

# **Math Note**

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My goal is to rewrite all undergraduate mathematics in my own language.  
And, for review, to draw diagrams illustrating the relationships between mathematical objects.

**Patch Note:**

- ~ 2025/9/28 - Drafted the initial framework of the paper, and Transcribed previous works.
- 2025/09/29 -
  - 1. Completed proof of Ring of Fractions.
  - 2. Transcribed Integral, Ratio, and Root Test.
  - 3. Transcribed Tube Lemma, Lindelöf and Countably Compact product Compact.
  - 4. Transcribed Coproduct with Continuous, open, closed map.
- 2025/09/30 -
  - 1. Proved Every open set in  $\mathbb{R}^n$  is countable union of closed cubes, disjoint of interiors remains.
  - 2. Transcribed Group action.
- 2025/10/01 -
  - 1. Transcribed One-point Compactification.
  - 2. Transcribed Definitions of subbasis, Borel set.
- 2025/10/02 -
  - 1. Proved Euclidean Domain
  - 2. Proving Existence of Nowhere-differentiable function.
- 2025/10/03 -
  - 1. Drafted definition and propositions of Quotient Space.
- 2025/10/04 -
  - 1. Studying Quotient Map.
- 2025/10/05 -
  - 1. Studied Basic Properties of the Quotient Map, and Drew quotient map diagram.
- 2025/10/06 -
  - 1. Drafted basic functions in a Metric space.
- 2025/10/07 -
  - 1. Proved basic properties of Completely regular space.
- 2025/10/08 -
  - 1. Proved Compact Hausdorff Space is Normal.
  - 2. Proved Equivalent Conditions of Completely Regular Space.
  - 3. Proving the Urysohn Metrization Theorem.
  - 4. Understanding relations and characteristic of Domains.
- 2025/10/10 -
  - 1. Transcribed basic statements of Polynomial Ring.
- 2025/10/11 -
  - 1. Proved Gauss's Lemma in Polynomial Ring, and Transcribed.
  - 2. Proving  $R$  U.F.D. iff  $R[x]$  U.F.D.
- 2025/10/13 -
  - 1. Understanding U.F.D and irreducible.
  - 2. Proved  $R$  U.F.D. iff  $R[x]$  U.F.D.
  - 3. Drew the Diagram of Domains.
- 2025/10/14 -
  - 1. Drew the Relations of Polynomial Ring Diagram.
  - 2. Study Quadratic Integer Ring.
  - 3. Study Nilpotent in Ring, and Nilradical Ring.
  - 4. Arranged Connected Space.
  - 5. Proved  $S^n$  is One-point Compactification of  $\mathbb{R}^n$ .
  - 6. Proving  $S^n$  is Connected via One-point Compactification.
- 2025/10/15 -
  - 1. Completed Irreducibility Criteria.
- 2025/10/16 -
  - 1. Observing Cyclotomic Polynomial.
- 2025/10/17 -
  - 1. Studying Quadratic Field and Quadratic Integer Ring.
- 2025/10/18 -
  - 1. Observed Nilradical Ideal.
- 2025/10/27 -
  - 1. Drew Diagram in Connectedness.
  - 2. Analyzed Topologist's Sine curve.

**Set Goal:**

1. Brouwer fixed-point theorem.
2. Abel-Ruffini theorem.
3. Stokes's Theorem.
4. Three-Body Problem has no Analytic general solution.

- 2025/11/6 -
  - 1. Completed Basic thoery of fields - Centre around Kronecker's Theorem.
- 2025/11/7 -
  - 1. Studying Algebraic Extension.
- 2025/11/14 -
  - 1. Proving  $B^n/S^{n-1} \cong S^n$ .
  - 2. Revised proof of basic field theory.
- 2025/11/18 -
  - 1. Revising the Basic Notation and Proving the Basic Properties of Vector Spaces.
  - 2. Proved Extension field is the Vector Space.
- 2025/11/23 -
  - 1. Wrote basic Definition of multivariable Derivative for PINN.
  - 2. Studying Typical Quotient Space: Cylinder, Torus, Möbius band, Klein bottle, Projective Space.
- 2025/11/24 -
  - 1. Studied Typical Quotient Space: Cylinder, Torus, Möbius band.
  - 2. Described Definition of Topological Manifold.

# Chapter 1

## Set Theory

### 1.1 Map

**Definition 1.1.0.1.** Let  $X, Y$  are sets. Define a **function**  $X$  to  $Y$  is a relation

$$f \subset X \times Y$$

such that

1. For any  $x \in X$ , there exists  $y \in Y$  such that  $(x, y) \in f$ .
2. If  $(x, y) \in f$  and  $(x, z) \in f$ , then  $y = z$ .

Denote  $f$  as:

$$f : X \rightarrow Y : x \mapsto f(x)$$

Define **Image** of  $f$  by  $A \subset X$ :

$$f[A] \stackrel{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, **Preimage** of  $f$  by  $B \subset Y$ :

$$f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \rightarrow Y$  is **Injective** if:  $f(x_1) = f(x_2) \implies x_1 = x_2$ .  
 $f : X \rightarrow Y$  is **Surjective** if:  $\forall y \in Y, \exists x \in X$  s.t.  $f(x) = y$ .  
If  $f$  is injective and surjective, called **bijective**.  
If  $f$  is bijective, then define **inverse** of  $f$  as:

$$f^{-1} : Y \rightarrow X : y \mapsto x$$

where  $x \in X$  is the unique elements of  $X$  such that  $f(x) = y$ .

**Theorem 1.1.0.1.** Let  $f : X \rightarrow Y$  be a function. Then,

1. There exists  $g : Y \rightarrow X$  such that  $g \circ f : X \rightarrow X$  be an identity function if and only if  $f$  is injective.
2. There exists  $h : Y \rightarrow X$  such that  $f \circ h : Y \rightarrow Y$  be an identity function if and only if  $f$  is surjective.

*Proof.*

1.  $\implies$ )

Assume that  $f(x_1) = f(x_2)$ . Then, existence of left inverse,  $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$ . Thus  $f$  injective.

1.  $\iff$ )

Since  $f$  is injection, for any  $y \in f[X]$ , there exists a unique element  $x_y \in X$  such that  $f(x) = y$ . Now, define

$$g : Y \rightarrow X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any  $x \in X$ ,  $g(f(x)) = g(y) = x$ .

2.  $\implies$ )

Let  $y \in Y$  be given. Since existence of right inverse,  $f(h(y)) = y$  where  $h(y) \in X$ . Thus,  $f$  is surjective.

2.  $\Leftarrow$ )

For any  $y \in Y$ , there exists a  $x_y \in X$  such that  $f(x_y) = y$ . Now, define

$$h : Y \rightarrow X : y \mapsto x_y$$

Then, for any  $y \in Y$ ,  $f \circ h(y) = f(x_y) = y$ . Thus,  $f \circ h$  is identity.  $\square$

**Corollary 1.1.0.1.** Let  $f : X \rightarrow Y$  be a function,  $\text{id}_X : X \rightarrow X : x \mapsto x$ , and  $\text{id}_Y : Y \rightarrow Y : y \mapsto y$ .

There exists a  $f^{-1} : Y \rightarrow X$  such that  $f^{-1} \circ f = \text{id}_X$  and  $f \circ f^{-1} = \text{id}_Y$  if and only if  $f$  is bijection.

*Proof.* If  $f$  is bijection, then there exists left inverse  $g$  and right inverse  $h$ .

Enough To Show that:  $g = h$ . Since  $g \circ f = \text{id}_X$  and  $f \circ h = \text{id}_Y$ ,

$g \circ f \circ h = g \circ \text{id}_Y$ , thus  $h = g$ .  $\square$

**Theorem 1.1.0.2.** Let  $X, Y, Z$  are sets,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  and  $A \subset X, B \subset Y, C \subset Z$ . Then followings are hold:

1.  $g[f[A]] = (g \circ f)[A]$ .
2.  $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$ .

*Proof.*

1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\stackrel{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \stackrel{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

$\square$

**Proposition 1.1.0.1.** Let  $f : X \rightarrow Y$  be a function,  $A, B \subset X$  and  $C, D \subset Y$ .

1. If  $A \subset B$ , then  $f[A] \subset f[B]$ .
2. If  $C \subset D$ , then  $f^{-1}[C] \subset f^{-1}[D]$

*Proof.*

$$\begin{aligned} y \in f[A] &\implies y = f(a) \text{ for some } a \in A \stackrel{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ x \in f^{-1}[C] &\implies f(x) \in C \stackrel{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

$\square$

**Lemma 1.1.0.1.** Let two set  $X, Y$  be given, and  $A \subset X$ ,  $B \subset Y$ ,  $f : X \rightarrow Y$ . Then followings are holds:

1.  $f^{-1}[f[A]] \supseteq A$ , and equality holds if  $f$  one-to-one.
2.  $f[f^{-1}[B]] \subseteq B$ , and equality holds if  $f$  onto.
3.  $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4.  $f[X] \setminus f[A] \subseteq f[X \setminus A]$ , and equality holds if  $f$  one-to-one.

**Proof.** Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A, y \neq f(x) \\ &\stackrel{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If  $f$  is injection, then Left Direction of  $(*)$  be true:  $\exists !x \in X \setminus A$  s.t.  $y = f(x)$ . □

# Chapter 2

## Group Theory

### 2.1 Isomorphism Theorems

**Definition 2.1.0.1.** Let  $G$  be a group, and  $A \subset G$ . Define **Subgroup generated by  $A$** :

$$\langle A \rangle \stackrel{\text{def}}{=} \bigcap_{A \subseteq H \leq G} H$$

**Lemma 2.1.0.1.** Let  $G$  be a group, and  $A \subset G$ .

$$\langle A \rangle = \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid n \in \mathbb{N}, \alpha_i \in \mathbb{Z}, a_i \in A\}$$

**Theorem 2.1.0.1. The First Isomorphism Theorem**

Let  $\varphi : G \rightarrow H$  be a Group-Homomorphism. Then,

$$G / \ker \varphi \cong \varphi[G]$$

$$\begin{array}{ccc} G & \xrightarrow{\varphi \text{ homomorphism}} & \varphi[G] \\ \pi \text{ homomorphism surjection} \swarrow & \uparrow \phi \text{ isomorphism} & \downarrow \\ G / \ker \varphi & & \end{array}$$

**Proof.** Let  $\pi : G \rightarrow G / \ker \varphi : x \mapsto x + \ker \varphi$ . Then, the map  $\phi : G / \ker \varphi \rightarrow \varphi[G] : a + \ker \varphi \mapsto \varphi(a)$  is isomorphism.

**Well-defined and Injective:**

$$a + \ker \varphi = b + \ker \varphi \iff a - b \in \ker \varphi \iff \varphi(a - b) = \varphi(a) - \varphi(b) = 0$$

**Surjective is clear.** □

**Theorem 2.1.0.2. The Second Isomorphism Theorem**

Let  $G$  be a Group, and  $H \leq G$ ,  $N \trianglelefteq G$ . Then,

$$HN/N \cong H/(H \cap N)$$

**Proof.**  $HK$  be a subgroup of  $G$ , being

$$HN = \bigcup_{h \in H} hN \stackrel{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And,  $N \leq HN$  is clear, thus  $N \trianglelefteq HN$ .

Meanwhile,  $H \cap N$  be a Normal Subgroup of  $H$ : for any  $h \in H, n \in H \cap N$ ,  $hn h^{-1} \in N$  because  $N$  is normal, and

$hn^{-1} \in H$  since  $h, n$  contained in  $H$ . Thus,  $hn^{-1} \in H \cap N$ , this implies  $H \cap N$  be a Normal of  $H$ . Now, Define a Map:

$$\varphi : H \rightarrow HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker \varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

**Theorem 2.1.0.3. The Third Isomorphism Theorem**

Let  $G$  be a Group, and  $H, K \trianglelefteq G$  with  $H \leq K$ . Then,  $K/H \trianglelefteq G/H$  and

$$(G/H)/(K/H) \cong (G/K)$$

**Proof.** First, show that  $K/H \trianglelefteq G/H$ . Let  $kH \in K/H$  and  $gH \in G/H$ . Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gk)H \in K/H$$

since  $gk \in K$ , being  $K \trianglelefteq G$ . Now, Define a map:

$$\varphi : G/H \rightarrow G/K : gH \mapsto gK$$

**1. Well-Defined.**

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \stackrel{H \trianglelefteq K}{\implies} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

**2. Homomorphism.**

Clearly, for any  $g_1H, g_2H \in G/H$ ,

$$\varphi(g_1Hg_2H) = \varphi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

**3. Surjection.** Let  $gK \in G/K$  be given. Then, clearly,  $\varphi(gH) = gK$ .

**4. Kernel.**

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

□

**Theorem 2.1.0.4. The Forth Isomorphism Theorem**

Let  $G$  be a Group, and  $N \trianglelefteq G$  be a Normal Subgroup. Then, there is a bijection between

$$D \stackrel{\text{def}}{=} \{H \leq G \mid N \leq H\}, \quad C \stackrel{\text{def}}{=} \{\overline{H} \leq G/N\}$$

**Proof.** Let  $\pi : G \rightarrow G/N : g \mapsto gN$  be a natural projection. And, Define

$$\Phi : D \rightarrow C : H \mapsto \pi[H]$$

This function is well-defined: For any  $H \in D$ , let  $aN, bN \in \pi[H]$ . Then,  $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$ , thus  $\pi[H] \leq G/N$ .

To show that one-to-one: Let  $\Phi(A) = \Phi(B)$ . Thus means,  $\pi[A] = \pi[B]$ . Let  $a \in A$ . Then,  $\pi(a) \in \pi[A] = \pi[B]$ , thus  $\pi(a) = \pi(b)$  for some  $b \in B$ . That is,  $aN = bN \iff a \in bN$ . Meanwhile,  $N \leq B$ , thus  $a \in bN \subset B$ ,  $A \subset B$ . Similarly,  $B \subset A$ , that is  $A = B$ .

To show that onto: Let  $K \in C$ . Then,  $N \leq \pi^{-1}[K] \leq G$ , thus clear.

□

## 2.2 Group Action

In this section, we follow that the notation of [Dummit and Foote, 2004, Abstract Algebra].

**Definition 2.2.0.1.** Let  $(G, *)$  be a Group, and  $A$  be a non-empty set.

Define **Group Action** of a group  $G$  on a set  $A$ :

$$\alpha : G \times A \rightarrow A : (g, a) \mapsto g \cdot a$$

satisfies

1. For all  $a \in A$ ,  $1_G \cdot a = a$ .
2. For all  $g_1, g_2 \in G$ ,  $a \in A$ ,  $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$

In this, we said to be ' $G$  acts on a set  $A$ '. Meanwhile, For each  $g \in G$ , Define a map

$$\sigma_g : A \rightarrow A : a \mapsto g \cdot a$$

Then, the *permutation representation*

$$\varphi : G \rightarrow S_A : g \mapsto \sigma_g$$

be a Homomorphism. Clearly, for each  $g \in G$ ,  $a \in A$ ,

$$\alpha(g, a) = g \cdot a = \sigma_g(a) = \varphi(g)(a)$$

Thus, there is one-to-one correspondence between group action and permutation representation.

For each  $a \in A$ , the **stabilizer** of  $a$  in  $G$ :

$$G_a \stackrel{\text{def}}{=} \{g \in G \mid g \cdot a = a\}$$

The **kernel of action**:

$$\ker \alpha \stackrel{\text{def}}{=} \{g \in G \mid g \cdot a = a, \forall a \in A\} = \bigcap_{a \in A} G_a$$

$G_a \leq G$  and  $\ker \alpha \trianglelefteq G$ .

If the kernel of action be trivial, the action is called **faithful**.

**Definition 2.2.0.2.** Let  $\alpha : G \times A \rightarrow A$  be a Group Action. Define a relation on  $A$ :

$$a \sim b \iff a = g \cdot b \text{ for some } g \in G$$

Then, this relation be equivalence relation. Denote the equivalence relation, called **orbit**:

$$\mathcal{C}_a \stackrel{\text{def}}{=} \{b \mid b = g \cdot a \text{ for some } g \in G\} = \{g \cdot a \mid g \in G\}$$

And, the action is called **transitive** if there is only one orbit.

**Lemma 2.2.0.1.** For each  $a \in A$ ,

$$|\mathcal{C}_a| = |G : G_a|$$

*Proof.* Since the map

$$\varphi_a : \mathcal{C}_a \rightarrow \{gG_a \mid g \in G\} : g \cdot a \mapsto gG_a$$

is well-defined, bijection. □

**Theorem 2.2.0.1.** Let  $G$  be a Group, let  $H \leq G$  and  $A = \{gH \mid g \in G\}$ ,  $G$  acts by left multiplication on the set  $A$ .

$$\pi_H : G \rightarrow S_A : g \mapsto \sigma_g$$

be a permutation representation afforded by this action. Then

1.  $G$  acts transitively on  $A$ .
2.  $G_{1H} = \{g \in G \mid gH = H\} = H$ .
3. The kernel of the action  $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$ , this is the largest normal subgroup of  $G$  contained in  $H$ .

**Proof.** Let  $aH, bH \in A$  be given. Then, for  $g = ba^{-1}$ ,  $g \cdot aH = (ga)H = bH$ . Thus,  $A = \mathcal{C}_a$  for any  $a \in G$ . 2 is clear, being  $gH = H \iff g \in H$ .

Now,

$$\begin{aligned}\ker \pi_H &= \{g \in G \mid gxH = xH, \forall x \in G\} \\ &= \{g \in G \mid (x^{-1}gx)H = H, \forall x \in G\} \\ &= \{g \in G \mid x^{-1}gx \in H, \forall x \in G\} \\ &= \{g \in G \mid g \in xHx^{-1}, \forall x \in G\} = \bigcap_{x \in G} xHx^{-1}\end{aligned}$$

And the second assertion given by:

Let  $N$  is a normal subgroup of  $G$  contained in  $H$ , then for any  $x \in G$ ,  $N = xNx^{-1} = xHx^{-1}$ . Thus,

$$N \leq \bigcap_{x \in G} xHx^{-1}$$

□

**Corollary 2.2.0.1.** If  $G$  is a finite group of order  $n$ ,  $p$  is the smallest prime dividing  $|G|$ . Then, any subgroup of index  $p$  is normal.

**Proof.** Let  $|G| = p^r p_1^{r_1} \cdots p_n^{r_n}$  be a prime decomposition,  $H \leq G$  with  $|G : H| = p$ .

Let  $K = \ker \pi_H \leq H$ ,  $k = |H : K|$ . Then,  $|G : K| = |G : H||H : K| = pk$ . By the First-Isomorphism Theorem,

$$G/\ker \pi_H \cong \pi_H[G] \leq S_A$$

and Since  $H$  has  $p$  left cosets,  $A \cong \mathbb{Z}_p$ , thus  $G/K$  is isomorphic to some subgroup of  $S_p$ .

Now, Lagrange's Theorem gives that  $|G/K| = pk$  divides  $|S_p| = p!$ . This implies  $k \mid (p-1)!$ .

$|G : K| = pk$  implies  $|G| = pk \cdot |K|$ . Since  $p$  is the minimal prime that divides  $|G|$ , thus every prime divisor of  $k$  is greater than or equal to  $p$ . This implies must be  $k = 1$ . Thus  $H = K \trianglelefteq G$ . □

**Definition 2.2.0.3.** Let a Group action as:

$$\alpha : G \times G \rightarrow G : (g, a) \mapsto gag^{-1}$$

Now, the orbit driven from this action  $[a] = \{b \in G \mid \exists g \in G \text{ s.t. } b = gag^{-1}\}$  is called be **Conjugacy Class**. More generally,

$$\alpha : G \times \mathcal{P}(G) \rightarrow \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$$

**Lemma 2.2.0.2.** Let  $\alpha : G \times \mathcal{P}(G) \rightarrow \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$  be a Group action acting as Conjugate.

Then,  $G_S = N_G(S)$  and  $|\mathcal{C}_S| = |G : N_G(S)|$ , for any  $S \subseteq G$ .

In particular, if  $S$  is singleton,  $S = \{g_i\}$ , then  $|\mathcal{C}_{\{g_i\}}| = |G : N_G(g_i)| = |G : C_G(g_i)|$ .

*Proof.*

$$G_S = \{g \in G \mid gSg^{-1} = S\} = N_G(S)$$

Thus, for any  $S \in \mathcal{P}(G)$ ,

$$|\mathcal{C}_S| = |G : N_G(S)|$$

□

### 2.2.1 Lagrange's Theorem

## **2.3 Generating subset of a Group**

## 2.4 Commutator Subgroup

# Chapter 3

## Finite Group Theory

### 3.1 The Class Equation

**Theorem 3.1.0.1. The Class Equation**

Let  $G$  be a finite group, and

$g_1, \dots, g_r$  be representatives of the distinct conjugacy classes of  $G$  not contained in the center  $Z(G)$  of  $G$ . Then,

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

### 3.2 Cauchy's Theorem

**Lemma 3.2.0.1. Cauchy's Theorem**

Let  $G$  be a finite group, and  $p$  be a prime dividing  $|G|$ . Then,  $G$  has order  $p$  element.

*Proof.* Define a set:

$$S \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_p) \mid x_i \in G, x_1 x_2 \cdots x_p = 1\}$$

Then,  $S$  has exactly  $|G|^{p-1}$  elements because there are  $|G|$  possible choices for each of the first  $p-1$  elements in  $G$ .

Once  $x_1, \dots, x_{p-1}$  are chosen, then  $x_p$  is uniquely determined by the uniqueness of inverses.

Then, let  $\sigma = (1, 2, \dots, p)$  be a permutation. Then, for any  $\alpha \in S$ ,  $\sigma^n(\alpha) \in S$  for all  $n \in \mathbb{Z}$ , being  $ab = 1 \iff ba = 1$ . More precisely, let  $n \in \mathbb{Z}$  be given,  $\alpha = (x_1, \dots, x_n)$ . Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \dots, x_p, x_1, x_2, \dots, x_n)$$

By  $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$ ,  $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$ . Thus  $\sigma^n(\alpha) \in S$ . Now, define a relation on  $S$  as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on  $S$ . Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is singleton if and only if } \alpha = (x, \dots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,

Suppose that  $\alpha = (x_1, \dots, x_n)$  has different coordinate elements, let  $x_i \neq x_j$ , for some  $i < j$ . Then clearly

$$(x_1, \dots, x_i, \dots, x_p) \neq \sigma^{i-j}(x_1, \dots, x_i, \dots, x_j, \dots, x_p) = (\dots, \underset{i\text{'th element}}{x_j}, \dots)$$

Meanwhile, if  $[\alpha]$  has elements more than 1,  $[\alpha]$  has exactly number of  $p$  elements. Because suppose that  $\alpha = (x_1, \dots, x_p)$  has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \dots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist  $1 \leq i < j < p$  such that  $\sigma^i(\alpha) = \sigma^j(\alpha)$ , that is,  $\sigma^{j-i}(\alpha) = \alpha$ . Now,  $j - i \mid p$ , this is contradiction with  $p$  is prime. Therefore, every equivalent class has order 1 or  $p$ . Consequently,

$$|G|^{p-1} = k + pd$$

where  $k$  is a number of classes of size 1, and  $d$  is a number of classes of size  $p$ . And  $(1, 1, \dots, 1) \in S$ ,  $k$  is at least 1.

Since  $p$  divides  $|G|^{p-1} = k + pd$ , thus  $k$  must be bigger than 1, thus there exists elements such that  $x^p = 1$ .  $\square$

### 3.3 Sylow's Theorem

#### Theorem 3.3.0.1. Sylow's Theorem

Let  $G$  be a group of order  $p^\alpha m$ , where  $p$  is a prime such that  $p \nmid m$ .

A group of order  $p^r$ , ( $r \geq 1$ ) is called a  $p$ -group, Subgroups of  $G$  which are  $p$ -groups are called  $p$ -subgroup.

In particular, subgroups of order  $p^\alpha$  is called Sylow  $p$ -subgroup of  $G$ . And, define a collection

$$\text{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\}, \quad n_p(G) \stackrel{\text{def}}{=} \text{Card}(\text{Syl}_p(G))$$

#### The First Sylow Theorem

There exists a Sylow  $p$ -subgroup of  $G$ . i.e.,  $\text{Syl}_p(G) \neq \emptyset$ .

#### The Second Sylow Theorem

If  $P \in \text{Syl}_p(G)$  and  $Q \leq G$  be a  $p$ -subgroup. Then, there exists  $g \in G$  such that  $Q \leq gPg^{-1}$ .

#### The Third Sylow Theorem

$n_p \equiv 1 \pmod{p}$ ,  $n_p = |G : N_G(P)|$  for any  $P \in \text{Syl}_p(G)$ , and  $n_p \mid m$ .

Before prove above statements, we show that:

**Lemma 3.3.0.1.** Let  $P \in \text{Syl}_p(G)$ . If  $Q$  is  $p$ -subgroup of  $G$ , then  $Q \cap N_G(P) = Q \cap P$ .

**Proof.** Put  $H = Q \cap N_G(P)$ . Since  $P \leq G$ , for any  $p \in P$ ,  $pPp^{-1} = P$ , thus  $p \in N_G(P)$ . i.e.,  $P \leq N_G(P)$ . Thus, Enough to Show that  $H \leq Q \cap P$ . Since  $H \leq N_G(P)$ ,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus,  $PH \leq G$ . And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem,  $H \leq P$  and  $P \cap H \leq P$  must have order of powers of  $p$ , so  $PH$  be a  $p$ -group. Clearly,  $P \leq PH$  and  $P$  is the largest  $p$ -group of  $G$ , thus,  $PH = P$ . This means,  $H \leq P$ .  $\square$

**Proof.** The First Theorem: The existence of Sylow  $p$ -subgroup. Proof by Induction:

If  $|G| = 1$ , there is nothing to prove.

Assume inductively the existence of Sylow  $p$ -subgroups for all groups of order less than  $|G|$ .

In case of  $p \mid |Z(G)|$ , then by Cauchy's Theorem,  $Z(G)$  has a subgroup  $N$  which has order of  $p$ .

Clearly  $N$  is Normal, and  $G/N = |G|/|N| = p^{a-1}m$ . By assumption,  $G/N$  has a subgroup  $P'$  of order  $p^{a-1}$ .

By The Fourth Isomorphism Theorem, Let  $P \leq G$  be a subgroup such that  $P/N = P'$ .

Then,  $|P| = |P/N| \cdot |N| = p^\alpha$ , Thus  $P$  be a Sylow  $p$ -subgroup of  $G$ .

In case of  $p \nmid |Z(G)|$ .

Let  $g_1, \dots, g_r$  be representatives of the distinct conjugacy classes of  $G$ , not contained in  $Z(G)$ . Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

Since  $p$  divides  $|G|$ , if for all  $i = 1, 2, \dots, r$ ,  $p \mid |G : C_G(g_i)|$  then  $p \mid |Z(G)|$ , this is contradiction.

Thus, for some  $j$ ,  $p \nmid |G : C_G(g_j)|$ . Put  $H = C_G(g_j) < G$ . Then,  $|H|$  has a factor of  $p^\alpha$ , by  $p \nmid |G : C_G(g_j)|$ . Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption,  $H$  has a Sylow  $p$ -group, order of  $p^\alpha$ .

Consequently, the existence of Sylow  $p$ -subgroup was shown.

The Second Theorem: Relation of  $p$ -subgroups.

The First Theorem gives existence of Sylow  $p$ -subgroups. Let  $P \in \text{Syl}_p(G)$ . Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \dots, P_r\}$$

Let  $Q \leq G$  be an any  $p$ -subgroup of  $G$ . And,  $Q$  acts by conjugation on  $S$ . i.e.,

$$\alpha : Q \times S \rightarrow S : (q, P_i) \mapsto qP_iq^{-1}$$

Write  $S$  as a disjoint union of orbits under this action by  $Q$ :

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_s$$

where  $r = |\mathcal{O}_1| + \dots + |\mathcal{O}_s|$ . Rearrange a set  $S$  as:  $P_i \in \mathcal{O}_i$ ,  $1 \leq i \leq s$ . Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \stackrel{\text{Thm}}{=} |Q : N_Q(P_i)| \stackrel{\text{def}}{=} |Q : N_G(P_i) \cap Q| \stackrel{\text{Lemma}}{=} |Q : P_i \cap Q|$$

for each  $1 \leq i \leq s$ . Since  $Q$  was arbitrary, Let  $Q = P_1$ , so that  $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$ . And, for each  $i \geq 2$ ,  $P_i \cap P_1 < P_1$ ,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since  $P_1 \in \text{Syl}_p(G)$ , that is  $|P_1| = p^\alpha$ ,  $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$  where  $1 \leq k < \alpha$ . This means for each  $2 \leq i \leq s$ ,  $p$  divides  $|\mathcal{O}_i|$ . Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \dots + |\mathcal{O}_s|) \equiv 1 \pmod{p}$$

Now, Proof by Contradiction: Let  $Q \leq G$  be a  $p$ -subgroup. Suppose that for any  $1 \leq i \leq r$ ,  $Q \not\leq P_i$ . Then,  $P_i \cap Q < Q$  for all  $i$ , this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any  $i$ ,  $p$  divides  $|\mathcal{O}_i|$ , this is Contradiction. This proved Relation of  $p$ -subgroups.

Finally, The Third Theorem:

Since Second Theorem, this gives that  $S = \text{Syl}_p(G)$ , thus  $n_p(G) = r$ . That is,  $n_p \equiv 1 \pmod{p}$ .

Since all Sylow  $p$ -subgroups are Conjugate, for any  $P \in \text{Syl}_p(G)$ ,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. □

## 3.4 More Theorems

**Theorem 3.4.0.1.** *n Factorial Theorem*

If  $G$  is simple and there is a subgroup  $H$  with  $|G : H| = n$ , then  $|G| \mid n!$ .

*Proof.* Let  $G$  act on  $A = \{gH \mid g \in G\}$  by left multiplication. ( $|A| = n$ .)

Let  $\varphi : G \rightarrow S_n$  be a homomorphism afforded above action. Then,  $G \xrightarrow{G \text{ simp.}} G/\ker \varphi \cong \varphi[G] \leq S_n$

□

## 3.5 Simple groups

### 3.6 Cyclic Group

### 3.7 Symmetric Group

### 3.8 Dihedral Group

## **Chapter 4**

### **Ring Theory**

## 4.1 Addition and Multiplication in $\mathbb{Z}$

### 4.1.1 $\mathbb{Z}_n^\times$

**Theorem 4.1.1.1.** For any integer  $n > 1$ ,  $(\mathbb{Z}_n^\times, \times_n)$  is a group where  $\begin{cases} \mathbb{Z}_n^\times \stackrel{\text{def}}{=} \{k \in \mathbb{Z}_n \mid \gcd(k, n) = 1\} \\ a \times_n b \stackrel{\text{def}}{=} ab \bmod n \end{cases}$

*Proof.* Let  $a \in \mathbb{Z}_n^\times$  be given. Then, the Bezout's identity gives  $ax + ny = 1$  for some  $x, y \in \mathbb{Z}$ . Now,  $ax = 1 - ny \equiv 1 \pmod{n}$ .  $\square$

**Theorem 4.1.1.2. Fermat's Little Theorem**

If  $a \in \mathbb{Z}$  and  $p$  is prime not dividing  $a$ , then  $a^{p-1} \equiv 1 \pmod{p}$ .

*Proof.* Let  $a \in \{1, 2, \dots, p-1\}$ . Put  $k = |a|$ . Then,  $\{1, a^1, a^2, \dots, a^{k-1}\}$  forms a subgroup of  $\mathbb{Z}_p$ . By Lagrange's Theorem,  $k \mid p-1$ . Now,  $a^{p-1} = a^{km} = (a^k)^m = 1 \pmod{p}$ .  $\square$

**Theorem 4.1.1.3.**  $a \in \mathbb{Z}_n$  is a zero divisor if and only if  $\gcd(a, n) \neq 1$ .

*Proof.* Suppose that  $\gcd(a, n) = 1$ . If  $ax + n\mathbb{Z}$  is zero for some  $x \in \mathbb{Z}_n$ , then  $ax = nk$  for some  $k \in \mathbb{Z}$ . Thus,  $n \mid ax$  and  $\gcd(a, n) = 1$ , this implies  $n \mid x$ . Conversely, Suppose that  $d = \gcd(a, n) \neq 1$ . Then,

$$a \cdot \frac{n}{d} = \frac{a}{d} \cdot n \in n\mathbb{Z}$$

But,  $a \notin n\mathbb{Z}$  and  $\frac{n}{d} \notin n\mathbb{Z}$ , thus  $a$  is zero divisor.  $\square$

**Corollary 4.1.1.1.** The subset  $G_n \stackrel{\text{def}}{=} \{a \in \mathbb{Z}_n \mid a \text{ is zero divisor}\}$  is a group under the multiplication modulo  $n$ .

**Theorem 4.1.1.4. Euler's Theorem**

If  $\gcd(a, n) = 1$ ,  $a^{\varphi(n)} \equiv 1 \pmod{n}$ .

*Proof.* Let  $a \in \mathbb{Z}$  with  $\gcd(a, n) = 1$ . Then, there exists  $0 < b \in a + n\mathbb{Z}$  such that  $b < n$  and  $\gcd(b, n) = 1$ . Then,  $a^{\varphi(n)} \equiv b^{\varphi(n)} \pmod{n}$ . Since  $b \in \mathbb{Z}_n^\times$  and order of  $\mathbb{Z}_n^\times$  is  $\varphi(n)$ , thus  $b^{\varphi(n)} \equiv 1 \pmod{n}$ .  $\square$

## 4.2 Ideal

**Definition 4.2.0.1.** Let  $R$  be a Ring. A subset  $I \subseteq R$  is called *ideal* of  $R$  if:

1.  $I \subseteq R$  is a subgroup of  $R$ .
2.  $I$  is closed under the multiplication.
3. For any  $r \in R$ ,  $rI \subseteq I$  and  $Ir \subseteq I$ . (In other word, for any  $r \in R, a \in I$ ,  $ra \in I$  and  $ar \in I$ .)

**Theorem 4.2.0.1.** Let  $R$  be a Ring. Then, TFAE:

1.  $I \subset R$  is an Ideal of  $R$ .
2. The additive Quotient Group  $R/I \stackrel{\text{def}}{=} \{r + I \mid r \in R\}$  be a Ring under the operation:

$$(r + I) \times (s + I) = (rs) + I$$

*Proof.* Observation:

$$r_1 + I = r_2 + I \iff r_1 - r_2 \in I \iff \exists a \in I \text{ s.t. } r_1 = r_2 + a$$

Now, for well-definedness, want to show that the equality

$$\begin{aligned} (r + I) \times (s + I) &= (rs) + I \\ \stackrel{(*)}{=} [(r + \alpha) + I] \times [(s + \beta) + I] &= (r + \alpha)(s + \beta) + I = (rs + r\beta + \alpha s + \alpha\beta) + I \end{aligned}$$

(\*) holds for any  $r, s \in R$ ,  $\alpha, \beta \in I$ .

If  $I$  is Ideal, then  $r\beta, \alpha s, \alpha\beta \in I$ . Thus closed under the addition gives (\*).

Conversely, if this operation is well-defined, then for any  $r, s \in R$ ,  $\alpha, \beta \in I$ , (\*) holds.

Substituting zero to each  $r, s, \alpha, \beta$  gives  $I$  is ideal. □

#### 4.2.1 Properties of Ideal in Ring with identity

**Definition 4.2.1.1.** Let  $R$  be a Ring with identity, and  $A \subseteq R$ . Define *Ideal generated by A* as:

$$(A) \stackrel{\text{def}}{=} \bigcap_{\substack{I \text{ ideal} \\ A \subseteq I}} I$$

And,

$$\begin{aligned} RA &\stackrel{\text{def}}{=} \{r_1a_1 + \cdots + r_na_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\} \\ AR &\stackrel{\text{def}}{=} \{a_1r_1 + \cdots + a_nr_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\} \\ RAR &\stackrel{\text{def}}{=} \{r_1a_1r'_1 + \cdots + r_na_nr'_n \mid n \in \mathbb{N}, r_i, r'_i \in R, a_i \in A\} \end{aligned}$$

**Lemma 4.2.1.1.** Let  $R$  be a Ring with identity, and  $A \subseteq R$ . Then,  $(A) = RAR$ .

*Proof.* Since  $RAR$  is ideal which contains  $A$ ,  $(A) \subseteq RAR$ .

And, conversely, if  $\sum_{i=1}^n r_i a_i r'_i \in RAR$ , then  $\sum_{i=1}^n r_i a_i r'_i \in (A)$  because each  $r_i a_i r'_i$  are contained in  $(A)$ , being  $(A)$  is ideal containing  $A$  and ideal is closed under the addition.  $\square$

**Theorem 4.2.1.1.** Let  $I$  be an ideal of Ring  $R$  with identity.

$$I = R \text{ if and only if } I \text{ contains a unit.}$$

*Proof.* Right direction is clear by  $1 \in R = I$ .

Denote  $u \in I$  be a unit with  $vu = 1$ , and Let  $r \in R$  be given. Then,

$$r = r1 = rvu \in I$$

$\square$

**Definition 4.2.1.2.** An Ideal  $M$  of  $R$  is *Maximal ideal* if: There is no Ideal  $I$  such that  $M \subsetneq I \subsetneq R$ .

**Theorem 4.2.1.2.** Let  $R$  be a Ring with identity.

Then, every proper ideal  $I \subsetneq R$  is contained in a maximal ideal.

*Proof.*

$\square$

**Lemma 4.2.1.2.** Let  $R$  be a commutative Ring with identity,  $M, P$  are proper ideals of  $R$ .

1.  $M$  is Maximal Ideal if and only if  $R/M$  is a field.
2.  $P$  is Prime Ideal if and only if  $R/P$  is an integral domain.

**Summary:**  $M$  maximal  $\iff R/M$  field  $\implies R/M$  integral domain  $\iff M$  prime.

*Proof.*

$M$  is maximal  $\iff$  There is no ideal  $I$  such that  $M \subsetneq I \subsetneq R$   
 $\iff$  There are Ideals of  $R/M$  only 0 and  $R/M$   
 $\iff$   $R/M$  is field

$P$  is Prime Ideal  $\iff$  If  $ab \in P$ , then  $a \in P$  or  $b \in P$   
 $\iff$  If  $ab + P = P$ , then  $a + P = P$  or  $b + P = P$   
 $\iff$  If  $\bar{a}\bar{b} = \bar{0}$ , then  $\bar{a} = \bar{0}$  or  $\bar{b} = \bar{0}$   
 $\iff$   $R/P$  is integral domain

□

### 4.3 Ring of Fractions

**Theorem 4.3.0.1.** Let  $R$  be a Commutative Ring,  $D \subset R$  be a subset such that  $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$ . Then, there exists a Commutative Ring  $Q$  with identity satisfies:

1.  $R$  can embed in  $Q$ , and every element of  $D$  becomes unit in  $Q$ . More precisely,  $Q = \{rd^{-1} \mid r \in R, d \in D\}$ .
2.  $Q$  is the smallest Ring containing  $R$  with identity such that every element of  $D$  becomes unit in  $Q$ .

**Proof.** Let  $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$  and the relation  $\sim$  on  $\mathcal{F}$  by  $(r_1, d_1) \sim (r_2, d_2) \iff r_1d_2 = r_2d_1$ . Then,  $\sim$  is equivalent relation: reflexive and symmetric are clear, and Suppose that  $(r_1, d_1) \sim (r_2, d_2)$  and  $(r_2, d_2) \sim (r_3, d_3)$ .

$$r_2d_3 = r_3d_2 \implies r_2d_1d_3 = r_3d_1d_2 \implies r_1d_2d_3 = r_3d_1d_2 \implies d_2(r_1d_3 - r_3d_1) \implies r_1d_3 = r_3d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations  $+, \times$  on  $Q$ :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1d_2 + r_2d_1}{d_1d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1r_2}{d_1d_2}$$

**Well-Definedness:** If  $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$  and  $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$ ,

$$\frac{r_1d_2 + r_2d_1}{d_1d_2} = \frac{r_1d_2d'_1d'_2 + r_2d_1d'_1d'_2}{d_1d_2d'_1d'_2} = \frac{(r_1d'_1)d_2d'_2 + (r_2d'_2)d_1d'_1}{d_1d_2d'_1d'_2} = \frac{(r'_1d_1)d_2d'_2 + (r'_2d_2)d_1d'_1}{d_1d_2d'_1d'_2} = \frac{(r'_1d'_2 + r'_2d'_1)d_1d_2}{d_1d_2d'_1d'_2} = \frac{r'_1d'_2 + r'_2d'_1}{d'_1d'_2}$$

$$\frac{r_1r_2}{d_1d_2} = \frac{r_1r_2d'_1d'_2}{d_1d_2d'_1d'_2} = \frac{(r_1d'_1)(r_2d'_2)}{d_1d_2d'_1d'_2} = \frac{(r'_1d_1)(r'_2d_2)}{d_1d_2d'_1d'_2} = \frac{r'_1r'_2d_1d_2}{d_1d_2d'_1d'_2} = \frac{r'_1r'_2}{d'_1d'_2}$$

Now,  $(Q, +, \times)$  constructs Commutative Ring with identity: for any  $d \in D$ , put  $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$ ,  $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$ . Then,

1.  $(R, +, \times)$  closed under the operations since  $D$  is closed under the multiplication.

2.  $(R, +)$  has a zero:  $\frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1d + 0d_1}{d_1d} = \frac{r_1d}{d_1d} = \frac{r_1}{d_1}$ .

3.  $(R, +)$  has an inverse:  $\frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1d_1 + (-r_1)d_1}{d_1d_1} = \frac{[(r_1) + (-r_1)]d_1}{d_1d_1} = \frac{0d_1}{d_1d_1} = \frac{0}{d_1d_1} = 0_Q$ .

4.  $(R, +, \times)$  satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left( \frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2d_3 + r_3d_2}{d_2d_3} = \frac{r_1r_2d_3 + r_1r_3d_2}{d_1d_2d_3} = \frac{r_1r_2d_1d_3 + r_1r_3d_1d_2}{d_1d_2d_1d_3} = \frac{r_1r_2}{d_1d_2} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left( \frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1d_2 + r_2d_1}{d_1d_2} \times \frac{r_3}{d_3} = \frac{r_1r_3d_2 + r_2r_3d_1}{d_1d_2d_3} = \frac{r_1r_3d_2d_3 + r_2r_3d_1d_3}{d_1d_3d_2d_3} = \frac{r_1r_3}{d_1d_3} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

5.  $(R, \times)$  has an identity:  $\frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1d}{d_1d} = \frac{r_1}{d_1}$ .

6. Elements of  $D$  become unit in  $Q$ : Define  $\iota : R \rightarrow Q : r \mapsto \frac{rp}{p}$  where  $p \in D$  is any fixed element in  $D$ . Then,  $\iota$  is Ring-Monomorphism because:

6-1. Well-Defined and Injective:  $\iota(r_1) = \iota(r_2) \iff \frac{r_1p}{p} = \frac{r_2p}{p} \iff (r_1 - r_2)p = 0 \iff r_1 = r_2$

6-2. For any  $d \in D$ ,  $\iota(d)$  is a unit of  $Q$ : Put  $(\iota(d))^{-1} \stackrel{\text{def}}{=} \frac{p}{dp}$ , then

$$\iota(d) \times (\iota(d))^{-1} = \frac{dp}{p} \times \frac{p}{dp} = \frac{dpp}{dpp} = 1_Q$$

That is,  $\iota$  is embedding from  $R$  into  $Q$  such that  $\iota[D]$  becomes units of  $Q$  except zero. Moreover, if  $D = R \setminus \{0\}$ , then  $Q$  is field.

7.  $Q$  is the smallest ring containing  $R$  with identity such that every element of  $D$  becomes units in  $Q$ .

Let  $S$  be an any commutative ring with identity,

and assume that  $\varphi : R \rightarrow S$  is a Ring-Monomorphism such that for any  $d \in D$ ,  $\varphi(d)$  is unit in  $S$ .

Define  $\phi : Q \rightarrow S : \frac{r}{d} \mapsto \varphi(r)\varphi(d)^{-1}$ . Then, this  $\phi$  is well-defined and injective:

$$\begin{aligned} \phi\left(\frac{r_1}{d_1}\right) = \phi\left(\frac{r_2}{d_2}\right) &\iff \varphi(r_1)\varphi(d_1)^{-1} = \varphi(r_2)\varphi(d_2)^{-1} \iff \varphi(r_1)\varphi(d_2) = \varphi(r_2)\varphi(d_1) \\ \xrightarrow{\text{homomor.}} \varphi(r_1d_2) = \varphi(r_2d_1) &\stackrel{\text{one-to-one}}{\iff} r_1d_2 = r_2d_1 \iff \frac{r_1}{d_1} = \frac{r_2}{d_2} \end{aligned}$$

That is, if a commutative ring  $S$  with identity contains a copy of  $R$  such that the denominator set  $D$  of  $R$  becomes unit in  $S$ , then  $S$  contains ring of fractions  $Q$  of  $R$ . Thus  $S = Q$  is the smallest ring that satisfies these conditions.

□

## 4.4 Commutative Ring with identity

**Theorem 4.4.0.1.** Finite integral domain is field.

*Proof.* Let  $R$  be a finite integral domain, and non-zero  $a \in R$  be given.

Then, the map from  $R$  into  $R$   $x \mapsto ax$  is injective: because  $ax = ay \implies a(x - y) = 0$  and  $R$  has no zero divisor.  $R$  is finite, so this map is surjective. Thus, there exists  $b \in R$  such that  $ab = 1$ ,  $a$  is unit.  $\square$

**Lemma 4.4.0.1.** Let  $R$  be a Commutative Ring,  $a, b \in R$  with  $b \neq 0$ .

$$a = bx \text{ for some } x \in R \stackrel{\text{def}}{\iff} b \mid a \iff a \in (b) \iff (a) \subseteq (b)$$

**Lemma 4.4.0.2.** Let  $a, b$  be non-zero elements in a Commutative Ring  $R$ .

If  $(a, b) = (d)$ , then  $d$  is the greatest common divisor of  $a$  and  $b$ .

**Theorem 4.4.0.2.** Let  $R$  be an integral domain. If  $(d) = (d')$ , then  $d' = ud$  for some unit  $u \in R$ . Particular,  $d$  and  $d'$  are greatest common divisor of  $a$  and  $b$ , then  $(d) = (d')$ , thus  $d' = ud$  for some unit  $u \in R$ .

*Proof.* If either  $d$  or  $d'$  is zero, then there is nothing to prove. Thus, Suppose that neither  $d$  nor  $d'$  is non-zero. Since  $(d) \subseteq (d')$  and  $(d) \supseteq (d')$ ,  $d' = dx$  for some  $x \in R$  and  $d = d'y$  for some  $y \in R$ . Combining above, then  $d' = dx = (d'y)x = d'(yx)$ , this implies  $d'(1 - yx) = 0$ . Since  $d'$  is non-zero and  $d'$  chosen in the integral domain,  $1 - xy = 0$ . Now, both  $x$  and  $y$  are unit, we obtain the result. Second assertion is clear by the First.  $\square$

### 4.4.1 Euclidean Domain

**Definition 4.4.1.1.** An integral domain  $R$  is called *Euclidean Domain* if: there exists a norm  $N$  such that:

for any  $a, b \in R$  with  $b \neq 0$ , there exist  $q, r \in R$  with  $a = qb + r$  with  $r = 0$  or  $N(r) < N(b)$ .

This definition allows us the *Euclidean Algorithm* on an integral domain  $R$ : for any  $a, b \in R$  with  $b \neq 0$ ,

$$\begin{aligned} a &= q_0b + r_0 \\ b &= q_1r_0 + r_1 \\ r_0 &= q_2r_1 + r_2 \\ r_1 &= q_3r_2 + r_3 \\ &\vdots \\ r_k &= q_{k+2}r_{k+1} + r_{k+2} \\ &\vdots \\ r_{n-2} &= q_n r_{n-1} + r_n \\ r_{n-1} &= q_{n+1} r_n \end{aligned}$$

This process gives a chain:

$$N(r_n) < N(r_{n-1}) < \cdots < N(r_2) < N(r_1) < N(r_0)$$

and this process terminates in finite iteration, since well-ordering principle.

**Theorem 4.4.1.1.** Let  $I$  be an ideal of a Euclidean Domain  $R$ . Then,  $I$  is principal ideal.

**Proof.** If  $I$  is zero ideal, there is nothing to prove. Let  $I$  be a non-zero ideal. Since the set  $\{N(a) \mid a \in I \setminus \{0\}\}$  has a minimum by Well-Ordering Principle, choose  $d \in I$  such that  $N(d) \leq N(a)$ ,  $\forall a \in I \setminus \{0\}$ . Clearly,  $(d) \subseteq I$ . Let  $a \in I$ . Then, there is  $q, r \in R$  such that

$$a = qd + r \text{ with } r = 0 \text{ or } N(r) < N(d)$$

Since  $r = a - qd \in I$  by  $a, d \in I$ , thus closed under the multiplication gives  $r \in I$ . But, by minimality of  $d$ ,  $r$  must be 0. Now,  $a = qd + r = qd \in (d)$ .  $\square$

#### Theorem 4.4.1.2. Euclidean Algorithm

Let  $R$  be a Euclidean Domain,  $a, b \in R$  be non-zero.

Denote  $d = r_n$  where  $r_n$  is the last nonzero remainder in the Euclidean Algorithm for  $a$  and  $b$ .

Then,  $d$  is the greatest common integer of  $a$  and  $b$ . And,  $(d) = (a, b)$ . That is, there exist  $x, y \in R$  such that

$$d = ax + by$$

**Proof.** Note that:  $(a, b)$  is principal in Euclidean Domain.

Moreover,  $(a, b)$  is the smallest ideal containing  $(a)$  and  $(b)$ . That is, If  $(a) \subseteq (x)$  and  $(b) \subseteq (x)$ , then  $(a, b) \subseteq (x)$ . Now, Enough to Show:

1.  $(a), (b) \subseteq (d)$ . (It follows that  $(a, b) \subseteq (d)$ )

2.  $(d) \subseteq (a, b)$ . (That is,  $(d) = (a, b)$ )

Since  $(a), (b) \subseteq (d)$  if and only if  $d \mid a, b$ , show that  $d$  divides  $a, b$ .

In the last equation,  $r_{n-1} = q_{n+1}r_n = q_{n+1}d$ . Thus,  $d \mid r_{n-1}$ .

Clearly,  $r_n \mid r_{n-1}$ , thus  $d \mid r_{n-2}$ . Repeat this to finite times, then we obtain:  $\forall 1 \leq i \leq n$ ,  $d \mid r_i$ . As result,  $d \mid a$  and  $d \mid b$ . This proved 1.

For to show that 2., we will prove  $d \in (a, b)$ .

The first equation gives directly  $r_0 \in (a, b)$ .

That is,  $(r_0) \subseteq (a, b)$ , thus  $r_1 = b - q_1r_0 \in (a, b)$ .

Inductively,  $r_n = d \in (a, b)$ , theorem completed.  $\square$

$$a = q_0b + r_0$$

$$b = q_1r_0 + r_1$$

$$r_0 = q_2r_1 + r_2$$

$$r_1 = q_3r_2 + r_3$$

⋮

$$r_{n-2} = q_n r_{n-1} + r_n$$

$$r_{n-1} = q_{n+1}r_n + 0$$

## 4.4.2 Principal Ideal Domain

**Definition 4.4.2.1.** An integral domain  $R$  is called *Principal Ideal Domain* if: every ideal of  $R$  is principal.

**Theorem 4.4.2.1.** Let  $R$  be a Principal Ideal Domain, and  $a, b \in R$  be non-zero.

Let  $d$  be a generator for the principal ideal  $(a, b)$ . Then,

$d$  is the greatest common divisor of  $a$  and  $b$ , and unique up to multiplication of unit of  $R$ .

**Theorem 4.4.2.2.** Every non-zero Prime Ideal in a Principal Domain is Maximal Ideal.

**Proof.** Let  $(p)$  be a non-zero Prime Ideal.

Let  $I = (m)$  be an Ideal such that  $(p) \subseteq (m)$ . Since  $p \in (m)$ , there is a  $x \in R$  such that  $p = mx$ .

But,  $p = mx \in (p)$ , Prime Ideal,  $m \in (p)$  or  $x \in (p)$ .

If  $m \in (p)$ ,  $m = py$  for some  $y \in R$ . That is,  $m = py \in (p)$ ,  $(p) = (m)$ .

If  $x \in (p)$ ,  $x = pz$  for some  $z \in R$ . That is,  $p = mx = mpz = p(mz)$ ,  $m$  becomes a unit.

The Ideal  $(m)$  containing unit implies  $(m) = R$ .  $\square$

#### 4.4.3 Noetherian Domain

**Definition 4.4.3.1.** The Ring  $R$  is said to be *Noetherian Ring* if:  $R$  satisfies *Ascending Chain Condition* on ideals.

The Integral Domain  $R$  with Noetherian is called *Noetherian Doomain*.

**Theorem 4.4.3.1.** Principal Ideal Domain is Noetherian Domain.

*Proof.* Suppose that there is an ascending chain of ideals,

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R$$

(Considering only countable Chain: Since m.stackexchange 4265544.)

Put  $I \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} I_i$ . Since for any  $r \in R$  and  $a \in I$ , there exists  $i \in \mathbb{N}$  such that  $a \in I_i$ . Thus,  $ra \in I_i \in I$ ,  $I$  is ideal.

Since  $R$  is Principal, for some  $a \in R$ ,  $(a) = I$ . That is,  $a \in I$ . This implies there exists  $n \in \mathbb{N}$  such that  $a \in I_n$ . Now,  $(a) \subseteq I_n \subseteq I = (a)$ , This  $I_n = (a) = I$ . Consequently,  $R$  is Noetherian.  $\square$

#### 4.4.4 Unique Factorization Domain

**Definition 4.4.4.1.** Let  $R$  be an Integral Domain.

1. A non-zero, not unit  $r \in R$  is called **irreducible** of  $R$  if: If  $r = ab$  for some  $a, b \in R$ , then either  $a$  or  $b$  is a unit.
2. A non-zero, not unit  $p \in R$  is called **prime** of  $R$  if: If  $p | ab$  for some  $a, b \in R$ , then either  $p | a$  or  $p | b$ .

Clearly,  $p$  is prime if and only if  $(p)$  is Prime Ideal.

**Theorem 4.4.4.1.** Let  $R$  be an Integral Domain. Then, every prime element is irreducible.

*Proof.* Let  $R$  be an Integral Domain,  $p \in R$  be a prime. Suppose that  $p = ab$  for some  $a, b \in R$ . Then, clearly  $p | ab$ , thus  $p | a$  or  $p | b$ . If  $p | a$ , then  $a = px$  for some  $x \in R$ . Now,  $p = ab = pxb$ ,  $p(1 - xb) = 0$ . Since  $R$  is integral domain and  $p$  is non-zero,  $xb = 1$ . That is,  $b$  is a unit, thus  $p$  is irreducible.  $\square$

**Definition 4.4.4.2.** Let  $R$  be an integral domain and let  $r \in R$  be a nonzero, nonunit element.

1. We say that  $r$  is **factorizable** if there exist irreducible elements  $p_1, \dots, p_n$  ( $n \geq 1$ ) such that

$$r = p_1 p_2 \cdots p_n.$$

Any such expression is called an **irreducible factorization** of  $r$ .

2. An irreducible factorization is **unique up to associates** if for any two irreducible factorizations

$$r = p_1 \cdots p_n = q_1 \cdots q_m,$$

we have  $n = m$  and there exist a permutation  $\sigma \in S_n$  and units  $u_1, \dots, u_n \in R^\times$  such that

$$q_i = u_i p_{\sigma(i)} \quad (i = 1, \dots, n),$$

equivalently,  $q_i$  is associate to  $p_{\sigma(i)}$  for each  $i$ .

The domain  $R$  is called a **factorization domain** (also: **atomic**) if every nonzero, nonunit element of  $R$  is factorizable. If, in addition, irreducible factorizations are unique up to associates, then  $R$  is called a **unique factorization domain (UFD)**.

**Theorem 4.4.4.2.** Noetherian Domain is Factorization Domain.

*Proof.* Let  $R$  be a Noetherian Domain. And, let  $r \in R$  be a non-zero, not unit.

There exist only two possibility:  $r$  is irreducible or not irreducible.

If  $r$  is irreducible, then there is nothing to prove. If  $r$  is not irreducible, then there exist not unit  $r_1, r_2 \in R$  such that  $r = r_1 r_2$ .

If  $r_1$  and  $r_2$  are irreducible, prove end. If  $r_1$  is reducible, then there exist not unit  $r_{1,1} r_{1,2} \in R$  such that  $r_1 = r_{1,1} r_{1,2}$ .

If this process never terminates, then, there is a infinite strictly ascending chain:

$$(r) \subsetneq (r_1) \subsetneq (r_{1,1}) \subsetneq \cdots \subsetneq R$$

Strictly given by  $r = r_1 r_2$  and  $r_2$  is not a unit.

More precisely, if  $(r) = (r_1)$ , then  $r_1 = rk$  for some  $k \in R$ ,  $r_1 = rk = r_1 r_2 k$ ,  $r_1$  becomes a unit. Contradiction.  $\square$

**Theorem 4.4.4.3.**

1. In Principal Ideal Domain, every irreducible element is prime.
2. In Unique Factorization Domain, every irreducible element is prime.

*Proof.* Let  $R$  be a Principal Ideal Domain, and  $r \in R$  be an irreducible.

Suppose that  $(m)$  is an ideal of  $R$  such that  $(r) \subseteq (m)$ .

Then,  $r \in (m)$  implies  $r = mx$  for some  $x \in R$ , now irreducibility gives either  $m$  or  $x$  is a unit.

If  $m$  is a unit, then  $(m) = R$ . If  $x$  is a unit,  $r = mx$  implies  $rx^{-1} = m$  implies  $m \in (r)$  implies  $(m) \subseteq (r)$  implies  $(m) = (r)$ .

Consequently,  $(r)$  is maximal ideal in the Principal Ideal Domain,

$$(r) \text{ is a maximal} \iff R/(r) \text{ is a field} \implies R/(r) \text{ is an integral domain} \iff (r) \text{ is Prime.}$$

Let  $R$  be a Unique Factorization Domain, and  $r \in R$  be an irreducible. Suppose that  $r | ab$  for some  $a, b \in R$ .

If either  $a$  or  $b$  is unit, then  $r | ab$  implies  $r$  divides  $a$  or  $b$ , there is nothing to prove.

If neither  $a$  nor  $b$  is a unit, write as factorization form:  $a = a_1 \cdots a_n$  and  $b = b_1 \cdots b_m$ , being  $a, b$  in U.F.D.

Since  $r$  divides  $ab = a_1 \cdots a_n b_1 \cdots b_m$ , there exists  $x \in R$  such that

$$rx = a_1 \cdots a_n b_1 \cdots b_m$$

If  $x$  is a unit, then  $r = x^{-1} a_1 \cdots a_n b_1 \cdots b_m$ . But, the uniqueness gives contradiction. Thus  $x$  is not unit.

Now,  $x$  has irreducible factorization, the uniqueness gives  $r = a_i$  for some  $1 \leq i \leq n$  or  $r = b_j$  for some  $1 \leq j \leq m$ .

This means  $r$  divides  $a$  or  $b$ .  $\square$

## 4.5 Examples

### 4.5.1 Matrix Ring

**Theorem 4.5.1.1.**  $\phi : \mathbb{C} \rightarrow M_2(\mathbb{R}) : a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$  is Ring-Embedding.

*Proof.* □

### 4.5.2 Group Ring

### 4.5.3 Integer Ring

**Definition 4.5.3.1.** Let  $D$  be a rational number such that  $\sqrt{D} \notin \mathbb{Q}$ .

Define *Quadratic field* for  $D$ :

$$\mathbb{Q}(\sqrt{D}) \stackrel{\text{def}}{=} \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$$

### 4.5.4 Boolean Ring

**Definition 4.5.4.1.** A Ring  $R$  is called *Boolean Ring* if: for any  $a \in R$ ,  $a^2 = a$ .

**Theorem 4.5.4.1.** Boolean Ring is Commutative Ring.

*Proof.* Let  $a, b \in R$  be given. Then,

$$a^2 + b^2 = a + b = (a + b)^2 = a^2 + ab + ba + b^2 \implies ab = -ba$$

Meanwhile, for any  $c \in R$ ,

$$c = c^2 = (-c)^2 = -c$$

Thus,  $ab = -ba = ba$ . □

#### 4.5.5 Nilradical Ideal

**Definition 4.5.5.1.** Let  $R$  be a Ring.

An element  $x \in R$  is called **Nilpotent** if:  $x^m = 0$  for some  $m \in \mathbb{N}$ .

**Proposition 4.5.5.1.** Let  $R$  be a Ring with identity.

- Let  $n = a^k b$  where  $a, b \in \mathbb{Z}$  and  $k \in \mathbb{N}$ . Then,  $ab + n\mathbb{Z}$  is nilpotent in  $\mathbb{Z}/n\mathbb{Z}$ .

**Proof.**  $(ab)^k = a^k b^k = a^k b \cdot b^{k-1} = n \cdot b^{k-1} \in n\mathbb{Z}$ . □

- Let  $a \in \mathbb{Z}$  and  $n \in \mathbb{N}$ . Then,

$a + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$  is nilpotent if and only if a prime  $p$  divides  $n$  implies  $p$  divides  $a$ .

**Proof.** Let  $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$  be a prime decomposition.

The right assertion means  $a = p_1 \cdots p_k \cdot m$  for some  $m \in \mathbb{Z}$ . Let  $\alpha$  be the least common multiple.

Then,  $a^\alpha \in n\mathbb{Z}$ . Conversely, if  $a^k \in n\mathbb{Z}$  for some  $k \in \mathbb{N}$ ,  $a^k = nm$  for some  $m \in \mathbb{Z}$ .

If a prime  $p$  divides  $n$ , then it divides  $nm = a^k$ , thus divides  $a$ . □

**Proposition 4.5.5.2.** Let  $R$  be a Commutative Ring with identity, and nonzero  $x \in R$  be a nilpotent element.

- $x$  is zero divisor. Hence, not unit.
- For any  $r \in R$ ,  $rx$  is nilpotent.
- $1+x$  is unit in  $R$ .

**Proof.**  $(1+x)(1-x+x^2-x^3+\cdots+(-x)^{m-1}) = (1-x+x^2-x^3+\cdots+(-x)^{m-1})+(x-x^2+x^3-x^4+\cdots-(-x)^m) = 1$ . □

- If  $u$  is unit and  $a$  is nilpotent, then  $u+a$  is unit.

**Proof.**  $u+a = u^{-1}(1+ua)$ . □

**Definition 4.5.5.2.** Let  $R$  be a Commutative Ring. Define **Nilradical** of  $R$ :

$$\mathfrak{N}(R) \stackrel{\text{def}}{=} \{x \in R \mid x \text{ is nilpotent in } R\}$$

**Lemma 4.5.5.1.** Nilradical of Commutative Ring  $R$  with identity is ideal.

**Proof.** Trivially,  $0 \in \mathfrak{N}(R)$ . Suppose  $x, y \in \mathfrak{N}(R)$  are nilpotents in  $R$  with  $x^n = y^m = 0$  for some  $n, m \in \mathbb{N}$ .

$$(x-y)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^{n+m-k} (-y)^k = x^n \sum_{k=0}^m \binom{n+m}{k} x^{m-k} (-y)^k + (-y)^m \sum_{k=m+1}^{n-1} \binom{n+m}{k} x^{n+m-k} (-y)^{-m+k} = 0$$

implies  $x-y \in \mathfrak{N}(R)$ , thus Nilradical is closed under the subtraction, thus it is subgroup.

Closed under the multiplication is trivial:  $(xy)^{nm} = x^{nm}y^{nm} = 0^m 0^n = 0$ .

And, since above discussion,  $rx$  is nilpotent for any  $r \in R$  and  $a \in \mathfrak{N}(R)$ . □

**Theorem 4.5.5.1.** Let  $R$  be a Commutative Ring with identity.  $R/\mathfrak{N}(R)$  has no nilpotent element except zero.

**Proof.** Let  $a + \mathfrak{N}(R) \in R/\mathfrak{N}(R)$  be a non-zero. If  $a^m + \mathfrak{N}(R) = \mathfrak{N}(R)$  for some  $m \in \mathbb{N}$ , that is,  $a^m \in \mathfrak{N}(R)$ . This means  $a^{nm} = 0$  for some  $n \in \mathbb{N}$ , thus  $a \in \mathfrak{N}(R)$ . This is contradiction with  $a + \mathfrak{N}(R)$  is non-zero. □

**Theorem 4.5.5.2.** Let  $R$  be a Commutative Ring and  $I \subset R$  be an ideal.

If  $\mathfrak{N}(I) = I$  and  $\mathfrak{N}(R/I) = R/I$ , then  $\mathfrak{N}(R) = R$ .

*Proof.* Let  $r \in R$  be given. Since  $\mathfrak{N}(R/I) = R/I$ ,  $r + I$  is nilpotent in  $R/I$ . That is,  $r^n + I = I$  for some  $n \in \mathbb{N}$ . Since  $\mathfrak{N}(I) = I$ ,  $r^n \in I$  is nilpotent, thus  $(r^n)^m = 0$  for some  $m \in \mathbb{N}$ . Thus  $r^{nm} = 0$ .  $\square$

#### 4.5.6 Annihilator Ideal

**Lemma 4.5.6.1.** Let  $R$  be a Ring. For any  $a \in R$ ,  $I_a \stackrel{\text{def}}{=} \{x \in R \mid ax = 0\}$  is a subgroup of  $R$ .

*Proof.*  $0 \in I_a$  is trivial. Let  $x, y \in I_a$ . Then,  $a(x - y) = ax - ay = 0 - 0 = 0$ . Thus  $x - y \in I_a$ . Moreover,  $a(xy) = (ax)y = 0y = 0$ . Thus  $xy \in I_a$ .  $\square$

**Definition 4.5.6.1.** Let  $R$  be a Ring with identity,  $a \in R$  be a fixed element, and  $L \subset R$  be right ideal in  $R$ .

1. Define **right annihilator** of  $a$  in  $R$ :  $\{x \in R \mid ax = 0\}$ .
2. Define **right annihilator** of  $L$  in  $R$ :  $\{x \in R \mid ax = 0, \forall a \in L\}$ .

**Lemma 4.5.6.2.** Let  $R$  be a Ring with identity. Then, the right annihilator of  $a$  in  $R$  is right ideal.

*Proof.* Let  $r \in R$  and  $s \in \{x \in R \mid ax = 0\}$ . Then,  $asr = (as)r = 0r = 0$ . Thus  $sr \in \{x \in R \mid ax = 0\}$ .  $\square$

**Lemma 4.5.6.3.** Let  $R$  be a Ring with identity. Then, the right annihilator of  $L$  in  $R$  is two-sided ideal.

*Proof.* Let  $r, r' \in R$  and  $s \in \{x \in R \mid ax = 0, \forall a \in L\}$ .

For any  $a \in L$ ,  $arsr' = (ar)sr' = 0r' = 0$ . Thus,  $rsr' \in \{x \in R \mid ax = 0, \forall a \in L\}$ .  $\square$

## 4.6 Homomorphisms

### 4.6.1 Formal polynomial differentiation map

**Definition 4.6.1.1.** Let  $F$  be a field with characteristic is 0. Define **formal polynomial differentiation map**:

$$D : F[x] \rightarrow F[x] : a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n \mapsto a_1 + 2 \cdot a_2x + \cdots + n \cdot a_nx^{n-1}$$

**Proposition 4.6.1.1.** Formal polynomial differentiation map  $D$  satisfies:

1.  $D$  preserves addition, but not multiplication.
2.  $\ker D = F$ .
3.  $D[F[x]] = F[x]$ .

**Proof. Proof of 3.**

Let  $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n \in F[x]$  be given. Then,

$$D \left( a_0x + \frac{a_1}{2 \cdot 1}x^2 + \cdots + \frac{a_n}{(n+1) \cdot 1}x^{n+1} \right) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$$

□

## 4.7 Theorems

**Theorem 4.7.0.1.** Let  $R$  be a Ring that contains at least two elements.

Suppose that for any non-zero  $a \in R$ , there exists a unique  $b \in R$  such that  $aba = a$ .

Then,  $R$  is a division Ring.

*Proof.* Proof consists by three-step.

1)  $R$  has no zero divisor.

Let  $a \in R$  be a non-zero element. Then, there exists a unique elements  $b \in R$  such that  $aba = a$ .

Suppose that  $a \in R$  is zero divisor. Then, there is a non-zero element  $c \in R$  such that  $ac = 0$  or  $ca = 0$ .

Either  $ac = 0$  or  $ca = 0$ ,  $aca = 0$  is true. Now,

$$a = aba = aba - 0 = aba - aca = a(b - c)a$$

Since the Uniqueness,  $b - c = b$ , thus  $c = 0$ . Contradiction.

2) If  $aba = a$ , then  $bab = a$ .

It is clear:

$$a = aba = ababa$$

Now, the Uniqueness gives  $b = bab$ .

3)  $R$  has a identity.

Let non-zero  $a \in R$  be fixed. Then, there is a unique  $b \in R$  such that  $aba = a$ .

Let  $c \in R$  be given. Then,

$$\begin{aligned} ca &= caba \xrightarrow{\text{cancel}} c = cab \\ bc &= babc \xrightarrow{\text{cancel}} c = abc \end{aligned}$$

Thus,  $(ab)$  is identity in  $R$ .

4) Every non-zero element in  $R$  is unit.

Let  $a \in R$ . Then, for some  $b \in R$ ,  $aba = a$ . Since  $1 \in R$ ,  $aba - a = a(ba - 1) = 0$ .

Since  $R$  has no zero divisor,  $ba = 1$ . □

**Theorem 4.7.0.2.** Every characteristic of an integral domain  $R$  is 0 or prime number  $p$ .

*Proof.* If characteristic of  $R$  is zero, then there is nothing to prove.

Suppose that characteristic of  $R$  is  $n \in \mathbb{N}$ .

If  $n \in \mathbb{R}$  is not a prime number, then there exist two integer  $a, b \geq 2$  such that  $n = ab$ .

Then,  $n \cdot 1 = (ab) \cdot 1 = (a \cdot 1)(b \cdot 1) = 0$ . Since  $R$  has no zero divisor, either  $a \cdot 1 = 0$  or  $b \cdot 1 = 0$ . Contradiction. □

**Theorem 4.7.0.3.** Let  $R$  and  $R'$  be a Ring, and  $\varphi : R \rightarrow R'$  is Ring-Homomorphism with  $\varphi[R] \neq \{0'\}$ .

If  $R$  has identity and  $R'$  has no zero divisor, then  $\varphi(1)$  is identity of  $R'$ .

*Proof.* Proof consists by two step:

1.  $\varphi[R]$  has a identity.

For any  $\varphi(r) \in \varphi[R]$ ,  $\varphi(1)\varphi(r) = \varphi(1r) = \varphi(r) = \varphi(r1) = \varphi(r)\varphi(1)$ .

2.  $\varphi(1)$  is identity of  $R'$ .

Since  $\varphi(1)\varphi(1) = \varphi(11) = \varphi(1)$ ,  $\varphi(1)$  is idempotent element in  $R'$ . By lemma,  $\varphi(1)$  is identity of  $R'$ . □

**Lemma 4.7.0.1.** Let  $R$  be a Ring which has no zero divisor. If non-zero  $a \in R$  s.t  $a^2 = a$ , then  $a$  is identity.

*Proof.* Let  $r \in R$  be given. Since  $R$  has no zero divisor,

$$\begin{aligned} ra &= ra^2 \implies (r - ra)a = 0 \implies r = ra \\ ar &= a^2r \implies a(r - ar) = 0 \implies r = ar \end{aligned}$$

Thus,  $a$  is identity. □

## 4.8 Operation of Ideals

## Chapter 5

# Polynomial Ring Theory

**Definition 5.0.0.1.** Let  $R$  be a Commutative Ring with unity. Define Polynomial Ring:

$$R[x] \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^n a_i x^i \mid n \geq 0, a_i \in R \right\}$$

Addition defined by pointwise, and Multiplication defined by:

$$\left( \sum_{i=0}^n a_i x^i \right) \times \left( \sum_{i=0}^m b_i x^i \right) = \sum_{k=0}^{n+m} \left( \sum_{i=0}^k a_i b_{k-i} \right) x^k$$

**Proposition 5.0.0.1.** Let  $R$  be an integral domain, and  $p, q \in R[x]$  be non-zero elements.

1.  $\deg(pq) = \deg p + \deg q$ .
2.  $R[x]$  is an integral domain.
3. If  $p \in R[x]$  is unit, then  $\deg p = 0$  and  $p$  is unit in  $R$ .

*Proof.*

$$\left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{i=0}^m b_i x^i \right) = a_n b_m x^{n+m} + \dots,$$

This proves the statement (1) immediately, and assume not zero, then proves 2).  
And, if  $p \in R[x]$  is unit, then

$$0 = \deg 1 = \deg(pp^{-1}) = \deg p + \deg p^{-1}$$

Thus,  $\deg p = \deg p^{-1} = 0$  and this implies  $p, p^{-1} \in R$ . □

## 5.1 Basic Theorems

**Theorem 5.1.0.1.** Let  $I$  be an ideal of the Commutative Ring  $R$  with unity, and  $(I) \subseteq R[x]$ . Then,  $R[x]/(I) \cong (R/I)[x]$ . In particular, if  $I$  is prime ideal in  $R$ , then  $(I)$  is prime ideal in  $R[x]$ .

**Proof.** First, establish that  $(I) = I[x]$ . Since properties of Ideal,  $(I) = IR[x] = I[x]$  directly. Now, define a map  $\varphi : R[x] \rightarrow (R/I)[x] : \sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n (a_i + I)x^i$ . Then,  $\varphi$  is homomorphism with  $\ker \varphi = I[x]$ . The first-iso. Thm gives  $R[x]/(I) = R[x]/I[x] \cong (R/I)[x]$ . Particular,

$$I \text{ prime ideal} \iff R/I \text{ integral domain} \implies (R/I)[x] = R[x]/(I) \text{ integral domain} \iff (I) \text{ prime ideal.}$$

□

**Theorem 5.1.0.2.** If  $F$  is a field, then  $F[x]$  is Euclidean domain.

Specifically, assume  $R$  is Commutative Ring with unity,  $f, g \in R[x]$  with  $\deg f, \deg g \geq 0$ . If leading coefficient of  $g$  is unit in  $R$ , then there exists unique  $q, r \in R[x]$  such that

$$f(x) = g(x)q(x) + r(x) \quad (\deg r(x) < \deg g(x))$$

**Proof.** If  $\deg f < \deg g$ , put  $g(x) = 0$  and  $r(x) = f(x)$ . Then proved. Suppose that  $\deg f \geq \deg g$ , and using induction. If  $\deg f = 0$ , then put  $g(x) = 0$ , write leading coefficient of  $g$  as  $b$  and of  $f$  as  $a$ . Then, put  $q = b^{-1}a$ ,  $r = 0$ . If  $\deg f \geq 1$ , put  $n = \deg f$ ,  $m = \deg g$ . Then  $n \geq m$ . Write:

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ g(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \end{aligned}$$

Then, by induction,

$$\begin{aligned} f(x) &= a_n b_m^{-1} x^{n-m} g(x) + f_1(x) && (\deg f_1 < n - 1) \\ &= a_n b_m^{-1} x^{n-m} g(x) + q_1(x)g(x) + r(x) && (\deg r < \deg g) \\ &= (a_n b_m^{-1} x^{n-m} q_1(x))g(x) + r(x) && (\deg r < \deg g) \end{aligned}$$

To show the uniqueness,

$$f = q_1 g + r_1 = q_2 g + r_2 \implies g(q_1 - q_2) = r_2 - r_1 \implies \deg g(q_1 - q_2) = \deg g + \deg(q_1 - q_2) = \deg(r_2 - r_1) < \deg g$$

□

## 5.2 Relations between Rings and Their Polynomial Rings

**Lemma 5.2.0.1. Gauss's Lemma**

Let  $R$  be a Unique Factorization Domain with field of fractions  $F$ .

If  $p(x) \in R[x]$  is reducible in  $F[x]$ , then  $p(x)$  is reducible in  $R[x]$ .

**Proof.** Let  $p(x) \in R[x]$  be reducible in  $F[x]$ . i.e.,

$p(x) = A(x)B(x)$  for some  $A(x), B(x) \in F[x]$  with  $A(x), B(x)$  are both non-zero and non-units.

Both  $\deg A$  and  $\deg B$  are at least 1: if either degree were zero, then lie in  $F$ , hence a unit - contradiction.

Write  $A(x) = \sum_{i=0}^n \frac{r_i}{a_i} x^i$  and  $B(x) = \sum_{i=0}^m \frac{s_i}{b_i} x^i$ , and put  $d_1 = a_1 \cdots a_n$ ,  $d_2 = b_1 \cdots b_m$ .

Now,  $d_1 d_2 p(x) = d_1 A(x) d_2 B(x)$  where  $d_1 A(x), d_2 B(x) \in R[x]$ .

If  $d = d_1 d_2$  is unit in  $R$ , then  $p(x) = (d^{-1} d_1 A(x))(d_2 B(x))$  where  $d^{-1} d_1 A(x), d_2 B(x) \in R[x]$ , both are non-unit.

Suppose that  $d$  is not unit. Write  $d = p_1 p_2 \cdots p_n$  is factorization of  $d$ .

$p_1$  is prime, being irreducible in U.F.D.  $(p_1) = p_1 R[x]$  is prime,  $R[x]/(p_1 R[x]) \cong (R/(p_1 R))[x]$  is an integral domain.

Since  $d p(x) = p_1 \cdots p_n p(x) \in p_1 R[x]$ ,

$$\bar{0} = d p(x) + p_1 R[x] = d_1 A(x) d_2 B(x) + p_1 R[x] = \overline{d_1 A(x)} \times \overline{d_2 B(x)}$$

Since  $p_1 R[x]$  is an integral domain, either  $\overline{d_1 A(x)}$  or  $\overline{d_2 B(x)}$  is zero. WLOG, let  $\overline{d_1 A(x)} = d_1 A(x) + p_1 R[x] = \bar{0}$ . This means all coefficient of  $d_1 A(x)$  lies in  $p_1 R$ . Thus, we can cancel  $p_1$  in the equation  $d p(x) = d_1 A(x) d_2 B(x)$ . In finite process, we obtain  $p(x) = A'(x) B'(x)$  where  $A'(x), B'(x) \in R[x]$  with

$$A'(x) = r A(x), \quad B'(x) = s B(x) \text{ where } r, s \in F$$

□

**Corollary 5.2.0.1.** Let  $R$  be a Unique Factorization Domain with field of fractions  $F$ .

Suppose that the greatest common divisor of the coefficients of  $p(x) \in R[x]$  is 1. Then,

$p(x)$  is irreducible in  $R[x]$  if and only if  $p(x)$  is irreducible in  $F[x]$

In particular, if  $p(x)$  is an irreducible monic polynomial in  $R[x]$ , then it is also irreducible in  $F[x]$ .

**Proof.** By Contraposition of Gauss's Lemma, if  $p(x)$  is irreducible in  $R[x]$ , then  $p(x)$  is irreducible in  $F[x]$ . Conversely, suppose that  $p(x)$  is reducible in  $R[x]$ , and the greatest common divisor of coefficients of  $p(x)$  is 1. Write  $p(x) = a(x)b(x)$  where neither  $a(x)$  nor  $b(x)$  are not unit in  $R[x]$ , being reducible.

And, both  $a(x)$  and  $b(x)$  are not constant: because g.c.d. is 1. Thus, both are not unit in  $F[x]$ .

□

**Theorem 5.2.0.1.**  $R$  is Unique Factorization Domain if and only if  $R[x]$  is Unique Factorization Domain.

*Proof.* Suppose that  $R$  is Unique Factorization Domain with field of fractions  $F$ .

Let  $p(x) \in R[x]$  be non-zero element, and  $d \in R$  be the greatest common divisor of coefficients of  $p(x)$ .

Then,  $p(x) = dp'(x)$  where g.c.d. of coefficient of  $p'(x)$  is 1. More precisely, write  $p(x) = \sum_{i=0}^n a_i x^i$ , ( $a_i \in R$ ).

$$p(x) = \sum_{i=0}^n a_i x^i = \sum_{i=0}^n d a'_i x^i = d \left( \sum_{i=0}^n a'_i x^i \right)$$

for some  $a'_i \in R$  such that  $a_i = da'_i$ . Put g.c.d of  $a'_i$ 's to  $d' \in R$ . Then,  $a_i = da'_i = dd'a''_i$ .

This implies  $dd'$  divides every  $a_i$ ; hence  $dd'$  divides  $d$ . That is,  $d'$  is unit, thus  $d'$  must be 1.

Since  $F[x]$  is U.F.D, let  $p'(x) = p_1(x)p_2(x) \cdots p_n(x)$  be a factorization of  $p(x)$  in  $F[x]$ .

The g.c.d of  $p'(x)$  is 1, thus g.c.d. of each  $p_i(x)$  is 1.

Now, the corollary of the Gauss's Lemma gives that every  $p_i(x)$  is irreducible in  $R[x]$ .

Hence,  $p'(x) = p_1(x)p_2(x) \cdots p_n(x)$  is irreducible factorization in  $R[x]$ . To show that uniqueness, let

$$p'(x) = p_1(x) \cdots p_n(x) = q_1(x) \cdots q_m(x)$$

are two irreducibles factorizations of  $p'(x)$  in  $R[x]$ . Since g.c.d of  $p'(x)$  is 1, each  $p_i(x)$  and  $q_j(x)$  have g.c.d. 1.

Since the corollary of the Gauss's Lemma, all factors are irreducibles in  $F[x]$  and  $F[x]$  is U.F.D,  $n = m$ .

Moreover, each  $p_i(x)$  and  $q_i(x)$  are associates in  $F[x]$  (index rearrangement). Since associates up to unit in  $F[x]$ ,

$$p_i(x) = \frac{a}{b} q_i(x) \quad \text{for some } a, b \in R^\times$$

That is,  $bp_i(x) = aq_i(x)$ ; g.c.d. of left polynomial is  $b$ , and g.c.d. of right polynomial is  $a$ .

In integral domain, g.c.d. is unique up to unit,  $a = ub$  for some unit  $u \in R^\times$ . That is,

$$bp_i(x) = aq_i(x) = ubq_i(x) \implies p_i(x) = uq_i(x)$$

Proof complete. □

**Theorem 5.2.0.2.** Let  $R$  be a Commutative Ring with identity.

If  $R[x]$  is Principal Ideal Domain, then  $R$  is a Field.

### 5.3 Irreducibility Criteria

**Lemma 5.3.0.1.** Let  $F$  be a field, and  $p(x) \in F[x]$ .

$p(x)$  has a factor of degree one if and only if  $p(x)$  has a root in  $F$

*Proof.* Trivial. □

**Lemma 5.3.0.2.** Let  $F$  be a field, and  $p(x) \in F[x]$  be a polynomial of degree 2 or 3.

$p(x)$  is reducible if and only if  $p(x)$  has a root in  $F$ .

*Proof.* Trivial. □

**Theorem 5.3.0.1.** Let  $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in \mathbb{Z}[x]$ .

If  $\frac{r}{s} \in \mathbb{Q}$  is a root of  $p(x)$  where  $r$  and  $s$  are relatively prime, then  $r \mid a_0$  and  $s \mid a_n$ .

In particular, if  $p(x) \in \mathbb{Z}[x]$  is monic polynomial and  $p(d) \neq 0$  for all  $d \mid p(0)$ , then  $p(x)$  has no root in  $\mathbb{Q}$ .

*Proof.* By hypothesis,

$$\begin{aligned} p\left(\frac{r}{s}\right) &= a_n\left(\frac{r}{s}\right)^n + a_{n-1}\left(\frac{r}{s}\right)^{n-1} + \dots + a_1\left(\frac{r}{s}\right) + a_0 = 0 \\ \implies a_nr^n + a_{n-1}r^{n-1}s + \dots + a_1rs^{n-1} + a_0s^n &= 0 \\ \implies a_nr^n &= -(a_{n-1}r^{n-1}s + \dots + a_1rs^{n-1} + a_0s^n) = s(a_{n-1}r^{n-1} + \dots + a_1rs^{n-2} + a_0s^{n-1}) \end{aligned}$$

Hence,  $s$  divides  $a_nr^n$  and  $s$  and  $r$  are relatively prime,  $s \mid a_n$ . Similarly,  $r \mid a_0$ .

And, the second assertion is clear by contraposition. □

**Theorem 5.3.0.2.** Let  $I \subsetneq R$  be a proper ideal of integral domain  $R$ , and  $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in R[x]$ . If the image of  $p(x)$

$$\tilde{p}(x) = x^n + (a_{n-1} + I)x^{n-1} + \dots + (a_1 + I)x + (a_0 + I) \in (R/I)[x]$$

cannot be factored two polynomials into two smaller degree,  
then  $p(x)$  is irreducible in  $R[x]$ .

*Proof.* Suppose that  $p(x)$  is reducible in  $R[x]$ .

Then, there exist two monic  $a(x), b(x) \in R[x]$  with  $\deg a, \deg b \geq 1$  such that  $p(x) = a(x)b(x)$ .

But,  $p(x) + I[x] = a(x)b(x) + I[x] = (a(x) + I[x]) \times (b(x) + I[x])$  is still reducible in  $(R/I)[x]$ , because:

leading coefficient of  $a_n$  is 1  $\implies a(x) + I[x]$  is not constant, being unit 1 cannot be in the proper ideal  $I$ . □

**Theorem 5.3.0.3. Eisenstein's Criterion**

Let  $P$  be a prime ideal of the integral domain  $R$ , and  $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in R[x]$ , ( $n \geq 1$ ).

Suppose  $a_{n-1}, \dots, a_1, a_0$  are contained in  $P$  and  $a_0$  is not contained in  $P^2$ . Then,  $f(x)$  is irreducible in  $R[x]$ .

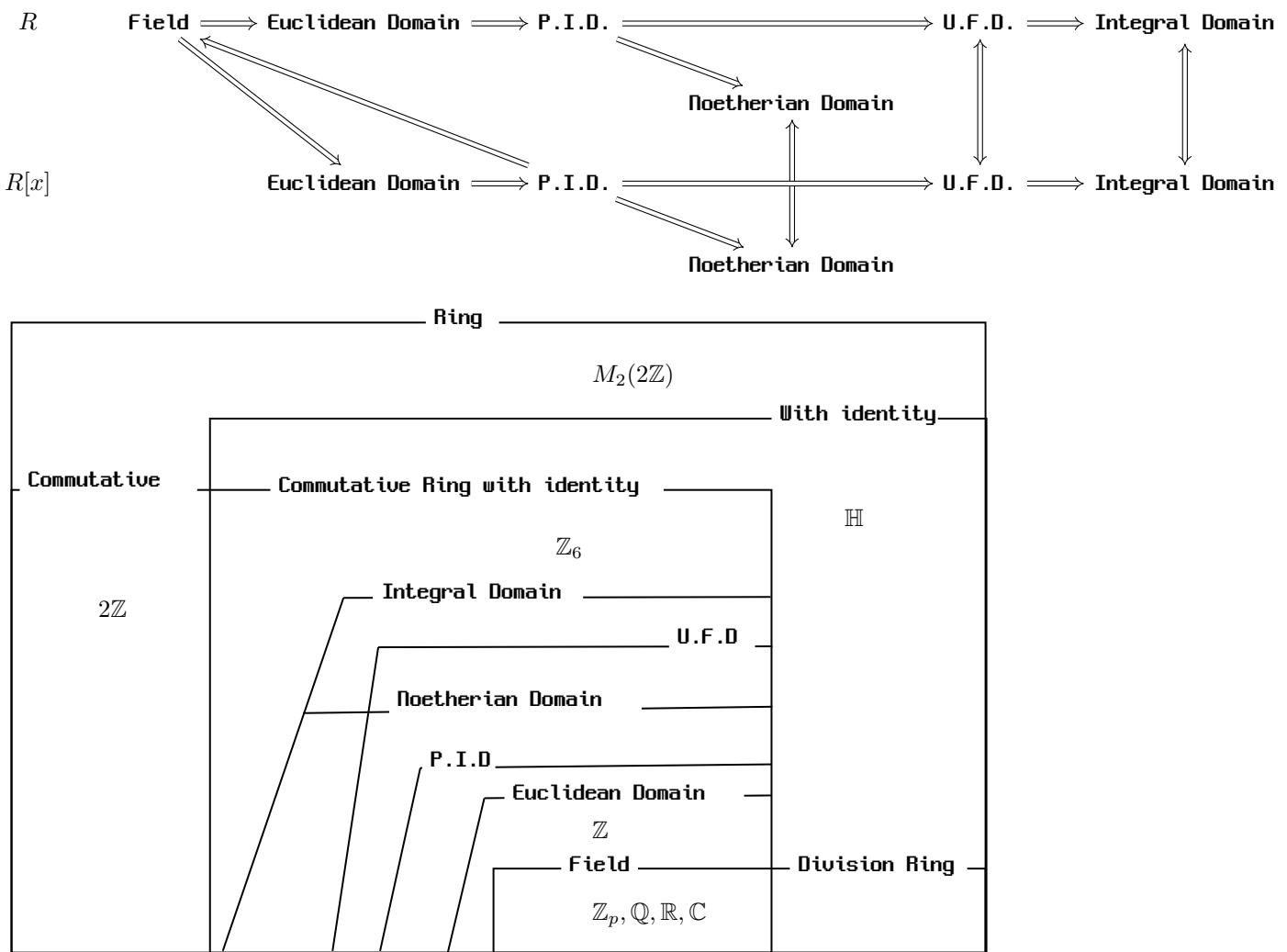
*Proof.* Proof by Contradiction. Suppose  $f(x)$  is reducible. Then,  $f(x) = a(x)b(x)$  for some nonconstant  $a(x), b(x)$  in  $R[x]$  (If it is constant, then contradicts to monic.) Observe that: In  $(R/P)[x]$ ,

$$(1+P)x^n + (a_{n-1}+P)x^{n-1} + \dots + (a_1+P)x + (a_0+P) = x^n = \overline{a(x)b(x)} \in (R/P)[x] \cong R[x]/P[x]$$

Thus, the constant terms of  $a(x)$  and  $b(x)$  both are contained in  $P$ .

But this implies that the product of these two constants is contained in  $P^2$ , contradiction. □

## 5.4 Summary and Diagram



In this section, we find and describe all examples and counterexamples in the diagram.

## 5.5 Examples

### 5.5.1 Quadratic Field and Quadratic integer Ring

**Definition 5.5.1.1.** Let  $D \in \mathbb{Q}$  be a rational number that is not a perfect square in  $\mathbb{Q}$ . Define the *Quadratic Field* for  $D$ :

$$\mathbb{Q}(\sqrt{D}) \stackrel{\text{def}}{=} \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$$

**Proposition 5.5.1.1.**  $\mathbb{Q}(\sqrt{D})$  is a Field.

**Proof.** Let  $a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ . Then,  $(a + b\sqrt{D})^{-1} = \frac{a - b\sqrt{D}}{a^2 - b^2D}$  is inverse. □

## 5.5.2 Cyclotomic Polynomial

**Definition 5.5.2.1.** Let  $p \in \mathbb{Z}$  be a prime. Define  $p^{\text{th}}$  Cyclotomic Polynomial:

$$\Phi_p(x) \stackrel{\text{def}}{=} \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}[x]$$

**Theorem 5.5.2.1.**  $p^{\text{th}}$  Cyclotomic Polynomial is irreducible in  $\mathbb{Z}[x]$ .

*Proof.* Observe that

$$\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = \frac{1}{x} \cdot \left( x^p + \binom{p}{1}x^{p-1} + \cdots + \binom{p}{p-1}x + 1 - 1 \right) = x^{p-1} + \binom{p}{1}x^{p-2} + \cdots + p$$

is irreducible in  $\mathbb{Q}[x]$  and  $\mathbb{Z}[x]$  by *Eisenstein's Criterion*. Using this fact, suppose that  $\Phi_p(x)$  is reducible. That is, there exist non-constant  $r(x), s(x) \in \mathbb{Z}[x]$  such that  $\Phi_p(x) = r(x)s(x)$ . Then,

$$\Phi_p(x+1) = r(x+1)s(x+1)$$

is reducible, contradiction. □

## 5.6 † Rigorously Definition

**Definition 5.6.0.1.** Suppose that  $(R, +, \cdot)$  is a Ring. Define a *Polynomial Ring* is:

$$R[x] \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} \{\{a_i\}_{i=0}^{\infty} \mid a_1, \dots, a_n \in R, a_{n+1} = a_{n+2} = \dots = 0_R\}$$

# Chapter 6

## Field Theory

**Lemma 6.0.0.1.** Suppose that  $\varphi : F \rightarrow F'$  is a Field-Homomorphism. If  $\varphi$  is not zero map, then  $\varphi$  is injective.

*Proof.* Since  $\varphi$  is not zero map,  $\varphi(a) \neq 0_{F'}$  for some  $a \in F$ . Thus,  $\ker \varphi \subsetneq F$ . But, the field  $F$  has only ideal for  $\{0\}$  or  $F$ , hence the ideal  $\ker \varphi$  must be  $\{0\}$ . Now,

$$\varphi(a) = \varphi(b) \implies \varphi(a - b) = 0 \implies a - b = 0$$

□

**Lemma 6.0.0.2.** Suppose that  $F$  is a field.

If  $p(x) \in F[x]$  is irreducible in  $F[x]$ , then  $F[x]/(p(x))$  is a field.

*Proof.* Since  $F$  is a field,  $F[x]$  is Euclidean domain, P.I.D., and U.F.D. Thus,

$$p(x) \text{ irreducible} \xrightarrow{\text{U.F.D.}} (p(x)) \text{ prime} \xrightarrow{\text{P.I.D.}} (p(x)) \text{ maximal in } F[x]$$

Hence,  $K = F[x]/(p(x))$  becomes a field.

□

**Theorem 6.0.0.1.** Suppose that  $F$  is a field, and  $p(x) \in F[x]$  is irreducible.

Then,  $K \stackrel{\text{def}}{=} F[x]/(p(x))$  is a field containing an isomorphic copy of  $F$ , and  $p(x)$  has a root  $\alpha$  in  $K$ .

*Proof.* Since  $F$  is a field,  $F[x]$  is Euclidean domain, P.I.D., and U.F.D. Thus,

$$p(x) \text{ irreducible} \xrightarrow{\text{U.F.D.}} (p(x)) \text{ prime} \xrightarrow{\text{P.I.D.}} (p(x)) \text{ maximal in } F[x]$$

Hence,  $K = F[x]/(p(x))$  becomes a field. Consider the Canonical projection

$$\pi : F[x] \rightarrow F[x]/(p(x)) : f(x) \mapsto f(x) + (p(x))$$

and restriction over,  $F$   $\varphi = \pi_F : F \rightarrow F[x]/(p(x))$  is Homomorphism with  $\ker \varphi = \{0\}$ , because  $\varphi(1) + (p(x)) \neq (p(x))$ . Now,  $F \cong F/\ker \varphi \cong \varphi[F] \subseteq K = F[x]/(p(x))$ . Further, denote  $\bar{x} \stackrel{\text{def}}{=} \pi(x) = x + (p(x))$ , and  $p(x) = \sum_{i=0}^n a_i x^i$ . Then,

$$p(\bar{x}) = \sum_{i=0}^n \bar{a}_i \bar{x}^i = \sum_{i=0}^n \pi(a_i) \pi(x)^i = \pi \left( \sum_{i=0}^n a_i x^i \right) = \pi(p(x)) = \overline{p(x)} = \bar{0} \in K$$

□

**Theorem 6.0.0.2.** Suppose that  $F$  is a field, and  $p(x) \in F[x]$  is irreducible. If  $K$  is an Extension field of  $F$  containing a root  $\alpha \in K$  of  $p(x)$ , then  $F(\alpha) \cong F[x]/(p(x))$ .

**Proof.** Define a map  $\varphi_\alpha : F[x]/(p(x)) \rightarrow F(\alpha) : f(x) + (p(x)) \mapsto f(\alpha)$ .

**Well-defined:** Suppose that  $f(x) + (p(x)) = g(x) + (p(x))$ . Then,  $\varphi_\alpha(f(x) + (p(x))) = f(\alpha)$  and  $\varphi_\alpha(g(x) + (p(x))) = g(\alpha)$ .

$$f(x) + (p(x)) = g(x) + (p(x)) \iff f(x) - g(x) \in (p(x))$$

$$f(x) - g(x) \in (p(x)) \implies f(x) - g(x) = p(x)r(x) \text{ for some } r(x) \in F[x] \implies f(\alpha) - g(\alpha) = p(\alpha)r(\alpha) = 0 \cdot r(\alpha) = 0.$$

**Ring-Homomorphism:**

$$1. \varphi([f(x) + (p(x))] + [g(x) + (p(x))]) = \varphi(f(x) + g(x) + (p(x))) = f(\alpha) + g(\alpha) = \varphi(f(x) + (p(x))) + \varphi(g(x) + (p(x))).$$

$$2. \varphi([f(x) + (p(x))] \cdot [g(x) + (p(x))]) = \varphi(f(x)g(x) + (p(x))) = f(\alpha)g(\alpha) = \varphi(f(x) + (p(x))) \cdot \varphi(g(x) + (p(x))).$$

**Injectivity:** Since  $\varphi$  is a non-trivial homomorphism from field into field, thus  $\ker \varphi = \{0\}$ .

(More precisely,  $\varphi(1) = 1 + (p(x)) \neq (p(x))$  because  $1 \notin (p(x))$ . Thus,  $\ker \varphi \subsetneq F[x]/(p(x)) \implies \ker \varphi = \{0\}$ .)

Now, the  $\varphi$  becomes an Embedding:

$$\varphi[F[x]/(p(x))] \cong F[x]/(p(x))$$

Since  $F[x]/(p(x))$  is a field, the image  $\varphi[F[x]/(p(x))]$  is a field, being it is isomorphic copy.

The definition of  $F(\alpha)$  gives  $\varphi[F[x]/(p(x))] \supset F(\alpha)$ , and  $\varphi[F[x]/(p(x))] \subset F(\alpha)$  is trivial. Consequently,

$$F(\alpha) = \varphi[F[x]/(p(x))] \cong F[x]/(p(x))$$

□

**Theorem 6.0.0.3.** Suppose that  $p(x) \in F[x]$  is an irreducible with  $\deg p(x) = n$ , and  $K = F[x]/(p(x))$ .

Put  $\bar{x} = x + (p(x)) \in K$ . Then,  $\{1, \bar{x}, \bar{x}^2, \dots, \bar{x}^{n-1}\}$  is basis for  $K$  as Vector space over  $F$ .

**Proof.** Above lemma guarantees  $K = F[x]/(p(x))$  is a field.

1) Linealy independent.

Suppose that  $a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1} = 0$  in  $K$  where  $a_i$  are not all zero. Then,

$$\begin{aligned} 0 &= a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1} \\ &= a_0 + a_1(x + (p(x))) + \dots + a_{n-1}(x^{n-1} + (p(x))) \\ &= a_0 + a_1x + \dots + a_{n-1}x^{n-1} + (p(x)) \\ \iff a_0 + a_1x + \dots + a_{n-1}x^{n-1} &= p(x)b(x) \text{ for some } b(x) \in F[x] \end{aligned}$$

But, the left term has a degree at most  $n-1$ , and the right term has a degree at least  $n$ . Contradiction.

2)  $F[x]$  is generated by  $\{1, \bar{x}, \dots, \bar{x}^{n-1}\}$ .

Let  $a(x) \in F[x]$ . Then, the Euclidean algorithm for  $F[x]$  gives that: there exist uniquely  $q(x), r(x) \in F[x]$  s.t

$$a(x) = q(x)p(x) + r(x) \text{ with } \deg r(x) < \deg p(x)$$

Since  $q(x)p(x) \in (p(x))$ ,  $a(x) + (p(x)) = r(x) + (p(x))$ , the degree of  $a(x) + (p(x))$  is smaller than  $n$ .

Now, for any  $a(x) + (p(x)) \in K = F[x]/(p(x))$ ,  $a(x) + (p(x)) \in \langle \{1, \bar{x}, \dots, \bar{x}^{n-1}\} \rangle$ .

□

**Corollary 6.0.0.1.** Suppose that  $F$  is a field, and  $p(x) \in F[x]$  is irreducible polynomial with  $\deg p(x) = n$ . If  $E$  is an Extension field of  $F$  containing  $\alpha$  as a root of  $p(x)$ . Then,

$$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1} \mid a_i \in F\} \subseteq E$$

**Proof.** Since  $F[x]/(p(x)) = \{a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1} \mid a_i \in F\} \cong F(\alpha)$ , being the isomorphism

$$\varphi_\alpha : F[x]/(p(x)) \rightarrow F(\alpha) : f(x) + (p(x)) = \overline{f(x)} \mapsto f(\alpha)$$

Thus, combining with above fact,

$$\begin{aligned} F(\alpha) &= \varphi_\alpha[F[x]/(p(x))] = \{\varphi_\alpha(a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1}) \mid a_i \in F\} \\ &= \{\varphi_\alpha(a_0) + \varphi_\alpha(a_1\bar{x}) + \dots + \varphi_\alpha(a_{n-1}\bar{x}^{n-1}) \mid a_i \in F\} \\ &= \{a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1} \mid a_i \in F\} \end{aligned}$$

□

## 6.1 Extension Field

**Definition 6.1.0.1.** Suppose that  $F$  is a field, and  $K$  be a field containing  $F$  as a subfield. The element  $\alpha \in K$  is called **algebraic** over  $F$  if:

There exists a non-zero polynomial  $f(x) \in F[x]$  such that  $f(\alpha) = 0$  in  $K$ .

If not algebraic, it is called **transcendental** over  $F$ .

**Theorem 6.1.0.1.** Suppose that  $\alpha \in K$  is algebraic over  $F$ .

Then, there exists a unique monic irreducible polynomial  $m_{\alpha,F}(x) \in F[x]$  such that  $m_{\alpha,F}(\alpha) = 0$ . Moreover,

$f(x) \in F[x]$  has  $\alpha$  as a root if and only if  $m_{\alpha,F}(x) \in F[x]$  divides  $f(x)$  in  $F[x]$ .

*Proof.* Since  $\alpha \in K$  is algebraic over  $F$ , the Well-Ordering principle gives the existence of minimal degree polynomial  $g(x) \in F[x]$  having  $\alpha$  as a root. More precisely,

$$S = \{\deg f(x) \in \mathbb{N} \mid f(x) \in F[x] \text{ s.t } f(\alpha) = 0\}$$

is non-empty since  $\alpha$  is algebraic. Therefore, the Well-Ordering Principle gives that  $S$  has a minimum  $n \in \mathbb{N}$ . Now, we can choose  $g(x) \in F[x]$  with  $\deg g(x) = n$  and  $g(\alpha) = 0$ .

Using Contradiction: Suppose that  $g(x)$  is reducible. That is,  $g(x) = a(x)b(x)$  with  $\deg a, \deg b < \deg g$ .

Since  $g(\alpha) = a(\alpha)b(\alpha) = 0$  and  $K$  is a field, either  $a(\alpha)$  or  $b(\alpha)$  must be zero.

This contradicts with  $n \in \mathbb{N}$  is the minimum degree of polynomial which has  $\alpha$  as a root.

In summary:

If  $g(x) \in F[x]$  is a polynomial of minimal degree which has  $\alpha$  as a root, then  $g(x)$  is irreducible over  $F$ .

Now, Suppose that  $f(x) \in F[x]$  is any polynomial having  $\alpha$  as a root. Then,  $\deg f(x) \geq \deg g(x)$  by setting.

By the Euclidean Algorithm in  $F[x]$ , there exist unique polynomials  $q(x), r(x) \in F[x]$  such that

$$f(x) = q(x)g(x) + r(x) \text{ with } \deg r(x) < \deg g(x)$$

Meanwhile,

$$0 = f(\alpha) = q(\alpha)g(\alpha) + r(\alpha) = q(\alpha) \cdot 0 + r(\alpha) = r(\alpha)$$

$r(\alpha) = 0$  implies  $r(x)$  must be zero, by minimality. Hence,  $g(x)$  divides  $f(x)$  in  $F[x]$ .

Finally, put  $a \in F$  be a leading coefficient of  $g(x)$ . Then,  $m_{\alpha,F}(x) = a^{-1}g(x)$  is monic polynomial.

And, Uniqueness given by: for any  $f(x) \in F[x]$ ,  $f(\alpha) = 0 \implies m_{\alpha,F}(x) \mid f(x)$ . □

**Corollary 6.1.0.1.** If  $L$  is an Extension of field  $F$  and  $\alpha \in L$  is algebraic over both  $F$  and  $L$ , then  $m_{\alpha,L}(x)$  divides  $m_{\alpha,F}(x)$  in  $L[x]$ .

Above theorem allows defining:

**Definition 6.1.0.2.** Suppose that  $K$  is an Extension field of a field  $F$ , and  $\alpha \in K$  is algebraic over  $F$ .

The monic irreducible polynomial which has  $\alpha$  as root is called **the minimal polynomial** for  $\alpha$  over  $F$ .

Denote this  $m_{\alpha,F}(x) \in F[x]$  or  $\text{irr}(\alpha, F)$ .

**Proposition 6.1.0.1.** Suppose that  $F$  is a field and  $E$  is an Extension of  $F$ .

If  $\alpha \in E$  is algebraic over  $F$ , then  $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$  and

$$[F(\alpha) : F] \stackrel{\text{def}}{=} \dim_F F(\alpha) = \deg m_{\alpha,F}(x) \stackrel{\text{def}}{=} \deg \alpha$$

**Theorem 6.1.0.2.** Suppose that  $E$  is an Extension field of a field  $F$ .

$\alpha \in E$  is algebraic over  $F$  if and only if  $[F(\alpha) : F]$  has finite dimension

*Proof.* If  $\alpha \in E$  is algebraic over  $F$ , then  $[F(\alpha) : F] = \deg m_{\alpha,F}(x)$ , thus finite.

Precisely, if  $f(x) \in F[x]$  satisfies  $\deg f(x) = n$  and  $f(\alpha) = 0$ , then  $[F(\alpha) : F] = \deg_F \alpha \leq n$ .

Meanwhile, suppose  $\alpha \in E$  where  $[E : F] = n$ . Then, the  $n+1$  elements

$$1, \alpha, \alpha^2, \dots, \alpha^n$$

must be linearly dependent, thus for some not all zero elements  $b_0, \dots, b_n \in F$ ,

$$b_0 + b_1\alpha + b_2\alpha^2 + \dots + b_n\alpha^n = 0$$

Now, the polynomial in  $F[x]$

$$f(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$$

satisfies  $f(\alpha) = 0$  in  $E$ . In summary, if  $[E : F] = n$ , then  $\alpha \in E$  is algebraic over  $F$  with  $\deg_F \alpha \leq n$ .  $\square$

**Corollary 6.1.0.2.** If  $E$  is an Extension field of  $F$  with  $[E : F]$  has finite, then  $E$  is algebraic over  $F$ .

**Theorem 6.1.0.3.** Suppose that  $F \subseteq L \subseteq E$  are fields. Then,

$$[E : F] = [E : L][L : F]$$

*Proof.* Suppose that  $[E : L]$  and  $[L : F]$  are finite. Put  $[E : L] = m$  and  $[L : F] = n$ .

Set  $A = \{\alpha_1, \alpha_2, \dots, \alpha_m\} \subset E$  and  $B = \{\beta_1, \beta_2, \dots, \beta_n\} \subset L$  are basis of  $E$  over  $L$  and  $L$  over  $F$ , respectively. Let  $x \in E$  be given. Since  $A$  is basis for  $E$  over  $L$ , there exists a unique linear combination: for  $a_i \in L$ ,

$$x = a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m = \sum_{i=1}^m a_i\alpha_i$$

Meanwhile, for each  $a_i \in L$ ,  $1 \leq i \leq m$ , there exist unique linear combinations: for  $b_{i,j} \in F$ ,

$$a_i = b_{i,1}\beta_1 + b_{i,2}\beta_2 + \dots + b_{i,n}\beta_n = \sum_{j=1}^n b_{i,j}\beta_j$$

Combining above, we obtain

$$x = \sum_{i=1}^m \left( \sum_{j=1}^n b_{i,j}\beta_j \right) \alpha_i = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} b_{i,j}\beta_j \alpha_i$$

Since each  $b_{i,j} \in F$  and  $\beta_j \alpha_i \in E$ , the  $mn$  elements subset  $\{\beta_j \alpha_i \mid 1 \leq i \leq m, 1 \leq j \leq n\} \subset E$  spans  $E$  over  $F$ .

$\square$

## **Chapter 7**

### **Galois Theory**

## Chapter 8

### Module Theory

**Definition 8.0.0.1.** Suppose that  $R$  be a Ring. An Abelian Group  $M$  is called **left-module over  $R$**  if:

The operation  $\cdot : R \times M \rightarrow M : (r, m) \mapsto rm$  satisfies: for any  $r, s \in R$ ,  $n, n \in M$ ,

$$\begin{cases} (r + s)m = rm + sm \\ (rs)m = r(sm) \\ r(m + n) = rm + rn \end{cases}$$

Moreover, if the Ring  $R$  contains identity 1, then  $1m = m$ .

# Chapter 9

## Linear Algebra

### 9.1 Vector Space

**Definition 9.1.0.1.** Suppose that  $F$  is a Field, and  $V$  is an Abelian Group.

And, the operation  $\cdot : F \times V \rightarrow V$  satisfies: For any  $a, b \in F$  and  $v, w \in V$ ,

$$\begin{cases} a \cdot (v + w) = a \cdot v + a \cdot w \\ (a + b) \cdot v = a \cdot v + b \cdot v \\ (ab) \cdot v = a \cdot (b \cdot v) \\ 1 \cdot v = v \end{cases}$$

The triple  $(V, +, \cdot)$  is called the *Vector Space* over  $F$ .

Equivalently, The Vector Space over a field  $F$  is  $F$ -Module.

**Definition 9.1.0.2.** Suppose that  $V, W$  are Vector Space over a field  $F$ .

A map  $\mathcal{L} : V \rightarrow W$  is called *Linear Map* if:

For any  $a \in F$  and  $v_1, v_2 \in V$ ,  $\mathcal{L}(a \cdot v_1 + v_2) = a \cdot \mathcal{L}(v_1) + \mathcal{L}(v_2)$

**Definition 9.1.0.3.** Suppose that  $V$  is a Vector Space over a field  $F$ , and  $W \subseteq V$  is a Subset.

The  $W$  is called *Subspace* of  $V$  if:  $\begin{cases} \text{For any } a \in F, w \in W, a \cdot w \in W \\ \text{For any } w_1, w_2 \in W, w_1 + w_2 \in W \end{cases}$ .

That is, the Subset of a Vector Space which is closed under the addition and scalar multiplication, then it is a Vector Space.

**Lemma 9.1.0.1.** Arbitrary intersection of Subspace is a Subspace.

*Proof.* Suppose that  $V$  is a Vector Space, and  $W_\alpha \leq V$ ,  $\alpha \in \Lambda$  are Subspaces.

Using Subspace Criterion: Let  $a \in F$ ,  $w_1, w_2 \in \bigcap_{\alpha \in \Lambda} W_\alpha$  be given. Then, for any  $\alpha \in \Lambda$ ,  $a \cdot w_1 + w_2 \in W_\alpha$ .  $\square$

This Lemma allows the definition:

**Definition 9.1.0.4.** Suppose that  $V$  is a Vector Space over a field  $F$ , and  $S \subseteq V$  be a Subset.  
Define a *Generated Subspace* by  $S$  is:

$$\langle S \rangle \stackrel{\text{def}}{=} \bigcap_{S \subseteq W \leq V} W$$

This  $\langle S \rangle$  is the *unique smallest Subspace* containing  $S$ . This  $S$  is called *Generating Subset* of  $\langle S \rangle$ .

**Lemma 9.1.0.2.** Suppose that  $V$  is a Vector Space over a field  $F$ , and  $S \subseteq V$  be a Subset. Then,

$$\langle S \rangle = \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$$

**Proof.** First,  $\langle S \rangle \supseteq \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$ , because  $\langle S \rangle$  is closed under the opearations.

And, the set  $\{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$  is a Subgroup of  $V$  containing  $S$ ,

Hence  $\langle S \rangle \subseteq \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$ . □

## 9.2 Linearly independent

**Definition 9.2.0.1.** Suppose that  $V$  is a Vector Space over a field  $F$ , and  $S \subseteq V$  is a Subset. A Subset  $S$  is called *Linearly independent* if: For any finite subset  $\{v_1, v_2, \dots, v_n\} \subseteq S$ ,

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \implies a_1 = a_2 = \dots = a_n = 0$$

If  $S$  is not Linearly independent, then it is called *Linearly dependent*.

**Lemma 9.2.0.1.** Suppose that  $S \subseteq V$  is a Linearly independent subset of a Vector Space  $V$  over  $F$ . If  $v \in V$  satisfies  $v = a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$  for some  $a_i \in F$  and  $v_i \in S$ , then this representation is unique. More precisely,

**Proof.** First,  $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n$  implies  $v \in \langle S \rangle$ .

Now, suppose that  $v \in V$  satisfies  $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n = b_1 \cdot w_1 + \dots + b_m \cdot w_m$ , WLOG  $n \leq m$ .

Put  $I \stackrel{\text{def}}{=} \{i \in \mathbb{N} \mid \exists j \in \mathbb{N} \text{ s.t. } v_i = w_j\}$ .

Then, there is a permutation  $\rho : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\}$  such that  $\forall i \in I, v_i = w_{\rho(i)}$ . Now,

$$\begin{aligned} \sum_{i \in I} a_i \cdot v_i + \sum_{j \notin I} a_j \cdot v_j &= \sum_{i \in I} b_{\rho(i)} \cdot w_{\rho(i)} + \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} = \sum_{i \in I} b_{\rho(i)} \cdot v_i + \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} \\ \implies \sum_{i \in I} (a_i - b_{\rho(i)}) \cdot v_i + \sum_{j \notin I} a_j \cdot v_j - \left( \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} \right) &= 0 \end{aligned}$$

Since  $S$  is linearly independent, for all  $j \notin I, a_j = b_{\rho(j)} = 0$  and for all  $i \in I, a_i = b_{\rho(i)}$ .  $\square$

This fact enables the definition in the next section.

## 9.3 Basis

**Definition 9.3.0.1.** Suppose that  $V$  is a Vector Space over a field  $F$ .

A subset  $\beta \subseteq V$  is called the *Basis* of  $V$  if:

1.  $\beta$  is Linearly independent.
2.  $\langle \beta \rangle = V$ .

**Lemma 9.3.0.1.** Suppose that  $V$  is a Vector Space over a field  $F$ . Then,

$\beta \subset V$  is a Basis of  $V \iff$  For any  $v \in V$ , there exists a Unique representation  $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n$ .

## **Chapter 10**

### **Category**

# Chapter 11

## Exercise

### 27. 16

Find a Prime ideal of  $\mathbb{Z} \times \mathbb{Z}$  that is not maximal.

**Solution.** Since  $S \subset R$  is Prime ideal if and only if  $R/S$  is an integral domain, and  $S \subset R$  is Maximal if and only if  $R/S$  is a field, we can choose  $S = \{0\} \times \mathbb{Z}$  as a prime but not maximal because

$$(\mathbb{Z} \times \mathbb{Z})/(\{0\} \times \mathbb{Z}) \cong \mathbb{Z}$$

is an integral domain, but not a field. This isomorphic guarantees by:

$$\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} : (a, b) \mapsto a$$

is surjective, homomorphism, and  $\ker \varphi = \{0\} \times \mathbb{Z}$ .

### 27. 33

Following two theorems are equivalent:

**Theorem 11.0.0.1. Fundamental Theorem of Algebra**

If  $f(x) \in \mathbb{C}[x]$  is non-constant polynomial, then  $f(x)$  has a root in  $\mathbb{C}$ .

**Theorem 11.0.0.2. Nullstellensatz for  $\mathbb{C}[x]$**

Suppose that  $f_1(x), \dots, f_r(x) \in \mathbb{C}[x]$  and  $g(x) \in \mathbb{C}[x]$ .

If  $\alpha \in \mathbb{C}$  is zero of all  $f_1, \dots, f_r$  implies  $g(\alpha) = 0$ , then for some  $n \in \mathbb{N}$ ,  $g(x)^n \in \langle f_1, \dots, f_r \rangle$ .

**Solution.** Suppose that the Fundamental Theorem of Algebra is true.

Since  $\mathbb{C}$  is a field, every ideal in  $\mathbb{C}[x]$  is principal. Thus, for some  $p(x) \in \mathbb{C}[x]$ ,

$$\langle f_1(x), \dots, f_r(x) \rangle = \langle p(x) \rangle$$

By the Fundamental Theorem of Algebra, we can write: (WLOG, suppose  $p(x)$  is monic)

$$p(x) = (x - \alpha_1)^{m_1} \cdots (x - \alpha_k)^{m_k}$$

where  $m_i \in \mathbb{N}$ . Now, since every  $f_i(x)$  is divided by  $p(x)$ , thus all  $f_i(x)$  has  $\alpha_1, \dots, \alpha_k$  as roots, this implies  $g(x)$  has  $\alpha_1, \dots, \alpha_k$  as roots. This means  $g(x) = q(x)(x - \alpha_1) \cdots (x - \alpha_k)$  for some  $q(x) \in \mathbb{C}[x]$ , thus  $g(x)^n \in \langle p(x) \rangle$  where  $n = m_1 m_2 \cdots m_r$ .

Conversely, suppose that Nullstellensatz is true.

Let non-constant  $f(x) \in \mathbb{C}[x]$  be given. Put  $f_1(x) = (x - \alpha)$ . Then,  $\alpha$  is zero of  $f_1(x)$  and zero of  $(x - \alpha)f(x)$ . By assumption, for some  $n \in \mathbb{N}$ ,  $f_1(x)^n \in \langle (x - \alpha)f(x) \rangle$ . That is,  $(x - \alpha)f(x)$  divides  $f_1(x)^n = (x - \alpha)^n$ . If  $n = 1$ , then  $f(x)$  must be contradiction, this contradicts with  $f$  is non-constant. If  $n > 1$ , then  $f(x)$  has  $(x - \alpha)$  as a factor, thus  $\alpha$  is zero of  $f(x)$ .

# Chapter 12

## General Topology

In this chapter, we follow the notations of [Steen et al., 1978, COUNTEREXAMPLES IN TOPOLOGY].

### 12.1 Basis

#### 12.1.1 Subbasis

**Definition 12.1.1.1.** Let  $X$  be a set.

A collection  $\mathcal{S} \subseteq \mathcal{P}(X)$  is called **subbasis** if:  $X = \bigcup_{S \in \mathcal{S}} S$ . (That is,  $\forall x \in X, \exists S \in \mathcal{S}$  s.t.  $x \in S$ )

$\beta_{\mathcal{S}}$  is called **Basis generated by the subbasis  $\mathcal{S}$** .

Note that:  $\mathcal{T}_{\beta_{\mathcal{S}}}$  is the smallest Topology such that containing  $\mathcal{S}$ .

## 12.2 Topological Map

**Definition 12.2.0.1.** Let  $X, Y$  are Topological Space.  $f : X \rightarrow Y$  is Continuous at  $x_0 \in X$  if:  
For any open  $V \in \mathcal{T}_Y$  with  $f(x_0) \in V$ , there is an open  $U \in \mathcal{T}_X$  with  $x_0 \in U$  such that  $f(U) \subset V$ .

**Definition 12.2.0.2.** Let  $X, Y$  are Topological Space. Define:

1.  $f : X \rightarrow Y$  is **Continuous Map** if: For any open  $V \subset Y$ ,  $f^{-1}[V] \subset X$  be open.
2.  $f : X \rightarrow Y$  is **Open Map** if: For any open subset  $A \subset X$ ,  $f[A] \subset Y$  be open.
3.  $f : X \rightarrow Y$  is **Closed Map** if: For any closed subset  $B \subset X$ ,  $f[B] \subset Y$  be closed.
4.  $f : X \rightarrow Y$  is **Homeomorphism** if:  $f$  is bijection, continuous, and  $f^{-1}$  is continuous.

**Theorem 12.2.0.1.** Let  $f : X \rightarrow Y$  be a Topological Map. Then, The Followings are Equivalent:

- a)  $f$  is Continuous Map.
- b) For any closed  $C \subset Y$ ,  $f^{-1}[C] \subset X$  be closed.
- c) For any subset  $A \subset X$ ,  $f[\overline{A}] \subset \overline{f[A]}$ .
- d) For any subset  $B \subset Y$ ,  $f^{-1}[B^\circ] \subset (f^{-1}[B])^\circ$ .

*Proof.*

- a)  $\Rightarrow$  b) Let  $C \subset Y$  is closed. Then,  $f^{-1}[Y \setminus C] = X \setminus f^{-1}[C]$  is open, thus  $f^{-1}[C]$  is closed.
- b)  $\Rightarrow$  c) Let  $A \subset X$ . Since  $A \subset f^{-1}[f[A]] \subset \underbrace{f^{-1}[f[A]]}_{\text{closed by b)}}$   $\Rightarrow \overline{A} \subset f^{-1}[f[A]] \Rightarrow f[\overline{A}] \subset f[f^{-1}[f[A]]] \subset \overline{f[A]}$ .
- c)  $\Rightarrow$  d) Let  $B \subset Y$ , set  $A = f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$ .  
Then,  $f[\overline{A}] = f[X \setminus f^{-1}[B]] = f[X \setminus (f^{-1}[B])^\circ]$  and  $f[\overline{A}] = \overline{f[f^{-1}[Y \setminus B]]} \subset \overline{Y \setminus B} = Y \setminus B^\circ$ .  
By c),

$$\begin{aligned} f[\overline{A}] &= f[X \setminus (f^{-1}[B])^\circ] \subset \overline{f[A]} \subset Y \setminus B^\circ \\ &\Rightarrow X \setminus (f^{-1}[B])^\circ \subset f^{-1}[f[X \setminus (f^{-1}[B])^\circ]] \subset f^{-1}[Y \setminus B^\circ] = X \setminus f^{-1}[B^\circ] \\ &\Rightarrow f^{-1}[B^\circ] \subset (f^{-1}[B])^\circ \end{aligned}$$

- d)  $\Rightarrow$  a) Let  $U \subset Y$  be an open set. By d),  $f^{-1}[U] \stackrel{U \text{ open}}{=} f^{-1}[U^\circ] \subset (f^{-1}[U])^\circ$ .  
Meanwhile, reverse inclusion is clear,  $f^{-1}[U] = (f^{-1}[U])^\circ$ , open.  $\square$

**Lemma 12.2.0.1.** Let  $X, Y$  are Topological Space. Then,

1.  $f : X \rightarrow Y$  is open map if and only if For any  $A \subset X$ ,  $f[A^\circ] \subset (f[A])^\circ$ .
2.  $f : X \rightarrow Y$  is closed map if and only if For any  $A \subset X$ ,  $\overline{f[A]} \subset f[\overline{A}]$ .

**Lemma 12.2.0.2.** Let  $X, Y$  are Topological Space, and  $f : X \rightarrow Y$  be a bijection. Then, TFAE:

1.  $f$  is open map.
2.  $f$  is closed map.
3.  $f^{-1} : Y \rightarrow X$  be continuous map.

Clearly, Homeomorphism is open, closed, continuous map.

**Lemma 12.2.0.3.** Let  $f : X \rightarrow Y$  be a Homeomorphism,  $A \subset X$ . Then, followings hold:

1.  $f[\bar{A}] = \bar{f[A]}$ .
2.  $f[A^\circ] = (f[A])^\circ$ .

*Proof.* 1. is clear by  $f$  is continuous, and closed map.

2.  $\supset$ )  $A^\circ \subset A \implies f[A^\circ] \subset f[A] \implies f[A^\circ] \subset (f[A])^\circ \subset f[A]$  by  $f$  open map.

2.  $\supset$ ) Let  $x \in (f[A])^\circ$  be given. Then, there is an open  $\mathcal{U} \in \mathcal{T}$  such that  $x \in \mathcal{U} \subset f[A]$ .

Now,  $f^{-1}[x] \subset f^{-1}[\mathcal{U}] \subset f^{-1}[f[A]] = A$  by  $f$  is bijection, this implies

$$(f^{-1}[\mathcal{U}])^\circ = f^{-1}[\mathcal{U}] \subset A^\circ \implies f^{-1}[x] \subset f^{-1}[\mathcal{U}] \subset A^\circ \implies f[f^{-1}[x]] = x \in f[A^\circ].$$

□

**Theorem 12.2.0.2.** Let  $X, Y, Z$  are Topological Space, and  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$ .

1. If  $f, g$  are Continuous map, then  $g \circ f$  is Continuous map.
2. If  $f, g$  are Open map, then  $g \circ f$  is Open map.
3. If  $f, g$  are Closed map, then  $g \circ f$  is Closed map.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$g \circ f$

Above three theorems are trivial.

1. If  $g \circ f$  is Open map,  $f$  is Continuous onto map. Then,  $g$  is Open map.
2. If  $g \circ f$  is Open map,  $g$  is Continuous one-to-one map. Then,  $f$  is Open map.

*Proof.* 1) Let  $U \in \mathcal{T}_Y$  be an open set. Since  $f$  is Continuous map,  $f^{-1}[U]$  is open of  $X$ .

Now,

$$(g \circ f)[\underbrace{f^{-1}[U]}_{\substack{\text{open} \\ \text{image of open map}}}] = g[f[f^{-1}[U]]] \stackrel{\text{onto}}{=} g[U] \in \mathcal{T}_Z$$

2) Let  $U \in \mathcal{T}_X$  be an open set. Since  $g \circ f$  is Open map,  $(g \circ f)[U]$  is open of  $Z$ .

Now, by  $g$  is Continuous one-to-one,

$$g^{-1}[(g \circ f)[U]] = g^{-1}[g[f[U]]] \stackrel{1 \text{ to } 1}{=} f[U] \in \mathcal{T}_Y$$

□

**Lemma 12.2.0.4. Pasting Lemma**

Suppose that  $X, Y$  are Topological Space, and  $A, B \subset X$  such that  $X = A \cup B$ .

If both  $A, B$  are Open or Closed,  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  are Continuous map such that  $f|_{A \cap B} = g|_{A \cap B}$ , then

$$f \cup g : X \rightarrow Y : \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is Continuous map.

*Proof.* Suppose that  $A, B \subset X$  are Closed in  $X$ . For any closed set  $C \subseteq Y$ ,

$$(f \cup g)^{-1}[C] = f^{-1}[C] \cup g^{-1}[C]$$

is closed in  $X$ , being  $A$  is closed in  $X$  and containing  $f^{-1}[C]$  as closed set, thus  $f^{-1}[C]$  is closed in  $X$ . □

## 12.3 Product Space

### 12.3.1 Finite Product Space

**Definition 12.3.1.1.** Let  $(X, \mathcal{T}_X)$  and  $(Y, \mathcal{T}_Y)$  are Topological Spaces. Define a Topology on  $X \times Y$ :

$$\mathcal{T}_P \stackrel{\text{def}}{=} \{U \times V \subset X \times Y \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

The Topological Space  $(X \times Y, \mathcal{T}_P)$  is called the *Product Space*.

**Lemma 12.3.1.1.** Suppose that  $(X \times Y, \mathcal{T}_P)$  is a Product Space for  $X$  and  $Y$ . Then, the Projection map

$$\pi_X : X \times Y \rightarrow X : (x, y) \mapsto x$$

is Continuous Open Onto map.

*Proof.* Surjection is Clear,

Continuous: For any open  $\mathcal{U} \in \mathcal{T}_X$ ,  $\pi_X^{-1}[\mathcal{U}] = \mathcal{U} \times Y \in \mathcal{T}_P$ .

Open: For any open  $O \times V \in \mathcal{T}_P$ ,  $\pi_X[O \times V] = O \in \mathcal{T}_X$ . Thus, it is Open map.  $\square$

**Theorem 12.3.1.1.**  $\mathcal{T}_P$  is the samllest Topology such that the Projection maps are Continuous.

*Proof.* Suppose that  $\mathcal{T}'$  is Topology on  $X \times Y$  such that  $\pi_X, \pi_Y$  are Continuous map.

Then, for any open  $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$ ,  $U \times Y = \pi_X^{-1}[U] \in \mathcal{T}'$  and  $X \times V = \pi_Y^{-1}[V] \in \mathcal{T}'$

Thus, the finite intersection  $U \times Y \cap X \times V = U \times V \in \mathcal{T}'$ . That is,  $\beta_p \subset \mathcal{T}'$ .  $\square$

**Theorem 12.3.1.2.** Let  $X$ , and  $\{Y_i\}_{i=1}^n$  are Topological Spaces. For a map

$$f : X \rightarrow \prod_{i=1}^n Y_i : x \mapsto (f_1(x), f_2(x), \dots, f_n(x))$$

The Followings Are Equivalent:

- a)  $f$  is Continuous map.
- b) For all  $i = 1, 2, \dots, n$ ,  $f_i : X \rightarrow Y_i$  is Continuous map.

*Proof.* a)  $\Rightarrow$  b). Since  $\pi_i$  is Continuous map,  $f_i = \pi_i \circ f$  be a Continuous map.

b)  $\Rightarrow$  a). Let  $B = U_1 \times U_2 \times \dots \times U_n \in \mathcal{T}_P$ , and  $\pi_i : \prod_{i=1}^n Y_i \rightarrow Y_i$  be a Projection for each  $i = 1, 2, \dots, n$ . Then,

$$\begin{aligned} B &= U_1 \times U_2 \times \dots \times U_n = (U_1 \times X_2 \times X_3 \times \dots \times X_n) \\ &\quad \cap (X_1 \times U_2 \times X_3 \times \dots \times X_n) \end{aligned}$$

⋮

$$\cap (X_1 \times X_2 \times X_3 \times \dots \times U_n) = \bigcap_{i=1}^n \pi_i^{-1}[U_i]$$

Now,

$$f^{-1}[B] = f^{-1} \left[ \bigcap_{i=1}^n \pi_i^{-1}[U_i] \right] = \bigcap_{i=1}^n f^{-1} [\pi_i^{-1}[U_i]] = \bigcap_{i=1}^n (\pi_i \circ f)^{-1}[U_i] = \overbrace{\bigcap_{i=1}^n f_i^{-1} \left[ \frac{U_i}{f_i \text{ conti}} \right]}^{\text{finite intersection}} \in \mathcal{T}_X$$

$\square$

**Theorem 12.3.1.3.** Let  $X_1, \dots, X_n$  are Topological Space and  $A_i \subset X_i$ . Then, for a Product Space  $\left(\prod_{i=1}^n X_i, \mathcal{T}_p\right)$ ,

$$1. \quad \overline{\prod_{i=1}^n A_i} = \prod_{i=1}^n \overline{A_i}.$$

$$2. \quad \left(\prod_{i=1}^n A_i\right)^\circ = \prod_{i=1}^n A_i^\circ.$$

*Proof.*

1.)

$$\begin{aligned} x \in \overline{\prod_{i=1}^n A_i} &\iff \forall U \in \mathcal{T}_p \text{ with } x \in U, \quad U \cap \prod_{i=1}^n A_i = \prod_{i=1}^n \pi_i[U] \cap \prod_{i=1}^n A_i = \prod_{i=1}^n (\pi_i[U] \cap A_i) \neq \emptyset \\ &\iff \forall U \in \mathcal{T}_p \text{ with } x \in U, \quad \forall i = 1, 2, \dots, n, \quad \pi_i[U] \cap A_i \neq \emptyset \\ &\iff \forall i = 1, 2, \dots, n, \quad \forall U_i \in \mathcal{T}_i \text{ with } \pi_i(x) \in U_i, \quad U_i \cap A_i \neq \emptyset \\ &\iff x \in \prod_{i=1}^n \overline{A_i} \end{aligned}$$

2.)

$$\begin{aligned} x \in \left(\prod_{i=1}^n A_i\right)^\circ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } x \in U \subset \prod_{i=1}^n A_i \\ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } x \in \prod_{i=1}^n \pi_i[U] \subset \prod_{i=1}^n A_i \\ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } \forall i = 1, 2, \dots, n, \quad \pi_i(x) \in \pi_i[U] \subset A_i \\ &\stackrel{(*)}{\iff} x \in \prod_{i=1}^n A_i^\circ \end{aligned}$$

However, the left direction of (\*) fails when the index set is infinite. □

## 12.4 Coproduct Space

**Definition 12.4.0.1.** Let  $(X_\alpha, \mathcal{T}_\alpha)$  ( $\alpha \in \Lambda$ ) are mutually disjoint Topological Spaces. Define a *Coproduct Topology*  $(X_{\text{II}}, \mathcal{T}_{\text{II}})$ :

$$X_{\text{II}} \stackrel{\text{def}}{=} \bigsqcup_{\alpha \in \Lambda} X_\alpha, \quad \mathcal{T}_{\text{II}} \stackrel{\text{def}}{=} \left\{ \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \mathcal{T}_\alpha \right\}$$

This actually be a Topology:

1.  $\emptyset, X_{\text{II}} \in \mathcal{T}_{\text{II}}$  is clear,
2. Closed under union is clear.
3. Closed under finite intersection, not infinite.

*Proof.* Proof of 3.

Let a finite collection

$$\left\{ \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^1, \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^2, \dots, \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^k \right\}$$

be given. Then, their intersection be:

$$\bigcap_{j=1}^k \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^j = \bigsqcup_{\alpha \in \Lambda} \bigcap_{j=1}^k \mathcal{U}_\alpha^j \in \mathcal{T}_{\text{II}}$$

□

**Theorem 12.4.0.1.** Let  $X_1, X_2, X_3$  and  $Y_1, Y_2, Y_3$  are mutually disjoint Topological Space, and for each  $i = 1, 2, 3$ ,

$$f_i : X_i \rightarrow Y_i : x \mapsto f_i(x)$$

Define a function

$$f = f_1 \amalg f_2 \amalg f_3 : \bigsqcup_{i=1}^3 X_i \rightarrow \bigsqcup_{i=1}^3 Y_i : x \mapsto \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \\ f_3(x) & x \in X_3 \end{cases}$$

where both Domain and Codomain are Coproduct Space. (Clearly, this function is well-defined.) Suppose that:

1.  $f_1$  is Open map, Closed map
2.  $f_2$  is Continuous map, Open map
3.  $f_3$  is Continuous map, Closed map

Then, The Followings hold:

1.  $f_1$  is Continuous map if and only if  $f$  is Continuous map.
2.  $f_2$  is Open map if and only if  $f$  is Open map.
3.  $f_3$  is Closed map if and only if  $f$  is Closed map.

*Proof.*

1. It follows that: For any open on Codomain  $U \in \mathcal{T}_{Y_{\text{II}}}$ ,

$$\begin{aligned} f^{-1}[U] &= \{x \in X \mid f(x) \in U\} = \{x \in X_1 \mid f_1(x) \in U\} \cup \{x \in X_2 \mid f_2(x) \in U\} \cup \{x \in X_3 \mid f_3(x) \in U\} \\ &= f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U] \end{aligned}$$

Thus, If  $f_1$  is Continuous, then  $f$  is Continuous map since  $f^{-1}[U]$  is the union of open sets.

And, If  $f$  is Continuous, then  $f^{-1}[U] \cap X_1$  be Open set and it is equal that  $(f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U]) \cap X_1 = f_1^{-1}[U]$ .

2. It follows that: For any open on Domain  $U \in \mathcal{T}_{X_{\Pi}}$ ,

$$f[U] = f_1[U] \cup f_2[U] \cup f_3[U]$$

This, if  $f_2$  is Open map, then  $f$  is Open map since  $f[U]$  is the union of open sets.

And, If  $f$  is Open, then  $f[U] \cap Y_2$  be Open set and it is equal that  $(f_1[U] \cup f_2[U] \cup f_3[U]) \cap Y_2 = f_2[U]$ .

3. Similar to the above. □

For a specific example, Define for each  $i = 1, 2, 3$ ,

$$X_i \stackrel{\text{def}}{=} \{a_i, b_i\}, \quad \begin{cases} \mathcal{T}_{i,D} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}, \{b_i\}\} \\ \mathcal{T}_{i,I} \stackrel{\text{def}}{=} \{\emptyset, X_i\} \\ \mathcal{T}_{i,a} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}\} \\ \mathcal{T}_{i,b} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{b_i\}\} \end{cases}$$

And define functions

1.  $f_1 : (X_1, \mathcal{T}_{1,I}) \rightarrow (X_1, \mathcal{T}_{1,D}) : x \mapsto x$  is Not Continuous, Open, Closed.
2.  $f_2 : (X_2, \mathcal{T}_{2,a}) \rightarrow (X_2, \mathcal{T}_{2,a}) : x \mapsto a_2$  is Continuous, Open, Not Closed.
3.  $f_3 : (X_1, \mathcal{T}_{3,a}) \rightarrow (X_1, \mathcal{T}_{3,b}) : x \mapsto a_3$  is Continuous, Not Open, Closed.
4.  $g_i : (X_i, \mathcal{T}_{i,D}) \rightarrow (X_i, \mathcal{T}_{i,D}) : x \mapsto x$  is Continuous, Open, Closed for each  $i = 1, 2, 3$ .

Now, from the above discussion,

1.  $g_1 \amalg g_2 \amalg g_3$  is Continuous, Open, Closed.
2.  $f_1 \amalg g_2 \amalg g_3$  is Not Continuous, Open, Closed.
3.  $g_1 \amalg f_2 \amalg g_3$  is Continuous, Not Open, Closed.
4.  $g_1 \amalg g_2 \amalg f_3$  is Continuous, Open, Not Closed.
5.  $f_1 \amalg f_2 \amalg f_3$  is Not Continuous, Not Open, Not Closed.
6.  $g_1 \amalg f_2 \amalg f_3$  is Continuous, Not Open, Not Closed.
7.  $f_1 \amalg f_2 \amalg g_3$  is Not Continuous, Not Open, Closed.
8.  $f_1 \amalg g_2 \amalg f_3$  is Not Continuous, Open, Not Closed.

No.	Map	Continuous	Open	Closed
1	$g_1 \amalg g_2 \amalg g_3$	Yes	Yes	Yes
2	$f_1 \amalg g_2 \amalg g_3$	No	No	No
3	$g_1 \amalg f_2 \amalg g_3$	Yes	No	Yes
4	$g_1 \amalg g_2 \amalg f_3$	Yes	Yes	No
5	$f_1 \amalg f_2 \amalg f_3$	No	No	No
6	$g_1 \amalg f_2 \amalg f_3$	Yes	No	No
7	$f_1 \amalg f_2 \amalg g_3$	No	No	Yes
8	$f_1 \amalg g_2 \amalg f_3$	No	Yes	No

## 12.5 Connected Space

**Definition 12.5.0.1.** Let  $X$  be a Topological Space. Define **Separation** of  $X$  be a tuple  $\{U, V\}$  satisfying:

$$U, V \in \mathcal{T}, U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset, U \cup V = X$$

If the separation exists, then  $X$  is called **disconnected**.

**Lemma 12.5.0.1.** Let  $X$  be a Topological Space. TFAE:

- a)  $X$  is disconnected.
- b) There exist closed sets  $C, D$  such that  $C, D \neq \emptyset, C \cap D = \emptyset, C \cup D = X$
- c) There exists a non-empty proper clopen subset  $\emptyset \neq A \subsetneq X$ .
- d) There exist subsets  $A, B \subseteq X$  such that  $A, B \neq \emptyset, \overline{A} \cap B = A \cap \overline{B} = \emptyset, A \cup B = X$ .
- e) There exists Continuous onto map  $f : X \rightarrow \{a, b\}$  where  $\{a, b\}$  is discrete space.

**Proof.** a)  $\iff$  b)  $\iff$  c)  $\implies$  d) given directly since the facts:

$$\begin{aligned} U \cap V = \emptyset &\iff U \subseteq X \setminus V \iff V \subseteq X \setminus U \\ U \cup V = X &\iff X \setminus V \subseteq U \iff X \setminus U \subseteq V \end{aligned}$$

d)  $\implies$  a) The tuple  $\{X \setminus \overline{A}, X \setminus \overline{B}\}$  becomes the Separation because:

$$\begin{aligned} A \cup B = X &\implies \overline{A} \cup \overline{B} = X \implies (X \setminus \overline{A}) \cap (X \setminus \overline{B}) = \emptyset \\ [(X \setminus \overline{A}) \cup (X \setminus \overline{B})] \cup [(X \setminus A) \cup (X \setminus B)] &= X \cup X = X \end{aligned}$$

□

**Theorem 12.5.0.1.** Let  $X$  be a Connected Space,  $Y$  is Topological Space and  $f : X \rightarrow Y$  be a Continuous map. Then,  $f[X]$  is Connected.

**Proof.** Suppose that  $f[X]$  is disconnected. Then, there exist non-empty open sets of  $f[X]$ ,  $\{U, V\}$  such that

$$U \cup V = f[X], U \cap V = \emptyset$$

Now,

$$f^{-1}[U] \cup f^{-1}[V] = f^{-1}[U \cup V] = f^{-1}[f[X]] = X$$

and

$$f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V] = f^{-1}[\emptyset] = \emptyset$$

Since  $f$  is continuous,  $\{f^{-1}[U], f^{-1}[V]\}$  be a separation of  $X$ , thus contradiction.

□

**Lemma 12.5.0.2.** Let  $A$  be a Connected Subspace of  $X$ . If  $A \subseteq B \subseteq \overline{A}$ , then  $B$  is Connected Subspace.

**Proof.** Using Contradiction: Suppose that  $B$  is Disconnected. Put  $\{U, V\}$  be a Separation of  $B$  such that

$$U, V \in \mathcal{T}_X, B \cap U \neq \emptyset, B \cap V \neq \emptyset, (B \cap U) \cap (B \cap V) = \emptyset, B \subseteq U \cup V$$

Meanwhile, since assumption,

$$B \subseteq \overline{A} \Rightarrow \begin{cases} \emptyset \neq B \cap U \subseteq \overline{A} \cap U \neq \emptyset \\ \emptyset \neq B \cap V \subseteq \overline{A} \cap V \neq \emptyset \end{cases}$$

To show  $A \cap U \neq \emptyset$ , Suppose that  $A \cap U = \emptyset$ . Then,  $A \subseteq X \setminus U$  and  $U$  is open implies  $A \subseteq \overline{A} \subseteq X \setminus U$ . This implies  $\overline{A} \cap U = \emptyset$ , Contradiction. Thus  $A \cap U \neq \emptyset$ , similarly,  $A \cap V \neq \emptyset$ .

On the other hand,

$$A \subset B \Rightarrow \begin{cases} (A \cap U) \cap (A \cap V) \subseteq (B \cap U) \cap (B \cap V) = \emptyset \\ A \subset B \subseteq U \cup V \end{cases}$$

Consequently,  $\{U, V\}$  be a Separation of  $A$ , Contradiction.  $\square$

**Theorem 12.5.0.2.** Let  $X$  be a Topological Space, and subspaces  $A_\alpha \subset X$ ,  $(\alpha \in \Lambda)$  are Connected.

If  $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$ , then  $\bigcup_{\alpha \in \Lambda} A_\alpha$  is Connected Space.

**Proof.** Proof by Contradiction.

Suppose that  $A = \bigcup_{\alpha \in \Lambda} A_\alpha$  is Disconnected. Let  $\{U, V\}$  be a separation of  $A$ . Choose  $a \in \bigcap_{\alpha \in \Lambda} A_\alpha$ , since assumption. Then, since  $A = U \cup V$ , WLOG, assume that  $a \in U$ . Set for each  $\alpha \in \Lambda$ ,  $U_\alpha = U \cap A_\alpha$  and  $V_\alpha = V \cap A_\alpha$ . Then,

$$a \in U_\alpha \neq \emptyset, U_\alpha \cap V_\alpha = U \cap V \cap A_\alpha = \emptyset, U_\alpha \cup V_\alpha = (U \cup V) \cap A_\alpha = A_\alpha$$

Thus,  $V_\alpha$  must be emptyset. Now,

$$V = V \cap A = V \cap \left( \bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (V \cap A_\alpha) = \bigcup_{\alpha \in \Lambda} V_\alpha = \emptyset$$

This is Contradiction.  $\square$

**Corollary 12.5.0.1.** Let  $X$  be a Topological Space, and subspaces  $A_n \subset X$  ( $n \in \mathbb{N}$ ) are Connected.

If for any  $n \in \mathbb{N}$ ,  $\left( \bigcup_{i=1}^n A_i \right) \cap A_{n+1} \neq \emptyset$ , then  $\bigcup_{n=1}^{\infty} A_n$  is Connected.

**Proof.** Put

$$B_n \stackrel{\text{def}}{=} \bigcup_{i=1}^n A_i$$

$B_1 = A_1$  is Connected, by assumption. Inductively, Suppose that  $B_n$  is Connected.

Then, above theorem and  $B_{n+1} = B_n \cap A_{n+1} \neq \emptyset$  gives  $B_{n+1}$  is Connected. Meanwhile,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

and

$$\bigcap_{n=1}^{\infty} B_n = B_1 = A_1 \neq \emptyset, \text{ because } A_1 \cap A_2 \neq \emptyset$$

Thus, above theorem gives  $\bigcup_{n=1}^{\infty} B_n$  Connected.  $\square$

**Theorem 12.5.0.3.** Let  $X, Y$  are Connected Spaces. Then,  $X \times Y$  is Connected.

*Proof.* For any  $x \in X$ ,  $y \in Y$ ,  $X \times \{y\}$  and  $\{x\} \times Y$  are Connected, being  $X \times \{y\} \cong X$  and  $\{x\} \times Y \cong Y$ .

And,  $X \times \{y\} \cap \{x\} \times Y = \{(x, y)\} \neq \emptyset$ , thus  $X \times \{y\} \cup \{x\} \times Y$  is Connected. Let  $b \in Y$  fixed.

For any  $x \in X$ , define  $T_x \stackrel{\text{def}}{=} X \times \{b\} \cup \{x\} \times Y$ .  $T_x$  is Connected by above discussion, and contains  $X \times \{b\}$  for any  $x \in X$ . Thus,  $X \times \{b\} \subset \bigcap_{x \in X} T_x \neq \emptyset$  implies:

$$X \times Y = \bigcup_{x \in X} T_x$$

is Connected. □

### 12.5.1 Connected Component

**Definition 12.5.1.1.** Let  $X$  be a Topological Space. Define a relation  $\sim$  of  $X$ :

$$x \sim y \iff \text{There exists a Connected Subspace of } A \subset X \text{ such that } x, y \in A$$

Then, this relation be an Equivalent Relation, and define Equivalent Class of this relation:

$$\mathcal{C}_x \stackrel{\text{def}}{=} \{y \in X \mid x \sim y\}$$

$\mathcal{C}_x$  is called Connected Component.

**Theorem 12.5.1.1.** Let  $X$  be a Topological Space. Then,

1. For each  $x \in X$ , there exists a unique Connected Component  $\mathcal{C}_x$  which is containing  $x$ .
2.  $\mathcal{C}_x$  be the largest Connected Subspace containing  $x$ .
3. If  $A \subset X$  be a non-empty Connected Subspace,  
then there exists a unique Connected Component  $\mathcal{C}_x$  such that  $A \subset \mathcal{C}_x$ .
4. Every Connected Component is Closed.
5.  $X$  is Connected if and only if There exists only one Connected Component.

**Proof.** 1. is clear:  $x \in \mathcal{C}_y \iff x \sim y \iff \mathcal{C}_x = \mathcal{C}_y$ .

2. Let  $x \in X$  be given.

Largestet: Let  $A \subseteq X$  be a Connected subset s.t  $x \in A$ . Then, for any  $y \in A$ ,  $x \sim y \implies y \in \mathcal{C}_x$ . Thus  $A \subseteq \mathcal{C}_x$ .

Connected: Let  $z \in \mathcal{C}_x$  be given. Then,  $z \sim x$  implies there exists a Connected subset  $A_z$  such that  $x, z \in A_z \subseteq \mathcal{C}_x$ . Now,

$$\mathcal{C}_x = \bigcup_{z \in \mathcal{C}_x} A_z, \quad x \in \bigcap_{z \in \mathcal{C}_x} A_z \neq \emptyset$$

Thus, theorem gives  $\mathcal{C}_x$  is Connected.

3. Let  $A \subset X$  be a non-empty subset, and Connected. Put Open sets tuple  $\{U, V\}$  be a Separation of  $A$ .

Existence: Fix  $x \in A$ . Then, for any  $a \in A$ ,  $x \sim a \implies a \in \mathcal{C}_x$ . Thus  $A \subseteq \mathcal{C}_x$ .

Uniqueness: If  $A \subseteq \mathcal{C}_x$  and  $A \subseteq \mathcal{C}_y$ , then for any  $a \in A$ ,  $a \sim x$  and  $a \sim y$  implies  $x \sim y$ , thus  $\mathcal{C}_x = \mathcal{C}_y$ .

4. Let  $\mathcal{C}_x$  be a Connected Component. Then,  $\overline{\mathcal{C}_x}$  is Connected by theorem, containing  $x$ . Thus  $\overline{\mathcal{C}_x} \subseteq \mathcal{C}_x$ .  $\square$

**Theorem 12.5.1.2.** Let  $X, Y$  are Topological Space such that  $X \cong Y$ . Then,  $X$  and  $Y$  have same number of Connected Component.

**Proof.** Let  $f : X \rightarrow Y$  be a Homeomorphism, and  $x \in X$ . Since Homeo- preserves Connectedness,  $f[\mathcal{C}_x]$  is Connected. Meanwhile,  $f(x) \in f[\mathcal{C}_x]$ , thus  $f[\mathcal{C}_x] \subset \mathcal{C}_{f(x)}$ .

Similarly,  $x = f^{-1}(f(x)) \in f^{-1}[\mathcal{C}_{f(x)}]$ , thus  $f^{-1}[\mathcal{C}_{f(x)}] \subset \mathcal{C}_x$ , implies  $f[f^{-1}[\mathcal{C}_{f(x)}]] = \mathcal{C}_{f(x)} \subset f[\mathcal{C}_x]$ .

That is, for any  $x \in X$ ,  $f[\mathcal{C}_x] = \mathcal{C}_{f(x)}$ . Similarly, for any  $y \in Y$ ,  $f^{-1}[\mathcal{C}_y] = \mathcal{C}_{f^{-1}(y)}$ .

**Claim:** the map

$$\phi : \{\mathcal{C}_x \mid x \in X\} \rightarrow \{\mathcal{C}_y \mid y \in Y\} : \mathcal{C}_x \mapsto \mathcal{C}_{f(x)}$$

be a One-to-One, Onto.

**Injection)** Let  $\phi(\mathcal{C}_{x_1}) = \phi(\mathcal{C}_{x_2})$ . That is,  $\mathcal{C}_{f(x_1)} = \mathcal{C}_{f(x_2)}$ . Since above discussion, we obtain  $f[\mathcal{C}_{x_1}] = f[\mathcal{C}_{x_2}]$ . Now,

$$\mathcal{C}_{x_1} = f^{-1}[f[\mathcal{C}_{x_1}]] = f^{-1}[f[\mathcal{C}_{x_2}]] = \mathcal{C}_{x_2}$$

**Surjection)** Let  $\mathcal{C}_y$  be given from codomain. Since  $f$  is surjection, there exists  $x \in X$  such that  $y = f(x)$ . Thus,  $\mathcal{C}_y = \mathcal{C}_{f(x)} = \phi(\mathcal{C}_x)$ , thus surjective.  $\square$

**Theorem 12.5.1.3.** If the Topological Space  $X$  has finite number of Connected Components, then for each Connected Component is Clopen.

## 12.5.2 Locally Connected

**Definition 12.5.2.1.** A Space  $X$  is called *Locally Connected* if:

For any  $x \in X$  and neighborhood  $N$  of  $x$ , there exists Connected Open set  $U \in \mathcal{T}$  such that  $x \in U \subseteq N$ .

**Theorem 12.5.2.1.** Let  $X$  be a Space. Then,

$X$  is Locally Connected if and only if For any open  $U$ , every Connected Component of subspace  $U$  is open.

Note that:  $U \subset X$  is open of  $X$ , then for any subset  $V \subseteq U$ ,

$V$  is open in  $X$  if and only if  $V$  is open in  $U$ .

*Proof.* Suppose that  $X$  is Locally Connected.

Let  $U$  be an open set of  $X$ , and  $\mathcal{C}$  be a Connected Component of  $U$ .

Since Locally Connectedness, For any  $x \in \mathcal{C}$ , there exists Connected Open set  $V_x$  such that  $x \in V_x \subseteq U$ .

Since  $\mathcal{C}$  is the largest Connected set containing  $x$ , thus  $V_x \subseteq \mathcal{C}$ . Now,  $\mathcal{C} = \bigcup_{x \in \mathcal{C}} V_x$ , thus open.

Conversely, Let  $x \in X$  and  $U$  be an open set containing  $x$ . Take  $\mathcal{C}_x$  is a Connected Component of  $U$  containing  $x$ .

By assumption,  $\mathcal{C}_x$  is Open, thus  $X$  is Locally Connected, as  $\mathcal{C}_x$  is Connected open neighborhood.  $\square$

**Corollary 12.5.2.1.** Every Connected Component of Locally Connected Space is Clopen.

### 12.5.3 Path Connected

**Definition 12.5.3.1.** Let  $I = [0, 1] \subset \mathbb{R}$ , and  $X$  be a Topological Space.

Define Path from  $x \in X$  to  $y \in X$  is: Continuous map  $p : I \rightarrow X$  such that  $p(0) = x$  and  $p(1) = y$ .

A Space  $X$  is called Path Connected Space if: For any  $x, y \in X$ , there exists a path from  $x$  to  $y$ .

**Theorem 12.5.3.1.** Path Connected Space is Connected.

*Proof.* Fix  $x_0 \in X$ . By Path Connectedness, for any  $x \in X$ , there exists a path  $p_x$  from  $x_0$  to  $x$ .

Since  $I$  is Connected,  $p_x[I]$  is Connected in  $X$  and contains  $x_0$  and  $x$ . Now,

$$\bigcup_{x \in X} p_x[I] = X, \quad x_0 \in \bigcap_{x \in X} p_x[I] \neq \emptyset$$

Thus,  $X$  is Connected.  $\square$

*Proof. Version 2.* Let  $X$  be a Path-Connected but not Connected. Let  $\{U, V\}$  be a Separation of  $X$ , and put  $x \in U$ ,  $y \in V$ .

Then, there is a path  $\alpha : [0, 1] \rightarrow X$  such that  $\alpha(0) = x$  and  $\alpha(1) = y$ . And,  $\alpha^{-1}[U], \alpha^{-1}[V]$  are disjoint opens.

Moreover,  $0 \in \alpha^{-1}[U]$  and  $1 \in \alpha^{-1}[V]$ , and  $\alpha^{-1}[U] \cup \alpha^{-1}[V] = \alpha^{-1}[U \cup V] = \alpha^{-1}[X] = [0, 1]$ .

Thus, these becomes Separation of  $[0, 1]$ , Contradiction.  $\square$

**Theorem 12.5.3.2.** Let  $X$  be a Topological Space, and subspaces  $A_\gamma \subset X (\gamma \in \Gamma)$  are Path-Connected.

If  $\bigcap_{\gamma \in \Gamma} A_\gamma \neq \emptyset$ , then  $\bigcup_{\gamma \in \Gamma} A_\gamma$  is Path-Connected.

*Proof.* Fix  $x^* \in \bigcap_{\gamma \in \Gamma} A_\gamma$ , and let  $x, y \in \bigcup_{\gamma \in \Gamma} A_\gamma$  be given. Then, for some  $\alpha, \beta \in \Gamma$ ,  $x \in A_\alpha$  and  $y \in A_\beta$ . Put  $p_\alpha$  is a path from  $x$  to  $x^*$  and  $p_\beta$  is a path from  $y$  to  $x^*$ . Define

$$p_{\alpha*\beta} : [0, 1] \rightarrow \bigcup_{\gamma \in \Gamma} A_\gamma : x \mapsto \begin{cases} p_\alpha(2x) & x \in [0, \frac{1}{2}] \\ p_\beta(2x - 1) & x \in [\frac{1}{2}, 1] \end{cases}$$

Since Pasting lemma, this function is Continuous, moreover path from  $x$  to  $y$  in  $\bigcup_{\gamma \in \Gamma} A_\gamma$ .  $\square$

**Corollary 12.5.3.1.** If  $A_n \subset X$ , ( $n \in \mathbb{N}$ ) are Path-Connected

**Theorem 12.5.3.3.** If  $X$  is Path-Connected and  $f : X \rightarrow Y$  is Continuous, then  $f[X]$  is Path-Connected.

*Proof.* Let  $f(x), f(y) \in f[X]$  be given. Put  $p : I \rightarrow X$  is a path from  $x$  to  $y$ .

Then,  $f \circ p : I \rightarrow f[X]$  is continuous,  $f(p(0)) = f(x)$  and  $f(p(1)) = f(y)$ , thus it is a path.  $\square$

#### 12.5.4 Path-Connected Component

**Definition 12.5.4.1.** Let  $X$  be a Topological Space. Define a relation  $\sim$  of  $X$ :

$$x \sim y \iff \text{There exists a path from } x \text{ to } y \text{ in } X$$

Then, this relation be an Equivalent Relation, and define Equivalent Class of this relation:

$$\mathcal{P}_x \stackrel{\text{def}}{=} \{y \in X \mid x \sim y\}$$

$\mathcal{P}_x$  is called Path-Connected Component.

#### 12.5.5 Locally Path Connected

**Definition 12.5.5.1.** A Space  $X$  is called Locally Path-Connected if:

For any  $x \in X$  and neighborhood  $N$  of  $x$ , there exists Path-Connected Open set  $P \in \mathcal{T}$  such that  $x \in P \subseteq N$ .

**Theorem 12.5.5.1.** A Space  $X$  is Locally Path-Connected if and only if:

For any open  $U$ , every Path-Connected Component of subspace  $U$  is open.

*Proof.* Suppose that  $X$  is Locally Path-Connected.

Let  $U$  be an open set of  $X$ , and  $\mathcal{P}$  be a Path-Connected Component of  $U$ .

Since Locally Path-Connectedness, For any  $x \in \mathcal{P}$ , there exists Path-Connected Open set  $V_x$  such that  $x \in V_x \subseteq U$ . Since  $\mathcal{P}$  is the largest Path-Connected set containing  $x$ , thus  $V_x \subseteq \mathcal{P}$ . Now,  $\mathcal{P} = \bigcup_{x \in \mathcal{P}} V_x$ , thus open.

Conversely, Let  $x \in X$  and  $U$  be an open set containing  $x$ . Take  $\mathcal{P}_x$  is a Connected Component of  $U$  containing  $x$ . By assumption,  $\mathcal{P}_x$  is Open, thus  $X$  is Locally Connected, as  $\mathcal{P}_x$  is Connected open neighborhood.  $\square$

**Theorem 12.5.5.2.** Let  $X$  be a Locally Path-Connected Space,

$\sim_c$  be a Connected relation, and  $\sim_p$  be a Path-Connected relation.

Then,  $X/\sim_c = X/\sim_p$ . (Moreover, every element in the collection is Clopen.)

*Proof.* Let  $x \in X$  be given. Put  $\mathcal{C}_x$  is Connected Component of  $x$ , and  $\mathcal{P}_x$  is Path-Connected Component of  $x$ .

Since Path-Connected Space is Connected,  $\mathcal{P}_x \subseteq \mathcal{C}_x$ . Using Contradiction: Suppose that  $\mathcal{P}_x \subsetneq \mathcal{C}_x$ .

Generally, Since for any  $y \in \mathcal{C}_x$ , there exists a Path-Connected Component  $\mathcal{P}_y$  such that  $x \in \mathcal{P}_y \subseteq \mathcal{C}_x$ .

That is,  $\mathcal{C}_x = \bigcup_{y \in \mathcal{C}_x} \mathcal{P}_y$ . Now,  $\mathcal{C}_x \setminus \mathcal{P}_x$  is non-empty, and Path-Connected Component being  $\sim_p$  is equivalent relation.

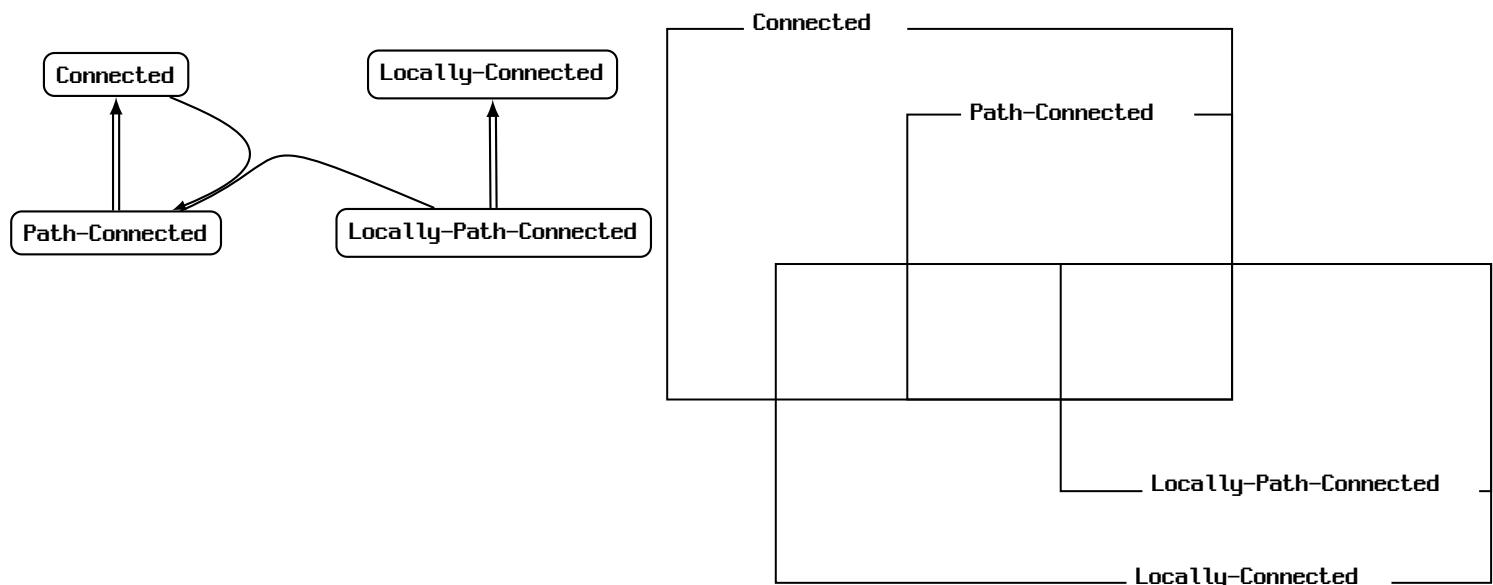
Since  $X$  is Locally Path-Connected Space,  $\mathcal{P}_x$  and  $\mathcal{C}_x \setminus \mathcal{P}_x$  are Path-Connected Open sets, thus these becomes Separation of  $\mathcal{C}_x$ . This is Contradiction.  $\square$

**Corollary 12.5.5.1.** Connected and Locally Path-Connected Space is Path-Connected.

*Proof.* If  $X$  is Connected, then  $X$  is also Connected Component.

Since above theorem,  $X$  is Path-Connected Component, thus Path-Connected.  $\square$

### 12.5.6 Summary and Diagram and Counterexamples

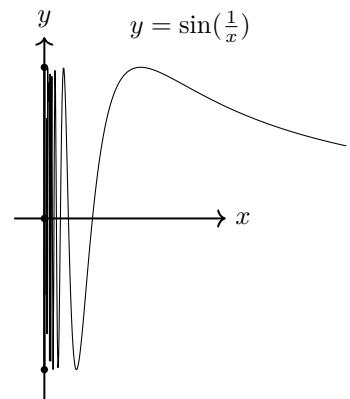


### 12.5.7 Topologist's Sine Curve

**Definition 12.5.7.1. Topologist's Sine Curve**

Let  $Y = \{(0, y) \mid -1 \leq y \leq 1\}$  and  $S^+ = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$ .

The union  $S = Y \cup S^+$  with Euclidean metric is called **Topologist's Sine Curve**.



**Theorem 12.5.7.1. Topologist's Sine Curve is Connected Space.**

**Proof.** Since  $f : (0, \infty) \rightarrow S^+ : x \mapsto (x, \sin(\frac{1}{x}))$  is Continuous and  $(0, \infty)$  is Connected,  $S^+ = f[(0, \infty)]$  is Connected.

Meanwhile, Let  $\varepsilon > 0$  be given. There exists  $0 < r < \varepsilon$  such that  $\sin(\frac{1}{r}) = y$ . That is,  $(r, \sin(\frac{1}{r})) \in B_\varepsilon(0, y)$ .

Now,  $S^+ \subseteq Y \cup S^+ \subseteq \overline{S^+}$ , thus  $S = Y \cup S^+$  is Connected.  $\square$

**Theorem 12.5.7.2. Topologist's Sine Curve is Not Locally Connected Space.**

**Proof.** Consider the Ball  $B_{\frac{1}{2}}((0, 0))$ . Then,  $B_{\frac{1}{2}}((0, 0)) \cap S$  is open neighborhood of  $(0, 0)$ .

Let  $U \cap S \subset B_{\frac{1}{2}}((0, 0)) \cap S$  be an open set of  $S$  containing  $(0, 0)$ .

Since  $(0, 0) \in U$ ,  $U \cap S^+ \neq \emptyset$ . Put  $(a, b) \in U \cap S^+$ . Then, there exists  $m \in \mathbb{N}$  such that

$$0 < \frac{1}{(2m + \frac{1}{2})\pi} < a$$

But,  $(\frac{1}{(2m + \frac{1}{2})\pi}, \sin((2m + \frac{1}{2})\pi)) = (\frac{1}{(2m + \frac{1}{2})\pi}, 1) \notin B_{\frac{1}{2}}((0, 0))$ , because  $|((0, 0) - (\frac{1}{(2m + \frac{1}{2})\pi}, 1))| > \frac{1}{2}$ .

Now,

$$((-\infty, \frac{1}{(2m + \frac{1}{2})\pi}) \times \mathbb{R}) \cap S, ((\frac{1}{(2m + \frac{1}{2})\pi}, \infty) \times \mathbb{R}) \cap S$$

becomes Separation of  $U \cap S$ . Thus, there is no Connected Open neighborhood.  $\square$

**Theorem 12.5.7.3. Topologist's Sine Curve is Not Path-Connected Space.**

**Proof.** Let  $(0, a) \in Y$ ,  $(b, \sin(\frac{1}{b})) \in S^+$  be given. Suppose that there exists a path  $p : I \rightarrow S$  from  $(0, a)$  to  $(b, \sin(\frac{1}{b}))$ .

Then,  $W = \{x \in I \mid p[[0, x]] \subseteq Y\}$  is non-empty, bounded above. Put  $r = \sup W$ . Clearly  $r \in \overline{W}$ .

Meanwhile,  $Y$  is closed in  $S$ , thus  $p(r) \in p[\overline{W}] \subseteq p[W] \subseteq \overline{Y} = Y$ , thus  $r < 1$ .

And, since  $p$  is Continuous, there exists  $\delta \in (0, 1 - r)$  such that  $|x - r| < \delta \implies |p(x) - p(r)| < \frac{1}{2}$ .

By definition of Supremum, for some  $t \in (r, r + \delta)$  such that  $p(t) \in S^+$  with

$$\text{diam}(p[[r, t]]) \leq \frac{1}{2}, \quad p(r) \in Y, \quad p(t) \in S^+$$

Meanwhile, Claim: If  $0 < c < t$  and  $(c, \sin(\frac{1}{c})) \in S^+$ , then  $(c, \sin(\frac{1}{c})) \in p[[r, t]]$ .

If  $(c, \sin(\frac{1}{c}))$  is not contained in  $p[[r, t]]$ , then

$$U = (-\infty, c) \times \mathbb{R}, \quad V = (c, \infty) \times \mathbb{R}$$

becomes Separation of  $p[[r, t]]$ , thus Contradiction. Now, for large enough  $m \in \mathbb{N}$ ,

$$a = \frac{1}{(2m + \frac{1}{2})\pi} < t, \quad b = \frac{1}{(2m + \frac{3}{2})\pi} < t$$

Each  $p, q \in p[[r, t]]$  and  $d(a, b) > 2$ , Contradiction.  $\square$

**Corollary 12.5.7.1.** Topologist's Sine Curve is *Not* Locally-Path-Connected Space.

## 12.6 Compact Space

**Definition 12.6.0.1.** A Topological Space  $X$  is *compact* if: every open cover contains a finite subcover. i.e.,

If  $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ , ( $U_\alpha \in \mathcal{T}$ ), then there is finite subcover such that  $X = \bigcup_{i=1}^N U_{\alpha_i}$

This is equivalent with:

If  $\emptyset = \bigcap_{\alpha \in \Lambda} C_\alpha$ , ( $C_\alpha$  closed), then there is finite subset such that  $\emptyset = \bigcap_{i=1}^N C_{\alpha_i}$

**Definition 12.6.0.2.** Let  $X$  be a set.  $A \subset \mathcal{P}(X)$  satisfies *finite intersection property* if:

For all finite subset of  $A$ ,  $\{A_i \mid i = 1, 2, \dots, n\} \subset A$  satisfies  $\bigcap_{i=1}^n A_i \neq \emptyset$ .

**Example.** 1.  $X = \mathbb{R}$ , and let  $A = \{(n, \infty) \mid n \in \mathbb{N}\}$ . Then,

$$\bigcap_{S \in A} S = \emptyset, \quad \bigcap_{\substack{S \in F \subset A \\ |F| < \infty}} S \neq \emptyset$$

2.  $X = \mathbb{R}$ , and let  $A = \{\mathbb{R} \setminus F \mid |F| < \aleph_0\}$ .

**Theorem 12.6.0.1.** Let  $X$  be a Topological Space, Then, TFAE:

a)  $X$  is Compact Space.

b) If  $A$  is a collection of closed subsets of  $X$  that satisfies FIP, then  $\bigcap_{C \in A} C \neq \emptyset$ .

c) If  $A$  is a collection of subsets of  $X$  that satisfies FIP, then  $\bigcap_{S \in A} \overline{S} \neq \emptyset$ .

**Proof.** a)  $\implies$  b). Proof by Contradiction:

Suppose that  $A \subset \mathcal{P}(X)$  be a collection of closed subsets such that FIP.

Assume that  $\bigcap_{C \in A} C = \emptyset$ . Since  $X$  is Compact,

$$\emptyset = \bigcap_{C \in A} C \text{ if and only if } X = \bigcup_{C \in A} (X \setminus C), \text{ where } X \setminus C \text{ is open.}$$

This implies that there is a finite subcover:

$$X = \bigcup_{i=1}^N (X \setminus C_i) \text{ if and only if } \emptyset = \bigcap_{i=1}^N C_i$$

This is Contradiction with  $A$  satisfies FIP.

b)  $\implies$  a). Proof by Contraposition:

Suppose that  $X$  is not Compact. Then, there exists an Open Cover  $\mathcal{O}$  with no finite subcover: i.e.,

$$X = \bigcup_{U \in \mathcal{O}} U \text{ if and only if } \emptyset = \bigcap_{U \in \mathcal{O}} (X \setminus U)$$

And,

For any finite subset of  $\mathcal{O}$ ,  $F = \{U_i \mid i = 1, \dots, N\}$  satisfies  $X \supseteq \bigcup_{i=1}^N U_i$  if and only if  $\emptyset \neq \bigcap_{i=1}^N (X \setminus U_i)$

Thus,  $\mathcal{K} = \{X \setminus \mathcal{U} \mid \mathcal{U} \in \mathcal{O}\}$  satisfies FIP, but  $\emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U}) = \bigcap_{\mathcal{C} \in \mathcal{K}} \mathcal{C}$ . Thus, not a) implies not b).  $\square$

**Theorem 12.6.0.2.** Let  $X$  is Compact Space,  $Y$  is Topological Space.  
If  $f : X \rightarrow Y$  is Continuous Map, then  $f[X]$  is Compact.

*Proof.* Let  $\mathcal{O}$  be an open cover of  $f[X]$ . i.e.,  $f[X] \subset \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U}$ . Now,

$$X \subset f^{-1}[f[X]] \subset f^{-1} \left[ \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \right] = \bigcup_{\substack{\mathcal{U} \in \mathcal{O} \text{ open, } f \text{ conti.}}} f^{-1}[\mathcal{U}]$$

Since  $X$  is compact, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^N f^{-1}[\mathcal{U}_i]$$

Consequently,

$$f[X] \subset f \left[ \bigcup_{i=1}^N f^{-1}[\mathcal{U}_i] \right] = \bigcup_{i=1}^N f[f^{-1}[\mathcal{U}_i]] \subset \bigcup_{i=1}^N \mathcal{U}_i$$

$\square$

**Theorem 12.6.0.3.** Closed set of compact space is compact.

*Proof.* Let  $X$  be a compact, and  $E \subset X$  be a closed subset. Let  $\mathcal{O}$  be an open over of  $E$ . Then,

$$X = E \cup (X \setminus E) \subset \left( \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \right) \cup (X \setminus E)$$

be an open cover of  $X$ . Thus, there is a finite subcover such that

$$X = \left( \bigcup_{i=1}^N \mathcal{U}_i \right) \cup (X \setminus E) \iff E \subset \bigcup_{i=1}^N \mathcal{U}_i$$

$\square$

**Theorem 12.6.0.4.** Let  $X$  be a Topological Space, and  $\beta$  be a basis of  $X$ . Then, TFAE:

- a)  $X$  is Compact Space.
- b) Every open cover consisting of basis elements has a finite subcover.

*Proof.* a)  $\implies$  b). Clear by definition of Compact.

b)  $\implies$  a). Let  $\{\mathcal{U}_\alpha \mid \alpha \in \Lambda\}$  be an Open cover of  $X$ . That is,

$$X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma$$

where  $\{B_\alpha^\gamma \mid \gamma \in \Gamma_\alpha\}$  is subset of basis such that  $\bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma = \mathcal{U}_\alpha$ . Now, by 2), there is finite subcover such that

$$X = \bigcup_{i=1}^n \bigcup_{j=1}^m B_{\alpha_i}^{\gamma_j} \subset \bigcup_{i=1}^n \mathcal{U}_{\alpha_i}$$

Thus,  $\{\mathcal{U}_{\alpha_i} \mid i = 1, 2, \dots, n\}$  be a finite subcover.  $\square$

**Theorem 12.6.0.5.** Let  $X, Y$  are Topological Space. Then, TFAE:

- a)  $X \times Y$  is Compact.
- b)  $X$  and  $Y$  both are Compact.

**Proof.** a)  $\Rightarrow$  b) is clear since projection preserves Compactness.

b)  $\Rightarrow$  a) Let  $\mathcal{O} \stackrel{\text{def}}{=} \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$  be an Open cover of  $X \times Y$ .

Let  $x \in X$  fix. Then,  $\{x\} \times Y$  be a Compact, being  $\{x\} \times Y \cong Y$  by Homeomorphism given by Projection.

Then, there is a finite subcover of  $\mathcal{O}$  such that

$$\{x\} \times Y \subset \bigcup_{i=1}^{n_x} (U_i^x \times V_i^x)$$

Now, for each  $x \in X$ , define  $U^x \stackrel{\text{def}}{=} \bigcap_{i=1}^{n_x} U_i^x$ . Then,  $U^x$  is an open set containing  $x$ , and for any  $i = 1, 2, \dots, n_x$ ,  $U^x \subset U_i^x$ .

Since  $\{U^x \mid x \in X\}$  be an open cover of  $X$ , there is a finite subcover such that

$$X = \bigcup_{i=1}^m U^{x_i}$$

being  $X$  is Compact. Now,

$$X \times Y = \left( \bigcup_{i=1}^m U^{x_i} \right) \times Y = \bigcup_{i=1}^m (U^{x_i} \times Y) \subset \bigcup_{i=1}^m \bigcup_{j=1}^{n_{x_i}} (U_j^{x_i} \times V_j^{x_i})$$

Thus,  $\{U_j^{x_i} \times V_j^{x_i} \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n_{x_i}\}$  be a finite subcover. □

### Tube Lemma

Let  $X$  be a Topological Space, and  $Y$  is Compact Space.

Then, for product space  $X \times Y$ , and fixed  $x_0 \in X$ , following statement holds:

For any open  $N \subset X \times Y$  with  $\{x_0\} \times Y \subset N$ , there is an open  $W \in \mathcal{T}_X$  such that  $\{x_0\} \times Y \subset W \times Y \subset N$ .

**Proof.** Clearly,  $\{x_0\} \times Y$  compact, being  $\{x_0\} \times Y \cong Y$ .

For any  $y \in Y$ ,  $(x_0, y) \in \{x_0\} \times Y \subset N$ , thus there exist opens  $U \in \mathcal{T}_X$  and  $V \in \mathcal{T}_Y$  such that  $(x_0, y) \in U \times V \subset N$ . Now, Clearly  $\{U_y \times V_y \subset X \times Y \mid y \in Y\}$  be an open cover of  $\{x_0\} \times Y$ , thus there is a finite subcover such that

$$\{x_0\} \times Y \subset \bigcup_{i=1}^N (U_{y_i} \times V_{y_i}) \subset N$$

Set  $W = \bigcap_{i=1}^N U_{y_i}$ . Then, clearly  $x_0 \in W$ , and

Let  $(x, y) \in W \times Y$ . Then, since  $Y = \bigcup_{i=1}^n V_{y_i}$ , there is  $1 \leq k \leq n$  such that  $y \in V_{y_k}$ .

Thus,  $(x, y) \in U_{y_k} \times V_{y_k} \subset N$ , this implies  $W \times Y \subset N$ . □

**Theorem 12.6.0.6.** Let  $Y$  be a Compact Space. Then, the following statements are true, but their converses are false:

1. If  $X$  be a Lindelöf Space, then the product Topology  $X \times Y$  be a Lindelöf Space.
2. If  $X$  be a Countable Compact Space, then the product Topology  $X \times Y$  be a Countable Compact Space.

**Proof.** 1. Let  $\mathcal{O}$  be an open cover of  $X \times Y$ .

For any  $x \in X$ ,  $\{x\} \times Y$  is compact set, being  $\{x\} \times Y \simeq Y$ . Thus, there is a finite subcover of  $\mathcal{O}$  such that

$$\{x\} \times Y \subset \bigcup_{j=1}^{N_x} U_j^x \quad (U_j^x \in \mathcal{O})$$

Since Tube Lemma, there is an open  $W_x \in \mathcal{T}_X$  such that

$$\{x\} \times Y \subset W_x \times Y \subset \bigcup_{j=1}^{N_x} U_j^x$$

Meanwhile, since  $X$  is Lindelöf, therefore for an open cover  $\{W_x \mid x \in X\}$  there exists a Countable subcover such that

$$X \subset \bigcup_{i=1}^{\infty} W_{x_i}$$

Consequently,

$$X \times Y \subset \left( \bigcup_{i=1}^{\infty} W_{x_i} \right) \times Y \subset \bigcup_{i=1}^{\infty} (W_{x_i} \times Y) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_{x_i}} U_j^{x_i}$$

Now,  $\{U_j^{x_i} \mid i \in \mathbb{N}, 1 \leq j \leq N_{x_i}\} \subset \mathcal{O}$  be a Countable Open Cover of  $X \times Y$ . □

**Proof.** 2. Let  $\{U_n \subset X \times Y \mid n \in \mathbb{N}\}$  be a Countable open cover of  $X \times Y$ . For each finite subset  $F \subset \mathbb{N}$ , define

$$V_F \stackrel{\text{def}}{=} \left\{ x \in X \mid \{x\} \times Y \subset \bigcup_{n \in F} U_n \right\}$$

Then  $V_F$  satisfies:

1)  $V_F$  is open: Let a finite subset  $F \subset \mathbb{N}$  fix. For each  $x \in V_F$ ,  $\{x\} \times Y \subset \bigcup_{n \in F} U_n$  by definition.

Then, there is an open  $W_x \in \mathcal{T}_X$  such that  $\{x\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$  by Tube Lemma.

Meanwhile,  $W_x \subset V_F$  because for all  $s \in W_x$ ,  $\{s\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$ , thus  $s \in V_F$ .

In summary, for any  $x \in V_F$ , there is an open  $W_x \in \mathcal{T}_X$  such that  $x \in W_x \subset V_F$ . Consequently,  $V_F$  is open of  $X$ .

2)  $\{V_F \mid F \subset \mathbb{N}, |F| < \infty\}$  is a Countable Open Cover of  $X$ :

Countability given by above set is collection of subsets of Countable set. Meanwhile,

For any  $x \in X$ , there is a finite subcover of  $\{U_n \mid n \in \mathbb{N}\}$  such that  $\{x\} \times Y \subset \bigcup_{n \in F} U_n$  where  $F$  finite.

That is,  $x \in V_F$ . Now, the open cover of  $X$ ,

$$\{V_{F_x} \mid x \in X\} \subset \{V_F \mid F \subset \mathbb{N}\}$$

at most Countable. Since  $X$  is Countably Compact Space, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^N V_{F_i}$$

Consequently,

$$X \times Y \subset \left( \bigcup_{i=1}^N V_{F_i} \right) \times Y = \bigcup_{i=1}^N (V_{F_i} \times Y) \subset \bigcup_{i=1}^N \bigcup_{n \in F_i} U_n$$

That is,  $\{U_i \mid i = 1, 2, \dots, N, n \in F_i\}$  be a finite subcover. □

### 12.6.1 Locally Compact

**Definition 12.6.1.1.** A Space  $X$  is called *Locally Compact* if:

For any  $x \in X$ , there exist open  $U$  and compact  $C$  such that  $x \in U \subseteq C$ .

**Lemma 12.6.1.1.** Let  $X$  be a Hausdorff Space. TFAE:

1.  $X$  is Locally-compact space.
2. For any  $x \in X$ , there exists an open  $U$  with  $x \in U$  such that the closure  $\overline{U}$  is Compact in  $X$ .

## 12.6.2 One-point Compactification

**Definition 12.6.2.1.** Let  $(X, \mathcal{T})$  be a Space.

Define  $X_\infty \stackrel{\text{def}}{=} X \sqcup \{\infty\}$  and  $\mathcal{T}_\infty \stackrel{\text{def}}{=} \mathcal{T} \sqcup \{U \subseteq X_\infty \mid \infty \in U, X_\infty \setminus U \text{ is compact in } X\}$ .  
This  $(X_\infty, \mathcal{T}_\infty)$  is called one-point compactification of  $X$ .

**Theorem 12.6.2.1.** Let  $(X, \infty)$  be a Locally-Compact Hausdorff Space, but not Compact. Then, one-point compactification  $(X_\infty, \mathcal{T}_\infty)$  of  $X$  is Compact Hausdorff Space.

*Proof.* This proof consisted of five steps.

1). **Claim:**  $\mathcal{T}_\infty$  is Topology on  $X_\infty$ . (Using  $X$  is Hausdorff)

Let  $U_\gamma \in \Gamma$ , ( $\gamma \in \Gamma$ ) be elements of  $\mathcal{T}_\infty$ .

Define  $\Gamma_1 \stackrel{\text{def}}{=} \{\alpha \in \Gamma \mid U_\alpha \in \mathcal{T}\}$ , and  $\Gamma_2 \stackrel{\text{def}}{=} \Gamma \setminus \Gamma_1 = \{\beta \in \Gamma \mid \infty \in U_\beta, X_\infty \setminus U_\beta \text{ is compact in } X\}$ .

Then,  $\bigcup_{\gamma \in \Gamma} U_\gamma = \left( \bigcup_{\alpha \in \Gamma_1} U_\alpha \right) \cup \left( \bigcup_{\beta \in \Gamma_2} U_\beta \right)$ . The left term is open in  $X$  clearly.

And, put  $C_\beta = X_\infty \setminus U_\beta$  for each  $\beta \in \Gamma_2$ . Then,  $C_\beta$  is Compact in  $X$  by definition, thus closed by  $X$  is Hausdorff.

$$\bigcup_{\beta \in \Gamma_2} U_\beta = \bigcup_{\beta \in \Gamma_2} X_\infty \setminus C_\beta = X_\infty \setminus \left( \bigcap_{\beta \in \Gamma_2} C_\beta \right)$$

This intersection of  $C_\beta$  is compact, being any intersection of closed is closed and closed subset of compact. That is, it is compact in  $X$ , therefore this union of  $U_\beta$  is contained in  $\mathcal{T}_\infty$ .

Let  $U_1, U_2 \in \mathcal{T}$ , and  $V_1, V_2 \in \mathcal{T}_\infty \setminus \mathcal{T}$ . Put  $C_i \stackrel{\text{def}}{=} X_\infty \setminus V_i$ , ( $i = 1, 2$ ). Then,  $C_i$  is compact. Now,

$$U_1 \cap U_2 \in \mathcal{T} \subset \mathcal{T}_\infty$$

$$U_1 \cap V_1 = U_1 \cap (X_\infty \setminus C_1) = U_1 \cap X_\infty \cap C_1^c = U_1 \cap C_1^c = U_1 \setminus C_1 \in \mathcal{T} \subset \mathcal{T}_\infty$$

$$V_1 \cap V_2 = (X_\infty \setminus C_1) \cap (X_\infty \setminus C_2) = X_\infty \setminus (C_1 \cap C_2) \in \mathcal{T}_\infty$$

Thus closed under the arbitrary union and finite intersection.

2). **Claim:**  $(X, \mathcal{T})$  is a Subspace of  $(X_\infty, \mathcal{T}_\infty)$ . That is,  $\mathcal{T} = \{U \cap X \mid U \in \mathcal{T}_\infty\}$ . (Using  $X$  is Hausdorff)

The right inclusion is clear:  $U \in \mathcal{T} \implies U \in \mathcal{T}_\infty$ . Thus  $U = X \cap U \in \{U \cap X \mid U \in \mathcal{T}_\infty\}$ .

To show the left inclusion: Let  $U \in \mathcal{T}_\infty$ . If  $U \in \mathcal{T}$ , then  $X \cap U = U \in \mathcal{T}$ .

If  $U \notin \mathcal{T}$ , then  $X_\infty \setminus U$  is compact in  $X$ . Now,  $X \cap U = X \setminus (X_\infty \setminus U) \in \mathcal{T}$ .

compact in  $\mathcal{T}_2 \implies$  closed

3). **Claim:**  $\overline{X} = X_\infty$ . That is, closure of  $X$  is  $X_\infty$ . (Using  $X$  is not compact)

Let  $U \in \mathcal{T}_\infty$  with  $\infty \in U$ . Then,  $X_\infty \setminus U$  is compact of  $X$ , thus  $X_\infty \setminus U \subsetneq X$  because  $X$  is not compact.

4). **Claim:**  $X_\infty$  is Compact Space.

Let  $\mathcal{O} = \{U_\alpha \mid \alpha \in \Lambda\}$  be an open cover of  $X_\infty$ . Since  $\infty \in X_\infty = \bigcup_{\alpha \in \Lambda} U_\alpha$ , there is  $\alpha_0 \in \Lambda$  such that  $\infty \in U_{\alpha_0}$ .

$C \stackrel{\text{def}}{=} X_\infty \setminus U_{\alpha_0}$  is compact in  $X$ , thus so in  $X_\infty$ . And,  $C \subseteq \bigcup_{\alpha \in \Lambda \setminus \{\alpha_0\}} U_\alpha$ , thus there is finite subcover of  $C$ .

Finally, union of finite subcover of  $C$  and  $U_{\alpha_0}$  is finite subcover of  $X_\infty$ .

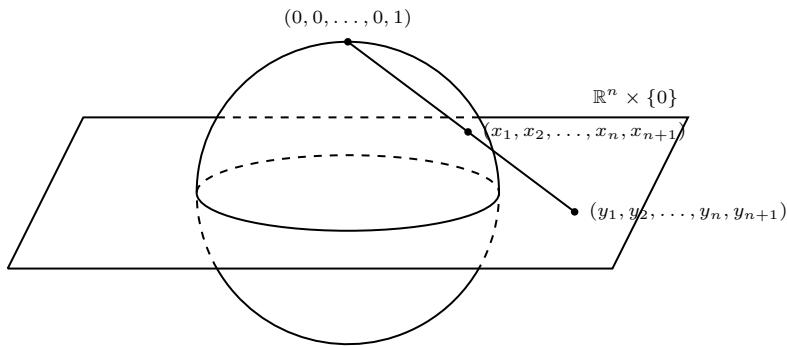
5). **Claim:**  $X_\infty$  is Hausdorff. (Using  $X$  is Locally-Compact)

Let  $x, y \in X_\infty$ . If both  $x, y$  are contained  $X$ , then there is nothing to prove, being  $X$  is hausdorff.

If  $x \in X$  and  $y = \infty$ , then there is open  $U$  and compact  $C$  of  $X$  such that  $x \in U \subseteq C$ , by Locally-Compact.

Now,  $x \in U$  and  $\infty \in X_\infty \setminus C$ , both are open of  $X_\infty$  with  $U \cap (X_\infty \setminus C) = \emptyset$ . □

### 12.6.3 Stereographic projection



**Definition 12.6.3.1.** Let  $\mathbb{R}^{n+1}$  be a Euclidean Space. Define **Unit Sphere** on  $\mathbb{R}^{n+1}$ :

$$S^n \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \dots + x_n + x_{n+1} = 1\}$$

**Theorem 12.6.3.1.** For any  $p \in S^n$ ,  $S^n \setminus \{p\} \cong \mathbb{R}^n$ .

*Proof.* WLOG, put  $p = (0, 0, \dots, 0, 1)$ . Define a map **Stereographic Projection**:

$$f : S^n \setminus \{p\} \rightarrow \mathbb{R}^n : (x_1, x_2, \dots, x_n, x_{n+1}) \mapsto \left( \frac{x_1}{1-x_{n+1}}, \frac{x_2}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right)$$

This map derived from:

$$(y_1, y_2, \dots, y_n, y_{n+1}) = t(\vec{x} - \vec{p}) + (0, 0, \dots, 0, 1) = (x_1t, x_2t, \dots, x_nt, (x_{n+1}-1)t+1)$$

$y_{n+1} = 0$  when  $t = \frac{1}{1-x_{n+1}}$ . Each  $(x_i, x_{n+1}) \mapsto \frac{x_i}{1-x_{n+1}}$  is Continuous map, so  $f$  is Continuous map. And, the map

$$g : \mathbb{R}^n \rightarrow S^n \setminus \{p\} : (x_i)_{i=1}^n \mapsto \left( \frac{2x_1}{1+\|(x_i)\|^2}, \frac{2x_2}{1+\|(x_i)\|^2}, \dots, \frac{2x_n}{1+\|(x_i)\|^2}, 1 - \frac{2}{1+\|(x_i)\|^2} \right)$$

is Continuous, and  $g = f^{-1}$ . Thus,  $f$  is Homeomorphism.  $\square$

**Theorem 12.6.3.2.** One Point Compactification of  $\mathbb{R}^n$  is  $S^n \subseteq \mathbb{R}^{n+1}$ .

*Proof.* Since  $\mathbb{R}$  is Locally-Compact Hausdorff Space, finite product Space  $\mathbb{R}^n$  is Locally-Compact Hausdorff Space. Let  $\mathbb{R}_\infty^n = \mathbb{R}^n \cup \{\infty\}$  be a One Point Compactification of  $\mathbb{R}^n$ . Then,  $\mathbb{R}_\infty^n$  is Compact Hausdorff Space.

Meanwhile, put  $p = (0, 0, \dots, 0, 1)$ . Since  $S^n \setminus \{p\} \cong \mathbb{R}^n$ , there exists a Homeomorphism  $f : S^n \setminus \{p\} \rightarrow \mathbb{R}$ . Define

$$\tilde{f} : S^n \rightarrow \mathbb{R}_\infty^n : \begin{cases} f(x) & x \neq p \\ \infty & x = p \end{cases}$$

Then,  $\tilde{f}$  is Bijective map, and Continuous because: for any open  $U \subseteq \mathbb{R}_\infty^n$  containing  $\infty$ ,

$$\tilde{f}^{-1}[\mathbb{R}_\infty^n \setminus U] = f^{-1}[\mathbb{R}^n \setminus U]$$

Since  $\mathbb{R}^n \setminus U$  is Compact and  $f^{-1}$  is continuous,  $\tilde{f}^{-1}[\mathbb{R}_\infty^n \setminus U] = f^{-1}[\mathbb{R}^n \setminus U]$  is compact in  $S^n$ , thus closed. Moreover, since  $S^n$  is Compact and  $\mathbb{R}_\infty^n$  is Hausdorff,  $f$  is Closed map. Consequently,  $\tilde{f}$  is Homeomorphism.  $\square$

## 12.7 Borel Set

**Definition 12.7.0.1.** Let  $X$  be a Topological Space.

1.  $F \subseteq X$  is called  $F_\sigma$ -set if:  $F$  can be represented as countable union of closed sets.
2.  $G \subseteq X$  is called  $G_\delta$ -set if:  $G$  can be represented as countable intersection of open sets.

**Proposition 12.7.0.1.** Let  $X$  be a Topological Space.

1. If  $F \subseteq X$  is  $F_\sigma$ -set, then there exists sequence of closed sets  $\{F_n\}_{n \in \mathbb{N}}$  such that

$$F = \bigcup_{n=1}^{\infty} F_n, \quad F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

2. If  $G \subseteq X$  is  $G_\delta$ -set, then there exists sequence of open sets  $\{G_n\}_{n \in \mathbb{N}}$  such that

$$G = \bigcap_{n=1}^{\infty} G_n, \quad G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

3. Countable union of  $F_\sigma$ -sets is  $F_\sigma$ .
4. Finite intersection of  $F_\sigma$ -sets is  $F_\sigma$ .
5. Countable intersection of  $G_\delta$ -sets is  $G_\delta$ .
6. Finite union of  $G_\delta$ -sets is  $G_\delta$ .
7. Complement of  $F_\sigma$ -set is  $G_\delta$ .

## 12.8 Baire Category

**Definition 12.8.0.1.** The Topological Space  $X$  is called *Baire Space* if:

If  $\{G_n \mid n \in \mathbb{N}\}$  be a Countable Collection of dense open sets of  $X$ , then  $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in  $X$ .

**Definition 12.8.0.2.** Let  $X$  be a Topological Space.

$A \subset X$  is said to be *nowhere dense subset* if  $(\overline{A})^\circ = \emptyset$ .

1.  $B \subset X$  is called *first category* if  $B$  can be representative by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be *second category*.

## 12.9 Locally Compact Hausdorff Space

**Theorem 12.9.0.1.** Locally Compact Hausdorff Space is Baire Space.

## 12.10 Complete Metric Space

**Definition 12.10.0.1.** Let  $(X, d)$  be a Metric Space, and  $\{p_n\}$  be a Sequence in  $X$ .

The Sequence  $\{p_n\}$  is called *Cauchy Sequence* if:

For any  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $m, n \geq N \implies d(p_m, p_n) < \varepsilon$ .

A Metric Space  $(X, d)$  is said to be *Complete* if every Cauchy Sequences Converge.

**Lemma 12.10.0.1.** Let  $\{E_n\}$  be a sequence of closed bounded non-empty subsets in a Complete Metric Space  $X$  such that

$E_n \supset E_{n+1}$ . If  $\lim_{n \rightarrow \infty} \text{diam } E_n = 0$ , then  $\bigcap_{n=1}^{\infty} E_n = \{p\}$  for some  $p \in X$ .

**Proof.** For each  $n \in \mathbb{N}$ , construct  $p_n \in E_n$ .

Let  $\varepsilon > 0$  be given. Since  $\text{diam } E_n \rightarrow 0$ , there is  $N \in \mathbb{N}$  such that  $\text{diam } E_n < \varepsilon$ .

For any  $m, n \geq N$ ,  $E_N$  contains  $p_m, p_n$ . That is,  $d(p_m, p_n) < \varepsilon$ . Thus,  $\{p_n\}$  be a Cauchy sequence of  $X$ .

Since  $X$  is complete, there is a unique point  $p \in X$  such that  $p_n \rightarrow p$ . Let  $N \in \mathbb{N}$  be an integer such that  $n \geq N \implies |p_n - p| < \varepsilon$ .

Now, for each  $n \geq N$ ,  $E_n$  has a limit point as  $p$ . And for any  $n \in \mathbb{N}$ ,  $E_n$  contains  $E_N, E_{N+1}, \dots$ , thus for all  $n \in \mathbb{N}$ ,  $E_n$  has a limit point as  $p$ . Meanwhile,  $E_n$  closed,  $p \in E_n, \forall n \in \mathbb{N}$ .

Consequently,  $p \in \bigcap_{n=1}^{\infty} E_n$ . If there is  $q \in X$  such that  $p \neq q$ ,  $q \in \bigcap_{n=1}^{\infty} E_n$ . Then,  $\text{diam } E_n \geq d(p, q) > 0, \forall n \in \mathbb{N}$ .  $\square$

**Theorem 12.10.0.1.** Complete Metric Space is Baire Space.

**Proof.** Suppose that  $\{G_n \mid n \in \mathbb{N}\}$  be a Countable Collection of dense open set of Complete Metric Space.

Let an open  $U \in \mathcal{T}$  be given. Since  $G_n$  is dense in the Space,  $U \cap G_1$  is non-empty open set.

Thus, there exists a  $p_1 \in U \cap G_1$  such that for some  $r_1 > 0$ ,  $B_{r_1}(p_1) \subset U \cap G_1$ .

Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

**Set**  $E_1 = U$ ,  $E_2 = B_{\frac{r_1}{2}}(p_1)$ .

**Suppose that**  $E_1, \dots, E_{n-1}$  **are chosen.** Then, since  $E_{n-1} \cap G_{n-1}$  is open, being intersection of opens.

Thus there exists a point  $p_{n-1} \in E_{n-1} \cap G_{n-1}$  and exists  $r_{n-1}$  such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

**Set**  $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$ . Since inductively construction of  $\{E_n\}$ ,  $E_{n+1} \subset E_n$  and  $\overline{E_n} \subset G_n$  for all  $n \in \mathbb{N}$ . Consequently,

$$U \cap \left( \bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left( \bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

### 12.10.1 Nowhere Differentiable function

**Theorem 12.10.1.1.** Let  $\mathcal{C}[\mathbb{R}] \stackrel{\text{def}}{=} \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous.}\}$  and  $d : \mathcal{C}[\mathbb{R}] \times \mathcal{C}[\mathbb{R}] \rightarrow \mathbb{R} : (f, g) \mapsto \sup_{t \in \mathbb{R}} |f(t) - g(t)|$ .

Then,  $(\mathcal{C}[\mathbb{R}], d)$  is Complete Metric Space, and set of Nowhere-Differentiable functions is dense in  $\mathcal{C}[\mathbb{R}]$ .

**Proof.** First, show that  $f$  satisfies triangle inequality: let  $f, g, h \in \mathcal{C}[\mathbb{R}]$  be given.

For any  $t \in \mathbb{R}$ ,  $|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$ . Thus,

$$d(f, g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)| \leq \sup_{t \in \mathbb{R}} [|f(t) - h(t)| + |h(t) - g(t)|] \leq \sup_{t \in \mathbb{R}} |f(t) - h(t)| + \sup_{t \in \mathbb{R}} |h(t) - g(t)| = d(f, h) + d(h, g)$$

□

## 12.10.2 Banach Fixed Point Theorem

**Definition 12.10.2.1.** Let  $f : X \rightarrow X$  be any function. A point  $x \in X$  is called a **fixed point** of  $f$  if  $f(x) = x$ .

**Definition 12.10.2.2.** Let  $X$  be a Metric Space. A map  $f : X \rightarrow X$  is called **Contractive** with respect to the metric  $d$  if:

There exists  $\alpha \in (0, 1)$  such that for all  $x, y \in X$ ,  $d(f(x), f(y)) \leq \alpha d(x, y)$ .

**Theorem 12.10.2.1. Banach Fixed point Theorem**

Let  $(X, d)$  be a Complete Metric Space, and  $f : X \rightarrow X$  be a Contractive map. Then, there exists a unique fixed point of  $f$ ,  $x^* \in X$ .

*Proof.* Clearly,

Contractive  $\implies$  Lipschitz Condition  $\implies$  Continuous.

Thus,  $f$  is Continuous.

Let  $x_0 \in X$  be arbitrary, and construct a sequence  $\{x_n\}$  recursively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} f(x_n), \quad n \geq 0$$

Then, for any  $n \geq 0$ ,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let  $\varepsilon > 0$  be given. Put  $N \in \mathbb{N}$  such that  $\alpha^N \cdot d(x_1, x_0) < \varepsilon(1 - \alpha)$ . Then,  $n \geq m \geq N$  implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \varepsilon(1 - \alpha) \frac{1}{1 - \alpha} = \varepsilon \end{aligned}$$

Therefore,  $\{x_n\}$  is Cauchy sequence. Since  $X$  is Complete, for some  $x^* \in X$ ,  $\lim_{n \rightarrow \infty} x_n = x^*$ . Consequently,

$$\lim_{n \rightarrow \infty} f(x_n) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x^*) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

□

## 12.11 Maps in Metric Space

In this section,  $(X, d_X)$  and  $(Y, d_Y)$  are metric spaces.

### 12.11.1 Metric

**Definition 12.11.1.1.** A **metric** on a set  $X$  is a map  $d : X \times X \rightarrow [0, \infty)$  such that for all  $x, y, z \in X$ :

1.  $d(x, y) = 0 \iff x = y$ .
2.  $d(x, y) = d(y, x)$ .
3.  $d(x, z) \leq d(x, y) + d(y, z)$ .

We call  $(X, d)$  a **metric space**.

**Theorem 12.11.1.1.** The map  $d : X \times X \rightarrow \mathbb{R}$  is continuous.

**Proof.** Let  $(x, y) \in X \times X$  and  $\varepsilon > 0$ . For  $U = B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y)$  and any  $(p, q) \in U$ ,

$$\begin{aligned}d(p, q) &\leq d(p, x) + d(x, y) + d(y, q) < d(x, y) + \varepsilon \\d(x, y) &\leq d(x, p) + d(p, q) + d(q, y) < d(p, q) + \varepsilon\end{aligned}$$

so  $|d(p, q) - d(x, y)| < \varepsilon$ . □

### 12.11.2 Diameter

**Definition 12.11.2.1.** For  $E \subseteq X$ , the **diameter** is

$$\text{diam } E \stackrel{\text{def}}{=} \sup_{x, y \in E} d(x, y).$$

**Theorem 12.11.2.1.** For any  $E \subseteq X$ ,  $\text{diam } E = \text{diam } \overline{E}$ .

**Proof.** Clearly,  $\text{diam } E \leq \text{diam } \overline{E}$ . Let  $\varepsilon > 0$  be given. Then, there exist  $a, b \in \overline{E}$  such that

$$\text{diam } \overline{E} - \frac{\varepsilon}{2} \leq d(a, b) < \text{diam } \overline{E}$$

Meanwhile,  $a, b \in \overline{E}$  implies:  $B_{\frac{\varepsilon}{2}}(a) \cap E \neq \emptyset$ ,  $B_{\frac{\varepsilon}{2}}(b) \cap E \neq \emptyset$ .

Put  $p \in B_{\frac{\varepsilon}{2}}(a) \cap E$  and  $q \in B_{\frac{\varepsilon}{2}}(b) \cap E$ . Now, the triangle inequality gives

$$\text{diam } \overline{E} - \frac{\varepsilon}{2} \leq d(a, b) \leq d(a, p) + d(p, q) + d(q, b) \leq \frac{\varepsilon}{2} + \text{diam } E + \frac{\varepsilon}{2} = \text{diam } E + \varepsilon$$

Since  $\varepsilon$  is chosen arbitrarily,  $\text{diam } \overline{E} \leq \text{diam } E$ . □

### 12.11.3 Distance

**Definition 12.11.3.1.** For nonempty  $E \subseteq X$ , define  $\rho_E : X \rightarrow [0, \infty)$  by

$$\rho_E(x) \stackrel{\text{def}}{=} \inf_{t \in E} d(x, t).$$

**Proposition 12.11.3.1.** For all  $x \in X$ ,  $\rho_E(x) = 0$  if and only if  $x \in \overline{E}$ .

*Proof.*

$$\rho_E(x) = 0 \stackrel{\text{by def.}}{\iff} \inf_{t \in E} d(x, t) = 0 \iff \forall \varepsilon > 0, \exists p \in E \text{ s.t. } 0 < d(x, p) \leq \varepsilon \iff \forall \varepsilon > 0, B_\varepsilon(x) \cap E \neq \emptyset$$

□

**Theorem 12.11.3.1.** The distance  $\rho_E$  satisfies Lipschitz Condition. Furthermore, Uniformly Continuous.

*Proof.* Let  $x, y \in X$  be given. Then, for any  $z \in E$ ,

$$\rho_E(x) = \inf_{t \in E} d(x, t) \leq d(x, z) \leq d(x, y) + d(y, z)$$

Since  $z \in E$  given arbitrarily,

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

Thus  $\rho_E(x) - \rho_E(y) \leq d(x, y)$ . Similarly,  $\rho_E(y) - \rho_E(x) \leq d(x, y)$ . That is, For any  $x, y \in X$ ,  $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$ . Now, for any  $\varepsilon > 0$ , put  $\delta = \varepsilon$ . Then,

$$d(x, y) < \delta \implies |\rho_E(x) - \rho_E(y)| \leq d(x, y) < \delta = \varepsilon$$

□

**Theorem 12.11.3.2.** Let  $C \subseteq X$  be compact,  $F \subseteq X$  closed, and  $C \cap F = \emptyset$ . Then there exists  $\delta > 0$  such that

$$d(p, q) \geq \delta \text{ for all } p \in C, q \in F.$$

*Proof.*

□

#### 12.11.4 Isometry

**Definition 12.11.4.1.** An onto map  $f : (X, d_X) \rightarrow (Y, d_Y)$  is an *isometry* if: for all  $x, y \in X$ ,

$$d_X(x, y) = d_Y(f(x), f(y))$$

## 12.12 Separation Axioms

## 12.13 Urysohn Metrization Theorem

### 12.13.1 Urysohn Lemma

Recall that:

**Definition 12.13.1.1.**  $X$  is  $T_4$  if: For any disjoint closed set  $A$  and  $B$ , there exist disjoint open  $U, V$  such that  $A \subseteq U$  and  $B \subseteq V$ .

**Lemma 12.13.1.1.**  $X$  is  $T_4$  Space if and only if For any closed  $C$  and open  $U$  with  $C \subseteq U$ , there exists open  $O$  such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\bar{O}} \subseteq \underset{\text{open}}{U}$$

*Proof.* Proof of the left direction only.

Let  $X$  be a  $T_4$  Space, and  $C \subset X$  be a closed,  $U$  be a open containing  $C$ . Then,  $C \subset U$  implies  $U^c \subset C^c$ , thus  $U^c$  is a closed set disjoint from  $C$ . By  $T_4$  condition, There exist disjoint opens  $O, O'$  such that  $C \subset O$  and  $U^c \subset O' \iff O'^c \subset U$ .

Since  $O \cap O' = \emptyset \iff O \subset O'^c$ ,  $O$  contained in  $U$ , this implies that  $C \subset O \subset U$ .

Since closure is the smallest closed set such that contains it, consequently  $C \subset O \subset \bar{O} \subset O'^c \subset U$ .  $\square$

**Definition 12.13.1.2.** Let  $X$  be a Toplogical Space, and  $A, B \subset X$  are disjoint closed subset.

A real-valued Continuous map  $f : X \rightarrow [a, b]$  is called **Urysohn function** for  $A$  and  $B$  if:  $f|_A = a$  and  $f|_B = b$ . In another form,

$$f : X \rightarrow [a, b] : x \rightarrow \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

**Lemma 12.13.1.2. Urysohn Lemma**

$T_4$  Space has an Urysohn function for any two disjoint closed subsets.

*Proof.* Generalization is the last thing to proven, first of all, prove in case of  $[a, b] = [0, 1]$ . This proof consists by three Step.

Let  $X$  be a  $T_4$  Space, and  $A, B \subset X$  be closed subsets.

*Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.*

Consider a set of **Dyadic Rationals**  $D \stackrel{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, k \leq 2^n - 1 \right\}$ . We will show that the following statement holds:

For any  $r, s \in D$  with  $r < s$ , there exist open sets  $U_r, U_s$  such that  $A \subseteq \bar{U}_r \subseteq U_s \subseteq X \setminus B$  (\*)

For this, Enough to Show that: For any  $k \in \mathbb{N}$ , there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \bar{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \bar{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^{k-1}}{2^k}} \subseteq \bar{U}_{\frac{2^{k-1}}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is Noetherian. That is,  $X$  satisfies Ascending Chain Condition for open sets.)

Let  $k = 1$ . Then, By  $T_4$  condition gives that: There exists an open set  $U_1$  such that

$$A \subseteq U_1 \subseteq \bar{U}_1 \subseteq X \setminus B$$

Now, naming this  $U_1$  as  $U_{\frac{1}{2}}$ , proved when  $k = 1$ .

Suppose that for some  $k > 1$ , the Chain exists as:

$$\underset{\text{closed}}{A} \stackrel{*1}{\subseteq} U_{\frac{1}{2^k}} \subseteq \underset{\text{closed}}{\bar{U}_{\frac{1}{2^k}}} \stackrel{*2}{\subseteq} U_{\frac{2}{2^k}} \subseteq \cdots \stackrel{*2^{k-1}}{\subseteq} U_{\frac{2^{k-1}}{2^k}} \subseteq \underset{\text{closed}}{\bar{U}_{\frac{2^{k-1}}{2^k}}} \stackrel{*2^k}{\subseteq} X \setminus B$$

By repeatedly applying the  $T_4$  condition  $2^k$  times, as indicated by the indices  $*1, *2, \dots, *2^k$ , we can construct  $2^k$  open sets such that:

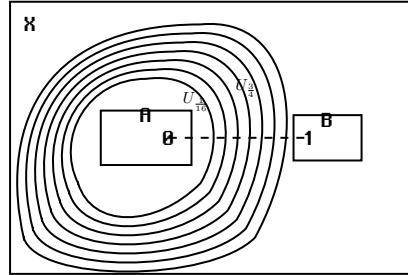
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map  $f : X \rightarrow [0, 1]$  as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map  $f$  is well-defined by (\*) and  $\sup D \leq 1$ . And  $f$  satisfies that:

1.  $\forall r \in D, x \in A \subset U_r$ . Thus,  $f(x) = 0$  if  $x \in A$ .
2.  $\forall r \in D, x \in B \subset X \setminus U_r$ . Thus,  $f(x) = \sup D = 1$  if  $x \in B$ .
3. If  $x \in \overline{U}_r$ , then for every  $s > r$ ,  $x \in \overline{U}_r \subset U_s$ . Thus,  $f(x) \leq r$ . In Contrapositive,  $f(x) > r \implies x \notin \overline{U}_r$ . (If  $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$ , then there is  $s \in D$  such that  $s > r$  and  $x \notin U_s$ , Contradiction.)
4. If  $x \notin U_r$ , then,  $f(x) \geq r$ . In Contrapositive,  $f(x) < r \implies x \in U_r$ .

Now, show that this map  $f$  is Continuous map: Let  $x \in X$  be fixed arbitrarily, and  $\varepsilon > 0$  be given.

In Case of  $0 < f(x) < 1$ .

Since Density of Dyadic Rationals, Choose  $r, s \in D$  such that  $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$ .

Now, we obtain that:

$$x \stackrel{(*)}{\in} U_s \setminus \overline{U}_r \stackrel{(**)}{\subseteq} f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$$

(\*) directly given by above properties, (\*\*) given applying the fact that  $x \in U_s \subset \overline{U}_s$  and  $x \notin \overline{U}_r$ .

In Case of  $f(x) = 0$ .

Choose  $r \in D$  such that  $f(x) = 0 < r < \varepsilon = f(x) + \varepsilon$ . Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \varepsilon)]$$

In Case of  $f(x) = 1$ .

Choose  $r \in D$  such that  $f(x) - \varepsilon = 1 - \varepsilon < r < 1 = f(x)$ . Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \varepsilon, f(x))]$$

Consequently,  $f$  is Continuous map on  $[0, 1]$  such that  $f|_A = 0$  and  $f|_B = 1$ .

Step 3. Generalization.

Since  $[0, 1] \cong [a, b]$  for any  $a < b$ , let  $g : [0, 1] \rightarrow [a, b] : x \mapsto (1-x)a + xb$  be a Homeomorphism.

Then,  $h = g \circ f : X \rightarrow [a, b]$  becomes a Continuous map such that  $h|_A = a$  and  $h|_B = b$ . □

## 12.13.2 Tietze Extension Theorem

**Theorem 12.13.2.1. Tietze Extension Theorem**

Let  $X$  be a  $T_4$  Space, and  $A \subseteq X$  be a closed subset.

For any Continuous map  $f : A \rightarrow \mathbb{R}$ , there exists a Continuous map:

$$g : X \rightarrow \mathbb{R} \text{ s.t. } g|_A = f$$

This  $g$  is called *extension* of  $f$ .

*Proof.* This proof consists by three steps.

**Step 1.** First, we will show that:

For any Continuous map  $f : A \rightarrow [-r, r]$ , there is a Continuous map  $h : X \rightarrow \mathbb{R}$  s.t.  $\begin{cases} \forall x \in X, |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, |f(a) - h(a)| \leq \frac{2}{3}r \end{cases}$  (\*)

Set

$$I_1 \stackrel{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \stackrel{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \stackrel{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and  $A$  is closed subspace of  $X$ ,  $f^{-1}[I_1]$  and  $f^{-1}[I_3]$  are closed of  $X$ .

And,  $I_1$  and  $I_3$  are disjoint, thus  $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$ .

Now, apply the Urysohn Lemma: There exists an Urysohn function  $h : X \rightarrow I_2$  for  $f^{-1}[I_1]$  and  $f^{-1}[I_3]$ .

Clearly, this map  $h$  satisfies the first condition in (\*). And, for show the second condition, let  $a \in A$  be given.

If  $a \in f^{-1}[I_1]$ , then  $f(a) \in I_1$  and  $h(a) = -\frac{1}{3}r$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

If  $a \in f^{-1}[I_3]$ , then  $f(a) \in I_3$  and  $h(a) = \frac{1}{3}r$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

If  $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$ , then  $f(a), h(a) \in I_2$ , thus  $|f(a) - h(a)| \leq \frac{2}{3}r$ .

Therefore, the second condition satisfied.

**Step 2.** We will show that: for any  $f : A \rightarrow [-1, 1]$ , there exists an extension of  $f$ .

Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of  $h_1$ , the continuous map  $f - h_1 : A \rightarrow [-\frac{2}{3}, \frac{2}{3}] : x \rightarrow f(x) - h_1(x)$  is well-defined.

Again, there exists a Continuous map:

$$h_2 : X \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a)| \leq (\frac{2}{3})^2 \end{cases}$$

Inductively, for any  $n \in \mathbb{N}$ , there exists a Continuous map:

$$h_n : X \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, |h_n(x)| \leq \frac{1}{3} \cdot (\frac{2}{3})^{n-1} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a) - \dots - h_n(a)| \leq (\frac{2}{3})^n \end{cases}$$

Define a map

$$g : X \rightarrow [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any  $x \in X$ ,

$$|g(x)| = \left| \sum_{n=1}^{\infty} h_n(x) \right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, Weierstrass M-test gives that  $\sum_{n=1}^{\infty} h_n(x)$  converges uniformly.

Moreover, for any  $a \in A$ ,

$$\left| f(a) - \sum_{k=1}^n h_k(a) \right| \leq \left(\frac{2}{3}\right)^n \implies \left| f(a) - \sum_{n=1}^{\infty} h_n(a) \right| = |f(a) - g(a)| = 0$$

That is,  $g$  is Continuous on  $X$  and  $g|_A = f$ . Therefore,  $g$  is extension of  $f$ .

**Step 3.** Finally, we generalize the result in Step 2.:

Let  $f : A \rightarrow [a, b]$  be a Continuous map on the closed subspace  $A$ . And, let  $\varphi : [a, b] \rightarrow [-1, 1]$  be a Homeomorphism.

Then,  $\varphi \circ f : A \rightarrow [-1, 1]$  is Continuous map, thus there exists an extension  $g : X \rightarrow [-1, 1]$  such that  $g|_A = \varphi \circ f$ .

Now,  $\varphi^{-1} \circ g : X \rightarrow [a, b]$  is Continuous, and  $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$ , Therefore this  $\varphi^{-1} \circ g$  is the extension of  $f$ .

Let  $f : A \rightarrow \mathbb{R}$  be a Continuous map on the closed subspace  $A$ .

And, let  $\varphi : \mathbb{R} \rightarrow (-1, 1)$  be a Homeomorphism. Then, the map  $\phi : \mathbb{R} \rightarrow [-1, 1] : x \mapsto \varphi(x)$  is still Continuous.

Now, The Continuous map  $\phi \circ f : A \rightarrow [-1, 1]$  has an extension  $g : X \rightarrow [-1, 1]$  such that  $g|_A = \phi \circ f$ .

Put  $B = g^{-1}[\{-1, 1\}]$ . Then  $B$  is Closed on  $X$ , and  $A \cap B = \emptyset$ . Now, apply the Urysohn Lemma to this, there exists an Urysohn function for  $A$  and  $B$ : Continuous map  $\gamma : X \rightarrow [0, 1]$  such that  $\gamma|_A = 1$  and  $\gamma|_B = 0$ .

Define a map  $\eta : X \rightarrow (-1, 1) : x \mapsto g(x)\gamma(x)$ . Then, if  $g(x) = 1$  or  $g(x) = -1$ , then  $x \in B$ , thus  $g(x)\gamma(x) = 0$ .

Therefore,  $\eta$  is well-defined. And, for any  $a \in A$ ,  $\eta(a) = g(a)\gamma(a) = g(a)$ , thus  $\eta|_A = \phi \circ f$ .

Consequently, the map  $\phi^{-1} \circ \eta$  is an extension of  $f$ , we wanted.  $\square$

Recall that:

**Definition 12.13.2.1.**  $X$  is  $T_1$  if:

For any distinct  $x, y \in X$ , there exist open sets  $U_x, U_y$  such that  $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$ .

**Lemma 12.13.2.1.**  $X$  is  $T_1$  if and only if For any  $x \in X$ , a singleton  $\{x\}$  is closed in  $X$ .

*Proof.* The left direction is clear.

Let  $x \in X$ . Then, for any  $y \in X$  with  $y \neq x$ ,  $T_1$  condition gives that there is an open set such that  $y \in U_y$  and  $x \notin U_y$ .

Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition.  $\square$

### 12.13.3 Urysohn Metrization Theorem

**Definition 12.13.3.1.** A space  $X$  is called *Completely Regular* if:  $X$  is  $T_1$  and  $T_{3\frac{1}{2}}$  where  $T_{3\frac{1}{2}}$  Condition: For any closed set  $C \subset X$  and  $x \in X \setminus C$ , there exists an *Urysohn function* for  $\{x\}$  and  $C$ . Completely regular space is sometimes called *Tychonoff Space*.

**Proposition 12.13.3.1.** Normal Space  $\Rightarrow$  Completely Regular Space  $\Rightarrow$  Regular Space.

*Proof.* If  $X$  is Normal space, then every singletone is closed by  $T_1$ . And, the *Urysohn Lemma* gives Urysohn map. If  $X$  is Completely Regular, then for closed  $C \subset X$  and  $x \in X \setminus C$ , there exists a continuous map  $f : X \rightarrow [0, 1]$  s.t

$$f[\{x\}] = 0 \text{ and } f[C] = \{1\}$$

Then,

$$\{x\} \subseteq f^{-1}\left[\left[0, \frac{1}{2}\right)\right], \quad C \subseteq f^{-1}\left[\left(\frac{1}{2}, 1\right]\right]$$

□

**Theorem 12.13.3.1.**  $T_{3\frac{1}{2}}$  is Hereditary. Furthermore, Completely Regular is hereditary since  $T_1$  is hereditary.

*Proof.* Let  $X$  be a  $T_{3\frac{1}{2}}$  Space, and  $Y \subseteq X$  be a subspace of  $X$ .

Let  $C \subseteq Y$  is closed set of  $Y$ , and  $x \in Y \setminus C$ . Note that:

$$C = \text{Closure of } C \text{ in } Y = \bigcap_{\substack{F \text{ closed in } Y \\ C \subseteq F}} F = \bigcap_{\substack{F' \text{ closed in } X \\ \text{s.t. } F = F' \cap Y}} F' \cap Y = (\text{Closure of } C \text{ in } X) \cap Y$$

Since  $x$  is contained in  $Y$  but not  $C$ , thus  $x$  is not contained in Closure of  $C$  in  $X$ . Now, since  $X$  is  $T_{3\frac{1}{2}}$ ,

There exists a Continuous map  $f : X \rightarrow [0, 1]$  s.t.  $f(x) = 0, f|_{\text{cl}_X(C)} = 1$

The restriction  $f_Y$  is continuous, and Urysohn function for  $x$  and  $C$ .

□

**Theorem 12.13.3.2.** Arbitrary product space of  $T_{3\frac{1}{2}}$  space is  $T_{3\frac{1}{2}}$ .

*Proof.* Let  $X_\gamma$  ( $\gamma \in \Gamma$ ) be  $T_{3\frac{1}{2}}$  Spaces. Put  $X = \prod_{\gamma \in \Gamma} X_\gamma$ . Suppose that  $C \subset X$  is closed set, and  $x \in X \setminus C$ .

Since  $X \setminus C$  is open, there exists an open  $U$  in  $X$  such that  $x \in U \subset X \setminus C$ .

Put  $F = \{\alpha \in \Gamma \mid X_\alpha \neq \pi_\alpha[U]\}$ . By definition of product space, this  $F$  is a finite index set. Note that:

$$\forall \alpha \in F, \pi_\alpha(x) \notin X_\alpha \setminus \pi_\alpha[U]$$

And, for each  $\alpha \in F$ ,  $X_\alpha \setminus \pi_\alpha[U]$  are non-empty closed set in  $X_\alpha$ , there exist continuous maps  $f_\alpha$  such that

$$f_\alpha : X_\alpha \rightarrow [0, 1], \quad f_\alpha|_{X_\alpha \setminus \pi_\alpha[U]} = 0, \quad f_\alpha|_{\pi_\alpha(U)} = 1$$

And, the composition  $f_\alpha \circ \pi_\alpha$  ( $\alpha \in F$ ) is continuous, and

$$(f_\alpha \circ \pi_\alpha)[X \setminus \pi_\alpha^{-1}[\pi_\alpha[U]]] = (f_\alpha \circ \pi_\alpha)[\pi_\alpha^{-1}[X_\alpha \setminus \pi_\alpha[U]]] \subseteq f_\alpha[X_\alpha \setminus \pi_\alpha[U]] = \{0\}$$

Now, the map

$$\Psi : X \rightarrow [0, 1] : t \mapsto \prod_{\alpha \in F} (f_\alpha \circ \pi_\alpha)(t)$$

is Continuous, and  $\Psi(x) = 1$  and  $\Psi[C] \subseteq \Psi[X \setminus U] = \{0\}$ .

□

**Theorem 12.13.3.3.** If  $X$  is Completely Regular, then for some index set  $\Lambda$ ,  $X$  can be embedded in  $[0, 1]^\Lambda$ .

*Proof.* Denote that:

$$\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \rightarrow [0, 1] \mid f \text{ is continuous}\}$$

**Claim:** the following function is embedding  $X$  into  $[0, 1]^\Lambda$ .

$$F : X \rightarrow [0, 1]^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$$

1.  $F$  is Continuous, since each  $f_\alpha$  is Continuous.

2.  $F$  is injective: Let  $x \neq y$  in  $X$ . Then,  $\{x\}$  and  $\{y\}$  are closed by  $T_1$ .

By  $T_{3\frac{1}{2}}$ , there exists a continuous map  $f : X \rightarrow [0, 1]$  such that  $f(x) = 0$  and  $f(y) = 1$ .

Since  $f = f_\beta$  for some  $\beta \in \Lambda$ ,  $F(x) \neq F(y)$ .

3.  $F$  is Open map: Let  $U \subseteq X$  be an open set, and let  $y \in F[U]$ . That is, for some  $x \in U$ ,  $F(x) = y$ .

Since  $X \setminus U$  is closed,  $x \notin X \setminus U$ , and  $T_{3\frac{1}{2}}$ , there exists a continuous map

$$f_\alpha : X \rightarrow [0, 1] \text{ s.t. } f_\alpha(x) = 0, f_\alpha|_{X \setminus U} = 1$$

Meanwhile, put  $V \stackrel{\text{def}}{=} \pi_\alpha^{-1}[[0, 1]] \subseteq [0, 1]^\Lambda$ , and  $W \stackrel{\text{def}}{=} V \cap F[X]$ . Then,  $W$  is open in the subspace  $F[X]$ .

And,  $\pi_\alpha(y) = \pi_\alpha(F(x)) = (\pi_\alpha \circ F)(x) = f_\alpha(x) = 0$ , thus  $y \in W$ . Now, there remains to prove:  $W \subseteq F[U]$ .

Let  $z \in W$ . Then,  $z \in V$  and  $z = F(x)$  for some  $x \in X$ , this implies  $\pi_\alpha(z) \in [0, 1]$ , i.e.,  $\pi_\alpha(F(x)) = f_\alpha(x) \neq 1$ .

Now,  $x \in U$ , that is  $F(x) = z \in F[U]$ . Thus,  $W \subseteq F[U]$ , consequently  $F$  is embedding.

□

**Corollary 12.13.3.1.** Let  $X$  is a Topological Space. TFAE:

a)  $X$  is Completely Regular Space.

b)  $X$  can be embedded in Compact Hausdorff Space.

c)  $X$  can be embedded in Normal Space.

*Proof.*

1. a)  $\Rightarrow$  b).  $X$  can be embedded in  $[0, 1]^\Lambda$ . And  $[0, 1]$  is Compact Hausdorff.

2. b)  $\Rightarrow$  c). Every Compact Hausdorff Space is Normal.

3. c)  $\Rightarrow$  a). Normal Space is Completely Regular, and Completely Regular is hereditary.

□

**Lemma 12.13.3.1.** Every Compact Hausdorff Space is Normal.

*Proof.* Let  $X$  be a Compact Hausdorff Space, and  $C, D \subset X$  be disjoint closed subsets.

Since  $X$  is Compact,  $C$  and  $D$  are Compact. Fix  $x \in C$ . Then, for any  $y \in D$ ,

There exist disjoint opens  $U_y, V_y$  such that  $x \in U_y$  and  $y \in V_y$ .

Since  $\{V_y \mid y \in D\}$  is open cover of  $D$ , there is a finite subcover  $\{V_y^i \mid 1 \leq i \leq n\}$ . That is,  $D \subseteq \bigcup_{i=1}^n V_y^i$ .

Now,  $\bigcap_{i=1}^n U_y^i$  is open set containing  $x$ , and

$$\left( \bigcup_{i=1}^n V_y^i \right) \cap \left( \bigcap_{i=1}^n U_y^i \right) = \bigcup_{i=1}^n \left( V_y^i \cap \left( \bigcap_{i=1}^n U_y^i \right) \right) = \bigcup_{i=1}^n \emptyset = \emptyset$$

In summary, for any  $x \in C$ , there exist disjoint open  $U_x, V_x$  such that  $x \in U_x$  and  $D \subset V_x$ .

Using this, Let  $\{U_x \mid x \in C\}$  be an open cover, then compactness gives the finite subcover  $\{U_x^i \mid 1 \leq i \leq n\}$ . Now,

$$C \subseteq \bigcup_{i=1}^n U_x^i, D \subseteq \bigcap_{i=1}^n V_x^i, \left( \bigcup_{i=1}^n U_x^i \right) \cap \left( \bigcap_{i=1}^n V_x^i \right) = \bigcup_{i=1}^n \left( U_x^i \cap \left( \bigcap_{i=1}^n V_x^i \right) \right) = \emptyset$$

□

**Theorem 12.13.3.4. Embedding Theorem**

Let  $X$  be a  $T_1$  Space. Denote  $\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$ .

Suppose that for any  $x \in X$  and open neighborhood  $U$  of  $x$ , there exists  $\alpha \in \Lambda$  such that  $f_\alpha(x) > 0$ ,  $f_\alpha|_{X \setminus U} = 0$ . Then, the map  $F : X \rightarrow \mathbb{R}^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$  is embedding.

**Theorem 12.13.3.5.** Suppose that  $X$  is Second-Countable Regular Space.

Then there exists a Countable collection  $\{f_n : X \rightarrow [0, 1] \mid n \in \mathbb{N}\}$  such that:

For any open  $U \subset X$  and  $x \in U$ , there exists  $n \in \mathbb{N}$  such that  $f_n(x) > 0$  and  $f_n|_{X \setminus U} = 0$ .

**Theorem 12.13.3.6. Urysohn Metrization Theorem**

If  $X$  is a Second-Countable Regular Space, then  $X$  is Metrizable.

## 12.14 Examples

**Proposition 12.14.0.1.** Lower Limit Topology  $(\mathbb{R}, \mathcal{T}_l)$  is  $T_1$  and  $T_4$  Space. Therefore, Normal Space.

*Proof.*  $T_1$  is clear, because: let  $x, y \in \mathbb{R}$  be a distinct two points. Without Loss of Generality, assume  $x < y$ . Then,

$$\begin{cases} x \in \left[x, \frac{x+y}{2}\right), \quad y \in [y, y+1) \\ y \notin \left[x, \frac{x+y}{2}\right), \quad x \notin [y, y+1) \end{cases}$$

Thus,  $T_1$  satisfied. And, to show  $T_4$ , Let  $C, D \subseteq \mathbb{R}$  be disjoint closed subsets. Let  $x \in C$  be given. Then, there exists a basis element of  $\mathcal{T}_l$ ,  $[a, p_x)$  such that

$$x \in [a, p_x) \subseteq \mathbb{R} \setminus D$$

since  $C \subseteq \mathbb{R} \setminus D$  and  $\mathbb{R} \setminus D$  is open. Now,

$$U = \bigcup_{x \in C} [x, p_x)$$

is open set containing  $C$ , and  $U \cap D = \emptyset$ .

Similarly, let  $y \in D$  be given. Then, there exists a basis element  $[a, q_y)$  such that

$$y \in [b, q_y) \subseteq \mathbb{R} \setminus C$$

Then, for each  $y \in D$ ,  $[y, q_y) \cap U = \emptyset$  because: Suppose that  $[y, q_y) \cap U \neq \emptyset$ . Choose  $p \in [y, q_y) \cap U$ .

That is,  $p \in [y, q_y)$  and for some  $x \in C$ ,  $p \in [x, p_x)$ .

Hence,  $[\max(x, y), \min(p_x, q_y))$  is non-empty set which containing  $x$  or  $y$ .

If either  $x$  or  $y$  contained in  $[\max(x, y), \min(p_x, q_y))$ , contradiction. Now, an union

$$V = \bigcup_{y \in D} [y, q_y)$$

is open set containing  $D$ , and  $U \cap V = \emptyset$ ,

□

## 12.15 Quotient Space

**Definition 12.15.0.1.** Let  $(X, \mathcal{T})$  be a Topological Space,  $Y$  be a set, and  $f : X \rightarrow Y$  be an onto map. Define Quotient Topology on  $Y$  induced by  $f$ :  $\mathcal{T}_Q \stackrel{\text{def}}{=} \{U \subseteq Y \mid f^{-1}[U] \in \mathcal{T}\}$ . This is the largest topology on  $Y$  such that  $f$  is Continuous map.

**Definition 12.15.0.2.** Let  $X$  be a Topological Space, and  $\sim$  be an equivalent relation on  $X$ . Define Canonical map on  $X$ :  $\pi : X \rightarrow X/\sim : x \mapsto [x]$ , and define Quotient Space  $(X/\sim, \mathcal{T}_Q)$  where  $\mathcal{T}_Q$  is quotient topology on  $X/\sim$  induced by  $\pi$ .

$X$  Topological Space,  $\sim$  equivalent relation on  $X$ ,  $\pi : X \rightarrow X/\sim : x \mapsto [x]$  canonical map.

**Lemma 12.15.0.1.** For any topological space  $Z$  and a map  $g : X/\sim \rightarrow Z$ ,

$g$  is Continuous if and only if  $g \circ \pi$  is Continuous.

*Proof.* Let  $g \circ \pi$  be Continuous map. Then, for any open  $U \subseteq Z$ ,

$$(g \circ \pi)^{-1}[U] = \pi^{-1}[g^{-1}[U]]$$

is open, thus  $g^{-1}[U]$  is open in  $X/\sim$ . That is,  $g$  is Continuous.  $\square$

**Lemma 12.15.0.2.** Let  $Z$  be a Topological space.

If given Continuous map  $f : X \rightarrow Z$  satisfies  $x \sim y \implies f(x) = f(y)$ , then  $\tilde{f} : X/\sim \rightarrow Z : [x] \mapsto f(x)$  is Continuous, and unique map such that  $\tilde{f} \circ \pi = f$ .

*Proof.* Well-Defined because:  $[x] = [y] \iff x \sim y \implies f(x) = f(y)$ .

$\tilde{f} \circ \pi = f$ : for any  $x \in X$ ,  $(\tilde{f} \circ \pi)(x) = \tilde{f}(\pi(x)) = \tilde{f}([x]) = f(x)$ , thus  $\tilde{f}$  is continuous since above lemma.

Uniqueness: if  $g : X/\sim \rightarrow Z$  satisfies  $g \circ \pi = f$ , then for any  $[x] \in X/\sim$ ,

$$g([x]) = g(\pi(x)) = (g \circ \pi)(x) = f(x) = \tilde{f}([x])$$

$\square$

**Lemma 12.15.0.3.** Let  $Z$  be a Topological space.

If given Continuous onto map  $f : X \rightarrow Z$  satisfies  $x \sim y \iff f(x) = f(y)$ , and  $f$  is either open or closed map, then  $\tilde{f} : X/\sim \rightarrow Z : [x] \mapsto f(x)$  is Homeomorphism.

*Proof.* Since  $[x] = [y] \iff x \sim y \iff f(x) = f(y)$ ,  $\tilde{f}$  is Well-defined and injective.

Continuity given by above Lemma, and Surjective:

$$\tilde{f}[X/\sim] = \tilde{f}[\pi[X]] = f[X] = Z$$

The last equality given by Surjectiveness of  $f$ .  $\square$

## 12.16 Quotient Map

**Definition 12.16.0.1.** Let  $X, Y$  be Topological Space.

A Continuous onto map  $f : X \rightarrow Y$  is called **quotient map** if:

$$U \subseteq Y \text{ is open if and only if } f^{-1}[U] \subseteq X \text{ is open.}$$

### 12.16.1 Basic Properties

**Proposition 12.16.1.1.** Composition of quotient maps is quotient map.

*Proof.* Suppose that  $X, Y, Z$  are Topological Space, and  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  are Quotient map. Then,

$$U \subseteq Z \text{ is open} \iff g^{-1}[U] \subseteq Y \text{ is open} \iff f^{-1}[g^{-1}[U]] \text{ is open}$$

implies  $g \circ f : X \rightarrow Z$  is Quotient map, being  $f^{-1}[g^{-1}[U]] = (g \circ f)^{-1}[U]$ .  $\square$

**Proposition 12.16.1.2.** Continuous onto map is quotient map if either open or closed map.

*Proof.* Suppose that  $f : X \rightarrow Y$  is Continuous onto map.

If  $f$  is an Open map,  $\square$

**Theorem 12.16.1.1.** If  $f : X \rightarrow Y$  is quotient map, then  $X/\sim \cong Y$  where

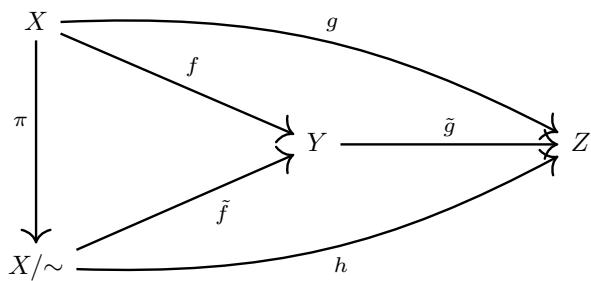
$$x \sim y \iff f(x) = f(y)$$

Moreover, if continuous map  $g : X \rightarrow Z$  satisfies

$$f(x) = f(y) \implies g(x) = g(y)$$

Then,  $\tilde{g} : Y \rightarrow Z : f(x) \mapsto g(x)$  is the unique continuous map such that  $\tilde{g} \circ f = g$ .

### 12.16.2 Quotient map Diagram



- $f : X \rightarrow Y$  is quotient map.
- $g : X \rightarrow Z$  is continuous map s.t.  $f(x) = f(y) \implies g(x) = g(y)$ .
- $\pi : X \rightarrow X/\sim : x \mapsto [x]$ .
- $X/\sim$  is quotient topology induced by  $\pi$ .

In this setting,  $\tilde{f}$  is Homeomorphism between  $X/\sim$  and  $Y$ , and  $h = \tilde{g} \circ \tilde{f}$  is Continuous map between  $X/\sim$  and  $Z$ .

## 12.17 Typical Quotient Spaces

We follow some convention:

'Equivalent relation on  $X$ : Condition' means the smallest Equivalent relation on  $X$  such that the Condition.  
In this section, put  $I \stackrel{\text{def}}{=} [0, 1] \subset \mathbb{R}$ .

### 12.17.1 Cylinder

**Definition 12.17.1.1.** Define an Equivalent Relation on  $I^2 = I \times I$ :

$$\forall y \in I, (0, y) \sim (1, y)$$

That is,

$$\sim^{\text{def}} \{((x, y), (x, y)) \mid x, y \in I\} \cup \{((0, y), (1, y)), ((1, y), (0, y)) \mid y \in I\}$$

The Quotient Space  $I^2 / \sim$  is called *Cylinder*.

**Theorem 12.17.1.1.** The Cylinder  $I^2 / \sim$  is Homeomorphic to  $S^1 \times I$ .

*Proof.* Define a map:

$$f : I^2 \rightarrow \frac{S^1 \times I}{\subset \mathbb{R}^3} : (x, y) \mapsto (\underbrace{(\cos(2\pi x), \sin(2\pi x))}_{\in S^1}, y)$$

$f$  is Continuous: Since  $x \mapsto \cos(2\pi x)$ ,  $x \mapsto \sin(2\pi x)$ , and  $y \mapsto y$  are all Continuous.

$f$  is Surjective, clearly. Thus, the  $f$  is Continuous onto map.

$f$  is Closed map:  $I^2$  is Compact, being Closed Bounded subset of  $\mathbb{R}^2$ , and  $S^1 \times I$  is Hausdorff, Hereditary Property.  
Finally, Check that Fact:

$$\begin{aligned} (x_1, y_1) \sim (x_2, y_2) &\iff (x_1, y_1) = (x_2, y_2) \text{ or } x_1 = 0, x_2 = 1, y_1 = y_2 \text{ or } x_1 = 1, x_2 = 0, y_1 = y_2 \\ &\iff f((x_1, y_1)) = f((x_2, y_2)) \end{aligned}$$

Now, The map  $\tilde{f} : I^2 / \sim \rightarrow S^1 \times I : [(x, y)] \mapsto f((x, y))$  induced by  $f$  is Homeomorphism,  $I^2 / \sim \cong S^1 \times I$ . □

### 12.17.2 Möbius band

**Definition 12.17.2.1.** Define an Equivalent Relation on  $I^2$ :

$$\forall y \in I, (0, y) \sim (1, 1 - y)$$

The Quotient Space  $I^2 / \sim$  is called *Möbius band*.

### 12.17.3 Torus

**Definition 12.17.3.1.** Define an Equivalent Relation on  $I^2$ :

$$\begin{cases} \forall x \in I, (x, 0) \sim (x, 1) \\ \forall y \in I, (0, y) \sim (1, y) \\ (0, 0) \sim (1, 0) \sim (0, 1) \sim (1, 1) \end{cases}$$

The Quotient Space  $I^2/\sim$  is called **Torus**.

**Theorem 12.17.3.1.** The Torus  $I^2/\sim$  is Homeomorphic to  $S^1 \times S^1 \subset \mathbb{C}^2$ .

*Proof.* Define a map:

$$f : \underbrace{I^2}_{\text{Compact}} \rightarrow \underbrace{S^1 \times S^1}_{\text{Hausdorff}} : (x, y) \mapsto (e^{2x\pi i}, e^{2y\pi i})$$

Then,  $f$  is Continuous Onto, and Closed map. Moreover,

$$(x_1, y_1) \sim (x_2, y_2) \iff (e^{2x_1\pi i}, e^{2y_1\pi i}) = (e^{2x_2\pi i}, e^{2y_2\pi i}) \iff f((x_1, y_1)) = f((x_2, y_2))$$

Now, The map  $\tilde{f} : I^2/\sim \rightarrow S^1 \times S^1 : [(x, y)] \mapsto f((x, y))$  induced by  $f$  is Homeomorphism,  $I^2/\sim \cong S^1 \times S^1$ .  $\square$

**Theorem 12.17.3.2.** The Torus  $I^2/\sim$  can embed into  $\mathbb{R}^3$ .

*Proof.* Define a map:

$$g : I^2 \rightarrow \mathbb{C} \times \mathbb{R} : (x, y) \mapsto ([2 + \cos(2\pi x) \cdot e^{2y\pi i}, \sin(2\pi x)])$$

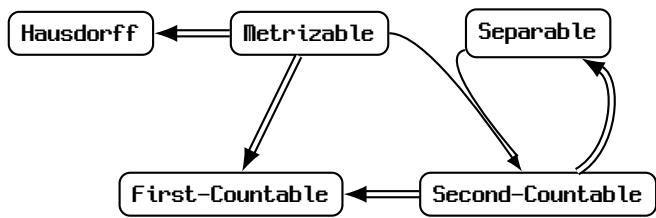
Then,  $g$  is Continuous map. Moreover,

$$\begin{aligned} g((x_1, y_1)) = g((x_2, y_2)) &\iff ([2 + \cos(2\pi x_1) \cdot e^{2y_1\pi i}, \sin(2\pi x_1)]) = ([2 + \cos(2\pi x_2) \cdot e^{2y_2\pi i}, \sin(2\pi x_2)]) \\ &\iff (x_1 = x_2 \text{ or } x_1, x_2 \in \{0, 1\}) \text{ and } (y_1 = y_2 \text{ or } y_1, y_2 \in \{0, 1\}) \\ &\iff (x_1, y_1) \sim (x_2, y_2) \end{aligned}$$

Meanwhile,  $g[I^2] \subset \mathbb{C} \times \mathbb{R}$  is Hausdorff, thus the map  $G : I^2 \rightarrow g[I^2] : (x, y) \mapsto g((x, y))$  is Continuous Onto Closed map.

Now, the map  $\tilde{G} : I^2/\sim \rightarrow g[I^2] \subset \mathbb{C} \times \mathbb{R}$  is Embedding.  $\square$

## 12.18 Diagrams



## 12.19 Manifold

### 12.19.1 Definition

**Definition 12.19.1.1.** A Topological Space  $X$  is called *n-Dimensional Manifold* if:  $X$  is Second-Countable Hausdorff Space, and satisfying

For all  $x \in X$ , there exists Open  $U_x$  with  $x \in U_x$  such that  $U_x \cong \mathbb{R}^n$  or  $U_x \cong \mathbb{H}^n \stackrel{\text{def}}{=} \{(x_i)_{i=1}^n \in \mathbb{R}^n \mid x_n \geq 0\}$ .

The 1-Dimensional Manifold is called *Curve*, and 2-Dimensional Manifold is called *Surface*.

If every Neighborhood Homeomorphic to  $\mathbb{R}^n$ , then it is called a *Manifold with no Boundary*, and if not, then it is called a *Manifold with Boundary*.

For *n*-Dimensional Manifold  $X$ , and  $x \in X$ ,

If there exists an Open  $U_x$  with  $x \in U_x$  s.t  $U_x \cong \mathbb{R}^n$ , then  $x$  is called *Interior*, Denote the set  $X^\circ \subset X$ .

And, the set  $\partial X \stackrel{\text{def}}{=} X \setminus X^\circ$  is called *Boundary* of  $X$ .

\*This Definition is temporary.

**Lemma 12.19.1.1.** For any  $r > 0$ , The Euclidean Space  $\mathbb{R}^n$  is Homeomorphic to  $B(0, r)$ , and The  $\mathbb{H}^n$  is Homeomorphic to  $B(0, r)$  to  $B(0, r) \cap \mathbb{H}^n$ .

### 12.19.2 Connected Sum

# Chapter 13

## Algebraic Topology

### 13.1 Orbit Space

**Definition 13.1.0.1.** Suppose that the set  $G$  is  $\left\{ \begin{array}{l} \text{Topological Space under the Topology } \mathcal{T} \\ \text{Group under the operation } \cdot : G \times G \rightarrow G \end{array} \right.$ .

The triple  $(G, \mathcal{T}, \cdot)$  is called *Topological Group* if:  $\left\{ \begin{array}{l} \mu : G \times G \rightarrow G : (g_1, g_2) \mapsto g_1 g_2 \\ \iota : G \rightarrow G : g \mapsto g^{-1} \end{array} \right.$  are Continuous maps.

#### 13.1.1 General Linear Group

**Definition 13.1.1.1.** Define a set of invertible matrices, *General Linear Group over field F*:

$$\mathrm{GL}(n, F) \stackrel{\mathrm{def}}{=} \{A \in \mathcal{M}_{n,n}(F) \mid \det A \neq 0\}$$

### 13.2 Homotopy

### 13.3 Fundamental Group

## **Chapter 14**

### **Basic Analysis**

## 14.1 Tests for Series

### 14.1.1 Integral Test

**Theorem 14.1.1.1.** Let  $f : [1, \infty) \rightarrow \mathbb{R}$  be a decreasing function which satisfies  $\begin{cases} \lim_{x \rightarrow \infty} f(x) = 0 \\ f > 0 \end{cases}$ . Then,

$$\int_1^\infty f(x)dx \text{ converges if and only if } \sum_{k=1}^{\infty} f(k) \text{ converges.}$$

Futhermore, put  $d_n \stackrel{\text{def}}{=} \sum_{k=1}^n f(k) - \int_1^n f(x)dx$ , then for any  $n \in \mathbb{N}$ ,  $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$ , and for any  $k \in \mathbb{N}$ ,  $0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k)$ . (Clearly,  $\lim_{n \rightarrow \infty} d_n$  exists.)

**Proof.** Since

$$\begin{aligned} \int_1^{n+1} f(x)dx &= \sum_{k=1}^n \int_k^{k+1} f(x)dx \stackrel{\text{decreasing}}{\downarrow} \sum_{k=1}^n f(k) = \sum_{k=1}^n f(k) \\ \implies f(n+1) &= \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^n f(k) \leq \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)dx = d_{n+1} \end{aligned}$$

And,

$$d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \geq \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Immediate  $d_n$  converges, being bounded and decreasing. That is,

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \left( \sum_{k=1}^n f(k) - \int_1^n f(x)dx \right)$$

converges. Meanwhile, since

$$0 \leq d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \leq \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

Now, telescope:

$$0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k) - \lim_{n \rightarrow \infty} f(n+1) = f(k)$$

□

### 14.1.2 Ratio Test

**Theorem 14.1.2.1.** Let  $\sum a_n$  be given.

$\sum_{n=1}^{\infty} a_n$  converges if:  $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ .

$\sum_{n=1}^{\infty} a_n$  diverges if:  $n_0 \in \mathbb{N}$  such that  $\forall n \geq n_0$ ,  $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ .

**Proof.** Choose  $\beta < 1$  such that for some  $N \in \mathbb{N}$ ,  $n \geq N \implies \left| \frac{a_{n+1}}{a_n} \right| < \beta < 1$ .

Then,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ &\vdots \\ |a_{N+p}| &< \beta^p |a_N| \quad (p \in \mathbb{N}) \end{aligned}$$

As a result, for all  $n \geq N$ ,  $|a_n| < \beta^{n-N} |a_N|$ . And,  $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \beta^{n-N} |a_N| < \infty$ .

□

### 14.1.3 Root Test

**Theorem 14.1.3.1.** Let  $\sum a_n$  be given.

$\sum_{n=1}^{\infty} a_n$  converges if:  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ .

$\sum_{n=1}^{\infty} a_n$  diverges if:  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ .

**Proof.** Put  $\beta \in \mathbb{R}$  such that  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < \beta < 1$ . Then, there is  $N \in \mathbb{N}$  such that  $n \geq N \implies \sqrt[n]{|a_n|} < \beta$ .

Now,  $\sum |a_n| < \sum \beta^n < \infty$ . But if  $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ , then  $a_n \not\rightarrow 0$ .

□

## 14.2 Arithmetic means

Let  $\{s_n\}$  be a Complex numbers Sequence. Define the **Arithmetic means** of  $\{s_n\}$ :

$$\sigma_n \stackrel{\text{def}}{=} \frac{s_0 + \cdots + s_n}{n+1} = \frac{1}{n+1} \left( \sum_{i=0}^n s_i \right)$$

Then, the Arithmetic means  $\sigma_n$  has the following properties:

1). If  $\lim_{n \rightarrow \infty} s_n = s$ , then  $\lim_{n \rightarrow \infty} \sigma_n = s$ .

*Proof.* Let  $\varepsilon > 0$  be given. Then, there exists  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|s_n - s| < \varepsilon$ .

Now, for  $n \geq N$ ,

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{s_0 + \cdots + s_n}{n+1} - \frac{(n+1)s}{n+1} \right| = \left| \frac{(s_0 - s) + \cdots + (s_n - s)}{n+1} \right| \\ &\stackrel{\text{tri. eq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^n |s_k - s|}{n+1} \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \varepsilon \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \varepsilon \end{aligned}$$

Now, put  $M \in \mathbb{N}$  satisfies  $M \geq N$  and  $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \varepsilon$ , using Archimedean property.  
Then,  $n \geq M$  implies  $|\sigma_n - s| < \varepsilon$ , thus  $\sigma_n \rightarrow s$ . □

2). Put  $a_n = s_n - s_{n-1}$ , for  $n \geq 1$ . If  $\lim_{n \rightarrow \infty} na_n = 0$  and  $\sigma_n$  converges, then  $s_n$  converges.

*Proof.* First,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + \cdots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^n s_k}{n+1} \\ &= \frac{1}{n+1} ((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \cdots + (ns_n - ns_{n-1})) \\ &= \frac{1}{n+1} \sum_{k=1}^n ka_k \end{aligned}$$

Now, if  $na_n \rightarrow 0$  and  $\sigma_n \rightarrow \sigma$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \left( \sigma_n + \frac{1}{n+1} \sum_{k=1}^n ka_k \right) \\ &= \lim_{n \rightarrow \infty} \sigma_n + \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n ka_k \stackrel{1)}{=} \sigma \end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:

3). The sequence  $\{na_n\}$  bounded by  $M < \infty$ , and  $\sigma_n \rightarrow \sigma$ . Then,  $s_n \rightarrow \sigma$ .

*Proof.* First, For positive integers  $m < n$ ,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^n s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left( \frac{1}{m+1} - \frac{1}{n+1} \right) \sum_{k=0}^n s_k \\ &= s_n - \frac{m+1}{n-m} \cdot \left( \frac{\sum_{k=0}^m s_k + \sum_{k=m+1}^n s_k}{m+1} - \frac{\sum_{k=0}^n s_k}{n+1} \right) \\ &= s_n - \frac{m+1}{n-m} \cdot \left( \sigma_m - \sigma_n + \frac{\sum_{k=m+1}^n s_k}{m+1} \right) \\ &= \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \end{aligned}$$

Meanwhile, since for any  $n \in \mathbb{N}$ ,  $|na_n| = n|s_n - s_{n-1}| < M$ , for  $k = m+1, \dots, n$ ,

$$\begin{aligned} |s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \dots + s_{k+1} - s_k| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{k+1} - s_k| \\ &\leq \frac{M}{n} + \frac{M}{n-1} + \dots + \frac{M}{k+1} \leq \frac{n-k}{k+1}M \leq \frac{n-k}{m+2}M \leq \frac{n-m-1}{m+2}M \end{aligned}$$

Let  $\varepsilon > 0$  be given. For each  $n \in \mathbb{N}$ , put  $m \in \mathbb{N}$  such that

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$$

Then,

$$m(1+\varepsilon) \leq n-\varepsilon \implies m+\varepsilon(1+m) \leq n \implies \frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$$

and

$$n-\varepsilon < (m+1)(1+\varepsilon) \implies n+1 < (m+2)(1+\varepsilon) \implies \frac{n+1}{m+2} - 1 < \varepsilon \implies \frac{n-m-1}{m+2} < \varepsilon$$

Now, for arbitrary  $n \in \mathbb{N}$ ,

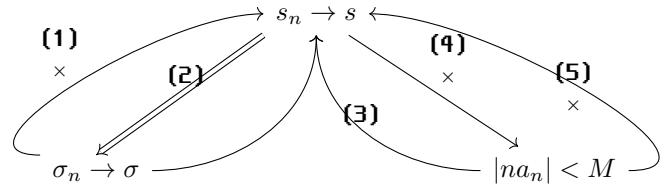
$$\begin{aligned} |s_n - \sigma| &\leq |s_n - \sigma| + |\sigma_n - \sigma| \\ &\implies \limsup_{n \rightarrow \infty} |s_n - \sigma| \leq \limsup_{n \rightarrow \infty} |s_n - \sigma_n| + \limsup_{n \rightarrow \infty} |\sigma_n - \sigma| \end{aligned}$$

And,

$$\begin{aligned} |s_n - \sigma_n| &= \frac{m+1}{n-m} |\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{k=m+1}^n |s_n - s_k| < \frac{1}{\varepsilon} |\sigma_n - \sigma_m| + M\varepsilon \\ &\implies \limsup_{n \rightarrow \infty} |s_n - \sigma_n| \leq \frac{1}{\varepsilon} \limsup_{n \rightarrow \infty} |\sigma_n - \sigma_m| + M\varepsilon = M\varepsilon \end{aligned}$$

Consequently,  $\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq (M+1)\varepsilon$ , thus  $s_n \rightarrow \sigma$ . □

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let  $s_n \stackrel{\text{def}}{=} \exp(\frac{inx\pi}{2})$ . Then,

- $s_n$  diverges.
- $na_n$  diverges.
- $\sigma_n \rightarrow 0$ .

(2) Let  $s_n \stackrel{\text{def}}{=} \frac{1}{n}$ ,  $s_0 = 0$ .

(3) Let  $s_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k}$ . Then,

- $s_n$  diverges.
- $a_n = \frac{1}{n}$ , thus  $na_n \rightarrow 1$ , bounded.
- If  $\sigma_n$  converges, then the diagram implies that  $s_n$  must converge, leading to a contradiction. Therefore,  $\sigma_n$  diverges.

(4)  $s_n = \sum_{k=1}^n \frac{(-1)^k}{\sqrt{k}}$ ,  $s_0 = 0$ . Then,

- $s_n$  converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$ , thus  $na_n$  diverges.

### 14.3 Taylor's Theorem

**Theorem 14.3.0.1.** *Taylor's Theorem*

Let  $f : [a, b] \rightarrow \mathbb{R}$ , and let  $n \in \mathbb{N}$  be fixed. Suppose that  $\begin{cases} f^{(n-1)} \text{ is continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$  Then, for any  $\alpha, \beta \in [a, b]$ , there exists  $x \in (\alpha, \beta)$  such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n$$

**Proof.** Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left( f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k - M(t - \alpha)^n, \quad (a \leq t \leq b)$$

If we differentiate the above equation  $n$  times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, \quad (a < t < b)$$

For each  $k = 0, 1, \dots, n-1$ ,

$$\begin{aligned} \frac{d^r}{dt^r} \left( \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} ((t - \alpha)^k) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting  $t = \alpha$ , only the  $f^{(r)}(\alpha)$  term remains. Therefore, for  $r = 0, \dots, n-1$ ,  $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$ . Since  $g(\beta) = 0$  by definition, the Mean-Value Theorem implies there exists a  $x_1 \in (\alpha, \beta)$  s.t.  $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$ . And similarly, there is  $x_2 \in (x_1, \beta)$  s.t.  $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$ .

Inductively, for some  $x_n \in (\alpha, \beta)$ ,  $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$ . That is,  $M = \frac{f^{(n)}(x_n)}{n!}$ .

Proof Complete by Initial Setting. □

**Corollary 14.3.0.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be an infinitely differentiable function.

Suppose that there exists a  $M > 0$  such that for any  $n \in \mathbb{N}$ ,  $\sup_{t \in [a, b]} |f^{(n)}(t)| \leq M$ . Then, for any  $x, \alpha \in [a, b]$ ,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (x - \alpha)^k$$

## 14.4 Convexity

### 14.4.1 Definition

**Definition 14.4.1.1.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Real-valued function.  $f$  is said to be **convex** if: For any  $x, y \in (a, b), \lambda \in (0, 1)$ ,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Convex function has following properties:

**Lemma 14.4.1.1.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Convex function, and  $a < x_1 < x_2 < x_3 < b$ . Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

*Proof.* To show that first inequality, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2 - x_1}{x_3 - x_1}(f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. □

#### 14.4.2 Properties

**Proposition 14.4.2.1.** If  $f : (a, b) \rightarrow \mathbb{R}$  is Convex, then  $f$  is Continuous.

**Proof.** Let  $\varepsilon > 0$  be given,  $s < t$  are fixed in  $(a, b)$ . For any  $x, y \in (s, t)$  with  $s < x < y < t$ ,

$$\frac{f(s) - f(a)}{s - a} \leq \frac{f(x) - f(s)}{x - s} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(t) - f(y)}{t - y} \leq \frac{f(b) - f(t)}{b - t}$$

Put  $M = \max \left\{ \left| \frac{f(s) - f(a)}{s - a} \right|, \left| \frac{f(b) - f(t)}{b - t} \right| \right\}$ . Then, for any  $x, y \in (s, t)$ ,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M$$

Now,

$$|f(y) - f(x)| \leq M|y - x| < \varepsilon$$

Since  $s, t \in (a, b)$  was arbitrary,  $f$  is continuous on  $(a, b)$ . □

**Proposition 14.4.2.2.** Let  $f$  is differentiable on  $(a, b)$ . Then,

$f$  is Convex if and only if  $f'$  is monotonically increasing on  $(a, b)$ .

**Proof.** Prove by showing both directions: right and left.

**Right Direction** Let  $x_1 < x_2$  in  $(a, b)$ . Then,

$$f'(x_1) = \lim_{t \rightarrow x_1} \frac{f(t) - f(x_1)}{t - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{\tau \rightarrow x_2} \frac{f(\tau) - f(x_2)}{\tau - x_2} = f'(x_2)$$

More rigorously, put  $\varepsilon = \left| f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right|$ . (If  $\varepsilon = 0$ , then there is nothing to prove.).

Now, there exists a  $\delta > 0$  such that  $|t - x_1| < \delta$  implies

$$\left| f'(x_1) - \frac{f(t) - f(x_1)}{t - x_1} \right| < \varepsilon \iff -\varepsilon + \frac{f(t) - f(x_1)}{t - x_1} < f'(x_1) \stackrel{(*)}{=} \varepsilon + \frac{f(t) - f(x_1)}{t - x_1}$$

If  $f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$ , then  $(*)$  gives

$$f'(x_1) < f'(x_1) + \frac{f(t) - f(x_1)}{t - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \iff \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(t) - f(x_1)}{t - x_1} \quad \forall t \text{ s.t. } |t - x_1| < \delta$$

If  $|t - x_1| < |x_2 - x_1|$ , then this contradicts to Convexity.

Consequently, we obtain the first inequality, similarly can prove the second inequality.

**Left Direction** Let  $x, y \in (a, b)$  and  $\lambda \in (0, 1)$  be given. The Mean Value Theorem gives that:

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) - f(x) &= f'(z_1)(\lambda x + (1 - \lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1 - \lambda)y) \\ f(y) - f(\lambda x + (1 - \lambda)y) &= f'(z_2)(y - \lambda x + (1 - \lambda)y) \text{ for some } z_2 \in (\lambda x + (1 - \lambda)y, y) \end{aligned}$$

Now, Monotonically increasing gives

$$\begin{aligned} \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{\lambda x + (1 - \lambda)y - x} &= f'(z_1) \leq f'(z_2) = \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} \\ \implies \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{(1 - x)(y - x)} &\leq \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} \\ \implies \lambda f(\lambda x + (1 - \lambda)y) - \lambda f(x) &\leq (1 - \lambda)f(y) - (1 - \lambda)\lambda x + (1 - \lambda)y \\ \implies f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

□

**Corollary 14.4.2.1.** If  $f : [a, b] \rightarrow \mathbb{R}$  is twice-differentiable, then

$$f \text{ is Convex if and only if } f''(x) \geq 0 \text{ for all } x \in (a, b).$$

**Theorem 14.4.2.1.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be given. Then,

$f$  is Convex if and only if  $f$  is Continuous, and Midpoint Convex.

Midpoint convex is that  $f$  satisfies  $\forall x, y \in (a, b), f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$ .

*Proof.* The right direction is clear. To show the left direction, we demonstrate that Midpoint Convexity implies Dyadic Rational Convexity. Claim: For any  $n \in \mathbb{N}$ ,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) \quad (*)$$

Using Induction: If  $n = 1$ , it is clear by Midpoint Convexity.

Assume that for  $n \in \mathbb{N}$ ,  $(*)$  is True. Then,

$$\begin{aligned} f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2} \cdot \left[ \frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n} \right]\right) \\ &\stackrel{\text{m.c.}}{\leq} \frac{1}{2} \left( f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right) \right) \\ &\stackrel{(*)}{\leq} \frac{1}{2} \left( \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} f(x_k) \right) \\ &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f(x_k) \end{aligned}$$

Consequently, we obtain the claim. Now, let  $n \in \mathbb{N}$ , and  $m$  be an integer such that  $1 \leq m \leq 2^n$ .

Put  $x_1 = x_2 = \dots = x_m = x$  and  $x_{m+1} = x_{m+2} = \dots = x_{2^n} = y$ . Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let  $x, y \in (a, b), \lambda \in (0, 1)$  be given.

Since  $\frac{\lfloor 2^n \lambda \rfloor}{2^n} \rightarrow \lambda$  as  $n \rightarrow \infty$ , for any  $n \in \mathbb{N}$ ,

$$f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n \lambda \rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n \rightarrow \infty} f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} \left[\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity.  $\square$

## 14.5 Lipschitz Condition

### 14.5.1 Definition

**Definition 14.5.1.1.** A real-valued function  $f : (a, b) \rightarrow \mathbb{R}$  is called **Lipschitz Continuous** if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant  $L$  is said to be **Lipschitz Constant** of  $f$ . In particular, the constant

$$D \stackrel{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called **dilation** of  $f$ . Clearly,

$$\forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if  $L > 0$  is Lipschitz Constant of  $f$ , then  $D \leq L$ . That is,  $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$ .

### 14.5.2 Properties

**Proposition 14.5.2.1.** If  $f : (a, b) \rightarrow \mathbb{R}$  is Lipschitz Continuous, then  $f$  is uniformly continuous.

*Proof.* Let  $L \geq 0$  be a Lipschitz Constant of  $f$ . Then, for any  $\varepsilon > 0$ ,

$$\forall x, y \in (a, b), |x - y| < \frac{\varepsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \varepsilon$$

□

**Proposition 14.5.2.2.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be a Differentiable function. Then,

$f$  is Lipschitz Continuous if and only if  $f'$  is bounded in  $(a, b)$ .

*Proof.*

*Right Direction*

Let  $L > 0$  be a Lipschitz constant of  $f$ , and  $x \in (a, b)$  be given. Since definition of derivative,

$$f'(x) \stackrel{\text{def}}{=} \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct  $x, t \in (a, b)$ ,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \rightarrow x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \rightarrow x} L = L$$

*Left Direction*

Let distinct  $x, y \in (a, b)$  be given. Then, the Mean-Value Theorem gives: There exists a  $z \in (x, y)$  such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If  $x = y$ , then there is nothing to prove. □

Note that:

$$\text{Lipschitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

## 14.6 Optimization Methods

### 14.6.1 Newton-Raphson Method

**Theorem 14.6.1.1. Newton-Raphson Method**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a twice-differentiable,  $f(a) < 0 < f(b)$ . Suppose that  $f$  satisfies: for all  $x \in [a, b]$ ,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is,  $f$  is strictly increasing convex function, and Lipschitz Continuous.

Further, there uniquely exists  $x^* \in (a, b)$  such that  $f(x^*) = 0$ .

Let  $x_1 \in (x^*, b)$  fixed. Define a sequence  $\{x_n\}$  inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then,  $\{x_n\}$  satisfies the following three conditions:

1.  $\{x_n\}$  is decreasing sequence.
2.  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$ .
3. For any  $n \in \mathbb{N}$ ,  $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$ .

Condition 3 means that for a suitable initial value  $x_1$ , we can establish an upper bound for the error.

*Proof.* This proof consists by three steps.

Since  $f''$  is non-negative, and  $f'$  is positive,  $f$  is strictly increasing convex function.

And Fundamental Theorem of Calculus gives: for any  $x \in (a, b)$ ,

$$f'(x) \stackrel{\text{FTC}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x Mdt + f'(a) = M(x-a) + f'(a) \leq M(b-a) + f'(a)$$

Thus,  $f'$  is bounded on  $(a, b)$ , thus  $f$  is Lipschitz Continuous.

**Step 1.**  $f$  has a unique root  $x^*$ .

The existence of root given directly by Intermediate-Value theorem.

Suppose that  $x^*, x' \in (a, b)$  are distinct root of  $f$ . i.e.,  $f(x^*) = f(x') = 0$ . Then, by Mean-value theorem, there is  $c \in (a, b)$  between  $x^*$  and  $x'$  such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is,  $f'(c) = 0$ . This is contradiction with  $f'$  is positive.

**Step 2.**  $\{x_n\}$  decrease.

**Proof by induction:**

For  $n = 1$ ,  $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$ , thus  $x_2 < x_1$ . And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since  $f(x_2) > 0 = f(x^*)$ , the Mean-Value Theorem implies that  $x_2 > x^*$ .

To use induction, suppose that for some  $n \geq 1$ ,  $x^* < x_{n+1} < x_n$ . Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus  $x_{n+2} < x_{n+1}$  and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that  $x_{n+2} > x^*$ . Therefore, induction completes.

Now,  $x_n \rightarrow x^*$  as  $n \rightarrow \infty$  for some  $x' \in [x^*, x_1]$  since  $\{x_n\}$  is Bounded below and Decreasing.

Still it remains that to show  $x' = x^*$ . By Continuity,

$$\begin{aligned} f'(x_n)(x_{n+1} - x_n) + f(x_n) &= 0 \\ \implies \lim_{n \rightarrow \infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] &= f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of  $f$  is unique, thus  $x' = x^*$ .

**Step 3. Establishing the error bound.**

The Taylor's Theorem implies that

$$\begin{aligned} f(x^*) &= f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n) \\ \implies x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2 \end{aligned}$$

Consequently,

$$\begin{aligned} 0 \leq x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left( \frac{f''(t_{n-1})}{2f'(x_{n-1})} \right)^2 (x_{n-1} - x^*)^4 = \dots \\ &= \prod_{i=1}^n \left[ \frac{f''(t_i)}{2f'(x_i)} \right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[ \frac{M}{2\delta} \right]^{2^{n+1}-1} [x_1 - x^*]^{2^n} \end{aligned}$$

□

## 14.6.2 Gradient Descent

**Theorem 14.6.2.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function that satisfies the following conditions:

1.  $f$  is Convex function.
2.  $f'$  is Lipschitz Continuous with Lipschitz constant of  $f$ ,  $L > 0$ . In this,  $f$  is called  $L$ -Smooth.
3.  $f$  has at least one local minimizer  $x^*$ .

Then,  $x^*$  is a Global minimizer of  $\mathbb{R}$ , and there exists a unique closed interval  $M$  containing  $x^*$  such that

$$\forall x \in M, t \notin M, f(x) = f(x^*) < f(t)$$

And, given initial point  $x_0 \in \mathbb{R}$  and  $0 < \gamma \leq \frac{1}{L}$ , define a sequence  $\{x_n\}$  inductively as follows:

$$x_{n+1} = x_t - \gamma \cdot f'(x_n)$$

Then, for any  $N \in \mathbb{N}$ ,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

**Proof.** Let  $x^* \in \mathbb{R}$  be a local minimizer. That is, there exists a  $\delta > 0$  such that  $\forall t \in (x^* - \delta, x^* + \delta)$ ,  $f(x^*) \leq f(t)$ . Then,

$$0 \leq \lim_{t \rightarrow x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \rightarrow x^-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus,  $f'(x^*) = 0$ . And, by convexity,  $f'$  is monotonically increasing. Now, The Fundamental Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, f(x) = \int_{x^*}^x f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore,  $x^*$  is a Global minimizer of  $f$ .

Now, establish the closed interval  $M$ . Since  $f'$  is Lipschitz Continuous, thus  $f'$  is Continuous.

Let  $D \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$ . (Note that:  $x^* \in D$ , thus  $D$  is not emptyset.)

$D$  is closed because: Let  $\{x_n\}$  be a convergent sequence in  $D$ . That is, for all  $n \in \mathbb{N}$ ,  $f(x_n) = 0$ . Then, by continuity,

$$f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = 0$$

The limit of  $\{x_n\}$  is contained in  $D$ , thus  $D$  is closed.

And,  $D$  is interval: i.e, for any  $x \in (\inf D, \sup D)$ ,  $x \in D$  because:

Suppose that there exists  $x \in (\inf D, \sup D)$  such that  $x \notin D$ . That is,  $f'(x) \neq 0$ . This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let  $x, y \in \mathbb{R}$  be given.

The Fundamental Theorem of Calculus and  $L$ -Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\stackrel{?}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any  $\lambda > 0$ , Put  $y = x - \lambda f'(x)$ . Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda \left(\frac{L\lambda}{2} - 1\right) |f'(x)|^2$$

Put  $\lambda = \frac{1}{L}$ , then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

**Meanwhile, the convexity gives: for any  $x, y \in \mathbb{R}$ ,**

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$$

**since derivative of convex function increase monotonically.** Put  $z = y - \frac{1}{L}(f'(y) - f'(x))$ . Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\leq f'(x)(x - z) + f'(y)(z - y) + \frac{L}{2}|z - y|^2 \\ &= f'(x) \left( x - y + \frac{1}{L}(f'(y) - f'(x)) \right) - f'(y) \left( \frac{1}{L}(f'(y) - f'(x)) \right) + \frac{L}{2} \left| \frac{1}{L}(f'(y) - f'(x)) \right|^2 \\ &= f'(x)(x - y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x - y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \leq f'(x)(x - y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \leq f'(y)(y - x) - (f(y) - f(z)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \leq (f'(y) - f'(x))(y - x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 &= |x_n - \gamma \cdot f'(x_n) - x^*|^2 = |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\ &\leq |x_n - x^*|^2 - 2\gamma \frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left( \gamma^2 - \frac{2\gamma}{L} \right) |f'(x_n)|^2 \leq |x_n - x^*|^2 \end{aligned}$$

Thus,  $|x_n - x^*|$  decrease as  $n \rightarrow \infty$ . That is,  $|x_n - x^*| \leq |x_0 - x^*|$  for all  $n \in \mathbb{N}$ .

Consider  $x_{n+1}$  and  $x_n$ . First, we obtain

$$\begin{aligned} f(x_{n+1}) &\leq f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\ &= f(x_n) - \left( \gamma - \frac{L}{2}\gamma^2 \right) |f'(x_n)|^2 \end{aligned}$$

Subtracting  $f(x^*)$  above, then

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left( \gamma - \frac{L}{2}\gamma^2 \right) |f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \leq f'(x_n)(x_n - x^*) \leq |f'(x_n)||x_n - x^*| \leq |f'(x_n)||x_0 - x^*|$$

Combining above two inequalities,

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by  $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$ ,

$$\begin{aligned} \frac{1}{f(x_n) - f(x^*)} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ \implies \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \sum_{n=0}^{N-1} \left[ \left( \gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] &\leq \sum_{n=0}^{N-1} \left[ \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

**Consequently,**

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[ \left( \gamma - \frac{L}{2} \gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

**Organizing the formula, as result:**

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

## 14.7 Integral

### 14.7.1 Inequality of Riemann-Stieltjes Integral

Let  $p, q \geq 1$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ , and functions lying on  $[a, b]$ .

**Lemma 14.7.1.1.** Let  $f, g \in \mathcal{R}(\alpha)$  with  $f, g \geq 0$ , and  $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$ . Then,  $\int_a^b f(x)g(x) d\alpha \leq 1$ .

*Proof.* For any  $x \in [a, b]$ , the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x) d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p} \int_a^b [f(x)]^p d\alpha + \frac{1}{q} \int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

**Definition 14.7.1.1.** Let  $f \in \mathcal{R}(\alpha)$ . Define a Norm of  $f$ :

$$\|f\|_p \stackrel{\text{def}}{=} \left( \int_a^b |f(x)|^p d\alpha \right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions,  $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \rightarrow \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$ .

### Lemma 14.7.1.2. Hölder's Inequality

Let  $f, g \in \mathcal{F}$ . Then,

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

*Proof.* Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[ \frac{|f(x)|}{\|f\|_p} \right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \int_a^b \left[ \frac{|g(x)|}{\|g\|_q} \right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \int_a^b |f(x)||g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[ \int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[ \int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

□

**Theorem 14.7.1.1. Minkowski inequality**

Let  $f, g \in \mathcal{F}$ . Then, for any  $p \geq 1$ ,  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

*Proof.*

$$\begin{aligned}
 \|f + g\|_p^p &= \int_a^b |f + g|^p d\alpha = \int_a^b |f + g| |f + g|^{p-1} d\alpha \\
 &\leq \int_a^b [|f| + |g|] |f + g|^{p-1} d\alpha \\
 &= \int_a^b |f| |f + g|^{p-1} d\alpha + \int_a^b |g| |f + g|^{p-1} d\alpha \\
 &\stackrel{\text{Hölder}}{\leq} \left[ \int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} \left[ \int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} + \left[ \int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \left[ \int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} \\
 &= \left[ \int_a^b |f + g|^p d\alpha \right]^{\frac{p-1}{p}} \left( \left[ \int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} + \left[ \int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \right) = \|f + g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
 \end{aligned}$$

Now,

$$\|f + g\|_p^p \cdot \|f + g\|_p^{1-p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

□

## **Chapter 15**

### **Measure**

# Chapter 16

## Complex Analysis

### 16.1 Series

**Theorem 16.1.0.1. Laurent's theorem**

Suppose that  $f$  is analytic on annular domain  $D = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$ , and  $C$  is simple closed contour around  $z_0$  and lying in that domain  $D$ . Then each point in  $D$ ,  $f(z)$  can express that:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right) + \frac{1}{2\pi i} \cdot \sum_{n=1}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{-n+1}} ds \cdot \frac{1}{(z - z_0)^n} \right) \\ &= \frac{1}{2\pi i} \cdot \sum_{n=-\infty}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right), \quad (R_1 < |z - z_0| < R_2) \end{aligned}$$

In particular, If  $f(s)$  is analytic inside and on circle  $C$ ,  $\forall n \in \mathbb{N}$ ,  $f(s) \cdot (s - z_0)^{n-1}$  is analytic too. then by Cauchy-Goursat Thm, term (2) is zero, thus we can write that:

$$f(z) = \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left( \int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right)$$

and, since  $f$  is analytic on  $C$ , applies Cauchy integral theorem:

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n$$

This is what we already know as the Taylor Series form. Therefore, we can say Laurent's theorem is generalization form of Taylor Theorem.

*Proof.*

In case of  $z_0 = 0$ .

First, since  $C$  is lying in annular  $R_1 < |z| < R_2$ ,

can construct annular  $A : r_1 < |z| < r_2$  such that  $A$  contains circle  $C$ .

Let write  $C_1 : |z| = r_1$ ,  $C_2 : |z| = r_2$ , each circles are positively oriented.

Now, construct circle  $\gamma$  such that positively oriented and lying in annular  $A : r_1 < |z| < r_2$ .

Then by multiply connected theorem, we get that:

$$\int_{C_2} \frac{f(s)}{s - z} ds = \int_{\gamma} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{s - z} ds$$

Inside and on  $\gamma$ ,  $f$  is analytic, thus we can apply Cauchy integral theorem:

$$\begin{aligned} \int_{\gamma} \frac{f(s)}{s - z} ds &= 2\pi i \cdot f(z) = \int_{C_2} \frac{f(s)}{s - z} ds - \int_{C_1} \frac{f(s)}{s - z} ds = \int_{C_2} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{z - s} ds \\ \implies f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z - s} ds \end{aligned}$$

And we already know in proof of Taylor theorem,

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}$$

and also

$$\begin{aligned}\frac{1}{z-s} &= \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \frac{s^N}{(z-s)z^N} \\ &= \sum_{n=1}^N \frac{s^{n-1}}{z^n} + \frac{s^N}{(z-s)z^N} \\ &= \sum_{n=1}^N \frac{1}{s^{-n+1} \cdot z^n} + \frac{s^N}{(z-s)z^N}\end{aligned}$$

Now we can write that:

$$\begin{aligned}f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z-s} ds \\ &= \frac{1}{2\pi i} \int_{C_2} \left( \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} f(s) + \frac{z^N}{(s-z)s^N} f(s) \right) ds + \frac{1}{2\pi i} \int_{C_1} \left( \sum_{n=1}^N \frac{f(s)}{s^{-n+1} \cdot z^n} + \frac{s^N}{(z-s)z^N} f(s) \right) ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i} \sum_{n=1}^N \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \cdot \frac{1}{z^n} + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i} \sum_{n=1}^N \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \cdot z^{-n} + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds\end{aligned}$$

And by construction of  $C$ ,  $C_1$ ,  $C_2$ ,  $f$  is analytic between  $C$  and  $C_1$ , also  $C$  and  $C_2$ .

Thus applies multiply connected:

$$\begin{aligned}&= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_C \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i} \sum_{n=1}^N \int_C \frac{f(s)}{s^{-n+1}} ds \cdot z^{-n} + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \\ &= \frac{1}{2\pi i} \sum_{n=-N}^{N-1} \int_C \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds\end{aligned}$$

Now, enough to show

$$\begin{aligned}\frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds &\rightarrow 0 \text{ as } N \rightarrow \infty \\ \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds &\rightarrow 0 \text{ as } N \rightarrow \infty\end{aligned}$$

Let  $|z| = r$ . Then  $r_1 < r < r_2$ . And, Let  $M = \max \left\{ \max_{z \in C_1} f(z), \max_{z \in C_2} f(z) \right\}$ . And,

for  $s$  on  $C_2$ ,  $|s-z| \geq ||s|-|z|| = r_2 - r$ , for  $s$  on  $C_1$ ,  $|z-s| \geq ||z|-|s|| = r - r_1$ .

Finally, since ML inequality,

$$\begin{aligned}\left| \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds \right| &\leq \frac{|z^N|}{2\pi} \int_{C_2} \left| \frac{f(s)}{(s-z)s^N} \right| ds \leq \frac{r^N}{2\pi} \frac{M \cdot 2\pi r_2}{(r_2 - r)(r_2)^N} = \frac{Mr_2}{r_2 - r} \left( \frac{r}{r_2} \right)^N \\ \left| \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds \right| &\leq \frac{1}{2\pi \cdot r^N} \int_{C_1} \left| \frac{s^N f(s)}{z-s} \right| ds \leq \frac{1}{2\pi \cdot r^N} \frac{(r_1)^N \cdot M \cdot 2\pi r_1}{r - r_1} = \frac{Mr_1}{r - r_1} \left( \frac{r_1}{r} \right)^N\end{aligned}$$

Consequently, since  $\left( \frac{r}{r_2} \right) < 1$ ,  $\left( \frac{r_1}{r} \right) < 1$ , we get result.

In case of  $z_0 \neq 0$ .

Let  $f$  be analytic throughout annular  $R_1 < |z - z_0| < R_2$ .

Then  $g(z) = f(z + z_0)$  is analytic throughout  $R_1 < |(z + z_0) - z_0| < R_2$ .

Now let  $C : z = z(t)$  ( $a \leq t \leq b$ ) is closed simple contour, following by statement.

Then  $\forall t \in [a, b]$ ,  $R_1 < |z(t) - z_0| < R_2$  and

for  $\Gamma : z = z(t) - z_0$  ( $a \leq t \leq b$ ) is lying in  $R_1 < |z| < R_2$ . Now since In  $z_0 = 0$  case,

$$g(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot z^n \quad (R_1 < |z| < R_2)$$

This is equal that:

$$f(z + z_0) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot z^n \quad (R_1 < |z| < R_2)$$

Finally, change  $z$  to  $z - z_0$  then:

$$f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \quad (R_1 < |z - z_0| < R_2)$$

And

$$\int_{\Gamma} \frac{g(s)}{s^{n+1}} ds = \int_a^b \frac{f(z(t) - z_0 + z_0)}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_a^b \frac{f(z(t))}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Consequently we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \\ &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \cdot (z - z_0)^n \quad (R_1 < |z - z_0| < R_2) \end{aligned}$$

□

# Chapter 17

## Multivariable Analysis

### 17.1 Differentiation

**Definition 17.1.0.1.** Suppose that  $A \subseteq \mathbb{R}^m$  is a subset, and  $x \in A^\circ$ . Given a non-zero vector  $u \in \mathbb{R}^m$ , Define *directional derivative of  $f$  at  $x$  with respect to the vector  $u$* :

$$f'(x; u) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \frac{f(x + tu) - f(x)}{t}$$

**Definition 17.1.0.2.** Suppose that  $A \subseteq \mathbb{R}^m$  is a subset, and  $x \in A^\circ$ .

The function  $f$  is called *differentiable* at  $x$  if: There exists a  $B \in \mathcal{M}_{n,m}(\mathbb{R})$  such that

$$\frac{f(x + h) - f(x) - B \cdot h}{\|h\|} \rightarrow 0 \in \mathbb{R}^n \text{ as } h \rightarrow 0 \in \mathbb{R}^m$$

More rigorously,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall h \in \mathbb{R}^m, 0 < \|h\| < \delta \implies \frac{\|f(x + h) - f(x) - B \cdot h\|}{\|h\|} < \varepsilon$$

If exists, this  $B$  is unique; The matrix  $B$  is denoted  $Df(x)$ , which is called *derivative of  $f$  at  $x$* .

$$\frac{f\left(\underset{\in \mathbb{R}^m}{x} + \underset{\in \mathbb{R}^m}{h}\right) - f\left(\underset{\in \mathbb{R}^m}{x}\right) - \underset{\in \mathcal{M}_{n,m}}{B} \cdot \underset{\in \mathbb{R}^m}{h}}{\underset{\in \mathbb{R}^n}{\|h\|}}$$

**Theorem 17.1.0.1.** Suppose that  $A \subseteq \mathbb{R}^m$  is a subset.

If  $f : A \rightarrow \mathbb{R}^n$  is differentiable at  $x \in A^\circ$ , then for any  $u \in \mathbb{R}^m$ ,  $f'(x; u)$  exists. Moreover,

$$f'(x; u) = Df(x) \cdot u$$

*Proof.* By assumption, the Derivative  $Df(x) \in \mathcal{M}_{n,m}(\mathbb{R})$  exists such that

$$\frac{f(x + h) - f(x) - B \cdot h}{\|h\|} \rightarrow 0 \text{ as } h \rightarrow 0$$

Let non-zero vector  $u \in \mathbb{R}^m$  be given. Choose  $\varepsilon > 0$  arbitrarily. Since assumption, there exists a  $\delta > 0$  such that

$$0 < \|h\| < \delta \implies \frac{\|f(x + h) - f(x) - B \cdot h\|}{\|h\|} < \frac{\varepsilon}{\|u\|}$$

**Put**  $\delta_0 > 0$  **such that**  $0 < t < \delta_0 \implies \|t\mathbf{u}\| < \delta$  (**Precisely, put**  $\delta_0 = \frac{\delta}{\|\mathbf{u}\|}$ ). **Now,**

$$\begin{aligned} 0 < t < \delta_0 &\implies \frac{\|f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) - B \cdot t\mathbf{u}\|}{\|t\mathbf{u}\|} = \frac{\|f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) - B \cdot t\mathbf{u}\|}{|t|\|\mathbf{u}\|} < \frac{\varepsilon}{\|\mathbf{u}\|} \\ &\implies \frac{\|f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) - B \cdot t\mathbf{u}\|}{t} = \left\| \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t} - B \cdot \mathbf{u} \right\| < \varepsilon \end{aligned}$$

**Thus,**  $B \cdot \mathbf{u} = Df(\mathbf{x}) = f(\mathbf{x}; \mathbf{u})$ .

□

## **Chapter 18**

# **Differential Geometry**

## **Chapter 19**

# **Differential Equation**

### **19.1 System of Differential Equation**

#### **19.1.1 Definitions**

#### **19.1.2 Basic Properties**

### **19.2 Lorenz system**

## **Chapter 20**

### **Differential Form**

# Chapter 21

## Spaces

### 21.1 $\mathbb{R}^n$

#### 21.1.1 Inner Product in $\mathbb{R}$

#### 21.1.2 $p$ -norm in $\mathbb{R}^n$

**Definition 21.1.2.1.** Let  $\mathbb{R}^n$  be given. Define  $p$ -norm on  $\mathbb{R}^n$  as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$$

where  $p \in [1, \infty]$ . In particular,  $p$ -norm is a Metric, being Minkowski inequality.

**Lemma 21.1.2.1. Young's inequality**

Let  $u, v > 0$ , and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ .  
Then,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

*Proof.* Since  $f(x) = \log x$  is concave, we obtain

$$\forall \lambda \in [0, 1], \quad \lambda f(x)(1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

thus,

$$\log\left(\frac{1}{p}u^p + \frac{1}{q}v^q\right) \geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) = \log(uv)$$

Since  $\exp(x)$  increasing, we get

$$\exp\left(\log\left(\frac{1}{p}u^p + \frac{1}{q}v^q\right)\right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

□

**Lemma 21.1.2.2. Hölder's inequality**

Let  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_n)$  be give, and  $p, q \in [1, \infty]$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

*Proof.* Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Then, since young's inequality, for each  $i \in \{1, 2, \dots, n\}$ ,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all  $i = 1, 2, \dots, n$ :

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^n |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n |x_i y_i| \leq \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left( \sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

□

**Theorem 21.1.2.1. Minkowski inequality**

Given complex-valued sequences  $\{x_i\}_{i=1}^n$  and  $\{y_i\}_{i=1}^n$ ,

$$\left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[ \sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} + \left[ \sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}}$$

*Proof.* Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\ &\stackrel{\text{Hölder}}{\leq} \left[ \sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[ \sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}} \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \left[ \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \end{aligned}$$

Now, Divide each side as  $[\sum_{i=1}^n |x_i + y_i|^p]^{\frac{p-1}{p}}$ , then we obtain

$$\left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{1-\frac{p-1}{p}} = \left[ \sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[ \left( \sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right]$$

□

**Theorem 21.1.2.2.** Let  $d_{p_1}, d_{p_2}$  are  $p$ -norm on  $\mathbb{R}^n$  with  $1 \leq p_1 < p_2 \leq \infty$ . Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, d_{p_2}(x, y) \leq d_{p_1}(x, y) \leq C d_{p_2}(x, y)$$

In particular,  $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$ .

**Proof.** Let  $p_1 < p_2$ .

For show that first-inequality,

$$1 = \sum_{i=1}^n \left[ \frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_2} \leq \sum_{i=1}^n \left[ \frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{p_1}{p_2}}} = \left[ \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[ \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_1} \iff 1 \leq \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \iff \left[ \sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}} \leq \left[ \sum_{i=1}^n |x_i - y_i|^{p_1} \right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned} (d_{p_1}(x, y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\stackrel{\text{Hölder}}{\leq} \left[ \sum_{i=1}^n \left( |x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}} \right)^{\frac{p_1}{p_2}} \right]^{\frac{p_1}{p_2}} \cdot \left[ \sum_{i=1}^n 1^{\frac{p_2}{p_2-p_1}} \right]^{1-\frac{p_1}{p_2}} = \left[ \sum_{i=1}^n (|x_i - y_i|^{p_2})^{\frac{p_1}{p_2}} \right]^{\frac{p_1}{p_2}} \cdot n^{1-\frac{p_1}{p_2}} \end{aligned}$$

Taking the  $\frac{1}{p_1}$ -th power of both sides, then

$$d_{p_1}(x, y) \leq \left[ \sum_{i=1}^n (|x_i - y_i|^{p_2})^{\frac{1}{p_2}} \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

**Corollary 21.1.2.1.** Let  $\mathbb{R}^n$  be given as a set, and  $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  are  $p$ -norm on  $\mathbb{R}^n$ . Then,

$$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every  $p \geq 1$ , the metric space  $(\mathbb{R}^n, d_p)$  induces the same topology as the product topology on  $\mathbb{R}^n$ . In particular,  $\mathbb{R}^n$  with the product topology coincides with  $\mathbb{R}^n$  endowed with any  $p$ -norm.

### 21.1.3 Open and Closed set in $\mathbb{R}^n$

**Definition 21.1.3.1.** For  $p \in [1, \infty]$ , define  $p$ -Ball in  $\mathbb{R}^n$  as:

$$B_p(x, r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all  $p$ -norms are equivalent, for any  $p \in [1, \infty]$ , the collection

$$\beta_p \stackrel{\text{def}}{=} \{B_p(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\}$$

is Countable basis of  $\mathbb{R}^n$ . Immediately, we obtain:

**Lemma 21.1.3.1.** Every open set in  $\mathbb{R}^n$  is a countable union of  $p$ -Balls.

We call 2-Ball the Ball, and  $\infty$ -Ball the Cube.

**Theorem 21.1.3.1.** Let  $U \subseteq \mathbb{R}^n$  be an open set. Then,  $U$  is a countable union of closed cubes with disjoint interiors.

*Proof.* Let  $U \subseteq \mathbb{R}^n$  be an open set, and define the collection of Dyadic Cubes on  $\mathbb{R}^n$  as: for each  $k \in \mathbb{N}$ ,

$$Q_k \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^n \left[ \frac{q_i}{2^k}, \frac{q_i + 1}{2^k} \right] \subset \mathbb{R}^n \mid q_i \in \mathbb{Z} \right\}$$

Each element of  $Q_k$  is product of closed intervals, and its interiors are disjoint. For each  $k \in \mathbb{N}$ , construct:

$$Q_k^* \stackrel{\text{def}}{=} \{Q \in Q_k \mid Q \subseteq U\}$$

Then, the union  $Q^* = \bigcup_{k \in \mathbb{N}} Q_k^*$  is a countable union of closed cubes, and  $Q^* = U$ :  $Q^* \subseteq U$  is clear, and let  $x \in U$ .

Since property of metric space, there exists  $\delta > 0$  such that  $x \in B_2(x, \delta) \subseteq U$ . Put  $k \in \mathbb{N}$  such that  $\frac{1}{2^k} < \frac{\delta}{\sqrt{n}}$ .

Then,  $x \in C \subset B_2(x, \delta) \subseteq U$  for some  $C \in Q_k$ , because  $\text{diam } C = \sqrt{n}2^{-k}$ . Since  $C \subset U$ ,  $C \in Q_k^* \subset Q^*$ . i.e.,  $U \subseteq Q^*$ . For disjointness of interiors, we will use the fact:

For any  $Q_1, Q_2 \in Q^*$ , either their interiors are disjoint, or one is contained in the other.

(Conti.)

□

## 21.2 Manifold

### 21.3 Topological Vector Space

## 21.4 Hilbert Space

**Definition 21.4.0.1.** Complete Inner product Vector Space is called *Hilbert Space*.

### 21.4.1 Hilbert Space in $\mathbb{R}^\omega$

**Definition 21.4.1.1.** Define  $\mathbb{R}^\omega \stackrel{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$  as the countable product of Euclidean space  $\mathbb{R}$  with product topology.  
 And define  $\mathbb{H} \stackrel{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$ , Metric on  $\mathbb{H}$  as  $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ .  
 The Metric Space  $(\mathbb{H}, \mu)$  is called *Hilbert Space* or  $l_2$  Space.

Define the operations elementwise; then  $(\mathbb{H}, +, \times)$  is a Vector Space over  $\mathbb{R}$ . Moreover,  $\mathbb{H}$  is Complete Metric Space and Inner product Vector Space.

**Lemma 21.4.1.1.**  $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$  is Metric function induced by the inner product.

**Proof.** We know that  $\mathbb{R}^\omega$  is Vector Space. Moreover,  $\mathbb{H} \subset \mathbb{R}^\omega$  is Subspace. Using subspace criteria:

$S \subset V$  is Subspace of Vector Space  $V$  if and only if  $0 \in S$  and For any  $x, y \in S$  and  $a \in F$ ,  $ax + y \in S$ .

Clearly,  $\{0\} \in \mathbb{H}$ . Let  $a \in \mathbb{R}$  and  $\{x_n\}, \{y_n\} \in \mathbb{H}$  be given. Then,  $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$  because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} [a^2 x_i^2 + 2ax_i y_i + y_i^2] \stackrel{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The  $(*)$  given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i| |y_i| \leq \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \leq \sum_{i=1}^{\infty} (x_i^2 + y_i^2) = \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\infty} y_i^2 < \infty \quad (*)$$

Thus  $\mathbb{H}$  is Vector Space over  $\mathbb{R}$ . Now, define inner product on  $\mathbb{H}$  as:

$$\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since  $(*)$ . And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a \langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus,  $(\mathbb{H}, \langle \cdot, \cdot \rangle)$  is inner product space.

Using inner product, define the Norm on  $\mathbb{H}$  as:

$$\|\cdot\| : \mathbb{H} \rightarrow \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define Metric on  $\mathbb{H}$  as:

$$\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

**Theorem 21.4.1.1. Hilbert Space is Separable.**

**Proof.** For each  $n \in \mathbb{N}$ , define  $D_n \stackrel{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q}, p_{n+1} = p_{n+2} = \dots = 0\}$  and  $D \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} D_n$ .

Then,  $D$  is countable set. We will show that  $\overline{D} = \mathbb{H}$ .

Let  $\epsilon > 0$  and  $\{x_n\} \in \mathbb{H}$  be given. Since convergence, there exists  $N \in \mathbb{N}$  such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^N x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each  $i = 1, 2, \dots, N$ ,  $p_i \in \mathbb{Q}$   $|x_i - p_i| < \frac{\epsilon}{\sqrt{2N}}$  and  $p_i = 0$  for  $i \geq N + 1$ .

Then,  $\{p_n\} \in D_n \subset D$  and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} (x_i - p_i)^2} = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

**Corollary 21.4.1.1. Hilbert Space is Second-Countable.**

**Theorem 21.4.1.2. Hilbert Space is Complete.**

**Proof.** Let  $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$  be a Cauchy sequence in  $\mathbb{H}$ . For any fixed  $n, m \in \mathbb{N}$  and for each  $j \in \mathbb{N}$ ,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2}$$

That is, for each  $j \in \mathbb{N}$ ,  $\{x_{n,j}\}$  is Cauchy sequence in  $\mathbb{R}$ . Since  $\mathbb{R}$  is Complete, put  $y_j \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} x_{n,j}$ , each  $j \in \mathbb{N}$ .

Let  $\epsilon > 0$  be given. Then, there exists  $N \in \mathbb{N}$  such that  $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \frac{\epsilon}{2}$ .

Meanwhile, for each  $k \in \mathbb{N}$ ,

$$\sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus,  $n, m \geq N \implies \sum_{i=1}^k (x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$ , for each  $k \in \mathbb{N}$ .

Taking limit to  $m$ , then  $n \geq N \implies \lim_{m \rightarrow \infty} \left( \sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \right) = \sum_{i=1}^k \left( x_{n,i} - \lim_{m \rightarrow \infty} x_{m,i} \right)^2 = \sum_{i=1}^k (x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$ .

And, for all  $k \in \mathbb{N}$ ,

$$\sum_{i=1}^k y_i^2 = \sum_{i=1}^k (2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus  $\{y_i\} \in \mathbb{H}$ . As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - y_i)^2} = \sqrt{\lim_{k \rightarrow \infty} \sum_{i=1}^k (x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

**Theorem 21.4.1.3.**  $\mathbb{H} \subset \mathbb{R}^\omega$  with subspace topology is Metrizable.

*Proof.* We will use two Lemmas:

**Lemma 21.4.1.2.** Countable Product of Metric Space is Metrizable.

*Proof.* Let  $(X_i, d_i)$  be a metric space, for each  $i \in \mathbb{N}$ .

If  $d : X \times X \rightarrow \mathbb{R}$  is a Metric, then  $\frac{d}{1+d}$  is also Metric, because

$$\frac{d(x, z)}{1+d(x, z)} \stackrel{\frac{x}{1+x} \text{ increasing}}{\uparrow} \frac{d(x, y) + d(y, z)}{1+d(x, y) + d(y, z)} \stackrel{d \geq 0}{\uparrow} \frac{d(x, y)}{1+d(x, y)} + \frac{d(y, z)}{1+d(y, z)} \quad (*)$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \rightarrow \mathbb{R} : ((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} \right]$$

Then  $d_\Pi$  is a Metric because: the triangle inequality is satisfied since

$$\begin{aligned} d_\Pi((x_n)_{n=1}^\infty, (z_n)_{n=1}^\infty) &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1+d_i(x_i, z_i)} \right] \\ &\stackrel{(*)}{\leq} \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \left( \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1+d_i(y_i, z_i)} \right) \right] \\ &= \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} \right] + \sum_{i=1}^\infty \left[ \frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1+d_i(y_i, z_i)} \right] \\ &= d_\Pi((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) + d_\Pi((y_n)_{n=1}^\infty, (z_n)_{n=1}^\infty) \end{aligned}$$

Reflexivity and symmetry are clear.

And, it remains to show that the metric  $d_\Pi$  generates the given product topology.

□

**Lemma 21.4.1.3.** Metrizable is Hereditary.

Proof omitted.

Consequently, since  $\mathbb{H} \subset \mathbb{R}^\omega$  is a subspace of a metric space, it is metrizable.

□

**21.5 Banach Space**

**21.6  $L_p$  Space**

**21.7  $l_p$  Space**

## **Chapter 22**

# ***N*-Body Problem**

### **22.1 Introduction**

#### **22.1.1 Definition**

### **22.2 Basic Tools**

### **22.3 Two-Body Problem**

### **22.4 Three-Body Problem**

### **22.5 *N*-Body Problem**

# Chapter 23

## Artificial Intelligence

### 23.1 Feedforward Neural Network

#### 23.1.1 Definition

**Definition 23.1.1.1.** For the  $(i-1)$ -th layer with  $l_{i-1}$  nodes and the  $i$ -th layer with  $l_i$  nodes, the weight matrix sending the  $k$ -th node of layer  $(i-1)$  to the  $j$ -th node of layer  $i$  is defined as

$$W_i = (w_{j,k}^i) = \begin{pmatrix} w_{1,1}^i & w_{1,2}^i & \cdots & w_{1,l_{i-1}}^i \\ w_{2,1}^i & w_{2,2}^i & \cdots & w_{2,l_{i-1}}^i \\ \vdots & \vdots & \ddots & \vdots \\ w_{l_i,1}^i & w_{l_i,2}^i & \cdots & w_{l_i,l_{i-1}}^i \end{pmatrix} \in \mathcal{M}_{l_i, l_{i-1}}.$$

**Definition 23.1.1.2.** For the  $i$ -th layer with  $l_i$  nodes, the bias vector is defined as

$$B_i = \begin{pmatrix} b_1^i \\ b_2^i \\ \vdots \\ b_{l_i}^i \end{pmatrix} \in \mathcal{M}_{l_i, 1}.$$

**Definition 23.1.1.3.** The activation function (ReLU) is the map  $\mathcal{R} : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$\mathcal{R}(x) = \begin{cases} x, & x \geq 0, \\ 0, & x < 0, \end{cases} = \max(x, 0).$$

**Definition 23.1.1.4.** The matrix ReLU operator  $\mathcal{A} : \mathcal{M}_{n,m} \rightarrow \mathcal{M}_{n,m}$  is defined by

$$\mathcal{A}((x_{i,j})) = (\mathcal{R}(x_{i,j})).$$

**Definition 23.1.1.5.** The output function (Softmax)  $\mathcal{F} : \mathcal{M}_{n,1} \rightarrow \mathcal{M}_{n,1}$  is defined by

$$\mathcal{F}(Y) = \frac{1}{\sum_{i=1}^n e^{Y_{i,1}}} \begin{pmatrix} e^{Y_{1,1}} \\ e^{Y_{2,1}} \\ \vdots \\ e^{Y_{n,1}} \end{pmatrix}.$$

**Definition 23.1.1.6.** The mean squared error (MSE) loss function  $\mathcal{L} : (\mathcal{M}_{n,1}, \mathcal{M}_{n,1}) \rightarrow \mathbb{R}$  is defined by

$$\mathcal{L}(Y, T) = \frac{1}{n} \sum_{i=1}^n (Y_i - T_i)^2.$$

**Definition 23.1.1.7.** For a neural network with  $(n + 1)$  layers, the layers are computed recursively by

$$\begin{cases} L_0 = X \in \mathcal{M}_{l_0,1}, \\ L_1 = W_1 L_0 + B_1 \in \mathcal{M}_{l_1,1}, \\ L_i = W_i \mathcal{A}(L_{i-1}) + B_i \in \mathcal{M}_{l_i,1}, \quad 2 \leq i \leq n. \end{cases}$$

**Definition 23.1.1.8. Feedforward Neural Network  $\mathcal{N}$ :**

$$\mathcal{N} : \mathbb{F}^{l_0 + \sum_{i=1}^n (l_i l_{i-1}) + \sum_{i=1}^n l_i} \longrightarrow \mathbb{F}^{l_n},$$

$$\mathcal{N}(X, W_1, \dots, W_n, B_1, \dots, B_n) = \mathcal{F}(L_n) = Y.$$

### 23.1.2 Forward Propagation

### 23.1.3 Back Propagation

### 23.1.4 Gradient Descent

# Chapter 24

## Cake Theory

### 24.1 Definition

**Definition 24.1.0.1.**

#### 24.1.1 Subcake

**Definition 24.1.1.1.** Let  $C$  be a Cake. The non-empty subset  $S \subseteq C$  is called *Subcake* of  $C$  if:  $S$  satisfies

1.  $S$  is closed under the *Cutting*.
2. If  $\text{Topp}(C) \neq \emptyset$ , then for some  $s \in \text{Topp}(C)$ ,  $s$  is contained in  $S$ .
3.  $\text{Lay}(S) = \text{Lay}(C)$ .

### 24.2 Typical Cakes

#### 24.2.1 Cheeses Cake

#### 24.2.2 Chocolate Cake

#### 24.2.3 Strawberry Cake

### 24.3 Piece of Cake

**Definition 24.3.0.1.**

**Theorem 24.3.0.1.** Every Piece of Cake is a Subcake of some Cake.

#### 24.3.1 Subcake of Piece of Cake

[Athreya et al., 2019] [Croom, 2002] [Dummit and Foote, 2004]

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