ideal.
Let $I = (m)$ be an Ideal such that $(p) \subseteq (m)$. Since $p \in (m)$, there is a $x \in R$ such that $p = mx$.
But, $p = mx \in (p)$, Prime Ideal, $m \in (p)$ or $x \in (p)$.
If $m \in (p)$, $m = py$ for some $y \in R$. That is, $m = py \in (p)$, $(p) = (m)$.
If $x \in (p)$, $x = pz$ for some $z \in R$. That is, $p = mx = mpz = p(mz)$, m becomes a unit.
The Ideal (m) containing unit implies $(m) = R$. □

4.4.3 Noetherian Domain

Definition 4.4.3.1. The Ring R is said to be *Noetherian Ring* if: R satisfies *Acsending Chain Condition* on ideals.
The Integral Domain R with Noetherian is called *Noetherian Doamain*.

Theorem 4.4.3.1. Principal Ideal Domain is Noetherian Domain.

Proof. Suppose that there is an ascending chain of ideals,

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R$$

(Considering only countable Chain: Since m.stackexchange 4265544.)

Put $I \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} I_i$. Since for any $r \in R$ and $a \in I$, there eis $i \in \mathbb{N}$ such that $a \in I_i$. Thus, $ra \in I_i \in I$, I is ideal.
Since R is Principal, for some $a \in R$, $(a) = I$. That is, $a \in I$. This implies there exists $n \in \mathbb{N}$ such that $a \in I_n$.
Now, $(a) \subseteq I_n \subseteq I = (a)$, This $I_n = (a) = I$. Consequently, R is Noetherian. □

4.4.4 Unique Factorization Domain

Definition 4.4.4.1. Let R be an Integral Domain.

1. A non-zero, not unit $r \in R$ is called ***irreducible*** of R if: If $r = ab$ for some $a, b \in R$, then either a or b is a unit.

2. A non-zero, not unit $p \in R$ is called ***prime*** of R if: If $p \mid ab$ for some $a, b \in R$, then either $p \mid a$ or $p \mid b$.

Clearly, p is prime if and only if (p) is Prime Ideal.

Theorem 4.4.4.1. Let R be an Integral Domain. Then, every prime element is irreducible.

Proof. Let R be an Integral Domain, $p \in R$ be a prime. Suppose that $p = ab$ for some $a, b \in R$.
Then, clearly $p \mid ab$, thus $p \mid a$ or $p \mid b$. If $p \mid a$, then $a = px$ for some $x \in R$. Now, $p = ab = pxb$, $p(1 - xb) = 0$.
Since R is integral domain and p is non-zero, $xb = 1$. That is, b is a unit, thus p is irreducible. □

Definition 4.4.4.2. Let R be an integral domain and let $r \in R$ be a nonzero, nonunit element.

1. We say that r is ***factorizable*** if there exist irreducible elements $p_1, \ldots, p_n$ ($n \geq 1$) such that

$$r = p_1 p_2 \cdots p_n.$$

   Any such expression is called an ***irreducible factorization*** of r.

2. An irreducible factorization is ***unique up to associates*** if for any two irreducible factorizations

$$r = p_1 \cdots p_n \ = \ q_1 \cdots q_m,$$

   we have $n = m$ and there exist a permutation $\sigma \in S_n$ and units $u_1, \ldots, u_n \in R^\times$ such that

$$q_i \ = \ u_i \, p_{\sigma(i)} \qquad (i = 1, \ldots, n),$$

   equivalently, q_i is associate to $p_{\sigma(i)}$ for each i.

The domain R is called a ***factorization domain*** (also: ***atomic***) if every nonzero, nonunit element of R is factorizable. If, in addition, irreducible factorizations are unique up to associates, then R is called a ***unique factorization domain (UFD)***.

Theorem 4.4.4.2. Noetherian Domain is Factorization Domain.

Proof. Let R be a Noetherian Domain. And, let $r \in R$ be a non-zero, not unit.
There exist onyl two possibility: r is irreducible or not irreducible.
If r is irreducible, then there is nothing to prove. If r is not irreducible, then there exist not unit $r_1, r_2 \in R$ such that $r = r_1 r_2$.
If r_1 and r_2 are irreducible, prove end. If r_1 is reducible, then there exist not unit $r_{1,1} r_{1,2} \in R$ such that $r_1 = r_{1,1} r_{1,2}$.
If this process never terminates, then, there is a infinite strictly ascending chain:

$$(r) \subsetneq (r_1) \subsetneq (r_{1,1}) \subsetneq \cdots \subsetneq R$$

Strictly given by $r = r_1 r_2$ and r_2 is not a unit.
More precisely, if $(r) = (r_1)$, then $r_1 = rk$ for some $k \in R$, $r_1 = rk = r_1 r_2 k$, r_1 becomes a unit. Contradiction. □

Theorem 4.4.4.3.

1. In Principal Ideal Domain, every irreducible element is prime.
2. In Unique Factorization Domain, every irreducible element is prime.

Proof. Let R be a Principal Ideal Domain, and $r \in R$ be an irreducible.
Suppose that (m) is an ideal of R such that $(r) \subseteq (m)$.
Then, $r \in (m)$ implies $r = mx$ for some $x \in R$, now irreducibility gives either m or x is a unit.
If m is a unit, then $(m) = R$. If x is a unit, $r = mx$ implies $rx^{-1} = m$ implies $m \in (r)$ implies $(m) \subseteq (r)$ implies $(m) = (r)$.
Consequently, (r) is maximal ideal in the Principal Ideal Domain,

$$(r) \text{ is a maximal} \iff R/(r) \text{ is a field} \implies R/(r) \text{ is an integral domain} \iff (r) \text{ is Prime.}$$

Let R be a Unique Factorization Domain, and $r \in R$ be an irreducible. Suppose that $r \mid ab$ for some $a, b \in R$.
If either a or b is unit, then $r \mid ab$ implies r divides a or b, there is nothing to prove.
If neither a nor b is a unit, write as factorization form: $a = a_1 \cdots a_n$ and $b = b_1 \cdots b_m$, being a, b in U.F.D.
Since r divides $ab = a_1 \cdots a_n b_1 \cdots b_m$, there exists $x \in R$ such that

$$rx = a_1 \cdots a_n b_1 \cdots b_m$$

If x is a unit, then $r = x^{-1} a_1 \cdots a_n b_1 \cdots b_m$. But, the uniqueness gives contradiction. Thus x is not unit.
Now, x has irreducible factorization, the uniqueness gives $r = a_i$ for some $1 \leq i \leq n$ or $r = b_j$ for some $1 \leq j \leq m$.
This means r divides a or b. $\square$

4.5 Examples

4.5.1 Matrix Ring

Theorem 4.5.1.1. $\phi : \mathbb{C} \to M_2(\mathbb{R}) : a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is Ring-Embedding.

Proof. □

4.5.2 Group Ring

4.5.3 Integer Ring

Definition 4.5.3.1. Let D be a rational number such that $\sqrt{D} \notin \mathbb{Q}$.
Define ***Quadratic field*** for D:

$$\mathbb{Q}(\sqrt{D}) \stackrel{\text{def}}{=} \{a + b\sqrt{D} \mid a, b \in Q\}$$

4.5.4 Boolean Ring

Definition 4.5.4.1. A Ring R is called ***Boolean Ring*** if: for any $a \in R$, $a^2 = a$.

Theorem 4.5.4.1. Boolean Ring is Commutative Ring.

Proof. Let $a, b \in R$ be given. Then,

$$a^2 + b^2 = a + b = (a + b)^2 = a^2 + ab + ba + b^2 \implies ab = -ba$$

Meanwhile, for any $c \in R$,

$$c = c^2 = (-c)^2 = -c$$

Thus, $ab = -ba = ba$. □

4.5.5 Nilradical Ideal

Definition 4.5.5.1. Let R be a Ring.
An element $x \in R$ is called ***Nilpotent*** if: $x^m = 0$ for some $m \in \mathbb{N}$.

Proposition 4.5.5.1. Let R be a Ring with identity.

1. Let $n = a^k b$ where $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, $ab + n\mathbb{Z}$ is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

   Proof. $(ab)^k = a^k b^k = a^k b \cdot b^{k-1} = n \cdot b^{k-1} \in n\mathbb{Z}$. □

2. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then,

   $$a + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z} \text{ is nilpotent } \textit{if and only if} \text{ a prime } p \text{ divides } n \text{ implies } p \text{ divides } a.$$

   Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be a prime decomposition.
   The right assertion means $a = p_1 \cdots p_k \cdot m$ for some $m \in \mathbb{Z}$. Let α be the least common multiple.
   Then, $a^\alpha \in n\mathbb{Z}$. Conversely, if $a^k \in n\mathbb{Z}$ for some $k \in \mathbb{N}$, $a^k = nm$ for some $m \in \mathbb{Z}$.
   If a prime p divides n, then it divides $nm = a^k$, thus divides a. □

Proposition 4.5.5.2. Let R be a Commutative Ring with identity, and nonzero $x \in R$ be a nilpotent element.

1. x is zero divisor. Hence, not unit.

2. For any $r \in R$, rx is nilpotent.

3. $1 + x$ is unit in R.

   Proof. $(1+x)(1-x+x^2-x^3+\cdots+(-x)^{m-1}) = (1-x+x^2-x^3+\cdots+(-x)^{m-1})+(x-x^2+x^3-x^4+\cdots-(-x)^m) = 1$. □

4. If u is unit and a is nilpotent, then $u + a$ is unit.
   Proof. $u + a = u^{-1}(1 + ua)$. □

Definition 4.5.5.2. Let R be a Commutative Ring. Define ***Nilradical*** of R:

$$\mathfrak{N}(R) \overset{\text{def}}{=} \{x \in R \mid x \text{ is nilpotent in } R\}$$

Lemma 4.5.5.1. Nilradical of Commutative Ring R with identity is ideal.

Proof. Trivially, $0 \in \mathfrak{N}(R)$. Suppose $x, y \in \mathfrak{N}(R)$ are nilpotents in R with $x^n = y^m = 0$ for some $n, m \in \mathbb{N}$.

$$(x-y)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^{n+m-k}(-y)^k = x^n \sum_{k=0}^{m} \binom{n+m}{k} x^{m-k}(-y)^k + (-y)^m \sum_{k=m+1}^{n-1} \binom{n+m}{k} x^{n+m-k}(-y)^{-m+k} = 0$$

implies $x - y \in \mathfrak{N}(R)$, thus Nilradical is closed under the substraction, thus it is subgroup.
Closed under the multiplication is trivial: $(xy)^{nm} = x^{nm} y^{nm} = 0^m 0^n = 0$.
And, since above discussion, rx is nilpotent for any $r \in R$ and $a \in \mathfrak{N}(R)$. □

Theorem 4.5.5.1. Let R be a Commutative Ring with identity. $R/\mathfrak{N}(R)$ has no nilpotent element except zero.

Proof. Let $a + \mathfrak{N}(R) \in R/\mathfrak{N}(R)$ be a non-zero. If $a^m + \mathfrak{N}(R) = \mathfrak{N}(R)$ for some $m \in \mathbb{N}$, that is, $a^m \in \mathfrak{N}(R)$.
This means $a^{nm} = 0$ for some $n \in \mathbb{N}$, thus $a \in \mathfrak{N}(R)$. This is contradiction with $a + \mathfrak{N}(R)$ is non-zero. □

Theorem 4.5.5.2. Let R be a Commutative Ring and $I \subset R$ be an ideal.
If $\mathfrak{N}(I) = I$ and $\mathfrak{N}(R/I) = R/I$, then $\mathfrak{N}(R) = R$.

Proof. Let $r \in R$ be given. Since $\mathfrak{N}(R/I) = R/I$, $r + I$ is nilpotent in R/I. That is, $r^n + I = I$ for some $n \in \mathbb{N}$. Since $\mathfrak{N}(I) = I$, $r^n \in I$ is nilpotent, thus $(r^n)^m = 0$ for some $m \in \mathbb{N}$. Thus $r^{nm} = 0$. □

4.5.6 Annihilator Ideal

Lemma 4.5.6.1. Let R be a Ring. For any $a \in R$, $I_a \overset{\text{def}}{=} \{x \in R \mid ax = 0\}$ is a subgroup of R.

Proof. $0 \in I_a$ is trivial. Let $x, y \in I_a$. Then, $a(x-y) = ax - ay = 0 - 0 = 0$. Thus $x - y \in I_a$.
Moreover, $a(xy) = (ax)y = 0y = 0$. Thus $xy \in R$. □

Definition 4.5.6.1. Let R be a Ring with identity, $a \in R$ be a fixed element, and $L \subset R$ be right ideal in R.

1. Define ***right annihilator*** of a in R: $\{x \in R \mid ax = 0\}$.
2. Define ***right annihilator*** of L in R: $\{x \in R \mid ax = 0,\ \forall a \in L\}$.

Lemma 4.5.6.2. Let R be a Ring with identity. Then, the right annihilator of a in R is right ideal.

Proof. Let $r \in R$ and $s \in \{x \in R \mid ax = 0\}$. Then, $asr = (as)r = 0r = 0$. Thus $sr \in \{x \in R \mid ax = 0\}$. □

Lemma 4.5.6.3. Let R be a Ring with identity. Then, the right annihilator of L in R is two-sided ideal.

Proof. Let $r, r' \in R$ and $s \in \{x \in R \mid ax = 0,\ \forall a \in L\}$.
For any $a \in L$, $arsr' = (ar)sr' = 0r' = 0$. Thus, $rsr' \in \{x \in R \mid ax = 0,\ \forall a \in L\}$. □

4.6 Homomorphisms

4.6.1 Formal polynomial differentiation map

Definition 4.6.1.1. Let F be a field with characteristic is 0. Define *formal polynomial differentiation map*:

$$D : F[x] \to F[x] : a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n \mapsto a_1 + 2 \cdot a_2 x + \cdots + n \cdot a_n x^{n-1}$$

Proposition 4.6.1.1. Formal polynomial differentiation map D satisfies:

1. D preserves addition, but not multiplcation.
2. $\ker D = F$.
3. $D[F[x]] = F[x]$.

Proof. Proof of 3.
Let $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n \in F[x]$ be given. Then,

$$D\left(a_0 x + \frac{a_1}{2 \cdot 1}x^2 + \cdots \frac{a_n}{(n+1) \cdot 1}x^{n+1}\right) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n$$

□

4.7 Theorems

Theorem 4.7.0.1. Let R be a Ring that contains at least two elements.
Suppose that for any non-zero $a \in R$, there exists a unique $b \in R$ such that $aba = a$.
Then, R is a division Ring.

Proof. Proof consists by three-step.
1) R has no zero divisor.
Let $a \in R$ be a non-zero element. Then, there exists a unique elements $b \in R$ such that $aba = a$.
Suppose that $a \in R$ is zero divisor. Then, there is a non-zero element $c \in R$ such that $ac = 0$ or $ca = 0$.
Either $ac = 0$ or $ca = 0$, $aca = 0$ is true. Now,

$$a = aba = aba - 0 = aba - aca = a(b - c)a$$

Since the Uniqueness, $b - c = b$, thus $c = 0$. Contradiction.
2) If $aba = a$, then $bab = a$.
It is clear:

$$a = aba = ababa$$

Now, the Uniqueness gives $b = bab$.
3) R has a identity.
Let non-zero $a \in R$ be fixed. Then, there is a unique $b \in R$ such that $aba = a$.
Let $c \in R$ be given. Then,

$$ca = caba \overset{\text{cancle}}{\Longrightarrow} c = cab$$
$$bc = babc \overset{\text{cancle}}{\Longrightarrow} c = abc$$

Thus, (ab) is identity in R.
4) Every non-zero element in R is unit.
Let $a \in R$. Then, for some $b \in R$, $aba = a$. Since $1 \in R$, $aba - a = a(ba - 1) = 0$.
Since R has no zero divisor, $ba = 1$. □

Theorem 4.7.0.2. Every chracteristic of an integral domain R is 0 or prime number p.

Proof. If chracteristic of R is zero, then there is nothing to prove.
Suppose that chracteristic of R is $n \in \mathbb{N}$.
If $n \in \mathbb{R}$ is not a prime number, then there exist two integer $a, b \geq 2$ such that $n = ab$.
Then, $n \cdot 1 = (ab) \cdot 1 = (a \cdot 1)(b \cdot 1) = 0$. Since R has no zero divisor, either $a \cdot 1 = 0$ or $b \cdot 1 = 0$. Contradiction. □

Theorem 4.7.0.3. Let R and R' be a Ring, and $\varphi : R \to R'$ is Ring-Homomorphism with $\varphi[R] \neq \{0'\}$.
If R has identity and R' has no zero divisor, then $\varphi(1)$ is identity of R'.

Proof. Proof consists by two step:
1. $\varphi[R]$ has a identity.
For any $\varphi(r) \in \varphi[R]$, $\varphi(1)\varphi(r) = \varphi(1r) = \varphi(r) = \varphi(r1) = \varphi(r)\varphi(1)$.
2. $\varphi(1)$ is identity of R'.
Since $\varphi(1)\varphi(1) = \varphi(11) = \varphi(1)$, $\varphi(1)$ is idempotent element in R'. By lemma, $\varphi(1)$ is identity of R'. □

Lemma 4.7.0.1. Let R be a Ring which has no zero divisor. If non-zero $a \in R$ s.t $a^2 = a$, then a is identity.

Proof. Let $r \in R$ be given. Since R has no zero divisor,

$$ra = ra^2 \implies (r - ra)a = 0 \implies r = ra$$
$$ar = a^2r \implies a(r - ar) = 0 \implies r = ar$$

Thus, a is identity. □

4.8 Operation of Ideals

Chapter 5

Polynomial Ring Theory

Definition 5.0.0.1. Let R be a Commutative Ring with unity. Define ***Polynomial Ring***:

$$R[x] \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; n \geq 0,\ \ a_i \in R \right\}$$

Addition defined by pointwise, and Multiplcation defined by:

$$\left(\sum_{i=0}^{n} a_i x^i \right) \times \left(\sum_{i=0}^{m} b_i x^i \right) = \sum_{k=0}^{n+m} \left(\sum_{i=0}^{k} a_i b_{k-i} \right) x^k$$

Proposition 5.0.0.1. Let R be an integral domain, and $p, q \in R[x]$ be non-zero elements.

1. $\deg(pq) = \deg p + \deg q$.
2. $R[x]$ is an integral domain.
3. If $p \in R[x]$ is unit, then $\deg p = 0$ and p is unit in R.

Proof.

$$\left(\sum_{i=0}^{n} a_i x^i \right) \left(\sum_{i=0}^{m} b_i x^i \right) = a_n b_m x^{n+m} + \cdots ,$$

This proves the statement (1) immediately, and assume not zero, then proves $2)$.
And, if $p \in R[x]$ is unit, then

$$0 = \deg 1 = \deg(pp^{-1}) = \deg p + \deg p^{-1}$$

Thus, $\deg p = \deg p^{-1} = 0$ and this implies $p, p^{-1} \in R$. □

5.1 Basic Theorems

Theorem 5.1.0.1. Let I be an ideal of the Commutative Ring R with unity, and $(I) \subseteq R[x]$.
Then, $R[x]/(I) \cong (R/I)[x]$. In particular, if I is prime ideal in R, then (I) is prime ideal in $R[x]$.

Proof. First, establish that $(I) = I[x]$. Since properties of Ideal, $(I) = IR[x] = I[x]$ directly.
Now, define a map $\varphi : R[x] \to (R/I)[x] : \sum_{i=0}^{n} a_i x^i \mapsto \sum_{i=0}^{n}(a_i + I)x^i$. Then, φ is homomorphism with $\ker\varphi = I[x]$.
The first-iso. Thm gives $R[x]/(I) = R[x]/I[x] \cong (R/I)[x]$. Particular,

$$I \text{ prime ideal} \iff R/I \text{ integral domain} \implies (R/I)[x] = R[x]/(I) \text{ integral domain} \iff (I) \text{ prime ideal.}$$

□

Theorem 5.1.0.2. If F is a field, then $F[x]$ is Euclidean domain.
Specifically, assume R is Commutative Ring with unity, $f, g \in R[x]$ with $\deg f, \deg g \geq 0$.
If leading coefficient of g is unit in R, then there exists unique $q, r \in R[x]$ such that

$$f(x) = g(x)q(x) + r(x) \quad (\deg r(x) < \deg g(x))$$

Proof. If $\deg f < \deg g$, put $g(x) = 0$ and $r(x) = f(x)$. Then proved. Suppose that $\deg f \geq \deg g$, and using induction.
If $\deg f = 0$, then put $g(x) = 0$, write leading coefficient of g as b and of f as a. Then, put $q = b^{-1}a$, $r = 0$.
If $\deg f \geq 1$, put $n = \deg f$, $m = \deg g$. Then $n \geq m$. Write:

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \\ g(x) &= b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0 \end{aligned}$$

Then, by induction,

$$\begin{aligned} f(x) &= a_n b_m^{-1} x^{n-m} g(x) + f_1(x) && (\deg f_1 < n-1) \\ &= a_n b_m^{-1} x^{n-m} g(x) + q_1(x)g(x) + r(x) && (\deg r < \deg g) \\ &= (a_n b_m^{-1} x^{n-m} q_1(x))g(x) + r(x) && (\deg r < \deg g) \end{aligned}$$

To show the uniqness,

$$f = q_1 g + r_1 = q_2 g + r_2 \implies g(q_1 - q_2) = r_2 - r_1 \implies \deg g(q_1 - q_2) = \deg g + \deg(q_1 - q_2) = \deg(r_2 - r_1) < \deg g$$

□

5.2 Relations between Rings and Their Polynomial Rings

Lemma 5.2.0.1. *Gauss's Lemma*
Let R be a Unique Factorization Domain with field of fractions F.
If $p(x) \in R[x]$ is reducible in $F[x]$, then $p(x)$ is reducible in $R[x]$.

Proof. Let $p(x) \in R[x]$ be reducible in $F[x]$. i.e.,

$$p(x) = A(x)B(x) \text{ for some } A(x), B(x) \in F[x] \text{ with } A(x), B(x) \text{ are both non-zero and non-units.}$$

Both $\deg A$ and $\deg B$ are at least 1: if either degree were zero, then lie in F, hence a unit - contradiction.
Write $A(x) = \sum_{i=0}^{n} \frac{r_i}{a_i} x^i$ and $B(x) = \sum_{i=0}^{m} \frac{s_i}{b_i} x^i$., and put $d_1 = a_1 \cdots a_n$, $d_2 = b_1 \cdots b_m$.
Now, $d_1 d_2 p(x) = d_1 A(x) d_2 B(x)$ where $d_1 A(x), d_2 B(x) \in R[x]$.
If $d = d_1 d_2$ is unit in R, then $p(x) = (d^{-1} d_1 A(x))(d_2 B(x))$ where $d^{-1} d_1 A(x), d_2 B(x) \in R[x]$, both are non-unit.
Suppose that d is not unit. Write $d = p_1 p_2 \cdots p_n$ is factorization of d.
p_1 is prime, being irreducible in U.F.D. $(p_1) = p_1 R[x]$ is prime, $R[x]/p_1 R[x] \cong (R/p_1 R)[x]$ is an integral domain.
Since $dp(x) = p_1 \cdots p_n p(x) \in p_1 R[x]$,

$$\overline{0} = dp(x) + p_1 R[x] = d_1 A(x) d_2 B(x) + p_1 R[x] = \overline{d_1 A(x)} \times \overline{d_2 B(x)}$$

Since $p_1 R[x]$ is an integral domain, either $\overline{d_1 A(x)}$ or $\overline{d_2 B(x)}$ is zero. WLOG, let $\overline{d_1 A(x)} = d_1 A(x) + p_1 R[x] = \overline{0}$.
This means all coefficient of $d_1 A(x)$ lies in $p_1 R$. Thus, we can cancle p_1 in the equation $dp(x) = d_1 A(x) d_2 B(x)$.
In finite process, we obtain $p(x) = A'(x) B'(x)$ where $A'(x), B'(x) \in R[x]$ with

$$A'(x) = rA(x), \;\; B'(x) = sB(x) \text{ where } r, s \in F$$

□

Corollary 5.2.0.1. Let R be a Unique Factorization Domain with field of fractions F.
Suppose that the greatest common divisor of the coefficients of $p(x) \in R[x]$ is 1. Then,

$$p(x) \text{ is irreducible in } R[x] \textit{ if and only if } p(x) \text{ is irreducible in } F[x]$$

In particular, if $p(x)$ is an irreducible monic polynomial in $R[x]$, then it is also irreducible in $F[x]$.

Proof. By Contraposition of Gauss's Lemma, if $p(x)$ is irreducible in $R[x]$, then $p(x)$ is irreducible in $F[x]$.
Conversely, suppose that $p(x)$ is reducible in $R[x]$, and the greatest common divisor of coefficients of $p(x)$ is 1.
Write $p(x) = a(x)b(x)$ where neitehr $a(x)$ nor $b(x)$ are not unit in $R[x]$, being reducible.
And, both $a(x)$ and $b(x)$ are not constant: because g.c.d. is 1. Thus, both are not unit in $F[x]$. □

> **Theorem 5.2.0.1.** R is Unique Factorization Domain *if and only if* $R[x]$ is Unique Factorization Domain.

Proof. Suppose that R is Unique Factoization Domain with field of fractions F.
Let $p(x) \in R[x]$ be non-zero element, and $d \in R$ be the greatest common divisor of coefficients of $p(x)$.
Then, $p(x) = dp'(x)$ where g.c.d. of coefficient of $p'(x)$ is 1. More precisely, write $p(x) = \sum_{i=0}^{n} a_i x^i, \ (a_i \in R)$.

$$p(x) = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} da_i' x^i = d\left(\sum_{i=0}^{n} a_i' x^i\right)$$

for some $a_i' \in R$ such that $a_i = da_i'$. Put g.c.d of a_i''s to $d' \in R$. Then, $a_i = da_i' = dd'a_i''$.
This implies dd' is divides every a_i; hence dd' divides d. That is, d' is unit, thus d' must be 1.
Since $F[x]$ is U.F.D, let $p'(x) = p_1(x)p_2(x)\cdots p_n(x)$ be a factorization of $p(x)$ in $F[x]$.
The g.c.d of $p'(x)$ is 1, thus g.c.d. of each $p_i(x)$ is 1.
Now, the corollary of the Gauss's Lemma gives that every $p_i(x)$ is irreducible in $R[x]$.
Hence, $p'(x) = p_1(x)p_2(x)\cdots p_n(x)$ is irreducible factorization in $R[x]$. To show that uniqueness, let

$$p'(x) = p_1(x)\cdots p_n(x) = q_1(x)\cdots q_m(x)$$

are two irreducibles factorizations of $p'(x)$ in $R[x]$. Since g.c.d of $p'(x)$ is 1, each $p_i(x)$ and $q_j(x)$ have g.c.d. 1.
Since the corollary of the Gauss's Lemma, all factors are irreducibles in $F[x]$ and $F[x]$ is U.F.D, $n = m$.
Moreover, each $p_i(x)$ and $q_i(x)$ are associates in $F[x]$(index rearrangement). Since associates up to unit in $F[x]$,

$$p_i(x) = \frac{a}{b} q_i(x) \quad \text{for some } a, b \in R^{\times}$$

That is, $bp_i(x) = aq_i(x)$; g.c.d. of left polynomial is b, and g.c.d. of right polynomial is a.
In integral domain, g.c.d. is unique up to unit, $a = ub$ for some unit $u \in R^{\times}$. That is,

$$bp_i(x) = aq_i(x) = ubq_i(x) \implies p_i(x) = uq_i(x)$$

Proof complete.

□

> **Theorem 5.2.0.2.** Let R be a Commutative Ring with identity.
> If $R[x]$ is Principal Ideal Domain, then R is a Field.

5.3 Irreducibility Criteria

Lemma 5.3.0.1. Let F be a field, and $p(x) \in F[x]$.

$p(x)$ has a factor of degree one ***if and only if*** $p(x)$ has a root in F

Proof. Trivial. $\square$

Lemma 5.3.0.2. Let F be a field, and $p(x) \in F[x]$ be a polynomial of degree 2 or 3.

$p(x)$ is reducible ***if and only if*** $p(x)$ has a root in F.

Proof. Trivial. $\square$

Theorem 5.3.0.1. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$.
If $\dfrac{r}{s} \in \mathbb{Q}$ is a root of $p(x)$ where r and s are relatively prime, then $r \mid a_0$ and $s \mid a_n$.
In particular, if $p(x) \in \mathbb{Z}[x]$ is ***monic polynomial*** and $p(d) \neq 0$ for all $d \mid p(0)$, then $p(x)$ has no root in $\mathbb{Q}$.

Proof. By hypothesis,

$$\begin{aligned}
&p\left(\frac{r}{s}\right) = a_n \left(\frac{r}{s}\right)^n + a_{n-1}\left(\frac{r}{s}\right)^{n-1} + \cdots + a_1\left(\frac{r}{s}\right) + a_0 = 0 \\
\Longrightarrow\ & a_n r^n + a_{n-1}r^{n-1}s + \cdots + a_1 r s^{n-1} + a_0 s^n = 0 \\
\Longrightarrow\ & a_n r^n = -(a_{n-1}r^{n-1}s + \cdots + a_1 r s^{n-1} + a_0 s^n = s(a_{n-1}r^{n-1} + \cdots + a_1 r s^{n-2} + a_0 s^{n-1})
\end{aligned}$$

Hence, s divides $a_n r^n$ and s and r are relatively prime, $s \mid a_n$. Similarly, $r \mid a_0$.
And, the second assertion is clear by contraposition. $\square$

Theorem 5.3.0.2. Let $I \subsetneq R$ be a proper ideal of integral domain R, and $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x]$.
If the image of $p(x)$

$$\tilde{p}(x) = x^n + (a_{n-1} + I)x^{n-1} + \cdots + (a_1 + I)x + (a_0 + I) \in (R/I)[x]$$

cannot be factored two polynomials into two smaller degree,
then $p(x)$ is irreducible in $R[x]$.

Proof. Suppose that $p(x)$ is reducible in $R[x]$.
Then, there exist two monic $a(x), b(x) \in R[x]$ with $\deg a, \deg b \geq 1$ such that $p(x) = a(x)b(x)$.
But, $p(x) + I[x] = a(x)b(x) + I[x] = (a(x) + I[x]) \times (b(x) + I[x])$ is still reducible in $(R/I)[x]$, because:
leading coefficient of a_n is $1 \implies a(x) + I[x]$ is not constant, being unit 1 cannot be in the proper ideal I. $\square$

Theorem 5.3.0.3. *Eisenstein's Criterion*
Let P be a prime ideal of the integral domain R, and $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[x], \ (n \geq 1)$.
Suppose $a_{n-1}, \ldots, a_1, a_0$ are contained in P and a_0 is not contained in P^2. Then, $f(x)$ is irreducible in $R[x]$.

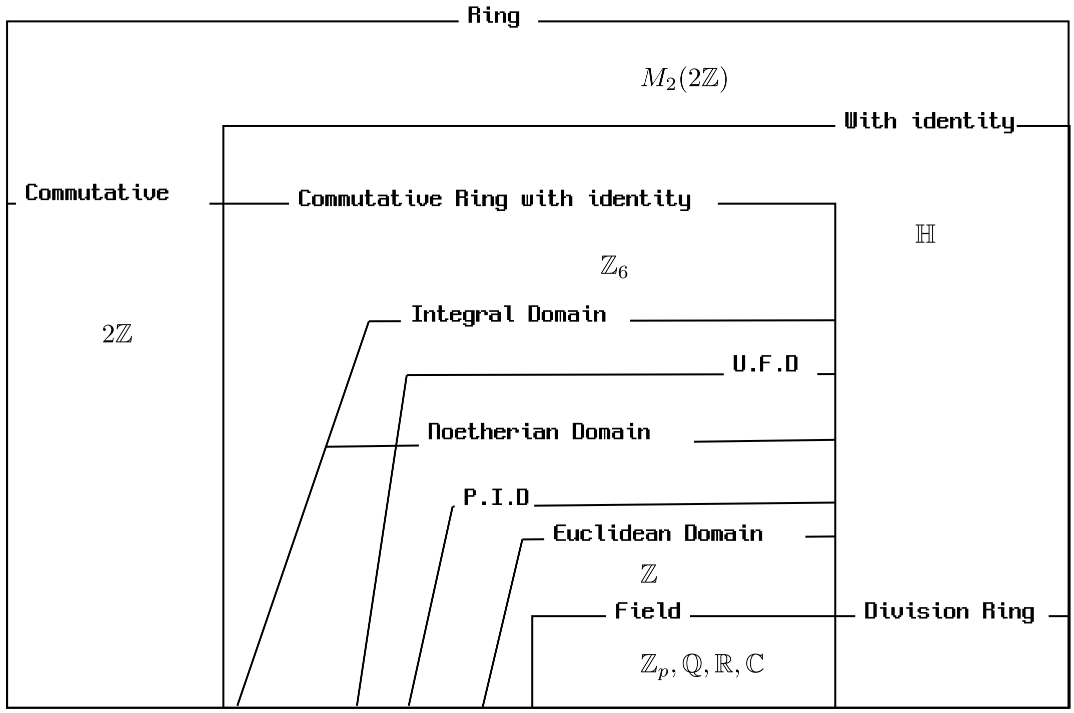
Proof. Proof by Contradiction. Suppose $f(x)$ is reducible. Then, $f(x) = a(x)b(x)$
for some nonconstant $a(x), b(x)$ in $R[x]$ (If it is constant, then contradicts to monic.)
Observe that: In $(R/P)[x]$,

$$(1 + P)x^n + (a_{n-1} + P)x^{n-1} + \cdots + (a_1 + P)x + (a_0 + P) = x^n = \overline{a(x)b(x)} \in (R/P)[x] \cong R[x]/P[x]$$

Thus, the constant terms of $a(x)$ and $b(x)$ both are contained in P.
But this implies that the product of these two constants is contained in P^2, contradiction. $\square$

5.4 Summary and Diagram

$$
\begin{tikzcd}
R & \text{Field} \arrow[r, Rightarrow] \arrow[ddr, Rightarrow] & \text{Euclidean Domain} \arrow[r, Rightarrow] & \text{P.I.D.} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] & & \text{U.F.D.} \arrow[r, Rightarrow] \arrow[dd, Leftrightarrow] & \text{Integral Domain} \arrow[dd, Leftrightarrow] \\
& & & & \text{Noetherian Domain} \arrow[dd, Leftrightarrow] & & \\
R[x] & & \text{Euclidean Domain} \arrow[r, Rightarrow] & \text{P.I.D.} \arrow[rr, Rightarrow] \arrow[dr, Rightarrow] \arrow[uull, Rightarrow] & & \text{U.F.D.} \arrow[r, Rightarrow] & \text{Integral Domain} \\
& & & & \text{Noetherian Domain} & &
\end{tikzcd}
$$



In this section, we find and describe all examples and counterexamples in the diagram.

5.5 Examples

5.5.1 Quadratic Field and Quadratic integer Ring

Definition 5.5.1.1. Let $D \in \mathbb{Q}$ be a rational number that is not a perfect square in $\mathbb{Q}$.
Define the *Quadratic Field* for D:

$$\mathbb{Q}(\sqrt{D}) \overset{\text{def}}{=} \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$$

Proposition 5.5.1.1. $\mathbb{Q}(\sqrt{D})$ is a Field.

Proof. Let $a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$. Then, $(a + b\sqrt{D})^{-1} = \dfrac{a - b\sqrt{D}}{a^2 - b^2 D}$ is inverse. □

5.5.2 Cyclotomic Polynomial

Definition 5.5.2.1. Let $p \in \mathbb{Z}$ be a prime. Define p^{th} ***Cyclotomic Polynomial***:

$$\Phi_p(x) \overset{\text{def}}{=} \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}[x]$$

Theorem 5.5.2.1. p^{th} Cyclotomic Polynomial is irreducible in $\mathbb{Z}[x]$.

Proof. Observe that

$$\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = \frac{1}{x} \cdot \left(x^p + \binom{p}{1} x^{p-1} + \cdots + \binom{p}{p-1} x + 1 - 1 \right) = x^{p-1} + \binom{p}{1} x^{p-2} + \cdots + p$$

is irreducible in $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$ by ***Eigenstein's Criterion***. Using this fact, suppose that $\Phi_p(x)$ is reducible. That is, there exist non-constant $r(x), s(x) \in \mathbb{Z}[x]$ such that $\Phi_p(x) = r(x)s(x)$. Then,

$$\Phi_p(x+1) = r(x+1)s(x+1)$$

is reducible, contradiction. $\square$

5.6 † Rigorously Definition

Definition 5.6.0.1. Suppose that $(R, +, \cdot)$ is a Ring. Define a ***Polynomial Ring*** is:

$$R[x] \overset{\textsf{def}}{=} \bigcup_{n\in\mathbb{N}} \{\{a_i\}_{i=0}^{\infty} \mid a_1, \ldots, a_n \in R,\ \ a_{n+1} = a_{n+2} = \cdots = 0_R\}$$

Definition 5.6.0.2. Suppose that $(R, +, \cdot)$ is a Ring. Define a ***Two-variables Polynomial Ring*** is:

$$\begin{aligned} R[x, y] \overset{\textsf{def}}{=} R[x][y] &= \bigcup_{n\in\mathbb{N}} \{\{a_i\}_{i=0}^{\infty} \mid a_1, \ldots, a_n \in R[x],\ \ a_{n+1} = a_{n+2} = \cdots = 0_R\} \\ &= \bigcup_{n\in\mathbb{N}} \{\{a_i\}_{i=0}^{\infty} \mid a_1, \ldots, a_n \in R[x],\ \ a_{n+1} = a_{n+2} = \cdots = 0_R\} \end{aligned}$$

Chapter 6

Field Theory

Lemma 6.0.0.1. Suppose that $\varphi : F \to F'$ is a Field-Homomorphism. If φ is not zero map, then φ is injective.

Proof. Since φ is not zero map, $\varphi(a) \neq 0_{F'}$ for some $a \in F$. Thus, $\ker\varphi \subsetneq F$.
But, the field F has only ideal for $\{0\}$ or F, hence the ideal $\ker\varphi$ must be $\{0\}$. Now,

$$\varphi(a) = \varphi(b) \implies \varphi(a-b) = 0 \implies a - b = 0$$

□

Lemma 6.0.0.2. Suppose that F is a field.
If $p(x) \in F[x]$ is irreducible in $F[x]$, then $F[x]/(p(x))$ is a field.

Proof. Since F is a field, $F[x]$ is Euclidean domain, P.I.D., and U.F.D. Thus,

$$p(x) \text{ irreducible } \underset{\text{U.F.D.}}{\implies} (p(x)) \text{ prime } \underset{\text{P.I.D.}}{\implies} (p(x)) \text{ maximal in } F[x]$$

Hence, $K = F[x]/(p(x))$ becomes a field. □

Theorem 6.0.0.1. Suppose that F is a field, and $p(x) \in F[x]$ is irreducible.
Then, $K \overset{\text{def}}{=} F[x]/(p(x))$ is a field containing an isomorphic copy of F, and $p(x)$ has a root α in K.

Proof. Since F is a field, $F[x]$ is Euclidean domain, P.I.D., and U.F.D. Thus,

$$p(x) \text{ irreducible } \underset{\text{U.F.D.}}{\implies} (p(x)) \text{ prime } \underset{\text{P.I.D.}}{\implies} (p(x)) \text{ maximal in } F[x]$$

Hence, $K = F[x]/(p(x))$ becomes a field. Consider the Canonical projection

$$\pi : F[x] \to F[x]/(p(x)) : f(x) \mapsto f(x) + (p(x))$$

and restriction over F, $\varphi = \pi_F : F \to F[x]/(p(x))$ is Homomorphism with $\ker\varphi = \{0\}$, because $\varphi(1) + (p(x)) \neq (p(x))$.
Now, $F \cong F/\ker\varphi \cong \varphi[F] \subseteq K = F[x]/(p(x))$. Further, denote $\overline{x} \overset{\text{def}}{=} \pi(x) = x + (p(x))$, and $p(x) = \sum_{i=0}^{n} a_i x^i$. Then,

$$p(\overline{x}) = \sum_{i=0}^{n} \overline{a_i x}^i = \sum_{i=0}^{n} \pi(a_i)\pi(x)^i = \pi\left(\sum_{i=0}^{n} a_i x_i\right) = \pi(p(x)) = \overline{p(x)} = \overline{0} \in K$$

□

> **Theorem 6.0.0.2.** Suppose that F is a field, and $p(x) \in F[x]$ is irreducible.
> If K is an Extension field of F containing a root $\alpha \in K$ of $p(x)$, then $F(\alpha) \cong F[x]/(p(x))$.

Proof. Define a map $\varphi_\alpha : F[x]/(p(x)) \to F(\alpha) : f(x) + (p(x)) \mapsto f(\alpha)$.
Well-defined: Suppose that $f(x)+(p(x)) = g(x)+(p(x))$. Then, $\varphi_\alpha(f(x)+(p(x))) = f(\alpha)$ and $\varphi_\alpha(g(x)+(p(x))) = g(\alpha)$.

$$f(x) + (p(x)) = g(x) + (p(x)) \iff f(x) - g(x) \in (p(x))$$

$$f(x) - g(x) \in (p(x)) \implies f(x) - g(x) = p(x)r(x) \text{ for some } r(x) \in F[x] \implies f(\alpha) - g(\alpha) = p(\alpha)r(\alpha) = 0 \cdot r(\alpha) = 0.$$

Ring-Homomorphism:

1. $\varphi([f(x) + (p(x))] + [g(x) + (p(x))]) = \varphi(f(x) + g(x) + (p(x))) = f(\alpha) + g(\alpha) = \varphi(f(x) + (p(x))) + \varphi(g(x) + (p(x)))$.
2. $\varphi([f(x) + (p(x))] \cdot [g(x) + (p(x))]) = \varphi(f(x)g(x) + (p(x))) = f(\alpha)g(\alpha) = \varphi(f(x) + (p(x))) \cdot \varphi(g(x) + (p(x)))$.

Injectivity: Since φ is a non-trivial homomorphism from field into field, thus $\ker \varphi = \{0\}$.
(More precisely, $\varphi(1) = 1 + (p(x)) \neq (p(x))$ because $1 \notin (p(x))$. Thus, $\ker \varphi \subsetneq F[x]/(p(x)) \implies \ker \varphi = \{0\}$.)
Now, the φ becomes an Embedding:

$$\varphi[F[x]/(p(x))] \cong F[x]/(p(x))$$

Since $F[x]/(p(x))$ is a field, the image $\varphi[F[x]/(p(x))]$ is a field, being it is isomorphic copy.
The definition of $F(\alpha)$ gives $\varphi[F[x]/(p(x))] \supset F(\alpha)$, and $\varphi[F[x]/(p(x))] \subset F(\alpha)$ is trivial. Consequently,

$$F(\alpha) = \varphi[F[x]/(p(x))] \cong F[x]/(p(x))$$

□

> **Theorem 6.0.0.3.** Suppose that $p(x) \in F[x]$ is an irreducible with $\deg p(x) = n$, and $K = F[x]/(p(x))$.
> Put $\overline{x} = x + (p(x)) \in K$. Then, $\{1, \overline{x}, \overline{x}^2, \ldots, \overline{x}^{n-1}\}$ is basis for K as Vector space over F.

Proof. Above lemma guarantees $K = F[x]/(p(x))$ is a field.
1) Linealy independent.
Suppose that $a_0 + a_1\overline{x} + \cdots + a_{n-1}\overline{x}^{n-1} = 0$ in K where a_i are not all zero. Then,

$$\begin{aligned}
0 =& a_0 + a_1\overline{x} + \cdots + a_{n-1}\overline{x}^{n-1} \\
=& a_0 + a_1(x + (p(x))) + \cdots + a_{n-1}(x^{n-1} + (p(x))) \\
=& a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + (p(x)) \\
\iff& a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} = p(x)b(x) \text{ for some } b(x) \in F[x]
\end{aligned}$$

But, the left term has a degree at most $n-1$, and the right term has a degree at least n. Contradiction.
2) $F[x]$ is generated by $\{1, \overline{x}, \ldots, \overline{x}^{n-1}\}$.
Let $a(x) \in F[x]$. Then, the Euclidean algorithm for $F[x]$ gives that: there exist uniquely $q(x), r(x) \in F[x]$ s.t

$$a(x) = q(x)p(x) + r(x) \text{ with } \deg r(x) < \deg p(x)$$

Since $q(x)p(x) \in (p(x))$, $a(x) + (p(x)) = r(x) + (p(x))$, the degree of $a(x) + (p(x))$ is smaller than n.
Now, for any $a(x) + (p(x)) \in K = F[x]/(p(x))$, $a(x) + (p(x)) \in \langle\{1, \overline{x}, \cdots, \overline{x}^{n-1}\}\rangle$. □

> **Corollary 6.0.0.1.** Suppose that F is a field, and $p(x) \in F[x]$ is irreducible polynomial with $\deg p(x) = n$.
> If E is an Extension field of F containing α as a root of $p(x)$. Then,
>
> $$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} \mid a_i \in F\} \subseteq E$$

Proof. Since $F[x]/(p(x)) = \{a_0 + a_1\overline{x} + \cdots + a_{n-1}\overline{x}^{n-1} \mid a_i \in F\} \cong F(\alpha)$, being the isomorphsim

$$\varphi_\alpha : F[x]/(p(x)) \to F(\alpha) : f(x) + (p(x)) = \overline{f(x)} \mapsto f(\alpha)$$

Thus, combining with above fact,

$$\begin{aligned}
F(\alpha) = \varphi_\alpha[F[x]/(p(x))] &= \{\varphi_\alpha(a_0 + a_1\overline{x} + \cdots + a_{n-1}\overline{x}^{n-1}) \mid a_i \in F\} \\
&= \{\varphi_\alpha(a_0) + \varphi_\alpha(a_1\overline{x}) + \cdots + \varphi_\alpha(a_{n-1}\overline{x}^{n-1}) \mid a_i \in F\} \\
&= \{a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_{n-1}\alpha^{n-1} \mid a_i \in F\}
\end{aligned}$$

□

6.1 Extension Field

Definition 6.1.0.1. Suppose that F is a field, and K be a field containing F as a subfield.
The element $\alpha \in K$ is called ***algebraic*** over F if:

There exists a non-zero polynomial $f(x) \in F[x]$ such that $f(\alpha) = 0$ in K.

If not algebraic, it is called ***transcendental*** over F.

Theorem 6.1.0.1. Suppose that $\alpha \in K$ is algebraic over F.
Then, there exists a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ such that $m_{\alpha,F}(\alpha) = 0$. Moreover,

$f(x) \in F[x]$ has α as a root ***if and only if*** $m_{\alpha,F}(x) \in F[x]$ divides $f(x)$ in $F[x]$.

Proof. Since $\alpha \in K$ is algebraic over F, the Well-Ordering principle gives the existence of minimal degree polynomial $g(x) \in F[x]$ having α as a root. More precisely,

$$S = \{\deg f(x) \in \mathbb{N} \mid f(x) \in F[x] \text{ s.t } f(\alpha) = 0\}$$

is non-emtpy since α is algebraic. Therefore, the Well-Ordering Principle gives that S has a minimum $n \in \mathbb{N}$.
Now, we can choose $g(x) \in F[x]$ with $\deg g(x) = n$ and $g(\alpha) = 0$.
Using Contradiction: Suppose that $g(x)$ is reducible. That is, $g(x) = a(x)b(x)$ with $\deg a, \deg b < \deg g$.
Since $g(\alpha) = a(\alpha)b(\alpha) = 0$ and K is a field, either $a(\alpha)$ or $b(\alpha)$ must be zero.
This contradicts with $n \in \mathbb{N}$ is the minimum degree of polynomial which has α as a root.
In summary:

If $g(x) \in F[x]$ is a polynomial of minimal degree which has α as a root, then $g(x)$ is irreducible over F.

Now, Suppose that $f(x) \in F[x]$ is any polynomial having α as a root. Then, $\deg f(x) \geq \deg g(x)$ by setting.
By the Euclidean Algorithm in $F[x]$, there exist unique polynoimals $q(x), r(x) \in F[x]$ such that

$$f(x) = q(x)g(x) + r(x) \text{ with } \deg r(x) < \deg g(x)$$

Meanwhile,

$$0 = f(\alpha) = q(\alpha)g(\alpha) + r(\alpha) = q(\alpha) \cdot 0 + r(\alpha) = r(\alpha)$$

$r(\alpha) = 0$ implies $r(x)$ must be zero, by minimality. Hence, $g(x)$ divides $f(x)$ in $F[x]$.
Finally, put $a \in F$ be a leading coefficient of $g(x)$. Then, $m_{\alpha,F}(x) = a^{-1}g(x)$ is monic polynomial.
And, Uniqueness given by: for any $f(x) \in F[x]$, $f(\alpha) = 0 \implies m_{\alpha,F}(x) \mid f(x)$. □

Corollary 6.1.0.1. If L is an Extension of field F and $\alpha \in L$ is algebraic over both F and L,
then $m_{\alpha,L}(x)$ divides $m_{\alpha,F}(x)$ in $L[x]$.

Above theorem allows defining:

Definition 6.1.0.2. Suppose that K is an Extension field of a field F, and $\alpha \in K$ is algebric over F.
The monic irreducible polynomial which has α as root is called ***the minial polynomial*** for α over F.
Denote this $m_{\alpha,F}(x) \in F[x]$ or $\operatorname{irr}(\alpha, F)$.

Proposition 6.1.0.1. Suppose that F is a field and E is an Extension of F.
If $\alpha \in E$ is algebraic over F, then $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$ and

$$[F(\alpha) : F] \overset{\text{def}}{=} \dim_F F(\alpha) = \deg m_{\alpha,F}(x) \overset{\text{def}}{=} \deg \alpha$$

> **Theorem 6.1.0.2.** Suppose that E is an Extension field of a field F, and $\alpha \in E$.
>
> $$\alpha \in E \text{ is algebraic over } F \textit{ if and only if } [F(\alpha):F] \text{ has finite dimension}$$

Proof. If $\alpha \in E$ is algebraic over F, then $[F(\alpha):F] = \deg m_{\alpha,F}(x)$, thus finite.
Precisely, if $f(x) \in F[x]$ satisfies $\deg f(x) = n$ and $f(\alpha) = 0$, then $[F(\alpha):F] = \deg_F \alpha \le n$.
Conversely, suppose $\alpha \in E$ where $[F(\alpha):F] = n$. Then, the $n+1$ elements

$$1, \alpha, \alpha^2, \ldots, \alpha^n$$

must be Linearly dependent, thus for some not all zero elements $b_0, \ldots, b_n \in F$,

$$b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_n\alpha^n = 0$$

Now, the polynomial in $F[x]$

$$f(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$

satisfies $f(\alpha) = 0$ in $F(\alpha)$. In summary, if $[F(\alpha):F] = n$, then $\alpha \in E$ is algebraic over F with $\deg_F \alpha \le n$. □

> **Corollary 6.1.0.2.** If E is an Extension field of F with $[E:F]$ has finite, then E is algebraic over F.

> **Theorem 6.1.0.3.** Suppose that $F \subseteq L \subseteq E$ are fields. Then,
>
> $$[E:F] = [E:L][L:F]$$

Proof. Suppose that $[E:L]$ and $[L:F]$ are finite. Put $[E:L] = m$ and $[L:F] = n$.
Set $A = \{\alpha_1, \alpha_2, \ldots, \alpha_m\} \subset E$ and $B = \{\beta_1, \beta_2, \ldots, \beta_n\} \subset L$ are basis of E over L and L over F, respectively.
Let $x \in E$ be given. Since A is basis for E over L, there exists a unique linear combination: for $a_i \in L$,

$$x = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m = \sum_{i=1}^{m} a_i\alpha_i$$

Meanwhile, for each $a_i \in L,\ 1 \le i \le m$, there exist unique linear combinations: for $b_{i,j} \in F$,

$$a_i = b_{i,1}\beta_1 + b_{i,2}\beta_2 + \cdots + b_{i,n}\beta_n = \sum_{j=1}^{n} b_{i,j}\beta_j$$

Combining above, we obtain

$$x = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} b_{i,j}\beta_j\right)\alpha_i = \sum_{\substack{1\le i\le m\\ 1\le j\le n}} b_{i,j}\beta_j\alpha_i$$

Since each $b_{i,j} \in F$ and $\beta_j\alpha_i \in E$, the subset $C = \{\beta_j\alpha_i \mid \alpha_i \in A,\ \beta_j \in B\} \subset E$ spans E over F.
Meanwhile, this C is linearly independent over F, because: Suppose that

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n} c_{i,j}\beta_j\right)\alpha_i = 0$$

Since each $c_{i,j}\beta_j \in L$ and $A = \{\alpha_i\}$ is linearly independent over L, for each $1 \le i \le m$,

$$\sum_{j=1}^{n} c_{i,j}\beta_j = 0$$

$c_{i,j} \in F$ and $B = \{\beta_j\}$ is linearly independent over F, all $c_{i,j} = 0$. Thus, C is basis which has mn elements.
Meanwhile, if $[L:F]$ is infinite, then $[E:F]$ is clearly infinite; being $E \subset L$. Similarly, $[E:L]$. □

Theorem 6.1.0.4. Suppose that E is an algebraic Extension field of F. Then,

$$E \text{ is finite Extension} \iff \text{There exist finite elements } \alpha_1, \dots, \alpha_n \in E \text{ such that } E = F(\alpha_1, \dots, \alpha_n).$$

Proof. Suppose that E is finite Extension.
If $[E:F] = 1$, then $F(1) = F = E$, claim proved.
If $[E:F] > 1$, then $E \neq F$. Thus, put $\alpha_1 \in E \setminus F$. If $F(\alpha_1) = E$, then there is nothing to prove. If $F(\alpha_1) \neq E$, then put again $\alpha_2 \in E \setminus F(\alpha_1)$. Since $[E:F]$ is finite integer, this process will terminate in finite.
Thus for some $n \in \mathbb{N}$, $E = F(\alpha_1, \dots, \alpha_n)$.
Conversely, suppose that $E = F(\alpha_1, \dots, \alpha_n)$ for some $\alpha_i \in E$. Then,

$$E = [E : F(\alpha_n)][F(\alpha_n) : F(\alpha_{n-1})] \cdots [F(\alpha_1) : F]$$

Since E is algebraic, $[E : F(\alpha_n)], [F(\alpha_n) : F(\alpha_{n-1})], \dots [F(\alpha_1) : F]$ are all finite. Thus $[E:F]$ is finite. □

Theorem 6.1.0.5. If E is Algebraic over K and K is Algebraic over F. Then, E is Algebraic over F.

Proof. Let $\alpha \in E$ be given. Then, there exist not all zero $a_0, \dots, a_n \in K$ such that

$$a_n \alpha^n + a_{n-1} \alpha^{n-1} + \cdots + a_1 \alpha + a_0 = 0$$

Since each $a_i \in K$ is algebraic over F, thus $F(\alpha_0, \dots, \alpha_n)$ is finite over F, being above theorem.
Further,

$$F(\alpha, a_0, \dots, a_n) = F(a_0, \dots, a_n)(\alpha)$$

is finite over $F(a_0, \dots, a_n)$, because above equation gives that α is algebraic over $F(a_0, \dots, a_n)$. Now,

$$[F(\alpha, a_0, \dots, a_n) : F] = [F(\alpha, a_0, \dots, a_n) : F(a_0, \dots, a_n)][F(a_0, \dots, a_n) : F]$$

is finite, thus algebraic over F. This means $\alpha \in E$ is algebraic over F. □

6.2 Algebraic Closure

Lemma 6.2.0.1. Suppose that E is an Extension field of F. Then,

$$\overline{F}_E \overset{\text{def}}{=} \{\alpha \in E \mid \alpha \text{ is algebraic over } F\}$$

is a subfield of E. This $\overline{F}_E$ is called ***Algebraic Closure*** of F in E.

Proof. Let $\alpha, \beta \in \overline{F}_E$ be given. Since α is algebraic over F, thus $F(\alpha)$ is finite extension of F.
Then, $F(\alpha, \beta) = F(\alpha)(\beta)$ is finite over $F(\alpha)$, thus $F(\alpha, \beta)$ is finite extension over F, thus algebraic over F.
This implies $F(\alpha, \beta) \subseteq \overline{F}_E$. Now, with assumption $\beta \neq 0$,

$$\alpha + \beta, \alpha\beta, \frac{\alpha}{\beta} \in F(\alpha, \beta) \subseteq \overline{F}_E$$

□

Definition 6.2.0.1. A field F is said to be ***Algebraically Closed*** if:

Every non-constant polynomial in $F[x]$ has a zero in F

Theorem 6.2.0.1. A field F is Algebraically Closed ***if and only if***

every non-constant polynomial in $F[x]$ factors into linear factors within $F[x]$.

Proof. Suppose that F is Algebraically Closed.
Let $f(x) \in F[x]$ be given. Since $f(x)$ has a root in F, put $\alpha_0 \in F$ such that $f(\alpha_0) = 0$. Then,

$$f(x) = (x - \alpha_0) f_1(x)$$

And, $f_1(x) \in F[x]$, it has a root in F, put $\alpha_1 \in F$ such that $f_1(\alpha_1) = 0$. Now,

$$f(x) = (x - \alpha_0)(x - \alpha_1) f_2(x)$$

In finite process, we get:

$$f(x) = c(x - \alpha_0)(x - \alpha_1) \cdots (x - \alpha_{\deg f(x)})$$

Conversely argument is clear. □

Lemma 6.2.0.2. Suppose that F is a field, and $f(x) \in F[x]$ is non-constant polynomial.
Then, there exists an Extension E such that $f(x)$ has a root in E.

Proof. Let $f(x) \in F[x]$. Since $F[x]$ is U.F.D., factors $f(x)$ into irreducibles in $F[x]$,

$$f(x) = f_1(x) \cdots f_n(x)$$

Now, for any $1 \leq j \leq n$, $F[x]/(f_j(x))$ is an Extension of F containing a root of $f(x)$. □

6.2.1 † Existence of Algebraic Closure

Definition 6.2.1.1. Suppose that F is a field. An Extension $\overline{F}$ is called *Algebraic Closure* of F if:

1. $\overline{F}$ is Algebraic over F.
2. Every nonconstant polynomial $f(x) \in F[x]$ factors completely into linear factors over $\overline{F}$.

Theorem 6.2.1.1. An Algebraic Clousre $\overline{F}$ is Algebraically Closed.

Proof. Lef $f(x) \in \overline{F}[x]$ be given. Then, there exists an Extension E of $\overline{F}$ containing a root $\alpha \in E$ of $f(x)$. Now, $\overline{F}(\alpha) \subset E$ is algebraic over $\overline{F}$, and by definition, $\overline{F}$ is algebraic over F. Thus $\overline{F}(\alpha)$ is algebraic over F. Particulary, α is algebraic over F, thus $\alpha \in \overline{F}$. □

Theorem 6.2.1.2. For any field F, there exists an Algebraically closed field K containing F.

Proof. Denote the set $M \overset{\text{def}}{=} \{f \in F[x] \mid \deg f \geq 1\}$. Set $S \overset{\text{def}}{=} \{x_f \mid f \in M\}$. Define the Polynomial Ring

$$F[S] \overset{\text{def}}{=} \bigcup_{\substack{\mathcal{F} \subset S \\ \mathcal{F} \text{ finite}}} F[\mathcal{F}]$$

Consider the ideal

$$I \overset{\text{def}}{=} (\{f(x_f) \mid f \in M\}) \subseteq F[S]$$

If $I = F[S]$, that is, I is Entire, then $1 \in I$. Thus, for some $g_1, \ldots, g_n \in F[S]$ such that

$$g_1 f_1(x_{f_1}) + g_2 f_1(x_{f_2}) + \cdots + g_n f_n(x_{f_n}) = 1$$

For simplicity, write $x_i = x_{f_i}$. Since each g_i has only a finite number of variables, for $m \geq n$, write

$$\sum_{i=1}^{n} g_i(x_1, \ldots, x_n, \ldots, x_m) f_i(x_i) = 1$$

For each $1 \leq i \leq n$, let F_i be an Extension of F such that f_i has a root α_i.
Then, in the Compostion Field $F_1 \cdots F_n$,

$$1 = \sum_{i=1}^{n} g_i(x_1, \ldots, x_n, \ldots, x_m) f_i(\alpha_i) = 0$$

But this is Contradiction, thus I is Proper Ideal. The Zorn's Lemma gives the Existence of a Maximal Ideal

$$I \subsetneq \mathcal{M} \subsetneq F[S]$$

Thus, the Quotient ring

$$K_1 \overset{\text{def}}{=} F[S]/\mathcal{M}$$

is a Field containing F, and each polynomial f has a root in K_1.
Inductively, construct K_2 as an Extension of K_1 containing a root of all polynomial in K_1. We obtain

$$F = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_j \subseteq \cdots$$

Finally, define

$$K = \bigcup_{j=0}^{\infty} K_j$$

Then, K contains F, and any polynomial $k(x) \in K[x]$, for some $j \geq 0$, $k(x) \in K_j[x]$. And, $k(x)$ has a root in $K_{j+1}[x]$ by construction, thus K is Algebraically Closed. □

Theorem 6.2.1.3. Suppose that E is an Extension field of F, and Q is a Field of Fractions of $F[x]$. Then, E is Not Algebraic over F ***if and only if*** There exists an Embedding $\varphi : Q \to E$ such that $\forall a \in F,\ \varphi(a) = a$.

6.3 Splitting Field

Definition 6.3.0.1. The extension field K of F is called a ***splitting field*** for $f(x) \in F[x]$ if:

K is the smallest extension of F s.t $f(x) \in F[x]$ factors completely into linear factors in $K[x]$

More rigorously, there exist $a, \alpha_1, \ldots, \alpha_n \in K$ such that $K = F(\alpha_1, \ldots, \alpha_n)$ where

$$f(x) = a(x - \alpha_1) \cdots (x - \alpha_n)$$

Theorem 6.3.0.1. Suppose that F is a field, and $f(x) \in F[x]$ is a polynomial of degree $n \geq 1$. Then, there exists a splitting field K for $f(x)$ of F.

Proof. First, Claim: there exists an Extension field such that $f(x)$ factors completely into linear factors.
Using induction: For $n = 1$, Put $K = F$. Then claim proved immediately.
Suppose that for some $n \in \mathbb{N}$, the claim is true. Let $f(x) \in F[x]$ with $\deg f(x) = n + 1$.
If every irreducible factor of $f(x)$ has degree 1, then there is nothing to prove, being to put $K = F$.
Thus, suppose $p(x)$ is an irreducible factor of $f(x)$ such that $\deg p(x) \geq 2$. That is, $f(x) = p(x)q(x)$.
By Theorem, there exists an Extension E of F such that $p(x) \in F[x]$ has a root $\alpha \in E$.
In the extension K, $p(x)$ has a linear factor, thus the remaining factor has degree smaller than $n + 1$.
Now, by assumption of induction, $p(x)$ and $q(x)$ are factors completely into linear factors, so $f(x)$.
The smallestness given by intersection, these K. $\square$

Chapter 7

Galois Theory

Chapter 8

Module Theory

Definition 8.0.0.1. Suppose that R be a Ring. An Abelian Group M is called ***left-module*** over R if:

The operation $\cdot : R \times M \to M : (r,m) \mapsto rm$ satisfies: for any $r, s \in R,\ \ n, n \in M,$ $\begin{cases} (r+s)m = rm + sm \\ (rs)m = r(sm) \\ r(m+n) = rm + rn \end{cases}$

Moreover, if the Ring R contains identity 1, then $1m = m$.

Chapter 9

Linear Algebra

9.1 Vector Space

Definition 9.1.0.1. Suppose that F is a Field, and V is an Abelian Group.

And, the operation $\cdot : F \times V \to V$ satisfies: For any $a, b \in F$ and $v, w \in V$, $\begin{cases} a \cdot (v + w) = a \cdot v + a \cdot w \\ (a + b) \cdot v = a \cdot v + b \cdot v \\ (ab) \cdot v = a \cdot (b \cdot v) \\ 1 \cdot v = v \end{cases}$

The triple $(V, +, \cdot)$ is called the *Vector Space* over F.

Equivalently, The Vector Space over a field F is F-Module.

Definition 9.1.0.2. Suppose that V, W are Vector Space over a field F.
A map $\mathcal{L} : V \to W$ is called *Linear Map* if:

$$\text{For any } a \in F \text{ and } v_1, v_2 \in V,\ \mathcal{L}(a \cdot v_1 + v_2) = a \cdot \mathcal{L}(v_1) + \mathcal{L}(v_2)$$

Definition 9.1.0.3. Suppose that V is a Vector Space over a field F, and $W \subseteq V$ is a Subset.
The W is called *Subspace* of V if: $\begin{cases} \text{For any } a \in F,\ \ w \in W,\ \ a \cdot w \in W \\ \text{For any } w_1, w_2 \in W,\ \ w_1 + w_2 \in W \end{cases}$.

That is, the Subset of a Vector Space which is closed under the addition and scalar multiplication, then it is a Vector Space.

Lemma 9.1.0.1. Arbitrary intersection of Subspace is a Subspace.

Proof. Suppose that V is a Vector Space, and $W_\alpha \leq V,\ \alpha \in \Lambda$ are Subspaces.
Using Subspace Criterion: Let $a \in F,\ w_1, w_2 \in \bigcap_{\alpha \in \Lambda} W_\alpha$ be given. Then, for any $\alpha \in \Lambda$, $a \cdot w_1 + w_2 \in W_\alpha$. □

This Lemma allows the definition:

Definition 9.1.0.4. Suppose that V is a Vector Space over a field F, and $S \subseteq V$ be a Subset.
Define a *Generated Subspace* by S is:

$$\langle S \rangle \overset{\text{def}}{=} \bigcap_{S \subseteq W \leq V} W$$

This $\langle S \rangle$ is the ***unique smallest*** Subspace containing S. This S is called *Generating Subset* of $\langle S \rangle$.

Lemma 9.1.0.2. Suppose that V is a Vector Space over a field F, and $S \subseteq V$ be a Subset. Then,

$$\langle S \rangle = \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N},\ \ a_i \in F,\ \ v_i \in S\}$$

Proof. First, $\langle S \rangle \supseteq \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N},\ \ a_i \in F,\ \ v_i \in S\}$, because $\langle S \rangle$ is closed under the opeartions.
And, the set $\{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N},\ \ a_i \in F,\ \ v_i \in S\}$ is a Subgroup of V containing S,
Hence $\langle S \rangle \subseteq \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N},\ \ a_i \in F,\ \ v_i \in S\}$. $\square$

9.2 Linearly independent

Definition 9.2.0.1. Suppose that V is a Vector Space over a field F, and $S \subseteq V$ is a Subset.
A Subset S is called *Linearly independent* if: For any finite subset $\{v_1, v_2, \ldots, v_n\} \subseteq S$,

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0$$

If S is not Linearly independent, then it is called *Linearly dependent*.

Lemma 9.2.0.1. Suppose that $S \subseteq V$ is a Linearly independent subset of a Vector Space V over F.
If $v \in V$ satisfies $v = a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n$ for some $a_i \in F$ and $v_i \in S$,
then this representation is unique. More precisely,

Proof. First, $v = a_1 \cdot v_1 + \cdots + a_n \cdot v_n$ implies $v \in \langle S \rangle$.
Now, suppose that $v \in V$ satisfies $v = a_1 \cdot v_1 + \cdots + a_n \cdot v_n = b_1 \cdot w_1 + \cdots + b_m \cdot w_m$, WLOG $n \leq m$.
Put $I \overset{\text{def}}{=} \{i \in \mathbb{N} \mid \exists j \in \mathbb{N} \text{ s.t. } v_i = w_j\}$.
Then, there is a permutation $\rho : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\}$ such that $\forall i \in I,\ v_i = w_{\rho(i)}$. Now,

$$\sum_{i \in I} a_i \cdot v_i + \sum_{j \notin I} a_j \cdot v_j = \sum_{i \in I} b_{\rho(i)} \cdot w_{\rho(i)} + \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} = \sum_{i \in I} b_{\rho(i)} \cdot v_i + \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)}$$

$$\implies \sum_{i \in I} (a_i - b_{\rho(i)}) \cdot v_i + \sum_{j \notin I} a_j \cdot v_j - \left(\sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} \right) = 0$$

Since S is linearly independent, for all $j \notin I,\ a_j = b_{\rho(j)} = 0$ and for all $i \in I,\ a_i = b_{\rho(i)}$. $\square$

This fact enables the definition in the next section.

9.3 Basis

Definition 9.3.0.1. Suppose that V is a Vector Space over a field F.
A subset $\beta \subseteq V$ is called the *Basis* of V if:

1. β is Linearly independent.
2. $\langle \beta \rangle = V$.

Lemma 9.3.0.1. Suppose that V is a Vector Space over a field F. Then,

$$\beta \subset V \text{ is a Basis of } V \iff \text{For any } v \in V\text{, there exists a Unique representation } v = a_1 \cdot v_1 + \cdots + a_n v_n.$$

9.4 † Existence of Basis

> **Theorem 9.4.0.1.** Every Vector Space has a Basis.

Chapter 10

Category

Chapter 11

Exercise

27. 16

Find a Prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not maxiaml.

Solution. Since $S \subset R$ is Prime ideal if and only if R/S is an integral domain, and $S \subset R$ is Maximal if and only if R/S is a field, we can choose $S = \{0\} \times \mathbb{Z}$ as a prime but not maximal becasue

$$(\mathbb{Z} \times \mathbb{Z})/(\{0\} \times \mathbb{Z}) \cong \mathbb{Z}$$

is an integral domain, but not a field. This isomorphic guarantees by:

$$\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} : (a, b) \mapsto a$$

is surjective, homomorphism, and $\ker \varphi = \{0\} \times \mathbb{Z}$.

27. 33

Following two theorems are equivalent:

> **Theorem 11.0.0.1.** *Fundamental Theorem of Algebra*
> If $f(x) \in \mathbb{C}[x]$ is non-constant polynomial, then $f(x)$ has a root in $\mathbb{C}$.

> **Theorem 11.0.0.2.** *Nullstellensatz for* $\mathbb{C}[x]$
> Suppose that $f_1(x), \dots, f_r(x) \in \mathbb{C}[x]$ and $g(x) \in \mathbb{C}[x]$.
> If $\alpha \in \mathbb{C}$ is zero of all $f_1, \dots, f_r$ implies $g(\alpha) = 0$, then for some $n \in \mathbb{N}$, $g(x)^n \in \langle f_1, \dots, f_r \rangle$.

Solution. Suppose that the Fundamental Theorem of Algebra is true.
Since $\mathbb{C}$ is a field, every ideal in $\mathbb{C}[x]$ is principal. Thus, for some $p(x) \in \mathbb{C}[x]$,

$$\langle f_1(x), \dots, f_r(x) \rangle = \langle p(x) \rangle$$

By the Fundamental Theorem of Algebra, we can write: (WLOG, suppose $p(x)$ is monic)

$$p(x) = (x - \alpha_1)^{m_1} \cdots (x - \alpha_k)^{m_k}$$

where $m_i \in \mathbb{N}$. Now, since every $f_i(x)$ is divided by $p(x)$, thus all $f_i(x)$ has $\alpha_1, \dots, \alpha_k$ as roots, this implies $g(x)$ has $\alpha_1, \dots, \alpha_k$ as roots. This means $g(x) = q(x)(x - \alpha_1) \cdots (x - \alpha_k)$ for some $q(x) \in \mathbb{C}[x]$, thus $g(x)^n \in \langle p(x) \rangle$ where $n = m_1 m_2 \cdots m_r$.
Conversely, suppose that Nullstellensatz is true.
Let non-constant $f(x) \in \mathbb{C}[x]$ be given. Put $f_1(x) = (x - \alpha)$. Then, α is zero of $f_1(x)$ and zero of $(x - \alpha)f(x)$. By assumption, for some $n \in \mathbb{N}$, $f_1(x)^n \in \langle (x - \alpha)f(x) \rangle$. That is, $(x - \alpha)f(x)$ divides $f_1(x)^n = (x - \alpha)^n$. If $n = 1$, then $f(x)$ must be contradiction, this contradicts with f is non-constant. If $n > 1$, then $f(x)$ has $(x - \alpha)$ as a factor, thus α is zero of $f(x)$.

Chapter 12

General Topology

In this chapter, we follow the notations of [Steen et al., 1978, COUNTEREXAMPLES IN TOPOLOGY].

12.1 Basis

12.1.1 Subbasis

Definition 12.1.1.1. Let X be a set.
A collection $\mathcal{S} \subseteq \mathcal{P}(X)$ is called *subbasis* if: $X = \bigcup_{S \in \mathcal{S}} S$. (That is, $\forall x \in X,\ \exists S \in \mathcal{S}$ s.t. $x \in S$)
$\beta_{\mathcal{S}}$ is called *Basis generated by the subbasis* $\mathcal{S}$.
Note that: $\mathcal{T}_{\beta_{\mathcal{S}}}$ is the smallest Topology such that containing $\mathcal{S}$.

12.2 Topological Map

Definition 12.2.0.1. Let X, Y are Topological Space. $f : X \to Y$ is Continuous at $x_0 \in X$ if:
For any open $V \in \mathcal{T}_Y$ with $f(x_0) \in V$, there is an open $U \in \mathcal{T}_X$ with $x_0 \in U$ such that $f(U) \subset V$.

Definition 12.2.0.2. Let X, Y are Topological Space. Define:

1. $f : X \to Y$ is **Continuous Map** if: For any open $V \subset Y$, $f^{-1}[Y] \subset X$ be open.
2. $f : X \to Y$ is **Open Map** if: For any open subset $A \subset X$, $f[A] \subset Y$ be open.
3. $f : X \to Y$ is **Closed Map** if: For any closed subset $B \subset X$, $f[B] \subset Y$ be closed.
4. $f : X \to Y$ is **Homeomorphsim** if: f is bijection, continuous, and f^{-1} is continuous.

Theorem 12.2.0.1. Let $f : X \to Y$ be a Topological Map. Then, The Followings are Equivalent:

a) f is Continuous Map.

b) For any closed $C \subset Y$, $f^{-1}[C] \subset X$ be closed.

c) For any subset $A \subset X$, $f[\overline{A}] \subset \overline{f[A]}$.

d) For any subset $B \subset Y$, $f^{-1}[A^\circ] \subset (f^{-1}[B])^\circ$.

Proof.
a) $\implies$ **b)** Let $C \subset Y$ is closed. Then, $f^{-1}[Y \setminus C] = X \setminus f^{-1}[C]$ is open, thus $f^{-1}[C]$ is closed.
b) $\implies$ **c)** Let $A \subset X$. Since $A \subset f^{-1}[f[A]] \subset \underbrace{f^{-1}[\overline{f[A]}]}_{\text{closed by } b)} \implies \overline{A} \subset f^{-1}[\overline{f[A]}] \implies f[\overline{A}] \subset f[f^{-1}[\overline{f[A]}]] \subset \overline{f[A]}$.
c) $\implies$ **d)** Let $B \subset Y$, set $A = f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$.
Then, $f[\overline{A}] = f[\overline{X \setminus f^{-1}[B]}] = f[X \setminus (f^{-1}[B])^\circ]$ and $\overline{f[A]} = \overline{f[f^{-1}[Y \setminus B]]} \subset \overline{Y \setminus B} = Y \setminus B^\circ$.
By c),

$$\begin{aligned} & f[\overline{A}] = f[X \setminus (f^{-1}[B])^\circ] \subset \overline{f[A]} \subset Y \setminus B^\circ \\ \implies & X \setminus (f^{-1}[B])^\circ \subset f^{-1}[f[X \setminus (f^{-1}[B])^\circ]] \subset f^{-1}[Y \setminus B^\circ] = X \setminus f^{-1}[B^\circ] \\ \implies & f^{-1}[B^\circ] \subset (f^{-1}[B])^\circ \end{aligned}$$

d) $\implies$ **a)** Let $U \subset Y$ be an open set. By d), $f^{-1}[U] \overset{U \text{ open}}{=} f^{-1}[U^\circ] \subset (f^{-1}[U])^\circ$.
Meanwhile, reverse inclusion is clear, $f^{-1}[U] = (f^{-1}[U])^\circ$, open. □

Lemma 12.2.0.1. Let X, Y are Topological Space. Then,

1. $f : X \to Y$ is open map if and only if For any $A \subset X$, $f[A^\circ] \subset (f[A])^\circ$.
2. $f : X \to Y$ is closed map if and only if For any $A \subset X$, $\overline{f[A]} \subset f[\overline{A}]$.

Lemma 12.2.0.2. Let X, Y are Topological Space, and $f : X \to Y$ be a bijection. Then, TFAE:

1. f is open map.
2. f is closed map.
3. $f^{-1} : Y \to X$ be continuous map.

Clearly, Homeomorphism is open, closed, continuous map.

Lemma 12.2.0.3. Let $f : X \to Y$ be a Homeomorphism, $A \subset X$. Then, followings hold:

1. $f[\overline{A}] = \overline{f[A]}$.
2. $f[A^\circ] = (f[A])^\circ$.

Proof. 1. is clear by f is continuous, and closed map.
2. $\subset$) $A^\circ \subset A \implies f[A^\circ] \subset f[A] \implies f[A^\circ] \subset (f[A])^\circ \subset f[A]$ by f open map.
2. $\supset$) Let $x \in (f[A])^\circ$ be given. Then, there is an open $\mathcal{U} \in \mathcal{T}$ such that $x \in \mathcal{U} \subset f[A]$.
Now, $f^{-1}[x] \subset f^{-1}[\mathcal{U}] \subset f^{-1}[f[A]] = A$ by f is bijection, this implies
$(f^{-1}[\mathcal{U}])^\circ = f^{-1}[\mathcal{U}] \subset A^\circ \implies f^{-1}[x] \subset f^{-1}[\mathcal{U}] \subset A^\circ \implies f[f^{-1}[x]] = x \in f[A^\circ]$. □

Theorem 12.2.0.2. Let X, Y, Z are Topological Space, and $f : X \to Y$, $g : Y \to Z$.

1. If f, g are Continuous map, then $g \circ f$ is Continuous map.
2. If f, g are Open map, then $g \circ f$ is Open map.
3. If f, g are Closed map, then $g \circ f$ is Closed map.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Above three theorems are trivial.

1. If $g \circ f$ is Open map, f is Continuous onto map. Then, g is Open map.
2. If $g \circ f$ is Open map, g is Continuous one-to-one map. Then, f is Open map.

Proof. 1) Let $U \in \mathcal{T}_Y$ be an open set. Since f is Continuous map, $f^{-1}[U]$ is open of X.
Now,

$$\underbrace{(g \circ f)[\underbrace{f^{-1}[U]}_{\text{open}}]}_{\text{image of open map}} = g[f[f^{-1}[U]]] \overset{\text{onto}}{=} g[U] \in \mathcal{T}_Z$$

2) Let $U \in \mathcal{T}_X$ be an open set. Since $g \circ f$ is Open map, $(g \circ f)[U]$ is open of Z.
Now, by g is Continuous one-to-one,

$$g^{-1}[(g \circ f)[U]] = g^{-1}[g[f[U]]] \overset{\text{1 to 1}}{=} f[U] \in \mathcal{T}_Y$$

□

Lemma 12.2.0.4. *Pasting Lemma*
Suppose that X, Y are Topological Space, and $A, B \subset X$ such that $X = A \cup B$.
If both A, B are Open or Closed, $f : A \to Y$ and $g : B \to Y$ are Continuous map such that $f|_{A\cap B} = g|_{A\cap B}$, then

$$f \cup g : X \to Y : \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is Continuous map.

Proof. Suppose that $A, B \subset X$ are Closed in X. For any closed set $C \subseteq Y$,

$$(f \cup g)^{-1}[C] = f^{-1}[C] \cup g^{-1}[C]$$

is closed in X, being A is closed in X and containing $f^{-1}[C]$ as closed set, thus $f^{-1}[C]$ is closed in X. □

12.3 Product Space

12.3.1 Finite Product Space

Definition 12.3.1.1. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are Topological Spaces. Define a Topology on $X \times Y$:

$$\mathcal{T}_P \overset{\text{def}}{=} \{U \times V \subset X \times Y \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

The Topological Space $(X \times Y, \mathcal{T}_P)$ is called the ***Product Space***.

Lemma 12.3.1.1. Suppose that $(X \times Y, \mathcal{T}_p)$ is a Product Space for X and Y. Then, the Projection map

$$\pi_X : X \times Y \to X : (x, y) \mapsto x$$

is Continuous Open Onto map.

Proof. Surjection is Clear,
Continuous: For any open $\mathcal{U} \in \mathcal{T}_X$, $\pi_X^{-1}[\mathcal{U}] = \mathcal{U} \times Y \in \mathcal{T}_P$.
Open: For any open $O \times V \in \mathcal{T}_P$, $\pi_X[O \times V] = O \in \mathcal{T}_X$. Thus, it is Open map. □

Theorem 12.3.1.1. $\mathcal{T}_p$ is the samllest Topology such that the Projection maps are Continuous.

Proof. Suppost that $\mathcal{T}'$ is Topology on $X \times Y$ such that π_X, π_Y are Continous map.
Then, for any open $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$, $U \times Y = \pi_X^{-1}[U] \in \mathcal{T}'$ and $X \times V = \pi_Y^{-1}[V] \in \mathcal{T}'$
Thus, the finite intersection $U \times Y \cap X \times V = U \times V \in \mathcal{T}'$. That is, $\beta_p \subset \mathcal{T}'$. □

Theorem 12.3.1.2. Let X, and $\{Y_i\}_{i=1}^n$ are Topological Spaces. For a map

$$f : X \to \prod_{i=1}^n Y_i : x \mapsto (f_1(x), f_2(x), \ldots, f_n(x))$$

The Follwings Are Equivalent:

a) f is Continuous map.

b) For all $i = 1, 2, \ldots, n$, $f_i : X \to Y_i$ is Continuous map.

Proof. $a) \implies b)$. Since π_i is Continuous map, $f_i = \pi_i \circ f$ be a Continuous map.
$b) \implies a)$. Let $B = U_1 \times U_2 \times \cdots \times U_n \in \mathcal{T}_P$, and $\pi_i : \prod_{i=1}^n Y_i \to Y_i$ be a Projection for each $i = 1, 2, \ldots, n$. Then,

$$\begin{aligned} B = U_1 \times U_2 \times \cdots \times U_n =& (U_1 \times X_2 \times X_3 \times \cdots \times X_n) \\ & \cap (X_1 \times U_2 \times X_3 \times \cdots \times X_n) \\ & \vdots \\ & \cap (X_1 \times X_2 \times X_3 \times \cdots \times U_n) = \bigcap_{i=1}^n \pi_i^{-1}[U_i] \end{aligned}$$

Now,

$$f^{-1}[B] = f^{-1}\left[\bigcap_{i=1}^n \pi_i^{-1}[U_i]\right] = \bigcap_{i=1}^n f^{-1}\left[\pi_i^{-1}[U_i]\right] = \bigcap_{i=1}^n (\pi_i \circ f)^{-1}[U_i] = \overbrace{\underbrace{\bigcap_{i=1}^n f_i^{-1}[\underbrace{U_i}_{\text{open}}]}_{f_i \text{ conti}}}^{\text{finite intersection}} \in \mathcal{T}_X$$

□

Theorem 12.3.1.3. Let $X_1, \dots, X_n$ are Topological Space and $A_i \subset X_i$. Then, for a Product Space $\left(\prod_{i=1}^{n} X_i, \mathcal{T}_p\right)$,

1. $\overline{\prod_{i=1}^{n} A_i} = \prod_{i=1}^{n} \overline{A_i}$.
2. $\left(\prod_{i=1}^{n} A_i\right)^{\circ} = \prod_{i=1}^{n} A_i^{\circ}$.

Proof.
1.)

$$\begin{aligned} x \in \overline{\prod_{i=1}^{n} A_i} &\iff \forall U \in \mathcal{T}_p \text{ with } x \in U,\ U \cap \prod_{i=1}^{n} A_i = \prod_{i=1}^{n} \pi_i[U] \cap \prod_{i=1}^{n} A_i = \prod_{i=1}^{n} (\pi_i[U] \cap A_i) \neq \emptyset \\ &\iff \forall U \in \mathcal{T}_p \text{ with } x \in U,\ \forall i = 1, 2, \dots, n,\ \pi_i[U] \cap A_i \neq \emptyset \\ &\iff \forall i = 1, 2, \dots, n,\ \forall U_i \in \mathcal{T}_i \text{ with } \pi_i(x) \in U_i,\ U_i \cap A_i \neq \emptyset \\ &\iff x \in \prod_{i=1}^{n} \overline{A_i} \end{aligned}$$

2.)

$$\begin{aligned} x \in \left(\prod_{i=1}^{n} A_i\right)^{\circ} &\iff \exists U \in \mathcal{T}_p \text{ s.t. } x \in U \subset \prod_{i=1}^{n} A_i \\ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } x \in \prod_{i=1}^{n} \pi_i[U] \subset \prod_{i=1}^{n} A_i \\ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } \forall i = 1, 2, \dots, n,\ \pi_i(x) \in \pi_i[U] \subset A_i \\ &\overset{(*)}{\iff} x \in \prod_{i=1}^{n} A_i^{\circ} \end{aligned}$$

However, the left direction of $(*)$ fails when the index set is infinite. □

12.4 Coproduct Space

Definition 12.4.0.1. Let $(X_\alpha, \mathcal{T}_\alpha)$ $(\alpha \in \Lambda)$ are mutually disjoint Topological Spaces.
Define a *Coproduct Topology* $(X_{\amalg}, \mathcal{T}_{\amalg})$:

$$X_{\amalg} \overset{\text{def}}{=} \bigsqcup_{\alpha\in\Lambda} X_\alpha, \;\; \mathcal{T}_{\amalg} \overset{\text{def}}{=} \left\{ \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha \,\middle|\, \mathcal{U}_\alpha \in \mathcal{T}_\alpha \right\}$$

This actually be a Topology:
1. $\emptyset, X_{\amalg} \in \mathcal{T}_{\amalg}$ is clear,
2. Closed under union is clear.
3. Closed under finite intersection, not infinite.

Proof. Proof of *3.*
Let a finite collection

$$\left\{ \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^1, \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^2, \cdots, \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^k \right\}$$

be given. Then, their intersection be:

$$\bigcap_{j=1}^{k} \bigsqcup_{\alpha\in\Lambda} \mathcal{U}_\alpha^j = \bigsqcup_{\alpha\in\Lambda} \bigcap_{j=1}^{k} \mathcal{U}_\alpha^j \in \mathcal{T}_{\amalg}$$

□

Theorem 12.4.0.1. Let X_1, X_2, X_3 and Y_1, Y_2, Y_3 are mutually disjoint Topological Space, and for each $i = 1, 2, 3$,

$$f_i : X_i \to Y_i : x \mapsto f_i(x)$$

Define a function

$$f = f_1 \amalg f_2 \amalg f_3 : \bigsqcup_{i=1}^{3} X_i \to \bigsqcup_{i=1}^{3} Y_i : x \mapsto \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \\ f_3(x) & x \in X_3 \end{cases}$$

where both Domain and Codomain are Coproduct Space. (Clearly, this function is well-defined.)
Suppose that:

1. f_1 is Open map, Closed map
2. f_2 is Continuous map, Open map
3. f_3 is Continuous map, Closed map

Then, The Follwings hold:

1. f_1 is Continuous map *if and only if* f is Continuous map.
2. f_2 is Open map *if and only if* f is Open map.
3. f_3 is Closed map *if and only if* f is Closed map.

Proof.
1. It follows that: For any open on Codomain $U \in \mathcal{T}_{Y_{\amalg}}$,

$$\begin{aligned} f^{-1}[U] &= \{x \in X \mid f(x) \in U\} = \{x \in X_1 \mid f_1(x) \in U\} \cup \{x \in X_2 \mid f_2(x) \in U\} \cup \{x \in X_3 \mid f_3(x) \in U\} \\ &= f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U] \end{aligned}$$

Thus, If f_1 is Continuous, then f is Continuous map since $f^{-1}[U]$ is the union of open sets.
And, If f is Continuous, then $f^{-1}[U] \cap X_1$ be Open set and it is equal that $(f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U]) \cap X_1 = f_1^{-1}[U]$.

2. It follows that: For any open on Domain $U \in \mathcal{T}_{X_\amalg}$,

$$f[U] = f_1[U] \cup f_2[U] \cup f_3[U]$$

This, if f_2 is Open map, then f is Open map since $f[U]$ is the union of open sets.
And, If f is Open, then $f[U] \cap Y_2$ be Open set and it is equal that $(f_1[U] \cup f_2[U] \cup f_3[U]) \cap Y_2 = f_2[U]$.
3. Similar to the above. $\square$

For a specific example, Define for each $i = 1, 2, 3$,

$$X_i \overset{\text{def}}{=} \{a_i, b_i\}, \qquad \begin{cases} \mathcal{T}_{i,D} \overset{\text{def}}{=} \{\emptyset, X_i, \{a_i\}, \{b_i\}\} \\ \mathcal{T}_{i,I} \overset{\text{def}}{=} \{\emptyset, X_i\} \\ \mathcal{T}_{i,a} \overset{\text{def}}{=} \{\emptyset, X_i, \{a_i\}\} \\ \mathcal{T}_{i,b} \overset{\text{def}}{=} \{\emptyset, X_i, \{b_i\}\} \end{cases}$$

And define functions

1. $f_1 : (X_1, \mathcal{T}_{1,I}) \to (X_1, \mathcal{T}_{1,D}) : x \mapsto x$ is Not Continuous, Open, Closed.
2. $f_2 : (X_2, \mathcal{T}_{2,a}) \to (X_2, \mathcal{T}_{2,a}) : x \mapsto a_2$ is Continuous, Open, Not Closed.
3. $f_3 : (X_1, \mathcal{T}_{3,a}) \to (X_1, \mathcal{T}_{3,b}) : x \mapsto a_3$ is Continuous, Not Open, Closed.
4. $g_i : (X_i, \mathcal{T}_{i,D}) \to (X_i, \mathcal{T}_{i,D}) : x \mapsto x$ is Continuous, Open, Closed for each $i = 1, 2, 3$.

Now, from the above discussion,

1. $g_1 \amalg g_2 \amalg g_3$ is Continuous, Open, Closed.
2. $f_1 \amalg g_2 \amalg g_3$ is Not Continuous, Open, Closed.
3. $g_1 \amalg f_2 \amalg g_3$ is Continuous, Not Open, Closed.
4. $g_1 \amalg g_2 \amalg f_3$ is Continuous, Open, Not Closed.
5. $f_1 \amalg f_2 \amalg f_3$ is Not Continuous, Not Open, Not Closed.
6. $g_1 \amalg f_2 \amalg f_3$ is Continuous, Not Open, Not Closed.
7. $f_1 \amalg f_2 \amalg g_3$ is Not Continuous, Not Open, Closed.
8. $f_1 \amalg g_2 \amalg f_3$ is Not Continuous, Open, Not Closed.

No.	Map	Continuous	Open	Closed
1	$g_1 \amalg g_2 \amalg g_3$	Yes	Yes	Yes
2	$f_1 \amalg g_2 \amalg g_3$	No	No	No
3	$g_1 \amalg f_2 \amalg g_3$	Yes	No	Yes
4	$g_1 \amalg g_2 \amalg f_3$	Yes	Yes	No
5	$f_1 \amalg f_2 \amalg f_3$	No	No	No
6	$g_1 \amalg f_2 \amalg f_3$	Yes	No	No
7	$f_1 \amalg f_2 \amalg g_3$	No	No	Yes
8	$f_1 \amalg g_2 \amalg f_3$	No	Yes	No

12.5 Connected Space

Definition 12.5.0.1. Let X be a Topological Space. Define *Separation* of X be a tuple $\{U, V\}$ satisfying:

$$U, V \in \mathcal{T},\ U \neq \emptyset,\ V \neq \emptyset,\ U \cap V = \emptyset,\ U \cup V = X$$

If the separation exists, then X is called *disconnected*.

Lemma 12.5.0.1. Let X be a Topological Space. TFAE:

a) X is disconnected.

b) There exist closed sets C, D such that $C, D \neq \emptyset$, $C \cap D = \emptyset$, $C \cap D = X$

c) There exists a non-empty proper clopen subset $\emptyset \neq A \subsetneq X$.

d) There exist subsets $A, B \subseteq X$ such that $A, B \neq \emptyset$, $\overline{A} \cap B = A \cap \overline{B} = \emptyset$, $A \cup B = X$.

e) There exists Continuous onto map $f : X \to \{a, b\}$ where $\{a, b\}$ is discrete space.

Proof. $a) \iff b) \iff c) \implies d)$ given directly since the facts:

$$U \cap V = \emptyset \iff U \subseteq X \setminus V \iff V \subseteq X \setminus U$$
$$U \cup V = X \iff X \setminus V \subseteq U \iff X \setminus U \subseteq V$$

$d) \implies a)$ The tuple $\{X \setminus \overline{A},\ X \setminus \overline{B}\}$ becomes the Separation because:

$$A \cup B = X \implies \overline{A} \cup \overline{B} = X \implies (X \setminus \overline{A}) \cap (X \setminus \overline{B}) = \emptyset$$
$$[(X \setminus \overline{A}) \cup (X \setminus B)] \cup [(X \setminus A) \cup (X \setminus \overline{B})] = X \cup X = X$$

$\square$

Theorem 12.5.0.1. Let X be a Connected Space, Y is Topological Space and $f : X \to Y$ be a Continuous map. Then, $f[X]$ is Connected.

Proof. Supoose that $f[X]$ is disconnected. Then, there exist non-empty open sets of $f[X]$, $\{U, V\}$ such that

$$U \cup V = f[X],\ U \cap V = \emptyset$$

Now,

$$f^{-1}[U] \cup f^{-1}[V] = f^{-1}[U \cup V] = f^{-1}[f[X]] = X$$

and

$$f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V] = f^{-1}[\emptyset] = \emptyset$$

Since f is continuous, $\{f^{-1}[U], f^{-1}[V]\}$ be a separation of X, thus contradiction. $\square$

Lemma 12.5.0.2. Let A be a Connected Subspace of X. If $A \subseteq B \subseteq \overline{A}$, then B is Connected Subspace.

Proof. Using Contradiction: Suppose that B is Disconnected. Put $\{U, V\}$ be a Separation of B such that

$$U, V \in \mathcal{T}_X,\ \ B \cap U \neq \emptyset,\ \ B \cap V \neq \emptyset,\ \ (B \cap U) \cap (B \cap V) = \emptyset,\ \ B \subseteq U \cup V$$

Meanwhile, since assumption,

$$B \subseteq \overline{A} \implies \begin{cases} \emptyset \neq B \cap U \subseteq \overline{A} \cap U \neq \emptyset \\ \emptyset \neq B \cap V \subseteq \overline{A} \cap V \neq \emptyset \end{cases}$$

To show $A \cap U \neq \emptyset$, Suppose that $A \cap U = \emptyset$. Then, $A \subseteq X \setminus U$ and U is open implies $A \subseteq \overline{A} \subseteq X \setminus U$.
This implies $\overline{A} \cap U = \emptyset$, Contradiction. Thus $A \cap U \neq \emptyset$, similarly, $A \cap V \neq \emptyset$.
On the other hand,

$$A \subset B \implies \begin{cases} (A \cap U) \cap (A \cap V) \subseteq (B \cap U) \cap (B \cap V) = \emptyset \\ A \subset B \subseteq U \cup V \end{cases}$$

Consequently, $\{U, V\}$ be a Separation of A, Contradiction. □

Theorem 12.5.0.2. Let X be a Topological Space, and subspaces $A_\alpha \subset X,\ (\alpha \in \Lambda)$ are Connected.
If $\bigcap\limits_{\alpha \in \Lambda} A_\alpha \neq \emptyset$, then $\bigcup\limits_{\alpha \in \Lambda} A_\alpha$ is Connected Space.

Proof. Proof by Contradiction.
Suppose that $A = \bigcup\limits_{\alpha \in \Lambda} A_\alpha$ is Disconnected. Let $\{U, V\}$ be a separation of A. Choose $a \in \bigcap\limits_{\alpha \in \Lambda} A_\alpha$, since assumption.
Then, since $A = U \cup V$, WLOG, assume that $a \in U$. Set for each $\alpha \in \Lambda$, $U_\alpha = U \cap A_\alpha$ and $V_\alpha = V \cap A_\alpha$. Then,

$$a \in U_\alpha \neq \emptyset,\ \ U_\alpha \cap V_\alpha = U \cap V \cap A_\alpha = \emptyset,\ \ U_\alpha \cup V_\alpha = (U \cup V) \cap A_\alpha = A_\alpha$$

Thus, V_α must be emptyset. Now,

$$V = V \cap A = V \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (V \cap A_\alpha) = \bigcup_{\alpha \in \Lambda} V_\alpha = \emptyset$$

This is Contradiction. □

Corollary 12.5.0.1. Let X be a Topological Space, and subspaces $A_n \subset X$ ($n \in \mathbb{N}$) are Connected.
If for any $n \in \mathbb{N}$, $\left(\bigcup\limits_{i=1}^{n} A_i \right) \cap A_{n+1} \neq \emptyset$, then $\bigcup\limits_{n=1}^{\infty} A_n$ is Connected.

Proof. Put

$$B_n \stackrel{\text{def}}{=} \bigcup_{i=1}^{n} A_i$$

$B_1 = A_1$ is Conneted, by assumption. Inductively, Suppose that B_n is Connected.
Then, above theorem and $B_{n+1} = B_n \cap A_{n+1} \neq \emptyset$ gives B_{n+1} is Connected. Meanwhile,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

and

$$\bigcap_{n=1}^{\infty} B_n = B_1 = A_1 \neq \emptyset, \text{ because } A_1 \cap A_2 \neq \emptyset$$

Thus, above theorem gives $\bigcup\limits_{n=1}^{\infty} B_n$ Connected. □

Theorem 12.5.0.3. Let X, Y are Connected Spaces. Then, $X \times Y$ is Connected.

Proof. For any $x \in X$, $y \in Y$, $X \times \{y\}$ and $\{x\} \times Y$ are Connected, being $X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$.
And, $X \times \{y\} \cap \{x\} \times Y = \{(x, y)\} \neq \emptyset$, thus $X \times \{y\} \cup \{x\} \times Y$ is Connected. Let $b \in Y$ fixed.
For any $x \in X$, define $T_x \overset{\text{def}}{=} X \times \{b\} \cup \{x\} \times Y$. T_x is Connected by above discussion, and contains $X \times \{b\}$ for any $x \in X$. Thus, $X \times \{b\} \subset \bigcap_{x \in X} T_x \neq \emptyset$ implies:

$$X \times Y = \bigcup_{x \in X} T_x$$

is Connected. $\square$

12.5.1 Connected Component

Definition 12.5.1.1. Let X be a Topological Space. Define a relation $\sim$ of X:

$$x \sim y \iff \text{There exists a Connected Subspace of } A \subset X \text{ such that } x, y \in A$$

Then, this relation be an Equivalent Relation, and define Equivalent Class of this relation:

$$\mathcal{C}_x \overset{\text{def}}{=} \{y \in X \mid x \sim y\}$$

$\mathcal{C}_x$ is called **Connected Component**.

Theorem 12.5.1.1. Let X be a Topological Space. Then,

1. For each $x \in X$, there exists a unique Connected Component $\mathcal{C}_x$ which is containing x.
2. $\mathcal{C}_x$ be the largest Connected Subspace containing x.
3. If $A \subset X$ be a non-empty Connected Subspace,
   then there exists a unique Connected Component $\mathcal{C}_x$ such that $A \subset \mathcal{C}_x$.
4. Every Connected Component is Closed.
5. X is Connected if and only if There exists only one Connected Component.

Proof. 1. is clear: $x \in \mathcal{C}_y \iff x \sim y \iff \mathcal{C}_x = \mathcal{C}_y$.
2. Let $x \in X$ be given.
Largeset: Let $A \subseteq X$ be a Connected subset s.t $x \in A$. Then, for any $y \in A$, $x \sim y \implies y \in \mathcal{C}_x$. Thus $A \subseteq \mathcal{C}_x$.
Connected: Let $z \in \mathcal{C}_x$ be given. Then, $z \sim x$ implies there exists a Connected subset A_x such that $x, z \in A_z \subseteq \mathcal{C}_x$.
Now,

$$\mathcal{C}_x = \bigcup_{z \in \mathcal{C}_x} A_z, \quad x \in \bigcap_{z \in \mathcal{C}_x} A_z \neq \emptyset$$

Thus, theorem gives $\mathcal{C}_x$ is Connected.
3. Let $A \subset X$ be a non-empty subset, and Connected. Put Open sets tuple $\{U, V\}$ be a Separation of A.
Existence: Fix $x \in A$. Then, for any $a \in A$, $x \sim a \implies a \in \mathcal{C}_x$. Thus $A \subseteq \mathcal{C}_x$.
Uniqueness: If $A \subseteq \mathcal{C}_x$ and $A \subseteq \mathcal{C}_y$, then for any $a \in A$, $a \sim x$ and $a \sim y$ implies $x \sim y$, thus $\mathcal{C}_x = \mathcal{C}_y$.
4. Let $\mathcal{C}_x$ be a Connected Component. Then, $\overline{\mathcal{C}_x}$ is Connected by theorem, containing x. Thus $\overline{\mathcal{C}_x} \subseteq \mathcal{C}_x$. $\square$

Theorem 12.5.1.2. Let X, Y are Topological Space such that $X \cong Y$. Then, X and Y have same number of Connected Component.

Proof. Let $f : X \to Y$ be a Homeomorphism, and $x \in X$. Since Homeo- preserves Connectedness, $f[\mathcal{C}_x]$ is Connected.
Meanwhile, $f(x) \in f[\mathcal{C}_x]$, thus $f[\mathcal{C}_x] \subset \mathcal{C}_{f(x)}$.
Similarly, $x = f^{-1}(f(x)) \in f^{-1}[\mathcal{C}_{f(x)}]$, thus $f^{-1}[\mathcal{C}_{f(x)}] \subset \mathcal{C}_x$, implies $f[f^{-1}[\mathcal{C}_{f(x)}]] = \mathcal{C}_{f(x)} \subset f[\mathcal{C}_x]$.
That is, for any $x \in X$, $f[\mathcal{C}_x] = \mathcal{C}_{f(x)}$. Similarly, for any $y \in Y$, $f^{-1}[\mathcal{C}_y] = \mathcal{C}_{f^{-1}(y)}$.
Claim: the map

$$\phi : \{\mathcal{C}_x \mid x \in X\} \to \{\mathcal{C}_y \mid y \in Y\} : \mathcal{C}_x \mapsto \mathcal{C}_{f(x)}$$

be a One-to-One, Onto.
Injection) Let $\phi(\mathcal{C}_{x_1}) = \phi(\mathcal{C}_{x_2})$. That is, $\mathcal{C}_{f(x_1)} = \mathcal{C}_{f(x_2)}$. Since above discussion, we obtain $f[\mathcal{C}_{x_1}] = f[\mathcal{C}_{x_2}]$. Now,

$$\mathcal{C}_{x_1} = f^{-1}[f[\mathcal{C}_{x_1}]] = f^{-1}[f[\mathcal{C}_{x_2}]] = \mathcal{C}_{x_2}$$

Surjection) Let $\mathcal{C}_y$ be given from codomain. Since f is surjection, there exists $x \in X$ such that $y = f(x)$. Thus, $\mathcal{C}_y = \mathcal{C}_{f(x)} = \phi(\mathcal{C}_x)$, thus surjective. $\square$

Theorem 12.5.1.3. If the Topological Space X has finite number of Connected Components, then for each Connected Compoent is Clopen.

12.5.2 Locally Connected

Definition 12.5.2.1. A Space X is called *Locally Connected* if:

For any $x \in X$ and neighborhood N of x, there exists Connected Open set $U \in \mathcal{T}$ such that $x \in U \subseteq N$.

Theorem 12.5.2.1. Let X be a Space. Then,

X is Locally Connected ***if and only if*** For any open U, every Connected Component of subspace U is open.

Note that: $U \subset X$ is open of X, then for any subset $V \subseteq U$,

V is open in X ***if and only if*** V is open in U.

Proof. Suppose that X is Locally Connected.
Let U be an open set of X, and $\mathcal{C}$ be a Connected Component of U.
Since Locally Connectedness, For any $x \in \mathcal{C}$, there exists Connected Open set V_x such that $x \in V_x \subseteq U$.
Since $\mathcal{C}$ is the largeset Connected set containing x, thus $V_x \subseteq \mathcal{C}$. Now, $\mathcal{C} = \bigcup_{x \in \mathcal{C}} V_x$, thus open.

Conversely, Let $x \in X$ and U be an open set containing x. Take $\mathcal{C}_x$ is a Connected Component of U containing x.
By assumption, $\mathcal{C}_x$ is Open, thus X is Locally Connected, as $\mathcal{C}_x$ is Connected open neiborhood. □

Corollary 12.5.2.1. Every Connected Component of Locally Connected Space is Clopen.

12.5.3 Path Connected

Definition 12.5.3.1. Let $I = [0, 1] \subset \mathbb{R}$, and X be a Topological Space.
Define ***Path*** from $x \in X$ to $y \in X$ is: Continuous map $p : I \to X$ such that $p(0) = x$ and $p(1) = y$.
A Space X is called ***Path Connected Space*** if: For any $x, y \in X$, there exists a path from x to y.

Theorem 12.5.3.1. Path Connected Space is Connected.

Proof. Fix $x_0 \in X$. By Path Connectedness, for any $x \in X$, there exists a path p_x from x_0 to x.
Since I is Connected, $p_x[I]$ is Connected in X and contains x_0 and x. Now,

$$\bigcup_{x \in X} \alpha_x[I] = X, \;\; x_0 \in \bigcap_{x \in X} \alpha_x[I] \neq \emptyset$$

Thus, X is Connected. $\square$

Proof. Version 2. Let X be a Path-Connected but not Connected. Let $\{U, V\}$ be a Separation of X, and put $x \in U$, $y \in V$.
Then, there is a path $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. And, $\alpha^{-1}[U], \alpha^{-1}[V]$ are disjoint opens.
Moreover, $0 \in \alpha^{-1}[U]$ and $1 \in \alpha^{-1}[V]$, and $\alpha^{-1}[U] \cup \alpha^{-1}[V] = \alpha^{-1}[U \cup V] = \alpha^{-1}[X] = [0, 1]$.
Thus, these becomes Separation of $[0, 1]$, Contradiction. $\square$

Theorem 12.5.3.2. Let X be a Topological Space, and subspaces $A_\gamma \subset X(\gamma \in \Gamma)$ are Path-Connected.
If $\displaystyle\bigcap_{\gamma \in \Gamma} A_\gamma \neq \emptyset$, then $\displaystyle\bigcup_{\gamma \in \Gamma} A_\gamma$ is Path-Connected.

Proof. Fix $x^* \in \bigcap_{\gamma \in \Gamma} A_\gamma$, and let $x, y \in \bigcup_{\gamma \in \Gamma} A_\gamma$ be given. Then, for some $\alpha, \beta \in \Gamma$, $x \in A_\alpha$ and $y \in A_\beta$.
Put p_α is a path from x to x^* and p_β is a path from y to x^*. Define

$$p_{\alpha * \beta} : [0, 1] \to \bigcup_{\gamma \in \Gamma} A_\gamma : x \mapsto \begin{cases} p_\alpha(2x) & x \in [0, \frac{1}{2}] \\ p_\beta(2x - 1) & x \in [\frac{1}{2}, 1] \end{cases}$$

Since Pasting lemma, this function is Continuous, moreover path from x to y in $\bigcup_{\gamma \in \Gamma} A_\gamma$. $\square$

Corollary 12.5.3.1. If $A_n \subset X, \;\; (n \in \mathbb{N})$ are Path-Connected

Theorem 12.5.3.3. If X is Path-Connected and $f : X \to Y$ is Continuous, then $f[X]$ is Path-Connected.

Proof. Let $f(x), f(y) \in f[X]$ be given. Put $p : I \to X$ is a path from x to y.
Then, $f \circ p : I \to f[X]$ is continuous, $f(p(0)) = f(x)$ and $f(p(1)) = f(y)$, thus it is a path. $\square$

12.5.4 Path-Connected Component

Definition 12.5.4.1. Let X be a Topological Space. Define a relation $\sim$ of X:

$$x \sim y \iff \text{There exists a path from } x \text{ to } y \text{ in } X$$

Then, this relation be an Equivalent Relation, and define Equivalent Class of this relation:

$$\mathcal{P}_x \overset{\text{def}}{=} \{y \in X \mid x \sim y\}$$

$\mathcal{P}_x$ is called **Path-Connected Component.**

12.5.5 Locally Path Connected

Definition 12.5.5.1. A Space X is called *Locally Path-Connected* if:

For any $x \in X$ and neighborhood N of x, there exists Path-Connected Open set $P \in \mathcal{T}$ such that $x \in P \subseteq N$.

Theorem 12.5.5.1. A Space X is Locally Path-Connected *if and only if*:
For any open U, every Path-Connected Component of subspace U is open.

Proof. Suppose that X is Locally Path-Connected.
Let U be an open set of X, and $\mathcal{P}$ be a Path-Connected Component of U.
Since Locally Path-Connectedness, For any $x \in \mathcal{P}$, there exists Path-Connected Open set V_x such that $x \in V_x \subseteq U$.
Since $\mathcal{P}$ is the largeset Path-Connected set containing x, thus $V_x \subseteq \mathcal{P}$. Now, $\mathcal{P} = \bigcup_{x \in \mathcal{P}} V_x$, thus open.

Conversely, Let $x \in X$ and U be an open set containing x. Take $\mathcal{P}_x$ is a Connected Component of U containing x.
By assumption, $\mathcal{P}_x$ is Open, thus X is Locally Connected, as $\mathcal{P}_x$ is Connected open neiborhood. □

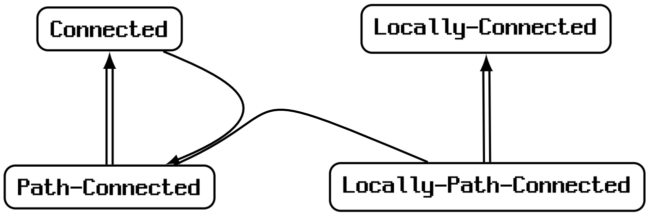
Theorem 12.5.5.2. Let X be a Locally Path-Connected Space,
$\sim_c$ be a Connected relation, and $\sim_p$ be a Path-Connected relation.
Then, $X/{\sim_c} = X/{\sim_p}$. (Moreover, every element in the collection is Clopen.)

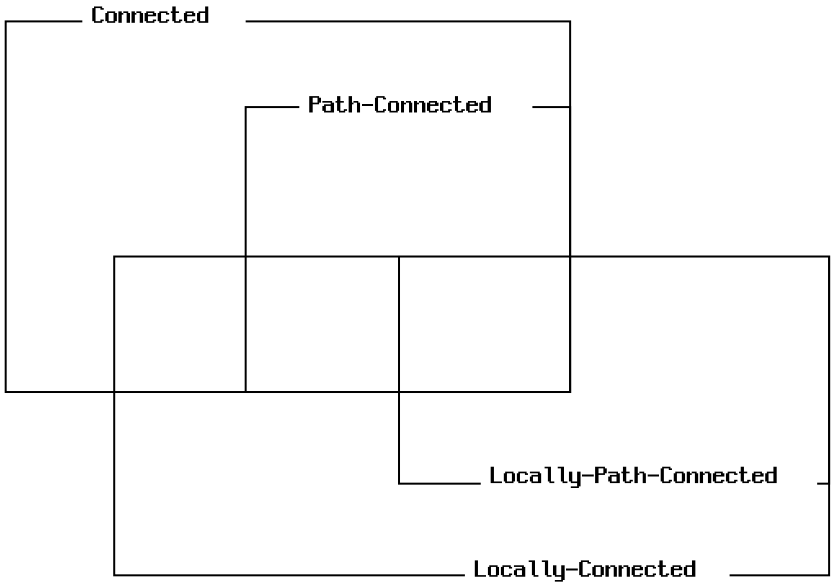
Proof. Let $x \in X$ be given. Put $\mathcal{C}_x$ is Connected Component of x, and $\mathcal{P}_x$ is Path-Connected Component of x.
Since Path-Connected Space is Connected, $\mathcal{P}_x \subseteq \mathcal{C}_x$. Using Contradiction: Suppose that $\mathcal{P}_x \subsetneq \mathcal{C}_x$.
Generally, Since for any $y \in \mathcal{C}_x$, there exists a Path-Connected Component $\mathcal{P}_y$ such that $x \in \mathcal{P}_y \subseteq \mathcal{C}_x$.
That is, $\mathcal{C}_x = \bigcup_{y \in \mathcal{C}_x} \mathcal{P}_y$. Now, $\mathcal{C}_x \setminus \mathcal{P}_x$ is non-empty, and Path-Connected Component being $\sim_p$ is equivalent relation.
Since X is Locally Path-Connected Space, $\mathcal{P}_x$ and $\mathcal{C}_x \setminus \mathcal{P}_x$ are Path-Connected Open sets,
thus these becomes Separation of $\mathcal{C}_x$. This is Contradiction. □

Corollary 12.5.5.1. Connected and Locally Path-Connected Space is Path-Connected.

Proof. If X is Connected, then X is also Connected Component.
Since above theorem, X is Path-Connected Component, thus Path-Connected. □

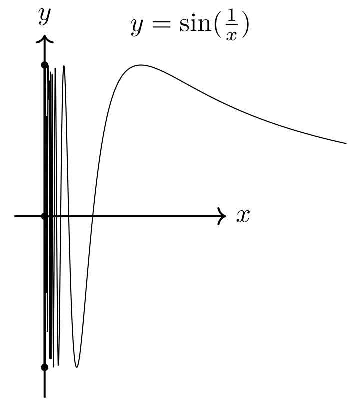
12.5.6 Summary and Diagram and Conuterexamples





12.5.7 Topologist's Sine Curve

Definition 12.5.7.1. *Topologist's Sine Curve*
Let $Y = \{(0, y) \mid -1 \leq y \leq 1\}$ and $S^+ = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$.
The union $S = Y \cup S^+$ with Euclidean metric is called *Topologist's Sine Curve*.



Theorem 12.5.7.1. Topologist's Since Curve is Connected Space.

Proof. Since $f : (0, \infty) \to S^+ : x \mapsto (x, \sin(\frac{1}{x}))$ is Continuous and $(0, \infty)$ is Connected, $S^+ = f[(0, \infty)]$ is Connected.
Meanwhile, Let $\varepsilon > 0$ be given. There exists $0 < r < \varepsilon$ such that $\sin(\frac{1}{r}) = y$. That is, $(r, \sin(\frac{1}{r})) \in B_\varepsilon(0, y)$.
Now, $S^+ \subseteq Y \cup S^+ \subseteq \overline{S^+}$, thus $S = Y \cup S^+$ is Connected. $\square$

Theorem 12.5.7.2. Topologist's Since Curve is ***Not*** Locally Connected Space.

Proof. Consider the Ball $B_{\frac{1}{2}}((0, 0))$. Then, $B_{\frac{1}{2}}((0, 0)) \cap S$ is open neighborhood of $(0, 0)$.
Let $U \cap S \subset B_{\frac{1}{2}}((0, 0)) \cap S$ be an open set of S containing $(0, 0)$.
Since $(0, 0) \in U$, $U \cap S^+ \neq \emptyset$. Put $(a, b) \in U \cap S^+$. Then, there exists $m \in \mathbb{N}$ such that

$$0 < \frac{1}{(2m + \frac{1}{2})\pi} < a$$

But, $(\frac{1}{(2m+\frac{1}{2})\pi}, \sin((2m + \frac{1}{2})\pi)) = (\frac{1}{(2m+\frac{1}{2})\pi}, 1) \notin B_{\frac{1}{2}}((0, 0))$, because $|(0, 0) - (\frac{1}{(2m+\frac{1}{2})\pi}, 1)| > \frac{1}{2}$.
Now,

$$((-\infty, \frac{1}{(2m + \frac{1}{2})\pi}) \times \mathbb{R}) \cap S, \;\; ((\frac{1}{(2m + \frac{1}{2})\pi}, \infty) \times \mathbb{R}) \cap S$$

becomes Separation of $U \cap S$. Thus, there is no Connected Open neighborhood. $\square$

Theorem 12.5.7.3. Topologist's Since Curve is ***Not*** Path-Connected Space.

Proof. Let $(0, a) \in Y$, $(b, \sin(\frac{1}{b})) \in S^+$ be given. Suppose that there exists a path $p : I \to S$ from $(0, a)$ to $(b, \sin(\frac{1}{b}))$.
Then, $W = \{x \in I \mid p[[0, x]] \subseteq Y\}$ is non-empty, bounded above. Put $r = \sup W$. Clearly $r \in \overline{W}$.
Meanwhile, Y is closed in S, thus $p(r) \in p[\overline{W}] \subseteq \overline{p[W]} \subseteq \overline{Y} = Y$, thus $r < 1$.
Ane, since p is Continuous, there exists $\delta \in (0, 1 - r)$ such that $|x - r| < \delta \implies |p(x) - p(r)| < \frac{1}{2}$.
By definition of Supremum, for some $t \in (r, r + \delta)$ such that $p(t) \in S^+$ with

$$\operatorname{diam}(p[[r, t]]) \leq \frac{1}{2}, \;\; p(r) \in Y, \;\; p(t) \in S^+$$

Meanwhile, Claim: If $0 < c < t$ and $(c, \sin(\frac{1}{c})) \in S^+$, then $(c, \sin(\frac{1}{c})) \in p[[r, t]]$.
If $(c, \sin(\frac{1}{c}))$ is not contained in $p[[r, t]]$, then

$$U = (-\infty, c) \times \mathbb{R}, \;\; V = (c, \times) \times \mathbb{R}$$

becomes Separation of $p[[r, t]]$, thus Contradiction. Now, for large enough $m \in \mathbb{N}$,

$$a = \frac{1}{(2m + \frac{1}{2})\pi} < t, \;\; b = \frac{1}{(2m + \frac{3}{2})\pi} < t$$

Each $p, q \in p[[r, t]]$ and $d(a, b) > 2$, Contradiction. $\square$

Corollary 12.5.7.1. Topologist's Since Curve is *Not* Locally-Path-Connected Space.

12.6 Compact Space

Definition 12.6.0.1. A Topological Space X is *compact* if: every open cover contains a finite subcover. i.e.,

$$\text{If } X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha,\ (\mathcal{U}_\alpha \in \mathcal{T}), \text{ then there is finite subcover such that } X = \bigcup_{i=1}^{N} \mathcal{U}_{\alpha_i}$$

This is equivalent with:

$$\text{If } \emptyset = \bigcap_{\alpha \in \Lambda} \mathcal{C}_\alpha,\ (\mathcal{C}_\alpha \text{ closed}), \text{ then there is finite subset such that } \emptyset = \bigcap_{i=1}^{N} \mathcal{C}_{\alpha_i}$$

Definition 12.6.0.2. Let X be a set. $A \subset \mathcal{P}(X)$ satisfies *finite intersection property* if:

$$\text{For all finite subset of } A,\ \{A_i \mid i = 1, 2, \ldots, n\} \subset A \text{ satisfies } \bigcap_{i=1}^{n} A_i \neq \emptyset.$$

Example.

1. $X = \mathbb{R}$, and let $A = \{(n, \infty) \mid n \in \mathbb{N}\}$. Then,

$$\bigcap_{S \in A} S = \emptyset, \quad \bigcap_{\substack{S \in F \subset A \\ |F| < \infty}} S \neq \emptyset$$

2. $X = \mathbb{R}$, and let $A = \{\mathbb{R} \setminus F \mid |F| < \aleph_0\}$.

Theorem 12.6.0.1. Let X be a Topological Space, Then, TFAE:

a) X is Compact Space.

b) If A is a collection of closed subsets of X that satisfies *FIP*, then $\bigcap_{\mathcal{C} \in A} \mathcal{C}_i \neq \emptyset$.

c) If A is a collection of subsets of X that satisfies *FIP*, then $\bigcap_{S \in A} \overline{S} \neq \emptyset$.

Proof. $a) \implies b)$. Proof by Contradiction:
Suppose that $A \subset \mathcal{P}(X)$ be a collection of closed subsets such that *FIP*.
Assume that $\bigcap_{\mathcal{C} \in A} \mathcal{C} = \emptyset$. Since X is Compact,

$$\emptyset = \bigcap_{\mathcal{C} \in A} \mathcal{C} \textit{ if and only if } X = \bigcup_{\mathcal{C} \in A} (X \setminus \mathcal{C}), \text{ where } X \setminus \mathcal{C} \text{ is open.}$$

This implies that there is a finite subcover:

$$X = \bigcup_{i=1}^{N} (X \setminus \mathcal{C}_i) \textit{ if and only if } \emptyset = \bigcap_{i=1}^{N} \mathcal{C}$$

This is Contradiction with A satisfies *FIP*.
$b) \implies a)$. Proof by Contraposition:
Suppose that X is not Compact. Then, there exists an Open Cover $\mathcal{O}$ with no finite subcover: i.e.,

$$X = \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \textit{ if and only if } \emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U})$$

And,

$$\text{For any finite subset of } \mathcal{O},\ F = \{\mathcal{U}_i \mid i = 1, \ldots, N\} \text{ satisfies } X \supsetneq \bigcup_{i=1}^{N} \mathcal{U}_i \textit{ if and only if } \emptyset \neq \bigcap_{i=1}^{N} (X \setminus \mathcal{U}_i)$$

Thus, $\mathcal{K} = \{X \setminus \mathcal{U} \mid \mathcal{U} \in \mathcal{O}\}$ satisfies **FIP**, but $\emptyset = \bigcap_{\mathcal{U}\in\mathcal{O}} (X \setminus \mathcal{U}) = \bigcap_{\mathcal{C}\in\mathcal{K}} \mathcal{C}$. Thus, not $a)$ implies not $b)$. □

> **Theorem 12.6.0.2.** Let X is Compact Space, Y is Topological Space.
> If $f : X \to Y$ is Continuous Map, then $f[X]$ is Compact.

Proof. Let $\mathcal{O}$ be an open cover of $f[X]$. i.e, $f[X] \subset \bigcup_{\mathcal{U}\in\mathcal{O}} \mathcal{U}$. Now,

$$X \subset f^{-1}[f[X]] \subset f^{-1}\left[\bigcup_{\mathcal{U}\in\mathcal{O}} \mathcal{U}\right] = \bigcup_{\mathcal{U}\in\mathcal{O}} \underbrace{f^{-1}[\mathcal{U}]}_{\text{open, } f \text{ conti.}}$$

Since X is compact, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^{N} f^{-1}[\mathcal{U}_i]$$

Consequently,

$$f[X] \subset f\left[\bigcup_{i=1}^{N} f^{-1}[\mathcal{U}_i]\right] = \bigcup_{i=1}^{N} f\left[f^{-1}[\mathcal{U}_i]\right] \subset \bigcup_{i=1}^{N} \mathcal{U}_i$$

□

> **Theorem 12.6.0.3.** Closed set of compact space is compact.

Proof. Let X be a compact, and $E \subset X$ be a closed subset. Let $\mathcal{O}$ be an open over of E.
Then,

$$X = E \cup (X \setminus E) \subset \left(\bigcup_{\mathcal{U}\in\mathcal{O}} \mathcal{U}\right) \cup (X \setminus E)$$

be an open cover of X. Thus, there is a finite subcover such that

$$X = \left(\bigcup_{i=1}^{N} \mathcal{U}_i\right) \cup (X \setminus E) \iff E \subset \bigcup_{i=1}^{N} \mathcal{U}_i$$

□

> **Theorem 12.6.0.4.** Let X be a Topological Space, and β be a basis of X. Then, TFAE:
>
> ***a)*** X is Compact Space.
>
> ***b)*** Every open cover consisting of basis elements has a finite subcover.

Proof. $a) \implies b)$. Clear by definition of Compact.
$b) \implies a)$. Let $\{\mathcal{U}_\alpha \mid \alpha \in \Lambda\}$ be an Open cover of X. That is,

$$X = \bigcup_{\alpha\in\Lambda} \mathcal{U}_\alpha = \bigcup_{\alpha\in\Lambda} \bigcup_{\gamma\in\Gamma_\alpha} B_\alpha^\gamma$$

where $\{B_\alpha^\gamma \mid \gamma \in \Gamma_\alpha\}$ is subset of basis such that $\bigcup_{\gamma\in\Gamma_\alpha} B_\alpha^\gamma = \mathcal{U}_\alpha$. Now, by $2)$, there is finite subcover such that

$$X = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_{\alpha_i}^{\gamma_j} \subset \bigcup_{i=1}^{n} \mathcal{U}_{\alpha_i}$$

Thus, $\{\mathcal{U}_{\alpha_i} \mid i = 1, 2, \ldots, n\}$ be a finite subcover. □

Theorem 12.6.0.5. Let X, Y are Topological Space. Then, TFAE:

a) $X \times Y$ is Compact.

b) X and Y both are Compact.

Proof. $a) \implies b)$ is clear since projection preserves Compactness.
$b) \implies a)$ Let $\mathcal{O} \overset{\text{def}}{=} \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ be an Open cover of $X \times Y$.
Let $x \in X$ fix. Then, $\{x\} \times Y$ be a Compact, being $\{x\} \times Y \cong Y$ by Homeomorphism given by Projection.
Then, there is a finite subcover of $\mathcal{O}$ such that

$$\{x\} \times Y \subset \bigcup_{i=1}^{n_x} (U_i^x \times V_i^x)$$

Now, for each $x \in X$, define $U^x \overset{\text{def}}{=} \bigcap_{i=1}^{n_x} U_i^x$. Then, U^x is an open set containing x, and for any $i = 1, 2, \ldots, n_x$, $U^x \subset U_i^x$.
Since $\{U^x \mid x \in X\}$ be an open cover of X, there is a finite subcover such that

$$X = \bigcup_{i=1}^{m} U^{x_i}$$

being X is Compact. Now,

$$X \times Y = \left(\bigcup_{i=1}^{m} U^{x_i} \right) \times Y = \bigcup_{i=1}^{m} (U^{x_i} \times Y) \subset \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_{x_i}} (U_j^{x_i} \times V_j^{x_i})$$

Thus, $\{U_j^{x_i} \times V_j^{x_i} \mid i = 1, 2, \ldots, m, \ \ j = 1, 2, \ldots, n_{x_i}\}$ be a finite subcover. $\square$

Tube Lemma

Let X be a Topological Space, and Y is Compact Space.
Then, for product space $X \times Y$, and fixed $x_0 \in X$, following statement holds:

For any open $N \subset X \times Y$ with $\{x_0\} \times Y \subset N$, there is an open $W \in \mathcal{T}_X$ such that $\{x_0\} \times Y \subset W \times Y \subset N$.

Proof. Clearly, $\{x_0\} \times Y$ compact, being $\{x_0\} \times Y \simeq Y$.
For any $y \in Y$, $(x_0, y) \in \{x_0\} \times Y \subset N$, thus there exist opens $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x_0, y) \in U_y \times V_y \subset N$.
Now, Clearly $\{U_y \times V_y \subset X \times Y \mid y \in Y\}$ be an open cover of $\{x_0\} \times Y$, thus there is a finite subcover such that

$$\{x_0\} \times Y \subset \bigcup_{i=1}^{N} (U_{y_i} \times V_{y_i}) \subset N$$

Set $W = \bigcap_{i=1}^{N} U_{y_i}$. Then, clearly $x_0 \in W$, and

Let $(x, y) \in W \times Y$. Then, since $Y = \bigcup_{i=1}^{n} V_{y_i}$, there is $1 \leq k \leq n$ such that $y \in V_{y_k}$.
Thus, $(x, y) \in U_{y_k} \times V_{y_k} \subset N$, this implies $W \times Y \subset N$. $\square$

> **Theorem 12.6.0.6.** Let Y be a Compact Space. Then, the following statements are true, but their converses are false:
>
> 1. If X be a Lindelöf Space, then the product Topology $X \times Y$ be a Lindel$\ddot{o}$f Space.
>
> 2. If X be a Countable Compact Space, then the product Topology $X \times Y$ be a Countable Compact Space.

Proof. 1. Let $\mathcal{O}$ be an open cover of $X \times Y$.
For any $x \in X$, $\{x\} \times Y$ is compact set, being $\{x\} \times Y \simeq Y$. Thus, there is a finite subcover of $\mathcal{O}$ such that

$$\{x\} \times Y \subset \bigcup_{j=1}^{N_x} U_j^x \quad (U_j^x \in \mathcal{O})$$

Since Tube Lemma, there is an open $W_x \in \mathcal{T}_X$ such that

$$\{x\} \times Y \subset W_x \times Y \subset \bigcup_{j=1}^{N_x} U_j^x$$

Meanwhile, since X is Lindel$\ddot{o}$f, therefore for an open cover $\{W_x \mid x \in X\}$
there exists a Countable subcover such that

$$X \subset \bigcup_{i=1}^{\infty} W_{x_i}$$

Consequently,

$$X \times Y \subset \left(\bigcup_{i=1}^{\infty} W_{x_i}\right) \times Y \subset \bigcup_{i=1}^{\infty} (W_{x_i} \times Y) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_{x_i}} U_j^{x_i}$$

Now, $\left\{U_j^{x_i} \mid i \in \mathbb{N}, 1 \le j \le N_{x_i}\right\} \subset \mathcal{O}$ be a Countable Open Cover of $X \times Y$. □

Proof. 2. Let $\{U_n \subset X \times Y \mid n \in \mathbb{N}\}$ be a Countable open cover of $X \times Y$. For each finite subset $F \subset \mathbb{N}$, define

$$V_F \overset{\text{def}}{=} \left\{ x \in X \;\middle|\; \{x\} \times Y \subset \bigcup_{n \in F} U_n \right\}$$

Then V_F satisfies:
1) V_F is open: Let a finite subset $F \subset \mathbb{N}$ fix. For each $x \in V_F$, $\{x\} \times Y \subset \bigcup_{n \in F} U_n$ by definition.

Then, there is an open $W_x \in \mathcal{T}_X$ such that $\{x\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$ by Tube Lemma.

Meanwhile, $W_x \subset V_F$ because for all $s \in W_x$, $\{s\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$, thus $s \in V_F$.
In summary, for any $x \in V_F$, there is an open $W_x \in \mathcal{T}_X$ such that $x \in W_x \subset V_F$. Consequently, V_F is open of X.
2) $\{V_F \mid F \subset \mathbb{N}, |F| < \infty\}$ is a Countable Open Cover of X:
Countability given by above set is collection of subsets of Countable set. Meanwhile,
For any $x \in X$, there is a finite subcover of $\{U_n \mid n \in \mathbb{N}\}$ such that $\{x\} \times Y \subset \bigcup_{n \in F} U_n$ where F finite.

That is, $x \in V_F$. Now, the open cover of X,

$$\{V_{F_x} \mid x \in X\} \subset \{V_F \mid F \subset \mathbb{N}\}$$

at most Countable. Since X is Countably Compact Space, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^{N} V_{F_i}$$

Consequently,

$$X \times Y \subset \left(\bigcup_{i=1}^{N} V_{F_i}\right) \times Y = \bigcup_{i=1}^{N} (V_{F_i} \times Y) \subset \bigcup_{i=1}^{N} \bigcup_{n \in F_i} U_n$$

That is, $\{U_i \mid i = 1, 2, \ldots, N,\ n \in F_i\}$ be a finite subcover. □

12.6.1 Locally Compact

Definition 12.6.1.1. A Space X is called *Locally Compact* if:

For any $x \in X$, there exist open U and compact C such that $x \in U \subseteq C$.

Lemma 12.6.1.1. Let X be a Hausdorff Space. TFAE:

1. X is Locally-compact space.
2. For any $x \in X$, there exists an open U with $x \in U$ such that the closure $\overline{U}$ is Compact in X.

12.6.2 One-point Compactification

Definition 12.6.2.1. Let $(X,\mathcal{T})$ be a Space.
Define $X_\infty \overset{\text{def}}{=} X \sqcup \{\infty\}$ and $\mathcal{T}_\infty \overset{\text{def}}{=} \mathcal{T} \sqcup \{U \subseteq X_\infty \mid \infty \in U,\ X_\infty \setminus U \text{ is compact in } X\}$.
This $(X_\infty, \mathcal{T}_\infty)$ is called *one-point compactification* of X.

Theorem 12.6.2.1. Let (X,∞) be a Locally-Compact Hausdorff Space, but not Compact.
Then, one-point compactification $(X_\infty, \mathcal{T}_\infty)$ of X is Compact Hausdorff Space.

Proof. This proof consisted of five steps.
1). Claim: $\mathcal{T}_\infty$ is Topology on X_∞. (Using X is Hausdorff)
Let $U_\gamma \in \Gamma,\ (\gamma \in \Gamma)$ be elements of $\mathcal{T}_\infty$.
Define $\Gamma_1 \overset{\text{def}}{=} \{\alpha \in \Gamma \mid U_\alpha \in \mathcal{T}\}$, and $\Gamma_2 \overset{\text{def}}{=} \Gamma \setminus \Gamma_1 = \{\beta \in \Gamma \mid \infty \in U,\ X_\infty \setminus U \text{ is compact in } X\}$.
Then, $\displaystyle\bigcup_{\gamma \in \Gamma} U_\gamma = \left(\bigcup_{\alpha \in \Gamma_1} U_\alpha\right) \cup \left(\bigcup_{\beta \in \Gamma_2} U_\beta\right)$. The left term is open in X clearly.
And, put $C_\beta = X_\infty \setminus U_\beta$ for each $\beta \in \Gamma_2$. Then, C_β is Compact in X by definition, thus closed by X is Hausdorff.

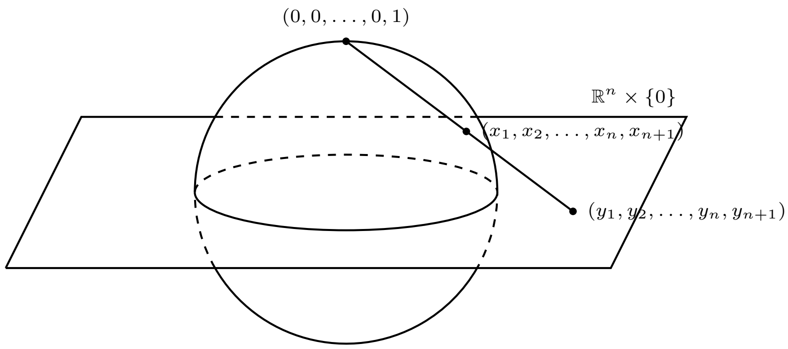
$$\bigcup_{\beta \in \Gamma_2} U_\beta = \bigcup_{\beta \in \Gamma_2} X_\infty \setminus C_\beta = X_\infty \setminus \left(\bigcap_{\beta \in \Gamma_2} C_\beta\right)$$

This intersection of C_β is compact, being any intersection of closed is closed and closed subset of compact.
That is, it is compact in X, therefore this union of U_β is contained in $\mathcal{T}_\infty$.
Let $U_1, U_2 \in \mathcal{T}$, and $V_1, V_2 \in \mathcal{T}_\infty \setminus \mathcal{T}$. Put $C_i \overset{\text{def}}{=} X_\infty \setminus V_i,\ (i = 1,2)$. Then, C_i is compact. Now,

$$\begin{aligned}
&U_1 \cap U_2 \in \mathcal{T} \subset \mathcal{T}_\infty \\
&U_1 \cap V_1 = U_1 \cap (X_\infty \setminus C_1) = U_1 \cap X_\infty \cap C_1^c = U_1 \cap C_1^c = U_1 \setminus C_1 \in \mathcal{T} \subset \mathcal{T}_\infty \\
&V_1 \cap V_2 = (X_\infty \setminus C_1) \cap (X_\infty \setminus C_2) = X_\infty \setminus (C_1 \cap C_2) \in \mathcal{T}_\infty
\end{aligned}$$

Thus closed under the arbitrary union and finite intersection.
2). Claim: $(X,\mathcal{T})$ is a Subspace of $(X_\infty, \mathcal{T}_\infty)$. That is, $\mathcal{T} = \{U \cap X \mid U \in \mathcal{T}_\infty\}$. (Using X is Hausdorff)
The right inclusion is clear: $U \in \mathcal{T} \implies U \in \mathcal{T}_\infty$. Thus $U = X \cap U \in \{U \cap X \mid U \in \mathcal{T}_\infty\}$.
To show the left inclusion: Let $U \in \mathcal{T}_\infty$. If $U \in \mathcal{T}$, then $X \cap U = U \in \mathcal{T}$.
If $U \notin \mathcal{T}$, then $X_\infty \setminus U$ is compact in X. Now, $X \cap U = X \setminus (\underbrace{X_\infty \setminus U}_{\text{compact in } T_2 \implies \text{closed}}) \in \mathcal{T}$.
3). Claim: $\overline{X} = X_\infty$. That is, closure of X is X_∞. (Using X is not compact)
Let $U \in \mathcal{T}_\infty$ with $\infty \in U$. Then, $X_\infty \setminus U$ is compact of X, thus $X_\infty \setminus U \subsetneq X$ because X is not compact.
4). Claim: X_∞ is Compact Space.
Let $\mathcal{O} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of X_∞. Since $\infty \in X_\infty = \displaystyle\bigcup_{\alpha \in \Lambda} U_\alpha$, there is $\alpha_0 \in \Lambda$ such that $\infty \in U_{\alpha_0}$.
$C \overset{\text{def}}{=} X_\infty \setminus U_{\alpha_0}$ is compact in X, thus so in X_∞. And, $C \subseteq \displaystyle\bigcup_{\alpha \in \Lambda \setminus \{\alpha_0\}} U_\alpha$, thus there is finite subcover of C.
Finally, union of finite subcover of C and U_{α_0} is finite subcover of X_∞.
5). Claim: X_∞ is Hausdorff. (Using X is Locally-Compact)
Let $x, y \in X_\infty$. If both x, y are contained X, then there is nothing to prove, being X is hausdorff.
If $x \in X$ and $y = \infty$, then there is open U and compact C of X such that $x \in U \subseteq C$, by Locaaly-Compact.
Now, $x \in U$ and $\infty \in X_\infty \setminus C$, both are open of X_∞ with $U \cap (X_\infty \setminus C) = \emptyset$. □

12.6.3 Stereographic projection



Definition 12.6.3.1. Let $\mathbb{R}^{n+1}$ be a Euclidean Space. Define *Unit Sphere* on $\mathbb{R}^{n+1}$:

$$S^n \overset{\text{def}}{=} \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \cdots + x_n + x_{n+1} = 1\}$$

Theorem 12.6.3.1. For any $p \in S^n$, $S^n \setminus \{p\} \cong \mathbb{R}^n$.

Proof. WLOG, put $p = (0,0,\ldots,0,1)$. Define a map *Stereographic Projection*:

$$f : S^n \setminus \{p\} \to \mathbb{R}^n : (x_1, x_2, \ldots, x_n, x_{n+1}) \mapsto \left(\frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \cdots, \frac{x_n}{1-x_{n+1}}\right)$$

This map derived from:

$$(y_1, y_2, \ldots, y_n, y_{n+1}) = t(\overrightarrow{x} - \overrightarrow{p}) + (0,0,\ldots,0,1) = (x_1 t, x_2 t, \ldots, x_n t, (x_{n+1}-1)t + 1)$$

$y_{n+1} = 0$ when $t = \dfrac{1}{1-x_{n+1}}$. Each $(x_i, x_{n+1}) \mapsto \dfrac{x_i}{1-x_{n+1}}$ is Continuous map, so f is Countinuous map.
And, the map

$$g : \mathbb{R}^n \to S^n \setminus \{p\} : (x_i)_{i=1}^n \mapsto \left(\frac{2x_1}{1+\|(x_i)\|^2}, \frac{2x_2}{1+\|(x_i)\|^2}, \cdots, \frac{2x_n}{1+\|(x_i)\|^2}, 1 - \frac{2}{1+\|(x_i)\|^2}\right)$$

is Continuous, and $g = f^{-1}$. Thus, f is Homeomorphism. $\square$

Theorem 12.6.3.2. One Point Compactification of $\mathbb{R}^n$ is $S^n \subseteq \mathbb{R}^{n+1}$.

Proof. Since $\mathbb{R}$ is Locally-Compact Hausdorff Space, finite product Space $\mathbb{R}^n$ is Locally-Compact Hausdroff Space.
Let $\mathbb{R}^n_\infty = \mathbb{R}^n \cup \{\infty\}$ be a One Point Compactification of $\mathbb{R}^n$. Then, $\mathbb{R}^n_\infty$ is Compact Hausdorff Space.
Meanwhile, put $p = (0,0,\ldots,0,1)$. Since $S^n \setminus \{p\} \cong \mathbb{R}^n$, there exists a Homeomorphism $f : S^n \setminus \{p\} \to \mathbb{R}$. Define

$$\tilde{f} : S^n \to \mathbb{R}^n_\infty : \begin{cases} f(x) & x \neq p \\ \infty & x = p \end{cases}$$

Then, $\tilde{f}$ is Bijective map, and Continuous because: for any open $U \subseteq \mathbb{R}^n_\infty$ containing ∞,

$$\tilde{f}^{-1}[\mathbb{R}^n_\infty \setminus U] = f^{-1}[\mathbb{R}^n \setminus U]$$

Since $\mathbb{R}^n \setminus U$ is Compact and f^{-1} is continuous, $\tilde{f}^{-1}[\mathbb{R}^n_\infty \setminus U] = f^{-1}[\mathbb{R}^n \setminus U]$ is compact in S^n, thus closed.
Moreover, since S^n is Compact and $\mathbb{R}^n_\infty$ is Hausdorff, $\tilde{f}$ is Closed map. Consequently, $\tilde{f}$ is Homeomorphism. $\square$

12.7 Borel Set

Definition 12.7.0.1. Let X be a Topological Space.

1. $F \subseteq X$ is called *F_σ-set* if: F can be represented as countable union of closed sets.
2. $G \subseteq X$ is called *G_δ-set* if: G can be represented as countable intersection of open sets.

Proposition 12.7.0.1. Let X be a Topological Space.

1. If $F \subseteq X$ is F_σ-set, then there exists sequence of closed sets $\{F_n\}_{n\in\mathbb{N}}$ such that
$$F = \bigcup_{n=1}^{\infty} F_n, \quad F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$$
2. If $G \subseteq X$ is G_δ-set, then there exists sequence of open sets $\{G_n\}_{n\in\mathbb{N}}$ such that
$$G = \bigcap_{n=1}^{\infty} G_n, \quad G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq\subseteq \cdots$$
3. Countable union of F_σ-sets is F_σ.
4. Finite intersection of F_σ-sets is F_σ.
5. Countable intersection of G_δ-sets is G_δ.
6. Finite union of G_δ-sets is G_δ.
7. Complement of F_σ-set is G_δ.

12.8 Baire Category

Definition 12.8.0.1. The Topological Space X is called ***Baire Space*** if:

$$\text{If } \{G_n \mid n \in \mathbb{N}\} \text{ be a Countable Collection of dense open sets of } X\text{, then } \overline{\bigcap_{n=1}^{\infty} G_n} = X$$

In brief, every Countable intersection of dense open sets be dense in X.

Definition 12.8.0.2. Let X be a Topological Space.
$A \subset X$ is said to be ***nowhere dense subset*** if $(\overline{A})^{\circ} = \emptyset$.

1. $B \subset X$ is called ***first category*** if B can be representive by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be ***second category***.

12.9 Locally Compact Hausdorff Space

Theorem 12.9.0.1. Locally Compact Hausdorff Space is Baire Space.

12.10 Complete Metric Space

Definition 12.10.0.1. Let (X,d) be a Metric Space, and $\{p_n\}$ be a Sequence in X.
The Sequence $\{p_n\}$ is called ***Cauchy Sequence*** if:

$$\text{For any } \varepsilon > 0\text{, there exists } N \in \mathbb{N} \text{ such that } m, n \geq N \implies d(p_m, p_n) < \varepsilon.$$

A Metric Space (X,d) is said to be ***Complete*** if every Cauchy Sequnces Converge.

Lemma 12.10.0.1. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that
$E_n \supset E_{n+1}$. If $\lim_{n\to\infty} \text{diam} E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.
Let $\varepsilon > 0$ be given. Since $\text{diam} E_n \to 0$, there is $N \in \mathbb{N}$ such that $\text{diam} E_n < \varepsilon$.
For any $m, n \geq M$, E_N contains p_m, p_n. That is, $d(p_m, p_n) < \varepsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X.
Since X is complete, there is a unique point $p \in X$ such taht $p_n \to p$. Let $N \in \mathbb{N}$ be an integer such that $n \geq N \implies |p_n - p| < \varepsilon$.
Now, for each $n \geq N$, E_n has a limit point as p. And for any $n \in \mathbb{N}$, E_n contains $E_N, E_{N+1}, \ldots$, thus for all $n \in \mathbb{N}$, E_n has a limit point as p. Meanwhile, E_n closed, $p \in E_n, \ \forall n \in \mathbb{N}$.
Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, $\text{diam} E_n \geq d(p,q) > 0, \ \forall n \in \mathbb{N}$. ∎

Theorem 12.10.0.1. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.
Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.
Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.
Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U,\ E_2 = B_{\frac{r_1}{2}}(p_1)$.
Suppose that $E_1, \ldots, E_{n-1}$ are chosen. Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.
Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$.
Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

12.10.1 Nowhere Differentiable function

> **Theorem 12.10.1.1.** Let $\mathcal{C}[\mathbb{R}] \stackrel{\text{def}}{=} \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous.}\}$ and $d : \mathcal{C}[\mathbb{R}] \times \mathcal{C}[\mathbb{R}] \to \mathbb{R} : (f, g) \mapsto \sup_{t \in \mathbb{R}} |f(t) - g(t)|$.
> Then, $(\mathcal{C}[\mathbb{R}], d)$ is Complete Metric Space, and set of Nowhere-Differentiable functions is dense in $\mathcal{C}[\mathbb{R}]$.

Proof. First, show that f satisfies triangle inequality: let $f, g, h \in \mathcal{C}[\mathbb{R}]$ be given.
For any $t \in \mathbb{R}$, $|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$. Thus,

$$d(f,g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)| \leq \sup_{t \in \mathbb{R}} [|f(t) - h(t)| + |h(t) - g(t)|] \leq \sup_{t \in \mathbb{R}} |f(t) - h(t)| + \sup_{t \in \mathbb{R}} |h(t) - g(t)| = d(f,h) + d(h,g)$$

□

12.10.2 Banach Fixed Point Theorem

Definition 12.10.2.1. Let $f : X \to X$ be any function. A point $x \in X$ is called a ***fixed point*** of f if $f(x) = x$.

Definition 12.10.2.2. Let X be a Metric Space. A map $f : X \to X$ is called *Contractive* with respect to the metric d if:

$$\text{There exsits } \alpha \in (0,1) \text{ such that for all } x, y \in X,\ d(f(x), f(y)) \le \alpha d(x,y).$$

Theorem 12.10.2.1. *Banach Fixed point Theorem*
Let (X,d) be a Complete Metric Space, and $f : X \to X$ be a Contractive map.
Then, there exists a unique fixed point of f, $x^* \in X$.

Proof. Clearly,

$$\text{Contractive} \implies \text{Lipschitz Condition} \implies \text{Continuous.}$$

Thus, f is Continuous.
Let $x_0 \in X$ be arbitrary, and construct a sequence $\{x_n\}$ recursively as follows:

$$x_{n+1} \overset{\text{def}}{=} f(x_n),\ \ n \ge 0$$

Then, for any $n \ge 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \le \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \le \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\ \ \vdots \\ &\le \alpha^n d(x_1, x_0) \end{aligned}$$

Let $\varepsilon > 0$ be given. Put $N \in \mathbb{N}$ such that $\alpha^N \cdot d(x_1, x_0) < \varepsilon(1-\alpha)$. Then, $n \ge m \ge N$ implies that

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\le \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \varepsilon(1-\alpha)\frac{1}{1-\alpha} = \varepsilon \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since X is Complete, for some $x^* \in X$, $\lim_{n\to\infty} x_n = x^*$. Consequently,

$$\lim_{n\to\infty} f(x_n) \overset{f \text{ conti.}}{=} f\left(\lim_{n\to\infty} x_n\right) = f(x^*) = \lim_{n\to\infty} x_{n+1} = x^*$$

□

12.11 Maps in Metric Space

In this section, (X, d_X) and (Y, d_Y) are metric spaces.

12.11.1 Metric

Definition 12.11.1.1. A ***metric*** on a set X is a map $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

We call (X, d) a ***metric space***.

Theorem 12.11.1.1. The map $d : X \times X \to \mathbb{R}$ is continuous.

Proof. Let $(x, y) \in X \times X$ and $\varepsilon > 0$. For $U = B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y)$ and any $(p, q) \in U$,

$$\begin{aligned} d(p,q) &\leq d(p,x) + d(x,y) + d(y,q) < d(x,y) + \varepsilon \\ d(x,y) &\leq d(x,p) + d(p,q) + d(q,y) < d(p,q) + \varepsilon \end{aligned}$$

so $|d(p, q) - d(x, y)| < \varepsilon$. □

12.11.2 Diameter

Definition 12.11.2.1. For $E \subseteq X$, the ***diameter*** is

$$\operatorname{diam} E \overset{\text{def}}{=} \sup_{x,y \in E} d(x, y).$$

Theorem 12.11.2.1. For any $E \subseteq X$, $\operatorname{diam} E = \operatorname{diam} \overline{E}$.

Proof. Clearly, $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$. Let $\varepsilon > 0$ be given. Then, there exist $a, b \in \overline{E}$ such that

$$\operatorname{diam} \overline{E} - \frac{\varepsilon}{2} \leq d(a, b) < \operatorname{diam} \overline{E}$$

Meanwhile, $a, b \in \overline{E}$ implies: $B_{\frac{\varepsilon}{2}}(a) \cap E \neq \emptyset$, $B_{\frac{\varepsilon}{2}}(b) \cap E \neq \emptyset$.
Put $p \in B_{\frac{\varepsilon}{2}}(a) \cap E$ and $q \in B_{\frac{\varepsilon}{2}}(b) \cap E$. Now, the triangle inequality gives

$$\operatorname{diam} \overline{E} - \frac{\varepsilon}{2} \leq d(a, b) \leq d(a, p) + d(p, q) + d(q, b) \leq \frac{\varepsilon}{2} + \operatorname{diam} E + \frac{\varepsilon}{2} = \operatorname{diam} E + \varepsilon$$

Since ε is chosen arbitrarily, $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$. □

12.11.3 Distance

Definition 12.11.3.1. For nonempty $E \subseteq X$, define $\rho_E : X \to [0, \infty)$ by

$$\rho_E(x) \overset{\text{def}}{=} \inf_{t \in E} d(x, t).$$

Proposition 12.11.3.1. *For all $x \in X$, $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.*

Proof.

$$\rho_E(x) = 0 \overset{\text{by def.}}{\iff} \inf_{t \in E} d(x, t) = 0 \iff \forall \varepsilon > 0,\ \exists p \in E \text{ s.t. } 0 < d(x, p) \leq \varepsilon \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap E \neq \emptyset$$

□

Theorem 12.11.3.1. The distance ρ_E satisfies *Lipschitz Condition*. Furthermore, *Uniformly Continuous*.

Proof. Let $x, y \in X$ be given. Then, for any $z \in E$,

$$\rho_E(x) = \inf_{t \in E} d(x, t) \leq d(x, z) \leq d(x, y) + d(y, z)$$

Since $z \in E$ given arbitrarily,

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

Thus $\rho_E(x) - \rho_E(y) \leq d(x, y)$. Similrarly, $\rho_E(y) - \rho_E(x) \leq d(x, y)$. That is, For any $x, y \in X$, $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$. Now, for any $\varepsilon > 0$, put $\delta = \varepsilon$. Then,

$$d(x, y) < \delta \implies |\rho_E(x) - \rho_E(y)| \leq d(x, y) < \delta = \varepsilon$$

□

Theorem 12.11.3.2. Let $C \subseteq X$ be compact, $F \subseteq X$ closed, and $C \cap F = \emptyset$. Then there exists $\delta > 0$ such that

$$d(p, q) \geq \delta \quad \text{for all } p \in C,\ q \in F.$$

Proof. □

12.11.4 Isometry

Definition 12.11.4.1. An onto map $f : (X, d_X) \to (Y, d_Y)$ is an *isometry* if: for all $x, y \in X$,

$$d_X(x, y) = d_Y(f(x), f(y))$$

12.12 Separation Axioms

12.13 Urysohn Metrization Theorem

12.13.1 Urysohn Lemma

Recall that:

Definition 12.13.1.1. X is T_4 if: For any disjoint closed set A and B, there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 12.13.1.1. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\overline{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.
Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C. Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C. By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.
Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U, this implies that $C \subset O \subset U$.
Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \overline{O} \subset O'^c \subset U$. □

Definition 12.13.1.2. Let X be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.
A real-valued Continuous map $f : X \to [a, b]$ is called *Urysohn function* for A and B if: $f|_A = a$ and $f|_B = b$.
In another form,

$$f : X \to [a, b] : x \to \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 12.13.1.2. *Urysohn Lemma*
T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.
Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.
Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.
Consider a set of *Dyadic Rationals* $D \overset{\text{def}}{=} \left\{ \frac{k}{2^n} \,\middle|\, n, k \in \mathbb{N},\ k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

$$\text{For any } r, s \in D \text{ with } r < s, \text{ there exist open sets } U_r, U_s \text{ such that } A \subseteq \overline{U}_r \subseteq U_s \subseteq X \setminus B \qquad (*)$$

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is *Noetherian*. That is, X satisfies Ascending Chain Condition for open sets.)
Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \overline{U_1} \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.
Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \overset{*1}{\subseteq} \underset{\text{open}}{U_{\frac{1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{1}{2^k}}} \overset{*2}{\subseteq} \underset{\text{open}}{U_{\frac{2}{2^k}}} \subseteq \cdots \overset{*2^k-1}{\subseteq} \underset{\text{open}}{U_{\frac{2^k-1}{2^k}}} \subseteq \underset{\text{closed}}{\overline{U}_{\frac{2^k-1}{2^k}}} \overset{*2^k}{\subseteq} \underset{\text{open}}{X \setminus B}$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \ldots, *2^k$, we can construct 2^k open sets such that:

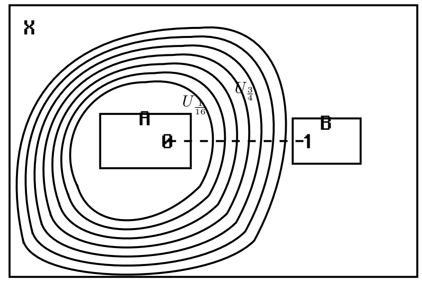
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \quad \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \to [0,1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap\limits_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap\limits_{t \in D} U_t \end{cases}$$



Then, this map f is well-defined by $(*)$ and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D,\ x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D,\ x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$.
   (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarlily, and $\varepsilon > 0$ be given.
In Case of $0 < f(x) < 1$.
Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$.
Now, we obtain that:

$$x \overset{(*)}{\in} U_s \setminus \overline{U}_r \overset{(**)}{\subseteq} f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$$

$(*)$ directly given by above properties, $(**)$ given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.
In Case of $f(x) = 0$.
Choose $r \in D$ such that $f(x) = 0 < r < \varepsilon = f(x) + \varepsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \varepsilon)]$$

In Case of $f(x) = 1$.
Choose $r \in D$ such that $f(x) - \varepsilon = 1 - \varepsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \varepsilon, f(x))]$$

Consequently, f is Continuous map on $[0,1]$ such that $f|_A = 0$ and $f|_B = 1$.

Step 3. Generalization.

Since $[0,1] \cong [a,b]$ for any $a < b$, let $g : [0,1] \to [a,b] : x \mapsto (1-x)a + xb$ be a Homeomorphism.
Then, $h = g \circ f : X \to [a,b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. $\square$

12.13.2 Tietze Extension Theorem

Theorem 12.13.2.1. *Tietze Extension Theroem*
Let X be a T_4 Space, and $A \subseteq X$ be a closed subset.
For any Continuous map $f : A \to \mathbb{R}$, there exists a Continuous map:

$$g : X \to \mathbb{R} \quad \textbf{s.t.} \quad g|_A = f$$

This g is called ***extension*** of f.

Proof. This proof consists by three steps.
Step 1. First, we will show that:

$$\text{For any Continuous map } f : A \to [-r, r]\text{, there is a Continuous map } h : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, & |f(a) - h(a)| \leq \frac{2}{3}r \end{cases} \tag{$*$}$$

Set

$$I_1 \overset{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \ I_2 \overset{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \ I_3 \overset{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and A is closed subspace of X, $f^{-1}[I_1]$ and $f^{-1}[I_3]$ are closed of X.
And, I_1 and I_3 are disjoint, thus $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$.
Now, apply the ***Urysohn Lemma***: There exists an Urysohn function $h : X \to I_2$ for $f^{-1}[I_1]$ and $f^{-1}[I_3]$.
Clearly, this map h satisfies the first condition in $(*)$. And, for show the second condition, let $a \in A$ be given.
If $a \in f^{-1}[I_1]$, then $f(a) \in I_1$ and $h(a) = -\frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \in f^{-1}[I_3]$, then $f(a) \in I_3$ and $h(a) = \frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
If $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$, then $f(a), h(a) \in I_2$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.
Therefore, the second condition satisfied.
Step 2. We will show that: for any $f : A \to [-1, 1]$, there exists an extension of f.
Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, & |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of h_1, the continuous map $f - h_1 : A \to \left[-\frac{2}{3}, \frac{2}{3}\right] : x \to f(x) - h_1(x)$ is well-defined.
Again, there exists a Continuous map:

$$h_2 : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, & |f(a) - h_1(a) - h_2(a)| \leq \left(\frac{2}{3}\right)^2 \end{cases}$$

Inductively, for any $n \in \mathbb{N}$, there exists a Continuous map:

$$h_n : X \to \mathbb{R} \textbf{ s.t. } \begin{cases} \forall x \in X, & |h_n(x)| \leq \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} \\ \forall a \in A, & |f(a) - h_1(a) - h_2(a) - \cdots - h_n(a)| \leq \left(\frac{2}{3}\right)^n \end{cases}$$

Define a map

$$g : X \to [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any $x \in X$,

$$|g(x)| = \left|\sum_{n=1}^{\infty} h_n(x)\right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, ***Weierstrass M-test*** gives that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly.
Moreover, for any $a \in A$,

$$\left|f(a) - \sum_{k=1}^{n} h_k(a)\right| \leq \left(\frac{2}{3}\right)^n \implies \left|f(a) - \sum_{n=1}^{\infty} h_n(a)\right| = |f(a) - g(a)| = 0$$

That is, g is Continuous on X and $g|_A = f$. Therefore, g is extension of f.
Step 3. Finally, we generalize the result in Step 2.:
Let $f : A \to [a,b]$ be a Continuous map on the closed subspace A. And, let $\varphi : [a,b] \to [-1,1]$ be a Homeomorphism.
Then, $\varphi \circ f : A \to [-1,1]$ is Continuous map, thus there exists an extension $g : X \to [-1,1]$ such that $g|_A = \varphi \circ f$.
Now, $\varphi^{-1} \circ g : X \to [a,b]$ is Continuous, and $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$, Therefore this $\varphi^{-1} \circ g$ is the extension of f.
Let $f : A \to \mathbb{R}$ be a Continuous map on the closed subspace A.
And, let $\varphi : \mathbb{R} \to (-1,1)$ be a Homeomorphism. Then, the map $\phi : \mathbb{R} \to [-1,1] : x \mapsto \varphi(x)$ is still Continuous.
Now, The Continuous map $\phi \circ f : A \to [-1,1]$ has an extension $g : X \to [-1,1]$ such that $g|_A = \phi \circ f$.
Put $B = g^{-1}[\{-1,1\}]$. Then B is Closed on X, and $A \cap B = \emptyset$. Now, apply the Urysohn Lemma to this,
there exists an Urysohn function for A and B: Continuous map $\gamma : X \to [0,1]$ such that $\gamma|_A = 1$ and $\gamma|_B = 0$.
Define a map $\eta : X \to (-1,1) : x \mapsto g(x)\gamma(x)$. Then, if $g(x) = 1$ or $g(x) = -1$, then $x \in B$, thus $g(x)\gamma(x) = 0$.
Therfore, η is well-defined. And, for any $a \in A$, $\eta(a) = g(a)\gamma(a) = g(a)$, thus $\eta|_A = \phi \circ f$.
Consequently, the map $\phi^{-1} \circ \eta$ is an extension of f, we wanted. □

Recall that:

Definition 12.13.2.1. X is T_1 if:
For any distinct $x, y \in X$, there exist open sets U_x, U_y such that $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$.

Lemma 12.13.2.1. X is T_1 if and only if For any $x \in X$, a singleton $\{x\}$ is closed in X.

Proof. The left direction is clear.
Let $x \in X$. Then, for any $y \in X$ with $y \neq x$, T_1 condition gives that there is an open set such that $y \in U_y$ and $x \notin U_y$.
Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. □

12.13.3 Urysohn Metrization Theorem

Definition 12.13.3.1. A space X is called ***Completely Regular*** if: X is T_1 and $T_{3\frac{1}{2}}$ where

$T_{3\frac{1}{2}}$ Condition: For any closed set $C \subset X$ and $x \in X \setminus C$, there exists an ***Urysohn funtion*** for $\{x\}$ and C.

Completely regular space is sometimes called ***Tychonoff Space***.

Proposition 12.13.3.1. Normal Space $\implies$ Completely Regular Space $\implies$ Regular Space.

Proof. If X is Normal space, then every singletone is closed by T_1. And, the ***Urysohn Lemma*** gives Urysohn map. If X is Completely Regular, then for closed $C \subset X$ and $x \in X \setminus C$, there exists a continuous map $f : X \to [0,1]$ s.t

$$f[\{x\}] = 0 \text{ and } f[C] = \{1\}$$

Then,

$$\{x\} \subseteq f^{-1}\left[\left[0, \frac{1}{2}\right)\right], \ C \subseteq f^{-1}\left[\left(\frac{1}{2}, 1\right]\right]$$

□

Theorem 12.13.3.1. $T_{3\frac{1}{2}}$ is Hereditary. Furthermore, ***Completely Regular*** is hereditary since T_1 is hereditary.

Proof. Let X be a $T_{3\frac{1}{2}}$ Space, and $Y \subseteq X$ be a subspace of X.
Let $C \subseteq Y$ is closed set of Y, and $x \in Y \setminus C$. Note that:

$$C = \text{Closure of } C \text{ in } Y = \bigcap_{\substack{F \text{ closed in } Y \\ C \subseteq F}} F = \bigcap_{\substack{F' \text{ closed in } X \\ \text{s.t. } F = F' \cap Y}} F' \cap Y = (\text{Closure of } C \text{ in } X) \cap Y$$

Since x is contained in Y but not C, thus x is not contained in Closure of C in X. Now, since X is $T_{3\frac{1}{2}}$,

$$\text{There exists a Continuous map } f : X \to [0,1] \text{ s.t. } f(x) = 0, \ f|_{\mathrm{cl}_X(C)} = 1$$

The restriction f_Y is continuous, and Urysohn function for x and C. □

Theorem 12.13.3.2. Arbitrary product space of $T_{3\frac{1}{2}}$ space is $T_{3\frac{1}{2}}$.

Proof. Let $X_\gamma \ (\gamma \in \Gamma)$ be $T_{3\frac{1}{2}}$ Spaces. Put $X = \prod_{\gamma \in \Gamma} X_\gamma$. Suppose that $C \subset X$ is closed set, and $x \in X \setminus C$.
Since $X \setminus C$ is open, there exists an open U in X such that $x \in U \subset X \setminus C$.
Put $F = \{\alpha \in \Gamma \mid X_\alpha \neq \pi_\alpha[U]\}$. By definition of product space, this F is a finite index set. Note that:

$$\forall \alpha \in F, \ \pi_\alpha(x) \notin X_\alpha \setminus \pi_\alpha[U]$$

And, for each $\alpha \in F$, $X_\alpha \setminus \pi_\alpha[U]$ are non-empty closed set in X_α, there exist continuous maps f_α such that

$$f_\alpha : X_\alpha \to [0,1], \ f_\alpha|_{X \setminus \pi_\alpha[U]} = 0, \ f_\alpha|_{\pi_\alpha(x)} = 1$$

And, the composition $f_\alpha \circ \pi_\alpha \ (\alpha \in F)$ is continuous, and

$$(f_\alpha \circ \pi_\alpha)[X \setminus \pi_\alpha^{-1}[\pi_\alpha[U]]] = (f_\alpha \circ \pi_\alpha)[\pi_\alpha^{-1}[X_\alpha \setminus \pi_\alpha[U]]] \subseteq f_\alpha[X_\alpha \setminus \pi_\alpha[U]] = \{0\}$$

Now, the map

$$\Psi : X \to [0,1] : t \mapsto \prod_{\alpha \in F} (f_\alpha \circ \pi_\alpha)(t)$$

is Continuous, and $\Psi(x) = 1$ and $\Psi[C] \subseteq \Psi[X \setminus U] = \{0\}$. □

Theorem 12.13.3.3. If X is *Completely Regular*, then for some index set Λ, X can be embedded in $[0,1]^{\Lambda}$.

Proof. Denote that:

$$\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \to [0,1] \mid f \text{ is continous}\}$$

Claim: the following function is embedding X into $[0,1]^{\Lambda}$.

$$F : X \to [0,1]^{\Lambda} : x \mapsto (f_\alpha(x))_{\alpha\in\Lambda}$$

1. F is Continuous, since each f_α is Continuous.
2. F is injective: Let $x \neq y$ in X. Then, $\{x\}$ and $\{y\}$ are closed by T_1.
   By $T_{3\frac{1}{2}}$, there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$.
   Since $f = f_\beta$ for some $\beta \in \Lambda$, $F(x) \neq F(y)$.
3. F is Open map: Let $U \subseteq X$ be an open set, and let $y \in F[U]$. That is, for some $x \in U$, $F(x) = y$.
   Since $X \setminus U$ is closed, $x \notin X \setminus U$, and $T_{3\frac{1}{2}}$, there exists a continuous map

   $$f_\alpha : X \to [0,1] \text{ s.t. } f_\alpha(x) = 0,\ \ f_\alpha|_{X\setminus U} = 1$$

   Meanwhile, put $V \overset{\text{def}}{=} \pi_\alpha^{-1}[[0,1)] \subseteq [0,1]^{\Lambda}$, and $W \overset{\text{def}}{=} V \cap F[X]$. Then, W is open in the subspace $F[X]$.
   And, $\pi_\alpha(y) = \pi_\alpha(F(x)) = (\pi_\alpha \circ F)(x) = f_\alpha(x) = 0$, thus $y \in W$. Now, there remains to prove: $W \subseteq F[U]$.
   Let $z \in W$. Then, $z \in V$ and $z = F(x)$ for some $x \in X$, this implies $\pi_\alpha(z) \in [0,1)$, i.e., $\pi_\alpha(F(x)) = f_\alpha(x) \neq 1$.
   Now, $x \in U$, that is $F(x) = z \in F[U]$. Thus, $W \subseteq F[U]$, consequently F is embedding.

□

Corollary 12.13.3.1. Let X is a Topological Space. TFAE:

a) X is *Completely Regular* Space.

b) X can be embedded in Compact Hausdorff Space.

c) X can be embedded in Normal Space.

Proof.

1. a) $\implies$ b). X can be embedded in $[0,1]^{\Lambda}$. And $[0,1]$ is Compact Hausdorff.
2. b) $\implies$ c). Every Compact Hausdorff Space is Normal.
3. c) $\implies$ a). Normal Space is Completely Regular, and Completely Regular is hereditary.

□

Lemma 12.13.3.1. Every Compact Hausdorff Space is Normal.

Proof. Let X be a Compact Hausdorff Space, and $C, D \subset X$ be disjoint closed subsets.
Since X is Compact, C and D are Compact. Fix $x \in C$. Then, for any $y \in D$,

There exist disjoint opens U_y, V_y such that $x \in U_y$ and $y \in V_y$.

Since $\{V_y \mid y \in D\}$ is open cover of D, there is a finite subcover $\{V_y^i \mid 1 \le i \le n\}$. That is, $D \subseteq \bigcup_{i=1}^{n} V_y^i$.

Now, $\bigcap_{i=1}^{n} U_y^i$ is open set containing x, and

$$\left(\bigcup_{i=1}^{n} V_y^i\right) \cap \left(\bigcap_{i=1}^{n} U_y^i\right) = \bigcup_{i=1}^{n}\left(V_y^i \cap \left(\bigcap_{i=1}^{n} U_y^i\right)\right) = \bigcup_{i=1}^{n} \emptyset = \emptyset$$

In summary, for any $x \in C$, there exist disjoint open U_x, V_x such that $x \in U_x$ and $D \subset V_x$.
Using this, Let $\{U_x \mid x \in C\}$ be an open cover, then compactness gives the finite subcover $\{U_x^i \mid 1 \le i \le n\}$. Now,

$$C \subseteq \bigcup_{i=1}^{n} U_x^i,\ \ D \subseteq \bigcap_{i=1}^{n} V_x^i,\ \ \left(\bigcup_{i=1}^{n} U_x^i\right) \cap \left(\bigcap_{i=1}^{n} V_x^i\right) = \bigcup_{i=1}^{n}\left(U_x^i \cap \left(\bigcap_{i=1}^{n} V_x^i\right)\right) = \emptyset$$

□

Theorem 12.13.3.4. *Embedding Theorem*
Let X be a T_1 Space. Denote $\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$.
Suppoe that for any $x \in X$ and open neighborhood U of x, there exists $\alpha \in \Lambda$ such that $f_\alpha(x) > 0,\ f_\alpha|_{X \setminus U} = 0$.
Then, the map $F : X \to \mathbb{R}^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$ is embedding.

Theorem 12.13.3.5. Suppose that X is Second-Countable Regular Space.
Then there exists a Countable collection $\{f_n : X \to [0,1] \mid n \in \mathbb{N}\}$ such that:

For any open $U \subset X$ and $x \in U$, there exists $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n|_{X \setminus U} = 0$.

Theorem 12.13.3.6. *Urysohn Metrization Theroem*
If X is a Second-Countable Regular Space, then X is Metrizable.

12.14 Examples

Proposition 12.14.0.1. Lower Limit Topology $(\mathbb{R}, \mathcal{T}_l)$ is T_1 and T_4 Space. Therefore, *Normal Space*.

Proof. T_1 is clear, because: let $x, y \in \mathbb{R}$ be a distinct two points. Without Loss of Generality, assume $x < y$. Then,

$$\begin{cases} x \in \left[x, \dfrac{x+y}{2}\right), & y \in [y, y+1) \\ \\ y \notin \left[x, \dfrac{x+y}{2}\right), & x \notin [y, y+1) \end{cases}$$

Thus, T_1 satisfied. And, to show T_4, Let $C, D \subseteq \mathbb{R}$ be disjoint closed subsets.
Let $x \in C$ be given. Then, there exists a basis element of $\mathcal{T}_l$, $[a, p_x)$ such that

$$x \in [a, p_x) \subseteq \mathbb{R} \setminus D$$

since $C \subseteq \mathbb{R} \setminus D$ and $\mathbb{R} \setminus D$ is open. Now,

$$U = \bigcup_{x \in C} [x, p_x)$$

is open set containing C, and $U \cap D = \emptyset$.
Similarly, let $y \in D$ be given. Then, there exists a basis element $[a, q_y)$ such that

$$y \in [b, q_y) \subseteq \mathbb{R} \setminus C$$

Then, for each $y \in D$, $[y, q_y) \cap U = \emptyset$ because: Suppose that $[y, q_y) \cap U \neq \emptyset$. Choose $p \in [y, q_y) \cap U$.
That is, $p \in [y, q_y)$ and for some $x \in C$, $p \in [x, p_x)$.
Hence, $[\max(x, y), \min(p_x, q_y))$ is non-empty set which containing x or y.
If either x or y contained in $[\max(x, y), \min(p_x, q_y))$, contradiction. Now, an union

$$V = \bigcup_{y \in D} [y, q_y)$$

is open set containing D, and $U \cap V = \emptyset$, $\square$

12.15 Quotient Space

Definition 12.15.0.1. Let $(X, \mathcal{T})$ be a Topological Space, Y be a set, and $f : X \to Y$ be an onto map.
Define *Quotient Toplogy on Y induced by f*: $\mathcal{T}_Q \overset{\text{def}}{=} \{U \subseteq Y \mid f^{-1}[U] \in \mathcal{T}\}$.
This is the largest topology on Y such that f is Continuous map.

Definition 12.15.0.2. Let X be a Topological Space, and $\sim$ be an equivalent relation on X.
Define *Canonical map on X*: $\pi : X \to X/_\sim : x \mapsto [x]$, and
define *Quotient Space* $(X/_\sim, \mathcal{T}_Q)$ where $\mathcal{T}_Q$ is quotient topology on $X/_\sim$ induced by π.

X Topological Space, $\sim$ equivalent relation on X, $\pi : X \to X/_\sim : x \mapsto [x]$ canonical map.

Lemma 12.15.0.1. For any topological space Z and a map $g : X/_\sim \to Z$,

$$g \text{ is Continuous } \textit{if and only if } g \circ \pi \text{ is Continuous.}$$

Proof. Let $g \circ \pi$ be Continuous map. Then, for any open $U \subseteq Z$,

$$(g \circ \pi)^{-1}[U] = \pi^{-1}[g^{-1}[U]]$$

is open, thus $g^{-1}[U]$ is open in $X/_\sim$. That is, g is Continuous. $\square$

Lemma 12.15.0.2. Let Z be a Topological space.
If given Continuous map $f : X \to Z$ satisfies $x \sim y \implies f(x) = f(y)$,
then $\tilde{f} : X/_\sim \to Z : [x] \mapsto f(x)$ is Continuous, and unique map such that $\tilde{f} \circ \pi = f$.

Proof. Well-Defined because: $[x] = [y] \iff x \sim y \implies f(x) = f(y)$.
$\tilde{f} \circ \pi = f$: for any $x \in X$, $(\tilde{f} \circ \pi)(x) = \tilde{f}(\pi(x)) = \tilde{f}([x]) = f(x)$, thus $\tilde{f}$ is continuous since above lemma.
Uniqueness: if $g : X/_\sim \to Z$ satisfies $g \circ \pi = f$, then for any $[x] \in X/_\sim$,

$$g([x]) = g(\pi(x)) = (g \circ \pi)(x) = f(x) = \tilde{f}([x])$$

$\square$

Lemma 12.15.0.3. Let Z be a Topological space.
If given Continuous onto map $f : X \to Z$ satisfies $x \sim y \iff f(x) = f(y)$, and f is either open or closed map,
then $\tilde{f} : X/_\sim \to Z : [x] \mapsto f(x)$ is Homeomorphism.

Proof. Since $[x] = [y] \iff x \sim y \iff f(x) = f(y)$, $\tilde{f}$ is Well-defined and injective.
Continuousness gave by above Lemma, and Surjective:

$$\tilde{f}[X/_\sim] = \tilde{f}[\pi[X]] = f[X] = Z$$

The last equality given by Surjectivenss of f. $\square$

12.16 Quotient Map

Definition 12.16.0.1. Let X, Y be Toplogical Space.
A Continuous onto map $f : X \to Y$ is called ***quotient map*** if:

$$U \subseteq Y \text{ is open } \textbf{\textit{if and only if}} \ f^{-1}[U] \subseteq X \text{ is open.}$$

12.16.1 Basic Properties

Proposition 12.16.1.1. Composition of quotient maps is quotient map.

Proof. Suppose that X, Y, Z are Topological Space, and $f : X \to Y$, $g : Y \to Z$ are Quotient map. Then,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$$U \subseteq Z \text{ is open} \iff g^{-1}[U] \subseteq Y \text{ is open} \iff f^{-1}[g^{-1}[U]] \text{ is open}$$

implies $g \circ f : X \to Z$ is Quotient map, being $f^{-1}[g^{-1}[U]] = (g \circ f)^{-1}[U]$. □

Proposition 12.16.1.2. Continuous onto map is quotient map if either open or closed map.

Proof. Suppose that $f : X \to Y$ is Continuous onto map.
If f is an Open map, □

Theorem 12.16.1.1. If $f : X \to Y$ is quotient map, then $X/_{\sim} \cong Y$ where

$$x \sim y \iff f(x) = f(y)$$

Moreover, if continuous map $g : X \to Z$ satisfies

$$f(x) = f(y) \implies g(x) = g(y)$$

Then, $\tilde{g} : Y \to Z : f(x) \mapsto g(x)$ is the unique continuous map such that $\tilde{g} \circ f = g$.

12.16.2 Quotient map Diagram

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[drr, bend left, "g"] \arrow[dd, "\pi"'] & & \\ & Y \arrow[r, "\tilde{g}"] & Z \\ X/\sim \arrow[ur, "\tilde{f}"'] \arrow[urr, bend right, "h"'] & & \end{tikzcd}$$

- $f : X \to Y$ is quotient map.
- $g : X \to Z$ is continuous map s.t $f(x) = f(y) \implies g(x) = g(y)$.
- $\pi : X \to X/_{\sim} : x \mapsto [x]$.
- $X/_{\sim}$ is quotient topology induced by π.

In this setting, $\tilde{f}$ is Homeomorphism between $X/_{\sim}$ and Y, and $h = \tilde{g} \circ \tilde{f}$ is Continuous map between $X/_{\sim}$ and Z.

12.17 Typical Quotient Spaces

We follow some convention:
'Equivalent relation on X: Condition' means the smallest Equivalent relation on X such that the Condition.
In this section, put $I \stackrel{\text{def}}{=} [0,1] \subset \mathbb{R}$.

12.17.1 Cylinder

Definition 12.17.1.1. Define an Equivalent Relation on $I^2 = I \times I$:

$$\forall y \in I, \ \ (0, y) \sim (1, y)$$

That is,

$$\sim \stackrel{\text{def}}{=} \{((x,y),(x,y)) \mid x, y \in I\} \cup \{((0,y),(1,y)),((1,y),(0,y)) \mid y \in I\}$$

The Quotient Space $I^2/_\sim$ is called ***Cylinder***.

Theorem 12.17.1.1. The Cylinder $I^2/_\sim$ is Homeomorphic to $S^1 \times I$.

Proof. Define a map:

$$f : I^2 \to \underbrace{S^1 \times I}_{\subset \mathbb{R}^3} : (x,y) \mapsto (\underbrace{(\cos(2\pi x), \sin(2\pi x))}_{\in S^1}, y)$$

f is Continuous: Since $x \mapsto \cos(2\pi x)$, $x \mapsto \sin(2\pi x)$, and $y \mapsto y$ are all Continuous.
f is Surjective, clearly. Thus, the f is Continuous onto map.
f is Closed map: I^2 is Compact, being Closed Bounded subset of $\mathbb{R}^2$, and $S^1 \times I$ is Hausdorff, Hereditary Property.
Finally, Check that Fact:

$$\begin{aligned}(x_1,y_1) \sim (x_2,y_2) &\iff (x_1,y_1) = (x_2,y_2) \text{ or } x_1 = 0, \ x_2 = 1, \ y_1 = y_2 \text{ or } x_1 = 1, \ x_2 = 0, \ y_1 = y_2 \\ &\iff f((x_1,y_1)) = f((x_2,y_2))\end{aligned}$$

Now, The map $\tilde{f} : I^2/_\sim \to S^1 \times I : [(x,y)] \mapsto f((x,y))$ induced by f is Homeomorpshim, $I^2/_\sim \cong S^1 \times I$. □

12.17.2 Möbius band

Definition 12.17.2.1. Define an Equivalent Relation on I^2:

$$\forall y \in I, \ \ (0, y) \sim (1, 1-y)$$

The Quotient Space $I^2/_\sim$ is called ***Möbius band***.

12.17.3 Torus

Definition 12.17.3.1. Define an Equivalent Relation on I^2:

$$\begin{cases} \forall x \in I, \ \ (x,0) \sim (x,1) \\ \forall y \in I, \ \ (0,y) \sim (1,y) \\ (0,0) \sim (1,0) \sim (0,1) \sim (1,1) \end{cases}$$

The Quotient Space $I^2/_\sim$ is called *Torus*.

Theorem 12.17.3.1. The Torus $I^2/_\sim$ is Homeomorphic to $S^1 \times S^1 \subset \mathbb{C}^2$.

Proof. Define a map:

$$f : \underbrace{I^2}_{\text{Compact}} \to \underbrace{S^1 \times S^1}_{\text{Hausdorff}} : (x,y) \mapsto (e^{2x\pi i}, e^{2y\pi i})$$

Then, f is Continuous Onto, and Closed map. Moreover,

$$(x_1,y_1) \sim (x_2,y_2) \iff (e^{2x_1\pi i}, e^{2y_1\pi i}) = (e^{2x_2\pi i}, e^{2y_2\pi i}) \iff f((x_1,y_1)) = f((x_2,y_2))$$

Now, The map $\tilde{f} : I^2/_\sim \to S^1 \times S^1 : [(x,y)] \mapsto f((x,y))$ induced by f is Homeomorpshim, $I^2/_\sim \cong S^1 \times S^1$. $\square$

Theorem 12.17.3.2. The Torus $I^2/_\sim$ can embed into $\mathbb{R}^3$.

Proof. Define a map:

$$g : I^2 \to \mathbb{C} \times \mathbb{R} : (x,y) \mapsto ([2 + \cos(2\pi x) \cdot e^{2y\pi i}, \sin(2\pi x)])$$

Then, g is Continuous map. Moreover,

$$\begin{aligned} g((x_1,y_1)) = g((x_2,y_2)) &\iff ([2 + \cos(2\pi x_1) \cdot e^{2y_1\pi i}, \sin(2\pi x_1)]) = ([2 + \cos(2\pi x_2) \cdot e^{2y_2\pi i}, \sin(2\pi x_2)]) \\ &\iff (x_1 = x_2 \text{ or } x_1, x_2 \in \{0,1\}) \text{ and } (y_1 = y_2 \text{ or } y_1, y_2 \in \{0,1\}) \\ &\iff (x_1,y_1) \sim (x_2,y_2) \end{aligned}$$

Meanwhile, $g[I^2] \subset \mathbb{C} \times \mathbb{R}$ is Hausdorff, thus the map $G : I^2 \to g[I^2] : (x,y) \mapsto g((x,y))$ is Continuous Onto Closed map. Now, the map $\tilde{G} : I^2/_\sim \to \underbrace{g[I^2]}_{\subset \mathbb{C} \times \mathbb{R}}$ is Embedding. $\square$

12.17.4 Klein Bottle

Definition 12.17.4.1. Define an Equivalent Relation on I^2:

$$\begin{cases} \forall x \in I, \ \ (x,0) \sim (x,1) \\ \forall y \in I, \ \ (0,y) \sim (1,1-y) \\ (0,0) \sim (1,0) \sim (0,1) \sim (1,1) \end{cases}$$

The Quotient Space $I^2/_\sim$ is called *Klein Bottle*.

Theorem 12.17.4.1. The Klein Bottle $I^2/_\sim$ can embed into $\mathbb{R}^4$.

Proof. Describes Sketch: Define a map:

$$g : I^2 \to \mathbb{C}^2 : (x,y) \mapsto ([2 + \cos(2\pi x)] \cdot e^{2y\pi i}, \sin(2\pi x) \cdot e^{2y\pi i})$$

Then, $\tilde{g}$ becomes embbeding. $\square$

12.17.5 Real Projective Space

Definition 12.17.5.1. Define an Equivalent Relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$:

$$x \sim y \iff \exists t \in \mathbb{R} \text{ s.t. } t \cdot x = y$$

The Quotient Space $(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})/_\sim$ is called ***Real Projective Space***, denoted $\mathbb{R}P^n$.

Lemma 12.17.5.1. The Real Projective Space $\mathbb{R}P^n$ is Hausdorff.

Proof. Let distinct two point $[x], [y] \in \mathbb{R}P^n$ be given.
Clearly, $x \neq y$, and note that: for a projective map $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n : x \mapsto [x]$,

$$U \subseteq \mathbb{R}P^n \text{ is open } \iff \pi^{-1}[U] \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \text{ is open}$$

Regularizing: $y' = \dfrac{\|x\|}{\|y\|} \cdot y$. Then, $[y'] = [y]$ and $\|y'\| = \|x\|$. And put $r = d(x, y')$. Now,

$$U = \pi[B_{\frac{r}{2}}(x)], \ \ V = \pi[B_{\frac{r}{2}}(y')]$$

are disjoint open set of $\mathbb{R}P^n$, and $[x] \in U$ and $[y] = [y'] \in V$. Becasue:
U, V are Opens: Let $a \in \pi^{-1}[U]$ be given. Then,

$$a \in \pi^{-1}[U] \iff \pi(a) \in U \iff \exists z \in B_{\frac{r}{2}}(x) \text{ s.t. } a \sim z$$

Put $\delta = \frac{r}{2} - d(x, z)$. Then, $a \in B_\delta(a) \subset \pi^{-1}[U]$. Thus $\pi^{-1}[U]$ is open, so is U. Similarly, V.
U and V are disjoint: Suppose that there exists $[z] \in \pi[B_{\frac{r}{2}}(x)] \cap \pi[B_{\frac{r}{2}}(y')]$.
Then, for some $a \in B_{\frac{r}{2}}(x)$ and $b \in B_{\frac{r}{2}}(y')$, $[z] = [a] = [b]$. But, since setting, $[a] \neq [b]$. Contradiction. $\square$

Theorem 12.17.5.1. Define an Equivalent Relation $\sim'$ on $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$:

$$\forall \mathbf{x} \in S^n, \ \mathbf{x} \sim' -\mathbf{x}$$

Then, the Real Projective Space $\mathbb{R}P^n$ is Homeomorphic to the Quotient space $S^n/_{\sim'}$.

$$\begin{tikzcd} S^n \arrow[r, "i"] \arrow[d, "\pi'"] & \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \arrow[d, "\pi"] \\ S^n/\sim' \arrow[r, "\varphi"] & \mathbb{R}P^n \end{tikzcd}$$

$$\begin{aligned} i &: S^n \hookrightarrow \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \\ \pi &: \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbb{R}P^n \\ \pi' &: S^n \to S^n/_{\sim'} \\ \varphi &: S^n/_{\sim'} \to \mathbb{R}P^n \end{aligned}$$

Proof. Prove by four stage:
1) $\pi \circ i : S^n \to \mathbb{R}P^n$ is Continuous onto map.
Each π and i are Continuous, thus the Composition $\pi \circ i$ is Continuous directly.
To show surjective, Let $[\mathbf{x}] \in \mathbb{R}P^n$ be given. Then, $\dfrac{\mathbf{x}}{\|\mathbf{x}\|} \in S^n$ and $\dfrac{\mathbf{x}}{\|\mathbf{x}\|} \in [\mathbf{x}]$, thus

$$(\pi \circ i)\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = \pi(i\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)) = \pi\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = [\mathbf{x}]$$

2) $\mathbf{x} \sim' \mathbf{y} \iff (\pi \circ i)(\mathbf{x}) = (\pi \circ i)(\mathbf{y})$.
Suppose that $\mathbf{x} \sim' \mathbf{y}$. Then, $\mathbf{x} \sim \mathbf{y}$, this implies

$$(\pi \circ i)(\mathbf{x}) = \pi(\mathbf{x}) = \pi(\mathbf{y}) = (\pi \circ i)(\mathbf{y})$$

Suppose that $(\pi \circ i)(\mathbf{x}) = (\pi \circ i)(\mathbf{y})$. This means $[\mathbf{x}] = [\mathbf{y}]$, for some $t \in \mathbb{R}$, $\mathbf{y} = t\mathbf{x}$.
But, since $\mathbf{x}, \mathbf{y} \in S^n$,

$$1 = \|\mathbf{y}\| = \|t\mathbf{x}\| = |t|\|\mathbf{x}\| = |t|$$

This means $t = \pm 1$, thus $\mathbf{x} \sim' \mathbf{y}$.
In summary, $(\pi \circ i)$ is Continuous Onto map from S^n onto $\mathbb{R}P^n$.
Meanwhile, S^n is Compact and $\mathbb{R}P^n$ is Hausdorff, thus $(\pi \circ i)$ induces a Homeomorphism

$$\varphi : S^n/_{\sim'} \to \mathbb{R}P^n : [\mathbf{x}]_{\sim'} \mapsto [\mathbf{x}]_{\sim}$$

□

Theorem 12.17.5.2. Define an Equivalent Relation $\sim''$ on $D^n \overset{\text{def}}{=} \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq 1\}$:

$$\forall \mathbf{x} \in S^{n-1} \subset D^n, \ \mathbf{x} \sim'' -\mathbf{x}$$

Then, the Real Projective Space $\mathbb{R}P^n$ is Homeomorphic to the Quotient space $D^n/_{\sim''}$.

Proof. First, we will show that D^n can be embedded in S^n. Define a map: $j : D^n \to S^n : \mathbf{x} \mapsto (\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2})$. j is

Using this embedding, we will construct the following diagram of quotient maps:

$$\begin{tikzcd} D^n \arrow[r, "j"] \arrow[d, "\pi''"'] & S^n \arrow[d, "\pi'"] \\ D^n/\sim'' \arrow[r, "\phi"'] & S^n/\sim' \end{tikzcd}$$

Embedding, becasue Continuous and Injective are clear. Moreover, $j[D^n]$ is the upper hemisphere of S^n.
1) □

Theorem 12.17.5.3. The Quotient Space D^n/S^{n-1} is Homeomorphic to S^n.

Proof. Define a map: denote $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$f : D^n \to S^n : \mathbf{x} \mapsto (2x_1\sqrt{1 - \|\mathbf{x}\|^2}, 2x_2\sqrt{1 - \|\mathbf{x}\|^2}, \ldots, 2x_n\sqrt{1 - \|\mathbf{x}\|^2}, 2\|\mathbf{x}\|^2 - 1)$$

Then, f is Continuous is clear, and Surjective, because:
Let $\mathbf{y} = (y_1, \ldots, y_n, y_{n+1}) \in S^n \subset \mathbb{R}^{n+1}$ be given. If $y_{n+1} \neq 1$, put $\mathbf{y} = \dfrac{1}{\sqrt{2(1 - y_{n+1})}} \cdot (y_1, \ldots, y_n)$.

$$\begin{aligned} \|\mathbf{y}\| &= \frac{1}{\sqrt{2(1 - y_{n+1})}} \sqrt{\sum_{i=1}^{n} y_i^2} = \frac{1}{\sqrt{2(1 - y_{n+1})}} \sqrt{1 - y_{n+1}^2} = \frac{1}{\sqrt{2}} \sqrt{1 + y_{n+1}} \\ \implies \sqrt{1 - \|\mathbf{y}\|^2} &= \sqrt{1 - \frac{1}{2}(1 + y_{n+1})} = \frac{1}{\sqrt{2}} \sqrt{1 - y_{n+1}} \\ \implies f(\mathbf{y}) &= \left(\frac{2y_1}{\sqrt{2(1 - y_{n+1})}} \frac{1}{\sqrt{2}} \sqrt{1 - y_{n+1}}, \ldots, \frac{2y_n}{\sqrt{2(1 - y_{n+1})}} \frac{1}{\sqrt{2}} \sqrt{1 - y_{n+1}}, 2\left(\frac{\sqrt{1 + y_{n+1}}}{\sqrt{2}} \right)^2 - 1 \right) \end{aligned}$$

And if $y_{n+1} = 1$, for every $1 \leq i \leq n$, $y_i = 0$, then $f((0, \ldots, 0, 1)) = (0, 0, \ldots, 0, 1)$. Thus injective.
Meanwhile, since $f^{-1}[(0, \ldots, 0, 1)] = S^{n-1}$, thus

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{x}, \mathbf{y} \in S^{n-1} \iff f(\mathbf{x}) = f(\mathbf{y})$$

Finally, Since D^n is Compact and S^n is Hausdorff, f induces a Homeomorphism. □

12.18 Manifold

12.18.1 Definition

Definition 12.18.1.1. A Topological Space X is called n-***Dimensional Manifold*** if:
X is Second-Countable Hausdorff Space, and satisfying

For all $x \in X$, there exists Open U_x with $x \in U_x$ such that $U_x \cong \mathbb{R}^n$ or $U_x \cong \mathbb{H}^n \overset{\text{def}}{=} \{(x_i)_{i=1}^n \in \mathbb{R}^n \mid x_n \geq 0\}$.

The 1-Dimensional Manifold is called ***Curve***, and 2-Dimensional Manifold is called ***Surface***.

Definition 12.18.1.2. Let M be a n-Dimensional Manifold.
M is called a ***Manifold with no Boundary*** if:

For all $x \in M$, there exists Open U_x with $x \in U_x$ such that $U_x \cong \mathbb{R}^n$

If not, it is called a ***Manifold with Boundary***.

Definition 12.18.1.3. Let M be a n-Dimensional Manifold.
A point $x \in M$ is called an ***Interior point of Manifold*** if:

There exists an Open U_x with $x \in U_x$ such that $U_x \cong \mathbb{R}^n$

The set $M^\circ \overset{\text{def}}{=} \{x \in M \mid x \text{ is Interior point}\}$ is called ***Interior*** of a Manifold M.
The set $\partial M \overset{\text{def}}{=} M \setminus M^\circ$ is called ***Boundary*** of a Manifold M.

Lemma 12.18.1.1. For any $r > 0$, The Euclidean Space $\mathbb{R}^n$ is Homeomorphic to $B(\mathbf{0}, r)$, and
The $\mathbb{H}^n$ is Homeomorphic to $B(\mathbf{0}, r)$ to $B(\mathbf{0}, r) \cap \mathbb{H}^n$.

Lemma 12.18.1.2. Every Manifold is Locally Compact.

Proof. Let M be a n-Dimensional Manifold, and $x \in M$. Then, there is an Open U with $x \in U$.
By definition, $U \cong \mathbb{R}^n$. Put $f : U \to \mathbb{R}^n$ be a Homeomorphism, and $B \subset \mathbb{R}^n$ be an Open ball containing $f(x)$.
The Heine-Borel Theorem gives $\overline{B}$ is Compact, and containing $f(x)$. Now,

$$x \in f^{-1}[B] \subset f^{-1}[\overline{B}]$$

and $f^{-1}[B]$ is Open and $f^{-1}[\overline{B}]$ is Compact, thus x has a Compact Neighborhood. □

Theorem 12.18.1.1. Every Manifold is Normal.

Proof. Let M be a Manifold.
If M is Compact, then Compact and Hausdorff implies M is Normal.
If M is Not Compact, then M is Locally Compact, Hausdorff, and Not Compact.
Hence, we can construct One-Point Compactification M^* which is Compact Hausdorff, contains M as a Subspace.
Now, M is Regular, and Second Countable, thus Normal. □

12.18.2 Connected Sum

Definition 12.18.2.1. Suppose $\{x_j \mid j \in J\}$ are Topological Spaces.
Define *Coproduct* of given Collection

$$\amalg_{j\in J} X_j \overset{\text{def}}{=} \bigcup_{j\in J} X_j \times \{j\}$$

Define for each $j \in J$,

$$\varphi_j : X_j \to \amalg_{j\in J} X_j : x \mapsto (x, j)$$

Immediately, φ_j is injective.
The Topology $\mathcal{T}_\amalg$ on $\amalg_{j\in J} X_j$ is called *Coproduct Topology* if:

$$\mathcal{T}_\amalg \text{ is the Largest Topology such that every } \varphi_j,\ (j \in J) \text{ is Continuous}$$

Lemma 12.18.2.1. Suppose that $\amalg_{j\in J} X_j$ is Coproduct Space. TFAE:

a) $U \subseteq \amalg_{j\in J} X_j$ is Open.

b) For each $j \in J$, $\varphi_j^{-1}[U]$ is Open set of X_j.

Proof. $a) \implies b)$ is Clear.
Suppose that for all $j \in J$, $\varphi_j^{-1}[U]$ is Open of X_j.
Thus, for any Topology $\mathcal{T}$ such that $\forall j \in J,\ \varphi_j$ is Continuous, $U \in \mathcal{T}$. Now, by definition, $\mathcal{T} \subseteq \mathcal{T}_\amalg$. □

Definition 12.18.2.2. Suppose X and Y are Disjoint Topological Space, and $A \subset X$ is a Subspace.
Given Continuous map $f : A \to Y$, define equivalent relation on the Coproduct $X \amalg Y$:

$$\forall a \in A,\ a \sim f(a)$$

The Quotient Space $X \cup_f Y \overset{\text{def}}{=} X \amalg Y/_\sim$ is called *Adjuction Space*, and f is called *Attaching Map*.

Lemma 12.18.2.2. Let S_1 and S_2 are Surfaces and $x_i \in S_i^\circ$. Then, there exists Open sets B_i such that

$$x_i \in B_i \subset S_i,\ \overline{B_i} \subset S_i^\circ,\ B_1 \cong B_2$$

Proof. Let $x_1 \in S_1^\circ$ and $x_2 \in S_2^\circ$. Then, there exist Opens $U_1 \subset S_1$ and $U_2 \subset S_2$ such that

$$x_1 \in U_1 \cong B_1(\mathbf{0}), \quad x_2 \in U_2 \cong B_1(\mathbf{0})$$

Put $f_1 : B_1(\mathbf{0}) \to U_1$ and $f_2 : B_1(\mathbf{0}) \to U_2$ are Homeomorphism. Then, the restriction:

$$f_1| : B_{\frac{1}{2}}(\mathbf{0}) \to f_1[B_{\frac{1}{2}}(\mathbf{0})] : x \mapsto f_1(x), \quad f_2| : B_{\frac{1}{2}}(\mathbf{0}) \to f_2[B_{\frac{1}{2}}(\mathbf{0})] : x \mapsto f_2(x)$$

are Homeomorphisms. Put $B_1 \overset{\text{def}}{=} f_1[B_{\frac{1}{2}}(\mathbf{0})]$ and $B_2 \overset{\text{def}}{=} f_2[B_{\frac{1}{2}}(\mathbf{0})]$. Now, above map becomes the Homeomorphism

$$h : \partial\overline{B}_1 \to \partial\overline{B}_2 : x \mapsto (f_2| \circ (f_1|)^{-1})(x)$$

□

Definition 12.18.2.3. Suppose that S_1 and S_2 are Surfaces.
Let $x_1 \in S_1^\circ$ and $x_2 \in S_2^\circ$. Put $B_1 \subset S_1$ and $B_2 \subset S_2$ are Open sets such that

$$x_i \in B_i \subset S_i,\ \ \overline{B_i} \subset S_i^\circ,\ \ B_1 \cong B_2$$

(The Existence of this open allowed from above lemma.) Now, Using the Homeomorphism

$$h : \partial\overline{B_1} \to \partial\overline{B_2}$$

Define *Connected Sum* of $S_1 \setminus B_1$ and $S_2 \setminus B_2$ is:

$$S_1 \sharp S_2 \overset{\text{def}}{=} (S_1 \setminus B_1) \cup_h (S_2 \setminus B_2)$$

Well-Defined of this Definition guaranteed by below Lemmas.

Lemma 12.18.2.3. Let X, Y, Z be Topological Space, $A \subset Y, B \subset Z$, and $f : A \to X$, $g : B \to X$ be Continuous map.
If there exists a Homeomorphism $h : Y \to Z$ such that

$$h[A] = B, \quad g \circ h = f$$

Then, $X \cup_f Y \cong X \cup_g Z$.

Proof. Define a map

$$\varphi : X \cup_f Y \to X \cup_g Z, \quad [x]_f \mapsto [\tilde{h}(x)]_g.$$

$$\begin{array}{ccc} X \amalg Y & \xrightarrow{\tilde{h}} & X \amalg Z \\ \downarrow \pi_f & & \downarrow \pi_g \\ X \cup_f Y & \xrightarrow[\varphi]{} & X \cup_g Z \end{array}$$

where

$$\tilde{h} : X \amalg Y \to X \amalg Z : x \mapsto \begin{cases} x & x \in X \\ h(x) & x \in Y \end{cases}$$

1) $\tilde{h}$ is Homeomorphism.
Bijection is Clear. Let $U \subseteq X \amalg Z$ be an Open. Then,

$$\begin{aligned} \tilde{h}^{-1}[U] &= \tilde{h}^{-1}[(X \cap U) \cup (Z \cap U)] \\ &= \tilde{h}^{-1}[(X \cap U)] \cup \tilde{h}^{-1}[(Z \cap U)] \\ &= (X \cap U) \cup h^{-1}[Z \cap U] \end{aligned}$$

Thus $\tilde{h}$ is Continuous, and similarly $\tilde{h}$ is Open map. Thus, homeomorphism.
2) φ is Well-Defined.
Let $[x]_f = [y]_f$. If $x \notin A$, then $x = y$. Thus, there is nothing to prove.
Suppose that $x \in A$. Then, $f(x) \sim_f x \sim_f y \sim_f f(y)$. Now,

$$\begin{aligned} \varphi([x]_f) = [\tilde{h}(x)]_g = [h(x)]_g &= \{z \in X \amalg Z \mid z \sim_g (g \circ h)(x)\} \\ &= \{z \in X \amalg Z \mid z \sim_g f(x)\} \\ &= \{z \in X \amalg Z \mid z \sim_g f(y)\} \\ &= \varphi([y]_f) \end{aligned}$$

3-1) φ is Injective.
Let $\varphi([x]_f) = \varphi([y]_f)$. That is, $[\tilde{h}(x)]_g = [\tilde{h}(y)]_g$.
If $x \in X$, then $\tilde{h}(x) = x$, this means $[\tilde{h}(x)]_g = [x]_g = \{x\}$. Thus, $[\tilde{h}(y)]_g$ must be subset of X, $y \in X$ and $y = x$.
If $x \notin X$, then $[\tilde{h}(x)]_g = [h(x)]_g$. And, from above discussion, $y \notin X$, thus $[\tilde{h}(y)]_g = [h(y)]_g$.
If $h(x) \notin B$, then $\{h(x)\} = [h(x)]_g = [h(y)]_g = \{h(y)\}$, thus $x = y$ being h is Bijection.
If $h(x) \in B$, then $f(x) = (g \circ h)(x) \sim_g h(x) \sim_g h(y) \sim_g (g \circ h)(y) = f(y)$ and $f(x) \notin B$, thus $f(x) = f(y)$.
(Conti.) $\square$

Chapter 13

Algebraic Topology

13.1 Orbit Space

Definition 13.1.0.1. Suppose that the set G is $\begin{cases} \text{Topological Space under the Topology } \mathcal{T} \\ \text{Group under the operation } \cdot : G \times G \to G \end{cases}$.

The triple $(G, \mathcal{T}, \cdot)$ is called ***Topological Group*** if: $\begin{cases} \mu : G \times G \to G : (g_1, g_2) \mapsto g_1 g_2 \\ \iota : G \to G : g \mapsto g^{-1} \end{cases}$ are Continuous maps.

13.1.1 General Linear Group

Definition 13.1.1.1. Define a set of invertible matrices, ***General Linear Group*** over field F:

$$\mathrm{GL}(n, F) \stackrel{\text{def}}{=} \{A \in \mathcal{M}_{n,n}(F) \mid \det A \neq 0\}$$

13.2 Homotopy

13.3 Fundamental Group

Chapter 14

Basic Analysis

14.1 Tests for Series

14.1.1 Integral Test

Theorem 14.1.1.1. Let $f : [1, \infty) \to \mathbb{R}$ be a decreasing function which satisfies $\begin{cases} \lim_{x\to\infty} f(x) = 0 \\ f > 0 \end{cases}$. Then,

$$\int_1^\infty f(x)dx \text{ converges } \textit{if and only if } \sum_{k=1}^\infty f(k) \text{ converges.}$$

Futhermore, put $d_n \overset{\text{def}}{=} \sum_{k=1}^n f(k) - \int_1^n f(x)dx$, then for any $n \in \mathbb{N}$, $0 < f(n+1) \le d_{n+1} \le d_n \le f(1)$, and for any $k \in \mathbb{N}$, $0 \le d_k - \lim_{n\to\infty} d_n \le f(k)$. (Clearly, $\lim_{n\to\infty} d_n$ exists.)

Proof. Since

$$\int_1^{n+1} f(x)dx = \sum_{k=1}^n \int_k^{k+1} f(x) \overset{\substack{\text{decreasing}\\ \downarrow}}{\le} \sum_{k=1}^n \int_k^{k+1} f(k) = \sum_{k=1}^n f(k)$$

$$\implies f(n+1) = \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^n f(k) \le \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)dx = d_{n+1}$$

And,

$$d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \ge \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Immediate d_n converges, being bounded and decreasing. That is,

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \left(\sum_{k=1}^n f(k) - \int_1^n f(x)dx \right)$$

converges. Meanwhile, since

$$0 \le d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \le \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

Now, telescope:

$$0 \le d_k - \lim_{n\to\infty} d_n \le f(k) - \lim_{n\to\infty} f(n+1) = f(k)$$

□

14.1.2 Ratio Test

Theorem 14.1.2.1. Let $\sum a_n$ be given.
$\sum_{n=1}^\infty a_n$ converges if: $\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.
$\sum_{n=1}^\infty a_n$ diverges if: $n_0 \in \mathbb{N}$ such that $\forall n \ge n_0$, $\left| \frac{a_{n+1}}{a_n} \right| \ge 1$.

Proof. Choose $\beta < 1$ such that for some $N \in \mathbb{N}$, $n \ge N \implies \left| \frac{a_{n+1}}{a_n} \right| < \beta < 1$.
Then,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ &\vdots \\ |a_{N+p}| &< \beta^p |a_N| \quad (p \in \mathbb{N}) \end{aligned}$$

As a result, for all $n \ge N$, $|a_n| < \beta^{n-N} |a_N|$. And, $\sum_{n=1}^\infty |a_n| \le \sum_{n=1}^\infty \beta^{n-N} |a_N| < \infty$. □

14.1.3 Root Test

Theorem 14.1.3.1. Let $\sum a_n$ be given.

$\sum_{n=1}^{\infty} a_n$ converges if: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

$\sum_{n=1}^{\infty} a_n$ diverges if: $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

Proof. Put $\beta \in \mathbb{R}$ such that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < \beta < 1$. Then, there is $N \in \mathbb{N}$ such that $n \geq N \implies \sqrt[n]{|a_n|} < \beta$. Now, $\sum |a_n| < \sum \beta^n < \infty$. But if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $a_n \nrightarrow 0$. $\square$

14.2 Arithmetic means

Let $\{s_n\}$ be a Complex numbers Sequence. Define the ***Arithmetic means*** of $\{s_n\}$:

$$\sigma_n \overset{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1}\left(\sum_{i=0}^{n} s_i\right)$$

Then, the Arithmetic means σ_n has the following properties:
1). If $\lim_{n\to\infty} s_n = s$, then $\lim_{n\to\infty} \sigma_n = s$.

Proof. Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$.
Now, for $n \geq N$,

$$\begin{aligned}
|\sigma_n - s| &= \left|\frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1}\right| = \left|\frac{(s_0 - s) + \cdots + (s_n - s)}{n+1}\right| \\
&\overset{\text{tri.ieq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^{n} |s_k - s|}{n+1} \\
&< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \varepsilon \\
&< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \varepsilon
\end{aligned}$$

Now, put $M \in \mathbb{N}$ satisfies $M \geq N$ and $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \varepsilon$, using Archimedean property.
Then, $n \geq M$ implies $|\sigma_n - s| < \varepsilon$, thus $\sigma_n \to s$. □

2). Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. If $\lim_{n\to\infty} na_n = 0$ and σ_n converges, then s_n converges.

Proof. First,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^{n} s_k}{n+1} \\
&= \frac{1}{n+1}((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\
&= \frac{1}{n+1}\sum_{k=1}^{n} ka_k
\end{aligned}$$

Now, if $na_n \to 0$ and $\sigma_n \to \sigma$,

$$\begin{aligned}
\lim_{n\to\infty} s_n &= \lim_{n\to\infty}\left(\sigma_n + \frac{1}{n+1}\sum_{k=1}^{n} ka_k\right) \\
&= \lim_{n\to\infty} \sigma_n + \lim_{n\to\infty} \frac{1}{n+1}\sum_{k=1}^{n} ka_k \overset{1)}{=} \sigma
\end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:
3). The sequence $\{na_n\}$ bounded by $M < \infty$, and $\sigma_n \to \sigma$. Then, $s_n \to \sigma$.

Proof. First, For positive integers $m < n$,

$$\begin{aligned}
s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^{n} s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left(\frac{1}{m+1} - \frac{1}{n+1}\right)\sum_{k=0}^{n} s_k \\
&= s_n - \frac{m+1}{n-m} \cdot \left(\frac{\sum_{k=0}^{m} s_k + \sum_{k=m+1}^{n} s_k}{m+1} - \frac{\sum_{k=0}^{n} s_k}{n+1}\right) \\
&= s_n - \frac{m+1}{n-m} \cdot \left(\sigma_m - \sigma_n + \frac{\sum_{k=m+1}^{n} s_k}{m+1}\right) \\
&= \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{k=m+1}^{n} (s_n - s_k)
\end{aligned}$$

Meanwhile, since for any $n \in \mathbb{N}$, $|na_n| = n|s_n - s_{n-1}| < M$, for $k = m+1, \ldots, n$,

$$
\begin{aligned}
|s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \cdots + s_{k+1} - s_k| \\
&\le |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \cdots + |s_{k+1} - s_k| \\
&\le \frac{M}{n} + \frac{M}{n-1} + \cdots + \frac{M}{k+1} \le \frac{n-k}{k+1}M \le \frac{n-k}{m+2}M \le \frac{n-m-1}{m+2}M
\end{aligned}
$$

Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, put $m \in \mathbb{N}$ such that

$$m \le \frac{n-\varepsilon}{1+\varepsilon} < m+1$$

Then,

$$m(1+\varepsilon) \le n - \varepsilon \implies m + \varepsilon(1+m) \le n \implies \frac{m+1}{n-m} \le \frac{1}{\varepsilon}$$

and

$$n - \varepsilon < (m+1)(1+\varepsilon) \implies n+1 < (m+2)(1+\varepsilon) \implies \frac{n+1}{m+2} - 1 < \varepsilon \implies \frac{n-m-1}{m+2} < \varepsilon$$

Now, for arbitrary $n \in \mathbb{N}$,

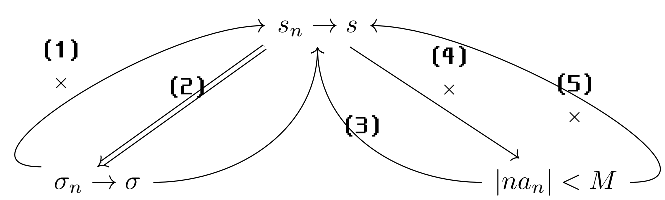
$$
\begin{aligned}
&|s_n - \sigma| \le |s_n - \sigma| + |\sigma_n - \sigma| \\
\implies &\limsup_{n\to\infty} |s_n - \sigma| \le \limsup_{n\to\infty} |s_n - \sigma_n| + \limsup_{n\to\infty} |\sigma_n - \sigma|
\end{aligned}
$$

And,

$$
\begin{aligned}
&|s_n - \sigma_n| = \frac{m+1}{n-m}|\sigma_n - \sigma_m| + \frac{1}{n-m}\sum_{k=m+1}^{n} |s_n - s_k| < \frac{1}{\varepsilon}|\sigma_n - \sigma_m| + M\varepsilon \\
\implies &\limsup_{n\to\infty} |s_n - \sigma_n| \le \frac{1}{\varepsilon}\limsup_{n\to\infty} |\sigma_n - \sigma_m| + M\varepsilon = M\varepsilon
\end{aligned}
$$

Consequently, $\limsup_{n\to\infty} |s_n - \sigma| \le (M+1)\varepsilon$, thus $s_n \to \sigma$. $\square$

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let $s_n \overset{\text{def}}{=} \exp(\frac{in\pi}{2})$. Then,

- s_n diverges.
- na_n diverges.
- $\sigma_n \to 0$.

(2) Let $s_n \overset{\text{def}}{=} \frac{1}{n}$, $s_0 = 0$.

(3) Let $s_n \overset{\text{def}}{=} \sum_{k=1}^{n} \frac{1}{k}$. Then,

- s_n diverges.
- $a_n = \frac{1}{n}$, thus $na_n \to 1$, bounded.
- If σ_n converges, then the diagram implies that s_n must converge, leading to a contradiction. Therefore, σ_n diverges.

(4) $s_n = \sum_{k=1}^{n} \frac{(-1)^k}{\sqrt{k}}$, $s_0 = 0$. Then,

- s_n converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$, thus na_n diverges.

14.3 Taylor's Theorem

Theorem 14.3.0.1. *Taylor's Theorem*

Let $f : [a, b] \to \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is Continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$

Then, for any $\alpha, \beta \in [a, b]$, there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

Proof. Put

$$M \overset{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left(f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \overset{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k - M(t - \alpha)^n, \quad (a \le t \le b)$$

If we differentiate the above equation n times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, (a < t < b)$$

For each $k = 0, 1, \ldots, n-1$,

$$\begin{aligned} \frac{d^r}{dt^r}\left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r}\left((t - \alpha)^k\right) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!}(t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \ldots, n-1$, $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$.
Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$.
And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$.

Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \dfrac{f^{(n)}(x_n)}{n!}$.
Proof Complete by Initial Setting. □

Corollary 14.3.0.1. Let $f : [a, b] \to \mathbb{R}$ be an infinitely differentiable function.
Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a,b]} |f^{(n)}(t)| \le M$. Then, for any $x, \alpha \in [a, b]$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(x - \alpha)^k$$

14.4 Convexity

14.4.1 Definition

Definition 14.4.1.1. Let $f : (a,b) \to \mathbb{R}$ be a Real-valued function. f is said to be *convex* if: For any $x, y \in (a,b), \lambda \in (0,1)$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

Convex function has follwing properties:

Lemma 14.4.1.1. Let $f : (a,b) \to \mathbb{R}$ be a Convex function, and $a < x_1 < x_2 < x_3 < b$. Then,

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2}$$

Proof. To show that first inequalty, note that

$$\frac{x_2-x_1}{x_3-x_1} \cdot x_3 + \left(1 - \frac{x_2-x_1}{x_3-x_1}\right) \cdot x_1 = \frac{(x_2-x_1)x_3 + (x_3-x_2)x_1}{x_3-x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2-x_1}{x_3-x_1} \cdot x_3 + \left(1 - \frac{x_2-x_1}{x_3-x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2-x_1}{x_3-x_1} \cdot f(x_3) + \left(1 - \frac{x_2-x_1}{x_3-x_1}\right) \cdot f(x_1) = \frac{x_2-x_1}{x_3-x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2-x_1}{x_3-x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2-x_1}{x_3-x_1}(f(x_3)-f(x_1)) \implies \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_1)}{x_3-x_1}$$

And similarly,

$$\frac{x_3-x_2}{x_3-x_1} \cdot x_1 + \left(1 - \frac{x_3-x_2}{x_3-x_1}\right) x_3 = x_2$$

gives the second inequality. □

14.4.2 Properties

Proposition 14.4.2.1. If $f:(a,b)\to\mathbb{R}$ is Convex, then f is Continuous.

Proof. Let $\varepsilon>0$ be given, $s<t$ are fixed in (a,b). For any $x,y\in(s,t)$ with $s<x<y<t$,

$$\frac{f(s)-f(a)}{s-a}\le\frac{f(x)-f(s)}{x-s}\le\frac{f(y)-f(x)}{y-x}\le\frac{f(t)-f(y)}{t-y}\le\frac{f(b)-f(t)}{b-t}$$

Put $M=\max\left\{\left|\frac{f(s)-f(a)}{s-a}\right|,\left|\frac{f(b)-f(t)}{b-t}\right|\right\}$. Then, for any $x,y\in(s,t)$,

$$\left|\frac{f(y)-f(x)}{y-x}\right|\le M$$

Now,

$$|f(y)-f(x)|\le M|y-x|<\varepsilon$$

Since $s,t\in(a,b)$ was arbitrary, f is continuous on (a,b). □

Proposition 14.4.2.2. Let f is differentiable on (a,b). Then,

f is Convex *if and only if* f' is monotonically increasing on (a,b).

Proof. Prove by showing both directions: right and left.
Right Direction Let $x_1<x_2$ in (a,b). Then,

$$f'(x_1)=\lim_{t\to x_1}\frac{f(t)-f(x_1)}{t-x_1}\le\frac{f(x_2)-f(x_1)}{x_2-x_1}\le\lim_{\tau\to x_2}\frac{f(\tau)-f(x_2)}{\tau-x_2}=f'(x_2)$$

More rigorously, put $\varepsilon=\left|f'(x_1)-\frac{f(x_2)-f(x_1)}{x_2-x_1}\right|$. (If $\varepsilon=0$, then there is nothing to prove.).
Now, there exists a $\delta>0$ such that $|t-x_1|<\delta$ implies

$$\left|f'(x_1)-\frac{f(t)-f(x_1)}{t-x_1}\right|<\varepsilon\iff-\varepsilon+\frac{f(t)-f(x_1)}{t-x_1}<f'(x_1)\overset{(*)}{=}\varepsilon+\frac{f(t)-f(x_1)}{t-x_1}$$

If $f'(x_1)-\frac{f(x_2)-f(x_1)}{x_2-x_1}>0$, then $(*)$ gives

$$f'(x_1)<f'(x_1)+\frac{f(t)-f(x_1)}{t-x_1}-\frac{f(x_2)-f(x_1)}{x_2-x_1}\iff\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(t)-f(x_1)}{t-x_1}\quad\forall t\ \text{s.t.}\ |t-x_1|<\delta$$

If $|t-x_1|<|x_2-x_1|$, then this contradicts to Convexity.
Consequently, we obtain the first inequality, similarly can prove the second inequality.
Left Direction Let $x,y\in(a,b)$ and $\lambda\in(0,1)$ be given. The Mean Value Theorem gives that:

$$f(\lambda x+(1-\lambda)y)-f(x)=f'(z_1)(\lambda x+(1-\lambda)y-x)\text{ for some }z_1\in(x,\lambda x+(1-\lambda)y)$$
$$f(y)-f(\lambda x+(1-\lambda)y)=f'(z_2)(y-\lambda x+(1-\lambda)y)\text{ for some }z_2\in(\lambda x+(1-\lambda)y,y)$$

Now, Monotonically increasing gives

$$\begin{aligned}\frac{f(\lambda x+(1-\lambda)y)-f(x)}{\lambda x+(1-\lambda)y-x}&=f'(z_1)\le f'(z_2)=\frac{f(y)-f(\lambda x+(1-\lambda)y)}{y-(\lambda x+(1-\lambda)y)}\\ \implies\frac{f(\lambda x+(1-\lambda)y)-f(x)}{(1-x)(y-x)}&\le\frac{f(y)-f(\lambda x+(1-\lambda)y)}{\lambda(y-x)}\\ \implies\lambda f(\lambda x+(1-\lambda)y)-\lambda f(x)&\le(1-\lambda)f(y)-(1-\lambda)\lambda x+(1-\lambda)y\\ \implies f(\lambda x+(1-\lambda)y)&\le\lambda f(x)+(1-\lambda)f(y)\end{aligned}$$

□

Corollary 14.4.2.1. If $f:[a,b]\to\mathbb{R}$ is twice-differentiable, then

$$f \text{ is Convex } \textit{if and only if } f''(x)\geq 0 \text{ for all } x\in(a,b).$$

Theorem 14.4.2.1. Let $f:[a.b]\to\mathbb{R}$ be given. Then,

$$f \text{ is Convex if and only if } f \text{ is Continuous, and Midpoint Convex.}$$

Midpoint convex is that f satisfies $\forall x,y\in(a,b),\ \ f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}$.

Proof. The right direction is clear. To show the left direction, we demonstrate that *Midpoint Convexity implies Dyadic Rational Convexity*. Claim: For any $n\in\mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n}x_k}{2^n}\right)\leq\frac{1}{2^n}\sum_{k=1}^{2^n}f(x_k) \tag{$*$}$$

Using Induction: If $n=1$, it is clear by Midpoint Convexity.
Assume that for $n\in\mathbb{N}$, $(*)$ is True. Then,

$$\begin{aligned}
f\left(\frac{\sum_{k=1}^{2^{n+1}}x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2}\cdot\left[\frac{\sum_{k=1}^{2^n}x_k}{2^n}+\frac{\sum_{k=2^n+1}^{2^{n+1}}x_k}{2^n}\right]\right)\\
&\overset{\text{m.c}}{\leq}\frac{1}{2}\left(f\left(\frac{\sum_{k=1}^{2^n}x_k}{2^n}\right)+f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}}x_k}{2^n}\right)\right)\\
&\overset{(*)}{\leq}\frac{1}{2}\left(\frac{1}{2^n}\sum_{k=1}^{2^n}f(x_k)+\frac{1}{2^n}\sum_{k=2^n+1}^{2^{n+1}}f(x_k)\right)\\
&=\frac{1}{2^{n+1}}\sum_{k=1}^{2^{n+1}}f(x_k)
\end{aligned}$$

Consequently, we obtain the claim. Now, let $n\in\mathbb{N}$, and m be an integer such that $1\leq m\leq 2^n$.
Put $x_1=x_2=\cdots=x_m=x$ and $x_{m+1}=x_{m+2}=\cdots=x_{2^n}=y$. Then

$$f\left(\frac{m}{2^n}x+\left(1-\frac{m}{2^n}\right)y\right)\leq\frac{m}{2^n}f(x)+\left(1-\frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x,y\in(a,b),\lambda\in(0,1)$ be given.
Since $\frac{\lfloor 2^n\lambda\rfloor}{2^n}\to\lambda$ as $n\to\infty$, for any $n\in\mathbb{N}$,

$$f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right)\leq\frac{\lfloor 2^n\lambda\rfloor}{2^n}f(x)+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n\to\infty}f\left(\frac{\lfloor 2^n\lambda\rfloor}{2^n}x+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right)\overset{f\ \text{conti.}}{=}f\left(\lim_{n\to\infty}\left[\frac{\lfloor 2^n\lambda\rfloor}{2^n}x+\left(1-\frac{\lfloor 2^n\lambda\rfloor}{2^n}\right)y\right]\right)=f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. $\square$

14.5 Lipschitz Condition

14.5.1 Definition

Definition 14.5.1.1. A real-vauled function $f:(a,b)\to\mathbb{R}$ is called *Lipschitz Continuous* if:

$$\exists L\geq 0 \text{ s.t. } \forall x_1,x_2\in(a,b),\ \ |f(x_1)-f(x_2)|\leq L\cdot|x_1-x_2|$$

The constant L is said to be *Lipschitz Constant* of f. In particular, the constant

$$D \overset{\text{def}}{=} \sup_{x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|}$$

is called *dilation* of f. Clearly,

$$\forall x_1,x_2\in(a,b),\ \ |f(x_1)-f(x_2)|\leq D\cdot|x_1-x_2|$$

and if $L>0$ is Lipschitz Constant of f, then $D\leq L$. That is, $D=\inf\{L>0\mid L \text{ is Lipschitz constant of } f\}$.

14.5.2 Properties

Proposition 14.5.2.1. If $f:(a,b)\to\mathbb{R}$ is Lipschitz Continuous, then f is uniformly continuous.

Proof. Let $L\geq 0$ be a Lipschitz Constant of f. Then, for any $\varepsilon>0$,

$$\forall x,y\in(a,b),\ \ |x-y|<\frac{\varepsilon}{L}\implies|f(x)-f(y)|\leq L|x-y|<\varepsilon$$

□

Proposition 14.5.2.2. Let $f:(a,b)\to\mathbb{R}$ be a Differentiable function. Then,

f is Lipschitz Continuous *if and only if* f' is bounded in (a,b).

Proof.
Right Direction
Let $L>0$ be a Lipschitz constant of f, and $x\in(a,b)$ be given. Since definition of derivative,

$$f'(x)\overset{\text{def}}{=}\lim_{t\to x}\frac{f(x)-f(t)}{x-t}$$

Meanwhile, the assumption gives: for any distinct $x,t\in(a,b)$,

$$\frac{|f(x)-f(t)|}{|x-t|}\leq L$$

Therefore,

$$f'(x)=\lim_{t\to x}\frac{f(x)-f(t)}{x-t}\leq\lim_{t\to x}\frac{|f(x)-f(t)|}{|x-t|}\leq\lim_{t\to x}L=L$$

Left Direction
Let distinct $x,y\in(a,b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z\in(x,y)$ such that

$$f(x)-f(y)=f'(z)(x-y)\implies f'(z)=\frac{f(x)-f(y)}{x-y}$$

Now,

$$\left|\frac{f(x)-f(y)}{x-y}\right|=|f'(z)|\leq L\implies|f(x)-f(y)|\leq L\cdot|x-y|$$

If $x=y$, then there is nothing to prove. □

Note that:

Lipschitz Continuous $\implies$ Uniformly Continuous $\implies$ Continuous

14.6 Optimization Methods

14.6.1 Newton-Raphson Method

Theorem 14.6.1.1. *Newton-Raphson Method*
Let $f : [a, b] \to \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that f satisfies: for all $x \in [a, b]$,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is, f is strictly increasing convex function, and Lipschitz Continuous.
Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.
Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then, $\{x_n\}$ satisfies the following three conditions:

1. $\{x_n\}$ is decreasing sequence.
2. $x_n \to x^*$ as $n \to \infty$.
3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.

Condition 3 means that for a suitable initial value x_1, we can establish an upper bound for the error.

Proof. This proof consists by three steps.
Since f'' is non-negative, and f' is positive, f is strictly increasing convex function.
And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \stackrel{\text{FTA}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x M dt + f'(a) = M(x - a) + f'(a) \leq M(b - a) + f'(a)$$

Thus, f' is bounded on (a, b), thus f is Lipschitz Continuous.
Step 1. f has a unique root x^*.
The existence of root given directly by Intermidate-Value theroem.
Suppose that $x^*, x' \in (a, b)$ are distinct root of f. i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theroem, there is $c \in (a, b)$ between x^* and x' such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with f' is positive.
Step 2. $\{x_n\}$ decrease.
Proof by induction:
For $n = 1$, $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.
To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.
Now, $x_n \to x'$ as $n \to \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.

Still it remains that to show $x' = x^*$. By Continuity,

$$
\begin{aligned}
& f'(x_n)(x_{n+1} - x_n) + f(x_n) = 0 \\
\Longrightarrow & \lim_{n\to\infty}[f'(x_n)(x_{n+1} - x_n) + f(x_n)] = f\left(\lim_{n\to\infty} x_n\right) = f(x') = 0
\end{aligned}
$$

Since the root of f is unique, thus $x' = x^*$.
Step 3. Establishing the error bound.
The Taylor's Theorem implies that

$$
\begin{aligned}
& f(x^*) = f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n) \\
\Longrightarrow & x_{n+1} - x^* = \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
0 \le x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})}\right)^2 (x_{n-1} - x^*)^4 = \cdots \\
&= \prod_{i=1}^{n} \left[\frac{f''(t_i)}{2f'(x_i)}\right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \le \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}
\end{aligned}
$$

$\square$

14.6.2 Gradient Descent

> **Theorem 14.6.2.1.** Let $f:\mathbb{R}\to\mathbb{R}$ be a differentiable function that satisfies the following conditions:
>
> 1. f is *Convex function*.
> 2. f' is *Lipschitz Continuous* with Lipschitz constant of f, $L>0$. In this, f is called L*-Smooth*.
> 3. f has at least one local minimizer x^*.
>
> Then, x^* is a Global minimizer of $\mathbb{R}$, and there exists a unique closed interval M containing x^* such that
>
> $$\forall x\in M, t\notin M,\ \ f(x)=f(x^*)<f(t)$$
>
> And, given initial point $x_0\in\mathbb{R}$ and $0<\gamma\le\frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:
>
> $$x_{n+1}=x_t-\gamma\cdot f'(x_n)$$
>
> Then, for any $N\in\mathbb{N}$,
>
> $$f(x_N)-f(x^*)\le\frac{|x_0-x^*|^2}{2\gamma N}$$

Proof. Let $x^*\in\mathbb{R}$ be a local minimizer. That is, there exists a $\delta>0$ such that $\forall t\in(x^*-\delta,x^*+\delta)$, $f(x^*)\le f(t)$. Then,

$$0\le\lim_{t\to x^*+}\frac{f(x^*)-f(t)}{x^*-t}=f'(x^*)=\lim_{t\to x^*-}\frac{f(x^*)-f(t)}{x^*-t}\le 0$$

thus, $f'(x^*)=0$. And, by convextiy, f' is monotonically inceasing. Now, The Fundametal Theorem of Calculus gives:

$$\forall x\in\mathbb{R},\ \ f(x)=\int_{x^*}^{x}f'(t)dt+f(x^*)\ge f(x^*)$$

Therefore, x^* is a Global minimizer of f.
Now, establish the closed interval M. Since f' is Lipschitz Continuous, thus f' is Continuous.
Let $D\overset{\text{def}}{=}\{x\in\mathbb{R}\mid f'(x)=0\}$. (Note that: $x^*\in D$, thus D is not emtpyset.)
D is closed because: Let $\{x_n\}$ be a convergent sequence in D. That is, for all $n\in\mathbb{N}$, $f(x_n)=0$. Then, by continuity,

$$f\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}f(x_n)=0$$

The limit of $\{x_n\}$ is contained in D, thus D is closed.
And, D is interval: i.e, for any $x\in(\inf D,\sup D)$, $x\in D$ because:
Suppose that there exists $x\in(\inf D,\sup D)$ such that $x\notin D$. That is, $f'(x)\ne 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x,y\in\mathbb{R}$ be given.
The Fundamental Theorem of Calculus and $L-$Smooth condition gives:

$$\begin{aligned}f(y)-f(x)&=\int_x^y f'(t)dt=\int_0^1 f'(x+(y-x)u)(y-x)du=f'(x)(y-x)+\int_0^1(f'(x+(y-x)u)-f'(x))(y-x)du\\&\overset{2.}{\le}f'(x)(y-x)+L\cdot|y-x|^2\int_0^1 u\ \,du=f'(x)(y-x)+\frac{L}{2}|y-x|^2\end{aligned}$$

For any $\lambda>0$, Put $y=x-\lambda f'(x)$. Then,

$$f(x-\lambda f'(x))\le f(x)-f'(x)(\lambda f'(x))+\frac{L}{2}|\lambda f'(x)|^2=f(x)+\lambda\left(\frac{L\lambda}{2}-1\right)|f'(x)|^2$$

Put $\lambda=\frac{1}{L}$, then

$$f\left(x-\frac{f'(x)}{L}\right)\le f(x)-\frac{L}{2}|f'(x)|^2\implies\frac{L}{2}|f'(x)|^2\le f(x)-f\left(x-\frac{f'(x)}{L}\right)\le f(x)-\inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y-x) \le f(y) - f(x) \le f'(y)(y-x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned}
f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\
&\le f'(x)(x-z) + f'(y)(z-y) + \frac{L}{2}|z-y|^2 \\
&= f'(x)\left(x - y + \frac{1}{L}(f'(y) - f'(x))\right) - f'(y)\left(\frac{1}{L}(f'(y) - f'(x))\right) + \frac{L}{2}\left|\frac{1}{L}(f'(y) - f'(x))\right|^2 \\
&= f'(x)(x-y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\
&= f'(x)(x-y) - \frac{1}{2L}|f'(y) - f'(x)|^2
\end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \le f'(x)(x-y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \le f'(y)(y-x) - (f(y) - f(z)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \le (f'(y) - f'(x))(y-x)$$

Since above inequalities, we obtain that

$$\begin{aligned}
|x_{n+1} - x^*|^2 = |x_n - \gamma \cdot f'(x_n) - x^*|^2 &= |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\
&= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\
&\le |x_n - x^*|^2 - 2\gamma\frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\
&= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L}\right)|f'(x_n)|^2 \le |x_n - x^*|^2
\end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \to \infty$. That is, $|x_n - x^*| \le |x_0 - x^*|$ for all $n \in \mathbb{N}$.
Consider x_{n+1} and x_n. First, we obtain

$$\begin{aligned}
f(x_{n+1}) &\le f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\
&= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\
&= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2
\end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right)|f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \le f'(x_n)(x_n - x^*) \le |f'(x_n)||x_n - x^*| \le |f'(x_n)||x_0 - x^*|$$

Combining abvoe two inequalities,

$$f(x_{n+1}) - f(x^*) \le f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned}
&\frac{1}{f(x_n) - f(x^*)} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)}\frac{1}{|x_0 - x^*|^2} \\
\implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)}\frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\
\implies &\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2} \le \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\
\implies &\sum_{n=0}^{N-1}\left[\left(\gamma - \frac{L}{2}\gamma^2\right) \cdot \frac{1}{|x_0 - x^*|^2}\right] \le \sum_{n=0}^{N-1}\left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)}\right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)}
\end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[\left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

14.7 Integral

14.7.1 Inequality of Riemann-Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

Lemma 14.7.1.1. Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\int_a^b f(x)g(x)d\alpha \leq 1$.

Proof. For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x)d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p}\int_a^b [f(x)]^p d\alpha + \frac{1}{q}\int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

Definition 14.7.1.1. Let $f \in \mathcal{R}(\alpha)$. Define a ***Norm*** of f:

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha\right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \to \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

Lemma 14.7.1.2. *Hölder's Inequality*
Let $f, g \in \mathcal{F}$. Then,

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Proof. Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \quad \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p}\right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \quad \int_a^b \left[\frac{|g(x)|}{\|g\|_q}\right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left|\int_a^b f(x)g(x)d\alpha\right| \leq \int_a^b |f(x)||g(x)|d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha\right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha\right]^{\frac{1}{q}}$$

□

Theorem 14.7.1.1. ***Minkowski inequality***
Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.

Proof.

$$\begin{aligned}
\|f+g\|_p^p &= \int_a^b |f+g|^p d\alpha = \int_a^b |f+g||f+g|^{p-1} d\alpha \\
&\leq \int_a^b [|f|+|g|]|f+g|^{p-1} d\alpha \\
&= \int_a^b |f||f+g|^{p-1} d\alpha + \int_a^b |g||f+g|^{p-1} d\alpha \\
&\overset{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}} \left[\int_a^b |f+g|^{(p-1)\frac{p}{p-1}} d\alpha\right]^{\frac{p-1}{p}} \\
&= \left[\int_a^b |f+g|^p d\alpha\right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha\right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha\right]^{\frac{1}{p}}\right) = \|f+g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
\end{aligned}$$

Now,

$$\|f+g\|_p^p \cdot \|f+g\|_p^{1-p} = \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

□

Chapter 15

Measure

Chapter 16

Complex Analysis

16.1 Series

Theorem 16.1.0.1. ***Laurent's theorem***
Suppose that f is analytic on annular domain $D = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$,
and C is simple closed contour around z_0 and lying in that domain D. Then each point in D, $f(z)$ can express that:

$$f(z) = \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right) + \frac{1}{2\pi i} \cdot \sum_{n=1}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{-n+1}} ds \cdot \frac{1}{(z - z_0)^n} \right)$$
$$= \frac{1}{2\pi i} \cdot \sum_{n=-\infty}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right), \qquad (R_1 < |z - z_0| < R_2)$$

In particular, If $f(s)$ is analytic inside and on circle C,
$\forall n \in \mathbb{N},\ f(s) \cdot (s - z_0)^{n-1}$ is analytic too. then by *Cauchy-Goursat Thm*, term (2) is zero, thus we can write that:

$$f(z) = \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right)$$

and, since f is analytic on C, applies *Cauchy integral theorem*:

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n$$

This is what we already know as the *Taylor Series* form. Therefore, we can say
Laurent's theorem is generalization form of *Taylor Theorem*.

Proof.
In case of $z_0 = 0$.
First, since C is lying in annular $R_1 < |z| < R_2$,
can construct annular $A : r_1 < |z| < r_2$ such that A contains circle C.
Let write $C_1 : |z| = r_1$, $C_2 : |z| = r_2$, each circles are positively oriented.
Now, construct circle γ such that positively oriented and lying in annular $A : r_1 < |z| < r_2$.
Then by *multiply connected theorem*, we get that:

$$\int_{C_2} \frac{f(s)}{s - z} ds = \int_{\gamma} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{s - z} ds$$

Inside and on γ, f is analytic, thus we can apply *Cauchy integral theorem*:

$$\int_{\gamma} \frac{f(s)}{s - z} ds = 2\pi i \cdot f(z) = \int_{C_2} \frac{f(s)}{s - z} ds - \int_{C_1} \frac{f(s)}{s - z} ds = \int_{C_2} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{z - s} ds$$
$$\implies f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z - s} ds$$

And we already know in proof of *Taylor theorem*,

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}$$

and also

$$\begin{aligned}
\frac{1}{z-s} &= \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \frac{s^N}{(z-s)z^N} \\
&= \sum_{n=1}^{N} \frac{s^{n-1}}{z^n} + \frac{s^N}{(z-s)z^N} \\
&= \sum_{n=1}^{N} \frac{1}{s^{-n+1}\cdot z^n} + \frac{s^N}{(z-s)z^N}
\end{aligned}$$

Now we can write that:

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{C_2} \frac{f(s)}{s-z}ds + \frac{1}{2\pi i}\int_{C_1} \frac{f(s)}{z-s}ds \\
&= \frac{1}{2\pi i}\int_{C_2} \left(\sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}}f(s) + \frac{z^N}{(s-z)s^N}f(s)\right)ds + \frac{1}{2\pi i}\int_{C_1}\left(\sum_{n=1}^{N} \frac{f(s)}{s^{-n+1}\cdot z^n} + \frac{s^N}{(z-s)z^N}f(s)\right)ds \\
&= \frac{1}{2\pi i}\sum_{n=0}^{N-1}\int_{C_2} \frac{f(s)}{s^{n+1}}ds\cdot z^n + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds + \frac{1}{2\pi i}\sum_{n=1}^{N}\int_{C_1}\frac{f(s)}{s^{-n+1}}ds\cdot\frac{1}{z^n} + \frac{1}{2\pi i\cdot z^N}\int_{C_1}\frac{s^N f(s)}{(z-s)}ds \\
&= \frac{1}{2\pi i}\sum_{n=0}^{N-1}\int_{C_2} \frac{f(s)}{s^{n+1}}ds\cdot z^n + \frac{1}{2\pi i}\sum_{n=1}^{N}\int_{C_1}\frac{f(s)}{s^{-n+1}}ds\cdot z^{-n} + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds + \frac{1}{2\pi i\cdot z^N}\int_{C_1}\frac{s^N f(s)}{(z-s)}ds
\end{aligned}$$

And by construction of C, C_1, C_2, f is analytic between C and C_1, also C and C_2.
Thus applies ***multiply connected***:

$$\begin{aligned}
&= \frac{1}{2\pi i}\sum_{n=0}^{N-1}\int_{C} \frac{f(s)}{s^{n+1}}ds\cdot z^n + \frac{1}{2\pi i}\sum_{n=1}^{N}\int_{C}\frac{f(s)}{s^{-n+1}}ds\cdot z^{-n} + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds + \frac{1}{2\pi i\cdot z^N}\int_{C_1}\frac{s^N f(s)}{(z-s)}ds \\
&= \frac{1}{2\pi i}\sum_{n=-N}^{N-1}\int_{C} \frac{f(s)}{s^{n+1}}ds\cdot z^n + \frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds + \frac{1}{2\pi i\cdot z^N}\int_{C_1}\frac{s^N f(s)}{(z-s)}ds
\end{aligned}$$

Now, enough to show

$$\frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds \to 0 \text{ as } N\to\infty$$
$$\frac{1}{2\pi i\cdot z^N}\int_{C_1}\frac{s^N f(s)}{z-s}ds \to 0 \text{ as } N\to\infty$$

Let $|z|=r$. Then $r_1<r<r_2$. And, Let $M = \max\left\{\max_{z\in C_1} f(z),\ \max_{z\in C_2} f(z)\right\}$. And,
for s on C_2, $|s-z|\ge ||s|-|z||=r_2-r$, for s on C_1, $|z-s|\ge ||z|-|s||=r-r_1$.
Finally, since ***ML inequality***,

$$\left|\frac{z^N}{2\pi i}\int_{C_2}\frac{f(s)}{(s-z)s^N}ds\right| \le \frac{|z^N|}{2\pi}\int_{C_2}\left|\frac{f(s)}{(s-z)s^N}\right|ds \le \frac{r^N}{2\pi}\frac{M\cdot 2\pi r_2}{(r_2-r)(r_2)^N} = \frac{Mr_2}{r_2-r}\left(\frac{r}{r_2}\right)^N$$
$$\left|\frac{1}{2\pi i\cdot z^N}\int_{C_1}\frac{s^N f(s)}{z-s}ds\right| \le \frac{1}{2\pi\cdot r^N}\int_{C_1}\left|\frac{s^N f(s)}{z-s}\right|ds \le \frac{1}{2\pi\cdot r^N}\frac{(r_1)^N\cdot M\cdot 2\pi r_1}{r-r_1} = \frac{Mr_1}{r-r_1}\left(\frac{r_1}{r}\right)^N$$

Consequently, since $\left(\dfrac{r}{r_2}\right)<1$, $\left(\dfrac{r_1}{r}\right)<1$, we get result.

In case of $z_0 \neq 0$.
Let f be analytic throughout annular $R_1 < |z - z_0| < R_2$.
Then $g(z) = f(z + z_0)$ is analytic throughout $R_1 < |(z + z_0) - z_0| < R_2$.
Now let $C : z = z(t) \quad (a \leq t \leq b)$ is closed simple contour, following by statement.
Then $\forall t \in [a, b],\ \ R_1 < |z(t) - z_0| < R_2$ and
for $\Gamma : z = z(t) - z_0 \quad (a \leq t \leq b)$ is lying in $R_1 < |z| < R_2$. Now since In $z_0 = 0$ case,

$$g(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_\Gamma \frac{g(s)}{s^{n+1}} ds \cdot z^n \qquad (R_1 < |z| < R_2)$$

This is equal that:

$$f(z + z_0) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_\Gamma \frac{g(s)}{s^{n+1}} ds \cdot z^n \qquad (R_1 < |z| < R_2)$$

Finally, change z to $z - z_0$ then:

$$f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_\Gamma \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \qquad (R_1 < |z - z_0| < R_2)$$

And

$$\int_\Gamma \frac{g(s)}{s^{n+1}} ds = \int_a^b \frac{f(z(t) - z_0 + z_0)}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_a^b \frac{f(z(t))}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Consequently we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_\Gamma \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \\ &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \cdot (z - z_0)^n \qquad (R_1 < |z - z_0| < R_2) \end{aligned}$$

$\square$

Chapter 17

Fourier Analysis

Chapter 18

Multivariable Analysis

18.1 Differentiation

Definition 18.1.0.1. Suppose that $A \subseteq \mathbb{R}^m$ is a subset, and $\mathbf{x} \in A^\circ$. Given a non-zero vector $\mathbf{u} \in \mathbb{R}^m$, Define ***directional derivative of f at $\mathbf{x}$ with respect to the vector $\mathbf{u}$***:

$$f'(\mathbf{x};\mathbf{u}) \overset{\text{def}}{=} \lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x})}{t}$$

Definition 18.1.0.2. Suppose that $A \subseteq \mathbb{R}^m$ is a subset, and $\mathbf{x} \in A^\circ$.
The function f is called ***differentiable*** at $\mathbf{x}$ if: There exists a $B \in \mathcal{M}_{n,m}(\mathbb{R})$ such that

$$\frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\cdot\mathbf{h}}{\|\mathbf{h}\|} \to \mathbf{0} \in \mathbb{R}^n \text{ as } \mathbf{h} \to \mathbf{0} \in \mathbb{R}^m$$

More rigorously,

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall \mathbf{h} \in \mathbb{R}^m,\ 0 < \|\mathbf{h}\| < \delta \implies \frac{\|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\cdot\mathbf{h}\|}{\|\mathbf{h}\|} < \varepsilon$$

If exists, this B is unique; The matrix B is denoted $Df(\mathbf{x})$, which is called ***derivative*** of f at $\mathbf{x}$.

$$\frac{\underbrace{f(\underbrace{\mathbf{x}}_{\in\mathbb{R}^m} + \underbrace{\mathbf{h}}_{\in\mathbb{R}^m}) - f(\underbrace{\mathbf{x}}_{\in\mathbb{R}^m})}_{\in\mathbb{R}^n} - \underbrace{\underbrace{B}_{\in\mathcal{M}_{n,m}} \cdot \underbrace{\mathbf{h}}_{\in\mathbb{R}^m}}_{\in\mathbb{R}^n}}{\|\mathbf{h}\|}$$

Theorem 18.1.0.1. Suppose that $A \subseteq \mathbb{R}^m$ is a subset.
If $f : A \to \mathbb{R}^n$ is differentiable at $\mathbf{x} \in A^\circ$, then for any $\mathbf{u} \in \mathbb{R}^m$, $f'(\mathbf{x};\mathbf{u})$ exists. Moreover,

$$f'(\mathbf{x};\mathbf{u}) = Df(\mathbf{x})\cdot\mathbf{u}$$

Proof. By assumption, the Derivative $Df(\mathbf{x}) \in \mathcal{M}_{n,m}(\mathbb{R})$ exists such that

$$\frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\cdot\mathbf{h}}{\|\mathbf{h}\|} \to \mathbf{0} \text{ as } \mathbf{h} \to \mathbf{0}$$

Let non-zero vector $\mathbf{u} \in \mathbb{R}^m$ be given. Choose $\varepsilon > 0$ arbitrarily. Since assumption, there exists a $\delta > 0$ such that

$$0 < \|\mathbf{h}\| < \delta \implies \frac{\|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - B\cdot\mathbf{h}\|}{\|\mathbf{h}\|} < \frac{\varepsilon}{\|\mathbf{u}\|}$$

Put $\delta_0 > 0$ such that $0 < t < \delta_0 \implies \|t\mathbf{u}\| < \delta$ (Precisely, put $\delta_0 = \frac{\delta}{\|\mathbf{u}\|}$). Now,

$$\begin{aligned} 0 < t < \delta_0 &\implies \frac{\|f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - B\cdot t\mathbf{u}\|}{\|t\mathbf{u}\|} = \frac{\|f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - B\cdot t\mathbf{u}\|}{|t|\|\mathbf{u}\|} < \frac{\varepsilon}{\|\mathbf{u}\|} \\ &\implies \frac{\|f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x}) - B\cdot t\mathbf{u}\|}{t} = \left\|\frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x})}{t} - B\cdot\mathbf{u}\right\| < \varepsilon \end{aligned}$$

Thus, $B\cdot\mathbf{u} = Df(\mathbf{x}) = f(\mathbf{x};\mathbf{u})$. □

Chapter 19

Differential Geometry

Chapter 20

Differential Equation

20.1 System of Differential Equation

20.1.1 Definitions

20.1.2 Basic Properties

20.2 Lorenz system

Chapter 21

Differential Form

Chapter 22

Spaces

22.1 $\mathbb{R}^n$

22.1.1 Inner Product in $\mathbb{R}$

22.1.2 p-norm in $\mathbb{R}^n$

Definition 22.1.2.1. Let $\mathbb{R}^n$ be given. Define ***p*-norm** on $\mathbb{R}^n$ as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n),\ \mathbf{y} = (y_1, \dots, y_n))$$

where $p \in [1, \infty]$. In particular, p-norm is a ***Metric***, being ***Minkowski inequality***.

Lemma 22.1.2.1. *Young's inequality*
Let $u, v > 0$, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
Then,

$$uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

Proof. Since $f(x) = \log x$ is concave, we obtain

$$\forall \lambda \in [0, 1],\ \ \lambda f(x)(1 - \lambda) f(y) \leq f(\lambda x + (1 - \lambda) y)$$

thus,

$$\log \left(\frac{1}{p} u^p + \frac{1}{q} v^q \right) \geq \frac{1}{p} \log(u^p) + \frac{1}{q} \log(v^q) = \log(uv)$$

Since $\exp(x)$ increasing, we get

$$\exp \left(\log \left(\frac{1}{p} u^p + \frac{1}{q} v^q \right) \right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p} u^p + \frac{1}{q} v^q$$

□

Lemma 22.1.2.2. *Holder's inequality*
Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be give, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$$

Then, since young's inequality, for each $i \in \{1, 2, \ldots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \ldots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

□

Theorem 22.1.2.1. *Minkowski inequality*
Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned}
\sum_{i=1}^{n} |x_i + y_i|^p &= \sum_{i=1}^{n} |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\
&= \sum_{i=1}^{n} |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \cdot |x_i + y_i|^{p-1} \\
&\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^{n} |y_i|^p\right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{p-1}{p}} \\
&= \left[\left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}\right] \cdot \left[\sum_{i=1}^{n} |x_i + y_i|^p\right]^{\frac{p-1}{p}}
\end{aligned}$$

Now, Divide each side as $\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}\right]$$

□

> **Theorem 22.1.2.2.** Let d_{p_1}, d_{p_2} are p-norm on $\mathbb{R}^n$ with $1 \leq p_1 < p_2 \leq \infty$. Then,
>
> $$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n,\ d_{p_2}(x,y) \leq d_{p_1}(x,y) \leq C d_{p_2}(x,y)$$
>
> In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.
For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_2} \leq \sum_{i=1}^n \left[\frac{|x_i - y_i|}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{p_1}{p_2}}} = \left[\frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[\frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}}\right]^{p_1} \iff 1 \leq \frac{\left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}}{\left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2}\right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1}\right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned}
(d_{p_1}(x,y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\
&\overset{\text{Hölder}}{\leq} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}}\right)\right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2 - p_1}}\right]^{1-\frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{p_1}{p_2}} \cdot n^{1-\frac{p_1}{p_2}}
\end{aligned}$$

Taking the $\frac{1}{p_1}$-th power of both sides, then

$$d_{p_1}(x,y) \leq \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})\right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x,y)$$

□

> **Corollary 22.1.2.1.** Let $\mathbb{R}^n$ be given as a set, and $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are p-norm on $\mathbb{R}^n$. Then,
>
> $$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every $p \geq 1$, the metric space $(\mathbb{R}^n, d_p)$ induces the same topology as the product topology on $\mathbb{R}^n$. In particular, $\mathbb{R}^n$ with the product topology coincides with $\mathbb{R}^n$ endowed with any p-norm.

22.1.3 Open and Closed set in $\mathbb{R}^n$

Definition 22.1.3.1. For $p \in [1, \infty]$, define p-***Ball*** in $\mathbb{R}^n$ as:

$$B_p(x, r) \overset{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all p-norms are equivalent, for any $p \in [1, \infty]$, the collection

$$\beta_p \overset{\text{def}}{=} \left\{B_p(x, r) \,\middle|\, x \in \mathbb{Q}^n, \ \ r \in \mathbb{Q}^+\right\}$$

is Countable basis of $\mathbb{R}^n$. Immediately, we obtain:

Lemma 22.1.3.1. Every open set in $\mathbb{R}^n$ is a countable union of p-Balls.

We call 2-Ball the ***Ball***, and ∞-Ball the ***Cube***.

Theorem 22.1.3.1. Let $U \subseteq \mathbb{R}^n$ be an open set. Then, U is a countable union of closed cubes with disjoint interiors.

Proof. Let $U \subseteq \mathbb{R}^n$ be an open set, and define the collection of ***Dyadic Cubes*** on $\mathbb{R}^n$ as: for each $k \in \mathbb{N}$,

$$Q_k \overset{\text{def}}{=} \left\{\prod_{i=1}^{n} \left[\frac{q_i}{2^k}, \frac{q_i + 1}{2^k}\right] \subset \mathbb{R}^n \,\middle|\, q_i \in \mathbb{Z}\right\}$$

Each element of Q_k is product of closed intervals, and its interiors are disjoint. For each $k \in \mathbb{N}$, construct:

$$Q_k^* \overset{\text{def}}{=} \{Q \in Q_k \mid Q \subseteq U\}$$

Then, the union $Q^* = \bigcup_{k \in \mathbb{N}} Q_k^*$ is a countable union of closed cubes, and $Q^* = U$: $Q^* \subseteq U$ is clear, and let $x \in U$. Since property of metric space, there exists $\delta > 0$ such that $x \in B_2(x, \delta) \subseteq U$. Put $k \in \mathbb{N}$ such that $\dfrac{1}{2^k} < \dfrac{\delta}{\sqrt{n}}$. Then, $x \in C \subset B_2(x, \delta) \subseteq U$ for some $C \in Q_k$, because $\operatorname{diam} C = \sqrt{n} 2^{-k}$. Since $C \subset U$, $C \in Q_k^* \subset Q^*$. i.e., $U \subseteq Q^*$. For disjointness of interiors, we will use the fact:

For any $Q_1, Q_2 \in Q^*$, either their interiors are disjoint, or one is contained in the other.

(Conti.) □

22.2 Manifold

22.3 Topological Vector Space

22.4 Hilbert Space

Definition 22.4.0.1. Complete Inner product Vector Space is called *Hilbert Space*.

22.4.1 Hilbert Space in $\mathbb{R}^\omega$

Definition 22.4.1.1. Define $\mathbb{R}^\omega \overset{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$ as the countable product of Euclidean space $\mathbb{R}$ with product topology.
And define $\mathbb{H} \overset{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \,\middle|\, \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$, ***Metric*** on $\mathbb{H}$ as $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$.
The Metric Space $(\mathbb{H}, \mu)$ is called *Hilbert Space* or *l_2 Space*.

Define the operations elementwise; then $(\mathbb{H}, +, \times)$ is a Vector Space over $\mathbb{R}$.
Moreover, $\mathbb{H}$ is Complete Metric Space and Inner product Vector Space.

Lemma 22.4.1.1. $\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ is Metric function induced by the inner product.

Proof. We know that $\mathbb{R}^\omega$ is Vector Space. Moreover, $\mathbb{H} \subset \mathbb{R}^\omega$ is Subspace. Using subspace criteria:

$S \subset V$ is Subspace of Vector Space V if and only if $0 \in S$ and For any $x, y \in S$ and $a \in F$, $ax + y \in S$.

Clearly, $\{0\} \in \mathbb{H}$. Let $a \in \mathbb{R}$ and $\{x_n\}, \{y_n\} \in \mathbb{H}$ be given. Then, $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$ because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} \left[a^2 x_i^2 + 2ax_i y_i + y_i^2\right] \overset{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The $(*)$ given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i||y_i| \le \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \le \sum_{i=1}^{\infty} (x_n^2 + y_n^2) = \sum_{i=1}^{\infty} x_n^2 + \sum_{i=1}^{\infty} y_n^2 < \infty \tag{$*$}$$

Thus $\mathbb{H}$ is Vector Space over $\mathbb{R}$. Now, define ***inner product*** on $\mathbb{H}$ as:

$$\langle \,\cdot\, , \,\cdot\, \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since $(*)$. And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a\langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus, $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is ***inner product space***.
Using ***ineer product***, define the ***Norm*** on $\mathbb{H}$ as:

$$\| \cdot \| : \mathbb{H} \to \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define ***Metric*** on $\mathbb{H}$ as:

$$\mu : \mathbb{H} \times \mathbb{H} \to \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

Theorem 22.4.1.1. *Hilbert Space* is Separable.

Proof. For each $n \in \mathbb{N}$, define $D_n \overset{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q},\ p_{n+1} = p_{n+1} = \cdots = 0\}$ and $D \overset{\text{def}}{=} \bigcup_{n \in \mathbb{N}} D_n$.

Then, D is countable set. We will show that $\overline{D} = \mathbb{H}$.
Let $\epsilon > 0$ and $\{x_n\} \in \mathbb{H}$ be given. Since convergence, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^{N} x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each $i = 1, 2, \ldots, N$, $p_i \in \mathbb{Q}$ $|x_i - p_i| < \dfrac{\epsilon}{\sqrt{2N}}$ and $p_i = 0$ for $i \geq N+1$.
Then, $\{p_n\} \in D_n \subset D$ and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^{N}(x_i - p_i)^2 + \sum_{i=N+1}^{\infty}(x_i - p_i)^2} = \sqrt{\sum_{i=1}^{N}(x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

Corollary 22.4.1.1. *Hilbert Space* is Second-Countable.

Theorem 22.4.1.2. *Hilbert Space* is Complete.

Proof. Let $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{H}$. For any fixed $n, m \in \mathbb{N}$ and for each $j \in \mathbb{N}$,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty}(x_{n,i} - x_{m,i})^2}$$

That is, for each $j \in \mathbb{N}$, $\{x_{n,j}\}$ is Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is Complete, put $y_j \overset{\text{def}}{=} \lim_{n \to \infty} x_{n,j}$, each $j \in \mathbb{N}$.
Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \dfrac{\epsilon}{2}$.
Meanwhile, for each $k \in \mathbb{N}$,

$$\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty}(x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus, $n, m \geq N \implies \displaystyle\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$, for each $k \in \mathbb{N}$.

Taking limit to m, then $n \geq N \implies \displaystyle\lim_{m \to \infty}\left(\sum_{i=1}^{k}(x_{n,i} - x_{m,i})^2\right) = \sum_{i=1}^{k}\left(x_{n,i} - \lim_{m \to \infty} x_{m,i}\right)^2 = \sum_{i=1}^{k}(x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$.

And, for all $k \in \mathbb{N}$,

$$\sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k}(2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus $\{y_i\} \in \mathbb{H}$. As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty}(x_{n,i} - y_i)^2} = \sqrt{\lim_{k \to \infty} \sum_{i=1}^{k}(x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

Theorem 22.4.1.3. $\mathbb{H} \subset \mathbb{R}^\omega$ with subbspace topology is Metrizable.

Proof. We will use two Lemmas:

Lemma 22.4.1.2. Countable Product of Metric Space is Metrizable.

Proof. Let (X_i, d_i) be a metric Space, for each $i \in \mathbb{N}$.
If $d : X \times X \to \mathbb{R}$ is a Metric, then $\dfrac{d}{1+d}$ is also Metric, because

$$\frac{d(x,z)}{\underbrace{1+d(x,z)}_{\frac{x}{1+x}}} \underset{\substack{\uparrow \\ \text{increasing}}}{\leq} \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)} \underset{\substack{\uparrow \\ d\geq 0}}{\leq} \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)} \tag{$*$}$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \to \mathbb{R} : (\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}\right]$$

Then d_Π is a Metric because: the triangle inequality is satisfied since

$$\begin{aligned}
d_\Pi\left(\{x_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty\right) &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i,z_i)}{1+d_i(x_i,z_i)}\right] \\
&\overset{(*)}{\leq} \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \left(\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} + \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)}\right)\right] \\
&= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)}\right] + \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)}\right] \\
&= d_\Pi\left(\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty\right) + d_\Pi\left(\{y_n\}_{n=1}^\infty, \{z_n\}_{n=1}^\infty\right)
\end{aligned}$$

Reflexivity and symmetry are clear.
And, it remains to show that the metric d_Π generates the given product topology.

□

Lemma 22.4.1.3. Metrizable is Hereditary.

Proof omitted.
Consequently, since $\mathbb{H} \subset \mathbb{R}^\omega$ is a subspace of a metric space, it is metrizable. □

22.5 Banach Space

22.6 L_p Space

22.7 l_p Space

Chapter 23

N-Body Problem

23.1 Introduction

23.1.1 Definition

23.2 Basic Tools

23.3 Two-Body Problem

23.4 Three-Body Problem

23.5 N-Body Problem

Chapter 24

Optimization

24.1 Convexity

Definition 24.1.0.1. Suppose that $f : X \to \mathbb{R}^n$ where X is Convex subset of Vector Space over a field $\mathbb{R}$. The function f is called *Convex function* if: For any $\mathbf{x}, \mathbf{y} \in X$, and $t \in [0, 1] \subseteq \mathbb{R}$

$$f(t \cdot \mathbf{x} + (1 - t) \cdot \mathbf{y}) \leq t f(\mathbf{x}) + (1 - t) f(\mathbf{y})$$

24.2 Gradient Descent

24.3 Gradient Flow

Theorem 24.3.0.1. Suppose that $f : \mathbb{R}^m \to \mathbb{R}$ is a Convex Differentiable function, and $\inf f > -\infty$. Define a map $\mathrm{x} : \mathbb{R} \to \mathbb{R}^m$ such that $\begin{cases} \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{x}'(t) = -\nabla f(\mathbf{x}(t)) \end{cases}$, where x_0 is given initial point. Then, the convexity gives the existence of a $\mathrm{x}^* \in \inf f$. And, for any $t \geq 0$,

$$f(\mathbf{x}(t)) - \inf f \leq \frac{1}{2t}\|\mathbf{x}_0 - \mathbf{x}^*\|^2$$

Proof. Let $\mathrm{x} \in \operatorname{argmin} f$ be given. Consider the Lyapunov function:

$$V(t) \overset{\text{def}}{=} t(f(\mathbf{x}(t)) - \inf f) + \frac{1}{2}\|\mathbf{x}(t) - \mathbf{x}^*\|^2$$

Differentiating with respect to t:

$$\begin{aligned} V'(t) &= (f(\mathbf{x}(t)) - \inf f) + t(\nabla f(\mathbf{x}(t)) \cdot \mathbf{x}'(t)) + \frac{1}{2}[(\mathbf{x}(t) - \mathbf{x}^*) \cdot \mathbf{x}'(t)] + \frac{1}{2}[\mathbf{x}'(t) \cdot (\mathbf{x}(t) - \mathbf{x}^*)] \\ &= (f(\mathbf{x}(t)) - \inf f) - t\|\nabla f(\mathbf{x}(t))\|^2 + (\mathbf{x}(t) - \mathbf{x}^*) \cdot \nabla f(\mathbf{x}(t)) \end{aligned}$$

Meanwhile, the Convexity gives:

$$\inf f = f(\mathbf{x}^*) \geq f(\mathbf{x}(t)) + \nabla f(\mathbf{x}(t)) \cdot (\mathbf{x}(t) - \mathbf{x}^*)$$

This implies

$$V'(t) = (f(\mathbf{x}(t)) - \inf f) + (\mathbf{x}(t) - \mathbf{x}^*) \cdot \nabla f(\mathbf{x}(t)) - t\|\nabla f(\mathbf{x}(t))\|^2 \leq -t\|\nabla f(\mathbf{x}(t))\|^2 \leq 0$$

Now, for all $t \geq 0$, $V(t) \leq V(0)$. This means

$$\begin{aligned} V(t) = t(f(\mathbf{x}(t)) - \inf f) + \frac{1}{2}\|\mathbf{x}(t) - \mathbf{x}^*\|^2 &\leq \frac{1}{2}\|\mathbf{x}(0) - \mathbf{x}^*\|^2 = V(0) \\ \implies f(\mathbf{x}(t)) - \inf f &\leq \frac{1}{2t}\|\mathbf{x}_0 - \mathbf{x}^*\|^2 \end{aligned}$$

□

Chapter 25

Artificial Inteligence

In this chapter, $\mathcal{M}_{n\times m}$ is a set of n by m Real-Valued Matrices.

25.1 Feedforward Neural Network

Definition 25.1.0.1. Given a function $f : \mathbb{R} \to \mathbb{R}$, Define an *Elementwise transformation for* f:

$$\mathbf{F}_{n\times m}(f;\ \cdot\ ) : \mathcal{M}_{n,m} \to \mathcal{M}_{n,m} : \begin{pmatrix} & a_{i,j} & \end{pmatrix}_{\substack{1\le i\le n\\ 1\le j\le m}} \mapsto \begin{pmatrix} & f(a_{i,j}) & \end{pmatrix}_{\substack{1\le i\le n\\ 1\le j\le m}}$$

If there is no confusion, we simply write $\mathbf{F} = \mathbf{F}_{n\times m}$.

Definition 25.1.0.2. Given integer sequence $\{l_i\}_{i=0}^n$, let $W_i \in \mathcal{M}_{l_i,l_{i-1}}$ and $B_i \in \mathcal{M}_{l_i,1}$.
Define a *Feedforward Neural Network* $\mathcal{N}$ as inductively:

$$\begin{cases} L_0 \overset{\text{def}}{=} X \in \mathcal{M}_{l_0,1} \\ L_1 \overset{\text{def}}{=} W_1 \cdot L_0 + B_1 \in \mathcal{M}_{l_1,1} \\ L_i \overset{\text{def}}{=} W_i \cdot \mathbf{F}(\max(x,0); L_{i-1}) + B_i \in \mathcal{M}_{l_i,1} \quad (2 \le i \le n) \end{cases}$$

For given *Output Function* $\mathcal{F} : \mathcal{M}_{l_n,1} \to \mathcal{M}_{l_n,1}$, and $\theta = (W_0, \dots, W_n, B_0, \dots, B_n)$,

$$\mathcal{N}(X;\theta) : \mathcal{M}_{l_0,1} \times \Omega \to \mathcal{M}_{l_n,1} : X \mapsto \mathcal{F}(L_n)$$

where $\Omega = \prod_{i=1}^n \mathcal{M}_{l_i,l_{i-1}} \times \prod_{i=1}^n \mathcal{M}_{l_i,1}$.

25.1.1 Forward Propagation

25.1.2 Back Propagation

From the definition, we can write the Neural Network as:

$$\mathcal{N}(X;\theta) = \mathcal{F} \circ L_n \circ L_{n-1} \circ \cdots \circ L_1 \circ L_0$$

The Chain Rule gives: For each $0 \le i \le n$

$$\frac{\partial \mathcal{N}}{\partial W_i} = \frac{\partial \mathcal{F}}{\partial L_n} \cdot \frac{\partial L_n}{\partial L_{n-1}} \cdots \frac{\partial L_{i+1}}{\partial L_i} \cdot \frac{\partial L_i}{\partial W_i}$$
$$\frac{\partial \mathcal{N}}{\partial B_i} = \frac{\partial \mathcal{F}}{\partial L_n} \cdot \frac{\partial L_n}{\partial L_{n-1}} \cdots \frac{\partial L_{i+1}}{\partial L_i} \cdot \frac{\partial L_i}{\partial B_i}$$

25.1.3 Gradient Descent

25.2 Automatic Differentiation

Chapter 26

Cake Theory

26.1 Definition

Definition 26.1.0.1.

26.1.1 Subcake

Definition 26.1.1.1. Let C be a Cake. The non-empty subset $S \subseteq C$ is called *Subcake* of C if: S satisfies

1. S is closed under the *Cutting*.
2. If $\text{Topp}(C) \neq \emptyset$, then for some $s \in \text{Topp}(C)$, s is contained in S.
3. $\text{Lay}(S) = \text{Lay}(C)$.

26.2 Typical Cakes

26.2.1 Cheeses Cake

26.2.2 Chocolate Cake

26.2.3 Strawberry Cake

26.3 Piece of Cake

Definition 26.3.0.1.

> **Theorem 26.3.0.1.** Every Piece of Cake is a Subcake of some Cake.

26.3.1 Subcake of Piece of Cake

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