

Math Note

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My goal is to rewrite all undergraduate mathematics in my own language.

And, for review, to draw diagrams illustrating the relationships between mathematical objects.

Patch Note:

- ~ 2025/9/28 - Drafted the initial framework of the paper, and Transcribed previous works.
- 2025/09/29 -
 - 1. Completed proof of Ring of Fractions.
 - 2. Transcribed Integral, Ratio, and Root Test.
 - 3. Transcribed Tube Lemma, Lindelöf and Countably Compact product Compact.
 - 4. Transcribed Coproduct with Continuous, open, closed map.
- 2025/09/30 -
 - 1. Proved Every open set in \mathbb{R}^n is countable union of closed cubes, disjoint of interiors remains.
 - 2. Transcribed Group action.
- 2025/10/01 -
 - 1. Transcribed One-point Compactification.
 - 2. Transcribed Definitions of subbasis, Borel set.
- 2025/10/02 -
 - 1. Proved Euclidean Domain
 - 2. Proving Existence of Nowhere-differentiable function.
- 2025/10/03 -
 - 1. Drafted definition and propositions of Quotient Space.
- 2025/10/04 -
 - 1. Studying Quotient Map.
- 2025/10/05 -
 - 1. Studied Basic Properties of the Quotient Map, and Drew quotient map diagram.
- 2025/10/06 -
 - 1. Drafted basic functions in a Metric space.
- 2025/10/07 -
 - 1. Proved basic properties of Completely regular space.
- 2025/10/08 -
 - 1. Proved Compact Hausdorff Space is Normal.
 - 2. Proved Equivalent Conditions of Completely Regular Space.
 - 3. Proving the Urysohn Metrization Theorem.
 - 4. Understanding relations and characteristic of Domains.
- 2025/10/10 -
 - 1. Transcribed basic statements of Polynomial Ring.
- 2025/10/11 -
 - 1. Proved Gauss's Lemma in Polynomial Ring, and Transcribed.
 - 2. Proving R U.F.D. iff $R[x]$ U.F.D.
- 2025/10/13 -
 - 1. Understanding U.F.D and irreducible.
 - 2. Proved R U.F.D. iff $R[x]$ U.F.D.
 - 3. Drew the Diagram of Domains.
- 2025/10/14 -
 - 1. Drew the Relations of Polynomial Ring Diagram.
 - 2. Study Quadratic Integer Ring.
 - 3. Study Nilpotent in Ring, and Nilradical Ring.
 - 4. Arranged Connected Space.
 - 5. Proved S^n is One-point Compactification of \mathbb{R}^n .
 - 6. Proving S^n is Connected via One-point Compactification.
- 2025/10/15 -
 - 1. Completed Irreducibility Criteria.
- 2025/10/16 -
 - 1. Observing Cyclotomic Polynomial.
- 2025/10/17 -
 - 1. Studying Quadratic Field and Quadratic Integer Ring.
- 2025/10/18 -
 - 1. Observed Nilradical Ideal.
- 2025/10/27 -
 - 1. Drew Diagram in Connectedness.
 - 2. Analyzed Topologist's Sine curve.

Goal:

1. Brouwer fixed-point theorem.
2. Abel-Ruffini theorem.
3. Stokes's Theorem.
4. Three-Body Problem has no Analytic general solution.

- 2025/11/6 -
 - 1. Completed Basic thoery of fields - Centre around Kronecker's Theorem.
- 2025/11/7 -
 - 1. Studying Algebraic Extension.
- 2025/11/14 -
 - 1. Proving $B^n/S^{n-1} \cong S^n$.
 - 2. Revised proof of basic field theory.
- 2025/11/18 -
 - 1. Revising the Basic Notation and Proving the Basic Properties of Vector Spaces.
 - 2. Proved Extension field is the Vector Space.
- 2025/11/23 -
 - 1. Wrote basic Definition of multivariable Derivative for PINN.
 - 2. Studying Typical Quotient Space: Cylinder, Torus, Möbius band, Klein bottle, Projective Space.
- 2025/11/24 -
 - 1. Studied Typical Quotient Space: Cylinder, Torus, Möbius band.
 - 2. Described Definition of Topological Manifold.
- 2025/11/27 -
 - 1. Studying Basic Theory of Partial Differential Equation.

Chapter 1

Set Theory

1.1 Map

Definition 1.1.0.1. Let X, Y are sets. Define a **function** X to Y is a relation

$$f \subset X \times Y$$

such that

1. For any $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$.
2. If $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Denote f as:

$$f : X \rightarrow Y : x \mapsto f(x)$$

Define **Image** of f by $A \subset X$:

$$f[A] \stackrel{\text{def}}{=} \{f(a) \mid a \in A\} \subset Y$$

And, **Preimage** of f by $B \subset Y$:

$$f^{-1}[B] \stackrel{\text{def}}{=} \{x \in X \mid f(x) \in B\} \subset X$$

$f : X \rightarrow Y$ is **Injective** if: $f(x_1) = f(x_2) \implies x_1 = x_2$.
 $f : X \rightarrow Y$ is **Surjective** if: $\forall y \in Y, \exists x \in X$ s.t. $f(x) = y$.
If f is injective and surjective, called **bijective**.
If f is bijective, then define **inverse** of f as:

$$f^{-1} : Y \rightarrow X : y \mapsto x$$

where $x \in X$ is the unique elements of X such that $f(x) = y$.

Theorem 1.1.0.1. Let $f : X \rightarrow Y$ be a function. Then,

1. There exists $g : Y \rightarrow X$ such that $g \circ f : X \rightarrow X$ be an identity function if and only if f is injective.
2. There exists $h : Y \rightarrow X$ such that $f \circ h : Y \rightarrow Y$ be an identity function if and only if f is surjective.

Proof.

1. \implies)

Assume that $f(x_1) = f(x_2)$. Then, existence of left inverse, $g(f(x_1)) = g(f(x_2)) \implies x_1 = x_2$. Thus f injective.

1. \iff)

Since f is injection, for any $y \in f[X]$, there exists a unique element $x_y \in X$ such that $f(x) = y$. Now, define

$$g : Y \rightarrow X : y \mapsto \begin{cases} x_y & y \in f[X] \\ \text{any element in } X & y \notin f[X] \end{cases}$$

Then, for any $x \in X$, $g(f(x)) = g(y) = x$.

2. \implies)

Let $y \in Y$ be given. Since existence of right inverse, $f(h(y)) = y$ where $h(y) \in X$. Thus, f is surjective.

2. \Leftarrow)

For any $y \in Y$, there exists a $x_y \in X$ such that $f(x_y) = y$. Now, define

$$h : Y \rightarrow X : y \mapsto x_y$$

Then, for any $y \in Y$, $f \circ h(y) = f(x_y) = y$. Thus, $f \circ h$ is identity. \square

Corollary 1.1.0.1. Let $f : X \rightarrow Y$ be a function, $\text{id}_X : X \rightarrow X : x \mapsto x$, and $\text{id}_Y : Y \rightarrow Y : y \mapsto y$.

There exists a $f^{-1} : Y \rightarrow X$ such that $f^{-1} \circ f = \text{id}_X$ and $f \circ f^{-1} = \text{id}_Y$ if and only if f is bijection.

Proof. If f is bijection, then there exists left inverse g and right inverse h .

Enough To Show that: $g = h$. Since $g \circ f = \text{id}_X$ and $f \circ h = \text{id}_Y$,

$g \circ f \circ h = g \circ \text{id}_Y$, thus $h = g$. \square

Theorem 1.1.0.2. Let X, Y, Z are sets, $f : X \rightarrow Y$, $g : Y \rightarrow Z$ and $A \subset X, B \subset Y, C \subset Z$. Then followings are hold:

1. $g[f[A]] = (g \circ f)[A]$.
2. $f^{-1}[g^{-1}[C]] = (g \circ f)^{-1}[C]$.

Proof.

1. It is clear by definition of image:

$$\begin{aligned} g[f[A]] &\stackrel{\text{def}}{=} g[\{f(a) \mid a \in A\}] = \{g(b) \mid b \in \{f(a) \mid a \in A\}\} \\ &= \{g(b) \mid b = f(a) \text{ for some } a \in A\} = \{g(f(a)) \mid \text{for some } a \in A\} = \{g(f(a)) \mid a \in A\} \end{aligned}$$

2. It is not clear,

$$f^{-1}[g^{-1}[C]] \stackrel{\text{def}}{=} f^{-1}[\{b \in Y \mid g(b) \in C\}] = \{a \in X \mid f(a) \in \{b \in Y \mid g(b) \in C\}\} = \{a \in X \mid g(f(a)) \in C\} = (g \circ f)^{-1}[C]$$

\square

Proposition 1.1.0.1. Let $f : X \rightarrow Y$ be a function, $A, B \subset X$ and $C, D \subset Y$.

1. If $A \subset B$, then $f[A] \subset f[B]$.
2. If $C \subset D$, then $f^{-1}[C] \subset f^{-1}[D]$

Proof.

$$\begin{aligned} y \in f[A] &\implies y = f(a) \text{ for some } a \in A \stackrel{A \subset B}{\implies} y = f(a) \text{ for some } a \in B \implies y \in f[B] \\ x \in f^{-1}[C] &\implies f(x) \in C \stackrel{C \subset D}{\implies} f(x) \in D \implies x \in f^{-1}[D] \end{aligned}$$

\square

Lemma 1.1.0.1. Let two set X, Y be given, and $A \subset X$, $B \subset Y$, $f : X \rightarrow Y$. Then followings are holds:

1. $f^{-1}[f[A]] \supseteq A$, and equality holds if f one-to-one.
2. $f[f^{-1}[B]] \subseteq B$, and equality holds if f onto.
3. $f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$
4. $f[X] \setminus f[A] \subseteq f[X \setminus A]$, and equality holds if f one-to-one.

Proof. Proof of 4.

$$\begin{aligned} y \in f[X] \setminus f[A] &\iff y \in f[X] \text{ and } y \notin f[A] \\ &\iff \exists x \in X \text{ s.t. } y = f(x) \text{ and } \forall x \in A, y \neq f(x) \\ &\stackrel{(*)}{\implies} \exists x \in X \setminus A \text{ s.t. } y = f(x) \\ &\iff y \in f[X \setminus A] \end{aligned}$$

If f is injection, then Left Direction of $(*)$ be true: $\exists !x \in X \setminus A$ s.t. $y = f(x)$. □

Chapter 2

Group Theory

2.1 Isomorphism Theorems

Definition 2.1.0.1. Let G be a group, and $A \subset G$. Define **Subgroup generated by A** :

$$\langle A \rangle \stackrel{\text{def}}{=} \bigcap_{A \subseteq H \leq G} H$$

Lemma 2.1.0.1. Let G be a group, and $A \subset G$.

$$\langle A \rangle = \{a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \mid n \in \mathbb{N}, \alpha_i \in \mathbb{Z}, a_i \in A\}$$

Theorem 2.1.0.1. The First Isomorphism Theorem

Let $\varphi : G \rightarrow H$ be a Group-Homomorphism. Then,

$$G / \ker \varphi \cong \varphi[G]$$

$$\begin{array}{ccc} G & \xrightarrow{\varphi \text{ homomorphism}} & \varphi[G] \\ \pi \text{ homomorphism surjection} \swarrow & \uparrow \phi \text{ isomorphism} & \downarrow \\ G / \ker \varphi & & \end{array}$$

Proof. Let $\pi : G \rightarrow G / \ker \varphi : x \mapsto x + \ker \varphi$. Then, the map $\phi : G / \ker \varphi \rightarrow \varphi[G] : a + \ker \varphi \mapsto \varphi(a)$ is isomorphism.

Well-defined and Injective:

$$a + \ker \varphi = b + \ker \varphi \iff a - b \in \ker \varphi \iff \varphi(a - b) = \varphi(a) - \varphi(b) = 0$$

Surjective is clear. □

Theorem 2.1.0.2. The Second Isomorphism Theorem

Let G be a Group, and $H \leq G$, $N \trianglelefteq G$. Then,

$$HN/N \cong H/(H \cap N)$$

Proof. HK be a subgroup of G , being

$$HN = \bigcup_{h \in H} hN \stackrel{N \trianglelefteq G}{=} \bigcup_{h \in H} Nh = NH$$

And, $N \leq HN$ is clear, thus $N \trianglelefteq HN$.

Meanwhile, $H \cap N$ be a Normal Subgroup of H : for any $h \in H, n \in H \cap N$, $hn h^{-1} \in N$ because N is normal, and

$hn^{-1} \in H$ since h, n contained in H . Thus, $hn^{-1} \in H \cap N$, this implies $H \cap N$ be a Normal of H . Now, Define a Map:

$$\varphi : H \rightarrow HN/N : h \mapsto hN$$

Clearly, this map is Well-Defined and Homomorphism. And,

$$\ker \varphi = \varphi^{-1}[1] = \{h \in H \mid hN = N\} = \{h \in H \mid h \in N\} = H \cap N$$

Thus, since The 1st Isomorphism Theorem,

$$HN/N \cong H/(H \cap N)$$

□

Theorem 2.1.0.3. The Third Isomorphism Theorem

Let G be a Group, and $H, K \trianglelefteq G$ with $H \leq K$. Then, $K/H \trianglelefteq G/H$ and

$$(G/H)/(K/H) \cong (G/K)$$

Proof. First, show that $K/H \trianglelefteq G/H$. Let $kH \in K/H$ and $gH \in G/H$. Then,

$$(gH)(kH)(gH)^{-1} = (gH)(kH)(g^{-1}H) = (gk)H \in K/H$$

since $gk \in K$, being $K \trianglelefteq G$. Now, Define a map:

$$\varphi : G/H \rightarrow G/K : gH \mapsto gK$$

1. Well-Defined.

$$g_1H = g_2H \iff g_1^{-1}g_2 \in H \stackrel{H \trianglelefteq K}{\implies} g_1^{-1}g_2 \in K \iff g_1K = g_2K$$

2. Homomorphism.

Clearly, for any $g_1H, g_2H \in G/H$,

$$\varphi(g_1Hg_2H) = \varphi(g_1g_2H) = g_1g_2K = g_1Kg_2K = \varphi(g_1H)\varphi(g_2H)$$

3. Surjection. Let $gK \in G/K$ be given. Then, clearly, $\varphi(gH) = gK$.

4. Kernel.

$$\ker \varphi = \{gH \in G/H \mid gK = 1\} = \{gH \in G/H \mid g \in K\} = K/H$$

Consequently, The 1st Isomorphism Theorem gives

$$(G/K) \cong (G/H)/\ker \varphi = (G/H)/(K/H)$$

□

Theorem 2.1.0.4. The Forth Isomorphism Theorem

Let G be a Group, and $N \trianglelefteq G$ be a Normal Subgroup. Then, there is a bijection between

$$D \stackrel{\text{def}}{=} \{H \leq G \mid N \leq H\}, \quad C \stackrel{\text{def}}{=} \{\overline{H} \leq G/N\}$$

Proof. Let $\pi : G \rightarrow G/N : g \mapsto gN$ be a natural projection. And, Define

$$\Phi : D \rightarrow C : H \mapsto \pi[H]$$

This function is well-defined: For any $H \in D$, let $aN, bN \in \pi[H]$. Then, $aN \cdot b^{-1}N = ab^{-1}N \in \pi[H]$, thus $\pi[H] \leq G/N$.

To show that one-to-one: Let $\Phi(A) = \Phi(B)$. Thus means, $\pi[A] = \pi[B]$. Let $a \in A$. Then, $\pi(a) \in \pi[A] = \pi[B]$, thus $\pi(a) = \pi(b)$ for some $b \in B$. That is, $aN = bN \iff a \in bN$. Meanwhile, $N \leq B$, thus $a \in bN \subset B$, $A \subset B$. Similarly, $B \subset A$, that is $A = B$.

To show that onto: Let $K \in C$. Then, $N \leq \pi^{-1}[K] \leq G$, thus clear.

□

2.2 Group Action

In this section, we follow that the notation of [Dummit and Foote, 2004, Abstract Algebra].

Definition 2.2.0.1. Let $(G, *)$ be a Group, and A be a non-empty set.

Define **Group Action** of a group G on a set A :

$$\alpha : G \times A \rightarrow A : (g, a) \mapsto g \cdot a$$

satisfies

1. For all $a \in A$, $1_G \cdot a = a$.
2. For all $g_1, g_2 \in G$, $a \in A$, $(g_1 * g_2) \cdot a = g_1 \cdot (g_2 \cdot a)$

In this, we said to be ' G acts on a set A '. Meanwhile, For each $g \in G$, Define a map

$$\sigma_g : A \rightarrow A : a \mapsto g \cdot a$$

Then, the *permutation representation*

$$\varphi : G \rightarrow S_A : g \mapsto \sigma_g$$

be a Homomorphism. Clearly, for each $g \in G$, $a \in A$,

$$\alpha(g, a) = g \cdot a = \sigma_g(a) = \varphi(g)(a)$$

Thus, there is one-to-one correspondence between group action and permutation representation.

For each $a \in A$, the **stabilizer** of a in G :

$$G_a \stackrel{\text{def}}{=} \{g \in G \mid g \cdot a = a\}$$

The **kernel of action**:

$$\ker \alpha \stackrel{\text{def}}{=} \{g \in G \mid g \cdot a = a, \forall a \in A\} = \bigcap_{a \in A} G_a$$

$G_a \leq G$ and $\ker \alpha \trianglelefteq G$.

If the kernel of action be trivial, the action is called **faithful**.

Definition 2.2.0.2. Let $\alpha : G \times A \rightarrow A$ be a Group Action. Define a relation on A :

$$a \sim b \iff a = g \cdot b \text{ for some } g \in G$$

Then, this relation be equivalence relation. Denote the equivalence relation, called **orbit**:

$$\mathcal{C}_a \stackrel{\text{def}}{=} \{b \mid b = g \cdot a \text{ for some } g \in G\} = \{g \cdot a \mid g \in G\}$$

And, the action is called **transitive** if there is only one orbit.

Lemma 2.2.0.1. For each $a \in A$,

$$|\mathcal{C}_a| = |G : G_a|$$

Proof. Since the map

$$\varphi_a : \mathcal{C}_a \rightarrow \{gG_a \mid g \in G\} : g \cdot a \mapsto gG_a$$

is well-defined, bijection. □

Theorem 2.2.0.1. Let G be a Group, let $H \leq G$ and $A = \{gH \mid g \in G\}$, G acts by left multiplication on the set A .

$$\pi_H : G \rightarrow S_A : g \mapsto \sigma_g$$

be a permutation representation afforded by this action. Then

1. G acts transitively on A .
2. $G_{1H} = \{g \in G \mid gH = H\} = H$.
3. The kernel of the action $\ker \pi_H = \bigcap_{x \in G} xHx^{-1}$, this is the largest normal subgroup of G contained in H .

Proof. Let $aH, bH \in A$ be given. Then, for $g = ba^{-1}$, $g \cdot aH = (ga)H = bH$. Thus, $A = \mathcal{C}_a$ for any $a \in G$. 2 is clear, being $gH = H \iff g \in H$.

Now,

$$\begin{aligned}\ker \pi_H &= \{g \in G \mid gxH = xH, \forall x \in G\} \\ &= \{g \in G \mid (x^{-1}gx)H = H, \forall x \in G\} \\ &= \{g \in G \mid x^{-1}gx \in H, \forall x \in G\} \\ &= \{g \in G \mid g \in xHx^{-1}, \forall x \in G\} = \bigcap_{x \in G} xHx^{-1}\end{aligned}$$

And the second assertion given by:

Let N is a normal subgroup of G contained in H , then for any $x \in G$, $N = xNx^{-1} = xHx^{-1}$. Thus,

$$N \leq \bigcap_{x \in G} xHx^{-1}$$

□

Corollary 2.2.0.1. If G is a finite group of order n , p is the smallest prime dividing $|G|$. Then, any subgroup of index p is normal.

Proof. Let $|G| = p^r p_1^{r_1} \cdots p_n^{r_n}$ be a prime decomposition, $H \leq G$ with $|G : H| = p$.

Let $K = \ker \pi_H \leq H$, $k = |H : K|$. Then, $|G : K| = |G : H||H : K| = pk$. By the First-Isomorphism Theorem,

$$G/\ker \pi_H \cong \pi_H[G] \leq S_A$$

and Since H has p left cosets, $A \cong \mathbb{Z}_p$, thus G/K is isomorphic to some subgroup of S_p .

Now, Lagrange's Theorem gives that $|G/K| = pk$ divides $|S_p| = p!$. This implies $k \mid (p-1)!$.

$|G : K| = pk$ implies $|G| = pk \cdot |K|$. Since p is the minimal prime that divides $|G|$, thus every prime divisor of k is greater than or equal to p . This implies must be $k = 1$. Thus $H = K \trianglelefteq G$. □

Definition 2.2.0.3. Let a Group action as:

$$\alpha : G \times G \rightarrow G : (g, a) \mapsto gag^{-1}$$

Now, the orbit driven from this action $[a] = \{b \in G \mid \exists g \in G \text{ s.t. } b = gag^{-1}\}$ is called be **Conjugacy Class**. More generally,

$$\alpha : G \times \mathcal{P}(G) \rightarrow \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$$

Lemma 2.2.0.2. Let $\alpha : G \times \mathcal{P}(G) \rightarrow \mathcal{P}(G) : (g, S) \mapsto gSg^{-1}$ be a Group action acting as Conjugate.

Then, $G_S = N_G(S)$ and $|\mathcal{C}_S| = |G : N_G(S)|$, for any $S \subseteq G$.

In particular, if S is singleton, $S = \{g_i\}$, then $|\mathcal{C}_{\{g_i\}}| = |G : N_G(g_i)| = |G : C_G(g_i)|$.

Proof.

$$G_S = \{g \in G \mid gSg^{-1} = S\} = N_G(S)$$

Thus, for any $S \in \mathcal{P}(G)$,

$$|\mathcal{C}_S| = |G : N_G(S)|$$

□

2.2.1 Lagrange's Theorem

2.3 Generating subset of a Group

2.4 Commutator Subgroup

Chapter 3

Finite Group Theory

3.1 The Class Equation

Theorem 3.1.0.1. The Class Equation

Let G be a finite group, and

g_1, \dots, g_r be representatives of the distinct conjugacy classes of G not contained in the center $Z(G)$ of G . Then,

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

3.2 Cauchy's Theorem

Lemma 3.2.0.1. Cauchy's Theorem

Let G be a finite group, and p be a prime dividing $|G|$. Then, G has order p element.

Proof. Define a set:

$$S \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_p) \mid x_i \in G, x_1 x_2 \cdots x_p = 1\}$$

Then, S has exactly $|G|^{p-1}$ elements because there are $|G|$ possible choices for each of the first $p-1$ elements in G .

Once x_1, \dots, x_{p-1} are chosen, then x_p is uniquely determined by the uniqueness of inverses.

Then, let $\sigma = (1, 2, \dots, p)$ be a permutation. Then, for any $\alpha \in S$, $\sigma^n(\alpha) \in S$ for all $n \in \mathbb{Z}$, being $ab = 1 \iff ba = 1$. More precisely, let $n \in \mathbb{Z}$ be given, $\alpha = (x_1, \dots, x_n)$. Then,

$$\sigma^n(\alpha) = (x_{n+1}, x_{n+2}, \dots, x_p, x_1, x_2, \dots, x_n)$$

By $x_1 \cdots x_n x_{n+1} \cdots x_p = 1$, $x_{n+1} \cdots x_p x_1 \cdots x_n = 1$. Thus $\sigma^n(\alpha) \in S$. Now, define a relation on S as:

$$\alpha \sim \beta \text{ if and only if } \beta = \sigma^n(\alpha) \text{ for some } n \in \mathbb{Z}$$

Then, this relation be equivalent relation, thus construct a partition on S . Claim:

$$[\alpha] = \{\beta \in S \mid \beta \sim \alpha\} \text{ is singleton if and only if } \alpha = (x, \dots, x) \text{ for some } x \in G.$$

Left direction is clear, and for show that Right direction,

Suppose that $\alpha = (x_1, \dots, x_n)$ has different coordinate elements, let $x_i \neq x_j$, for some $i < j$. Then clearly

$$(x_1, \dots, x_i, \dots, x_p) \neq \sigma^{i-j}(x_1, \dots, x_i, \dots, x_j, \dots, x_p) = (\dots, \underset{i\text{'th element}}{x_j}, \dots)$$

Meanwhile, if $[\alpha]$ has elements more than 1, $[\alpha]$ has exactly number of p elements. Because suppose that $\alpha = (x_1, \dots, x_p)$ has at least one different coordinate. Then,

$$\sigma^1(\alpha), \sigma^2(\alpha), \dots, \sigma^{p-1}(\alpha)$$

are mutually different: If there exist $1 \leq i < j < p$ such that $\sigma^i(\alpha) = \sigma^j(\alpha)$, that is, $\sigma^{j-i}(\alpha) = \alpha$. Now, $j - i \mid p$, this is contradiction with p is prime. Therefore, every equivalent class has order 1 or p . Consequently,

$$|G|^{p-1} = k + pd$$

where k is a number of classes of size 1, and d is a number of classes of size p . And $(1, 1, \dots, 1) \in S$, k is at least 1.

Since p divides $|G|^{p-1} = k + pd$, thus k must be bigger than 1, thus there exists elements such that $x^p = 1$. \square

3.3 Sylow's Theorem

Theorem 3.3.0.1. Sylow's Theorem

Let G be a group of order $p^\alpha m$, where p is a prime such that $p \nmid m$.

A group of order p^r , ($r \geq 1$) is called a p -group, Subgroups of G which are p -groups are called p -subgroup.

In particular, subgroups of order p^α is called Sylow p -subgroup of G . And, define a collection

$$\text{Syl}_p(G) \stackrel{\text{def}}{=} \{P \leq G \mid |P| = p^\alpha\}, \quad n_p(G) \stackrel{\text{def}}{=} \text{Card}(\text{Syl}_p(G))$$

The First Sylow Theorem

There exists a Sylow p -subgroup of G . i.e., $\text{Syl}_p(G) \neq \emptyset$.

The Second Sylow Theorem

If $P \in \text{Syl}_p(G)$ and $Q \leq G$ be a p -subgroup. Then, there exists $g \in G$ such that $Q \leq gPg^{-1}$.

The Third Sylow Theorem

$n_p \equiv 1 \pmod{p}$, $n_p = |G : N_G(P)|$ for any $P \in \text{Syl}_p(G)$, and $n_p \mid m$.

Before prove above statements, we show that:

Lemma 3.3.0.1. Let $P \in \text{Syl}_p(G)$. If Q is p -subgroup of G , then $Q \cap N_G(P) = Q \cap P$.

Proof. Put $H = Q \cap N_G(P)$. Since $P \leq G$, for any $p \in P$, $pPp^{-1} = P$, thus $p \in N_G(P)$. i.e., $P \leq N_G(P)$. Thus, Enough to Show that $H \leq Q \cap P$. Since $H \leq N_G(P)$,

$$PH = \bigcup_{h \in H} Ph = \bigcup_{h \in H} hP = HP$$

Thus, $PH \leq G$. And,

$$|PH| = \frac{|P||H|}{|P \cap H|}$$

By Lagrange's Theorem, $H \leq P$ and $P \cap H \leq P$ must have order of powers of p , so PH be a p -group. Clearly, $P \leq PH$ and P is the largest p -group of G , thus, $PH = P$. This means, $H \leq P$. \square

Proof. The First Theorem: The existence of Sylow p -subgroup. Proof by Induction:

If $|G| = 1$, there is nothing to prove.

Assume inductively the existence of Sylow p -subgroups for all groups of order less than $|G|$.

In case of $p \mid |Z(G)|$, then by Cauchy's Theorem, $Z(G)$ has a subgroup N which has order of p .

Clearly N is Normal, and $G/N = |G|/|N| = p^{a-1}m$. By assumption, G/N has a subgroup P' of order p^{a-1} .

By The Fourth Isomorphism Theorem, Let $P \leq G$ be a subgroup such that $P/N = P'$.

Then, $|P| = |P/N| \cdot |N| = p^\alpha$, Thus P be a Sylow p -subgroup of G .

In case of $p \nmid |Z(G)|$.

Let g_1, \dots, g_r be representatives of the distinct conjugacy classes of G , not contained in $Z(G)$. Then, The Class Equation gives

$$|G| = |Z(G)| + \sum_{i=1}^r |G : C_G(g_i)|$$

Since p divides $|G|$, if for all $i = 1, 2, \dots, r$, $p \mid |G : C_G(g_i)|$ then $p \mid |Z(G)|$, this is contradiction.

Thus, for some j , $p \nmid |G : C_G(g_j)|$. Put $H = C_G(g_j) < G$. Then, $|H|$ has a factor of p^α , by $p \nmid |G : C_G(g_j)|$. Now,

$$|H| = p^\alpha m' \quad (m' < m)$$

By assumption, H has a Sylow p -group, order of p^α .

Consequently, the existence of Sylow p -subgroup was shown.

The Second Theorem: Relation of p -subgroups.

The First Theorem gives existence of Sylow p -subgroups. Let $P \in \text{Syl}_p(G)$. Denote that:

$$S \stackrel{\text{def}}{=} \{gPg^{-1} \mid g \in G\} = \{P_1, \dots, P_r\}$$

Let $Q \leq G$ be an any p -subgroup of G . And, Q acts by conjugation on S . i.e.,

$$\alpha : Q \times S \rightarrow S : (q, P_i) \mapsto qP_iq^{-1}$$

Write S as a disjoint union of orbits under this action by Q :

$$S = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_s$$

where $r = |\mathcal{O}_1| + \dots + |\mathcal{O}_s|$. Rearrange a set S as: $P_i \in \mathcal{O}_i$, $1 \leq i \leq s$. Now, using Definition, Lemma, and above Theorem,

$$|\mathcal{O}_i| \stackrel{\text{Thm}}{=} |Q : N_Q(P_i)| \stackrel{\text{def}}{=} |Q : N_G(P_i) \cap Q| \stackrel{\text{Lemma}}{=} |Q : P_i \cap Q|$$

for each $1 \leq i \leq s$. Since Q was arbitrary, Let $Q = P_1$, so that $|\mathcal{O}_1| = |P_1 : P_1 \cap P_1| = 1$. And, for each $i \geq 2$, $P_i \cap P_1 < P_1$,

$$|\mathcal{O}_i| = |P_1 : P_i \cap P_1| > 1$$

Since $P_1 \in \text{Syl}_p(G)$, that is $|P_1| = p^\alpha$, $|P_1 : P_i \cap P_1| = |P_1|/|P_i \cap P_1| = p^k$ where $1 \leq k < \alpha$. This means for each $2 \leq i \leq s$, p divides $|\mathcal{O}_i|$. Thus,

$$r = |\mathcal{O}_1| + (|\mathcal{O}_2| + \dots + |\mathcal{O}_s|) \equiv 1 \pmod{p}$$

Now, Proof by Contradiction: Let $Q \leq G$ be a p -subgroup. Suppose that for any $1 \leq i \leq r$, $Q \not\leq P_i$. Then, $P_i \cap Q < Q$ for all i , this means

$$|\mathcal{O}_i| = |Q : P_i \cap Q| > 1$$

Thus for any i , p divides $|\mathcal{O}_i|$, this is Contradiction. This proved Relation of p -subgroups.

Finally, The Third Theorem:

Since Second Theorem, this gives that $S = \text{Syl}_p(G)$, thus $n_p(G) = r$. That is, $n_p \equiv 1 \pmod{p}$.

Since all Sylow p -subgroups are Conjugate, for any $P \in \text{Syl}_p(G)$,

$$n_p = r = |\mathcal{O}_1| = |G : N_G(P)|$$

Consequently, Completing the Sylow Theorem. □

3.4 More Theorems

Theorem 3.4.0.1. *n Factorial Theorem*

If G is simple and there is a subgroup H with $|G : H| = n$, then $|G| \mid n!$.

Proof. Let G act on $A = \{gH \mid g \in G\}$ by left multiplication. ($|A| = n$.)

Let $\varphi : G \rightarrow S_n$ be a homomorphism afforded above action. Then, $G \xrightarrow{G \text{ simp.}} G/\ker \varphi \cong \varphi[G] \leq S_n$

□

3.5 Simple groups

3.6 Cyclic Group

3.7 Symmetric Group

3.8 Dihedral Group

Chapter 4

Ring Theory

4.1 Addition and Multiplication in \mathbb{Z}

4.1.1 \mathbb{Z}_n^\times

Theorem 4.1.1.1. For any integer $n > 1$, $(\mathbb{Z}_n^\times, \times_n)$ is a group where $\begin{cases} \mathbb{Z}_n^\times \stackrel{\text{def}}{=} \{k \in \mathbb{Z}_n \mid \gcd(k, n) = 1\} \\ a \times_n b \stackrel{\text{def}}{=} ab \bmod n \end{cases}$

Proof. Let $a \in \mathbb{Z}_n^\times$ be given. Then, the Bezout's identity gives $ax + ny = 1$ for some $x, y \in \mathbb{Z}$. Now, $ax = 1 - ny \equiv 1 \pmod{n}$. \square

Theorem 4.1.1.2. Fermat's Little Theorem

If $a \in \mathbb{Z}$ and p is prime not dividing a , then $a^{p-1} \equiv 1 \pmod{p}$.

Proof. Let $a \in \{1, 2, \dots, p-1\}$. Put $k = |a|$. Then, $\{1, a^1, a^2, \dots, a^{k-1}\}$ forms a subgroup of \mathbb{Z}_p . By Lagrange's Theorem, $k \mid p-1$. Now, $a^{p-1} = a^{km} = (a^k)^m = 1 \pmod{p}$. \square

Theorem 4.1.1.3. $a \in \mathbb{Z}_n$ is a zero divisor if and only if $\gcd(a, n) \neq 1$.

Proof. Suppose that $\gcd(a, n) = 1$. If $ax + n\mathbb{Z}$ is zero for some $x \in \mathbb{Z}_n$, then $ax = nk$ for some $k \in \mathbb{Z}$. Thus, $n \mid ax$ and $\gcd(a, n) = 1$, this implies $n \mid x$. Conversely, Suppose that $d = \gcd(a, n) \neq 1$. Then,

$$a \cdot \frac{n}{d} = \frac{a}{d} \cdot n \in n\mathbb{Z}$$

But, $a \notin n\mathbb{Z}$ and $\frac{n}{d} \notin n\mathbb{Z}$, thus a is zero divisor. \square

Corollary 4.1.1.1. The subset $G_n \stackrel{\text{def}}{=} \{a \in \mathbb{Z}_n \mid a \text{ is zero divisor}\}$ is a group under the multiplication modulo n .

Theorem 4.1.1.4. Euler's Theorem

If $\gcd(a, n) = 1$, $a^{\varphi(n)} \equiv 1 \pmod{n}$.

Proof. Let $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$. Then, there exists $0 < b \in a + n\mathbb{Z}$ such that $b < n$ and $\gcd(b, n) = 1$. Then, $a^{\varphi(n)} \equiv b^{\varphi(n)} \pmod{n}$. Since $b \in \mathbb{Z}_n^\times$ and order of \mathbb{Z}_n^\times is $\varphi(n)$, thus $b^{\varphi(n)} \equiv 1 \pmod{n}$. \square

4.2 Ideal

Definition 4.2.0.1. Let R be a Ring. A subset $I \subseteq R$ is called *ideal* of R if:

1. $I \subseteq R$ is a subgroup of R .
2. I is closed under the multiplication.
3. For any $r \in R$, $rI \subseteq I$ and $Ir \subseteq I$. (In other word, for any $r \in R, a \in I$, $ra \in I$ and $ar \in I$.)

Theorem 4.2.0.1. Let R be a Ring. Then, TFAE:

1. $I \subset R$ is an Ideal of R .
2. The additive Quotient Group $R/I \stackrel{\text{def}}{=} \{r + I \mid r \in R\}$ be a Ring under the operation:

$$(r + I) \times (s + I) = (rs) + I$$

Proof. Observation:

$$r_1 + I = r_2 + I \iff r_1 - r_2 \in I \iff \exists a \in I \text{ s.t. } r_1 = r_2 + a$$

Now, for well-definedness, want to show that the equality

$$\begin{aligned} (r + I) \times (s + I) &= (rs) + I \\ \stackrel{(*)}{=} [(r + \alpha) + I] \times [(s + \beta) + I] &= (r + \alpha)(s + \beta) + I = (rs + r\beta + \alpha s + \alpha\beta) + I \end{aligned}$$

(*) holds for any $r, s \in R$, $\alpha, \beta \in I$.

If I is Ideal, then $r\beta, \alpha s, \alpha\beta \in I$. Thus closed under the addition gives (*).

Conversely, if this operation is well-defined, then for any $r, s \in R$, $\alpha, \beta \in I$, (*) holds.

Substituting zero to each r, s, α, β gives I is ideal. □

4.2.1 Properties of Ideal in Ring with identity

Definition 4.2.1.1. Let R be a Ring with identity, and $A \subseteq R$. Define *Ideal generated by A* as:

$$(A) \stackrel{\text{def}}{=} \bigcap_{\substack{I \text{ ideal} \\ A \subseteq I}} I$$

And,

$$\begin{aligned} RA &\stackrel{\text{def}}{=} \{r_1a_1 + \cdots + r_na_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\} \\ AR &\stackrel{\text{def}}{=} \{a_1r_1 + \cdots + a_nr_n \mid n \in \mathbb{N}, r_i \in R, a_i \in A\} \\ RAR &\stackrel{\text{def}}{=} \{r_1a_1r'_1 + \cdots + r_na_nr'_n \mid n \in \mathbb{N}, r_i, r'_i \in R, a_i \in A\} \end{aligned}$$

Lemma 4.2.1.1. Let R be a Ring with identity, and $A \subseteq R$. Then, $(A) = RAR$.

Proof. Since RAR is ideal which contains A , $(A) \subseteq RAR$.

And, conversely, if $\sum_{i=1}^n r_i a_i r'_i \in RAR$, then $\sum_{i=1}^n r_i a_i r'_i \in (A)$ because each $r_i a_i r'_i$ are contained in (A) , being (A) is ideal containing A and ideal is closed under the addition. \square

Theorem 4.2.1.1. Let I be an ideal of Ring R with identity.

$$I = R \text{ if and only if } I \text{ contains a unit.}$$

Proof. Right direction is clear by $1 \in R = I$.

Denote $u \in I$ be a unit with $vu = 1$, and Let $r \in R$ be given. Then,

$$r = r1 = rvu \in I$$

\square

Definition 4.2.1.2. An Ideal M of R is *Maximal ideal* if: There is no Ideal I such that $M \subsetneq I \subsetneq R$.

Theorem 4.2.1.2. Let R be a Ring with identity.

Then, every proper ideal $I \subsetneq R$ is contained in a maximal ideal.

Proof.

\square

Lemma 4.2.1.2. Let R be a commutative Ring with identity, M, P are proper ideals of R .

1. M is Maximal Ideal if and only if R/M is a field.
2. P is Prime Ideal if and only if R/P is an integral domain.

Summary: M maximal $\iff R/M$ field $\implies R/M$ integral domain $\iff M$ prime.

Proof.

M is maximal \iff There is no ideal I such that $M \subsetneq I \subsetneq R$
 \iff There are Ideals of R/M only 0 and R/M
 \iff R/M is field

P is Prime Ideal \iff If $ab \in P$, then $a \in P$ or $b \in P$
 \iff If $ab + P = P$, then $a + P = P$ or $b + P = P$
 \iff If $\bar{a}\bar{b} = \bar{0}$, then $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$
 \iff R/P is integral domain

□

4.3 Ring of Fractions

Theorem 4.3.0.1. Let R be a Commutative Ring, $D \subset R$ be a subset such that $\begin{cases} \text{no zero, no zero divisors} \\ \text{closed under multiplication} \end{cases}$. Then, there exists a Commutative Ring Q with identity satisfies:

1. R can embed in Q , and every element of D becomes unit in Q . More precisely, $Q = \{rd^{-1} \mid r \in R, d \in D\}$.
2. Q is the smallest Ring containing R with identity such that every element of D becomes unit in Q .

Proof. Let $\mathcal{F} \stackrel{\text{def}}{=} \{(r, d) \mid r \in R, d \in D\}$ and the relation \sim on \mathcal{F} by $(r_1, d_1) \sim (r_2, d_2) \iff r_1d_2 = r_2d_1$. Then, \sim is equivalent relation: reflexive and symmetric are clear, and Suppose that $(r_1, d_1) \sim (r_2, d_2)$ and $(r_2, d_2) \sim (r_3, d_3)$.

$$r_2d_3 = r_3d_2 \implies r_2d_1d_3 = r_3d_1d_2 \implies r_1d_2d_3 = r_3d_1d_2 \implies d_2(r_1d_3 - r_3d_1) \implies r_1d_3 = r_3d_1$$

Thus transitivity shown. Define

$$\frac{r}{d} \stackrel{\text{def}}{=} [(r, d)] = \{(a, b) \mid (a, b) \sim (r, d)\}, Q \stackrel{\text{def}}{=} \left\{ \frac{r}{d} \mid r \in R, d \in D \right\}$$

And define operations $+, \times$ on Q :

$$\frac{r_1}{d_1} + \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1d_2 + r_2d_1}{d_1d_2}, \quad \frac{r_1}{d_1} \times \frac{r_2}{d_2} \stackrel{\text{def}}{=} \frac{r_1r_2}{d_1d_2}$$

Well-Definedness: If $\frac{r_1}{d_1} = \frac{r'_1}{d'_1}$ and $\frac{r_2}{d_2} = \frac{r'_2}{d'_2}$,

$$\frac{r_1d_2 + r_2d_1}{d_1d_2} = \frac{r_1d_2d'_1d'_2 + r_2d_1d'_1d'_2}{d_1d_2d'_1d'_2} = \frac{(r_1d'_1)d_2d'_2 + (r_2d'_2)d_1d'_1}{d_1d_2d'_1d'_2} = \frac{(r'_1d_1)d_2d'_2 + (r'_2d_2)d_1d'_1}{d_1d_2d'_1d'_2} = \frac{(r'_1d'_2 + r'_2d'_1)d_1d_2}{d_1d_2d'_1d'_2} = \frac{r'_1d'_2 + r'_2d'_1}{d'_1d'_2}$$

$$\frac{r_1r_2}{d_1d_2} = \frac{r_1r_2d'_1d'_2}{d_1d_2d'_1d'_2} = \frac{(r_1d'_1)(r_2d'_2)}{d_1d_2d'_1d'_2} = \frac{(r'_1d_1)(r'_2d_2)}{d_1d_2d'_1d'_2} = \frac{r'_1r'_2d_1d_2}{d_1d_2d'_1d'_2} = \frac{r'_1r'_2}{d'_1d'_2}$$

Now, $(Q, +, \times)$ constructs Commutative Ring with identity: for any $d \in D$, put $0_Q \stackrel{\text{def}}{=} \frac{0}{d}$, $1_Q \stackrel{\text{def}}{=} \frac{d}{d}$. Then,

1. $(R, +, \times)$ closed under the operations since D is closed under the multiplication.

2. $(R, +)$ has a zero: $\frac{r_1}{d_1} + 0_Q = \frac{r_1}{d_1} + \frac{0}{d} = \frac{r_1d + 0d_1}{d_1d} = \frac{r_1d}{d_1d} = \frac{r_1}{d_1}$.

3. $(R, +)$ has an inverse: $\frac{r_1}{d_1} + \frac{-r_1}{d_1} = \frac{r_1d_1 + (-r_1)d_1}{d_1d_1} = \frac{[(r_1) + (-r_1)]d_1}{d_1d_1} = \frac{0d_1}{d_1d_1} = \frac{0}{d_1d_1} = 0_Q$.

4. $(R, +, \times)$ satisfies distributive law:

4-1. The left law:

$$\begin{aligned} \frac{r_1}{d_1} \times \left(\frac{r_2}{d_2} + \frac{r_3}{d_3} \right) &= \frac{r_1}{d_1} \times \frac{r_2d_3 + r_3d_2}{d_2d_3} = \frac{r_1r_2d_3 + r_1r_3d_2}{d_1d_2d_3} = \frac{r_1r_2d_1d_3 + r_1r_3d_1d_2}{d_1d_2d_1d_3} = \frac{r_1r_2}{d_1d_2} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_2}{d_2} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

4-2. The right law:

$$\begin{aligned} \left(\frac{r_1}{d_1} + \frac{r_2}{d_2} \right) \times \frac{r_3}{d_3} &= \frac{r_1d_2 + r_2d_1}{d_1d_2} \times \frac{r_3}{d_3} = \frac{r_1r_3d_2 + r_2r_3d_1}{d_1d_2d_3} = \frac{r_1r_3d_2d_3 + r_2r_3d_1d_3}{d_1d_3d_2d_3} = \frac{r_1r_3}{d_1d_3} + \frac{r_2r_3}{d_2d_3} \\ &= \frac{r_1}{d_1} \times \frac{r_3}{d_3} + \frac{r_2}{d_2} \times \frac{r_3}{d_3} \end{aligned}$$

5. (R, \times) has an identity: $\frac{r_1}{d_1} \times 1_Q = \frac{r_1}{d_1} \times \frac{d}{d} = \frac{r_1d}{d_1d} = \frac{r_1}{d_1}$.

6. Elements of D become unit in Q : Define $\iota : R \rightarrow Q : r \mapsto \frac{rp}{p}$ where $p \in D$ is any fixed element in D . Then, ι is Ring-Monomorphism because:

6-1. Well-Defined and Injective: $\iota(r_1) = \iota(r_2) \iff \frac{r_1p}{p} = \frac{r_2p}{p} \iff (r_1 - r_2)p = 0 \iff r_1 = r_2$

6-2. For any $d \in D$, $\iota(d)$ is a unit of Q : Put $(\iota(d))^{-1} \stackrel{\text{def}}{=} \frac{p}{dp}$, then

$$\iota(d) \times (\iota(d))^{-1} = \frac{dp}{p} \times \frac{p}{dp} = \frac{dpp}{dpp} = 1_Q$$

That is, ι is embedding from R into Q such that $\iota[D]$ becomes units of Q except zero. Moreover, if $D = R \setminus \{0\}$, then Q is field.

7. Q is the smallest ring containing R with identity such that every element of D becomes units in Q .

Let S be an any commutative ring with identity,

and assume that $\varphi : R \rightarrow S$ is a Ring-Monomorphism such that for any $d \in D$, $\varphi(d)$ is unit in S .

Define $\phi : Q \rightarrow S : \frac{r}{d} \mapsto \varphi(r)\varphi(d)^{-1}$. Then, this ϕ is well-defined and injective:

$$\begin{aligned} \phi\left(\frac{r_1}{d_1}\right) = \phi\left(\frac{r_2}{d_2}\right) &\iff \varphi(r_1)\varphi(d_1)^{-1} = \varphi(r_2)\varphi(d_2)^{-1} \iff \varphi(r_1)\varphi(d_2) = \varphi(r_2)\varphi(d_1) \\ \xrightarrow{\text{homomor.}} \varphi(r_1d_2) = \varphi(r_2d_1) &\stackrel{\text{one-to-one}}{\iff} r_1d_2 = r_2d_1 \iff \frac{r_1}{d_1} = \frac{r_2}{d_2} \end{aligned}$$

That is, if a commutative ring S with identity contains a copy of R such that the denominator set D of R becomes unit in S , then S contains ring of fractions Q of R . Thus $S = Q$ is the smallest ring that satisfies these conditions.

□

4.4 Commutative Ring with identity

Theorem 4.4.0.1. Finite integral domain is field.

Proof. Let R be a finite integral domain, and non-zero $a \in R$ be given.

Then, the map from R into R $x \mapsto ax$ is injective: because $ax = ay \implies a(x - y) = 0$ and R has no zero divisor. R is finite, so this map is surjective. Thus, there exists $b \in R$ such that $ab = 1$, a is unit. \square

Lemma 4.4.0.1. Let R be a Commutative Ring, $a, b \in R$ with $b \neq 0$.

$$a = bx \text{ for some } x \in R \stackrel{\text{def}}{\iff} b \mid a \iff a \in (b) \iff (a) \subseteq (b)$$

Lemma 4.4.0.2. Let a, b be non-zero elements in a Commutative Ring R .

If $(a, b) = (d)$, then d is the greatest common divisor of a and b .

Theorem 4.4.0.2. Let R be an integral domain. If $(d) = (d')$, then $d' = ud$ for some unit $u \in R$. Particular, d and d' are greatest common divisor of a and b , then $(d) = (d')$, thus $d' = ud$ for some unit $u \in R$.

Proof. If either d or d' is zero, then there is nothing to prove. Thus, Suppose that neither d nor d' is non-zero. Since $(d) \subseteq (d')$ and $(d) \supseteq (d')$, $d' = dx$ for some $x \in R$ and $d = d'y$ for some $y \in R$. Combining above, then $d' = dx = (d'y)x = d'(yx)$, this implies $d'(1 - yx) = 0$. Since d' is non-zero and d' chosen in the integral domain, $1 - xy = 0$. Now, both x and y are unit, we obtain the result. Second assertion is clear by the First. \square

4.4.1 Euclidean Domain

Definition 4.4.1.1. An integral domain R is called *Euclidean Domain* if: there exists a norm N such that:

for any $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ with $a = qb + r$ with $r = 0$ or $N(r) < N(b)$.

This definition allows us the *Euclidean Algorithm* on an integral domain R : for any $a, b \in R$ with $b \neq 0$,

$$\begin{aligned} a &= q_0b + r_0 \\ b &= q_1r_0 + r_1 \\ r_0 &= q_2r_1 + r_2 \\ r_1 &= q_3r_2 + r_3 \\ &\vdots \\ r_k &= q_{k+2}r_{k+1} + r_{k+2} \\ &\vdots \\ r_{n-2} &= q_n r_{n-1} + r_n \\ r_{n-1} &= q_{n+1} r_n \end{aligned}$$

This process gives a chain:

$$N(r_n) < N(r_{n-1}) < \cdots < N(r_2) < N(r_1) < N(r_0)$$

and this process terminates in finite iteration, since well-ordering principle.

Theorem 4.4.1.1. Let I be an ideal of a Euclidean Domain R . Then, I is principal ideal.

Proof. If I is zero ideal, there is nothing to prove. Let I be a non-zero ideal. Since the set $\{N(a) \mid a \in I \setminus \{0\}\}$ has a minimum by Well-Ordering Principle, choose $d \in I$ such that $N(d) \leq N(a)$, $\forall a \in I \setminus \{0\}$. Clearly, $(d) \subseteq I$. Let $a \in I$. Then, there is $q, r \in R$ such that

$$a = qd + r \text{ with } r = 0 \text{ or } N(r) < N(d)$$

Since $r = a - qd \in I$ by $a, d \in I$, thus closed under the multiplication gives $r \in I$. But, by minimality of d , r must be 0. Now, $a = qd + r = qd \in (d)$. \square

Theorem 4.4.1.2. Euclidean Algorithm

Let R be a Euclidean Domain, $a, b \in R$ be non-zero.

Denote $d = r_n$ where r_n is the last nonzero remainder in the Euclidean Algorithm for a and b .

Then, d is the greatest common integer of a and b . And, $(d) = (a, b)$. That is, there exist $x, y \in R$ such that

$$d = ax + by$$

Proof. Note that: (a, b) is principal in Euclidean Domain.

Moreover, (a, b) is the smallest ideal containing (a) and (b) . That is, If $(a) \subseteq (x)$ and $(b) \subseteq (x)$, then $(a, b) \subseteq (x)$. Now, Enough to Show:

1. $(a), (b) \subseteq (d)$. (It follows that $(a, b) \subseteq (d)$)

2. $(d) \subseteq (a, b)$. (That is, $(d) = (a, b)$)

Since $(a), (b) \subseteq (d)$ if and only if $d \mid a, b$, show that d divides a, b .

In the last equation, $r_{n-1} = q_{n+1}r_n = q_{n+1}d$. Thus, $d \mid r_{n-1}$.

Clearly, $r_n \mid r_{n-1}$, thus $d \mid r_{n-2}$. Repeat this to finite times, then we obtain: $\forall 1 \leq i \leq n$, $d \mid r_i$. As result, $d \mid a$ and $d \mid b$. This proved 1.

For to show that 2., we will prove $d \in (a, b)$.

The first equation gives directly $r_0 \in (a, b)$.

That is, $(r_0) \subseteq (a, b)$, thus $r_1 = b - q_1r_0 \in (a, b)$.

Inductively, $r_n = d \in (a, b)$, theorem completed. \square

$$a = q_0b + r_0$$

$$b = q_1r_0 + r_1$$

$$r_0 = q_2r_1 + r_2$$

$$r_1 = q_3r_2 + r_3$$

⋮

$$r_{n-2} = q_n r_{n-1} + r_n$$

$$r_{n-1} = q_{n+1}r_n + 0$$

4.4.2 Principal Ideal Domain

Definition 4.4.2.1. An integral domain R is called *Principal Ideal Domain* if: every ideal of R is principal.

Theorem 4.4.2.1. Let R be a Principal Ideal Domain, and $a, b \in R$ be non-zero.

Let d be a generator for the principal ideal (a, b) . Then,

d is the greatest common divisor of a and b , and unique up to multiplication of unit of R .

Theorem 4.4.2.2. Every non-zero Prime Ideal in a Principal Domain is Maximal Ideal.

Proof. Let (p) be a non-zero Prime Ideal.

Let $I = (m)$ be an Ideal such that $(p) \subseteq (m)$. Since $p \in (m)$, there is a $x \in R$ such that $p = mx$.

But, $p = mx \in (p)$, Prime Ideal, $m \in (p)$ or $x \in (p)$.

If $m \in (p)$, $m = py$ for some $y \in R$. That is, $m = py \in (p)$, $(p) = (m)$.

If $x \in (p)$, $x = pz$ for some $z \in R$. That is, $p = mx = mpz = p(mz)$, m becomes a unit.

The Ideal (m) containing unit implies $(m) = R$. \square

4.4.3 Noetherian Domain

Definition 4.4.3.1. The Ring R is said to be *Noetherian Ring* if: R satisfies *Ascending Chain Condition* on ideals.

The Integral Domain R with Noetherian is called *Noetherian Doomain*.

Theorem 4.4.3.1. Principal Ideal Domain is Noetherian Domain.

Proof. Suppose that there is an ascending chain of ideals,

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq R$$

(Considering only countable Chain: Since m.stackexchange 4265544.)

Put $I \stackrel{\text{def}}{=} \bigcup_{i=1}^{\infty} I_i$. Since for any $r \in R$ and $a \in I$, there exists $i \in \mathbb{N}$ such that $a \in I_i$. Thus, $ra \in I_i \in I$, I is ideal.

Since R is Principal, for some $a \in R$, $(a) = I$. That is, $a \in I$. This implies there exists $n \in \mathbb{N}$ such that $a \in I_n$. Now, $(a) \subseteq I_n \subseteq I = (a)$, This $I_n = (a) = I$. Consequently, R is Noetherian. \square

4.4.4 Unique Factorization Domain

Definition 4.4.4.1. Let R be an Integral Domain.

1. A non-zero, not unit $r \in R$ is called **irreducible** of R if: If $r = ab$ for some $a, b \in R$, then either a or b is a unit.
2. A non-zero, not unit $p \in R$ is called **prime** of R if: If $p | ab$ for some $a, b \in R$, then either $p | a$ or $p | b$.

Clearly, p is prime if and only if (p) is Prime Ideal.

Theorem 4.4.4.1. Let R be an Integral Domain. Then, every prime element is irreducible.

Proof. Let R be an Integral Domain, $p \in R$ be a prime. Suppose that $p = ab$ for some $a, b \in R$. Then, clearly $p | ab$, thus $p | a$ or $p | b$. If $p | a$, then $a = px$ for some $x \in R$. Now, $p = ab = pxb$, $p(1 - xb) = 0$. Since R is integral domain and p is non-zero, $xb = 1$. That is, b is a unit, thus p is irreducible. \square

Definition 4.4.4.2. Let R be an integral domain and let $r \in R$ be a nonzero, nonunit element.

1. We say that r is **factorizable** if there exist irreducible elements p_1, \dots, p_n ($n \geq 1$) such that

$$r = p_1 p_2 \cdots p_n.$$

Any such expression is called an **irreducible factorization** of r .

2. An irreducible factorization is **unique up to associates** if for any two irreducible factorizations

$$r = p_1 \cdots p_n = q_1 \cdots q_m,$$

we have $n = m$ and there exist a permutation $\sigma \in S_n$ and units $u_1, \dots, u_n \in R^\times$ such that

$$q_i = u_i p_{\sigma(i)} \quad (i = 1, \dots, n),$$

equivalently, q_i is associate to $p_{\sigma(i)}$ for each i .

The domain R is called a **factorization domain** (also: **atomic**) if every nonzero, nonunit element of R is factorizable. If, in addition, irreducible factorizations are unique up to associates, then R is called a **unique factorization domain (UFD)**.

Theorem 4.4.4.2. Noetherian Domain is Factorization Domain.

Proof. Let R be a Noetherian Domain. And, let $r \in R$ be a non-zero, not unit.

There exist only two possibility: r is irreducible or not irreducible.

If r is irreducible, then there is nothing to prove. If r is not irreducible, then there exist not unit $r_1, r_2 \in R$ such that $r = r_1 r_2$.

If r_1 and r_2 are irreducible, prove end. If r_1 is reducible, then there exist not unit $r_{1,1} r_{1,2} \in R$ such that $r_1 = r_{1,1} r_{1,2}$.

If this process never terminates, then, there is a infinite strictly ascending chain:

$$(r) \subsetneq (r_1) \subsetneq (r_{1,1}) \subsetneq \cdots \subsetneq R$$

Strictly given by $r = r_1 r_2$ and r_2 is not a unit.

More precisely, if $(r) = (r_1)$, then $r_1 = rk$ for some $k \in R$, $r_1 = rk = r_1 r_2 k$, r_1 becomes a unit. Contradiction. \square

Theorem 4.4.4.3.

1. In Principal Ideal Domain, every irreducible element is prime.
2. In Unique Factorization Domain, every irreducible element is prime.

Proof. Let R be a Principal Ideal Domain, and $r \in R$ be an irreducible.

Suppose that (m) is an ideal of R such that $(r) \subseteq (m)$.

Then, $r \in (m)$ implies $r = mx$ for some $x \in R$, now irreducibility gives either m or x is a unit.

If m is a unit, then $(m) = R$. If x is a unit, $r = mx$ implies $rx^{-1} = m$ implies $m \in (r)$ implies $(m) \subseteq (r)$ implies $(m) = (r)$.

Consequently, (r) is maximal ideal in the Principal Ideal Domain,

$$(r) \text{ is a maximal } \iff R/(r) \text{ is a field } \implies R/(r) \text{ is an integral domain } \iff (r) \text{ is Prime.}$$

Let R be a Unique Factorization Domain, and $r \in R$ be an irreducible. Suppose that $r | ab$ for some $a, b \in R$.

If either a or b is unit, then $r | ab$ implies r divides a or b , there is nothing to prove.

If neither a nor b is a unit, write as factorization form: $a = a_1 \cdots a_n$ and $b = b_1 \cdots b_m$, being a, b in U.F.D.

Since r divides $ab = a_1 \cdots a_n b_1 \cdots b_m$, there exists $x \in R$ such that

$$rx = a_1 \cdots a_n b_1 \cdots b_m$$

If x is a unit, then $r = x^{-1} a_1 \cdots a_n b_1 \cdots b_m$. But, the uniqueness gives contradiction. Thus x is not unit.

Now, x has irreducible factorization, the uniqueness gives $r = a_i$ for some $1 \leq i \leq n$ or $r = b_j$ for some $1 \leq j \leq m$.

This means r divides a or b . \square

4.5 Examples

4.5.1 Matrix Ring

Theorem 4.5.1.1. $\phi : \mathbb{C} \rightarrow M_2(\mathbb{R}) : a + bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ is Ring-Embedding.

Proof. □

4.5.2 Group Ring

4.5.3 Integer Ring

Definition 4.5.3.1. Let D be a rational number such that $\sqrt{D} \notin \mathbb{Q}$.

Define *Quadratic field* for D :

$$\mathbb{Q}(\sqrt{D}) \stackrel{\text{def}}{=} \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\}$$

4.5.4 Boolean Ring

Definition 4.5.4.1. A Ring R is called *Boolean Ring* if: for any $a \in R$, $a^2 = a$.

Theorem 4.5.4.1. Boolean Ring is Commutative Ring.

Proof. Let $a, b \in R$ be given. Then,

$$a^2 + b^2 = a + b = (a + b)^2 = a^2 + ab + ba + b^2 \implies ab = -ba$$

Meanwhile, for any $c \in R$,

$$c = c^2 = (-c)^2 = -c$$

Thus, $ab = -ba = ba$. □

4.5.5 Nilradical Ideal

Definition 4.5.5.1. Let R be a Ring.

An element $x \in R$ is called **Nilpotent** if: $x^m = 0$ for some $m \in \mathbb{N}$.

Proposition 4.5.5.1. Let R be a Ring with identity.

- Let $n = a^k b$ where $a, b \in \mathbb{Z}$ and $k \in \mathbb{N}$. Then, $ab + n\mathbb{Z}$ is nilpotent in $\mathbb{Z}/n\mathbb{Z}$.

Proof. $(ab)^k = a^k b^k = a^k b \cdot b^{k-1} = n \cdot b^{k-1} \in n\mathbb{Z}$. □

- Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then,

$a + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if a prime p divides n implies p divides a .

Proof. Let $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be a prime decomposition.

The right assertion means $a = p_1 \cdots p_k \cdot m$ for some $m \in \mathbb{Z}$. Let α be the least common multiple.

Then, $a^\alpha \in n\mathbb{Z}$. Conversely, if $a^k \in n\mathbb{Z}$ for some $k \in \mathbb{N}$, $a^k = nm$ for some $m \in \mathbb{Z}$.

If a prime p divides n , then it divides $nm = a^k$, thus divides a . □

Proposition 4.5.5.2. Let R be a Commutative Ring with identity, and nonzero $x \in R$ be a nilpotent element.

- x is zero divisor. Hence, not unit.

- For any $r \in R$, rx is nilpotent.

- $1+x$ is unit in R .

Proof. $(1+x)(1-x+x^2-x^3+\cdots+(-x)^{m-1}) = (1-x+x^2-x^3+\cdots+(-x)^{m-1})+(x-x^2+x^3-x^4+\cdots-(-x)^m) = 1$. □

- If u is unit and a is nilpotent, then $u+a$ is unit.

Proof. $u+a = u^{-1}(1+ua)$. □

Definition 4.5.5.2. Let R be a Commutative Ring. Define **Nilradical** of R :

$$\mathfrak{N}(R) \stackrel{\text{def}}{=} \{x \in R \mid x \text{ is nilpotent in } R\}$$

Lemma 4.5.5.1. Nilradical of Commutative Ring R with identity is ideal.

Proof. Trivially, $0 \in \mathfrak{N}(R)$. Suppose $x, y \in \mathfrak{N}(R)$ are nilpotents in R with $x^n = y^m = 0$ for some $n, m \in \mathbb{N}$.

$$(x-y)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} x^{n+m-k} (-y)^k = x^n \sum_{k=0}^m \binom{n+m}{k} x^{m-k} (-y)^k + (-y)^m \sum_{k=m+1}^{n-1} \binom{n+m}{k} x^{n+m-k} (-y)^{-m+k} = 0$$

implies $x-y \in \mathfrak{N}(R)$, thus Nilradical is closed under the subtraction, thus it is subgroup.

Closed under the multiplication is trivial: $(xy)^{nm} = x^{nm}y^{nm} = 0^m 0^n = 0$.

And, since above discussion, rx is nilpotent for any $r \in R$ and $a \in \mathfrak{N}(R)$. □

Theorem 4.5.5.1. Let R be a Commutative Ring with identity. $R/\mathfrak{N}(R)$ has no nilpotent element except zero.

Proof. Let $a + \mathfrak{N}(R) \in R/\mathfrak{N}(R)$ be a non-zero. If $a^m + \mathfrak{N}(R) = \mathfrak{N}(R)$ for some $m \in \mathbb{N}$, that is, $a^m \in \mathfrak{N}(R)$. This means $a^{nm} = 0$ for some $n \in \mathbb{N}$, thus $a \in \mathfrak{N}(R)$. This is contradiction with $a + \mathfrak{N}(R)$ is non-zero. □

Theorem 4.5.5.2. Let R be a Commutative Ring and $I \subset R$ be an ideal.

If $\mathfrak{N}(I) = I$ and $\mathfrak{N}(R/I) = R/I$, then $\mathfrak{N}(R) = R$.

Proof. Let $r \in R$ be given. Since $\mathfrak{N}(R/I) = R/I$, $r + I$ is nilpotent in R/I . That is, $r^n + I = I$ for some $n \in \mathbb{N}$. Since $\mathfrak{N}(I) = I$, $r^n \in I$ is nilpotent, thus $(r^n)^m = 0$ for some $m \in \mathbb{N}$. Thus $r^{nm} = 0$. \square

4.5.6 Annihilator Ideal

Lemma 4.5.6.1. Let R be a Ring. For any $a \in R$, $I_a \stackrel{\text{def}}{=} \{x \in R \mid ax = 0\}$ is a subgroup of R .

Proof. $0 \in I_a$ is trivial. Let $x, y \in I_a$. Then, $a(x - y) = ax - ay = 0 - 0 = 0$. Thus $x - y \in I_a$. Moreover, $a(xy) = (ax)y = 0y = 0$. Thus $xy \in I_a$. \square

Definition 4.5.6.1. Let R be a Ring with identity, $a \in R$ be a fixed element, and $L \subset R$ be right ideal in R .

1. Define **right annihilator** of a in R : $\{x \in R \mid ax = 0\}$.
2. Define **right annihilator** of L in R : $\{x \in R \mid ax = 0, \forall a \in L\}$.

Lemma 4.5.6.2. Let R be a Ring with identity. Then, the right annihilator of a in R is right ideal.

Proof. Let $r \in R$ and $s \in \{x \in R \mid ax = 0\}$. Then, $asr = (as)r = 0r = 0$. Thus $sr \in \{x \in R \mid ax = 0\}$. \square

Lemma 4.5.6.3. Let R be a Ring with identity. Then, the right annihilator of L in R is two-sided ideal.

Proof. Let $r, r' \in R$ and $s \in \{x \in R \mid ax = 0, \forall a \in L\}$.

For any $a \in L$, $arsr' = (ar)sr' = 0r' = 0$. Thus, $rsr' \in \{x \in R \mid ax = 0, \forall a \in L\}$. \square

4.6 Homomorphisms

4.6.1 Formal polynomial differentiation map

Definition 4.6.1.1. Let F be a field with characteristic is 0. Define **formal polynomial differentiation map**:

$$D : F[x] \rightarrow F[x] : a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n \mapsto a_1 + 2 \cdot a_2x + \cdots + n \cdot a_nx^{n-1}$$

Proposition 4.6.1.1. Formal polynomial differentiation map D satisfies:

1. D preserves addition, but not multiplication.
2. $\ker D = F$.
3. $D[F[x]] = F[x]$.

Proof. Proof of 3.

Let $a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n \in F[x]$ be given. Then,

$$D \left(a_0x + \frac{a_1}{2 \cdot 1}x^2 + \cdots + \frac{a_n}{(n+1) \cdot 1}x^{n+1} \right) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$$

□

4.7 Theorems

Theorem 4.7.0.1. Let R be a Ring that contains at least two elements.

Suppose that for any non-zero $a \in R$, there exists a unique $b \in R$ such that $aba = a$.

Then, R is a division Ring.

Proof. Proof consists by three-step.

1) R has no zero divisor.

Let $a \in R$ be a non-zero element. Then, there exists a unique elements $b \in R$ such that $aba = a$.

Suppose that $a \in R$ is zero divisor. Then, there is a non-zero element $c \in R$ such that $ac = 0$ or $ca = 0$.

Either $ac = 0$ or $ca = 0$, $aca = 0$ is true. Now,

$$a = aba = aba - 0 = aba - aca = a(b - c)a$$

Since the Uniqueness, $b - c = b$, thus $c = 0$. Contradiction.

2) If $aba = a$, then $bab = a$.

It is clear:

$$a = aba = ababa$$

Now, the Uniqueness gives $b = bab$.

3) R has a identity.

Let non-zero $a \in R$ be fixed. Then, there is a unique $b \in R$ such that $aba = a$.

Let $c \in R$ be given. Then,

$$\begin{aligned} ca &= caba \xrightarrow{\text{cancel}} c = cab \\ bc &= babc \xrightarrow{\text{cancel}} c = abc \end{aligned}$$

Thus, (ab) is identity in R .

4) Every non-zero element in R is unit.

Let $a \in R$. Then, for some $b \in R$, $aba = a$. Since $1 \in R$, $aba - a = a(ba - 1) = 0$.

Since R has no zero divisor, $ba = 1$. □

Theorem 4.7.0.2. Every characteristic of an integral domain R is 0 or prime number p .

Proof. If characteristic of R is zero, then there is nothing to prove.

Suppose that characteristic of R is $n \in \mathbb{N}$.

If $n \in \mathbb{R}$ is not a prime number, then there exist two integer $a, b \geq 2$ such that $n = ab$.

Then, $n \cdot 1 = (ab) \cdot 1 = (a \cdot 1)(b \cdot 1) = 0$. Since R has no zero divisor, either $a \cdot 1 = 0$ or $b \cdot 1 = 0$. Contradiction. □

Theorem 4.7.0.3. Let R and R' be a Ring, and $\varphi : R \rightarrow R'$ is Ring-Homomorphism with $\varphi[R] \neq \{0'\}$.

If R has identity and R' has no zero divisor, then $\varphi(1)$ is identity of R' .

Proof. Proof consists by two step:

1. $\varphi[R]$ has a identity.

For any $\varphi(r) \in \varphi[R]$, $\varphi(1)\varphi(r) = \varphi(1r) = \varphi(r) = \varphi(r1) = \varphi(r)\varphi(1)$.

2. $\varphi(1)$ is identity of R' .

Since $\varphi(1)\varphi(1) = \varphi(11) = \varphi(1)$, $\varphi(1)$ is idempotent element in R' . By lemma, $\varphi(1)$ is identity of R' . □

Lemma 4.7.0.1. Let R be a Ring which has no zero divisor. If non-zero $a \in R$ s.t $a^2 = a$, then a is identity.

Proof. Let $r \in R$ be given. Since R has no zero divisor,

$$\begin{aligned} ra &= ra^2 \implies (r - ra)a = 0 \implies r = ra \\ ar &= a^2r \implies a(r - ar) = 0 \implies r = ar \end{aligned}$$

Thus, a is identity. □

4.8 Operation of Ideals

Chapter 5

Polynomial Ring Theory

Definition 5.0.0.1. Let R be a Commutative Ring with unity. Define Polynomial Ring:

$$R[x] \stackrel{\text{def}}{=} \left\{ \sum_{i=0}^n a_i x^i \mid n \geq 0, a_i \in R \right\}$$

Addition defined by pointwise, and Multiplication defined by:

$$\left(\sum_{i=0}^n a_i x^i \right) \times \left(\sum_{i=0}^m b_i x^i \right) = \sum_{k=0}^{n+m} \left(\sum_{i=0}^k a_i b_{k-i} \right) x^k$$

Proposition 5.0.0.1. Let R be an integral domain, and $p, q \in R[x]$ be non-zero elements.

1. $\deg(pq) = \deg p + \deg q$.
2. $R[x]$ is an integral domain.
3. If $p \in R[x]$ is unit, then $\deg p = 0$ and p is unit in R .

Proof.

$$\left(\sum_{i=0}^n a_i x^i \right) \left(\sum_{i=0}^m b_i x^i \right) = a_n b_m x^{n+m} + \dots,$$

This proves the statement (1) immediately, and assume not zero, then proves 2).
And, if $p \in R[x]$ is unit, then

$$0 = \deg 1 = \deg(pp^{-1}) = \deg p + \deg p^{-1}$$

Thus, $\deg p = \deg p^{-1} = 0$ and this implies $p, p^{-1} \in R$. □

5.1 Basic Theorems

Theorem 5.1.0.1. Let I be an ideal of the Commutative Ring R with unity, and $(I) \subseteq R[x]$. Then, $R[x]/(I) \cong (R/I)[x]$. In particular, if I is prime ideal in R , then (I) is prime ideal in $R[x]$.

Proof. First, establish that $(I) = I[x]$. Since properties of Ideal, $(I) = IR[x] = I[x]$ directly. Now, define a map $\varphi : R[x] \rightarrow (R/I)[x] : \sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n (a_i + I)x^i$. Then, φ is homomorphism with $\ker \varphi = I[x]$. The first-iso. Thm gives $R[x]/(I) = R[x]/I[x] \cong (R/I)[x]$. Particular,

$$I \text{ prime ideal} \iff R/I \text{ integral domain} \implies (R/I)[x] = R[x]/(I) \text{ integral domain} \iff (I) \text{ prime ideal.}$$

□

Theorem 5.1.0.2. If F is a field, then $F[x]$ is Euclidean domain.

Specifically, assume R is Commutative Ring with unity, $f, g \in R[x]$ with $\deg f, \deg g \geq 0$. If leading coefficient of g is unit in R , then there exists unique $q, r \in R[x]$ such that

$$f(x) = g(x)q(x) + r(x) \quad (\deg r(x) < \deg g(x))$$

Proof. If $\deg f < \deg g$, put $g(x) = 0$ and $r(x) = f(x)$. Then proved. Suppose that $\deg f \geq \deg g$, and using induction. If $\deg f = 0$, then put $g(x) = 0$, write leading coefficient of g as b and of f as a . Then, put $q = b^{-1}a$, $r = 0$. If $\deg f \geq 1$, put $n = \deg f$, $m = \deg g$. Then $n \geq m$. Write:

$$\begin{aligned} f(x) &= a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \\ g(x) &= b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0 \end{aligned}$$

Then, by induction,

$$\begin{aligned} f(x) &= a_n b_m^{-1} x^{n-m} g(x) + f_1(x) && (\deg f_1 < n - 1) \\ &= a_n b_m^{-1} x^{n-m} g(x) + q_1(x)g(x) + r(x) && (\deg r < \deg g) \\ &= (a_n b_m^{-1} x^{n-m} q_1(x))g(x) + r(x) && (\deg r < \deg g) \end{aligned}$$

To show the uniqueness,

$$f = q_1 g + r_1 = q_2 g + r_2 \implies g(q_1 - q_2) = r_2 - r_1 \implies \deg g(q_1 - q_2) = \deg g + \deg(q_1 - q_2) = \deg(r_2 - r_1) < \deg g$$

□

5.2 Relations between Rings and Their Polynomial Rings

Lemma 5.2.0.1. Gauss's Lemma

Let R be a Unique Factorization Domain with field of fractions F .

If $p(x) \in R[x]$ is reducible in $F[x]$, then $p(x)$ is reducible in $R[x]$.

Proof. Let $p(x) \in R[x]$ be reducible in $F[x]$. i.e.,

$p(x) = A(x)B(x)$ for some $A(x), B(x) \in F[x]$ with $A(x), B(x)$ are both non-zero and non-units.

Both $\deg A$ and $\deg B$ are at least 1: if either degree were zero, then lie in F , hence a unit - contradiction.

Write $A(x) = \sum_{i=0}^n \frac{r_i}{a_i} x^i$ and $B(x) = \sum_{i=0}^m \frac{s_i}{b_i} x^i$, and put $d_1 = a_1 \cdots a_n$, $d_2 = b_1 \cdots b_m$.

Now, $d_1 d_2 p(x) = d_1 A(x) d_2 B(x)$ where $d_1 A(x), d_2 B(x) \in R[x]$.

If $d = d_1 d_2$ is unit in R , then $p(x) = (d^{-1} d_1 A(x))(d_2 B(x))$ where $d^{-1} d_1 A(x), d_2 B(x) \in R[x]$, both are non-unit.

Suppose that d is not unit. Write $d = p_1 p_2 \cdots p_n$ is factorization of d .

p_1 is prime, being irreducible in U.F.D. $(p_1) = p_1 R[x]$ is prime, $R[x]/(p_1 R[x]) \cong (R/(p_1 R))[x]$ is an integral domain.

Since $d p(x) = p_1 \cdots p_n p(x) \in p_1 R[x]$,

$$\bar{0} = d p(x) + p_1 R[x] = d_1 A(x) d_2 B(x) + p_1 R[x] = \overline{d_1 A(x)} \times \overline{d_2 B(x)}$$

Since $p_1 R[x]$ is an integral domain, either $\overline{d_1 A(x)}$ or $\overline{d_2 B(x)}$ is zero. WLOG, let $\overline{d_1 A(x)} = d_1 A(x) + p_1 R[x] = \bar{0}$. This means all coefficient of $d_1 A(x)$ lies in $p_1 R$. Thus, we can cancel p_1 in the equation $d p(x) = d_1 A(x) d_2 B(x)$. In finite process, we obtain $p(x) = A'(x) B'(x)$ where $A'(x), B'(x) \in R[x]$ with

$$A'(x) = r A(x), \quad B'(x) = s B(x) \text{ where } r, s \in F$$

□

Corollary 5.2.0.1. Let R be a Unique Factorization Domain with field of fractions F .

Suppose that the greatest common divisor of the coefficients of $p(x) \in R[x]$ is 1. Then,

$p(x)$ is irreducible in $R[x]$ if and only if $p(x)$ is irreducible in $F[x]$

In particular, if $p(x)$ is an irreducible monic polynomial in $R[x]$, then it is also irreducible in $F[x]$.

Proof. By Contraposition of Gauss's Lemma, if $p(x)$ is irreducible in $R[x]$, then $p(x)$ is irreducible in $F[x]$. Conversely, suppose that $p(x)$ is reducible in $R[x]$, and the greatest common divisor of coefficients of $p(x)$ is 1. Write $p(x) = a(x)b(x)$ where neither $a(x)$ nor $b(x)$ are not unit in $R[x]$, being reducible. And, both $a(x)$ and $b(x)$ are not constant: because g.c.d. is 1. Thus, both are not unit in $F[x]$.

□

Theorem 5.2.0.1. R is Unique Factorization Domain if and only if $R[x]$ is Unique Factorization Domain.

Proof. Suppose that R is Unique Factorization Domain with field of fractions F .

Let $p(x) \in R[x]$ be non-zero element, and $d \in R$ be the greatest common divisor of coefficients of $p(x)$.

Then, $p(x) = dp'(x)$ where g.c.d. of coefficient of $p'(x)$ is 1. More precisely, write $p(x) = \sum_{i=0}^n a_i x^i$, ($a_i \in R$).

$$p(x) = \sum_{i=0}^n a_i x^i = \sum_{i=0}^n d a'_i x^i = d \left(\sum_{i=0}^n a'_i x^i \right)$$

for some $a'_i \in R$ such that $a_i = da'_i$. Put g.c.d of a'_i 's to $d' \in R$. Then, $a_i = da'_i = dd'a''_i$.

This implies dd' divides every a_i ; hence dd' divides d . That is, d' is unit, thus d' must be 1.

Since $F[x]$ is U.F.D, let $p'(x) = p_1(x)p_2(x) \cdots p_n(x)$ be a factorization of $p(x)$ in $F[x]$.

The g.c.d of $p'(x)$ is 1, thus g.c.d. of each $p_i(x)$ is 1.

Now, the corollary of the Gauss's Lemma gives that every $p_i(x)$ is irreducible in $R[x]$.

Hence, $p'(x) = p_1(x)p_2(x) \cdots p_n(x)$ is irreducible factorization in $R[x]$. To show that uniqueness, let

$$p'(x) = p_1(x) \cdots p_n(x) = q_1(x) \cdots q_m(x)$$

are two irreducibles factorizations of $p'(x)$ in $R[x]$. Since g.c.d of $p'(x)$ is 1, each $p_i(x)$ and $q_j(x)$ have g.c.d. 1.

Since the corollary of the Gauss's Lemma, all factors are irreducibles in $F[x]$ and $F[x]$ is U.F.D, $n = m$.

Moreover, each $p_i(x)$ and $q_i(x)$ are associates in $F[x]$ (index rearrangement). Since associates up to unit in $F[x]$,

$$p_i(x) = \frac{a}{b} q_i(x) \quad \text{for some } a, b \in R^\times$$

That is, $bp_i(x) = aq_i(x)$; g.c.d. of left polynomial is b , and g.c.d. of right polynomial is a .

In integral domain, g.c.d. is unique up to unit, $a = ub$ for some unit $u \in R^\times$. That is,

$$bp_i(x) = aq_i(x) = ubq_i(x) \implies p_i(x) = uq_i(x)$$

Proof complete. □

Theorem 5.2.0.2. Let R be a Commutative Ring with identity.

If $R[x]$ is Principal Ideal Domain, then R is a Field.

5.3 Irreducibility Criteria

Lemma 5.3.0.1. Let F be a field, and $p(x) \in F[x]$.

$p(x)$ has a factor of degree one if and only if $p(x)$ has a root in F

Proof. Trivial. □

Lemma 5.3.0.2. Let F be a field, and $p(x) \in F[x]$ be a polynomial of degree 2 or 3.

$p(x)$ is reducible if and only if $p(x)$ has a root in F .

Proof. Trivial. □

Theorem 5.3.0.1. Let $p(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in \mathbb{Z}[x]$.

If $\frac{r}{s} \in \mathbb{Q}$ is a root of $p(x)$ where r and s are relatively prime, then $r \mid a_0$ and $s \mid a_n$.

In particular, if $p(x) \in \mathbb{Z}[x]$ is monic polynomial and $p(d) \neq 0$ for all $d \mid p(0)$, then $p(x)$ has no root in \mathbb{Q} .

Proof. By hypothesis,

$$\begin{aligned} p\left(\frac{r}{s}\right) &= a_n\left(\frac{r}{s}\right)^n + a_{n-1}\left(\frac{r}{s}\right)^{n-1} + \dots + a_1\left(\frac{r}{s}\right) + a_0 = 0 \\ \implies a_nr^n + a_{n-1}r^{n-1}s + \dots + a_1rs^{n-1} + a_0s^n &= 0 \\ \implies a_nr^n &= -(a_{n-1}r^{n-1}s + \dots + a_1rs^{n-1} + a_0s^n) = s(a_{n-1}r^{n-1} + \dots + a_1rs^{n-2} + a_0s^{n-1}) \end{aligned}$$

Hence, s divides a_nr^n and s and r are relatively prime, $s \mid a_n$. Similarly, $r \mid a_0$.

And, the second assertion is clear by contraposition. □

Theorem 5.3.0.2. Let $I \subsetneq R$ be a proper ideal of integral domain R , and $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in R[x]$. If the image of $p(x)$

$$\tilde{p}(x) = x^n + (a_{n-1} + I)x^{n-1} + \dots + (a_1 + I)x + (a_0 + I) \in (R/I)[x]$$

cannot be factored two polynomials into two smaller degree,
then $p(x)$ is irreducible in $R[x]$.

Proof. Suppose that $p(x)$ is reducible in $R[x]$.

Then, there exist two monic $a(x), b(x) \in R[x]$ with $\deg a, \deg b \geq 1$ such that $p(x) = a(x)b(x)$.

But, $p(x) + I[x] = a(x)b(x) + I[x] = (a(x) + I[x]) \times (b(x) + I[x])$ is still reducible in $(R/I)[x]$, because:

leading coefficient of a_n is 1 $\implies a(x) + I[x]$ is not constant, being unit 1 cannot be in the proper ideal I . □

Theorem 5.3.0.3. Eisenstein's Criterion

Let P be a prime ideal of the integral domain R , and $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 \in R[x]$, ($n \geq 1$).

Suppose a_{n-1}, \dots, a_1, a_0 are contained in P and a_0 is not contained in P^2 . Then, $f(x)$ is irreducible in $R[x]$.

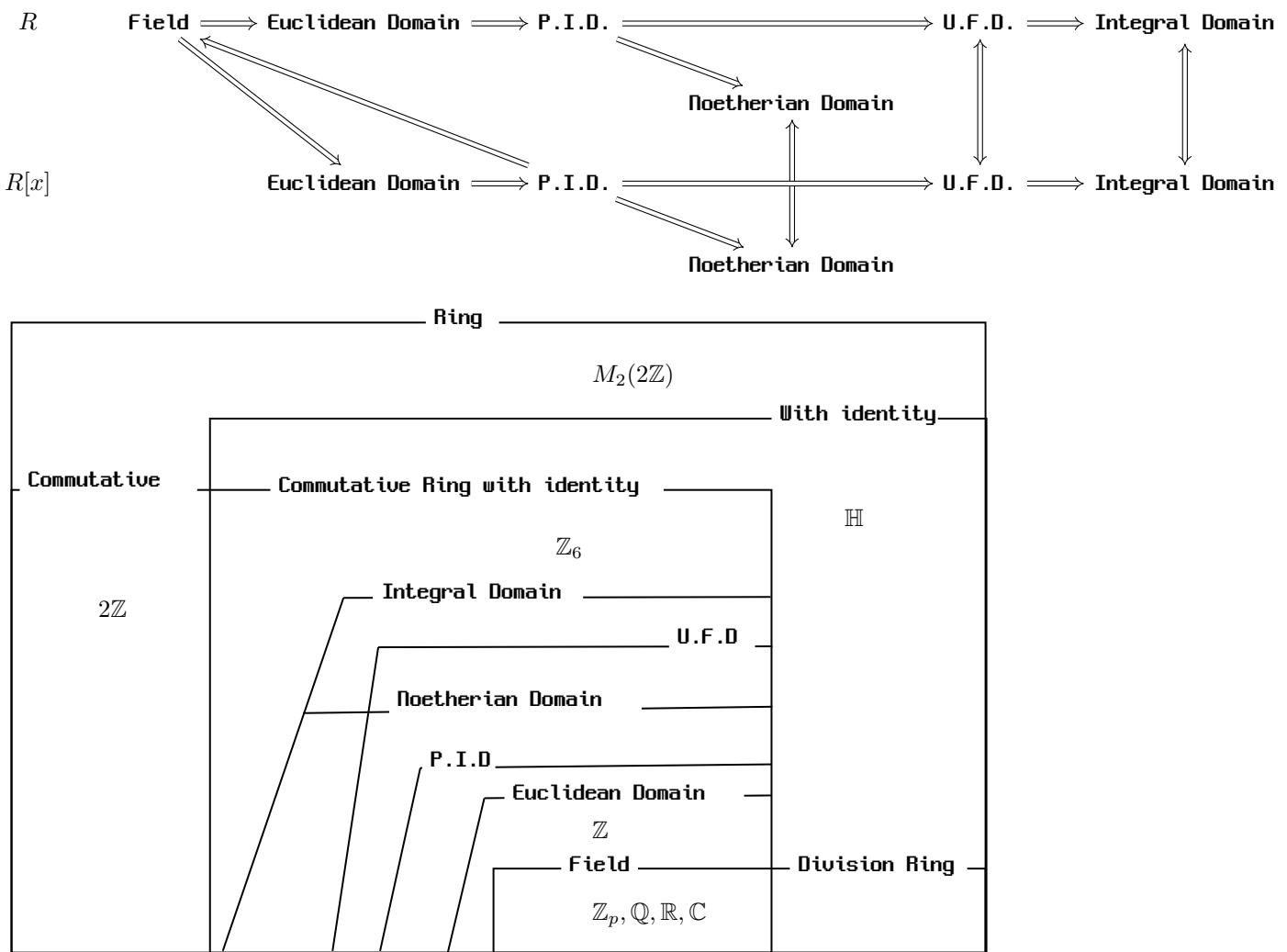
Proof. Proof by Contradiction. Suppose $f(x)$ is reducible. Then, $f(x) = a(x)b(x)$ for some nonconstant $a(x), b(x)$ in $R[x]$ (If it is constant, then contradicts to monic.) Observe that: In $(R/P)[x]$,

$$(1+P)x^n + (a_{n-1}+P)x^{n-1} + \dots + (a_1+P)x + (a_0+P) = x^n = \overline{a(x)b(x)} \in (R/P)[x] \cong R[x]/P[x]$$

Thus, the constant terms of $a(x)$ and $b(x)$ both are contained in P .

But this implies that the product of these two constants is contained in P^2 , contradiction. □

5.4 Summary and Diagram



In this section, we find and describe all examples and counterexamples in the diagram.

5.5 Examples

5.5.1 Quadratic Field and Quadratic integer Ring

Definition 5.5.1.1. Let $D \in \mathbb{Q}$ be a rational number that is not a perfect square in \mathbb{Q} . Define the *Quadratic Field* for D :

$$\mathbb{Q}(\sqrt{D}) \stackrel{\text{def}}{=} \{a + b\sqrt{D} \mid a, b \in \mathbb{Q}\} \subset \mathbb{C}$$

Proposition 5.5.1.1. $\mathbb{Q}(\sqrt{D})$ is a Field.

Proof. Let $a + b\sqrt{D} \in \mathbb{Q}(\sqrt{D})$. Then, $(a + b\sqrt{D})^{-1} = \frac{a - b\sqrt{D}}{a^2 - b^2D}$ is inverse. □

5.5.2 Cyclotomic Polynomial

Definition 5.5.2.1. Let $p \in \mathbb{Z}$ be a prime. Define p^{th} Cyclotomic Polynomial:

$$\Phi_p(x) \stackrel{\text{def}}{=} \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1 \in \mathbb{Z}[x]$$

Theorem 5.5.2.1. p^{th} Cyclotomic Polynomial is irreducible in $\mathbb{Z}[x]$.

Proof. Observe that

$$\Phi_p(x+1) = \frac{(x+1)^p - 1}{x} = \frac{1}{x} \cdot \left(x^p + \binom{p}{1}x^{p-1} + \cdots + \binom{p}{p-1}x + 1 - 1 \right) = x^{p-1} + \binom{p}{1}x^{p-2} + \cdots + p$$

is irreducible in $\mathbb{Q}[x]$ and $\mathbb{Z}[x]$ by *Eisenstein's Criterion*. Using this fact, suppose that $\Phi_p(x)$ is reducible. That is, there exist non-constant $r(x), s(x) \in \mathbb{Z}[x]$ such that $\Phi_p(x) = r(x)s(x)$. Then,

$$\Phi_p(x+1) = r(x+1)s(x+1)$$

is reducible, contradiction. □

5.6 † Rigorously Definition

Definition 5.6.0.1. Suppose that $(R, +, \cdot)$ is a Ring. Define a *Polynomial Ring* is:

$$R[x] \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} \{\{a_i\}_{i=0}^{\infty} \mid a_1, \dots, a_n \in R, a_{n+1} = a_{n+2} = \dots = 0_R\}$$

Chapter 6

Field Theory

Lemma 6.0.0.1. Suppose that $\varphi : F \rightarrow F'$ is a Field-Homomorphism. If φ is not zero map, then φ is injective.

Proof. Since φ is not zero map, $\varphi(a) \neq 0_{F'}$ for some $a \in F$. Thus, $\ker \varphi \subsetneq F$. But, the field F has only ideal for $\{0\}$ or F , hence the ideal $\ker \varphi$ must be $\{0\}$. Now,

$$\varphi(a) = \varphi(b) \implies \varphi(a - b) = 0 \implies a - b = 0$$

□

Lemma 6.0.0.2. Suppose that F is a field.

If $p(x) \in F[x]$ is irreducible in $F[x]$, then $F[x]/(p(x))$ is a field.

Proof. Since F is a field, $F[x]$ is Euclidean domain, P.I.D., and U.F.D. Thus,

$$p(x) \text{ irreducible} \xrightarrow{\text{U.F.D.}} (p(x)) \text{ prime} \xrightarrow{\text{P.I.D.}} (p(x)) \text{ maximal in } F[x]$$

Hence, $K = F[x]/(p(x))$ becomes a field.

□

Theorem 6.0.0.1. Suppose that F is a field, and $p(x) \in F[x]$ is irreducible.

Then, $K \stackrel{\text{def}}{=} F[x]/(p(x))$ is a field containing an isomorphic copy of F , and $p(x)$ has a root α in K .

Proof. Since F is a field, $F[x]$ is Euclidean domain, P.I.D., and U.F.D. Thus,

$$p(x) \text{ irreducible} \xrightarrow{\text{U.F.D.}} (p(x)) \text{ prime} \xrightarrow{\text{P.I.D.}} (p(x)) \text{ maximal in } F[x]$$

Hence, $K = F[x]/(p(x))$ becomes a field. Consider the Canonical projection

$$\pi : F[x] \rightarrow F[x]/(p(x)) : f(x) \mapsto f(x) + (p(x))$$

and restriction over F , $\varphi = \pi_F : F \rightarrow F[x]/(p(x))$ is Homomorphism with $\ker \varphi = \{0\}$, because $\varphi(1) + (p(x)) \neq (p(x))$. Now, $F \cong F/\ker \varphi \cong \varphi[F] \subseteq K = F[x]/(p(x))$. Further, denote $\bar{x} \stackrel{\text{def}}{=} \pi(x) = x + (p(x))$, and $p(x) = \sum_{i=0}^n a_i x^i$. Then,

$$p(\bar{x}) = \sum_{i=0}^n \bar{a}_i \bar{x}^i = \sum_{i=0}^n \pi(a_i) \pi(x)^i = \pi \left(\sum_{i=0}^n a_i x^i \right) = \pi(p(x)) = \overline{p(x)} = \bar{0} \in K$$

□

Theorem 6.0.0.2. Suppose that F is a field, and $p(x) \in F[x]$ is irreducible. If K is an Extension field of F containing a root $\alpha \in K$ of $p(x)$, then $F(\alpha) \cong F[x]/(p(x))$.

Proof. Define a map $\varphi_\alpha : F[x]/(p(x)) \rightarrow F(\alpha) : f(x) + (p(x)) \mapsto f(\alpha)$.

Well-defined: Suppose that $f(x) + (p(x)) = g(x) + (p(x))$. Then, $\varphi_\alpha(f(x) + (p(x))) = f(\alpha)$ and $\varphi_\alpha(g(x) + (p(x))) = g(\alpha)$.

$$f(x) + (p(x)) = g(x) + (p(x)) \iff f(x) - g(x) \in (p(x))$$

$$f(x) - g(x) \in (p(x)) \implies f(x) - g(x) = p(x)r(x) \text{ for some } r(x) \in F[x] \implies f(\alpha) - g(\alpha) = p(\alpha)r(\alpha) = 0 \cdot r(\alpha) = 0.$$

Ring-Homomorphism:

$$1. \varphi([f(x) + (p(x))] + [g(x) + (p(x))]) = \varphi(f(x) + g(x) + (p(x))) = f(\alpha) + g(\alpha) = \varphi(f(x) + (p(x))) + \varphi(g(x) + (p(x))).$$

$$2. \varphi([f(x) + (p(x))] \cdot [g(x) + (p(x))]) = \varphi(f(x)g(x) + (p(x))) = f(\alpha)g(\alpha) = \varphi(f(x) + (p(x))) \cdot \varphi(g(x) + (p(x))).$$

Injectivity: Since φ is a non-trivial homomorphism from field into field, thus $\ker \varphi = \{0\}$.

(More precisely, $\varphi(1) = 1 + (p(x)) \neq (p(x))$ because $1 \notin (p(x))$. Thus, $\ker \varphi \subsetneq F[x]/(p(x)) \implies \ker \varphi = \{0\}$.)

Now, the φ becomes an Embedding:

$$\varphi[F[x]/(p(x))] \cong F[x]/(p(x))$$

Since $F[x]/(p(x))$ is a field, the image $\varphi[F[x]/(p(x))]$ is a field, being it is isomorphic copy.

The definition of $F(\alpha)$ gives $\varphi[F[x]/(p(x))] \supset F(\alpha)$, and $\varphi[F[x]/(p(x))] \subset F(\alpha)$ is trivial. Consequently,

$$F(\alpha) = \varphi[F[x]/(p(x))] \cong F[x]/(p(x))$$

□

Theorem 6.0.0.3. Suppose that $p(x) \in F[x]$ is an irreducible with $\deg p(x) = n$, and $K = F[x]/(p(x))$.

Put $\bar{x} = x + (p(x)) \in K$. Then, $\{1, \bar{x}, \bar{x}^2, \dots, \bar{x}^{n-1}\}$ is basis for K as Vector space over F .

Proof. Above lemma guarantees $K = F[x]/(p(x))$ is a field.

1) Linealy independent.

Suppose that $a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1} = 0$ in K where a_i are not all zero. Then,

$$\begin{aligned} 0 &= a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1} \\ &= a_0 + a_1(x + (p(x))) + \dots + a_{n-1}(x^{n-1} + (p(x))) \\ &= a_0 + a_1x + \dots + a_{n-1}x^{n-1} + (p(x)) \\ &\iff a_0 + a_1x + \dots + a_{n-1}x^{n-1} = p(x)b(x) \text{ for some } b(x) \in F[x] \end{aligned}$$

But, the left term has a degree at most $n-1$, and the right term has a degree at least n . Contradiction.

2) $F[x]$ is generated by $\{1, \bar{x}, \dots, \bar{x}^{n-1}\}$.

Let $a(x) \in F[x]$. Then, the Euclidean algorithm for $F[x]$ gives that: there exist uniquely $q(x), r(x) \in F[x]$ s.t

$$a(x) = q(x)p(x) + r(x) \text{ with } \deg r(x) < \deg p(x)$$

Since $q(x)p(x) \in (p(x))$, $a(x) + (p(x)) = r(x) + (p(x))$, the degree of $a(x) + (p(x))$ is smaller than n .

Now, for any $a(x) + (p(x)) \in K = F[x]/(p(x))$, $a(x) + (p(x)) \in \langle \{1, \bar{x}, \dots, \bar{x}^{n-1}\} \rangle$.

□

Corollary 6.0.0.1. Suppose that F is a field, and $p(x) \in F[x]$ is irreducible polynomial with $\deg p(x) = n$.

If E is an Extension field of F containing α as a root of $p(x)$. Then,

$$F(\alpha) = \{a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1} \mid a_i \in F\} \subseteq E$$

Proof. Since $F[x]/(p(x)) = \{a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1} \mid a_i \in F\} \cong F(\alpha)$, being the isomorphism

$$\varphi_\alpha : F[x]/(p(x)) \rightarrow F(\alpha) : f(x) + (p(x)) = \overline{f(x)} \mapsto f(\alpha)$$

Thus, combining with above fact,

$$\begin{aligned} F(\alpha) &= \varphi_\alpha[F[x]/(p(x))] = \{\varphi_\alpha(a_0 + a_1\bar{x} + \dots + a_{n-1}\bar{x}^{n-1}) \mid a_i \in F\} \\ &= \{\varphi_\alpha(a_0) + \varphi_\alpha(a_1\bar{x}) + \dots + \varphi_\alpha(a_{n-1}\bar{x}^{n-1}) \mid a_i \in F\} \\ &= \{a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_{n-1}\alpha^{n-1} \mid a_i \in F\} \end{aligned}$$

□

6.1 Extension Field

Definition 6.1.0.1. Suppose that F is a field, and K be a field containing F as a subfield. The element $\alpha \in K$ is called **algebraic** over F if:

There exists a non-zero polynomial $f(x) \in F[x]$ such that $f(\alpha) = 0$ in K .

If not algebraic, it is called **transcendental** over F .

Theorem 6.1.0.1. Suppose that $\alpha \in K$ is algebraic over F .

Then, there exists a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ such that $m_{\alpha,F}(\alpha) = 0$. Moreover,

$f(x) \in F[x]$ has α as a root if and only if $m_{\alpha,F}(x) \in F[x]$ divides $f(x)$ in $F[x]$.

Proof. Since $\alpha \in K$ is algebraic over F , the Well-Ordering principle gives the existence of minimal degree polynomial $g(x) \in F[x]$ having α as a root. More precisely,

$$S = \{\deg f(x) \in \mathbb{N} \mid f(x) \in F[x] \text{ s.t } f(\alpha) = 0\}$$

is non-empty since α is algebraic. Therefore, the Well-Ordering Principle gives that S has a minimum $n \in \mathbb{N}$. Now, we can choose $g(x) \in F[x]$ with $\deg g(x) = n$ and $g(\alpha) = 0$.

Using Contradiction: Suppose that $g(x)$ is reducible. That is, $g(x) = a(x)b(x)$ with $\deg a, \deg b < \deg g$.

Since $g(\alpha) = a(\alpha)b(\alpha) = 0$ and K is a field, either $a(\alpha)$ or $b(\alpha)$ must be zero.

This contradicts with $n \in \mathbb{N}$ is the minimum degree of polynomial which has α as a root.

In summary:

If $g(x) \in F[x]$ is a polynomial of minimal degree which has α as a root, then $g(x)$ is irreducible over F .

Now, Suppose that $f(x) \in F[x]$ is any polynomial having α as a root. Then, $\deg f(x) \geq \deg g(x)$ by setting.

By the Euclidean Algorithm in $F[x]$, there exist unique polynomials $q(x), r(x) \in F[x]$ such that

$$f(x) = q(x)g(x) + r(x) \text{ with } \deg r(x) < \deg g(x)$$

Meanwhile,

$$0 = f(\alpha) = q(\alpha)g(\alpha) + r(\alpha) = q(\alpha) \cdot 0 + r(\alpha) = r(\alpha)$$

$r(\alpha) = 0$ implies $r(x)$ must be zero, by minimality. Hence, $g(x)$ divides $f(x)$ in $F[x]$.

Finally, put $a \in F$ be a leading coefficient of $g(x)$. Then, $m_{\alpha,F}(x) = a^{-1}g(x)$ is monic polynomial.

And, Uniqueness given by: for any $f(x) \in F[x]$, $f(\alpha) = 0 \implies m_{\alpha,F}(x) \mid f(x)$. □

Corollary 6.1.0.1. If L is an Extension of field F and $\alpha \in L$ is algebraic over both F and L , then $m_{\alpha,L}(x)$ divides $m_{\alpha,F}(x)$ in $L[x]$.

Above theorem allows defining:

Definition 6.1.0.2. Suppose that K is an Extension field of a field F , and $\alpha \in K$ is algebraic over F .

The monic irreducible polynomial which has α as root is called **the minimal polynomial** for α over F .

Denote this $m_{\alpha,F}(x) \in F[x]$ or $\text{irr}(\alpha, F)$.

Proposition 6.1.0.1. Suppose that F is a field and E is an Extension of F .

If $\alpha \in E$ is algebraic over F , then $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$ and

$$[F(\alpha) : F] \stackrel{\text{def}}{=} \dim_F F(\alpha) = \deg m_{\alpha,F}(x) \stackrel{\text{def}}{=} \deg \alpha$$

Theorem 6.1.0.2. Suppose that E is an Extension field of a field F , and $\alpha \in E$.

$\alpha \in E$ is algebraic over F if and only if $[F(\alpha) : F]$ has finite dimension

Proof. If $\alpha \in E$ is algebraic over F , then $[F(\alpha) : F] = \deg m_{\alpha, F}(x)$, thus finite.

Precisely, if $f(x) \in F[x]$ satisfies $\deg f(x) = n$ and $f(\alpha) = 0$, then $[F(\alpha) : F] = \deg_F \alpha \leq n$.

Conversely, suppose $\alpha \in E$ where $[F(\alpha) : F] = n$. Then, the $n+1$ elements

$$1, \alpha, \alpha^2, \dots, \alpha^n$$

must be linearly dependent, thus for some not all zero elements $b_0, \dots, b_n \in F$,

$$b_0 + b_1\alpha + b_2\alpha^2 + \dots + b_n\alpha^n = 0$$

Now, the polynomial in $F[x]$

$$f(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$$

satisfies $f(\alpha) = 0$ in $F(\alpha)$. In summary, if $[F(\alpha) : F] = n$, then $\alpha \in E$ is algebraic over F with $\deg_F \alpha \leq n$. \square

Corollary 6.1.0.2. If E is an Extension field of F with $[E : F]$ has finite, then E is algebraic over F .

Theorem 6.1.0.3. Suppose that $F \subseteq L \subseteq E$ are fields. Then,

$$[E : F] = [E : L][L : F]$$

Proof. Suppose that $[E : L]$ and $[L : F]$ are finite. Put $[E : L] = m$ and $[L : F] = n$.

Set $A = \{\alpha_1, \alpha_2, \dots, \alpha_m\} \subset E$ and $B = \{\beta_1, \beta_2, \dots, \beta_n\} \subset L$ are basis of E over L and L over F , respectively.

Let $x \in E$ be given. Since A is basis for E over L , there exists a unique linear combination: for $a_i \in L$,

$$x = a_1\alpha_1 + a_2\alpha_2 + \dots + a_m\alpha_m = \sum_{i=1}^m a_i\alpha_i$$

Meanwhile, for each $a_i \in L$, $1 \leq i \leq m$, there exist unique linear combinations: for $b_{i,j} \in F$,

$$a_i = b_{i,1}\beta_1 + b_{i,2}\beta_2 + \dots + b_{i,n}\beta_n = \sum_{j=1}^n b_{i,j}\beta_j$$

Combining above, we obtain

$$x = \sum_{i=1}^m \left(\sum_{j=1}^n b_{i,j}\beta_j \right) \alpha_i = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} b_{i,j}\beta_j \alpha_i$$

Since each $b_{i,j} \in F$ and $\beta_j \alpha_i \in E$, the subset $C = \{\beta_j \alpha_i \mid \alpha_i \in A, \beta_j \in B\} \subset E$ spans E over F .

Meanwhile, this C is linearly independent over F , because: Suppose that

$$\sum_{i=1}^m \left(\sum_{j=1}^n c_{i,j}\beta_j \right) \alpha_i = 0$$

Since each $c_{i,j}\beta_j \in L$ and $A = \{\alpha_i\}$ is linearly independent over L , for each $1 \leq i \leq m$,

$$\sum_{j=1}^n c_{i,j}\beta_j = 0$$

$c_{i,j} \in F$ and $B = \{\beta_j\}$ is linearly independent over F , all $c_{i,j} = 0$. Thus, C is basis which has mn elements.

Meanwhile, if $[L : F]$ is infinite, then $[E : F]$ is clearly infinite; being $E \subset L$. Similarly, $[E : L]$. \square

Theorem 6.1.0.4. Suppose that E is an algebraic Extension field of F . Then,

$$E \text{ is finite Extension} \iff \text{There exist finite elements } \alpha_1, \dots, \alpha_n \in E \text{ such that } E = F(\alpha_1, \dots, \alpha_n).$$

Proof. Suppose that E is finite Extension.

If $[E : F] = 1$, then $F(1) = F = E$, claim proved.

If $[E : F] > 1$, then $E \neq F$. Thus, put $\alpha_1 \in E \setminus F$. If $F(\alpha_1) = E$, then there is nothing to prove. If $F(\alpha_1) \neq E$, then put again $\alpha_2 \in E \setminus F(\alpha_1)$. Since $[E : F]$ is finite integer, this process will terminate in finite.

Thus for some $n \in \mathbb{N}$, $E = F(\alpha_1, \dots, \alpha_n)$.

Conversely, suppose that $E = F(\alpha_1, \dots, \alpha_n)$ for some $\alpha_i \in E$. Then,

$$E = [E : F(\alpha_n)][F(\alpha_n) : F(\alpha_{n-1})] \cdots [F(\alpha_1) : F]$$

Since E is algebraic, $[E : F(\alpha_n)], [F(\alpha_n) : F(\alpha_{n-1})], \dots, [F(\alpha_1) : F]$ are all finite. Thus $[E : F]$ is finite. \square

Theorem 6.1.0.5. If E is Algebraic over K and K is Algebraic over F . Then, E is Algebraic over F .

Proof. Let $\alpha \in E$ be given. Then, there exist not all zero $a_0, \dots, a_n \in K$ such that

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_1\alpha + a_0 = 0$$

Since each $a_i \in K$ is algebraic over F , thus $F(a_0, \dots, a_n)$ is finite over F , being above theorem. Further,

$$F(\alpha, a_0, \dots, a_n) = F(a_0, \dots, a_n)(\alpha)$$

is finite over $F(a_0, \dots, a_n)$, because above equation gives that α is algebraic over $F(a_0, \dots, a_n)$. Now,

$$[F(\alpha, a_0, \dots, a_n) : F] = [F(\alpha, a_0, \dots, a_n) : F(a_0, \dots, a_n)][F(a_0, \dots, a_n) : F]$$

is finite, thus algebraic over F . This means $\alpha \in E$ is algebraic over F . \square

6.2 Algebraic Closure

Lemma 6.2.0.1. Suppose that E is an Extension field of F . Then,

$$\overline{F}_E \stackrel{\text{def}}{=} \{\alpha \in E \mid \alpha \text{ is algebraic over } F\}$$

is a subfield of E . This \overline{F}_E is called *Algebraic Closure* of F in E .

Proof. Let $\alpha, \beta \in \overline{F}_E$ be given. Since α is algebraic over F , thus $F(\alpha)$ is finite extension of F . Then, $F(\alpha, \beta) = F(\alpha)(\beta)$ is finite over $F(\alpha)$, thus $F(\alpha, \beta)$ is finite extension over F , thus algebraic over F . This implies $F(\alpha, \beta) \subseteq \overline{F}_E$. Now, with assumption $\beta \neq 0$,

$$\alpha + \beta, \alpha\beta, \frac{\alpha}{\beta} \in F(\alpha, \beta) \subseteq \overline{F}_E$$

□

Definition 6.2.0.1. A field F is said to be *Algebraically Closed* if:

Every non-constant polynomial in $F[x]$ has a zero in F

Theorem 6.2.0.1. A field F is Algebraically Closed if and only if

every non-constant polynomial in $F[x]$ factors into linear factors within $F[x]$.

Proof. Suppose that F is Algebraically Closed.

Let $f(x) \in F[x]$ be given. Since $f(x)$ has a root in F , put $\alpha_0 \in F$ such that $f(\alpha_0) = 0$. Then,

$$f(x) = (x - \alpha_0)f_1(x)$$

And, $f_1(x) \in F[x]$, it has a root in F , put $\alpha_1 \in F$ such that $f_1(\alpha_1) = 0$. Now,

$$f(x) = (x - \alpha_0)(x - \alpha_1)f_2(x)$$

In finite process, we get:

$$f(x) = c(x - \alpha_0)(x - \alpha_1) \cdots (x - \alpha_{\deg f(x)})$$

Conversely argument is clear. □

Lemma 6.2.0.2. Suppose that F is a field, and $f(x) \in F[x]$ is non-constant polynomial. Then, there exists an Extension E such that $f(x)$ has a root in E .

Proof. Let $f(x) \in F[x]$. Since $F[x]$ is U.F.D., factors $f(x)$ into irreducibles in $F[x]$,

$$f(x) = f_1(x) \cdots f_n(x)$$

Now, for any $1 \leq j \leq n$, $F[x]/(f_j(x))$ is an Extension of F containing a root of $f(x)$. □

6.2.1 † Existence of Algebraic Closure

Definition 6.2.1.1. Suppose that F is a field. An Extension \bar{F} is called *Algebraic Closure* of F if:

1. \bar{F} is Algebraic over F .
2. Every nonconstant polynomial $f(x) \in F[x]$ factors completely into linear factors over \bar{F} .

Theorem 6.2.1.1. An Algebraic Clousre \bar{F} is Algebraically Closed.

Proof. Let $f(x) \in \bar{F}[x]$ be given. Then, there exists an Extension E of \bar{F} containing a root $\alpha \in E$ of $f(x)$. Now, $\bar{F}(\alpha) \subset E$ is algebraic over \bar{F} , and by definition, \bar{F} is algebraic over F . Thus $\bar{F}(\alpha)$ is algebraic over F . Particulary, α is algebraic over F , thus $\alpha \in \bar{F}$. \square

Theorem 6.2.1.2. For any field F , there exists an Algebraically closed field K containing F .

Proof. For every non-constant monic polynomial $f = f(x) \in F[x]$, let x_f denote an indeterminate. Denote that $M = \{f(x) \in F[x] \mid f(x) \text{ is monic and nonconstant}\}$. Consider the Polynomial Ring

$$F[\dots, x_f, \dots] \stackrel{\text{def}}{=} \bigcup_{\substack{\mathcal{F} \subset M \\ \mathcal{F} \text{ finite}}} F[x_f \mid f \in \mathcal{F}]$$

Consider the ideal $(\{f(x_f) \mid f \in M\}) \subseteq F[\dots, x_f, \dots]$. If this ideal is no proper, then

$$g_1(x_{f_1})f_1(x_{f_1}) + g_2(x_{f_2})f_1(x_{f_2}) + \dots + g_n(x_{f_n})f_n(x_{f_n}) = 1$$

for some g_1, \dots, g_n . \square

Theorem 6.2.1.3. Suppose that E is an Extension field of F , and Q is a Field of Fractions of $F[x]$. Then,

E is Not Algebraic over F if and only if There exists an Embedding $\varphi: Q \rightarrow E$ such that $\forall a \in F, \varphi(a) = a$.

6.3 Splitting Field

Definition 6.3.0.1. The extension field K of F is called a *splitting field* for $f(x) \in F[x]$ if:

K is the smallest extension of F s.t. $f(x) \in F[x]$ factors completely into linear factors in $K[x]$

More rigorously, there exist $a, \alpha_1, \dots, \alpha_n \in K$ such that $K = F(\alpha_1, \dots, \alpha_n)$ where

$$f(x) = a(x - \alpha_1) \cdots (x - \alpha_n)$$

Theorem 6.3.0.1. Suppose that F is a field, and $f(x) \in F[x]$ is a polynomial of degree $n \geq 1$. Then, there exists a splitting field K for $f(x)$ of F .

Proof. First, Claim: there exists an Extension field such that $f(x)$ factors completely into linear factors.

Using induction: For $n = 1$, Put $K = F$. Then claim proved immediately.

Suppose that for some $n \in \mathbb{N}$, the claim is true. Let $f(x) \in F[x]$ with $\deg f(x) = n + 1$.

If every irreducible factor of $f(x)$ has degree 1, then there is nothing to prove, being to put $K = F$.

Thus, suppose $p(x)$ is an irreducible factor of $f(x)$ such that $\deg p(x) \geq 2$. That is, $f(x) = p(x)q(x)$.

By Theorem, there exists an Extension E of F such that $p(x) \in E[x]$ has a root $\alpha \in E$.

In the extension K , $p(x)$ has a linear factor, thus the remaining factor has degree smaller than $n + 1$.

Now, by assumption of induction, $p(x)$ and $q(x)$ are factors completely into linear factors, so $f(x)$.

The smallestness given by intersection, these K . □

Chapter 7

Galois Theory

Chapter 8

Module Theory

Definition 8.0.0.1. Suppose that R be a Ring. An Abelian Group M is called **left-module over R** if:

The operation $\cdot : R \times M \rightarrow M : (r, m) \mapsto rm$ satisfies: for any $r, s \in R$, $n, n \in M$,

$$\begin{cases} (r + s)m = rm + sm \\ (rs)m = r(sm) \\ r(m + n) = rm + rn \end{cases}$$

Moreover, if the Ring R contains identity 1, then $1m = m$.

Chapter 9

Linear Algebra

9.1 Vector Space

Definition 9.1.0.1. Suppose that F is a Field, and V is an Abelian Group.

And, the operation $\cdot : F \times V \rightarrow V$ satisfies: For any $a, b \in F$ and $v, w \in V$,

$$\begin{cases} a \cdot (v + w) = a \cdot v + a \cdot w \\ (a + b) \cdot v = a \cdot v + b \cdot v \\ (ab) \cdot v = a \cdot (b \cdot v) \\ 1 \cdot v = v \end{cases}$$

The triple $(V, +, \cdot)$ is called the *Vector Space* over F .

Equivalently, The Vector Space over a field F is F -Module.

Definition 9.1.0.2. Suppose that V, W are Vector Space over a field F .

A map $\mathcal{L} : V \rightarrow W$ is called *Linear Map* if:

For any $a \in F$ and $v_1, v_2 \in V$, $\mathcal{L}(a \cdot v_1 + v_2) = a \cdot \mathcal{L}(v_1) + \mathcal{L}(v_2)$

Definition 9.1.0.3. Suppose that V is a Vector Space over a field F , and $W \subseteq V$ is a Subset.

The W is called *Subspace* of V if: $\begin{cases} \text{For any } a \in F, w \in W, a \cdot w \in W \\ \text{For any } w_1, w_2 \in W, w_1 + w_2 \in W \end{cases}$.

That is, the Subset of a Vector Space which is closed under the addition and scalar multiplication, then it is a Vector Space.

Lemma 9.1.0.1. Arbitrary intersection of Subspace is a Subspace.

Proof. Suppose that V is a Vector Space, and $W_\alpha \leq V$, $\alpha \in \Lambda$ are Subspaces.

Using Subspace Criterion: Let $a \in F$, $w_1, w_2 \in \bigcap_{\alpha \in \Lambda} W_\alpha$ be given. Then, for any $\alpha \in \Lambda$, $a \cdot w_1 + w_2 \in W_\alpha$. \square

This Lemma allows the definition:

Definition 9.1.0.4. Suppose that V is a Vector Space over a field F , and $S \subseteq V$ be a Subset.
Define a *Generated Subspace* by S is:

$$\langle S \rangle \stackrel{\text{def}}{=} \bigcap_{S \subseteq W \leq V} W$$

This $\langle S \rangle$ is the *unique smallest Subspace* containing S . This S is called *Generating Subset* of $\langle S \rangle$.

Lemma 9.1.0.2. Suppose that V is a Vector Space over a field F , and $S \subseteq V$ be a Subset. Then,

$$\langle S \rangle = \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$$

Proof. First, $\langle S \rangle \supseteq \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$, because $\langle S \rangle$ is closed under the opearations.

And, the set $\{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$ is a Subgroup of V containing S ,

Hence $\langle S \rangle \subseteq \{a_1 \cdot v_1 + \cdots + a_n \cdot v_n \mid n \in \mathbb{N}, a_i \in F, v_i \in S\}$. □

9.2 Linearly independent

Definition 9.2.0.1. Suppose that V is a Vector Space over a field F , and $S \subseteq V$ is a Subset. A Subset S is called *Linearly independent* if: For any finite subset $\{v_1, v_2, \dots, v_n\} \subseteq S$,

$$a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n = 0 \implies a_1 = a_2 = \dots = a_n = 0$$

If S is not Linearly independent, then it is called *Linearly dependent*.

Lemma 9.2.0.1. Suppose that $S \subseteq V$ is a Linearly independent subset of a Vector Space V over F . If $v \in V$ satisfies $v = a_1 \cdot v_1 + a_2 \cdot v_2 + \dots + a_n \cdot v_n$ for some $a_i \in F$ and $v_i \in S$, then this representation is unique. More precisely,

Proof. First, $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n$ implies $v \in \langle S \rangle$.

Now, suppose that $v \in V$ satisfies $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n = b_1 \cdot w_1 + \dots + b_m \cdot w_m$, WLOG $n \leq m$.

Put $I \stackrel{\text{def}}{=} \{i \in \mathbb{N} \mid \exists j \in \mathbb{N} \text{ s.t. } v_i = w_j\}$.

Then, there is a permutation $\rho : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\}$ such that $\forall i \in I, v_i = w_{\rho(i)}$. Now,

$$\begin{aligned} \sum_{i \in I} a_i \cdot v_i + \sum_{j \notin I} a_j \cdot v_j &= \sum_{i \in I} b_{\rho(i)} \cdot w_{\rho(i)} + \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} = \sum_{i \in I} b_{\rho(i)} \cdot v_i + \sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} \\ \implies \sum_{i \in I} (a_i - b_{\rho(i)}) \cdot v_i + \sum_{j \notin I} a_j \cdot v_j - \left(\sum_{j \notin I} b_{\rho(j)} \cdot w_{\rho(j)} \right) &= 0 \end{aligned}$$

Since S is linearly independent, for all $j \notin I, a_j = b_{\rho(j)} = 0$ and for all $i \in I, a_i = b_{\rho(i)}$. \square

This fact enables the definition in the next section.

9.3 Basis

Definition 9.3.0.1. Suppose that V is a Vector Space over a field F .

A subset $\beta \subseteq V$ is called the *Basis* of V if:

1. β is Linearly independent.
2. $\langle \beta \rangle = V$.

Lemma 9.3.0.1. Suppose that V is a Vector Space over a field F . Then,

$\beta \subset V$ is a Basis of $V \iff$ For any $v \in V$, there exists a Unique representation $v = a_1 \cdot v_1 + \dots + a_n \cdot v_n$.

9.4 † Existence of Basis

Theorem 9.4.0.1. Every Vector Space has a Basis.

Chapter 10

Category

Chapter 11

Exercise

27. 16

Find a Prime ideal of $\mathbb{Z} \times \mathbb{Z}$ that is not maximal.

Solution. Since $S \subset R$ is Prime ideal if and only if R/S is an integral domain, and $S \subset R$ is Maximal if and only if R/S is a field, we can choose $S = \{0\} \times \mathbb{Z}$ as a prime but not maximal because

$$(\mathbb{Z} \times \mathbb{Z})/(\{0\} \times \mathbb{Z}) \cong \mathbb{Z}$$

is an integral domain, but not a field. This isomorphic guarantees by:

$$\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} : (a, b) \mapsto a$$

is surjective, homomorphism, and $\ker \varphi = \{0\} \times \mathbb{Z}$.

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Following two theorems are equivalent:

Theorem 11.0.0.1. Fundamental Theorem of Algebra

If $f(x) \in \mathbb{C}[x]$ is non-constant polynomial, then $f(x)$ has a root in \mathbb{C} .

Theorem 11.0.0.2. Nullstellensatz for $\mathbb{C}[x]$

Suppose that $f_1(x), \dots, f_r(x) \in \mathbb{C}[x]$ and $g(x) \in \mathbb{C}[x]$.

If $\alpha \in \mathbb{C}$ is zero of all f_1, \dots, f_r implies $g(\alpha) = 0$, then for some $n \in \mathbb{N}$, $g(x)^n \in \langle f_1, \dots, f_r \rangle$.

Solution. Suppose that the Fundamental Theorem of Algebra is true.

Since \mathbb{C} is a field, every ideal in $\mathbb{C}[x]$ is principal. Thus, for some $p(x) \in \mathbb{C}[x]$,

$$\langle f_1(x), \dots, f_r(x) \rangle = \langle p(x) \rangle$$

By the Fundamental Theorem of Algebra, we can write: (WLOG, suppose $p(x)$ is monic)

$$p(x) = (x - \alpha_1)^{m_1} \cdots (x - \alpha_k)^{m_k}$$

where $m_i \in \mathbb{N}$. Now, since every $f_i(x)$ is divided by $p(x)$, thus all $f_i(x)$ has $\alpha_1, \dots, \alpha_k$ as roots, this implies $g(x)$ has $\alpha_1, \dots, \alpha_k$ as roots. This means $g(x) = q(x)(x - \alpha_1) \cdots (x - \alpha_k)$ for some $q(x) \in \mathbb{C}[x]$, thus $g(x)^n \in \langle p(x) \rangle$ where $n = m_1 m_2 \cdots m_r$.

Conversely, suppose that Nullstellensatz is true.

Let non-constant $f(x) \in \mathbb{C}[x]$ be given. Put $f_1(x) = (x - \alpha)$. Then, α is zero of $f_1(x)$ and zero of $(x - \alpha)f(x)$. By assumption, for some $n \in \mathbb{N}$, $f_1(x)^n \in \langle (x - \alpha)f(x) \rangle$. That is, $(x - \alpha)f(x)$ divides $f_1(x)^n = (x - \alpha)^n$. If $n = 1$, then $f(x)$ must be contradiction, this contradicts with f is non-constant. If $n > 1$, then $f(x)$ has $(x - \alpha)$ as a factor, thus α is zero of $f(x)$.

Chapter 12

General Topology

In this chapter, we follow the notations of [Steen et al., 1978, COUNTEREXAMPLES IN TOPOLOGY].

12.1 Basis

12.1.1 Subbasis

Definition 12.1.1.1. Let X be a set.

A collection $\mathcal{S} \subseteq \mathcal{P}(X)$ is called **subbasis** if: $X = \bigcup_{S \in \mathcal{S}} S$. (That is, $\forall x \in X, \exists S \in \mathcal{S}$ s.t. $x \in S$)

$\beta_{\mathcal{S}}$ is called **Basis generated by the subbasis \mathcal{S}** .

Note that: $\mathcal{T}_{\beta_{\mathcal{S}}}$ is the smallest Topology such that containing \mathcal{S} .

12.2 Topological Map

Definition 12.2.0.1. Let X, Y are Topological Space. $f : X \rightarrow Y$ is Continuous at $x_0 \in X$ if:
For any open $V \in \mathcal{T}_Y$ with $f(x_0) \in V$, there is an open $U \in \mathcal{T}_X$ with $x_0 \in U$ such that $f(U) \subset V$.

Definition 12.2.0.2. Let X, Y are Topological Space. Define:

1. $f : X \rightarrow Y$ is **Continuous Map** if: For any open $V \subset Y$, $f^{-1}[V] \subset X$ be open.
2. $f : X \rightarrow Y$ is **Open Map** if: For any open subset $A \subset X$, $f[A] \subset Y$ be open.
3. $f : X \rightarrow Y$ is **Closed Map** if: For any closed subset $B \subset X$, $f[B] \subset Y$ be closed.
4. $f : X \rightarrow Y$ is **Homeomorphism** if: f is bijection, continuous, and f^{-1} is continuous.

Theorem 12.2.0.1. Let $f : X \rightarrow Y$ be a Topological Map. Then, The Followings are Equivalent:

- a) f is Continuous Map.
- b) For any closed $C \subset Y$, $f^{-1}[C] \subset X$ be closed.
- c) For any subset $A \subset X$, $f[\overline{A}] \subset \overline{f[A]}$.
- d) For any subset $B \subset Y$, $f^{-1}[B^\circ] \subset (f^{-1}[B])^\circ$.

Proof.

- a) \Rightarrow b) Let $C \subset Y$ is closed. Then, $f^{-1}[Y \setminus C] = X \setminus f^{-1}[C]$ is open, thus $f^{-1}[C]$ is closed.
- b) \Rightarrow c) Let $A \subset X$. Since $A \subset f^{-1}[f[A]] \subset \underbrace{f^{-1}[f[A]]}_{\text{closed by b)}}$ $\Rightarrow \overline{A} \subset f^{-1}[f[A]] \Rightarrow f[\overline{A}] \subset f[f^{-1}[f[A]]] \subset \overline{f[A]}$.
- c) \Rightarrow d) Let $B \subset Y$, set $A = f^{-1}[Y \setminus B] = X \setminus f^{-1}[B]$.
Then, $f[\overline{A}] = f[X \setminus f^{-1}[B]] = f[X \setminus (f^{-1}[B])^\circ]$ and $f[\overline{A}] = \overline{f[f^{-1}[Y \setminus B]]} \subset \overline{Y \setminus B} = Y \setminus B^\circ$.
By c),

$$\begin{aligned} f[\overline{A}] &= f[X \setminus (f^{-1}[B])^\circ] \subset \overline{f[A]} \subset Y \setminus B^\circ \\ &\Rightarrow X \setminus (f^{-1}[B])^\circ \subset f^{-1}[f[X \setminus (f^{-1}[B])^\circ]] \subset f^{-1}[Y \setminus B^\circ] = X \setminus f^{-1}[B^\circ] \\ &\Rightarrow f^{-1}[B^\circ] \subset (f^{-1}[B])^\circ \end{aligned}$$

- d) \Rightarrow a) Let $U \subset Y$ be an open set. By d), $f^{-1}[U] \stackrel{U \text{ open}}{=} f^{-1}[U^\circ] \subset (f^{-1}[U])^\circ$.
Meanwhile, reverse inclusion is clear, $f^{-1}[U] = (f^{-1}[U])^\circ$, open. \square

Lemma 12.2.0.1. Let X, Y are Topological Space. Then,

1. $f : X \rightarrow Y$ is open map if and only if For any $A \subset X$, $f[A^\circ] \subset (f[A])^\circ$.
2. $f : X \rightarrow Y$ is closed map if and only if For any $A \subset X$, $\overline{f[A]} \subset f[\overline{A}]$.

Lemma 12.2.0.2. Let X, Y are Topological Space, and $f : X \rightarrow Y$ be a bijection. Then, TFAE:

1. f is open map.
2. f is closed map.
3. $f^{-1} : Y \rightarrow X$ be continuous map.

Clearly, Homeomorphism is open, closed, continuous map.

Lemma 12.2.0.3. Let $f : X \rightarrow Y$ be a Homeomorphism, $A \subset X$. Then, followings hold:

1. $f[\bar{A}] = \bar{f[A]}$.
2. $f[A^\circ] = (f[A])^\circ$.

Proof. 1. is clear by f is continuous, and closed map.

2. \supset) $A^\circ \subset A \implies f[A^\circ] \subset f[A] \implies f[A^\circ] \subset (f[A])^\circ \subset f[A]$ by f open map.

2. \subset) Let $x \in (f[A])^\circ$ be given. Then, there is an open $\mathcal{U} \in \mathcal{T}$ such that $x \in \mathcal{U} \subset f[A]$.

Now, $f^{-1}[x] \subset f^{-1}[\mathcal{U}] \subset f^{-1}[f[A]] = A$ by f is bijection, this implies

$$(f^{-1}[\mathcal{U}])^\circ = f^{-1}[\mathcal{U}] \subset A^\circ \implies f^{-1}[x] \subset f^{-1}[\mathcal{U}] \subset A^\circ \implies f[f^{-1}[x]] = x \in f[A^\circ].$$

□

Theorem 12.2.0.2. Let X, Y, Z are Topological Space, and $f : X \rightarrow Y$, $g : Y \rightarrow Z$.

1. If f, g are Continuous map, then $g \circ f$ is Continuous map.
2. If f, g are Open map, then $g \circ f$ is Open map.
3. If f, g are Closed map, then $g \circ f$ is Closed map.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

$g \circ f$

Above three theorems are trivial.

1. If $g \circ f$ is Open map, f is Continuous onto map. Then, g is Open map.
2. If $g \circ f$ is Open map, g is Continuous one-to-one map. Then, f is Open map.

Proof. 1) Let $U \in \mathcal{T}_Y$ be an open set. Since f is Continuous map, $f^{-1}[U]$ is open of X .

Now,

$$(g \circ f)[\underbrace{f^{-1}[U]}_{\substack{\text{open} \\ \text{image of open map}}}] = g[f[f^{-1}[U]]] \stackrel{\text{onto}}{=} g[U] \in \mathcal{T}_Z$$

2) Let $U \in \mathcal{T}_X$ be an open set. Since $g \circ f$ is Open map, $(g \circ f)[U]$ is open of Z .

Now, by g is Continuous one-to-one,

$$g^{-1}[(g \circ f)[U]] = g^{-1}[g[f[U]]] \stackrel{1 \text{ to } 1}{=} f[U] \in \mathcal{T}_Y$$

□

Lemma 12.2.0.4. Pasting Lemma

Suppose that X, Y are Topological Space, and $A, B \subset X$ such that $X = A \cup B$.

If both A, B are Open or Closed, $f : A \rightarrow Y$ and $g : B \rightarrow Y$ are Continuous map such that $f|_{A \cap B} = g|_{A \cap B}$, then

$$f \cup g : X \rightarrow Y : \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

is Continuous map.

Proof. Suppose that $A, B \subset X$ are Closed in X . For any closed set $C \subseteq Y$,

$$(f \cup g)^{-1}[C] = f^{-1}[C] \cup g^{-1}[C]$$

is closed in X , being A is closed in X and containing $f^{-1}[C]$ as closed set, thus $f^{-1}[C]$ is closed in X . □

12.3 Product Space

12.3.1 Finite Product Space

Definition 12.3.1.1. Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) are Topological Spaces. Define a Topology on $X \times Y$:

$$\mathcal{T}_P \stackrel{\text{def}}{=} \{U \times V \subset X \times Y \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$$

The Topological Space $(X \times Y, \mathcal{T}_P)$ is called the *Product Space*.

Lemma 12.3.1.1. Suppose that $(X \times Y, \mathcal{T}_P)$ is a Product Space for X and Y . Then, the Projection map

$$\pi_X : X \times Y \rightarrow X : (x, y) \mapsto x$$

is Continuous Open Onto map.

Proof. Surjection is Clear,

Continuous: For any open $\mathcal{U} \in \mathcal{T}_X$, $\pi_X^{-1}[\mathcal{U}] = \mathcal{U} \times Y \in \mathcal{T}_P$.

Open: For any open $O \times V \in \mathcal{T}_P$, $\pi_X[O \times V] = O \in \mathcal{T}_X$. Thus, it is Open map. \square

Theorem 12.3.1.1. \mathcal{T}_P is the samllest Topology such that the Projection maps are Continuous.

Proof. Suppose that \mathcal{T}' is Topology on $X \times Y$ such that π_X, π_Y are Continuous map.

Then, for any open $U \in \mathcal{T}_X, V \in \mathcal{T}_Y$, $U \times V = \pi_X^{-1}[U] \in \mathcal{T}'$ and $X \times V = \pi_Y^{-1}[V] \in \mathcal{T}'$

Thus, the finite intersection $U \times V \cap X \times V = U \times V \in \mathcal{T}'$. That is, $\beta_p \subset \mathcal{T}'$. \square

Theorem 12.3.1.2. Let X , and $\{Y_i\}_{i=1}^n$ are Topological Spaces. For a map

$$f : X \rightarrow \prod_{i=1}^n Y_i : x \mapsto (f_1(x), f_2(x), \dots, f_n(x))$$

The Followings Are Equivalent:

- a) f is Continuous map.
- b) For all $i = 1, 2, \dots, n$, $f_i : X \rightarrow Y_i$ is Continuous map.

Proof. a) \Rightarrow b). Since π_i is Continuous map, $f_i = \pi_i \circ f$ be a Continuous map.

b) \Rightarrow a). Let $B = U_1 \times U_2 \times \dots \times U_n \in \mathcal{T}_P$, and $\pi_i : \prod_{i=1}^n Y_i \rightarrow Y_i$ be a Projection for each $i = 1, 2, \dots, n$. Then,

$$\begin{aligned} B &= U_1 \times U_2 \times \dots \times U_n = (U_1 \times X_2 \times X_3 \times \dots \times X_n) \\ &\quad \cap (X_1 \times U_2 \times X_3 \times \dots \times X_n) \end{aligned}$$

⋮

$$\cap (X_1 \times X_2 \times X_3 \times \dots \times U_n) = \bigcap_{i=1}^n \pi_i^{-1}[U_i]$$

Now,

$$f^{-1}[B] = f^{-1} \left[\bigcap_{i=1}^n \pi_i^{-1}[U_i] \right] = \bigcap_{i=1}^n f^{-1} [\pi_i^{-1}[U_i]] = \bigcap_{i=1}^n (\pi_i \circ f)^{-1}[U_i] = \overbrace{\bigcap_{i=1}^n f_i^{-1} \left[\frac{U_i}{f_i \text{ conti}} \right]}^{\text{finite intersection}} \in \mathcal{T}_X$$

\square

Theorem 12.3.1.3. Let X_1, \dots, X_n are Topological Space and $A_i \subset X_i$. Then, for a Product Space $\left(\prod_{i=1}^n X_i, \mathcal{T}_p\right)$,

$$1. \quad \overline{\prod_{i=1}^n A_i} = \prod_{i=1}^n \overline{A_i}.$$

$$2. \quad \left(\prod_{i=1}^n A_i\right)^\circ = \prod_{i=1}^n A_i^\circ.$$

Proof.

1.)

$$\begin{aligned} x \in \overline{\prod_{i=1}^n A_i} &\iff \forall U \in \mathcal{T}_p \text{ with } x \in U, \quad U \cap \prod_{i=1}^n A_i = \prod_{i=1}^n \pi_i[U] \cap \prod_{i=1}^n A_i = \prod_{i=1}^n (\pi_i[U] \cap A_i) \neq \emptyset \\ &\iff \forall U \in \mathcal{T}_p \text{ with } x \in U, \quad \forall i = 1, 2, \dots, n, \quad \pi_i[U] \cap A_i \neq \emptyset \\ &\iff \forall i = 1, 2, \dots, n, \quad \forall U_i \in \mathcal{T}_i \text{ with } \pi_i(x) \in U_i, \quad U_i \cap A_i \neq \emptyset \\ &\iff x \in \prod_{i=1}^n \overline{A_i} \end{aligned}$$

2.)

$$\begin{aligned} x \in \left(\prod_{i=1}^n A_i\right)^\circ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } x \in U \subset \prod_{i=1}^n A_i \\ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } x \in \prod_{i=1}^n \pi_i[U] \subset \prod_{i=1}^n A_i \\ &\iff \exists U \in \mathcal{T}_p \text{ s.t. } \forall i = 1, 2, \dots, n, \quad \pi_i(x) \in \pi_i[U] \subset A_i \\ &\stackrel{(*)}{\iff} x \in \prod_{i=1}^n A_i^\circ \end{aligned}$$

However, the left direction of (*) fails when the index set is infinite. □

12.4 Coproduct Space

Definition 12.4.0.1. Let $(X_\alpha, \mathcal{T}_\alpha)$ ($\alpha \in \Lambda$) are mutually disjoint Topological Spaces. Define a *Coproduct Topology* $(X_{\text{II}}, \mathcal{T}_{\text{II}})$:

$$X_{\text{II}} \stackrel{\text{def}}{=} \bigsqcup_{\alpha \in \Lambda} X_\alpha, \quad \mathcal{T}_{\text{II}} \stackrel{\text{def}}{=} \left\{ \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha \mid \mathcal{U}_\alpha \in \mathcal{T}_\alpha \right\}$$

This actually be a Topology:

1. $\emptyset, X_{\text{II}} \in \mathcal{T}_{\text{II}}$ is clear,
2. Closed under union is clear.
3. Closed under finite intersection, not infinite.

Proof. Proof of 3.

Let a finite collection

$$\left\{ \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^1, \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^2, \dots, \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^k \right\}$$

be given. Then, their intersection be:

$$\bigcap_{j=1}^k \bigsqcup_{\alpha \in \Lambda} \mathcal{U}_\alpha^j = \bigsqcup_{\alpha \in \Lambda} \bigcap_{j=1}^k \mathcal{U}_\alpha^j \in \mathcal{T}_{\text{II}}$$

□

Theorem 12.4.0.1. Let X_1, X_2, X_3 and Y_1, Y_2, Y_3 are mutually disjoint Topological Space, and for each $i = 1, 2, 3$,

$$f_i : X_i \rightarrow Y_i : x \mapsto f_i(x)$$

Define a function

$$f = f_1 \amalg f_2 \amalg f_3 : \bigsqcup_{i=1}^3 X_i \rightarrow \bigsqcup_{i=1}^3 Y_i : x \mapsto \begin{cases} f_1(x) & x \in X_1 \\ f_2(x) & x \in X_2 \\ f_3(x) & x \in X_3 \end{cases}$$

where both Domain and Codomain are Coproduct Space. (Clearly, this function is well-defined.) Suppose that:

1. f_1 is Open map, Closed map
2. f_2 is Continuous map, Open map
3. f_3 is Continuous map, Closed map

Then, The Followings hold:

1. f_1 is Continuous map if and only if f is Continuous map.
2. f_2 is Open map if and only if f is Open map.
3. f_3 is Closed map if and only if f is Closed map.

Proof.

1. It follows that: For any open on Codomain $U \in \mathcal{T}_{Y_{\text{II}}}$,

$$\begin{aligned} f^{-1}[U] &= \{x \in X \mid f(x) \in U\} = \{x \in X_1 \mid f_1(x) \in U\} \cup \{x \in X_2 \mid f_2(x) \in U\} \cup \{x \in X_3 \mid f_3(x) \in U\} \\ &= f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U] \end{aligned}$$

Thus, If f_1 is Continuous, then f is Continuous map since $f^{-1}[U]$ is the union of open sets.

And, If f is Continuous, then $f^{-1}[U] \cap X_1$ be Open set and it is equal that $(f_1^{-1}[U] \cup f_2^{-1}[U] \cup f_3^{-1}[U]) \cap X_1 = f_1^{-1}[U]$.

2. It follows that: For any open on Domain $U \in \mathcal{T}_{X_{\Pi}}$,

$$f[U] = f_1[U] \cup f_2[U] \cup f_3[U]$$

This, if f_2 is Open map, then f is Open map since $f[U]$ is the union of open sets.

And, If f is Open, then $f[U] \cap Y_2$ be Open set and it is equal that $(f_1[U] \cup f_2[U] \cup f_3[U]) \cap Y_2 = f_2[U]$.

3. Similar to the above. □

For a specific example, Define for each $i = 1, 2, 3$,

$$X_i \stackrel{\text{def}}{=} \{a_i, b_i\}, \quad \begin{cases} \mathcal{T}_{i,D} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}, \{b_i\}\} \\ \mathcal{T}_{i,I} \stackrel{\text{def}}{=} \{\emptyset, X_i\} \\ \mathcal{T}_{i,a} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{a_i\}\} \\ \mathcal{T}_{i,b} \stackrel{\text{def}}{=} \{\emptyset, X_i, \{b_i\}\} \end{cases}$$

And define functions

1. $f_1 : (X_1, \mathcal{T}_{1,I}) \rightarrow (X_1, \mathcal{T}_{1,D}) : x \mapsto x$ is Not Continuous, Open, Closed.
2. $f_2 : (X_2, \mathcal{T}_{2,a}) \rightarrow (X_2, \mathcal{T}_{2,a}) : x \mapsto a_2$ is Continuous, Open, Not Closed.
3. $f_3 : (X_1, \mathcal{T}_{3,a}) \rightarrow (X_1, \mathcal{T}_{3,b}) : x \mapsto a_3$ is Continuous, Not Open, Closed.
4. $g_i : (X_i, \mathcal{T}_{i,D}) \rightarrow (X_i, \mathcal{T}_{i,D}) : x \mapsto x$ is Continuous, Open, Closed for each $i = 1, 2, 3$.

Now, from the above discussion,

1. $g_1 \amalg g_2 \amalg g_3$ is Continuous, Open, Closed.
2. $f_1 \amalg g_2 \amalg g_3$ is Not Continuous, Open, Closed.
3. $g_1 \amalg f_2 \amalg g_3$ is Continuous, Not Open, Closed.
4. $g_1 \amalg g_2 \amalg f_3$ is Continuous, Open, Not Closed.
5. $f_1 \amalg f_2 \amalg f_3$ is Not Continuous, Not Open, Not Closed.
6. $g_1 \amalg f_2 \amalg f_3$ is Continuous, Not Open, Not Closed.
7. $f_1 \amalg f_2 \amalg g_3$ is Not Continuous, Not Open, Closed.
8. $f_1 \amalg g_2 \amalg f_3$ is Not Continuous, Open, Not Closed.

No.	Map	Continuous	Open	Closed
1	$g_1 \amalg g_2 \amalg g_3$	Yes	Yes	Yes
2	$f_1 \amalg g_2 \amalg g_3$	No	No	No
3	$g_1 \amalg f_2 \amalg g_3$	Yes	No	Yes
4	$g_1 \amalg g_2 \amalg f_3$	Yes	Yes	No
5	$f_1 \amalg f_2 \amalg f_3$	No	No	No
6	$g_1 \amalg f_2 \amalg f_3$	Yes	No	No
7	$f_1 \amalg f_2 \amalg g_3$	No	No	Yes
8	$f_1 \amalg g_2 \amalg f_3$	No	Yes	No

12.5 Connected Space

Definition 12.5.0.1. Let X be a Topological Space. Define **Separation** of X be a tuple $\{U, V\}$ satisfying:

$$U, V \in \mathcal{T}, U \neq \emptyset, V \neq \emptyset, U \cap V = \emptyset, U \cup V = X$$

If the separation exists, then X is called **disconnected**.

Lemma 12.5.0.1. Let X be a Topological Space. TFAE:

- a) X is disconnected.
- b) There exist closed sets C, D such that $C, D \neq \emptyset, C \cap D = \emptyset, C \cup D = X$
- c) There exists a non-empty proper clopen subset $\emptyset \neq A \subsetneq X$.
- d) There exist subsets $A, B \subseteq X$ such that $A, B \neq \emptyset, \overline{A} \cap B = A \cap \overline{B} = \emptyset, A \cup B = X$.
- e) There exists Continuous onto map $f : X \rightarrow \{a, b\}$ where $\{a, b\}$ is discrete space.

Proof. a) \iff b) \iff c) \implies d) given directly since the facts:

$$\begin{aligned} U \cap V = \emptyset &\iff U \subseteq X \setminus V \iff V \subseteq X \setminus U \\ U \cup V = X &\iff X \setminus V \subseteq U \iff X \setminus U \subseteq V \end{aligned}$$

d) \implies a) The tuple $\{X \setminus \overline{A}, X \setminus \overline{B}\}$ becomes the Separation because:

$$\begin{aligned} A \cup B = X &\implies \overline{A} \cup \overline{B} = X \implies (X \setminus \overline{A}) \cap (X \setminus \overline{B}) = \emptyset \\ [(X \setminus \overline{A}) \cup (X \setminus \overline{B})] \cup [(X \setminus A) \cup (X \setminus B)] &= X \cup X = X \end{aligned}$$

□

Theorem 12.5.0.1. Let X be a Connected Space, Y is Topological Space and $f : X \rightarrow Y$ be a Continuous map. Then, $f[X]$ is Connected.

Proof. Suppose that $f[X]$ is disconnected. Then, there exist non-empty open sets of $f[X]$, $\{U, V\}$ such that

$$U \cup V = f[X], U \cap V = \emptyset$$

Now,

$$f^{-1}[U] \cup f^{-1}[V] = f^{-1}[U \cup V] = f^{-1}[f[X]] = X$$

and

$$f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V] = f^{-1}[\emptyset] = \emptyset$$

Since f is continuous, $\{f^{-1}[U], f^{-1}[V]\}$ be a separation of X , thus contradiction.

□

Lemma 12.5.0.2. Let A be a Connected Subspace of X . If $A \subseteq B \subseteq \overline{A}$, then B is Connected Subspace.

Proof. Using Contradiction: Suppose that B is Disconnected. Put $\{U, V\}$ be a Separation of B such that

$$U, V \in \mathcal{T}_X, B \cap U \neq \emptyset, B \cap V \neq \emptyset, (B \cap U) \cap (B \cap V) = \emptyset, B \subseteq U \cup V$$

Meanwhile, since assumption,

$$B \subseteq \overline{A} \Rightarrow \begin{cases} \emptyset \neq B \cap U \subseteq \overline{A} \cap U \neq \emptyset \\ \emptyset \neq B \cap V \subseteq \overline{A} \cap V \neq \emptyset \end{cases}$$

To show $A \cap U \neq \emptyset$, Suppose that $A \cap U = \emptyset$. Then, $A \subseteq X \setminus U$ and U is open implies $A \subseteq \overline{A} \subseteq X \setminus U$. This implies $\overline{A} \cap U = \emptyset$, Contradiction. Thus $A \cap U \neq \emptyset$, similarly, $A \cap V \neq \emptyset$.

On the other hand,

$$A \subset B \Rightarrow \begin{cases} (A \cap U) \cap (A \cap V) \subseteq (B \cap U) \cap (B \cap V) = \emptyset \\ A \subset B \subseteq U \cup V \end{cases}$$

Consequently, $\{U, V\}$ be a Separation of A , Contradiction. \square

Theorem 12.5.0.2. Let X be a Topological Space, and subspaces $A_\alpha \subset X$, ($\alpha \in \Lambda$) are Connected.

If $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$, then $\bigcup_{\alpha \in \Lambda} A_\alpha$ is Connected Space.

Proof. Proof by Contradiction.

Suppose that $A = \bigcup_{\alpha \in \Lambda} A_\alpha$ is Disconnected. Let $\{U, V\}$ be a separation of A . Choose $a \in \bigcap_{\alpha \in \Lambda} A_\alpha$, since assumption.

Then, since $A = U \cup V$, WLOG, assume that $a \in U$. Set for each $\alpha \in \Lambda$, $U_\alpha = U \cap A_\alpha$ and $V_\alpha = V \cap A_\alpha$. Then,

$$a \in U_\alpha \neq \emptyset, U_\alpha \cap V_\alpha = U \cap V \cap A_\alpha = \emptyset, U_\alpha \cup V_\alpha = (U \cup V) \cap A_\alpha = A_\alpha$$

Thus, V_α must be emptyset. Now,

$$V = V \cap A = V \cap \left(\bigcup_{\alpha \in \Lambda} A_\alpha \right) = \bigcup_{\alpha \in \Lambda} (V \cap A_\alpha) = \bigcup_{\alpha \in \Lambda} V_\alpha = \emptyset$$

This is Contradiction. \square

Corollary 12.5.0.1. Let X be a Topological Space, and subspaces $A_n \subset X$ ($n \in \mathbb{N}$) are Connected.

If for any $n \in \mathbb{N}$, $\left(\bigcup_{i=1}^n A_i \right) \cap A_{n+1} \neq \emptyset$, then $\bigcup_{n=1}^{\infty} A_n$ is Connected.

Proof. Put

$$B_n \stackrel{\text{def}}{=} \bigcup_{i=1}^n A_i$$

$B_1 = A_1$ is Connected, by assumption. Inductively, Suppose that B_n is Connected.

Then, above theorem and $B_{n+1} = B_n \cap A_{n+1} \neq \emptyset$ gives B_{n+1} is Connected. Meanwhile,

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

and

$$\bigcap_{n=1}^{\infty} B_n = B_1 = A_1 \neq \emptyset, \text{ because } A_1 \cap A_2 \neq \emptyset$$

Thus, above theorem gives $\bigcup_{n=1}^{\infty} B_n$ Connected. \square

Theorem 12.5.0.3. Let X, Y are Connected Spaces. Then, $X \times Y$ is Connected.

Proof. For any $x \in X$, $y \in Y$, $X \times \{y\}$ and $\{x\} \times Y$ are Connected, being $X \times \{y\} \cong X$ and $\{x\} \times Y \cong Y$.

And, $X \times \{y\} \cap \{x\} \times Y = \{(x, y)\} \neq \emptyset$, thus $X \times \{y\} \cup \{x\} \times Y$ is Connected. Let $b \in Y$ fixed.

For any $x \in X$, define $T_x \stackrel{\text{def}}{=} X \times \{b\} \cup \{x\} \times Y$. T_x is Connected by above discussion, and contains $X \times \{b\}$ for any $x \in X$. Thus, $X \times \{b\} \subset \bigcap_{x \in X} T_x \neq \emptyset$ implies:

$$X \times Y = \bigcup_{x \in X} T_x$$

is Connected. □

12.5.1 Connected Component

Definition 12.5.1.1. Let X be a Topological Space. Define a relation \sim of X :

$$x \sim y \iff \text{There exists a Connected Subspace of } A \subset X \text{ such that } x, y \in A$$

Then, this relation be an Equivalent Relation, and define Equivalent Class of this relation:

$$\mathcal{C}_x \stackrel{\text{def}}{=} \{y \in X \mid x \sim y\}$$

\mathcal{C}_x is called Connected Component.

Theorem 12.5.1.1. Let X be a Topological Space. Then,

1. For each $x \in X$, there exists a unique Connected Component \mathcal{C}_x which is containing x .
2. \mathcal{C}_x be the largest Connected Subspace containing x .
3. If $A \subset X$ be a non-empty Connected Subspace,
then there exists a unique Connected Component \mathcal{C}_x such that $A \subset \mathcal{C}_x$.
4. Every Connected Component is Closed.
5. X is Connected if and only if There exists only one Connected Component.

Proof. 1. is clear: $x \in \mathcal{C}_y \iff x \sim y \iff \mathcal{C}_x = \mathcal{C}_y$.

2. Let $x \in X$ be given.

Largestet: Let $A \subseteq X$ be a Connected subset s.t $x \in A$. Then, for any $y \in A$, $x \sim y \implies y \in \mathcal{C}_x$. Thus $A \subseteq \mathcal{C}_x$.

Connected: Let $z \in \mathcal{C}_x$ be given. Then, $z \sim x$ implies there exists a Connected subset A_z such that $x, z \in A_z \subseteq \mathcal{C}_x$. Now,

$$\mathcal{C}_x = \bigcup_{z \in \mathcal{C}_x} A_z, \quad x \in \bigcap_{z \in \mathcal{C}_x} A_z \neq \emptyset$$

Thus, theorem gives \mathcal{C}_x is Connected.

3. Let $A \subset X$ be a non-empty subset, and Connected. Put Open sets tuple $\{U, V\}$ be a Separation of A .

Existence: Fix $x \in A$. Then, for any $a \in A$, $x \sim a \implies a \in \mathcal{C}_x$. Thus $A \subseteq \mathcal{C}_x$.

Uniqueness: If $A \subseteq \mathcal{C}_x$ and $A \subseteq \mathcal{C}_y$, then for any $a \in A$, $a \sim x$ and $a \sim y$ implies $x \sim y$, thus $\mathcal{C}_x = \mathcal{C}_y$.

4. Let \mathcal{C}_x be a Connected Component. Then, $\overline{\mathcal{C}_x}$ is Connected by theorem, containing x . Thus $\overline{\mathcal{C}_x} \subseteq \mathcal{C}_x$. \square

Theorem 12.5.1.2. Let X, Y are Topological Space such that $X \cong Y$. Then, X and Y have same number of Connected Component.

Proof. Let $f : X \rightarrow Y$ be a Homeomorphism, and $x \in X$. Since Homeo- preserves Connectedness, $f[\mathcal{C}_x]$ is Connected.

Meanwhile, $f(x) \in f[\mathcal{C}_x]$, thus $f[\mathcal{C}_x] \subset \mathcal{C}_{f(x)}$.

Similarly, $x = f^{-1}(f(x)) \in f^{-1}[\mathcal{C}_{f(x)}]$, thus $f^{-1}[\mathcal{C}_{f(x)}] \subset \mathcal{C}_x$, implies $f[f^{-1}[\mathcal{C}_{f(x)}]] = \mathcal{C}_{f(x)} \subset f[\mathcal{C}_x]$.

That is, for any $x \in X$, $f[\mathcal{C}_x] = \mathcal{C}_{f(x)}$. Similarly, for any $y \in Y$, $f^{-1}[\mathcal{C}_y] = \mathcal{C}_{f^{-1}(y)}$.

Claim: the map

$$\phi : \{\mathcal{C}_x \mid x \in X\} \rightarrow \{\mathcal{C}_y \mid y \in Y\} : \mathcal{C}_x \mapsto \mathcal{C}_{f(x)}$$

be a One-to-One, Onto.

Injection) Let $\phi(\mathcal{C}_{x_1}) = \phi(\mathcal{C}_{x_2})$. That is, $\mathcal{C}_{f(x_1)} = \mathcal{C}_{f(x_2)}$. Since above discussion, we obtain $f[\mathcal{C}_{x_1}] = f[\mathcal{C}_{x_2}]$. Now,

$$\mathcal{C}_{x_1} = f^{-1}[f[\mathcal{C}_{x_1}]] = f^{-1}[f[\mathcal{C}_{x_2}]] = \mathcal{C}_{x_2}$$

Surjection) Let \mathcal{C}_y be given from codomain. Since f is surjection, there exists $x \in X$ such that $y = f(x)$. Thus, $\mathcal{C}_y = \mathcal{C}_{f(x)} = \phi(\mathcal{C}_x)$, thus surjective. \square

Theorem 12.5.1.3. If the Topological Space X has finite number of Connected Components, then for each Connected Component is Clopen.

12.5.2 Locally Connected

Definition 12.5.2.1. A Space X is called *Locally Connected* if:

For any $x \in X$ and neighborhood N of x , there exists Connected Open set $U \in \mathcal{T}$ such that $x \in U \subseteq N$.

Theorem 12.5.2.1. Let X be a Space. Then,

X is Locally Connected if and only if For any open U , every Connected Component of subspace U is open.

Note that: $U \subset X$ is open of X , then for any subset $V \subseteq U$,

V is open in X if and only if V is open in U .

Proof. Suppose that X is Locally Connected.

Let U be an open set of X , and \mathcal{C} be a Connected Component of U .

Since Locally Connectedness, For any $x \in \mathcal{C}$, there exists Connected Open set V_x such that $x \in V_x \subseteq U$.

Since \mathcal{C} is the largest Connected set containing x , thus $V_x \subseteq \mathcal{C}$. Now, $\mathcal{C} = \bigcup_{x \in \mathcal{C}} V_x$, thus open.

Conversely, Let $x \in X$ and U be an open set containing x . Take \mathcal{C}_x is a Connected Component of U containing x .

By assumption, \mathcal{C}_x is Open, thus X is Locally Connected, as \mathcal{C}_x is Connected open neighborhood. \square

Corollary 12.5.2.1. Every Connected Component of Locally Connected Space is Clopen.

12.5.3 Path Connected

Definition 12.5.3.1. Let $I = [0, 1] \subset \mathbb{R}$, and X be a Topological Space.

Define Path from $x \in X$ to $y \in X$ is: Continuous map $p : I \rightarrow X$ such that $p(0) = x$ and $p(1) = y$.

A Space X is called Path Connected Space if: For any $x, y \in X$, there exists a path from x to y .

Theorem 12.5.3.1. Path Connected Space is Connected.

Proof. Fix $x_0 \in X$. By Path Connectedness, for any $x \in X$, there exists a path p_x from x_0 to x .

Since I is Connected, $p_x[I]$ is Connected in X and contains x_0 and x . Now,

$$\bigcup_{x \in X} p_x[I] = X, \quad x_0 \in \bigcap_{x \in X} p_x[I] \neq \emptyset$$

Thus, X is Connected. \square

Proof. Version 2. Let X be a Path-Connected but not Connected. Let $\{U, V\}$ be a Separation of X , and put $x \in U$, $y \in V$.

Then, there is a path $\alpha : [0, 1] \rightarrow X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. And, $\alpha^{-1}[U], \alpha^{-1}[V]$ are disjoint opens.

Moreover, $0 \in \alpha^{-1}[U]$ and $1 \in \alpha^{-1}[V]$, and $\alpha^{-1}[U] \cup \alpha^{-1}[V] = \alpha^{-1}[U \cup V] = \alpha^{-1}[X] = [0, 1]$.

Thus, these becomes Separation of $[0, 1]$, Contradiction. \square

Theorem 12.5.3.2. Let X be a Topological Space, and subspaces $A_\gamma \subset X (\gamma \in \Gamma)$ are Path-Connected.

If $\bigcap_{\gamma \in \Gamma} A_\gamma \neq \emptyset$, then $\bigcup_{\gamma \in \Gamma} A_\gamma$ is Path-Connected.

Proof. Fix $x^* \in \bigcap_{\gamma \in \Gamma} A_\gamma$, and let $x, y \in \bigcup_{\gamma \in \Gamma} A_\gamma$ be given. Then, for some $\alpha, \beta \in \Gamma$, $x \in A_\alpha$ and $y \in A_\beta$. Put p_α is a path from x to x^* and p_β is a path from y to x^* . Define

$$p_{\alpha*\beta} : [0, 1] \rightarrow \bigcup_{\gamma \in \Gamma} A_\gamma : x \mapsto \begin{cases} p_\alpha(2x) & x \in [0, \frac{1}{2}] \\ p_\beta(2x - 1) & x \in [\frac{1}{2}, 1] \end{cases}$$

Since Pasting lemma, this function is Continuous, moreover path from x to y in $\bigcup_{\gamma \in \Gamma} A_\gamma$. \square

Corollary 12.5.3.1. If $A_n \subset X$, ($n \in \mathbb{N}$) are Path-Connected

Theorem 12.5.3.3. If X is Path-Connected and $f : X \rightarrow Y$ is Continuous, then $f[X]$ is Path-Connected.

Proof. Let $f(x), f(y) \in f[X]$ be given. Put $p : I \rightarrow X$ is a path from x to y .

Then, $f \circ p : I \rightarrow f[X]$ is continuous, $f(p(0)) = f(x)$ and $f(p(1)) = f(y)$, thus it is a path. \square

12.5.4 Path-Connected Component

Definition 12.5.4.1. Let X be a Topological Space. Define a relation \sim of X :

$$x \sim y \iff \text{There exists a path from } x \text{ to } y \text{ in } X$$

Then, this relation be an Equivalent Relation, and define Equivalent Class of this relation:

$$\mathcal{P}_x \stackrel{\text{def}}{=} \{y \in X \mid x \sim y\}$$

\mathcal{P}_x is called Path-Connected Component.

12.5.5 Locally Path Connected

Definition 12.5.5.1. A Space X is called Locally Path-Connected if:

For any $x \in X$ and neighborhood N of x , there exists Path-Connected Open set $P \in \mathcal{T}$ such that $x \in P \subseteq N$.

Theorem 12.5.5.1. A Space X is Locally Path-Connected if and only if:

For any open U , every Path-Connected Component of subspace U is open.

Proof. Suppose that X is Locally Path-Connected.

Let U be an open set of X , and \mathcal{P} be a Path-Connected Component of U .

Since Locally Path-Connectedness, For any $x \in \mathcal{P}$, there exists Path-Connected Open set V_x such that $x \in V_x \subseteq U$. Since \mathcal{P} is the largest Path-Connected set containing x , thus $V_x \subseteq \mathcal{P}$. Now, $\mathcal{P} = \bigcup_{x \in \mathcal{P}} V_x$, thus open.

Conversely, Let $x \in X$ and U be an open set containing x . Take \mathcal{P}_x is a Connected Component of U containing x . By assumption, \mathcal{P}_x is Open, thus X is Locally Connected, as \mathcal{P}_x is Connected open neighborhood. \square

Theorem 12.5.5.2. Let X be a Locally Path-Connected Space,

\sim_c be a Connected relation, and \sim_p be a Path-Connected relation.

Then, $X/\sim_c = X/\sim_p$. (Moreover, every element in the collection is Clopen.)

Proof. Let $x \in X$ be given. Put \mathcal{C}_x is Connected Component of x , and \mathcal{P}_x is Path-Connected Component of x .

Since Path-Connected Space is Connected, $\mathcal{P}_x \subseteq \mathcal{C}_x$. Using Contradiction: Suppose that $\mathcal{P}_x \subsetneq \mathcal{C}_x$.

Generally, Since for any $y \in \mathcal{C}_x$, there exists a Path-Connected Component \mathcal{P}_y such that $x \in \mathcal{P}_y \subseteq \mathcal{C}_x$.

That is, $\mathcal{C}_x = \bigcup_{y \in \mathcal{C}_x} \mathcal{P}_y$. Now, $\mathcal{C}_x \setminus \mathcal{P}_x$ is non-empty, and Path-Connected Component being \sim_p is equivalent relation.

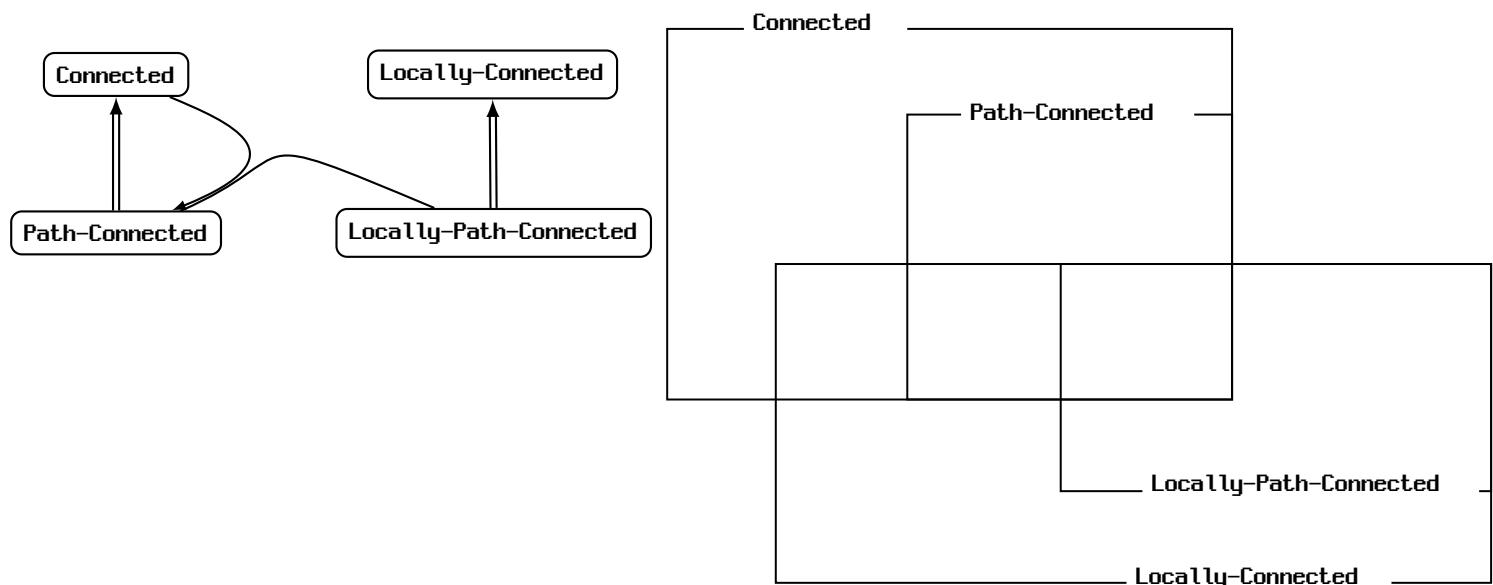
Since X is Locally Path-Connected Space, \mathcal{P}_x and $\mathcal{C}_x \setminus \mathcal{P}_x$ are Path-Connected Open sets, thus these becomes Separation of \mathcal{C}_x . This is Contradiction. \square

Corollary 12.5.5.1. Connected and Locally Path-Connected Space is Path-Connected.

Proof. If X is Connected, then X is also Connected Component.

Since above theorem, X is Path-Connected Component, thus Path-Connected. \square

12.5.6 Summary and Diagram and Counterexamples

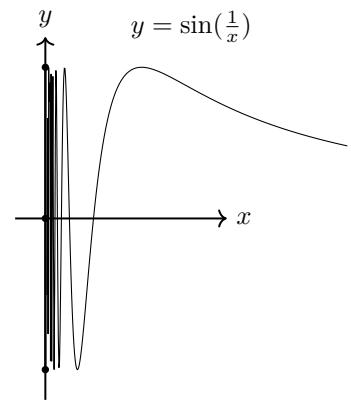


12.5.7 Topologist's Sine Curve

Definition 12.5.7.1. Topologist's Sine Curve

Let $Y = \{(0, y) \mid -1 \leq y \leq 1\}$ and $S^+ = \{(x, \sin(\frac{1}{x})) \mid x > 0\}$.

The union $S = Y \cup S^+$ with Euclidean metric is called **Topologist's Sine Curve**.



Theorem 12.5.7.1. Topologist's Sine Curve is Connected Space.

Proof. Since $f : (0, \infty) \rightarrow S^+ : x \mapsto (x, \sin(\frac{1}{x}))$ is Continuous and $(0, \infty)$ is Connected, $S^+ = f[(0, \infty)]$ is Connected.

Meanwhile, Let $\varepsilon > 0$ be given. There exists $0 < r < \varepsilon$ such that $\sin(\frac{1}{r}) = y$. That is, $(r, \sin(\frac{1}{r})) \in B_\varepsilon(0, y)$.

Now, $S^+ \subseteq Y \cup S^+ \subseteq \overline{S^+}$, thus $S = Y \cup S^+$ is Connected. \square

Theorem 12.5.7.2. Topologist's Sine Curve is Not Locally Connected Space.

Proof. Consider the Ball $B_{\frac{1}{2}}((0, 0))$. Then, $B_{\frac{1}{2}}((0, 0)) \cap S$ is open neighborhood of $(0, 0)$.

Let $U \cap S \subset B_{\frac{1}{2}}((0, 0)) \cap S$ be an open set of S containing $(0, 0)$.

Since $(0, 0) \in U$, $U \cap S^+ \neq \emptyset$. Put $(a, b) \in U \cap S^+$. Then, there exists $m \in \mathbb{N}$ such that

$$0 < \frac{1}{(2m + \frac{1}{2})\pi} < a$$

But, $(\frac{1}{(2m + \frac{1}{2})\pi}, \sin((2m + \frac{1}{2})\pi)) = (\frac{1}{(2m + \frac{1}{2})\pi}, 1) \notin B_{\frac{1}{2}}((0, 0))$, because $|((0, 0) - (\frac{1}{(2m + \frac{1}{2})\pi}, 1))| > \frac{1}{2}$.

Now,

$$((-\infty, \frac{1}{(2m + \frac{1}{2})\pi}) \times \mathbb{R}) \cap S, ((\frac{1}{(2m + \frac{1}{2})\pi}, \infty) \times \mathbb{R}) \cap S$$

becomes Separation of $U \cap S$. Thus, there is no Connected Open neighborhood. \square

Theorem 12.5.7.3. Topologist's Sine Curve is Not Path-Connected Space.

Proof. Let $(0, a) \in Y$, $(b, \sin(\frac{1}{b})) \in S^+$ be given. Suppose that there exists a path $p : I \rightarrow S$ from $(0, a)$ to $(b, \sin(\frac{1}{b}))$.

Then, $W = \{x \in I \mid p[[0, x]] \subseteq Y\}$ is non-empty, bounded above. Put $r = \sup W$. Clearly $r \in \overline{W}$.

Meanwhile, Y is closed in S , thus $p(r) \in p[\overline{W}] \subseteq p[W] \subseteq \overline{Y} = Y$, thus $r < 1$.

And, since p is Continuous, there exists $\delta \in (0, 1 - r)$ such that $|x - r| < \delta \implies |p(x) - p(r)| < \frac{1}{2}$.

By definition of Supremum, for some $t \in (r, r + \delta)$ such that $p(t) \in S^+$ with

$$\text{diam}(p[[r, t]]) \leq \frac{1}{2}, \quad p(r) \in Y, \quad p(t) \in S^+$$

Meanwhile, Claim: If $0 < c < t$ and $(c, \sin(\frac{1}{c})) \in S^+$, then $(c, \sin(\frac{1}{c})) \in p[[r, t]]$.

If $(c, \sin(\frac{1}{c}))$ is not contained in $p[[r, t]]$, then

$$U = (-\infty, c) \times \mathbb{R}, \quad V = (c, \infty) \times \mathbb{R}$$

becomes Separation of $p[[r, t]]$, thus Contradiction. Now, for large enough $m \in \mathbb{N}$,

$$a = \frac{1}{(2m + \frac{1}{2})\pi} < t, \quad b = \frac{1}{(2m + \frac{3}{2})\pi} < t$$

Each $p, q \in p[[r, t]]$ and $d(a, b) > 2$, Contradiction. \square

Corollary 12.5.7.1. Topologist's Sine Curve is *Not* Locally-Path-Connected Space.

12.6 Compact Space

Definition 12.6.0.1. A Topological Space X is *compact* if: every open cover contains a finite subcover. i.e.,

If $X = \bigcup_{\alpha \in \Lambda} U_\alpha$, ($U_\alpha \in \mathcal{T}$), then there is finite subcover such that $X = \bigcup_{i=1}^N U_{\alpha_i}$

This is equivalent with:

If $\emptyset = \bigcap_{\alpha \in \Lambda} C_\alpha$, (C_α closed), then there is finite subset such that $\emptyset = \bigcap_{i=1}^N C_{\alpha_i}$

Definition 12.6.0.2. Let X be a set. $A \subset \mathcal{P}(X)$ satisfies *finite intersection property* if:

For all finite subset of A , $\{A_i \mid i = 1, 2, \dots, n\} \subset A$ satisfies $\bigcap_{i=1}^n A_i \neq \emptyset$.

Example. 1. $X = \mathbb{R}$, and let $A = \{(n, \infty) \mid n \in \mathbb{N}\}$. Then,

$$\bigcap_{S \in A} S = \emptyset, \quad \bigcap_{\substack{S \in F \subset A \\ |F| < \infty}} S \neq \emptyset$$

2. $X = \mathbb{R}$, and let $A = \{\mathbb{R} \setminus F \mid |F| < \aleph_0\}$.

Theorem 12.6.0.1. Let X be a Topological Space, Then, TFAE:

a) X is Compact Space.

b) If A is a collection of closed subsets of X that satisfies FIP, then $\bigcap_{C \in A} C \neq \emptyset$.

c) If A is a collection of subsets of X that satisfies FIP, then $\bigcap_{S \in A} \overline{S} \neq \emptyset$.

Proof. a) \implies b). Proof by Contradiction:

Suppose that $A \subset \mathcal{P}(X)$ be a collection of closed subsets such that FIP.

Assume that $\bigcap_{C \in A} C = \emptyset$. Since X is Compact,

$$\emptyset = \bigcap_{C \in A} C \text{ if and only if } X = \bigcup_{C \in A} (X \setminus C), \text{ where } X \setminus C \text{ is open.}$$

This implies that there is a finite subcover:

$$X = \bigcup_{i=1}^N (X \setminus C_i) \text{ if and only if } \emptyset = \bigcap_{i=1}^N C_i$$

This is Contradiction with A satisfies FIP.

b) \implies a). Proof by Contraposition:

Suppose that X is not Compact. Then, there exists an Open Cover \mathcal{O} with no finite subcover: i.e.,

$$X = \bigcup_{U \in \mathcal{O}} U \text{ if and only if } \emptyset = \bigcap_{U \in \mathcal{O}} (X \setminus U)$$

And,

For any finite subset of \mathcal{O} , $F = \{U_i \mid i = 1, \dots, N\}$ satisfies $X \supseteq \bigcup_{i=1}^N U_i$ if and only if $\emptyset \neq \bigcap_{i=1}^N (X \setminus U_i)$

Thus, $\mathcal{K} = \{X \setminus \mathcal{U} \mid \mathcal{U} \in \mathcal{O}\}$ satisfies FIP, but $\emptyset = \bigcap_{\mathcal{U} \in \mathcal{O}} (X \setminus \mathcal{U}) = \bigcap_{\mathcal{C} \in \mathcal{K}} \mathcal{C}$. Thus, not a) implies not b). \square

Theorem 12.6.0.2. Let X is Compact Space, Y is Topological Space.
If $f : X \rightarrow Y$ is Continuous Map, then $f[X]$ is Compact.

Proof. Let \mathcal{O} be an open cover of $f[X]$. i.e., $f[X] \subset \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U}$. Now,

$$X \subset f^{-1}[f[X]] \subset f^{-1} \left[\bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \right] = \bigcup_{\substack{\mathcal{U} \in \mathcal{O} \text{ open, } f \text{ conti.}}} f^{-1}[\mathcal{U}]$$

Since X is compact, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^N f^{-1}[\mathcal{U}_i]$$

Consequently,

$$f[X] \subset f \left[\bigcup_{i=1}^N f^{-1}[\mathcal{U}_i] \right] = \bigcup_{i=1}^N f[f^{-1}[\mathcal{U}_i]] \subset \bigcup_{i=1}^N \mathcal{U}_i$$

\square

Theorem 12.6.0.3. Closed set of compact space is compact.

Proof. Let X be a compact, and $E \subset X$ be a closed subset. Let \mathcal{O} be an open over of E . Then,

$$X = E \cup (X \setminus E) \subset \left(\bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{U} \right) \cup (X \setminus E)$$

be an open cover of X . Thus, there is a finite subcover such that

$$X = \left(\bigcup_{i=1}^N \mathcal{U}_i \right) \cup (X \setminus E) \iff E \subset \bigcup_{i=1}^N \mathcal{U}_i$$

\square

Theorem 12.6.0.4. Let X be a Topological Space, and β be a basis of X . Then, TFAE:

- a) X is Compact Space.
- b) Every open cover consisting of basis elements has a finite subcover.

Proof. a) \implies b). Clear by definition of Compact.

b) \implies a). Let $\{\mathcal{U}_\alpha \mid \alpha \in \Lambda\}$ be an Open cover of X . That is,

$$X = \bigcup_{\alpha \in \Lambda} \mathcal{U}_\alpha = \bigcup_{\alpha \in \Lambda} \bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma$$

where $\{B_\alpha^\gamma \mid \gamma \in \Gamma_\alpha\}$ is subset of basis such that $\bigcup_{\gamma \in \Gamma_\alpha} B_\alpha^\gamma = \mathcal{U}_\alpha$. Now, by 2), there is finite subcover such that

$$X = \bigcup_{i=1}^n \bigcup_{j=1}^m B_{\alpha_i}^{\gamma_j} \subset \bigcup_{i=1}^n \mathcal{U}_{\alpha_i}$$

Thus, $\{\mathcal{U}_{\alpha_i} \mid i = 1, 2, \dots, n\}$ be a finite subcover. \square

Theorem 12.6.0.5. Let X, Y are Topological Space. Then, TFAE:

- a) $X \times Y$ is Compact.
- b) X and Y both are Compact.

Proof. a) \Rightarrow b) is clear since projection preserves Compactness.

b) \Rightarrow a) Let $\mathcal{O} \stackrel{\text{def}}{=} \{U \times V \mid U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ be an Open cover of $X \times Y$.

Let $x \in X$ fix. Then, $\{x\} \times Y$ be a Compact, being $\{x\} \times Y \cong Y$ by Homeomorphism given by Projection.

Then, there is a finite subcover of \mathcal{O} such that

$$\{x\} \times Y \subset \bigcup_{i=1}^{n_x} (U_i^x \times V_i^x)$$

Now, for each $x \in X$, define $U^x \stackrel{\text{def}}{=} \bigcap_{i=1}^{n_x} U_i^x$. Then, U^x is an open set containing x , and for any $i = 1, 2, \dots, n_x$, $U^x \subset U_i^x$.

Since $\{U^x \mid x \in X\}$ be an open cover of X , there is a finite subcover such that

$$X = \bigcup_{i=1}^m U^{x_i}$$

being X is Compact. Now,

$$X \times Y = \left(\bigcup_{i=1}^m U^{x_i} \right) \times Y = \bigcup_{i=1}^m (U^{x_i} \times Y) \subset \bigcup_{i=1}^m \bigcup_{j=1}^{n_{x_i}} (U_j^{x_i} \times V_j^{x_i})$$

Thus, $\{U_j^{x_i} \times V_j^{x_i} \mid i = 1, 2, \dots, m, j = 1, 2, \dots, n_{x_i}\}$ be a finite subcover. □

Tube Lemma

Let X be a Topological Space, and Y is Compact Space.

Then, for product space $X \times Y$, and fixed $x_0 \in X$, following statement holds:

For any open $N \subset X \times Y$ with $\{x_0\} \times Y \subset N$, there is an open $W \in \mathcal{T}_X$ such that $\{x_0\} \times Y \subset W \times Y \subset N$.

Proof. Clearly, $\{x_0\} \times Y$ compact, being $\{x_0\} \times Y \cong Y$.

For any $y \in Y$, $(x_0, y) \in \{x_0\} \times Y \subset N$, thus there exist opens $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ such that $(x_0, y) \in U \times V \subset N$. Now, Clearly $\{U_y \times V_y \subset X \times Y \mid y \in Y\}$ be an open cover of $\{x_0\} \times Y$, thus there is a finite subcover such that

$$\{x_0\} \times Y \subset \bigcup_{i=1}^N (U_{y_i} \times V_{y_i}) \subset N$$

Set $W = \bigcap_{i=1}^N U_{y_i}$. Then, clearly $x_0 \in W$, and

Let $(x, y) \in W \times Y$. Then, since $Y = \bigcup_{i=1}^n V_{y_i}$, there is $1 \leq k \leq n$ such that $y \in V_{y_k}$.

Thus, $(x, y) \in U_{y_k} \times V_{y_k} \subset N$, this implies $W \times Y \subset N$. □

Theorem 12.6.0.6. Let Y be a Compact Space. Then, the following statements are true, but their converses are false:

1. If X be a Lindelöf Space, then the product Topology $X \times Y$ be a Lindelöf Space.
2. If X be a Countable Compact Space, then the product Topology $X \times Y$ be a Countable Compact Space.

Proof. 1. Let \mathcal{O} be an open cover of $X \times Y$.

For any $x \in X$, $\{x\} \times Y$ is compact set, being $\{x\} \times Y \simeq Y$. Thus, there is a finite subcover of \mathcal{O} such that

$$\{x\} \times Y \subset \bigcup_{j=1}^{N_x} U_j^x \quad (U_j^x \in \mathcal{O})$$

Since Tube Lemma, there is an open $W_x \in \mathcal{T}_X$ such that

$$\{x\} \times Y \subset W_x \times Y \subset \bigcup_{j=1}^{N_x} U_j^x$$

Meanwhile, since X is Lindelöf, therefore for an open cover $\{W_x \mid x \in X\}$ there exists a Countable subcover such that

$$X \subset \bigcup_{i=1}^{\infty} W_{x_i}$$

Consequently,

$$X \times Y \subset \left(\bigcup_{i=1}^{\infty} W_{x_i} \right) \times Y \subset \bigcup_{i=1}^{\infty} (W_{x_i} \times Y) \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{N_{x_i}} U_j^{x_i}$$

Now, $\{U_j^{x_i} \mid i \in \mathbb{N}, 1 \leq j \leq N_{x_i}\} \subset \mathcal{O}$ be a Countable Open Cover of $X \times Y$. □

Proof. 2. Let $\{U_n \subset X \times Y \mid n \in \mathbb{N}\}$ be a Countable open cover of $X \times Y$. For each finite subset $F \subset \mathbb{N}$, define

$$V_F \stackrel{\text{def}}{=} \left\{ x \in X \mid \{x\} \times Y \subset \bigcup_{n \in F} U_n \right\}$$

Then V_F satisfies:

1) V_F is open: Let a finite subset $F \subset \mathbb{N}$ fix. For each $x \in V_F$, $\{x\} \times Y \subset \bigcup_{n \in F} U_n$ by definition.

Then, there is an open $W_x \in \mathcal{T}_X$ such that $\{x\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$ by Tube Lemma.

Meanwhile, $W_x \subset V_F$ because for all $s \in W_x$, $\{s\} \times Y \subset W_x \times Y \subset \bigcup_{n \in F} U_n$, thus $s \in V_F$.

In summary, for any $x \in V_F$, there is an open $W_x \in \mathcal{T}_X$ such that $x \in W_x \subset V_F$. Consequently, V_F is open of X .

2) $\{V_F \mid F \subset \mathbb{N}, |F| < \infty\}$ is a Countable Open Cover of X :

Countability given by above set is collection of subsets of Countable set. Meanwhile,

For any $x \in X$, there is a finite subcover of $\{U_n \mid n \in \mathbb{N}\}$ such that $\{x\} \times Y \subset \bigcup_{n \in F} U_n$ where F finite.

That is, $x \in V_F$. Now, the open cover of X ,

$$\{V_{F_x} \mid x \in X\} \subset \{V_F \mid F \subset \mathbb{N}\}$$

at most Countable. Since X is Countably Compact Space, there is a finite subcover such that

$$X \subset \bigcup_{i=1}^N V_{F_i}$$

Consequently,

$$X \times Y \subset \left(\bigcup_{i=1}^N V_{F_i} \right) \times Y = \bigcup_{i=1}^N (V_{F_i} \times Y) \subset \bigcup_{i=1}^N \bigcup_{n \in F_i} U_n$$

That is, $\{U_i \mid i = 1, 2, \dots, N, n \in F_i\}$ be a finite subcover. □

12.6.1 Locally Compact

Definition 12.6.1.1. A Space X is called *Locally Compact* if:

For any $x \in X$, there exist open U and compact C such that $x \in U \subseteq C$.

Lemma 12.6.1.1. Let X be a Hausdorff Space. TFAE:

1. X is Locally-compact space.
2. For any $x \in X$, there exists an open U with $x \in U$ such that the closure \overline{U} is Compact in X .

12.6.2 One-point Compactification

Definition 12.6.2.1. Let (X, \mathcal{T}) be a Space.

Define $X_\infty \stackrel{\text{def}}{=} X \sqcup \{\infty\}$ and $\mathcal{T}_\infty \stackrel{\text{def}}{=} \mathcal{T} \sqcup \{U \subseteq X_\infty \mid \infty \in U, X_\infty \setminus U \text{ is compact in } X\}$.
This $(X_\infty, \mathcal{T}_\infty)$ is called one-point compactification of X .

Theorem 12.6.2.1. Let (X, ∞) be a Locally-Compact Hausdorff Space, but not Compact. Then, one-point compactification $(X_\infty, \mathcal{T}_\infty)$ of X is Compact Hausdorff Space.

Proof. This proof consisted of five steps.

1). **Claim:** \mathcal{T}_∞ is Topology on X_∞ . (Using X is Hausdorff)

Let $U_\gamma \in \Gamma$, ($\gamma \in \Gamma$) be elements of \mathcal{T}_∞ .

Define $\Gamma_1 \stackrel{\text{def}}{=} \{\alpha \in \Gamma \mid U_\alpha \in \mathcal{T}\}$, and $\Gamma_2 \stackrel{\text{def}}{=} \Gamma \setminus \Gamma_1 = \{\beta \in \Gamma \mid \infty \in U_\beta, X_\infty \setminus U_\beta \text{ is compact in } X\}$.

Then, $\bigcup_{\gamma \in \Gamma} U_\gamma = \left(\bigcup_{\alpha \in \Gamma_1} U_\alpha \right) \cup \left(\bigcup_{\beta \in \Gamma_2} U_\beta \right)$. The left term is open in X clearly.

And, put $C_\beta = X_\infty \setminus U_\beta$ for each $\beta \in \Gamma_2$. Then, C_β is Compact in X by definition, thus closed by X is Hausdorff.

$$\bigcup_{\beta \in \Gamma_2} U_\beta = \bigcup_{\beta \in \Gamma_2} X_\infty \setminus C_\beta = X_\infty \setminus \left(\bigcap_{\beta \in \Gamma_2} C_\beta \right)$$

This intersection of C_β is compact, being any intersection of closed is closed and closed subset of compact. That is, it is compact in X , therefore this union of U_β is contained in \mathcal{T}_∞ .

Let $U_1, U_2 \in \mathcal{T}$, and $V_1, V_2 \in \mathcal{T}_\infty \setminus \mathcal{T}$. Put $C_i \stackrel{\text{def}}{=} X_\infty \setminus V_i$, ($i = 1, 2$). Then, C_i is compact. Now,

$$U_1 \cap U_2 \in \mathcal{T} \subset \mathcal{T}_\infty$$

$$U_1 \cap V_1 = U_1 \cap (X_\infty \setminus C_1) = U_1 \cap X_\infty \cap C_1^c = U_1 \cap C_1^c = U_1 \setminus C_1 \in \mathcal{T} \subset \mathcal{T}_\infty$$

$$V_1 \cap V_2 = (X_\infty \setminus C_1) \cap (X_\infty \setminus C_2) = X_\infty \setminus (C_1 \cap C_2) \in \mathcal{T}_\infty$$

Thus closed under the arbitrary union and finite intersection.

2). **Claim:** (X, \mathcal{T}) is a Subspace of $(X_\infty, \mathcal{T}_\infty)$. That is, $\mathcal{T} = \{U \cap X \mid U \in \mathcal{T}_\infty\}$. (Using X is Hausdorff)

The right inclusion is clear: $U \in \mathcal{T} \implies U \in \mathcal{T}_\infty$. Thus $U = X \cap U \in \{U \cap X \mid U \in \mathcal{T}_\infty\}$.

To show the left inclusion: Let $U \in \mathcal{T}_\infty$. If $U \in \mathcal{T}$, then $X \cap U = U \in \mathcal{T}$.

If $U \notin \mathcal{T}$, then $X_\infty \setminus U$ is compact in X . Now, $X \cap U = X \setminus (X_\infty \setminus U) \in \mathcal{T}$.

compact in $\mathcal{T}_2 \implies$ closed

3). **Claim:** $\overline{X} = X_\infty$. That is, closure of X is X_∞ . (Using X is not compact)

Let $U \in \mathcal{T}_\infty$ with $\infty \in U$. Then, $X_\infty \setminus U$ is compact of X , thus $X_\infty \setminus U \subsetneq X$ because X is not compact.

4). **Claim:** X_∞ is Compact Space.

Let $\mathcal{O} = \{U_\alpha \mid \alpha \in \Lambda\}$ be an open cover of X_∞ . Since $\infty \in X_\infty = \bigcup_{\alpha \in \Lambda} U_\alpha$, there is $\alpha_0 \in \Lambda$ such that $\infty \in U_{\alpha_0}$.

$C \stackrel{\text{def}}{=} X_\infty \setminus U_{\alpha_0}$ is compact in X , thus so in X_∞ . And, $C \subseteq \bigcup_{\alpha \in \Lambda \setminus \{\alpha_0\}} U_\alpha$, thus there is finite subcover of C .

Finally, union of finite subcover of C and U_{α_0} is finite subcover of X_∞ .

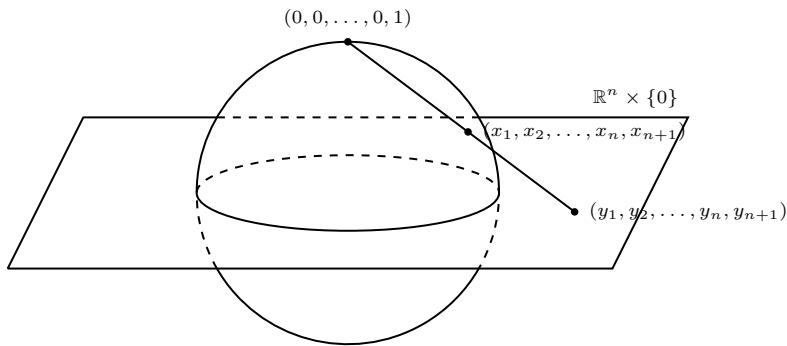
5). **Claim:** X_∞ is Hausdorff. (Using X is Locally-Compact)

Let $x, y \in X_\infty$. If both x, y are contained X , then there is nothing to prove, being X is hausdorff.

If $x \in X$ and $y = \infty$, then there is open U and compact C of X such that $x \in U \subseteq C$, by Locally-Compact.

Now, $x \in U$ and $\infty \in X_\infty \setminus C$, both are open of X_∞ with $U \cap (X_\infty \setminus C) = \emptyset$. □

12.6.3 Stereographic projection



Definition 12.6.3.1. Let \mathbb{R}^{n+1} be a Euclidean Space. Define **Unit Sphere** on \mathbb{R}^{n+1} :

$$S^n \stackrel{\text{def}}{=} \{(x_1, x_2, \dots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + x_2 + \dots + x_n + x_{n+1} = 1\}$$

Theorem 12.6.3.1. For any $p \in S^n$, $S^n \setminus \{p\} \cong \mathbb{R}^n$.

Proof. WLOG, put $p = (0, 0, \dots, 0, 1)$. Define a map **Stereographic Projection**:

$$f : S^n \setminus \{p\} \rightarrow \mathbb{R}^n : (x_1, x_2, \dots, x_n, x_{n+1}) \mapsto \left(\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right)$$

This map derived from:

$$(y_1, y_2, \dots, y_n, y_{n+1}) = t(\vec{x} - \vec{p}) + (0, 0, \dots, 0, 1) = (x_1t, x_2t, \dots, x_nt, (x_{n+1} - 1)t + 1)$$

$y_{n+1} = 0$ when $t = \frac{1}{1 - x_{n+1}}$. Each $(x_i, x_{n+1}) \mapsto \frac{x_i}{1 - x_{n+1}}$ is Continuous map, so f is Continuous map.
And, the map

$$g : \mathbb{R}^n \rightarrow S^n \setminus \{p\} : (x_i)_{i=1}^n \mapsto \left(\frac{2x_1}{1 + \| (x_i) \|^2}, \frac{2x_2}{1 + \| (x_i) \|^2}, \dots, \frac{2x_n}{1 + \| (x_i) \|^2}, 1 - \frac{2}{1 + \| (x_i) \|^2} \right)$$

is Continuous, and $g = f^{-1}$. Thus, f is Homeomorphism. □

Theorem 12.6.3.2. One Point Compactification of \mathbb{R}^n is $S^n \subseteq \mathbb{R}^{n+1}$.

Proof. Since \mathbb{R} is Locally-Compact Hausdorff Space, finite product Space \mathbb{R}^n is Locally-Compact Hausdorff Space. Let $\mathbb{R}_\infty^n = \mathbb{R}^n \cup \{\infty\}$ be a One Point Compactification of \mathbb{R}^n . Then, \mathbb{R}_∞^n is Compact Hausdorff Space. Meanwhile, put $p = (0, 0, \dots, 0, 1)$. Since $S^n \setminus \{p\} \cong \mathbb{R}^n$, there exists a Homeomorphism $f : S^n \setminus \{p\} \rightarrow \mathbb{R}$. Define

$$\tilde{f} : S^n \rightarrow \mathbb{R}_\infty^n : \begin{cases} f(x) & x \neq p \\ \infty & x = p \end{cases}$$

Then, \tilde{f} is Bijective map, and Continuous because: for any open $U \subseteq \mathbb{R}_\infty^n$ containing ∞ ,

$$\tilde{f}^{-1}[\mathbb{R}_\infty^n \setminus U] = f^{-1}[\mathbb{R}^n \setminus U]$$

Since $\mathbb{R}^n \setminus U$ is Compact and f^{-1} is continuous, $\tilde{f}^{-1}[\mathbb{R}_\infty^n \setminus U] = f^{-1}[\mathbb{R}^n \setminus U]$ is compact in S^n , thus closed. Moreover, since S^n is Compact and \mathbb{R}_∞^n is Hausdorff, f is Closed map. Consequently, \tilde{f} is Homeomorphism. □

12.7 Borel Set

Definition 12.7.0.1. Let X be a Topological Space.

1. $F \subseteq X$ is called F_σ -set if: F can be represented as countable union of closed sets.
2. $G \subseteq X$ is called G_δ -set if: G can be represented as countable intersection of open sets.

Proposition 12.7.0.1. Let X be a Topological Space.

1. If $F \subseteq X$ is F_σ -set, then there exists sequence of closed sets $\{F_n\}_{n \in \mathbb{N}}$ such that

$$F = \bigcup_{n=1}^{\infty} F_n, \quad F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \cdots$$

2. If $G \subseteq X$ is G_δ -set, then there exists sequence of open sets $\{G_n\}_{n \in \mathbb{N}}$ such that

$$G = \bigcap_{n=1}^{\infty} G_n, \quad G_1 \supseteq G_2 \supseteq \cdots \supseteq G_n \supseteq \cdots$$

3. Countable union of F_σ -sets is F_σ .
4. Finite intersection of F_σ -sets is F_σ .
5. Countable intersection of G_δ -sets is G_δ .
6. Finite union of G_δ -sets is G_δ .
7. Complement of F_σ -set is G_δ .

12.8 Baire Category

Definition 12.8.0.1. The Topological Space X is called *Baire Space* if:

If $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open sets of X , then $\overline{\bigcap_{n=1}^{\infty} G_n} = X$

In brief, every Countable intersection of dense open sets be dense in X .

Definition 12.8.0.2. Let X be a Topological Space.

$A \subset X$ is said to be *nowhere dense subset* if $(\overline{A})^\circ = \emptyset$.

1. $B \subset X$ is called *first category* if B can be representative by union of countable nowhere dense subsets.
2. If the subset is not first category, then it is said to be *second category*.

12.9 Locally Compact Hausdorff Space

Theorem 12.9.0.1. Locally Compact Hausdorff Space is Baire Space.

12.10 Complete Metric Space

Definition 12.10.0.1. Let (X, d) be a Metric Space, and $\{p_n\}$ be a Sequence in X .

The Sequence $\{p_n\}$ is called *Cauchy Sequence* if:

For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $m, n \geq N \implies d(p_m, p_n) < \varepsilon$.

A Metric Space (X, d) is said to be *Complete* if every Cauchy Sequences Converge.

Lemma 12.10.0.1. Let $\{E_n\}$ be a sequence of closed bounded non-empty subsets in a Complete Metric Space X such that

$E_n \supset E_{n+1}$. If $\lim_{n \rightarrow \infty} \text{diam } E_n = 0$, then $\bigcap_{n=1}^{\infty} E_n = \{p\}$ for some $p \in X$.

Proof. For each $n \in \mathbb{N}$, construct $p_n \in E_n$.

Let $\varepsilon > 0$ be given. Since $\text{diam } E_n \rightarrow 0$, there is $N \in \mathbb{N}$ such that $\text{diam } E_n < \varepsilon$.

For any $m, n \geq N$, E_N contains p_m, p_n . That is, $d(p_m, p_n) < \varepsilon$. Thus, $\{p_n\}$ be a Cauchy sequence of X .

Since X is complete, there is a unique point $p \in X$ such that $p_n \rightarrow p$. Let $N \in \mathbb{N}$ be an integer such that $n \geq N \implies |p_n - p| < \varepsilon$.

Now, for each $n \geq N$, E_n has a limit point as p . And for any $n \in \mathbb{N}$, E_n contains E_N, E_{N+1}, \dots , thus for all $n \in \mathbb{N}$, E_n has a limit point as p . Meanwhile, E_n closed, $p \in E_n, \forall n \in \mathbb{N}$.

Consequently, $p \in \bigcap_{n=1}^{\infty} E_n$. If there is $q \in X$ such that $p \neq q$, $q \in \bigcap_{n=1}^{\infty} E_n$. Then, $\text{diam } E_n \geq d(p, q) > 0, \forall n \in \mathbb{N}$. \square

Theorem 12.10.0.1. Complete Metric Space is Baire Space.

Proof. Suppose that $\{G_n \mid n \in \mathbb{N}\}$ be a Countable Collection of dense open set of Complete Metric Space.

Let an open $U \in \mathcal{T}$ be given. Since G_n is dense in the Space, $U \cap G_1$ is non-empty open set.

Thus, there exists a $p_1 \in U \cap G_1$ such that for some $r_1 > 0$, $B_{r_1}(p_1) \subset U \cap G_1$.

Then, automatically,

$$B_{\frac{r_1}{2}}(p_1) \subset \overline{B_{\frac{r_1}{2}}(p_1)} \subset B_{r_1}(p_1) \subset U \cap G_1$$

Set $E_1 = U$, $E_2 = B_{\frac{r_1}{2}}(p_1)$.

Suppose that E_1, \dots, E_{n-1} **are chosen.** Then, since $E_{n-1} \cap G_{n-1}$ is open, being intersection of opens.

Thus there exists a point $p_{n-1} \in E_{n-1} \cap G_{n-1}$ and exists r_{n-1} such that

$$B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

This implies that

$$B_{\frac{r_{n-1}}{2}}(p_{n-1}) \subset \overline{B_{\frac{r_{n-1}}{2}}(p_{n-1})} \subset B_{r_{n-1}}(p_{n-1}) \subset E_{n-1} \cap G_{n-1} \subset E_{n-1}$$

Set $E_n = B_{\frac{r_{n-1}}{2}}(p_{n-1})$. Since inductively construction of $\{E_n\}$, $E_{n+1} \subset E_n$ and $\overline{E_n} \subset G_n$ for all $n \in \mathbb{N}$. Consequently,

$$U \cap \left(\bigcap_{n=1}^{\infty} G_n \right) = \bigcap_{n=1}^{\infty} (U \cap G_n) \supset \bigcap_{n=1}^{\infty} (U \cap \overline{E_n}) = U \cap \left(\bigcap_{n=1}^{\infty} \overline{E_n} \right) = \bigcap_{n=1}^{\infty} \overline{E_n} \neq \emptyset$$

□

12.10.1 Nowhere Differentiable function

Theorem 12.10.1.1. Let $\mathcal{C}[\mathbb{R}] \stackrel{\text{def}}{=} \{f : \mathbb{R} \rightarrow \mathbb{R} \mid f \text{ is continuous.}\}$ and $d : \mathcal{C}[\mathbb{R}] \times \mathcal{C}[\mathbb{R}] \rightarrow \mathbb{R} : (f, g) \mapsto \sup_{t \in \mathbb{R}} |f(t) - g(t)|$.

Then, $(\mathcal{C}[\mathbb{R}], d)$ is Complete Metric Space, and set of Nowhere-Differentiable functions is dense in $\mathcal{C}[\mathbb{R}]$.

Proof. First, show that f satisfies triangle inequality: let $f, g, h \in \mathcal{C}[\mathbb{R}]$ be given.

For any $t \in \mathbb{R}$, $|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$. Thus,

$$d(f, g) = \sup_{t \in \mathbb{R}} |f(t) - g(t)| \leq \sup_{t \in \mathbb{R}} [|f(t) - h(t)| + |h(t) - g(t)|] \leq \sup_{t \in \mathbb{R}} |f(t) - h(t)| + \sup_{t \in \mathbb{R}} |h(t) - g(t)| = d(f, h) + d(h, g)$$

□

12.10.2 Banach Fixed Point Theorem

Definition 12.10.2.1. Let $f : X \rightarrow X$ be any function. A point $x \in X$ is called a **fixed point** of f if $f(x) = x$.

Definition 12.10.2.2. Let X be a Metric Space. A map $f : X \rightarrow X$ is called **Contractive** with respect to the metric d if:

There exists $\alpha \in (0, 1)$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha d(x, y)$.

Theorem 12.10.2.1. Banach Fixed point Theorem

Let (X, d) be a Complete Metric Space, and $f : X \rightarrow X$ be a Contractive map. Then, there exists a unique fixed point of f , $x^* \in X$.

Proof. Clearly,

Contractive \implies Lipschitz Condition \implies Continuous.

Thus, f is Continuous.

Let $x_0 \in X$ be arbitrary, and construct a sequence $\{x_n\}$ recursively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} f(x_n), \quad n \geq 0$$

Then, for any $n \geq 0$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \leq \alpha d(x_n, x_{n-1}) \\ &= d(f(x_{n-1}), f(x_{n-2})) \leq \alpha^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \alpha^n d(x_1, x_0) \end{aligned}$$

Let $\varepsilon > 0$ be given. Put $N \in \mathbb{N}$ such that $\alpha^N \cdot d(x_1, x_0) < \varepsilon(1 - \alpha)$. Then, $n \geq m \geq N$ implies that

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\ &\leq \alpha^n d(x_1, x_0) + \alpha^{n-1} d(x_1, x_0) + \cdots + \alpha^{m+1} d(x_1, x_0) \\ &= \alpha^{m+1} d(x_1, x_0) \sum_{r=0}^{n-m-1} \alpha^r < \alpha^N d(x_1, x_0) \sum_{r=0}^{\infty} \alpha^r < \varepsilon(1 - \alpha) \frac{1}{1 - \alpha} = \varepsilon \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy sequence. Since X is Complete, for some $x^* \in X$, $\lim_{n \rightarrow \infty} x_n = x^*$. Consequently,

$$\lim_{n \rightarrow \infty} f(x_n) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x^*) = \lim_{n \rightarrow \infty} x_{n+1} = x^*$$

□

12.11 Maps in Metric Space

In this section, (X, d_X) and (Y, d_Y) are metric spaces.

12.11.1 Metric

Definition 12.11.1.1. A **metric** on a set X is a map $d : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, z) \leq d(x, y) + d(y, z)$.

We call (X, d) a **metric space**.

Theorem 12.11.1.1. The map $d : X \times X \rightarrow \mathbb{R}$ is continuous.

Proof. Let $(x, y) \in X \times X$ and $\varepsilon > 0$. For $U = B_{\varepsilon/2}(x) \times B_{\varepsilon/2}(y)$ and any $(p, q) \in U$,

$$\begin{aligned}d(p, q) &\leq d(p, x) + d(x, y) + d(y, q) < d(x, y) + \varepsilon \\d(x, y) &\leq d(x, p) + d(p, q) + d(q, y) < d(p, q) + \varepsilon\end{aligned}$$

so $|d(p, q) - d(x, y)| < \varepsilon$. □

12.11.2 Diameter

Definition 12.11.2.1. For $E \subseteq X$, the **diameter** is

$$\text{diam } E \stackrel{\text{def}}{=} \sup_{x, y \in E} d(x, y).$$

Theorem 12.11.2.1. For any $E \subseteq X$, $\text{diam } E = \text{diam } \overline{E}$.

Proof. Clearly, $\text{diam } E \leq \text{diam } \overline{E}$. Let $\varepsilon > 0$ be given. Then, there exist $a, b \in \overline{E}$ such that

$$\text{diam } \overline{E} - \frac{\varepsilon}{2} \leq d(a, b) < \text{diam } \overline{E}$$

Meanwhile, $a, b \in \overline{E}$ implies: $B_{\frac{\varepsilon}{2}}(a) \cap E \neq \emptyset$, $B_{\frac{\varepsilon}{2}}(b) \cap E \neq \emptyset$.

Put $p \in B_{\frac{\varepsilon}{2}}(a) \cap E$ and $q \in B_{\frac{\varepsilon}{2}}(b) \cap E$. Now, the triangle inequality gives

$$\text{diam } \overline{E} - \frac{\varepsilon}{2} \leq d(a, b) \leq d(a, p) + d(p, q) + d(q, b) \leq \frac{\varepsilon}{2} + \text{diam } E + \frac{\varepsilon}{2} = \text{diam } E + \varepsilon$$

Since ε is chosen arbitrarily, $\text{diam } \overline{E} \leq \text{diam } E$. □

12.11.3 Distance

Definition 12.11.3.1. For nonempty $E \subseteq X$, define $\rho_E : X \rightarrow [0, \infty)$ by

$$\rho_E(x) \stackrel{\text{def}}{=} \inf_{t \in E} d(x, t).$$

Proposition 12.11.3.1. For all $x \in X$, $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

Proof.

$$\rho_E(x) = 0 \stackrel{\text{by def.}}{\iff} \inf_{t \in E} d(x, t) = 0 \iff \forall \varepsilon > 0, \exists p \in E \text{ s.t. } 0 < d(x, p) \leq \varepsilon \iff \forall \varepsilon > 0, B_\varepsilon(x) \cap E \neq \emptyset$$

□

Theorem 12.11.3.1. The distance ρ_E satisfies Lipschitz Condition. Furthermore, Uniformly Continuous.

Proof. Let $x, y \in X$ be given. Then, for any $z \in E$,

$$\rho_E(x) = \inf_{t \in E} d(x, t) \leq d(x, z) \leq d(x, y) + d(y, z)$$

Since $z \in E$ given arbitrarily,

$$\rho_E(x) \leq d(x, y) + \rho_E(y)$$

Thus $\rho_E(x) - \rho_E(y) \leq d(x, y)$. Similarly, $\rho_E(y) - \rho_E(x) \leq d(x, y)$. That is, For any $x, y \in X$, $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$. Now, for any $\varepsilon > 0$, put $\delta = \varepsilon$. Then,

$$d(x, y) < \delta \implies |\rho_E(x) - \rho_E(y)| \leq d(x, y) < \delta = \varepsilon$$

□

Theorem 12.11.3.2. Let $C \subseteq X$ be compact, $F \subseteq X$ closed, and $C \cap F = \emptyset$. Then there exists $\delta > 0$ such that

$$d(p, q) \geq \delta \text{ for all } p \in C, q \in F.$$

Proof.

□

12.11.4 Isometry

Definition 12.11.4.1. An onto map $f : (X, d_X) \rightarrow (Y, d_Y)$ is an *isometry* if: for all $x, y \in X$,

$$d_X(x, y) = d_Y(f(x), f(y))$$

12.12 Separation Axioms

12.13 Urysohn Metrization Theorem

12.13.1 Urysohn Lemma

Recall that:

Definition 12.13.1.1. X is T_4 if: For any disjoint closed set A and B , there exist disjoint open U, V such that $A \subseteq U$ and $B \subseteq V$.

Lemma 12.13.1.1. X is T_4 Space if and only if For any closed C and open U with $C \subseteq U$, there exists open O such that

$$\underset{\text{closed}}{C} \subseteq \underset{\text{open}}{O} \subseteq \underset{\text{closed}}{\bar{O}} \subseteq \underset{\text{open}}{U}$$

Proof. Proof of the left direction only.

Let X be a T_4 Space, and $C \subset X$ be a closed, U be a open containing C . Then, $C \subset U$ implies $U^c \subset C^c$, thus U^c is a closed set disjoint from C . By T_4 condition, There exist disjoint opens O, O' such that $C \subset O$ and $U^c \subset O' \iff O'^c \subset U$.

Since $O \cap O' = \emptyset \iff O \subset O'^c$, O contained in U , this implies that $C \subset O \subset U$.

Since closure is the smallest closed set such that contains it, consequently $C \subset O \subset \bar{O} \subset O'^c \subset U$. \square

Definition 12.13.1.2. Let X be a Toplogical Space, and $A, B \subset X$ are disjoint closed subset.

A real-valued Continuous map $f : X \rightarrow [a, b]$ is called **Urysohn function** for A and B if: $f|_A = a$ and $f|_B = b$. In another form,

$$f : X \rightarrow [a, b] : x \rightarrow \begin{cases} a & x \in A \\ b & x \in B \\ f(x) & x \notin A \cup B \end{cases}$$

Lemma 12.13.1.2. Urysohn Lemma

T_4 Space has an Urysohn function for any two disjoint closed subsets.

Proof. Generalization is the last thing to proven, first of all, prove in case of $[a, b] = [0, 1]$. This proof consists by three Step.

Let X be a T_4 Space, and $A, B \subset X$ be closed subsets.

Step 1. Construct a Chain of Open sets with Dyadic Rational Indices.

Consider a set of **Dyadic Rationals** $D \stackrel{\text{def}}{=} \left\{ \frac{k}{2^n} \mid n, k \in \mathbb{N}, k \leq 2^n - 1 \right\}$. We will show that the following statement holds:

For any $r, s \in D$ with $r < s$, there exist open sets U_r, U_s such that $A \subseteq \bar{U}_r \subseteq U_s \subseteq X \setminus B$ (*)

For this, Enough to Show that: For any $k \in \mathbb{N}$, there exists a Chain as:

$$A \subseteq U_{\frac{1}{2^k}} \subseteq \bar{U}_{\frac{1}{2^k}} \subseteq U_{\frac{2}{2^k}} \subseteq \bar{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^{k-1}}{2^k}} \subseteq \bar{U}_{\frac{2^{k-1}}{2^k}} \subseteq X \setminus B$$

(Note that this opens in the Chain are not necessary distinct: For instance, if Ambient Space is Finite, then the Space is Noetherian. That is, X satisfies Ascending Chain Condition for open sets.)

Let $k = 1$. Then, By T_4 condition gives that: There exists an open set U_1 such that

$$A \subseteq U_1 \subseteq \bar{U}_1 \subseteq X \setminus B$$

Now, naming this U_1 as $U_{\frac{1}{2}}$, proved when $k = 1$.

Suppose that for some $k > 1$, the Chain exists as:

$$\underset{\text{closed}}{A} \stackrel{*1}{\subseteq} U_{\frac{1}{2^k}} \subseteq \underset{\text{closed}}{\bar{U}_{\frac{1}{2^k}}} \stackrel{*2}{\subseteq} U_{\frac{2}{2^k}} \subseteq \cdots \stackrel{*2^{k-1}}{\subseteq} U_{\frac{2^{k-1}}{2^k}} \subseteq \underset{\text{closed}}{\bar{U}_{\frac{2^{k-1}}{2^k}}} \stackrel{*2^k}{\subseteq} X \setminus B$$

By repeatedly applying the T_4 condition 2^k times, as indicated by the indices $*1, *2, \dots, *2^k$, we can construct 2^k open sets such that:

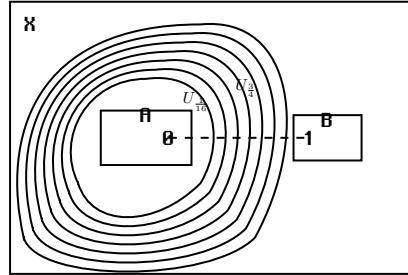
$$A \subseteq U_{\frac{1}{2^{k+1}}} \subseteq \overline{U}_{\frac{1}{2^{k+1}}} \subseteq U_{\frac{1}{2^k}} \subseteq \overline{U}_{\frac{1}{2^k}} \subseteq U_{\frac{3}{2^{k+1}}} \subseteq \overline{U}_{\frac{3}{2^{k+1}}} \subseteq U_{\frac{2}{2^k}} \subseteq \overline{U}_{\frac{2}{2^k}} \subseteq \cdots \subseteq U_{\frac{2^k-1}{2^k}} \subseteq \overline{U}_{\frac{2^k-1}{2^k}} \subseteq U_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq \overline{U}_{\frac{2^{k+1}-1}{2^{k+1}}} \subseteq X \setminus B$$

Finally, Step 1 proved.

Step 2. Construct an Urysohn Function.

Define a map $f : X \rightarrow [0, 1]$ as:

$$f(x) = \begin{cases} 0 & x \in \bigcap_{t \in D} U_t \\ \sup\{t \in D \mid x \notin U_t\} & x \notin \bigcap_{t \in D} U_t \end{cases}$$



Then, this map f is well-defined by (*) and $\sup D \leq 1$. And f satisfies that:

1. $\forall r \in D, x \in A \subset U_r$. Thus, $f(x) = 0$ if $x \in A$.
2. $\forall r \in D, x \in B \subset X \setminus U_r$. Thus, $f(x) = \sup D = 1$ if $x \in B$.
3. If $x \in \overline{U}_r$, then for every $s > r$, $x \in \overline{U}_r \subset U_s$. Thus, $f(x) \leq r$. In Contrapositive, $f(x) > r \implies x \notin \overline{U}_r$. (If $f(x) = \sup\{t \in D \mid x \notin U_t\} > r$, then there is $s \in D$ such that $s > r$ and $x \notin U_s$, Contradiction.)
4. If $x \notin U_r$, then, $f(x) \geq r$. In Contrapositive, $f(x) < r \implies x \in U_r$.

Now, show that this map f is Continuous map: Let $x \in X$ be fixed arbitrarily, and $\varepsilon > 0$ be given.

In Case of $0 < f(x) < 1$.

Since Density of Dyadic Rationals, Choose $r, s \in D$ such that $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$.

Now, we obtain that:

$$x \stackrel{(*)}{\in} U_s \setminus \overline{U}_r \stackrel{(**)}{\subseteq} f^{-1}[(f(x) - \varepsilon, f(x) + \varepsilon)]$$

(*) directly given by above properties, (**) given applying the fact that $x \in U_s \subset \overline{U}_s$ and $x \notin \overline{U}_r$.

In Case of $f(x) = 0$.

Choose $r \in D$ such that $f(x) = 0 < r < \varepsilon = f(x) + \varepsilon$. Then,

$$x \in U_r \subset f^{-1}[(f(x), f(x) + \varepsilon)]$$

In Case of $f(x) = 1$.

Choose $r \in D$ such that $f(x) - \varepsilon = 1 - \varepsilon < r < 1 = f(x)$. Then,

$$x \in X \setminus U_r \subset f^{-1}[(f(x) - \varepsilon, f(x))]$$

Consequently, f is Continuous map on $[0, 1]$ such that $f|_A = 0$ and $f|_B = 1$.

Step 3. Generalization.

Since $[0, 1] \cong [a, b]$ for any $a < b$, let $g : [0, 1] \rightarrow [a, b] : x \mapsto (1-x)a + xb$ be a Homeomorphism.

Then, $h = g \circ f : X \rightarrow [a, b]$ becomes a Continuous map such that $h|_A = a$ and $h|_B = b$. □

12.13.2 Tietze Extension Theorem

Theorem 12.13.2.1. Tietze Extension Theorem

Let X be a T_4 Space, and $A \subseteq X$ be a closed subset.

For any Continuous map $f : A \rightarrow \mathbb{R}$, there exists a Continuous map:

$$g : X \rightarrow \mathbb{R} \text{ s.t. } g|_A = f$$

This g is called *extension* of f .

Proof. This proof consists by three steps.

Step 1. First, we will show that:

For any Continuous map $f : A \rightarrow [-r, r]$, there is a Continuous map $h : X \rightarrow \mathbb{R}$ s.t. $\begin{cases} \forall x \in X, |h(x)| \leq \frac{1}{3}r \\ \forall a \in A, |f(a) - h(a)| \leq \frac{2}{3}r \end{cases}$ (*)

Set

$$I_1 \stackrel{\text{def}}{=} \left[-r, -\frac{1}{3}r\right], \quad I_2 \stackrel{\text{def}}{=} \left[-\frac{1}{3}r, \frac{1}{3}r\right], \quad I_3 \stackrel{\text{def}}{=} \left[\frac{1}{3}r, r\right]$$

Then, the preimage of continuous map preserves closed and A is closed subspace of X , $f^{-1}[I_1]$ and $f^{-1}[I_3]$ are closed of X .

And, I_1 and I_3 are disjoint, thus $f^{-1}[I_1 \cap I_3] = f^{-1}[I_1] \cap f^{-1}[I_3] = \emptyset$.

Now, apply the Urysohn Lemma: There exists an Urysohn function $h : X \rightarrow I_2$ for $f^{-1}[I_1]$ and $f^{-1}[I_3]$.

Clearly, this map h satisfies the first condition in (*). And, for show the second condition, let $a \in A$ be given.

If $a \in f^{-1}[I_1]$, then $f(a) \in I_1$ and $h(a) = -\frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.

If $a \in f^{-1}[I_3]$, then $f(a) \in I_3$ and $h(a) = \frac{1}{3}r$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.

If $a \notin (f^{-1}[I_1] \cup f^{-1}[I_3])$, then $f(a), h(a) \in I_2$, thus $|f(a) - h(a)| \leq \frac{2}{3}r$.

Therefore, the second condition satisfied.

Step 2. We will show that: for any $f : A \rightarrow [-1, 1]$, there exists an extension of f .

Apply the result in Step 1, there exists a Continuous map:

$$h_1 : X \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, |h_1(x)| \leq \frac{1}{3} \\ \forall a \in A, |f(a) - h_1(a)| \leq \frac{2}{3} \end{cases}$$

Now, the second condition of h_1 , the continuous map $f - h_1 : A \rightarrow [-\frac{2}{3}, \frac{2}{3}] : x \rightarrow f(x) - h_1(x)$ is well-defined.

Again, there exists a Continuous map:

$$h_2 : X \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, |h_2(x)| \leq \frac{1}{3} \cdot \frac{2}{3} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a)| \leq (\frac{2}{3})^2 \end{cases}$$

Inductively, for any $n \in \mathbb{N}$, there exists a Continuous map:

$$h_n : X \rightarrow \mathbb{R} \text{ s.t. } \begin{cases} \forall x \in X, |h_n(x)| \leq \frac{1}{3} \cdot (\frac{2}{3})^{n-1} \\ \forall a \in A, |f(a) - h_1(a) - h_2(a) - \dots - h_n(a)| \leq (\frac{2}{3})^n \end{cases}$$

Define a map

$$g : X \rightarrow [-1, 1] : x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

For any $x \in X$,

$$|g(x)| = \left| \sum_{n=1}^{\infty} h_n(x) \right| \leq \sum_{n=1}^{\infty} |h_n(x)| \leq \sum_{n=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} = 1$$

Therefore, this map is well-defined. And, Weierstrass M-test gives that $\sum_{n=1}^{\infty} h_n(x)$ converges uniformly.

Moreover, for any $a \in A$,

$$\left| f(a) - \sum_{k=1}^n h_k(a) \right| \leq \left(\frac{2}{3}\right)^n \implies \left| f(a) - \sum_{n=1}^{\infty} h_n(a) \right| = |f(a) - g(a)| = 0$$

That is, g is Continuous on X and $g|_A = f$. Therefore, g is extension of f .

Step 3. Finally, we generalize the result in Step 2.:

Let $f : A \rightarrow [a, b]$ be a Continuous map on the closed subspace A . And, let $\varphi : [a, b] \rightarrow [-1, 1]$ be a Homeomorphism.

Then, $\varphi \circ f : A \rightarrow [-1, 1]$ is Continuous map, thus there exists an extension $g : X \rightarrow [-1, 1]$ such that $g|_A = \varphi \circ f$.

Now, $\varphi^{-1} \circ g : X \rightarrow [a, b]$ is Continuous, and $(\varphi^{-1} \circ g)|_A = \varphi^{-1} \circ \varphi \circ f = f$, Therefore this $\varphi^{-1} \circ g$ is the extension of f .

Let $f : A \rightarrow \mathbb{R}$ be a Continuous map on the closed subspace A .

And, let $\varphi : \mathbb{R} \rightarrow (-1, 1)$ be a Homeomorphism. Then, the map $\phi : \mathbb{R} \rightarrow [-1, 1] : x \mapsto \varphi(x)$ is still Continuous.

Now, The Continuous map $\phi \circ f : A \rightarrow [-1, 1]$ has an extension $g : X \rightarrow [-1, 1]$ such that $g|_A = \phi \circ f$.

Put $B = g^{-1}[\{-1, 1\}]$. Then B is Closed on X , and $A \cap B = \emptyset$. Now, apply the Urysohn Lemma to this, there exists an Urysohn function for A and B : Continuous map $\gamma : X \rightarrow [0, 1]$ such that $\gamma|_A = 1$ and $\gamma|_B = 0$.

Define a map $\eta : X \rightarrow (-1, 1) : x \mapsto g(x)\gamma(x)$. Then, if $g(x) = 1$ or $g(x) = -1$, then $x \in B$, thus $g(x)\gamma(x) = 0$.

Therefore, η is well-defined. And, for any $a \in A$, $\eta(a) = g(a)\gamma(a) = g(a)$, thus $\eta|_A = \phi \circ f$.

Consequently, the map $\phi^{-1} \circ \eta$ is an extension of f , we wanted. \square

Recall that:

Definition 12.13.2.1. X is T_1 if:

For any distinct $x, y \in X$, there exist open sets U_x, U_y such that $\begin{cases} x \in U_x, & x \notin U_y \\ y \notin U_x, & y \in U_y \end{cases}$.

Lemma 12.13.2.1. X is T_1 if and only if For any $x \in X$, a singleton $\{x\}$ is closed in X .

Proof. The left direction is clear.

Let $x \in X$. Then, for any $y \in X$ with $y \neq x$, T_1 condition gives that there is an open set such that $y \in U_y$ and $x \notin U_y$.

Now, the union

$$\bigcup_{\substack{y \in X \\ y \neq x}} U_y = X \setminus \{x\}$$

is open by definition. \square

12.13.3 Urysohn Metrization Theorem

Definition 12.13.3.1. A space X is called *Completely Regular* if: X is T_1 and $T_{3\frac{1}{2}}$ where $T_{3\frac{1}{2}}$ Condition: For any closed set $C \subset X$ and $x \in X \setminus C$, there exists an *Urysohn function* for $\{x\}$ and C . Completely regular space is sometimes called *Tychonoff Space*.

Proposition 12.13.3.1. Normal Space \Rightarrow Completely Regular Space \Rightarrow Regular Space.

Proof. If X is Normal space, then every singletone is closed by T_1 . And, the *Urysohn Lemma* gives Urysohn map. If X is Completely Regular, then for closed $C \subset X$ and $x \in X \setminus C$, there exists a continuous map $f : X \rightarrow [0, 1]$ s.t

$$f[\{x\}] = 0 \text{ and } f[C] = \{1\}$$

Then,

$$\{x\} \subseteq f^{-1}\left[\left[0, \frac{1}{2}\right)\right], \quad C \subseteq f^{-1}\left[\left(\frac{1}{2}, 1\right]\right]$$

□

Theorem 12.13.3.1. $T_{3\frac{1}{2}}$ is Hereditary. Furthermore, Completely Regular is hereditary since T_1 is hereditary.

Proof. Let X be a $T_{3\frac{1}{2}}$ Space, and $Y \subseteq X$ be a subspace of X .

Let $C \subseteq Y$ is closed set of Y , and $x \in Y \setminus C$. Note that:

$$C = \text{Closure of } C \text{ in } Y = \bigcap_{\substack{F \text{ closed in } Y \\ C \subseteq F}} F = \bigcap_{\substack{F' \text{ closed in } X \\ \text{s.t. } F = F' \cap Y}} F' \cap Y = (\text{Closure of } C \text{ in } X) \cap Y$$

Since x is contained in Y but not C , thus x is not contained in Closure of C in X . Now, since X is $T_{3\frac{1}{2}}$,

There exists a Continuous map $f : X \rightarrow [0, 1]$ s.t. $f(x) = 0, f|_{\text{cl}_X(C)} = 1$

The restriction f_Y is continuous, and Urysohn function for x and C .

□

Theorem 12.13.3.2. Arbitrary product space of $T_{3\frac{1}{2}}$ space is $T_{3\frac{1}{2}}$.

Proof. Let X_γ ($\gamma \in \Gamma$) be $T_{3\frac{1}{2}}$ Spaces. Put $X = \prod_{\gamma \in \Gamma} X_\gamma$. Suppose that $C \subset X$ is closed set, and $x \in X \setminus C$.

Since $X \setminus C$ is open, there exists an open U in X such that $x \in U \subset X \setminus C$.

Put $F = \{\alpha \in \Gamma \mid X_\alpha \neq \pi_\alpha[U]\}$. By definition of product space, this F is a finite index set. Note that:

$$\forall \alpha \in F, \pi_\alpha(x) \notin X_\alpha \setminus \pi_\alpha[U]$$

And, for each $\alpha \in F$, $X_\alpha \setminus \pi_\alpha[U]$ are non-empty closed set in X_α , there exist continuous maps f_α such that

$$f_\alpha : X_\alpha \rightarrow [0, 1], \quad f_\alpha|_{X_\alpha \setminus \pi_\alpha[U]} = 0, \quad f_\alpha|_{\pi_\alpha(U)} = 1$$

And, the composition $f_\alpha \circ \pi_\alpha$ ($\alpha \in F$) is continuous, and

$$(f_\alpha \circ \pi_\alpha)[X \setminus \pi_\alpha^{-1}[\pi_\alpha[U]]] = (f_\alpha \circ \pi_\alpha)[\pi_\alpha^{-1}[X_\alpha \setminus \pi_\alpha[U]]] \subseteq f_\alpha[X_\alpha \setminus \pi_\alpha[U]] = \{0\}$$

Now, the map

$$\Psi : X \rightarrow [0, 1] : t \mapsto \prod_{\alpha \in F} (f_\alpha \circ \pi_\alpha)(t)$$

is Continuous, and $\Psi(x) = 1$ and $\Psi[C] \subseteq \Psi[X \setminus U] = \{0\}$.

□

Theorem 12.13.3.3. If X is Completely Regular, then for some index set Λ , X can be embedded in $[0, 1]^\Lambda$.

Proof. Denote that:

$$\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \rightarrow [0, 1] \mid f \text{ is continuous}\}$$

Claim: the following function is embedding X into $[0, 1]^\Lambda$.

$$F : X \rightarrow [0, 1]^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$$

1. F is Continuous, since each f_α is Continuous.

2. F is injective: Let $x \neq y$ in X . Then, $\{x\}$ and $\{y\}$ are closed by T_1 .

By $T_{3\frac{1}{2}}$, there exists a continuous map $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$.

Since $f = f_\beta$ for some $\beta \in \Lambda$, $F(x) \neq F(y)$.

3. F is Open map: Let $U \subseteq X$ be an open set, and let $y \in F[U]$. That is, for some $x \in U$, $F(x) = y$.

Since $X \setminus U$ is closed, $x \notin X \setminus U$, and $T_{3\frac{1}{2}}$, there exists a continuous map

$$f_\alpha : X \rightarrow [0, 1] \text{ s.t. } f_\alpha(x) = 0, f_\alpha|_{X \setminus U} = 1$$

Meanwhile, put $V \stackrel{\text{def}}{=} \pi_\alpha^{-1}[[0, 1]] \subseteq [0, 1]^\Lambda$, and $W \stackrel{\text{def}}{=} V \cap F[X]$. Then, W is open in the subspace $F[X]$.

And, $\pi_\alpha(y) = \pi_\alpha(F(x)) = (\pi_\alpha \circ F)(x) = f_\alpha(x) = 0$, thus $y \in W$. Now, there remains to prove: $W \subseteq F[U]$.

Let $z \in W$. Then, $z \in V$ and $z = F(x)$ for some $x \in X$, this implies $\pi_\alpha(z) \in [0, 1]$, i.e., $\pi_\alpha(F(x)) = f_\alpha(x) \neq 1$.

Now, $x \in U$, that is $F(x) = z \in F[U]$. Thus, $W \subseteq F[U]$, consequently F is embedding.

□

Corollary 12.13.3.1. Let X is a Topological Space. TFAE:

a) X is Completely Regular Space.

b) X can be embedded in Compact Hausdorff Space.

c) X can be embedded in Normal Space.

Proof.

1. a) \Rightarrow b). X can be embedded in $[0, 1]^\Lambda$. And $[0, 1]$ is Compact Hausdorff.

2. b) \Rightarrow c). Every Compact Hausdorff Space is Normal.

3. c) \Rightarrow a). Normal Space is Completely Regular, and Completely Regular is hereditary.

□

Lemma 12.13.3.1. Every Compact Hausdorff Space is Normal.

Proof. Let X be a Compact Hausdorff Space, and $C, D \subset X$ be disjoint closed subsets.

Since X is Compact, C and D are Compact. Fix $x \in C$. Then, for any $y \in D$,

There exist disjoint opens U_y, V_y such that $x \in U_y$ and $y \in V_y$.

Since $\{V_y \mid y \in D\}$ is open cover of D , there is a finite subcover $\{V_y^i \mid 1 \leq i \leq n\}$. That is, $D \subseteq \bigcup_{i=1}^n V_y^i$.

Now, $\bigcap_{i=1}^n U_y^i$ is open set containing x , and

$$\left(\bigcup_{i=1}^n V_y^i \right) \cap \left(\bigcap_{i=1}^n U_y^i \right) = \bigcup_{i=1}^n \left(V_y^i \cap \left(\bigcap_{i=1}^n U_y^i \right) \right) = \bigcup_{i=1}^n \emptyset = \emptyset$$

In summary, for any $x \in C$, there exist disjoint open U_x, V_x such that $x \in U_x$ and $D \subset V_x$.

Using this, Let $\{U_x \mid x \in C\}$ be an open cover, then compactness gives the finite subcover $\{U_x^i \mid 1 \leq i \leq n\}$. Now,

$$C \subseteq \bigcup_{i=1}^n U_x^i, D \subseteq \bigcap_{i=1}^n V_x^i, \left(\bigcup_{i=1}^n U_x^i \right) \cap \left(\bigcap_{i=1}^n V_x^i \right) = \bigcup_{i=1}^n \left(U_x^i \cap \left(\bigcap_{i=1}^n V_x^i \right) \right) = \emptyset$$

□

Theorem 12.13.3.4. Embedding Theorem

Let X be a T_1 Space. Denote $\{f_\alpha \mid \alpha \in \Lambda\} = \{f : X \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$.

Suppose that for any $x \in X$ and open neighborhood U of x , there exists $\alpha \in \Lambda$ such that $f_\alpha(x) > 0$, $f_\alpha|_{X \setminus U} = 0$. Then, the map $F : X \rightarrow \mathbb{R}^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$ is embedding.

Theorem 12.13.3.5. Suppose that X is Second-Countable Regular Space.

Then there exists a Countable collection $\{f_n : X \rightarrow [0, 1] \mid n \in \mathbb{N}\}$ such that:

For any open $U \subset X$ and $x \in U$, there exists $n \in \mathbb{N}$ such that $f_n(x) > 0$ and $f_n|_{X \setminus U} = 0$.

Theorem 12.13.3.6. Urysohn Metrization Theorem

If X is a Second-Countable Regular Space, then X is Metrizable.

12.14 Examples

Proposition 12.14.0.1. Lower Limit Topology $(\mathbb{R}, \mathcal{T}_l)$ is T_1 and T_4 Space. Therefore, Normal Space.

Proof. T_1 is clear, because: let $x, y \in \mathbb{R}$ be a distinct two points. Without Loss of Generality, assume $x < y$. Then,

$$\begin{cases} x \in \left[x, \frac{x+y}{2}\right), \quad y \in [y, y+1) \\ y \notin \left[x, \frac{x+y}{2}\right), \quad x \notin [y, y+1) \end{cases}$$

Thus, T_1 satisfied. And, to show T_4 , Let $C, D \subseteq \mathbb{R}$ be disjoint closed subsets. Let $x \in C$ be given. Then, there exists a basis element of \mathcal{T}_l , $[a, p_x)$ such that

$$x \in [a, p_x) \subseteq \mathbb{R} \setminus D$$

since $C \subseteq \mathbb{R} \setminus D$ and $\mathbb{R} \setminus D$ is open. Now,

$$U = \bigcup_{x \in C} [x, p_x)$$

is open set containing C , and $U \cap D = \emptyset$.

Similarly, let $y \in D$ be given. Then, there exists a basis element $[a, q_y)$ such that

$$y \in [b, q_y) \subseteq \mathbb{R} \setminus C$$

Then, for each $y \in D$, $[y, q_y) \cap U = \emptyset$ because: Suppose that $[y, q_y) \cap U \neq \emptyset$. Choose $p \in [y, q_y) \cap U$.

That is, $p \in [y, q_y)$ and for some $x \in C$, $p \in [x, p_x)$.

Hence, $[\max(x, y), \min(p_x, q_y))$ is non-empty set which containing x or y .

If either x or y contained in $[\max(x, y), \min(p_x, q_y))$, contradiction. Now, an union

$$V = \bigcup_{y \in D} [y, q_y)$$

is open set containing D , and $U \cap V = \emptyset$,

□

12.15 Quotient Space

Definition 12.15.0.1. Let (X, \mathcal{T}) be a Topological Space, Y be a set, and $f : X \rightarrow Y$ be an onto map. Define Quotient Topology on Y induced by f : $\mathcal{T}_Q \stackrel{\text{def}}{=} \{U \subseteq Y \mid f^{-1}[U] \in \mathcal{T}\}$. This is the largest topology on Y such that f is Continuous map.

Definition 12.15.0.2. Let X be a Topological Space, and \sim be an equivalent relation on X . Define Canonical map on X : $\pi : X \rightarrow X/\sim : x \mapsto [x]$, and define Quotient Space $(X/\sim, \mathcal{T}_Q)$ where \mathcal{T}_Q is quotient topology on X/\sim induced by π .

X Topological Space, \sim equivalent relation on X , $\pi : X \rightarrow X/\sim : x \mapsto [x]$ canonical map.

Lemma 12.15.0.1. For any topological space Z and a map $g : X/\sim \rightarrow Z$,

g is Continuous if and only if $g \circ \pi$ is Continuous.

Proof. Let $g \circ \pi$ be Continuous map. Then, for any open $U \subseteq Z$,

$$(g \circ \pi)^{-1}[U] = \pi^{-1}[g^{-1}[U]]$$

is open, thus $g^{-1}[U]$ is open in X/\sim . That is, g is Continuous. \square

Lemma 12.15.0.2. Let Z be a Topological space.

If given Continuous map $f : X \rightarrow Z$ satisfies $x \sim y \implies f(x) = f(y)$, then $\tilde{f} : X/\sim \rightarrow Z : [x] \mapsto f(x)$ is Continuous, and unique map such that $\tilde{f} \circ \pi = f$.

Proof. Well-Defined because: $[x] = [y] \iff x \sim y \implies f(x) = f(y)$.

$\tilde{f} \circ \pi = f$: for any $x \in X$, $(\tilde{f} \circ \pi)(x) = \tilde{f}(\pi(x)) = \tilde{f}([x]) = f(x)$, thus \tilde{f} is continuous since above lemma.

Uniqueness: if $g : X/\sim \rightarrow Z$ satisfies $g \circ \pi = f$, then for any $[x] \in X/\sim$,

$$g([x]) = g(\pi(x)) = (g \circ \pi)(x) = f(x) = \tilde{f}([x])$$

\square

Lemma 12.15.0.3. Let Z be a Topological space.

If given Continuous onto map $f : X \rightarrow Z$ satisfies $x \sim y \iff f(x) = f(y)$, and f is either open or closed map, then $\tilde{f} : X/\sim \rightarrow Z : [x] \mapsto f(x)$ is Homeomorphism.

Proof. Since $[x] = [y] \iff x \sim y \iff f(x) = f(y)$, \tilde{f} is Well-defined and injective.

Continuity given by above Lemma, and Surjective:

$$\tilde{f}[X/\sim] = \tilde{f}[\pi[X]] = f[X] = Z$$

The last equality given by Surjectiveness of f . \square

12.16 Quotient Map

Definition 12.16.0.1. Let X, Y be Topological Space.

A Continuous onto map $f : X \rightarrow Y$ is called **quotient map** if:

$$U \subseteq Y \text{ is open if and only if } f^{-1}[U] \subseteq X \text{ is open.}$$

12.16.1 Basic Properties

Proposition 12.16.1.1. Composition of quotient maps is quotient map.

Proof. Suppose that X, Y, Z are Topological Space, and $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are Quotient map. Then,

$$U \subseteq Z \text{ is open} \iff g^{-1}[U] \subseteq Y \text{ is open} \iff f^{-1}[g^{-1}[U]] \subseteq X \text{ is open}$$

implies $g \circ f : X \rightarrow Z$ is Quotient map, being $f^{-1}[g^{-1}[U]] = (g \circ f)^{-1}[U]$. \square

Proposition 12.16.1.2. Continuous onto map is quotient map if either open or closed map.

Proof. Suppose that $f : X \rightarrow Y$ is Continuous onto map.

If f is an Open map, \square

Theorem 12.16.1.1. If $f : X \rightarrow Y$ is quotient map, then $X/\sim \cong Y$ where

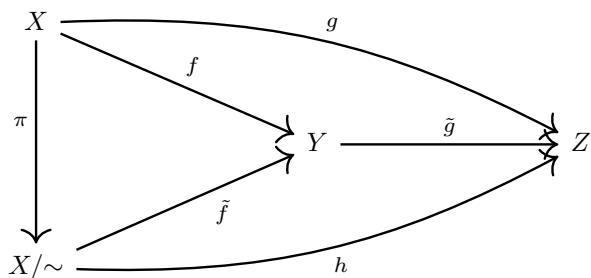
$$x \sim y \iff f(x) = f(y)$$

Moreover, if continuous map $g : X \rightarrow Z$ satisfies

$$f(x) = f(y) \implies g(x) = g(y)$$

Then, $\tilde{g} : Y \rightarrow Z : f(x) \mapsto g(x)$ is the unique continuous map such that $\tilde{g} \circ f = g$.

12.16.2 Quotient map Diagram



- $f : X \rightarrow Y$ is quotient map.
- $g : X \rightarrow Z$ is continuous map s.t. $f(x) = f(y) \implies g(x) = g(y)$.
- $\pi : X \rightarrow X/\sim : x \mapsto [x]$.
- X/\sim is quotient topology induced by π .

In this setting, \tilde{f} is Homeomorphism between X/\sim and Y , and $h = \tilde{g} \circ \tilde{f}$ is Continuous map between X/\sim and Z .

12.17 Typical Quotient Spaces

We follow some convention:

'Equivalent relation on X : Condition' means the smallest Equivalent relation on X such that the Condition.
In this section, put $I \stackrel{\text{def}}{=} [0, 1] \subset \mathbb{R}$.

12.17.1 Cylinder

Definition 12.17.1.1. Define an Equivalent Relation on $I^2 = I \times I$:

$$\forall y \in I, (0, y) \sim (1, y)$$

That is,

$$\sim^{\text{def}} \{((x, y), (x, y)) \mid x, y \in I\} \cup \{((0, y), (1, y)), ((1, y), (0, y)) \mid y \in I\}$$

The Quotient Space I^2 / \sim is called *Cylinder*.

Theorem 12.17.1.1. The Cylinder I^2 / \sim is Homeomorphic to $S^1 \times I$.

Proof. Define a map:

$$f : I^2 \rightarrow \frac{S^1 \times I}{\subset \mathbb{R}^3} : (x, y) \mapsto (\underbrace{(\cos(2\pi x), \sin(2\pi x))}_{\in S^1}, y)$$

f is Continuous: Since $x \mapsto \cos(2\pi x)$, $x \mapsto \sin(2\pi x)$, and $y \mapsto y$ are all Continuous.

f is Surjective, clearly. Thus, the f is Continuous onto map.

f is Closed map: I^2 is Compact, being Closed Bounded subset of \mathbb{R}^2 , and $S^1 \times I$ is Hausdorff, Hereditary Property.
Finally, Check that Fact:

$$\begin{aligned} (x_1, y_1) \sim (x_2, y_2) &\iff (x_1, y_1) = (x_2, y_2) \text{ or } x_1 = 0, x_2 = 1, y_1 = y_2 \text{ or } x_1 = 1, x_2 = 0, y_1 = y_2 \\ &\iff f((x_1, y_1)) = f((x_2, y_2)) \end{aligned}$$

Now, The map $\tilde{f} : I^2 / \sim \rightarrow S^1 \times I : [(x, y)] \mapsto f((x, y))$ induced by f is Homeomorphism, $I^2 / \sim \cong S^1 \times I$. □

12.17.2 Möbius band

Definition 12.17.2.1. Define an Equivalent Relation on I^2 :

$$\forall y \in I, (0, y) \sim (1, 1 - y)$$

The Quotient Space I^2 / \sim is called *Möbius band*.

12.17.3 Torus

Definition 12.17.3.1. Define an Equivalent Relation on I^2 :

$$\begin{cases} \forall x \in I, (x, 0) \sim (x, 1) \\ \forall y \in I, (0, y) \sim (1, y) \\ (0, 0) \sim (1, 0) \sim (0, 1) \sim (1, 1) \end{cases}$$

The Quotient Space I^2/\sim is called **Torus**.

Theorem 12.17.3.1. The Torus I^2/\sim is Homeomorphic to $S^1 \times S^1 \subset \mathbb{C}^2$.

Proof. Define a map:

$$f : \frac{I^2}{\text{Compact}} \rightarrow \frac{S^1 \times S^1}{\text{Hausdorff}} : (x, y) \mapsto (e^{2x\pi i}, e^{2y\pi i})$$

Then, f is Continuous Onto, and Closed map. Moreover,

$$(x_1, y_1) \sim (x_2, y_2) \iff (e^{2x_1\pi i}, e^{2y_1\pi i}) = (e^{2x_2\pi i}, e^{2y_2\pi i}) \iff f((x_1, y_1)) = f((x_2, y_2))$$

Now, The map $\tilde{f} : I^2/\sim \rightarrow S^1 \times S^1 : [(x, y)] \mapsto f((x, y))$ induced by f is Homeomorphism, $I^2/\sim \cong S^1 \times S^1$. \square

Theorem 12.17.3.2. The Torus I^2/\sim can embed into \mathbb{R}^3 .

Proof. Define a map:

$$g : I^2 \rightarrow \mathbb{C} \times \mathbb{R} : (x, y) \mapsto ([2 + \cos(2\pi x) \cdot e^{2y\pi i}, \sin(2\pi x)])$$

Then, g is Continuous map. Moreover,

$$\begin{aligned} g((x_1, y_1)) = g((x_2, y_2)) &\iff ([2 + \cos(2\pi x_1) \cdot e^{2y_1\pi i}, \sin(2\pi x_1)]) = ([2 + \cos(2\pi x_2) \cdot e^{2y_2\pi i}, \sin(2\pi x_2)]) \\ &\iff (x_1 = x_2 \text{ or } x_1, x_2 \in \{0, 1\}) \text{ and } (y_1 = y_2 \text{ or } y_1, y_2 \in \{0, 1\}) \\ &\iff (x_1, y_1) \sim (x_2, y_2) \end{aligned}$$

Meanwhile, $g[I^2] \subset \mathbb{C} \times \mathbb{R}$ is Hausdorff, thus the map $G : I^2 \rightarrow g[I^2] : (x, y) \mapsto g((x, y))$ is Continuous Onto Closed map.

Now, the map $\tilde{G} : I^2/\sim \rightarrow \frac{g[I^2]}{\subset \mathbb{C} \times \mathbb{R}}$ is Embedding. \square

12.17.4 Klein Bottle

Definition 12.17.4.1. Define an Equivalent Relation on I^2 :

$$\begin{cases} \forall x \in I, (x, 0) \sim (x, 1) \\ \forall y \in I, (0, y) \sim (1, 1 - y) \\ (0, 0) \sim (1, 0) \sim (0, 1) \sim (1, 1) \end{cases}$$

The Quotient Space I^2/\sim is called **Klein Bottle**.

Theorem 12.17.4.1. The Klein Bottle I^2/\sim can embed into \mathbb{R}^4 .

Proof. Describes Sketch: Define a map:

$$g : I^2 \rightarrow \mathbb{C}^2 : (x, y) \mapsto ([2 + \cos(2\pi x)] \cdot e^{2y\pi i}, \sin(2\pi x) \cdot e^{2y\pi i})$$

Then, \tilde{g} becomes embedding. \square

12.17.5 Real Projective Space

Definition 12.17.5.1. Define an Equivalent Relation on $\mathbb{R}^{n+1} \setminus \{\mathbf{0}\}$:

$$x \sim y \iff \exists t \in \mathbb{R} \text{ s.t. } t \cdot x = y$$

The Quotient Space $(\mathbb{R}^{n+1} \setminus \{\mathbf{0}\})/\sim$ is called **Real Projective Space**, denoted \mathbb{RP}^n .

Lemma 12.17.5.1. The Real Projective Space \mathbb{RP}^n is Hausdorff.

Proof. Let distinct two point $[x], [y] \in \mathbb{RP}^n$ be given.

Clearly, $x \neq y$, and note that: for a projective map $\pi : \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \rightarrow \mathbb{RP}^n : x \mapsto [x]$,

$$U \subseteq \mathbb{RP}^n \text{ is open} \iff \pi^{-1}[U] \subseteq \mathbb{R}^{n+1} \setminus \{\mathbf{0}\} \text{ is open}$$

Regularizing: $y' = \frac{\|x\|}{\|y\|} \cdot y$. Then, $[y'] = [y]$ and $\|y'\| = \|x\|$. And put $r = d(x, y')$. Now,

$$U = \pi[B_{\frac{r}{2}}(x)], \quad V = \pi[B_{\frac{r}{2}}(y')]$$

are disjoint open set of \mathbb{RP}^n , and $[x] \in U$ and $[y] = [y'] \in V$. And, $U \cap V = \emptyset$, because:

U, V are Opens: Let $a \in \pi^{-1}[U]$ be given. Then,

$$a \in \pi^{-1}[U] \iff \pi(a) \in U \iff \exists z \in B_{\frac{r}{2}}(x) \text{ s.t. } a \sim z$$

Put $\delta = \frac{r}{2} - d(x, z)$. Then, $a \in B_\delta(a) \subset \pi^{-1}[U]$. Thus $\pi^{-1}[U]$ is open, so is U . Similarly, V .

U and V are disjoint: Suppose that there exists $[z] \in \pi[B_{\frac{r}{2}}(x)] \cap \pi[B_{\frac{r}{2}}(y')]$.

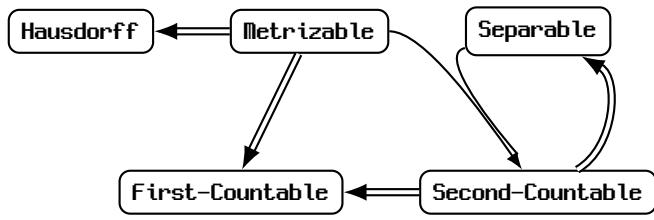
Then, for some $a \in B_{\frac{r}{2}}(x)$ and $b \in B_{\frac{r}{2}}(y')$, $[z] = [a] = [b]$. But, since setting, $[a] \neq [b]$. Contradiction. \square

Theorem 12.17.5.1. Define an Equivalent Relation \sim' on $S^n = \{\mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\| = 1\}$:

$$\forall \mathbf{x} \in S^n, \quad \mathbf{x} \sim' -\mathbf{x}$$

Then, the Real Projective Space \mathbb{RP}^n is Homeomorphic to the Quotient space S^n / \sim' .

12.18 Diagrams



12.19 Manifold

12.19.1 Definition

Definition 12.19.1.1. A Topological Space X is called *n-Dimensional Manifold* if: X is Second-Countable Hausdorff Space, and satisfying

For all $x \in X$, there exists Open U_x with $x \in U_x$ such that $U_x \cong \mathbb{R}^n$ or $U_x \cong \mathbb{H}^n \stackrel{\text{def}}{=} \{(x_i)_{i=1}^n \in \mathbb{R}^n \mid x_n \geq 0\}$.

The 1-Dimensional Manifold is called *Curve*, and 2-Dimensional Manifold is called *Surface*.

If every Neighborhood Homeomorphic to \mathbb{R}^n , then it is called a *Manifold with no Boundary*, and if not, then it is called a *Manifold with Boundary*.

For *n*-Dimensional Manifold X , and $x \in X$,

If there exists an Open U_x with $x \in U_x$ s.t $U_x \cong \mathbb{R}^n$, then x is called *Interior*, Denote the set $X^\circ \subset X$.

And, the set $\partial X \stackrel{\text{def}}{=} X \setminus X^\circ$ is called *Boundary* of X .

*This Definition is temporary.

Lemma 12.19.1.1. For any $r > 0$, The Euclidean Space \mathbb{R}^n is Homeomorphic to $B(0, r)$, and The \mathbb{H}^n is Homeomorphic to $B(0, r)$ to $B(0, r) \cap \mathbb{H}^n$.

12.19.2 Connected Sum

Chapter 13

Algebraic Topology

13.1 Orbit Space

Definition 13.1.0.1. Suppose that the set G is $\left\{ \begin{array}{l} \text{Topological Space under the Topology } \mathcal{T} \\ \text{Group under the operation } \cdot : G \times G \rightarrow G \end{array} \right.$.

The triple (G, \mathcal{T}, \cdot) is called *Topological Group* if: $\left\{ \begin{array}{l} \mu : G \times G \rightarrow G : (g_1, g_2) \mapsto g_1 g_2 \\ \iota : G \rightarrow G : g \mapsto g^{-1} \end{array} \right.$ are Continuous maps.

13.1.1 General Linear Group

Definition 13.1.1.1. Define a set of invertible matrices, *General Linear Group over field F*:

$$\mathrm{GL}(n, F) \stackrel{\mathrm{def}}{=} \{A \in \mathcal{M}_{n,n}(F) \mid \det A \neq 0\}$$

13.2 Homotopy

13.3 Fundamental Group

Chapter 14

Basic Analysis

14.1 Tests for Series

14.1.1 Integral Test

Theorem 14.1.1.1. Let $f : [1, \infty) \rightarrow \mathbb{R}$ be a decreasing function which satisfies $\begin{cases} \lim_{x \rightarrow \infty} f(x) = 0 \\ f > 0 \end{cases}$. Then,

$$\int_1^\infty f(x)dx \text{ converges if and only if } \sum_{k=1}^{\infty} f(k) \text{ converges.}$$

Futhermore, put $d_n \stackrel{\text{def}}{=} \sum_{k=1}^n f(k) - \int_1^n f(x)dx$, then for any $n \in \mathbb{N}$, $0 < f(n+1) \leq d_{n+1} \leq d_n \leq f(1)$, and for any $k \in \mathbb{N}$, $0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k)$. (Clearly, $\lim_{n \rightarrow \infty} d_n$ exists.)

Proof. Since

$$\begin{aligned} \int_1^{n+1} f(x)dx &= \sum_{k=1}^n \int_k^{k+1} f(x)dx \stackrel{\text{decreasing}}{\downarrow} \sum_{k=1}^n f(k) = \sum_{k=1}^n f(k) \\ \implies f(n+1) &= \sum_{k=1}^{n+1} f(k) - \sum_{k=1}^n f(k) \leq \sum_{k=1}^{n+1} f(k) - \int_1^{n+1} f(x)dx = d_{n+1} \end{aligned}$$

And,

$$d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \geq \int_n^{n+1} f(n+1)dx - f(n+1) = 0$$

Immediate d_n converges, being bounded and decreasing. That is,

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n f(k) - \int_1^n f(x)dx \right)$$

converges. Meanwhile, since

$$0 \leq d_n - d_{n+1} = \int_n^{n+1} f(x)dx - f(n+1) \leq \int_n^{n+1} f(n)dx - f(n+1) = f(n) - f(n+1)$$

Now, telescope:

$$0 \leq d_k - \lim_{n \rightarrow \infty} d_n \leq f(k) - \lim_{n \rightarrow \infty} f(n+1) = f(k)$$

□

14.1.2 Ratio Test

Theorem 14.1.2.1. Let $\sum a_n$ be given.

$\sum_{n=1}^{\infty} a_n$ converges if: $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.

$\sum_{n=1}^{\infty} a_n$ diverges if: $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$.

Proof. Choose $\beta < 1$ such that for some $N \in \mathbb{N}$, $n \geq N \implies \left| \frac{a_{n+1}}{a_n} \right| < \beta < 1$.

Then,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N| \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N| \\ &\vdots \\ |a_{N+p}| &< \beta^p |a_N| \quad (p \in \mathbb{N}) \end{aligned}$$

As a result, for all $n \geq N$, $|a_n| < \beta^{n-N} |a_N|$. And, $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} \beta^{n-N} |a_N| < \infty$.

□

14.1.3 Root Test

Theorem 14.1.3.1. Let $\sum a_n$ be given.

$\sum_{n=1}^{\infty} a_n$ converges if: $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$.

$\sum_{n=1}^{\infty} a_n$ diverges if: $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$.

Proof. Put $\beta \in \mathbb{R}$ such that $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} < \beta < 1$. Then, there is $N \in \mathbb{N}$ such that $n \geq N \implies \sqrt[n]{|a_n|} < \beta$.

Now, $\sum |a_n| < \sum \beta^n < \infty$. But if $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$, then $a_n \not\rightarrow 0$.

□

14.2 Arithmetic means

Let $\{s_n\}$ be a Complex numbers Sequence. Define the *Arithmetic means* of $\{s_n\}$:

$$\sigma_n \stackrel{\text{def}}{=} \frac{s_0 + \dots + s_n}{n+1} = \frac{1}{n+1} \left(\sum_{i=0}^n s_i \right)$$

Then, the Arithmetic means σ_n has the following properties:

1). If $\lim_{n \rightarrow \infty} s_n = s$, then $\lim_{n \rightarrow \infty} \sigma_n = s$.

Proof. Let $\varepsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|s_n - s| < \varepsilon$.

Now, for $n \geq N$,

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{s_0 + \dots + s_n}{n+1} - \frac{(n+1)s}{n+1} \right| = \left| \frac{(s_0 - s) + \dots + (s_n - s)}{n+1} \right| \\ &\stackrel{\text{tri. eq}}{\leq} \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{\sum_{k=N}^n |s_k - s|}{n+1} \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \frac{n+1-N}{n+1} \cdot \varepsilon \\ &< \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} + \varepsilon \end{aligned}$$

Now, put $M \in \mathbb{N}$ satisfies $M \geq N$ and $n \geq M \implies \frac{\sum_{k=0}^{N-1} |s_k - s|}{n+1} < \varepsilon$, using Archimedean property.
Then, $n \geq M$ implies $|\sigma_n - s| < \varepsilon$, thus $\sigma_n \rightarrow s$. □

2). Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. If $\lim_{n \rightarrow \infty} n a_n = 0$ and σ_n converges, then s_n converges.

Proof. First,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + \dots + s_n}{n+1} = \frac{(n+1)s_n - \sum_{k=0}^n s_k}{n+1} \\ &= \frac{1}{n+1} ((s_1 - s_0) + (2s_2 - 2s_1) + (3s_3 - 3s_2) + \dots + (ns_n - ns_{n-1})) \\ &= \frac{1}{n+1} \sum_{k=1}^n k a_k \end{aligned}$$

Now, if $na_n \rightarrow 0$ and $\sigma_n \rightarrow \sigma$,

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \left(\sigma_n + \frac{1}{n+1} \sum_{k=1}^n k a_k \right) \\ &= \lim_{n \rightarrow \infty} \sigma_n + \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{k=1}^n k a_k \stackrel{1)}{=} \sigma \end{aligned}$$

□

2) is conditional converse of 1). But, there is more weak version of the converse proposition:

3). The sequence $\{na_n\}$ bounded by $M < \infty$, and $\sigma_n \rightarrow \sigma$. Then, $s_n \rightarrow \sigma$.

Proof. First, For positive integers $m < n$,

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{\sum_{k=0}^n s_k}{n+1} = s_n - \frac{m+1}{n-m} \cdot \left(\frac{1}{m+1} - \frac{1}{n+1} \right) \sum_{k=0}^n s_k \\ &= s_n - \frac{m+1}{n-m} \cdot \left(\frac{\sum_{k=0}^m s_k + \sum_{k=m+1}^n s_k}{m+1} - \frac{\sum_{k=0}^n s_k}{n+1} \right) \\ &= s_n - \frac{m+1}{n-m} \cdot \left(\sigma_m - \sigma_n + \frac{\sum_{k=m+1}^n s_k}{m+1} \right) \\ &= \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{k=m+1}^n (s_n - s_k) \end{aligned}$$

Meanwhile, since for any $n \in \mathbb{N}$, $|na_n| = n|s_n - s_{n-1}| < M$, for $k = m+1, \dots, n$,

$$\begin{aligned} |s_n - s_k| &= |s_n - s_{n-1} + s_{n-1} - s_{n-2} + \dots + s_{k+1} - s_k| \\ &\leq |s_n - s_{n-1}| + |s_{n-1} - s_{n-2}| + \dots + |s_{k+1} - s_k| \\ &\leq \frac{M}{n} + \frac{M}{n-1} + \dots + \frac{M}{k+1} \leq \frac{n-k}{k+1}M \leq \frac{n-k}{m+2}M \leq \frac{n-m-1}{m+2}M \end{aligned}$$

Let $\varepsilon > 0$ be given. For each $n \in \mathbb{N}$, put $m \in \mathbb{N}$ such that

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1$$

Then,

$$m(1+\varepsilon) \leq n-\varepsilon \implies m+\varepsilon(1+m) \leq n \implies \frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$$

and

$$n-\varepsilon < (m+1)(1+\varepsilon) \implies n+1 < (m+2)(1+\varepsilon) \implies \frac{n+1}{m+2} - 1 < \varepsilon \implies \frac{n-m-1}{m+2} < \varepsilon$$

Now, for arbitrary $n \in \mathbb{N}$,

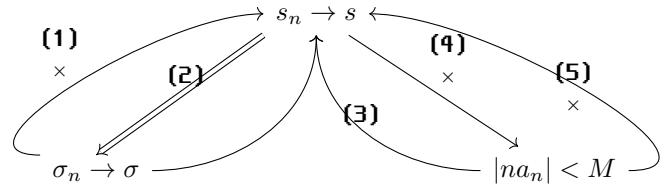
$$\begin{aligned} |s_n - \sigma| &\leq |s_n - \sigma| + |\sigma_n - \sigma| \\ &\implies \limsup_{n \rightarrow \infty} |s_n - \sigma| \leq \limsup_{n \rightarrow \infty} |s_n - \sigma_n| + \limsup_{n \rightarrow \infty} |\sigma_n - \sigma| \end{aligned}$$

And,

$$\begin{aligned} |s_n - \sigma_n| &= \frac{m+1}{n-m} |\sigma_n - \sigma_m| + \frac{1}{n-m} \sum_{k=m+1}^n |s_n - s_k| < \frac{1}{\varepsilon} |\sigma_n - \sigma_m| + M\varepsilon \\ &\implies \limsup_{n \rightarrow \infty} |s_n - \sigma_n| \leq \frac{1}{\varepsilon} \limsup_{n \rightarrow \infty} |\sigma_n - \sigma_m| + M\varepsilon = M\varepsilon \end{aligned}$$

Consequently, $\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq (M+1)\varepsilon$, thus $s_n \rightarrow \sigma$. □

In brief, the diagram of the above conditions like this:



Examples and Counterexamples of the Diagram:

(1) Let $s_n \stackrel{\text{def}}{=} \exp(\frac{inx\pi}{2})$. Then,

- s_n diverges.
- na_n diverges.
- $\sigma_n \rightarrow 0$.

(2) Let $s_n \stackrel{\text{def}}{=} \frac{1}{n}$, $s_0 = 0$.

(3) Let $s_n \stackrel{\text{def}}{=} \sum_{k=1}^n \frac{1}{k}$. Then,

- s_n diverges.
- $a_n = \frac{1}{n}$, thus $na_n \rightarrow 1$, bounded.
- If σ_n converges, then the diagram implies that s_n must converge, leading to a contradiction. Therefore, σ_n diverges.

(4) $s_n = \sum_{k=1}^n \frac{(-1)^k}{\sqrt{k}}$, $s_0 = 0$. Then,

- s_n converges, being the Alternating series Test.
- $a_n = \frac{(-1)^n}{\sqrt{n}}$, thus na_n diverges.

14.3 Taylor's Theorem

Theorem 14.3.0.1. *Taylor's Theorem*

Let $f : [a, b] \rightarrow \mathbb{R}$, and let $n \in \mathbb{N}$ be fixed. Suppose that $\begin{cases} f^{(n-1)} \text{ is continuous.} \\ f^{(n)}(t) \text{ exists for every } t \in (a, b). \end{cases}$ Then, for any $\alpha, \beta \in [a, b]$, there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n$$

Proof. Put

$$M \stackrel{\text{def}}{=} \frac{1}{(\beta - \alpha)^n} \cdot \left(f(\beta) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k \right)$$

That is,

$$f(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k + M(\beta - \alpha)^n$$

and put

$$g(t) \stackrel{\text{def}}{=} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k - M(t - \alpha)^n, \quad (a \leq t \leq b)$$

If we differentiate the above equation n times,

$$g^{(n)}(t) = f^{(n)}(t) - n!M, \quad (a < t < b)$$

For each $k = 0, 1, \dots, n-1$,

$$\begin{aligned} \frac{d^r}{dt^r} \left(\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k \right) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{d^r}{dt^r} ((t - \alpha)^k) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{k!} \cdot \frac{k!}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \\ &= \sum_{k=r+1}^{n-1} \frac{f^{(k)}(\alpha)}{(k-r)!} (t - \alpha)^{k-r} + f^{(r)}(\alpha) \end{aligned}$$

Substituting $t = \alpha$, only the $f^{(r)}(\alpha)$ term remains. Therefore, for $r = 0, \dots, n-1$, $g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0$.

Since $g(\beta) = 0$ by definition, the Mean-Value Theorem implies there exists a $x_1 \in (\alpha, \beta)$ s.t. $g'(x_1) = \frac{g(\beta) - g(\alpha)}{\beta - \alpha} = 0$.

And similarly, there is $x_2 \in (x_1, \beta)$ s.t. $g''(x_2) = \frac{g'(x_1) - g'(\alpha)}{\beta - \alpha} = 0$.

Inductively, for some $x_n \in (\alpha, \beta)$, $g^{(n)}(x_n) = f^{(n)}(x_n) - n!M = 0$. That is, $M = \frac{f^{(n)}(x_n)}{n!}$.

Proof Complete by Initial Setting. □

Corollary 14.3.0.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an infinitely differentiable function.

Suppose that there exists a $M > 0$ such that for any $n \in \mathbb{N}$, $\sup_{t \in [a, b]} |f^{(n)}(t)| \leq M$. Then, for any $x, \alpha \in [a, b]$,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} (x - \alpha)^k$$

14.4 Convexity

14.4.1 Definition

Definition 14.4.1.1. Let $f : (a, b) \rightarrow \mathbb{R}$ be a Real-valued function. f is said to be **convex** if: For any $x, y \in (a, b), \lambda \in (0, 1)$,

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

Convex function has following properties:

Lemma 14.4.1.1. Let $f : (a, b) \rightarrow \mathbb{R}$ be a Convex function, and $a < x_1 < x_2 < x_3 < b$. Then,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Proof. To show that first inequality, note that

$$\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1 = \frac{(x_2 - x_1)x_3 + (x_3 - x_2)x_1}{x_3 - x_1} = x_2$$

Now,

$$\begin{aligned} f(x_2) &= f\left(\frac{x_2 - x_1}{x_3 - x_1} \cdot x_3 + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot x_1\right) \\ &\leq \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + \left(1 - \frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) = \frac{x_2 - x_1}{x_3 - x_1} \cdot f(x_3) + f(x_1) - \left(\frac{x_2 - x_1}{x_3 - x_1}\right) \cdot f(x_1) \end{aligned}$$

In brief,

$$f(x_2) - f(x_1) \leq \frac{x_2 - x_1}{x_3 - x_1}(f(x_3) - f(x_1)) \implies \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1}$$

And similarly,

$$\frac{x_3 - x_2}{x_3 - x_1} \cdot x_1 + \left(1 - \frac{x_3 - x_2}{x_3 - x_1}\right) x_3 = x_2$$

gives the second inequality. □

14.4.2 Properties

Proposition 14.4.2.1. If $f : (a, b) \rightarrow \mathbb{R}$ is Convex, then f is Continuous.

Proof. Let $\varepsilon > 0$ be given, $s < t$ are fixed in (a, b) . For any $x, y \in (s, t)$ with $s < x < y < t$,

$$\frac{f(s) - f(a)}{s - a} \leq \frac{f(x) - f(s)}{x - s} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(t) - f(y)}{t - y} \leq \frac{f(b) - f(t)}{b - t}$$

Put $M = \max \left\{ \left| \frac{f(s) - f(a)}{s - a} \right|, \left| \frac{f(b) - f(t)}{b - t} \right| \right\}$. Then, for any $x, y \in (s, t)$,

$$\left| \frac{f(y) - f(x)}{y - x} \right| \leq M$$

Now,

$$|f(y) - f(x)| \leq M|y - x| < \varepsilon$$

Since $s, t \in (a, b)$ was arbitrary, f is continuous on (a, b) . □

Proposition 14.4.2.2. Let f is differentiable on (a, b) . Then,

f is Convex if and only if f' is monotonically increasing on (a, b) .

Proof. Prove by showing both directions: right and left.

Right Direction Let $x_1 < x_2$ in (a, b) . Then,

$$f'(x_1) = \lim_{t \rightarrow x_1} \frac{f(t) - f(x_1)}{t - x_1} \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \lim_{\tau \rightarrow x_2} \frac{f(\tau) - f(x_2)}{\tau - x_2} = f'(x_2)$$

More rigorously, put $\varepsilon = \left| f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right|$. (If $\varepsilon = 0$, then there is nothing to prove.).

Now, there exists a $\delta > 0$ such that $|t - x_1| < \delta$ implies

$$\left| f'(x_1) - \frac{f(t) - f(x_1)}{t - x_1} \right| < \varepsilon \iff -\varepsilon + \frac{f(t) - f(x_1)}{t - x_1} < f'(x_1) \stackrel{(*)}{=} \varepsilon + \frac{f(t) - f(x_1)}{t - x_1}$$

If $f'(x_1) - \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$, then $(*)$ gives

$$f'(x_1) < f'(x_1) + \frac{f(t) - f(x_1)}{t - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} \iff \frac{f(x_2) - f(x_1)}{x_2 - x_1} < \frac{f(t) - f(x_1)}{t - x_1} \quad \forall t \text{ s.t. } |t - x_1| < \delta$$

If $|t - x_1| < |x_2 - x_1|$, then this contradicts to Convexity.

Consequently, we obtain the first inequality, similarly can prove the second inequality.

Left Direction Let $x, y \in (a, b)$ and $\lambda \in (0, 1)$ be given. The Mean Value Theorem gives that:

$$\begin{aligned} f(\lambda x + (1 - \lambda)y) - f(x) &= f'(z_1)(\lambda x + (1 - \lambda)y - x) \text{ for some } z_1 \in (x, \lambda x + (1 - \lambda)y) \\ f(y) - f(\lambda x + (1 - \lambda)y) &= f'(z_2)(y - \lambda x + (1 - \lambda)y) \text{ for some } z_2 \in (\lambda x + (1 - \lambda)y, y) \end{aligned}$$

Now, Monotonically increasing gives

$$\begin{aligned} \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{\lambda x + (1 - \lambda)y - x} &= f'(z_1) \leq f'(z_2) = \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{y - (\lambda x + (1 - \lambda)y)} \\ \implies \frac{f(\lambda x + (1 - \lambda)y) - f(x)}{(1 - x)(y - x)} &\leq \frac{f(y) - f(\lambda x + (1 - \lambda)y)}{\lambda(y - x)} \\ \implies \lambda f(\lambda x + (1 - \lambda)y) - \lambda f(x) &\leq (1 - \lambda)f(y) - (1 - \lambda)\lambda x + (1 - \lambda)y \\ \implies f(\lambda x + (1 - \lambda)y) &\leq \lambda f(x) + (1 - \lambda)f(y) \end{aligned}$$

□

Corollary 14.4.2.1. If $f : [a, b] \rightarrow \mathbb{R}$ is twice-differentiable, then

$$f \text{ is Convex if and only if } f''(x) \geq 0 \text{ for all } x \in (a, b).$$

Theorem 14.4.2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be given. Then,

f is Convex if and only if f is Continuous, and Midpoint Convex.

Midpoint convex is that f satisfies $\forall x, y \in (a, b), f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$.

Proof. The right direction is clear. To show the left direction, we demonstrate that Midpoint Convexity implies Dyadic Rational Convexity. Claim: For any $n \in \mathbb{N}$,

$$f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) \leq \frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) \quad (*)$$

Using Induction: If $n = 1$, it is clear by Midpoint Convexity.

Assume that for $n \in \mathbb{N}$, $(*)$ is True. Then,

$$\begin{aligned} f\left(\frac{\sum_{k=1}^{2^{n+1}} x_k}{2^{n+1}}\right) &= f\left(\frac{1}{2} \cdot \left[\frac{\sum_{k=1}^{2^n} x_k}{2^n} + \frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n} \right]\right) \\ &\stackrel{\text{m.c.}}{\leq} \frac{1}{2} \left(f\left(\frac{\sum_{k=1}^{2^n} x_k}{2^n}\right) + f\left(\frac{\sum_{k=2^n+1}^{2^{n+1}} x_k}{2^n}\right) \right) \\ &\stackrel{(*)}{\leq} \frac{1}{2} \left(\frac{1}{2^n} \sum_{k=1}^{2^n} f(x_k) + \frac{1}{2^n} \sum_{k=2^n+1}^{2^{n+1}} f(x_k) \right) \\ &= \frac{1}{2^{n+1}} \sum_{k=1}^{2^{n+1}} f(x_k) \end{aligned}$$

Consequently, we obtain the claim. Now, let $n \in \mathbb{N}$, and m be an integer such that $1 \leq m \leq 2^n$.

Put $x_1 = x_2 = \dots = x_m = x$ and $x_{m+1} = x_{m+2} = \dots = x_{2^n} = y$. Then

$$f\left(\frac{m}{2^n}x + \left(1 - \frac{m}{2^n}\right)y\right) \leq \frac{m}{2^n}f(x) + \left(1 - \frac{m}{2^n}\right)f(y)$$

For complete this discussion, Let $x, y \in (a, b), \lambda \in (0, 1)$ be given.

Since $\frac{\lfloor 2^n \lambda \rfloor}{2^n} \rightarrow \lambda$ as $n \rightarrow \infty$, for any $n \in \mathbb{N}$,

$$f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \leq \frac{\lfloor 2^n \lambda \rfloor}{2^n}f(x) + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)f(y)$$

Finally, taking limits then

$$\lim_{n \rightarrow \infty} f\left(\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right) \stackrel{f \text{ conti.}}{=} f\left(\lim_{n \rightarrow \infty} \left[\frac{\lfloor 2^n \lambda \rfloor}{2^n}x + \left(1 - \frac{\lfloor 2^n \lambda \rfloor}{2^n}\right)y\right]\right) = f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

In brief, Midpoint Convexity implies Dyadic Rational Convexity, and with Continuous implies Convexity. \square

14.5 Lipschitz Condition

14.5.1 Definition

Definition 14.5.1.1. A real-valued function $f : (a, b) \rightarrow \mathbb{R}$ is called **Lipschitz Continuous** if:

$$\exists L \geq 0 \text{ s.t. } \forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq L \cdot |x_1 - x_2|$$

The constant L is said to be **Lipschitz Constant** of f . In particular, the constant

$$D \stackrel{\text{def}}{=} \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}$$

is called **dilation** of f . Clearly,

$$\forall x_1, x_2 \in (a, b), |f(x_1) - f(x_2)| \leq D \cdot |x_1 - x_2|$$

and if $L > 0$ is Lipschitz Constant of f , then $D \leq L$. That is, $D = \inf\{L > 0 \mid L \text{ is Lipschitz constant of } f\}$.

14.5.2 Properties

Proposition 14.5.2.1. If $f : (a, b) \rightarrow \mathbb{R}$ is Lipschitz Continuous, then f is uniformly continuous.

Proof. Let $L \geq 0$ be a Lipschitz Constant of f . Then, for any $\varepsilon > 0$,

$$\forall x, y \in (a, b), |x - y| < \frac{\varepsilon}{L} \implies |f(x) - f(y)| \leq L|x - y| < \varepsilon$$

□

Proposition 14.5.2.2. Let $f : (a, b) \rightarrow \mathbb{R}$ be a Differentiable function. Then,

f is Lipschitz Continuous if and only if f' is bounded in (a, b) .

Proof.

Right Direction

Let $L > 0$ be a Lipschitz constant of f , and $x \in (a, b)$ be given. Since definition of derivative,

$$f'(x) \stackrel{\text{def}}{=} \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t}$$

Meanwhile, the assumption gives: for any distinct $x, t \in (a, b)$,

$$\frac{|f(x) - f(t)|}{|x - t|} \leq L$$

Therefore,

$$f'(x) = \lim_{t \rightarrow x} \frac{f(x) - f(t)}{x - t} \leq \lim_{t \rightarrow x} \frac{|f(x) - f(t)|}{|x - t|} \leq \lim_{t \rightarrow x} L = L$$

Left Direction

Let distinct $x, y \in (a, b)$ be given. Then, the Mean-Value Theorem gives: There exists a $z \in (x, y)$ such that

$$f(x) - f(y) = f'(z)(x - y) \implies f'(z) = \frac{f(x) - f(y)}{x - y}$$

Now,

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(z)| \leq L \implies |f(x) - f(y)| \leq L \cdot |x - y|$$

If $x = y$, then there is nothing to prove. □

Note that:

$$\text{Lipschitz Continuous} \implies \text{Uniformly Continuous} \implies \text{Continuous}$$

14.6 Optimization Methods

14.6.1 Newton-Raphson Method

Theorem 14.6.1.1. Newton-Raphson Method

Let $f : [a, b] \rightarrow \mathbb{R}$ be a twice-differentiable, $f(a) < 0 < f(b)$. Suppose that f satisfies: for all $x \in [a, b]$,

$$f'(x) \geq \delta > 0 \text{ and } 0 \leq f''(x) \leq M$$

That is, f is strictly increasing convex function, and Lipschitz Continuous.

Further, there uniquely exists $x^* \in (a, b)$ such that $f(x^*) = 0$.

Let $x_1 \in (x^*, b)$ fixed. Define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} \stackrel{\text{def}}{=} x_n - \frac{f(x_n)}{f'(x_n)}$$

Then, $\{x_n\}$ satisfies the following three conditions:

1. $\{x_n\}$ is decreasing sequence.
2. $x_n \rightarrow x^*$ as $n \rightarrow \infty$.
3. For any $n \in \mathbb{N}$, $0 \leq x_{n+1} - x^* \leq \left[\frac{M}{2\delta}\right]^{2^{n+1}-1} [x_1 - x^*]^{2^n}$.

Condition 3 means that for a suitable initial value x_1 , we can establish an upper bound for the error.

Proof. This proof consists by three steps.

Since f'' is non-negative, and f' is positive, f is strictly increasing convex function.

And Fundamental Theorem of Calculus gives: for any $x \in (a, b)$,

$$f'(x) \stackrel{\text{FTC}}{=} \int_a^x f''(t)dt + f'(a) \leq \int_a^x Mdt + f'(a) = M(x-a) + f'(a) \leq M(b-a) + f'(a)$$

Thus, f' is bounded on (a, b) , thus f is Lipschitz Continuous.

Step 1. f has a unique root x^* .

The existence of root given directly by Intermediate-Value theorem.

Suppose that $x^*, x' \in (a, b)$ are distinct root of f . i.e., $f(x^*) = f(x') = 0$. Then, by Mean-value theorem, there is $c \in (a, b)$ between x^* and x' such that

$$f'(c)(x^* - x') = f(x^*) - f(x') = 0$$

That is, $f'(c) = 0$. This is contradiction with f' is positive.

Step 2. $\{x_n\}$ decrease.

Proof by induction:

For $n = 1$, $f'(x_1)(x_1 - x_2) \stackrel{\text{def}}{=} f(x_1) > f(x^*) = 0$, thus $x_2 < x_1$. And,

$$\begin{aligned} f(x_2) &\stackrel{\text{MVT}}{=} f(x_1) + f'(c_1)(x_2 - x_1) \quad \text{for some } c_1 \in (x_2, x_1) \\ &> f(x_1) + f'(x_1)(x_2 - x_1) = f'(x_1)(x_1 - x_2) + f'(x_1)(x_2 - x_1) = 0 \end{aligned}$$

Now, since $f(x_2) > 0 = f(x^*)$, the Mean-Value Theorem implies that $x_2 > x^*$.

To use induction, suppose that for some $n \geq 1$, $x^* < x_{n+1} < x_n$. Then,

$$f(x_{n+1}) = f'(x_{n+1})(x_{n+1} - x_{n+2}) > 0$$

Thus $x_{n+2} < x_{n+1}$ and

$$\begin{aligned} f(x_{n+2}) &\stackrel{\text{MVT}}{=} f(x_{n+1}) + f'(c_{n+1})(x_{n+2} - x_{n+1}) \quad \text{for some } c_{n+1} \in (x_{n+2}, x_{n+1}) \\ &\geq f(x_{n+1}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) \\ &= f(x_{n+1})(x_{n+1} - x_{n+2}) + f'(x_{n+1})(x_{n+2} - x_{n+1}) = 0 \end{aligned}$$

Again, the Mean-Value Theorem implies that $x_{n+2} > x^*$. Therefore, induction completes.

Now, $x_n \rightarrow x^*$ as $n \rightarrow \infty$ for some $x' \in [x^*, x_1]$ since $\{x_n\}$ is Bounded below and Decreasing.

Still it remains that to show $x' = x^*$. By Continuity,

$$\begin{aligned} f'(x_n)(x_{n+1} - x_n) + f(x_n) &= 0 \\ \implies \lim_{n \rightarrow \infty} [f'(x_n)(x_{n+1} - x_n) + f(x_n)] &= f\left(\lim_{n \rightarrow \infty} x_n\right) = f(x') = 0 \end{aligned}$$

Since the root of f is unique, thus $x' = x^*$.

Step 3. Establishing the error bound.

The Taylor's Theorem implies that

$$\begin{aligned} f(x^*) &= f(x_n) + f'(x_n)(x^* - x_n) + \frac{f''(t_n)}{2}(x^* - x_n)^2 \quad \text{for some } t_n \in (x^*, x_n) \\ \implies x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x^* - x_n)^2 \end{aligned}$$

Consequently,

$$\begin{aligned} 0 \leq x_{n+1} - x^* &= \frac{f''(t_n)}{2f'(x_n)}(x_n - x^*)^2 = \frac{f''(t_n)}{2f'(x_n)} \cdot \left(\frac{f''(t_{n-1})}{2f'(x_{n-1})} \right)^2 (x_{n-1} - x^*)^4 = \dots \\ &= \prod_{i=1}^n \left[\frac{f''(t_i)}{2f'(x_i)} \right]^{2^{(n+1-i)}} [x_1 - x^*]^{2^n} \leq \left[\frac{M}{2\delta} \right]^{2^{n+1}-1} [x_1 - x^*]^{2^n} \end{aligned}$$

□

14.6.2 Gradient Descent

Theorem 14.6.2.1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function that satisfies the following conditions:

1. f is Convex function.
2. f' is Lipschitz Continuous with Lipschitz constant of f , $L > 0$. In this, f is called L -Smooth.
3. f has at least one local minimizer x^* .

Then, x^* is a Global minimizer of \mathbb{R} , and there exists a unique closed interval M containing x^* such that

$$\forall x \in M, t \notin M, f(x) = f(x^*) < f(t)$$

And, given initial point $x_0 \in \mathbb{R}$ and $0 < \gamma \leq \frac{1}{L}$, define a sequence $\{x_n\}$ inductively as follows:

$$x_{n+1} = x_t - \gamma \cdot f'(x_n)$$

Then, for any $N \in \mathbb{N}$,

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

Proof. Let $x^* \in \mathbb{R}$ be a local minimizer. That is, there exists a $\delta > 0$ such that $\forall t \in (x^* - \delta, x^* + \delta)$, $f(x^*) \leq f(t)$. Then,

$$0 \leq \lim_{t \rightarrow x^*+} \frac{f(x^*) - f(t)}{x^* - t} = f'(x^*) = \lim_{t \rightarrow x^-} \frac{f(x^*) - f(t)}{x^* - t} \leq 0$$

thus, $f'(x^*) = 0$. And, by convexity, f' is monotonically increasing. Now, The Fundamental Theorem of Calculus gives:

$$\forall x \in \mathbb{R}, f(x) = \int_{x^*}^x f'(t)dt + f(x^*) \geq f(x^*)$$

Therefore, x^* is a Global minimizer of f .

Now, establish the closed interval M . Since f' is Lipschitz Continuous, thus f' is Continuous.

Let $D \stackrel{\text{def}}{=} \{x \in \mathbb{R} \mid f'(x) = 0\}$. (Note that: $x^* \in D$, thus D is not emptyset.)

D is closed because: Let $\{x_n\}$ be a convergent sequence in D . That is, for all $n \in \mathbb{N}$, $f(x_n) = 0$. Then, by continuity,

$$f\left(\lim_{n \rightarrow \infty} x_n\right) = \lim_{n \rightarrow \infty} f(x_n) = 0$$

The limit of $\{x_n\}$ is contained in D , thus D is closed.

And, D is interval: i.e, for any $x \in (\inf D, \sup D)$, $x \in D$ because:

Suppose that there exists $x \in (\inf D, \sup D)$ such that $x \notin D$. That is, $f'(x) \neq 0$. This is Contradiction with Monotonicity.

To set error of upper bound, we make inequalities: Let $x, y \in \mathbb{R}$ be given.

The Fundamental Theorem of Calculus and L -Smooth condition gives:

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt = \int_0^1 f'(x + (y-x)u)(y-x)du = f'(x)(y-x) + \int_0^1 (f'(x + (y-x)u) - f'(x))(y-x)du \\ &\stackrel{?}{\leq} f'(x)(y-x) + L \cdot |y-x|^2 \int_0^1 u du = f'(x)(y-x) + \frac{L}{2}|y-x|^2 \end{aligned}$$

For any $\lambda > 0$, Put $y = x - \lambda f'(x)$. Then,

$$f(x - \lambda f'(x)) \leq f(x) - f'(x)(\lambda f'(x)) + \frac{L}{2}|\lambda f'(x)|^2 = f(x) + \lambda \left(\frac{L\lambda}{2} - 1\right) |f'(x)|^2$$

Put $\lambda = \frac{1}{L}$, then

$$f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \frac{L}{2}|f'(x)|^2 \implies \frac{L}{2}|f'(x)|^2 \leq f(x) - f\left(x - \frac{f'(x)}{L}\right) \leq f(x) - \inf f$$

Meanwhile, the convexity gives: for any $x, y \in \mathbb{R}$,

$$f'(x)(y - x) \leq f(y) - f(x) \leq f'(y)(y - x)$$

since derivative of convex function increase monotonically. Put $z = y - \frac{1}{L}(f'(y) - f'(x))$. Then,

$$\begin{aligned} f(x) - f(y) &= f(x) - f(z) + f(z) - f(y) \\ &\leq f'(x)(x - z) + f'(y)(z - y) + \frac{L}{2}|z - y|^2 \\ &= f'(x) \left(x - y + \frac{1}{L}(f'(y) - f'(x)) \right) - f'(y) \left(\frac{1}{L}(f'(y) - f'(x)) \right) + \frac{L}{2} \left| \frac{1}{L}(f'(y) - f'(x)) \right|^2 \\ &= f'(x)(x - y) - \frac{1}{L}|f'(y) - f'(x)|^2 + \frac{1}{2L}|f'(y) - f'(x)|^2 \\ &= f'(x)(x - y) - \frac{1}{2L}|f'(y) - f'(x)|^2 \end{aligned}$$

Now,

$$\begin{cases} \frac{1}{2L}|f'(y) - f'(x)|^2 \leq f'(x)(x - y) - (f(x) - f(y)) \\ \frac{1}{2L}|f'(x) - f'(y)|^2 \leq f'(y)(y - x) - (f(y) - f(z)) \end{cases} \implies \frac{1}{L}|f'(y) - f'(x)|^2 \leq (f'(y) - f'(x))(y - x)$$

Since above inequalities, we obtain that

$$\begin{aligned} |x_{n+1} - x^*|^2 &= |x_n - \gamma \cdot f'(x_n) - x^*|^2 = |(x_n - x^*) - \gamma \cdot f'(x_n)|^2 \\ &= |x_n - x^*|^2 - 2\gamma|f'(x_n)| \cdot |x_n - x^*| + \gamma^2|f'(x_n)|^2 \\ &\leq |x_n - x^*|^2 - 2\gamma \frac{1}{L}|f'(x_n)|^2 + \gamma^2|f'(x_n)|^2 \\ &= |x_n - x^*|^2 + \left(\gamma^2 - \frac{2\gamma}{L} \right) |f'(x_n)|^2 \leq |x_n - x^*|^2 \end{aligned}$$

Thus, $|x_n - x^*|$ decrease as $n \rightarrow \infty$. That is, $|x_n - x^*| \leq |x_0 - x^*|$ for all $n \in \mathbb{N}$.

Consider x_{n+1} and x_n . First, we obtain

$$\begin{aligned} f(x_{n+1}) &\leq f(x_n) + f'(x_n)(x_{n+1} - x_n) + \frac{L}{2}|x_{n+1} - x_n|^2 \\ &= f(x_n) - \gamma|f'(x_n)|^2 + \frac{L}{2}\gamma^2|f'(x_n)|^2 \\ &= f(x_n) - \left(\gamma - \frac{L}{2}\gamma^2 \right) |f'(x_n)|^2 \end{aligned}$$

Subtracting $f(x^*)$ above, then

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2 \right) |f'(x_n)|^2$$

Meanwhile, Convexity gives

$$f(x_n) - f(x^*) \leq f'(x_n)(x_n - x^*) \leq |f'(x_n)||x_n - x^*| \leq |f'(x_n)||x_0 - x^*|$$

Combining above two inequalities,

$$f(x_{n+1}) - f(x^*) \leq f(x_n) - f(x^*) - \left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{|f(x_n) - f(x^*)|^2}{|x_0 - x^*|^2}$$

Dividing Both Sides by $(f(x_{n+1}) - f(x^*))(f(x_n) - f(x^*))$,

$$\begin{aligned} \frac{1}{f(x_n) - f(x^*)} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} \\ \implies \left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{f(x_n) - f(x^*)}{f(x_{n+1}) - f(x^*)} \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} &\leq \frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \\ \implies \sum_{n=0}^{N-1} \left[\left(\gamma - \frac{L}{2}\gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] &\leq \sum_{n=0}^{N-1} \left[\frac{1}{f(x_{n+1}) - f(x^*)} - \frac{1}{f(x_n) - f(x^*)} \right] = \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \end{aligned}$$

Consequently,

$$\frac{2\gamma N}{|x_0 - x^*|^2} \leq N \cdot \left[\left(\gamma - \frac{L}{2} \gamma^2 \right) \cdot \frac{1}{|x_0 - x^*|^2} \right] \leq \frac{1}{f(x_N) - f(x^*)} - \frac{1}{f(x_0) - f(x^*)} \leq \frac{1}{f(x_N) - f(x^*)}$$

Organizing the formula, as result:

$$f(x_N) - f(x^*) \leq \frac{|x_0 - x^*|^2}{2\gamma N}$$

□

14.7 Integral

14.7.1 Inequality of Riemann-Stieltjes Integral

Let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, and functions lying on $[a, b]$.

Lemma 14.7.1.1. Let $f, g \in \mathcal{R}(\alpha)$ with $f, g \geq 0$, and $\int_a^b [f(x)]^p d\alpha = \int_a^b [g(x)]^q d\alpha = 1$. Then, $\int_a^b f(x)g(x) d\alpha \leq 1$.

Proof. For any $x \in [a, b]$, the Young's Inequality gives

$$0 \leq f(x)g(x) \leq \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q}$$

Now,

$$\int_a^b f(x)g(x) d\alpha \leq \int_a^b \frac{[f(x)]^p}{p} + \frac{[g(x)]^q}{q} d\alpha = \frac{1}{p} \int_a^b [f(x)]^p d\alpha + \frac{1}{q} \int_a^b [g(x)]^q d\alpha = \frac{1}{p} + \frac{1}{q} = 1$$

□

Definition 14.7.1.1. Let $f \in \mathcal{R}(\alpha)$. Define a Norm of f :

$$\|f\|_p \stackrel{\text{def}}{=} \left(\int_a^b |f(x)|^p d\alpha \right)^{\frac{1}{p}}$$

This becomes actually norm of set of Stieltjes Integrable functions, $\mathcal{F} \stackrel{\text{def}}{=} \{f : [a, b] \rightarrow \mathbb{C} \mid f \in \mathcal{R}(\alpha)\}$.

Lemma 14.7.1.2. Hölder's Inequality

Let $f, g \in \mathcal{F}$. Then,

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Proof. Use above definition, Rewrite:

$$\|f\|_p^p = \int_a^b |f(x)|^p d\alpha, \|g\|_q^q = \int_a^b |g(x)|^q d\alpha$$

Now, we can make the condition of above lemma,

$$\int_a^b \left[\frac{|f(x)|}{\|f\|_p} \right]^p d\alpha = \frac{1}{\|f\|_p^p} \cdot \int_a^b |f(x)|^p d\alpha = 1, \int_a^b \left[\frac{|g(x)|}{\|g\|_q} \right]^q d\alpha = \frac{1}{\|g\|_q^q} \cdot \int_a^b |g(x)|^q d\alpha = 1$$

And apply this,

$$\int_a^b \frac{|f(x)| \cdot |g(x)|}{\|f\|_p \|g\|_q} d\alpha \leq 1 \implies \int_a^b |f(x)||g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

Finally, the general property of integral of product gives

$$\left| \int_a^b f(x)g(x) d\alpha \right| \leq \int_a^b |f(x)||g(x)| d\alpha \leq \|f\|_p \|g\|_q = \left[\int_a^b |f(x)|^p d\alpha \right]^{\frac{1}{p}} \cdot \left[\int_a^b |g(x)|^q d\alpha \right]^{\frac{1}{q}}$$

□

Theorem 14.7.1.1. Minkowski inequality

Let $f, g \in \mathcal{F}$. Then, for any $p \geq 1$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Proof.

$$\begin{aligned}
 \|f + g\|_p^p &= \int_a^b |f + g|^p d\alpha = \int_a^b |f + g| |f + g|^{p-1} d\alpha \\
 &\leq \int_a^b [|f| + |g|] |f + g|^{p-1} d\alpha \\
 &= \int_a^b |f| |f + g|^{p-1} d\alpha + \int_a^b |g| |f + g|^{p-1} d\alpha \\
 &\stackrel{\text{Hölder}}{\leq} \left[\int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} \left[\int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} + \left[\int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \left[\int_a^b |f + g|^{(p-1)\frac{p}{p-1}} d\alpha \right]^{\frac{p-1}{p}} \\
 &= \left[\int_a^b |f + g|^p d\alpha \right]^{\frac{p-1}{p}} \left(\left[\int_a^b |f|^p d\alpha \right]^{\frac{1}{p}} + \left[\int_a^b |g|^p d\alpha \right]^{\frac{1}{p}} \right) = \|f + g\|_p^{p-1} \cdot (\|f\|_p + \|g\|_p)
 \end{aligned}$$

Now,

$$\|f + g\|_p^p \cdot \|f + g\|_p^{1-p} = \|f + g\|_p \leq \|f\|_p + \|g\|_p$$

□

Chapter 15

Measure

Chapter 16

Complex Analysis

16.1 Series

Theorem 16.1.0.1. Laurent's theorem

Suppose that f is analytic on annular domain $D = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$, and C is simple closed contour around z_0 and lying in that domain D . Then each point in D , $f(z)$ can express that:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right) + \frac{1}{2\pi i} \cdot \sum_{n=1}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{-n+1}} ds \cdot \frac{1}{(z - z_0)^n} \right) \\ &= \frac{1}{2\pi i} \cdot \sum_{n=-\infty}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right), \quad (R_1 < |z - z_0| < R_2) \end{aligned}$$

In particular, If $f(s)$ is analytic inside and on circle C , $\forall n \in \mathbb{N}$, $f(s) \cdot (s - z_0)^{n-1}$ is analytic too. then by Cauchy-Goursat Thm, term (2) is zero, thus we can write that:

$$f(z) = \frac{1}{2\pi i} \cdot \sum_{n=0}^{\infty} \left(\int_C \frac{f(s)}{(s - z_0)^{n+1}} ds \cdot (z - z_0)^n \right)$$

and, since f is analytic on C , applies Cauchy integral theorem:

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n$$

This is what we already know as the Taylor Series form. Therefore, we can say Laurent's theorem is generalization form of Taylor Theorem.

Proof.

In case of $z_0 = 0$.

First, since C is lying in annular $R_1 < |z| < R_2$,

can construct annular $A : r_1 < |z| < r_2$ such that A contains circle C .

Let write $C_1 : |z| = r_1$, $C_2 : |z| = r_2$, each circles are positively oriented.

Now, construct circle γ such that positively oriented and lying in annular $A : r_1 < |z| < r_2$.

Then by multiply connected theorem, we get that:

$$\int_{C_2} \frac{f(s)}{s - z} ds = \int_{\gamma} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{s - z} ds$$

Inside and on γ , f is analytic, thus we can apply Cauchy integral theorem:

$$\begin{aligned} \int_{\gamma} \frac{f(s)}{s - z} ds &= 2\pi i \cdot f(z) = \int_{C_2} \frac{f(s)}{s - z} ds - \int_{C_1} \frac{f(s)}{s - z} ds = \int_{C_2} \frac{f(s)}{s - z} ds + \int_{C_1} \frac{f(s)}{z - s} ds \\ \implies f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s - z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z - s} ds \end{aligned}$$

And we already know in proof of *Taylor theorem*,

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} + \frac{z^N}{(s-z)s^N}$$

and also

$$\begin{aligned}\frac{1}{z-s} &= \sum_{n=0}^{N-1} \frac{s^n}{z^{n+1}} + \frac{s^N}{(z-s)z^N} \\ &= \sum_{n=1}^N \frac{s^{n-1}}{z^n} + \frac{s^N}{(z-s)z^N} \\ &= \sum_{n=1}^N \frac{1}{s^{-n+1} \cdot z^n} + \frac{s^N}{(z-s)z^N}\end{aligned}$$

Now we can write that:

$$\begin{aligned}f(z) &= \frac{1}{2\pi i} \int_{C_2} \frac{f(s)}{s-z} ds + \frac{1}{2\pi i} \int_{C_1} \frac{f(s)}{z-s} ds \\ &= \frac{1}{2\pi i} \int_{C_2} \left(\sum_{n=0}^{N-1} \frac{z^n}{s^{n+1}} f(s) + \frac{z^N}{(s-z)s^N} f(s) \right) ds + \frac{1}{2\pi i} \int_{C_1} \left(\sum_{n=1}^N \frac{f(s)}{s^{-n+1} \cdot z^n} + \frac{s^N}{(z-s)z^N} f(s) \right) ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i} \sum_{n=1}^N \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \cdot \frac{1}{z^n} + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \\ &= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_{C_2} \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i} \sum_{n=1}^N \int_{C_1} \frac{f(s)}{s^{-n+1}} ds \cdot z^{-n} + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds\end{aligned}$$

And by construction of C , C_1 , C_2 , f is analytic between C and C_1 , also C and C_2 .

Thus applies multiply connected:

$$\begin{aligned}&= \frac{1}{2\pi i} \sum_{n=0}^{N-1} \int_C \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{1}{2\pi i} \sum_{n=1}^N \int_C \frac{f(s)}{s^{-n+1}} ds \cdot z^{-n} + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds \\ &= \frac{1}{2\pi i} \sum_{n=-N}^{N-1} \int_C \frac{f(s)}{s^{n+1}} ds \cdot z^n + \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds + \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{(z-s)} ds\end{aligned}$$

Now, enough to show

$$\begin{aligned}\frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds &\rightarrow 0 \text{ as } N \rightarrow \infty \\ \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds &\rightarrow 0 \text{ as } N \rightarrow \infty\end{aligned}$$

Let $|z| = r$. Then $r_1 < r < r_2$. And, Let $M = \max \left\{ \max_{z \in C_1} f(z), \max_{z \in C_2} f(z) \right\}$. And,

for s on C_2 , $|s-z| \geq ||s|-|z|| = r_2 - r$, for s on C_1 , $|z-s| \geq ||z|-|s|| = r - r_1$.

Finally, since ML inequality,

$$\begin{aligned}\left| \frac{z^N}{2\pi i} \int_{C_2} \frac{f(s)}{(s-z)s^N} ds \right| &\leq \frac{|z^N|}{2\pi} \int_{C_2} \left| \frac{f(s)}{(s-z)s^N} \right| ds \leq \frac{r^N}{2\pi} \frac{M \cdot 2\pi r_2}{(r_2 - r)(r_2)^N} = \frac{Mr_2}{r_2 - r} \left(\frac{r}{r_2} \right)^N \\ \left| \frac{1}{2\pi i \cdot z^N} \int_{C_1} \frac{s^N f(s)}{z-s} ds \right| &\leq \frac{1}{2\pi \cdot r^N} \int_{C_1} \left| \frac{s^N f(s)}{z-s} \right| ds \leq \frac{1}{2\pi \cdot r^N} \frac{(r_1)^N \cdot M \cdot 2\pi r_1}{r - r_1} = \frac{Mr_1}{r - r_1} \left(\frac{r_1}{r} \right)^N\end{aligned}$$

Consequently, since $\left(\frac{r}{r_2} \right) < 1$, $\left(\frac{r_1}{r} \right) < 1$, we get result.

In case of $z_0 \neq 0$.

Let f be analytic throughout annular $R_1 < |z - z_0| < R_2$.

Then $g(z) = f(z + z_0)$ is analytic throughout $R_1 < |(z + z_0) - z_0| < R_2$.

Now let $C : z = z(t)$ ($a \leq t \leq b$) is closed simple contour, following by statement.

Then $\forall t \in [a, b]$, $R_1 < |z(t) - z_0| < R_2$ and

for $\Gamma : z = z(t) - z_0$ ($a \leq t \leq b$) is lying in $R_1 < |z| < R_2$. Now since In $z_0 = 0$ case,

$$g(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot z^n \quad (R_1 < |z| < R_2)$$

This is equal that:

$$f(z + z_0) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot z^n \quad (R_1 < |z| < R_2)$$

Finally, change z to $z - z_0$ then:

$$f(z) = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \quad (R_1 < |z - z_0| < R_2)$$

And

$$\int_{\Gamma} \frac{g(s)}{s^{n+1}} ds = \int_a^b \frac{f(z(t) - z_0 + z_0)}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_a^b \frac{f(z(t))}{(z(t) - z_0)^{n+1}} \cdot z'(t) dt = \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Consequently we get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_{\Gamma} \frac{g(s)}{s^{n+1}} ds \cdot (z - z_0)^n \\ &= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \cdot (z - z_0)^n \quad (R_1 < |z - z_0| < R_2) \end{aligned}$$

□

Chapter 17

Fourier Analysis

Chapter 18

Multivariable Analysis

18.1 Differentiation

Definition 18.1.0.1. Suppose that $A \subseteq \mathbb{R}^m$ is a subset, and $x \in A^\circ$. Given a non-zero vector $u \in \mathbb{R}^m$, Define *directional derivative of f at x with respect to the vector u* :

$$f'(x; u) \stackrel{\text{def}}{=} \lim_{t \rightarrow 0} \frac{f(x + tu) - f(x)}{t}$$

Definition 18.1.0.2. Suppose that $A \subseteq \mathbb{R}^m$ is a subset, and $x \in A^\circ$.

The function f is called *differentiable* at x if: There exists a $B \in \mathcal{M}_{n,m}(\mathbb{R})$ such that

$$\frac{f(x + h) - f(x) - B \cdot h}{\|h\|} \rightarrow 0 \in \mathbb{R}^n \text{ as } h \rightarrow 0 \in \mathbb{R}^m$$

More rigorously,

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall h \in \mathbb{R}^m, 0 < \|h\| < \delta \implies \frac{\|f(x + h) - f(x) - B \cdot h\|}{\|h\|} < \varepsilon$$

If exists, this B is unique; The matrix B is denoted $Df(x)$, which is called *derivative of f at x* .

$$\frac{f\left(\underset{\in \mathbb{R}^m}{x} + \underset{\in \mathbb{R}^m}{h}\right) - f\left(\underset{\in \mathbb{R}^m}{x}\right) - \underset{\in \mathcal{M}_{n,m}}{B} \cdot \underset{\in \mathbb{R}^m}{h}}{\underset{\in \mathbb{R}^n}{\|h\|}}$$

Theorem 18.1.0.1. Suppose that $A \subseteq \mathbb{R}^m$ is a subset.

If $f : A \rightarrow \mathbb{R}^n$ is differentiable at $x \in A^\circ$, then for any $u \in \mathbb{R}^m$, $f'(x; u)$ exists. Moreover,

$$f'(x; u) = Df(x) \cdot u$$

Proof. By assumption, the Derivative $Df(x) \in \mathcal{M}_{n,m}(\mathbb{R})$ exists such that

$$\frac{f(x + h) - f(x) - B \cdot h}{\|h\|} \rightarrow 0 \text{ as } h \rightarrow 0$$

Let non-zero vector $u \in \mathbb{R}^m$ be given. Choose $\varepsilon > 0$ arbitrarily. Since assumption, there exists a $\delta > 0$ such that

$$0 < \|h\| < \delta \implies \frac{\|f(x + h) - f(x) - B \cdot h\|}{\|h\|} < \frac{\varepsilon}{\|u\|}$$

Put $\delta_0 > 0$ **such that** $0 < t < \delta_0 \implies \|t\mathbf{u}\| < \delta$ (**Precisely, put** $\delta_0 = \frac{\delta}{\|\mathbf{u}\|}$). **Now,**

$$\begin{aligned} 0 < t < \delta_0 &\implies \frac{\|f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) - B \cdot t\mathbf{u}\|}{\|t\mathbf{u}\|} = \frac{\|f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) - B \cdot t\mathbf{u}\|}{|t|\|\mathbf{u}\|} < \frac{\varepsilon}{\|\mathbf{u}\|} \\ &\implies \frac{\|f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}) - B \cdot t\mathbf{u}\|}{t} = \left\| \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t} - B \cdot \mathbf{u} \right\| < \varepsilon \end{aligned}$$

Thus, $B \cdot \mathbf{u} = Df(\mathbf{x}) = f(\mathbf{x}; \mathbf{u})$. □

Chapter 19

Differential Geometry

Chapter 20

Differential Equation

20.1 System of Differential Equation

20.1.1 Definitions

20.1.2 Basic Properties

20.2 Lorenz system

Chapter 21

Differential Form

Chapter 22

Spaces

22.1 \mathbb{R}^n

22.1.1 Inner Product in \mathbb{R}

22.1.2 p -norm in \mathbb{R}^n

Definition 22.1.2.1. Let \mathbb{R}^n be given. Define p -norm on \mathbb{R}^n as:

$$d_p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} : (\mathbf{x}, \mathbf{y}) \mapsto \left(\sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}, \quad (\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n))$$

where $p \in [1, \infty]$. In particular, p -norm is a metric, being Minkowski inequality.

Lemma 22.1.2.1. Young's inequality

Let $u, v > 0$, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$.
Then,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

Proof. Since $f(x) = \log x$ is concave, we obtain

$$\forall \lambda \in [0, 1], \quad \lambda f(x)(1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

thus,

$$\log\left(\frac{1}{p}u^p + \frac{1}{q}v^q\right) \geq \frac{1}{p}\log(u^p) + \frac{1}{q}\log(v^q) = \log(uv)$$

Since $\exp(x)$ increasing, we get

$$\exp\left(\log\left(\frac{1}{p}u^p + \frac{1}{q}v^q\right)\right) \geq \exp(\log(uv))$$

i.e.,

$$uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$$

□

Lemma 22.1.2.2. Hölder's inequality

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be given, and $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

Proof. Denote that

$$\|x\|_p \stackrel{\text{def}}{=} \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

Then, since Young's inequality, for each $i \in \{1, 2, \dots, n\}$,

$$\frac{|x_i|}{\|x\|_p} \cdot \frac{|y_i|}{\|y\|_p} \leq \frac{1}{p} \cdot \frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q} \cdot \frac{|y_i|^q}{\|y\|_q^q}$$

Summing for all $i = 1, 2, \dots, n$:

$$\frac{1}{\|x\|_p \|y\|_q} \cdot \sum_{i=1}^n |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1$$

Therefore,

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \cdot \left(\sum_{i=1}^n |y_i|^q \right)^{\frac{1}{q}}$$

□

Theorem 22.1.2.1. Minkowski inequality

Given complex-valued sequences $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$,

$$\left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}}$$

Proof. Denote

$$|x_i + y_i|^p = |x_i + y_i| \cdot |x_i + y_i|^{p-1}$$

Then,

$$\begin{aligned} \sum_{i=1}^n |x_i + y_i|^p &= \sum_{i=1}^n |x_i + y_i| \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n (|x_i| + |y_i|) \cdot |x_i + y_i|^{p-1} \\ &= \sum_{i=1}^n |x_i| \cdot |x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i| \cdot |x_i + y_i|^{p-1} \\ &\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^n |x_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} + \left[\sum_{i=1}^n |y_i|^p \right]^{\frac{1}{p}} \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right] \cdot \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{p-1}{p}} \end{aligned}$$

Now, Divide each side as $[\sum_{i=1}^n |x_i + y_i|^p]^{\frac{p-1}{p}}$, then we obtain

$$\left[\sum_{i=1}^n |x_i + y_i|^p \right]^{1-\frac{p-1}{p}} = \left[\sum_{i=1}^n |x_i + y_i|^p \right]^{\frac{1}{p}} \leq \left[\left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p \right)^{\frac{1}{p}} \right]$$

□

Theorem 22.1.2.2. Let d_{p_1}, d_{p_2} are p -norm on \mathbb{R}^n with $1 \leq p_1 < p_2 \leq \infty$. Then,

$$\exists C > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^n, d_{p_2}(x, y) \leq d_{p_1}(x, y) \leq C d_{p_2}(x, y)$$

In particular, $C = n^{\frac{1}{p_1} - \frac{1}{p_2}}$.

Proof. Let $p_1 < p_2$.

For show that first-inequality,

$$1 = \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_2} \leq \sum_{i=1}^n \left[\frac{|x_i - y_i|}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_1} = \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{p_1}{p_2}}} = \left[\frac{\sum_{i=1}^n |x_i - y_i|^{p_1}^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_1}$$

Thus, we obtain that:

$$1 \leq \left[\frac{\sum_{i=1}^n |x_i - y_i|^{p_1}^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \right]^{p_1} \iff 1 \leq \frac{\sum_{i=1}^n |x_i - y_i|^{p_1}^{\frac{1}{p_1}}}{[\sum_{i=1}^n |x_i - y_i|^{p_2}]^{\frac{1}{p_2}}} \iff \left[\sum_{i=1}^n |x_i - y_i|^{p_2} \right]^{\frac{1}{p_2}} \leq \left[\sum_{i=1}^n |x_i - y_i|^{p_1} \right]^{\frac{1}{p_1}}$$

For show that second-inequality, using Hölder's inequality.

$$\begin{aligned} (d_{p_1}(x, y))^{p_1} &= \sum_{i=1}^n |x_i - y_i|^{p_1} = \sum_{i=1}^n |x_i - y_i|^{p_1} \cdot 1 \\ &\stackrel{\text{Hölder}}{\leq} \left[\sum_{i=1}^n \left(|x_i - y_i|^{p_1 \cdot \frac{p_2}{p_1}} \right)^{\frac{p_1}{p_2}} \right]^{\frac{p_1}{p_2}} \cdot \left[\sum_{i=1}^n 1^{\frac{p_2}{p_2-p_1}} \right]^{1-\frac{p_1}{p_2}} = \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})^{\frac{p_1}{p_2}} \right]^{\frac{p_1}{p_2}} \cdot n^{1-\frac{p_1}{p_2}} \end{aligned}$$

Taking the $\frac{1}{p_1}$ -th power of both sides, then

$$d_{p_1}(x, y) \leq \left[\sum_{i=1}^n (|x_i - y_i|^{p_2})^{\frac{1}{p_2}} \right]^{\frac{1}{p_2}} \cdot n^{\frac{1}{p_1} - \frac{1}{p_2}} = n^{\frac{1}{p_1} - \frac{1}{p_2}} \cdot d_{p_2}(x, y)$$

□

Corollary 22.1.2.1. Let \mathbb{R}^n be given as a set, and $d_{p_1}, d_{p_2} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ are p -norm on \mathbb{R}^n . Then,

$$\mathcal{T}_{d_{p_1}} = \mathcal{T}_{d_{p_2}}$$

For every $p \geq 1$, the metric space (\mathbb{R}^n, d_p) induces the same topology as the product topology on \mathbb{R}^n . In particular, \mathbb{R}^n with the product topology coincides with \mathbb{R}^n endowed with any p -norm.

22.1.3 Open and Closed set in \mathbb{R}^n

Definition 22.1.3.1. For $p \in [1, \infty]$, define p -Ball in \mathbb{R}^n as:

$$B_p(x, r) \stackrel{\text{def}}{=} \{y \in \mathbb{R}^n : \|x - y\|_p < r\}$$

Since all p -norms are equivalent, for any $p \in [1, \infty]$, the collection

$$\beta_p \stackrel{\text{def}}{=} \{B_p(x, r) \mid x \in \mathbb{Q}^n, r \in \mathbb{Q}^+\}$$

is Countable basis of \mathbb{R}^n . Immediately, we obtain:

Lemma 22.1.3.1. Every open set in \mathbb{R}^n is a countable union of p -Balls.

We call 2-Ball the Ball, and ∞ -Ball the Cube.

Theorem 22.1.3.1. Let $U \subseteq \mathbb{R}^n$ be an open set. Then, U is a countable union of closed cubes with disjoint interiors.

Proof. Let $U \subseteq \mathbb{R}^n$ be an open set, and define the collection of Dyadic Cubes on \mathbb{R}^n as: for each $k \in \mathbb{N}$,

$$Q_k \stackrel{\text{def}}{=} \left\{ \prod_{i=1}^n \left[\frac{q_i}{2^k}, \frac{q_i + 1}{2^k} \right] \subset \mathbb{R}^n \mid q_i \in \mathbb{Z} \right\}$$

Each element of Q_k is product of closed intervals, and its interiors are disjoint. For each $k \in \mathbb{N}$, construct:

$$Q_k^* \stackrel{\text{def}}{=} \{Q \in Q_k \mid Q \subseteq U\}$$

Then, the union $Q^* = \bigcup_{k \in \mathbb{N}} Q_k^*$ is a countable union of closed cubes, and $Q^* = U$: $Q^* \subseteq U$ is clear, and let $x \in U$.

Since property of metric space, there exists $\delta > 0$ such that $x \in B_2(x, \delta) \subseteq U$. Put $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \frac{\delta}{\sqrt{n}}$.

Then, $x \in C \subset B_2(x, \delta) \subseteq U$ for some $C \in Q_k$, because $\text{diam } C = \sqrt{n}2^{-k}$. Since $C \subset U$, $C \in Q_k^* \subset Q^*$. i.e., $U \subseteq Q^*$. For disjointness of interiors, we will use the fact:

For any $Q_1, Q_2 \in Q^*$, either their interiors are disjoint, or one is contained in the other.

(Conti.)

□

22.2 Manifold

22.3 Topological Vector Space

22.4 Hilbert Space

Definition 22.4.0.1. Complete Inner product Vector Space is called *Hilbert Space*.

22.4.1 Hilbert Space in \mathbb{R}^ω

Definition 22.4.1.1. Define $\mathbb{R}^\omega \stackrel{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$ as the countable product of Euclidean space \mathbb{R} with product topology.
 And define $\mathbb{H} \stackrel{\text{def}}{=} \left\{ \{x_n\}_{n=1}^{\infty} \mid \sum_{n=1}^{\infty} x_n^2 < \infty \right\} \subset \mathbb{R}^\omega$, Metric on \mathbb{H} as $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$.
 The Metric Space (\mathbb{H}, μ) is called *Hilbert Space* or l_2 Space.

Define the operations elementwise; then $(\mathbb{H}, +, \times)$ is a Vector Space over \mathbb{R} .

Moreover, \mathbb{H} is Complete Metric Space and Inner product Vector Space.

Lemma 22.4.1.1. $\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$ is Metric function induced by the inner product.

Proof. We know that \mathbb{R}^ω is Vector Space. Moreover, $\mathbb{H} \subset \mathbb{R}^\omega$ is Subspace. Using subspace criteria:

$S \subset V$ is Subspace of Vector Space V if and only if $0 \in S$ and For any $x, y \in S$ and $a \in F$, $ax + y \in S$.

Clearly, $\{0\} \in \mathbb{H}$. Let $a \in \mathbb{R}$ and $\{x_n\}, \{y_n\} \in \mathbb{H}$ be given. Then, $a\{x_n\} + \{y_n\} = \{ax_n + y_n\} \in \mathbb{H}$ because:

$$\sum_{i=1}^{\infty} (ax_i + y_i)^2 = \sum_{i=1}^{\infty} [a^2 x_i^2 + 2ax_i y_i + y_i^2] \stackrel{(*)}{=} a^2 \sum_{i=1}^{\infty} x_i^2 + 2a \sum_{i=1}^{\infty} x_i y_i + \sum_{i=1}^{\infty} y_i^2 < \infty$$

The $(*)$ given by:

$$\sum_{i=1}^{\infty} |x_i y_i| = \sum_{i=1}^{\infty} |x_i| |y_i| \leq \sum_{i=1}^{\infty} (\max(|x_i|, |y_i|))^2 \leq \sum_{i=1}^{\infty} (x_i^2 + y_i^2) = \sum_{i=1}^{\infty} x_i^2 + \sum_{i=1}^{\infty} y_i^2 < \infty \quad (*)$$

Thus \mathbb{H} is Vector Space over \mathbb{R} . Now, define inner product on \mathbb{H} as:

$$\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \sum_{i=1}^{\infty} x_i y_i$$

This definition is well-defined since $(*)$. And, Linearity in first:

$$\langle a\{x_n\} + \{y_n\}, \{z_n\} \rangle = \langle \{ax_n + y_n\}, \{z_n\} \rangle = \sum_{i=1}^{\infty} (ax_i + y_i) z_i = a \sum_{i=1}^{\infty} x_i z_i + \sum_{i=1}^{\infty} y_i z_i = a \langle \{x_n\}, \{z_n\} \rangle + \langle \{y_n\}, \{z_n\} \rangle$$

The other conditions are clear. Thus, $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ is inner product space.

Using inner product, define the Norm on \mathbb{H} as:

$$\|\cdot\| : \mathbb{H} \rightarrow \mathbb{R} : \{x_n\} \mapsto \sqrt{\langle \{x_n\}, \{x_n\} \rangle}$$

Finally, define Metric on \mathbb{H} as:

$$\mu : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R} : (\{x_n\}, \{y_n\}) \mapsto \|\{x_n\} - \{y_n\}\| = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2}$$

□

Theorem 22.4.1.1. Hilbert Space is Separable.

Proof. For each $n \in \mathbb{N}$, define $D_n \stackrel{\text{def}}{=} \{\{p_n\} \mid p_i \in \mathbb{Q}, p_{n+1} = p_{n+2} = \dots = 0\}$ and $D \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} D_n$.

Then, D is countable set. We will show that $\overline{D} = \mathbb{H}$.

Let $\epsilon > 0$ and $\{x_n\} \in \mathbb{H}$ be given. Since convergence, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} x_i^2 = \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^N x_i^2 < \frac{\epsilon^2}{2}$$

Since density of Rationals, put each $i = 1, 2, \dots, N$, $p_i \in \mathbb{Q}$ $|x_i - p_i| < \frac{\epsilon}{\sqrt{2N}}$ and $p_i = 0$ for $i \geq N + 1$.

Then, $\{p_n\} \in D_n \subset D$ and

$$\mu(\{x_n\}, \{p_n\}) = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} (x_i - p_i)^2} = \sqrt{\sum_{i=1}^N (x_i - p_i)^2 + \sum_{i=N+1}^{\infty} x_i^2} < \sqrt{N \cdot \frac{\epsilon^2}{2N} + \frac{\epsilon^2}{2}} = \epsilon$$

□

Corollary 22.4.1.1. Hilbert Space is Second-Countable.

Theorem 22.4.1.2. Hilbert Space is Complete.

Proof. Let $\{\{x_{n,i}\}_{i=1}^{\infty}\}_{n=1}^{\infty}$ be a Cauchy sequence in \mathbb{H} . For any fixed $n, m \in \mathbb{N}$ and for each $j \in \mathbb{N}$,

$$|x_{n,j} - x_{m,j}| < \mu(\{x_{n,i}\}, \{x_{m,i}\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2}$$

That is, for each $j \in \mathbb{N}$, $\{x_{n,j}\}$ is Cauchy sequence in \mathbb{R} . Since \mathbb{R} is Complete, put $y_j \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} x_{n,j}$, each $j \in \mathbb{N}$.

Let $\epsilon > 0$ be given. Then, there exists $N \in \mathbb{N}$ such that $n, m \geq N \implies \mu(\{x_{n,i}\}, \{x_{m,i}\}) < \frac{\epsilon}{2}$.

Meanwhile, for each $k \in \mathbb{N}$,

$$\sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \leq \sum_{i=1}^{\infty} (x_{n,i} - x_{m,i})^2 = [\mu(\{x_{n,i}\}, \{x_{m,i}\})]^2$$

Thus, $n, m \geq N \implies \sum_{i=1}^k (x_{n,i} - x_{m,i})^2 < \left(\frac{\epsilon}{2}\right)^2$, for each $k \in \mathbb{N}$.

Taking limit to m , then $n \geq N \implies \lim_{m \rightarrow \infty} \left(\sum_{i=1}^k (x_{n,i} - x_{m,i})^2 \right) = \sum_{i=1}^k \left(x_{n,i} - \lim_{m \rightarrow \infty} x_{m,i} \right)^2 = \sum_{i=1}^k (x_{n,i} - y_i)^2 < \left(\frac{\epsilon}{2}\right)^2$.

And, for all $k \in \mathbb{N}$,

$$\sum_{i=1}^k y_i^2 = \sum_{i=1}^k (2(x_{n,i}^2 + (x_{n,i} - y_i)^2)) \leq 2\|\{x_{n,i}\}_{i=1}^{\infty}\|^2 + \left(\frac{\epsilon}{2}\right)^2$$

Thus $\{y_i\} \in \mathbb{H}$. As a result,

$$n \geq N \implies \mu(\{x_n\}, \{y_n\}) = \sqrt{\sum_{i=1}^{\infty} (x_{n,i} - y_i)^2} = \sqrt{\lim_{k \rightarrow \infty} \sum_{i=1}^k (x_{n,i} - y_i)^2} < \frac{\epsilon}{2}$$

□

Theorem 22.4.1.3. $\mathbb{H} \subset \mathbb{R}^\omega$ with subspace topology is Metrizable.

Proof. We will use two Lemmas:

Lemma 22.4.1.2. Countable Product of Metric Space is Metrizable.

Proof. Let (X_i, d_i) be a metric space, for each $i \in \mathbb{N}$.

If $d : X \times X \rightarrow \mathbb{R}$ is a Metric, then $\frac{d}{1+d}$ is also Metric, because

$$\frac{d(x, z)}{1+d(x, z)} \stackrel{\frac{x}{1+x} \text{ increasing}}{\uparrow} \frac{d(x, y) + d(y, z)}{1+d(x, y) + d(y, z)} \stackrel{d \geq 0}{\uparrow} \frac{d(x, y)}{1+d(x, y)} + \frac{d(y, z)}{1+d(y, z)} \quad (*)$$

Using this fact, define

$$d_\Pi : \prod X_i \times \prod X_i \rightarrow \mathbb{R} : ((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \mapsto \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} \right]$$

Then d_Π is a Metric because: the triangle inequality is satisfied since

$$\begin{aligned} d_\Pi((x_n)_{n=1}^\infty, (z_n)_{n=1}^\infty) &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, z_i)}{1+d_i(x_i, z_i)} \right] \\ &\stackrel{(*)}{\leq} \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \left(\frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} + \frac{d_i(y_i, z_i)}{1+d_i(y_i, z_i)} \right) \right] \\ &= \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(x_i, y_i)}{1+d_i(x_i, y_i)} \right] + \sum_{i=1}^\infty \left[\frac{1}{2^i} \cdot \frac{d_i(y_i, z_i)}{1+d_i(y_i, z_i)} \right] \\ &= d_\Pi((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) + d_\Pi((y_n)_{n=1}^\infty, (z_n)_{n=1}^\infty) \end{aligned}$$

Reflexivity and symmetry are clear.

And, it remains to show that the metric d_Π generates the given product topology.

□

Lemma 22.4.1.3. Metrizable is Hereditary.

Proof omitted.

Consequently, since $\mathbb{H} \subset \mathbb{R}^\omega$ is a subspace of a metric space, it is metrizable.

□

22.5 Banach Space

22.6 L_p Space

22.7 l_p Space

Chapter 23

***N*-Body Problem**

23.1 Introduction

23.1.1 Definition

23.2 Basic Tools

23.3 Two-Body Problem

23.4 Three-Body Problem

23.5 *N*-Body Problem

Chapter 24

Optimization

24.1 Convexity

Definition 24.1.0.1. Suppose that $f : X \rightarrow \mathbb{R}^n$ where X is Convex subset of Vector Space over a field \mathbb{R} . The function f is called *Convex function* if: For any $\mathbf{x}, \mathbf{y} \in X$, and $t \in [0, 1] \subseteq \mathbb{R}$

$$f(t \cdot \mathbf{x} + (1 - t) \cdot \mathbf{y}) \leq tf(\mathbf{x}) + (1 - t)f(\mathbf{y})$$

24.2 Gradient Descent

24.3 Gradient Flow

Theorem 24.3.0.1. Suppose that $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is a Convex Differentiable function, and $\inf f > -\infty$.

Define a map $x : \mathbb{R} \rightarrow \mathbb{R}^m$ such that $\begin{cases} x(0) = x_0 \\ x'(t) = -\nabla f(x(t)) \end{cases}$, where x_0 is given initial point.

Then, the convexity gives the existence of a $x^* \in \inf f$. And, for any $t \geq 0$,

$$f(x(t)) - \inf f \leq \frac{1}{2t} \|x_0 - x^*\|^2$$

Proof. Let $x \in \operatorname{argmin} f$ be given. Consider the Lyapunov function:

$$V(t) \stackrel{\text{def}}{=} t(f(x(t)) - \inf f) + \frac{1}{2} \|x(t) - x^*\|^2$$

Differentiating with respect to t :

$$\begin{aligned} V'(t) &= (f(x(t)) - \inf f) + t(\nabla f(x(t)) \cdot x'(t)) + \frac{1}{2}[(x(t) - x^*) \cdot x'(t)] + \frac{1}{2}[x'(t) \cdot (x(t) - x^*)] \\ &= (f(x(t)) - \inf f) - t\|\nabla f(x(t))\|^2 + (x(t) - x^*) \cdot \nabla f(x(t)) \end{aligned}$$

Meanwhile, the Convexity gives:

$$\inf f = f(x^*) \geq f(x(t)) + \nabla f(x(t)) \cdot (x(t) - x^*)$$

This implies

$$V'(t) = (f(x(t)) - \inf f) + (x(t) - x^*) \cdot \nabla f(x(t)) - t\|\nabla f(x(t))\|^2 \leq -t\|\nabla f(x(t))\|^2 \leq 0$$

Now, for all $t \geq 0$, $V(t) \leq V(0)$. This means

$$\begin{aligned} V(t) &= t(f(x(t)) - \inf f) + \frac{1}{2}\|x(t) - x^*\|^2 \leq \frac{1}{2}\|x(0) - x^*\|^2 = V(0) \\ \implies f(x(t)) - \inf f &\leq \frac{1}{2t}\|x_0 - x^*\|^2 \end{aligned}$$

□

Chapter 25

Artificial Intelligence

In this chapter, $\mathcal{M}_{n \times m}$ is a set of n by m Real-Valued Matrices.

25.1 Feedforward Neural Network

Definition 25.1.0.1. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, Define an Elementwise transformation for f :

$$\mathbf{F}_{n \times m}(f; \cdot) : \mathcal{M}_{n,m} \rightarrow \mathcal{M}_{n,m} : \left(\begin{array}{c} a_{i,j} \\ \hline 1 \leq i \leq n \\ 1 \leq j \leq m \end{array} \right) \mapsto \left(\begin{array}{c} f(a_{i,j}) \\ \hline 1 \leq i \leq n \\ 1 \leq j \leq m \end{array} \right)$$

If there is no confusion, we simply write $\mathbf{F} = \mathbf{F}_{n \times m}$.

Definition 25.1.0.2. Given integer sequence $\{l_i\}_{i=0}^n$, let $W_i \in \mathcal{M}_{l_i, l_{i-1}}$ and $B_i \in \mathcal{M}_{l_i, 1}$. Define a Feedforward Neural Network \mathcal{N} as inductively:

$$\begin{cases} L_0 \stackrel{\text{def}}{=} X \in \mathcal{M}_{l_0, 1} \\ L_1 \stackrel{\text{def}}{=} W_1 \cdot L_0 + B_1 \in \mathcal{M}_{l_1, 1} \\ L_i \stackrel{\text{def}}{=} W_i \cdot \mathbf{F}(\max(x, 0); L_{i-1}) + B_i \in \mathcal{M}_{l_i, 1} \quad (2 \leq i \leq n) \end{cases}$$

For given Output Function $\mathcal{F} : \mathcal{M}_{l_n, 1} \rightarrow \mathcal{M}_{l_n, 1}$, and $\theta = (W_0, \dots, W_n, B_0, \dots, B_n)$,

$$\mathcal{N}(X; \theta) : \mathcal{M}_{l_0, 1} \times \Omega \rightarrow \mathcal{M}_{l_n, 1} : X \mapsto \mathcal{F}(L_n)$$

where $\Omega = \prod_{i=1}^n \mathcal{M}_{l_i, l_{i-1}} \times \prod_{i=1}^n \mathcal{M}_{l_i, 1}$.

25.1.1 Forward Propagation

25.1.2 Back Propagation

From the definition, we can write the Neural Network as:

$$\mathcal{N}(X; \theta) = \mathcal{F} \circ L_n \circ L_{n-1} \circ \dots \circ L_1 \circ L_0$$

The Chain Rule gives: For each $0 \leq i \leq n$

$$\begin{aligned} \frac{\partial \mathcal{N}}{\partial W_i} &= \frac{\partial \mathcal{F}}{\partial L_n} \cdot \frac{\partial L_n}{\partial L_{n-1}} \dots \frac{\partial L_{i+1}}{\partial L_i} \cdot \frac{\partial L_i}{\partial W_i} \\ \frac{\partial \mathcal{N}}{\partial B_i} &= \frac{\partial \mathcal{F}}{\partial L_n} \cdot \frac{\partial L_n}{\partial L_{n-1}} \dots \frac{\partial L_{i+1}}{\partial L_i} \cdot \frac{\partial L_i}{\partial B_i} \end{aligned}$$

25.1.3 Gradient Descent

25.2 Automatic Differentiation

Chapter 26

Cake Theory

26.1 Definition

Definition 26.1.0.1.

26.1.1 Subcake

Definition 26.1.1.1. Let C be a Cake. The non-empty subset $S \subseteq C$ is called *Subcake* of C if: S satisfies

1. S is closed under the *Cutting*.
2. If $\text{Topp}(C) \neq \emptyset$, then for some $s \in \text{Topp}(C)$, s is contained in S .
3. $\text{Lay}(S) = \text{Lay}(C)$.

26.2 Typical Cakes

26.2.1 Cheeses Cake

26.2.2 Chocolate Cake

26.2.3 Strawberry Cake

26.3 Piece of Cake

Definition 26.3.0.1.

Theorem 26.3.0.1. Every Piece of Cake is a Subcake of some Cake.

26.3.1 Subcake of Piece of Cake

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