

**Linear Algebra (MT1004)**

BS(AI), BS(CS), BS(CY), BS(DS), BS(SE)

**Course Instructors**

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**Sessional-I Exam**

Total Time (Hrs): 1

Total Marks: 45

Total Questions: 4

Date: September 22, 2025

Roll No

Section

Student Signature

DO NOT WRITE BELOW THIS LINE

Attempt all the questions

[CLO #1: Apply elementary row operations to solve linear systems of equations.]

Question # 1

[17 Marks]

(a) Let the solution of  $Ax = b$  be

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

The matrix  $B$  and vector  $d$  are obtained by performing the elementary row operation  $R_2 \leftrightarrow R_2 + 2R_1$  on  $A$  and  $b$  respectively. What will be the solution of the equation  $Bx = d$ ? Determine the order of  $A$ . (2)

(b) For any matrices  $A$  of size  $3 \times 4$  and  $B$  of size  $4 \times 3$ , what is the maximum possible number of pivot positions in  $A$  and in  $B$ ? If both  $A$  and  $B$  have 3 pivot positions, comment on the nature of the solution sets of the systems  $Ax = b$  and  $Bx = d$ . (3)

(c) Consider the following matrix  $A$  and vector  $b$ :

$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ -2 & 4 & -6 & -2 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 2 \\ -10 \end{bmatrix}.$$

(i) Find the relation between the solution sets of  $Ax = b$  and  $Ax = 0$ . (5)

(ii) Determine whether the columns of  $A$  span the whole of  $\mathbb{R}^3$ . If not, find the equation of span of the columns of  $A$ . (3)

(iii) Find the basis and the dimension of the  $\text{Nul}(A)$  and the  $\text{Col}(A)$ . (4)

CLO #2: Evaluate mathematical expressions to compute quantities that deal with linear systems and eigenvalue problems.]

Question # 2

[7 Marks]

a) Suppose we represent a dataset by a matrix  $D_{100 \times 50}$  (100 samples, 50 features). If  $\text{rank}(D) = 20$ , what is the dimension of the column space, row space, null space, and left null space of  $D$ ? (2)

- (b) Consider the following set of polynomials in  $\mathbb{P}_2$  (the space of polynomials of degree 2 or less than 2)

$$S = \{1 + x, x + 2x^2, 2x\}.$$

Find a basis for the subspace of  $\mathbb{P}_2$  spanned by  $S$ .

[CLO #2: Evaluate mathematical expressions to compute quantities that deal with linear systems and eigenvalue problems.]

[15 Marks]

Question # 3

(a) Suppose  $T : M_{2 \times 2} \rightarrow \mathbb{R}^5$  be a linear transformation.

(i) If  $\dim(\text{Range}(T)) = 3$ , can we conclude that  $T$  is onto? What can you say about its nullity and rank?

(ii) If  $T$  is one-to-one, what must be the dimension of  $\text{Range}(T)$ ?

(b) Let  $T : \mathbb{P}_1 \rightarrow \mathbb{R}^3$  be a linear transformation defined by

$$T((p(x))) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}, \quad \text{where } p(x) = ax + b.$$

(i) Find  $T((p(x)))$  when  $p(x) = 2x + 1$ .

(ii) Find the matrix of the transformation  $T$ .

(iii) Compute the kernel of  $T$ . Explain how this relates to one-to-one.

(iv) Compute the range (image) of  $T$ . Determine whether  $T$  is onto.

[CLO #2: Evaluate mathematical expressions to compute quantities that deal with linear systems and eigenvalue problems.]

[6 Marks]

Question # 4

Let  $\mathcal{B} = \{b_1, b_2\}$  and  $\mathcal{C} = \{c_1, c_2\}$  be the basis of  $\mathbb{R}^2$ , where

$$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad b_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Find the change-of-coordinates matrix from  $\mathcal{B}$  to  $\mathcal{C}$ .

(4)

(b) Find the  $\mathcal{C}$ -coordinate vector for  $x$  when  $[x]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .

(2)

Good Luck!