Masterarbeit

Turing machines and irrational values of l^2-Betti numbers

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1 Introduction

2 l^2-Betti numbers

In this whole chapter G is always a discrete countable group with multiplication $\cdot_G$.

2.1 Von Neumann Dimension

Definition 2.1. *The group ring RG over a Ring R is the free module over G with the additional multiplication*

$$\sum_{h\in H}\alpha_h\cdot h\cdot_{RG}\sum_{k\in K}\beta_k\cdot k=\sum_{h\in H}\sum_{k\in K}\alpha_h\beta_k\cdot h\cdot_G k$$

with $\alpha_h,\beta_k\in R$ and $H,K\in G$ finite subsets.

Definition 2.2. *$l^2(G)$ is the Hilbert space consisting of formal sums $\sum_{g\in G}\lambda_g\cdot g$ such that $\lambda_g\in\mathbb{C}$ and $\sum_{g\in G}|\lambda_g|^2<\infty$. The multiplication is defined as in 2.1 and the scalar product is defined through*

$$\langle\sum_{g\in G}\alpha_g\cdot g,\sum_{g\in G}\beta_g\cdot g\rangle:=\sum_{g\in G}\alpha_g\overline{\beta_g}.$$

Definition 2.3. *The group von Neumann algebra $\mathcal{N}(G)$ is defined as the algebra of G-equivariant bounded operators from $l^2(G)$ to $l^2(G)$ where G acts on $l^2(G)$ by left multiplication.*

We will denote the bounded operators of a Hilbert space H_1 with $\mathcal{B}(H_1)$. If G acts on a Hilbert Space H_2 we will call the subspace of G-equivariant elements of H_2 by H_2^G. Therefore we can also define the group von Neumann algebra through $\mathcal{N}(G):=\mathcal{B}(l^2(G))^G$.

Definition 2.4. *Let $e_G\in G$ be the unit element of G. Then*

$$tr_{\mathcal{N}(G)}:\mathcal{N}(G)\to\mathbb{C},A\mapsto\langle A(1\cdot e_G),1\cdot e_G\rangle$$

is called the von Neumann trace on $\mathcal{N}(G)$.

Remark 2.5. *Right multiplication with an element of the Group ring $\mathbb{C}G$ is a G-equivariant operator on $l^2(G)$ and therefore has a well defined von Neumann trace.*

Definition 2.6. *A Hilbert $\mathcal{N}(G)$-module is a Hilbert space H with a linear isometric G-action such that there exists an isometric linear G-embedding from H to $l^2(G)^n$ for some $n\in\mathbb{N}$.*

Definition 2.7. *Let H be a Hilbert $\mathcal{N}(G)$-module with corresponding embedding $\iota:H\to l^2(G)^n$ and let $pr_{\iota(H)}:l^2(G)^n\to H$ be the corresponding G-equivariant projection. Let*

$\iota_i : l^2(G) \to l^2(G)^n, y \mapsto (x_1, \dots, x_n)$ with $x_j = \delta_{i,j} \cdot y$ the inclusion of the i-th $l^2(G)$ and $pr_i : l^2(G)^n \to l^2(G), (x_1, \dots, x_n) \mapsto x_i$ the corresponding projection. Then

$$tr_{\mathcal{N}(G)} : \mathcal{B}(H)^G \to \mathbb{C}, A \mapsto \sum_{i=1}^{n} tr_{\mathcal{N}(G)}(pr_i \circ \iota \circ A \circ pr_{\iota(H)} \circ \iota_i)$$

is called the von Neumann trace on H.

One can show that the trace is independent of the chosen embedding and therefore well-defined.

Definition 2.8. *Let H be a Hilbert $\mathcal{N}(G)$-module. We define the von Neumann dimension of H as*

$$dim_{\mathcal{N}(G)}(H) = tr_{\mathcal{N}(G)}(id_H).$$

2.2 l^2-Betti numbers of G-CW-complexes

In this chapter we will require some basic knowledge of CW-complexes and regular homology. For more information on this topic see e.g.[3].

Definition 2.9. *Let X be a CW-complex and $\rho : G \to Homeo(X)$ an action of G on X by homeomorphism such that for every open cell $E \subset X$ and every $g \in G$ it holds that $\rho(g)(E)$ is again an open cell and $\rho(g)|_E = id_E$ if $\rho(g)(E) \cap E \neq \emptyset$. Then ρ is called a cellular action.*

Definition 2.10. *A G-CW-complex is a CW-complex together with a cellular G-action.*

Definition 2.11. *The orbit of an n-cell in a G-CW-complex is called a G-equivariant n-cell.*

Definition 2.12. *Let X be a G-CW complex. X is called*

- *proper if all stabilizer groups are finite,*
- *free if all stabilizer groups are trivial,*
- *finite type if for every $n \in \mathbb{N}$ it has only finitely many G-equivariant n-cell.*

Remark 2.13. *One can show that the chain complex of a G-CW complex X is a chain complex $C_*(X)$ of left $\mathbb{Z}G$-modules, i.e. each $C_n(X)$ is a left $\mathbb{Z}G$-module and the differentials are homomorphisms of these modules.*

Definition 2.14. *The l^2-chain complex of a G-CW complex X is*

$$C_*^{(2)}(X) = l^2G \otimes_{\mathbb{Z}G} C_*(X).$$

Definition 2.15. *Let X be a proper, finite type G-CW complex and $C_*^{(2)}(X)$ the corresponding l^2-chain complex with its differential $d_*^{(2)}$. Then*

$$H_n^{(2)}(X) := ker(d_n^{(2)})/\overline{im(d_{n+1}^{(2)})}$$

is called the n-th l^2-homology and

$$b_n^{(2)}(X) = dim_{\mathcal{N}(G)} H_n^{(2)}(X)$$

the n-th l^2-Betti number.
Remark 2.16. *One can show that for a proper, finite type G-CW complex $H_n^{(2)}(X)$ is a Hilbert $\mathcal{N}(G)$-module and therefore the l^2-Betti number is in fact well-defined.*

2.3 l^2-Betti numbers arising from a group

Remark 2.17. *A Matrix $A \in \mathbb{C}G^{n\times m}$ can be seen as a map from $l^2G^n \to l^2G^m$ via $x \mapsto x^\top \cdot A$. Therefore $ker(A)$ is a Hilbert $\mathcal{N}(G)$-module and has a well-defined von-Neumann-dimension.*
Theorem 2.18. *Let $x \in \mathbb{R}$ and G be a discrete, countable, finitely generated group. The following are equivalent:*

i) There exists a cocompact free finite type G-CW-complex X and a $n \in \mathbb{N}$ such that $b_n^{(2)}(X) = x$

ii) There exists a Matrix $A \in \mathbb{Q}G^{n\times n}$ for a natural numbers $n \in \mathbb{N}$ such that $dim_{\mathcal{N}(G)}(ker(A)) = x$

Proof. "*i)* $\Rightarrow$ *ii)*" Let X be a cocompact free finite type G-CW-complex and $n \in \mathbb{N}$ such that $b_n^{(2)}(X) = x$. We define

$$\Delta_n^{(2)} = d_n^{(2)*} d_n^{(2)} + d_{n+1}^{(2)} d_{n+1}^{(2)*},$$

which is a map from $C_n^{(2)}(X)$ to $C_n^{(2)}(X)$ and for which $ker(\Delta_n^{(2)}) = H_n^{(2)}$ holds. With 2.17 *ii)* follows.

"*ii)* $\Rightarrow$ *i)*" Let $n \in \mathbb{N}$ and $A \in \mathbb{Q}G^{n\times n}$. Because $k \cdot A$ has the same kernel as A for every $k \in \mathbb{Z}$ we can assume that $A \in \mathbb{Z}G^{n\times n}$. The corresponding map will be also called A. We will construct a G-CW-complex X with $d_3^{(2)} = A$ and $d_4^{(2)}$ trivial. Therefore $H_3^{(2)}(X) = ker(A)$ and *i)* follows.

First we can see that a graph can always be seen as a CW-complex with the edges as 1-cells and the vertices as 0-cells. Let $g_1 \dots g_m$ be generators of G and let Y be the corresponding Cayley graph of G which has a natural cellular free G-action and is therefore a free, finite type G-CW-complex. In addition it has exactly 1 G-equivariant 0-cell and m G-equivariant 1-cell and no other cells. We then glue n many 2-cells on each 0-cell, i.e. we attach them via the attaching map which sends all of S^1 to a single point.

We now glue in n many G-equivariant 3-cells in such a way that A corresponds to $d_3^{(2)}$. Let $A_{ij} = \sum_{g\in G} z_{(g,i,j)} g \in \mathbb{Z}G$ denote the entry in the i-th row and the j-th column of A. We fix one 0-cell and call it e and we call the corresponding 2-cells $e_1 \dots e_n$. We glue in

$G \times D^3$ where each element $h \times D^3$ is glued $z_{(g,1,j)}$ many times to $hg \cdot e_j$ for each $g \in G$ and $j \in \{1, \dots, n\}$. We then see that $d_3^{(2)}$ maps the vector $(\sum_{g \in G} \lambda_g g, 0, \dots, 0) \in (l^2G)^n$ to $(\sum_{g \in G} \sum_{h \in H} \lambda_g z_{(h,1,1)} gh, \dots, \sum_{g \in G} \sum_{h \in H} \lambda_g z_{(h,1,n)}) \in (l^2G)^n$. We then repeat this for every row of A and get that $d_3^{(2)} = A$. The resulting G-CW-complex X is still free, finite type and because X/G is finite X is cocompact. $\square$

Remark 2.19. *One can show that 2.18 also holds if G is not finitely generated.*
Definition 2.20. *We say that a number $x \in \mathbb{R}$ arises from a discrete countable group G if it fulfills one of the above conditions.*
Lemma 2.21. *If $x \in \mathbb{R}$ and $z \in \mathbb{R}$ both arise from a discrete, countable Group G, then so does $x + z$.*
Lemma 2.22. *Let G is a discrete countable Group and H a discrete finite Group. If $x \in \mathbb{R}$ arises from $G \times H$ then $x \cdot |H|$ arises from G.*

3 Turing dynamical systems

Let us first give a definition of a Turing machine. For convenience we will only allow 1 and 0 as symbols on our tape.
Definition 3.1. *A Turing machine is a 5-tuple $T = (S, \delta, A, R, I)$ where S is a finite set of states and $\delta : S \setminus (A \cup R) \times \mathbb{Z}/2\mathbb{Z} \to S \times \mathbb{Z}/2\mathbb{Z} \times \{-1, 0, 1\}$ is called the transition function. $A \subset S$ is called the set of accepting states, $R \subset S$ is called the set of rejecting states and $I \in S$ is called the initial state.*

In addition each Turing machine uses a "tape", i.e. a set $\mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}}$ with a "head" on the element corresponding to the index 0. A Turing machine can operate on an element of $Y \in \mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}}$ which is called an input. We start with the initial state and use the transition function δ to get $\delta(I, Y_0)$. The first coordinate corresponds to the new element Y_0, the second to the new state and the third corresponds to shifting the tape to the left or the right and therefore getting a new Y_0, e.g. if $\delta(I, Y_0)_3 = 1$ we shift the whole tape one to the left, therefore our new Y_0 is our old Y_1. We repeat this step until we get a state in $A \cup R$. We say, that the Turing machine accepts Y if we get a state in A after finitely many steps and it rejects Y if we get a state in R after finitely many steps. We say that the Turing machine holds for Y if it accepts or rejects it.

For the purpose of calculating l^2-Betti numbers of groups we need to extend this definition. Let (X, μ) be a probability measure space divided into finitely many disjoint measurable subsets X_i. Let Γ be a countable discrete group and ρ be a right measure preserving action of Γ on X. We now choose 3 disjoint subsets A, R, I, where each of them is a union of certain X_i. They will be called the accepting set, the rejecting set and the initial set. In addition we choose a $\gamma_i \in \Gamma$ for each $X_i \subset X$ such that for each i with $X_i \subset A$ or $X_i \subset R$ it holds that $\gamma_i = e$ where e is the neutral element in Γ. Let $Ind : X \to \mathbb{N}$ be the map which assigns to each $x \in X$ the corresponding index of the X_i it is contained in.

Definition 3.2. *The map*

$$T_X : X \to X, x \mapsto \rho(\gamma_{Ind(x)})(x)$$

is called the Turing map. The group Γ and the space X (with all the choices of subsets and corresponding γ_i made in the last paragraph) together with the Turing map will be called a Turing dynamical system and will be denoted by (T_X)

Let $x \in X$. If there is a $k \in \mathbb{N}$ such that $T_X^k(x) = T_X^{k+1}(x)$ there also holds $T_X^h(x) = T_X^{h+1}(x)$ $\forall h \in \mathbb{N}$ with $h \geq k$. We then define $T_X^\infty(x) = T_X^k(x)$. Mostly we don't look at the map T_X but the map T_X^∞. We say that the Turing dynamical system (T_X) accepts an input $y \in I$ if $T_X^\infty(y) \in A$ and it rejects it if $T_X^\infty(y) \in R$. In addition we say that (T_X) holds for y if it accepts or rejects it.

We can see that a Turing machine can be emulated with a Turing dynamical system. But first lets fix notation

Remark 3.3. *Let $M \subset N^{\mathbb{Z}}$ be defined through $M = \{(n_i)_{i\in\mathbb{Z}} \in N^{\mathbb{Z}} | n_{-k} = m_{-k}, \cdots, n_l = m_l\}$ $k, l \in \mathbb{N}$, i.e. a set where $k+l$ coordinates including the 0-coordinate are fixed. We then simply write $[m_{-k} \cdots \underline{m_0} \cdots m_l]$ for M and $[m_{-k} \cdots \underline{m_0} \cdots m_l][\sigma]$ for $M \times \{\sigma\}$.*

Example 3.4. *Let $T = (S, \delta, \tilde{A}, \tilde{R}, \tilde{I})$ be a Turing machine. We define $X = \mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}} \times S$ and $\Gamma = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \times Bij(S)$. The action of Γ on X is defined by the following rules:*

- *The generator $\overline{1}$ of the group $\mathbb{Z}/2\mathbb{Z}$ acts on the $\mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}}$ part of X by adding $\overline{1}$ to the element with index 0.*
- *The generator 1 of the group $\mathbb{Z}$ acts also on the $\mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}}$ part by shifting every element 1 to the left, i.e. decreasing the index of every element by 1.*
- *$Bij(S)$ acts on S in the natural way.*

We will now use the transition function to construct the Turing map. We choose the following devision of X:

$$X = \bigcup_{x\in\mathbb{Z}/2\mathbb{Z}, \sigma\in S} [\underline{x}][\sigma]$$

For an arbitrary $X_i = [\underline{x}][\sigma]$ we can use the transition function of the Turing machine and get $\delta(\sigma, x) = (\tilde{\sigma}, \alpha, \beta)$.Let $\tau \in Bij(S)$ be some map which sends σ to $\tilde{\sigma}$ and let $y \in \mathbb{Z}/2\mathbb{Z}$ be $y = x - \alpha$. We can then see y as a map from $\mathbb{Z}/2\mathbb{Z}$ to itself which sends x to α. Our element γ_i corresponding to X_i is then given by

$$\gamma_i = \begin{cases} (id, \overline{0}, 0) & \text{if } X_i \subset A \cup R \\ (y, \beta, \tau) & \text{else} \end{cases}$$

We only need to define A, R and I. A is given by

$$A = \bigcup_{x\in\mathbb{Z}/2\mathbb{Z}, \sigma\in\tilde{A}} [\underline{x}][\sigma]$$

which is of course a union of some X_i. R and I can be defined analogous. The resulting Turing dynamical system emulates the Turing machine in the following way: If we have an input $Y \in \mathbb{Z}/2\mathbb{Z}$ of the Turing machine we can transform it to an element $Y \times \tilde{I} \in X$. The Turing dynamical system accepts (rejects) $Y \times \tilde{I}$ exactly when the Turing machine accepts (rejects) Y.

Definition 3.5. *The first fundamental set $\mathcal{F}_1(T_X)$ is the subset of I consisting of those points x with $T_X^\infty(x) \in A$ and there is no point y with $T_X(y) = x$. The second fundamental set is the subset of A defined as $\mathcal{F}_2(T_X) = T_X^\infty(\mathcal{F}_1(T_X))$. The first (second) fundamental value $\Omega_1(T_X)(\Omega_2(T_X))$ is the measure of the corresponding fundamental set.*

Definition 3.6. *We say that a Turing dynamical system (T_X)*

- *stops at any configuration if $T_X^\infty(x) \in A \cup R$ for almost all $x \in X$*
- *has disjoint accepting chains if $T_X^\infty(x) \neq T_X^\infty(y)$ for almost all $x, y \in I$ with $x \neq y$*
- *does not restart if the set $T_X(X) \cap I$ has measure 0*

Definition 3.7. *Let (T_X) be a Turing dynamical system where $X = \Pi_{j\in J}\mathbb{Z}/2\mathbb{Z}$ is an infinite product of $\mathbb{Z}/2\mathbb{Z}$ and each X_i has only finitely many fixed elements, i.e. is of the form $X_i = \{(x_j)_{j\in J} \in X \mid x_k = 0\ \forall k \in I_1, x_h = 1\ \forall h \in I_2\ \ I_1, I_2 \subset J \text{ finite}\}$. If in addition the action of Γ is by continuous group automorphisms and (T_X) stops at any configuration, has disjoint accepting chains and does not restart it is called a computing Touring dynamical system.*

Before we give an example of a computing Turing dynamical system we fix the notation of the shifting operation, because we will need it later on.

Definition 3.8. *An element $z \in \mathbb{Z}$ acts on $\mathbb{Z}/2\mathbb{Z}^\mathbb{Z} = \prod_{i\in\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ by decreasing every index by z. We call this action the shift action and denote it by ζ. $\mathbb{Z}$ acts on $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ in the same way which we will also call ζ.*

Example 3.9. *Let T be a read-only Turing machine. Let (T_X) be the corresponding Turing dynamical system as constructed in 3.4. We assume that T is in such a way that (T_X) has disjoint accepting chains and stops at any configuration (which is not always the case). We will construct a computable Turing dynamical system from T. We can easily assure that (T_X) does not restart by adding a new state I' to T with $\delta(I', x) = \delta(I, x)\ \forall x \in \{0, 1\}$ and setting I' as the initial state of T. Because we do not change the symbols of the tape it suffices to use $\Gamma = \mathbb{Z} \times Bij(S)$ where $\mathbb{Z}$ operates on $X = \mathbb{Z}/2\mathbb{Z}^\mathbb{Z} \times S$ via shifting and $Bij(S)$ acts on S in the natural way. Because a Turing machine has only finitely many states there exists a number $n \in \mathbb{N}$ such that $|S| < 2^n$. We identify every state with a different element $z \in \mathbb{Z}/2\mathbb{Z}^n$ with $z \neq 0$.*

We can then assume that $X = \mathbb{Z}/2\mathbb{Z}^\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}^n$ by filling $\mathbb{Z}/2\mathbb{Z}^n$ with "dummy states" which will never be used. Because $Aut(\mathbb{Z}/2\mathbb{Z}^n)$ acts transitively on all of $\mathbb{Z}/2\mathbb{Z}^n/\{0\}$ we can assume that $\Gamma = \mathbb{Z} \times Aut(S)$. Because in the sets X_i in 3.4 there are only $n+1$ fixed components they are of the desired form and therefore the resulting Turing dynamical system is computing.

4 Computing l^2-Betti numbers

In this chapter we want to prove the following central theorem which connects l^2-Betti numbers to the concept of Turing dynamical systems.

Theorem 4.1. *Let (T_X) be a computing Touring dynamical system. Then $\mu(I)-\Omega_1(T_X)$ is a l^2-Betti number arising from $\hat{X} \rtimes_{\hat{\rho}} \Gamma$ where $\hat{X}$ and $\hat{\rho}$ are the Pontryagin duals of the corresponding group or map.*

We will give the definition of Pontryagin duals and the semidirect product later in this chapter. The rest of this chapter is used to prove this theorem and give the needed definitions.

4.1 Groupoids

We begin by giving some algebraic definitions, namely the construct of groupoids which extends the definition of groups. Therefore we require some basic knowledge about categories.

Definition 4.2. *A groupoid is a small category whose morphisms are all invertible.*

Example 4.3. *Let G be a group. We want to see how we can express G as a groupoid. Let $\mathcal{G}_0 = \{\bullet\}$ be the set with only one element and $\mathcal{G}$ be the category with objects G_0 and morphisms G from $\bullet$ to $\bullet$.The composition of morphisms in $\mathcal{G}$ is the same as the multiplication in G. Then $\mathcal{G}$ is a small category and every morphism is invertible because every group element has an inverse. Therefore $\mathcal{G}$ is a groupoid.*

We will always denote the set of objects of a groupoid $\mathcal{G}$ by $\mathcal{G}_0$ and we can identify it with a subset of the set of morphisms of $\mathcal{G}$ by the embedding $\mathbf{1} : \mathcal{G}_0 \to mor(\mathcal{G}), x \mapsto [id : x \to x]$ which sends every object to the corresponding identity morphism. Therefore we will only only look at $mor(\mathcal{G})$ and also call it $\mathcal{G}$.

Definition 4.4. *For every Groupoid $\mathcal{G}$ we define the maps $s : \mathcal{G} \to \mathcal{G}_0, [f : X \to Y] \mapsto X$ and $r : \mathcal{G} \to \mathcal{G}_0, [f : X \to Y] \mapsto Y$ which will be called source and range map.*

Definition 4.5. *A discrete measurable groupoid is a groupoid where $\mathcal{G}$ is also a measurable space, s, r and the maps gained through inverting or composition are all measurable and the fibers of s and r are countable. If in addition we have a measure μ such that*

$$\int_{\mathcal{G}_0} |r^{-1}(x) \cap U| d\mu(x) = \int_{\mathcal{G}_0} |s^{-1}(x) \cap U| d\mu(x) \qquad \forall U \subset \mathcal{G}$$

holds we call $\mathcal{G}$ discrete measured

4.2 Groupoid ring

A fundamental concept used for the calculation of l^2-Betti numbers of groups was the group ring. We will transfer this to the notion of groupoids.

Definition 4.6. *Let U be a subset of $\mathcal{G}_0$. A measurable edge is a map $\Phi : U \to \mathcal{G}$ such that $s \circ \Phi = id$ and $r \circ \Phi$ is injective.*

From the definition we see, that defining a measurable edge means taking a subset of $\mathcal{G}_0$ and associating a morphism to every object such that the morphism starts in this object and no two such morphisms end in the same object. We want to define the inverse of a measurable edge in such a way, that it is also a measurable edge. Therefore it does not suffice to take the inverse of the map Φ. Instead we invert every morphism in the image of Φ.

Definition 4.7. *Let $\Phi : U \to \mathcal{G}$ be a measurable edge. The inverse of Φ will be called Φ^{-1} and is defined as $\Phi^{-1} : r(Im(\Phi)) \to \mathcal{G}$ such that Φ^{-1} is a measurable edge and $\Phi^{-1} \circ r \circ \Phi(x) = \Phi(x)^{-1}$ $\forall x \in r(Im(\Phi))$ where $\Phi(x)^{-1}$ is the inverse of the morphism $\Phi(x)$.*

We now come to the definition of the groupoid ring. We want it to be ring of operators of $L^2(\mathcal{G})$, such that it is in a way generated by measurable edges. For a measurable edge Φ we define a operator $\tilde{\Phi}$ through

$$\tilde{\Phi} : L^2(\mathcal{G}) \to L^2(\mathcal{G}), F \mapsto \left[\tilde{\Phi}(F) : \mathcal{G} \to \mathbb{C}, \gamma \mapsto \begin{cases} F(\gamma \cdot_{\mathcal{G}} \Phi^{-1}(r(\gamma)) & \text{if } r(\gamma) \in Dom(\Phi^{-1}) \\ 0 & \text{otherwise} \end{cases}\right].$$

In addition for a map $f \in L^\infty(\mathcal{G}_0)$ we also define a operator $\tilde{f}$ on $L^2(\mathcal{G})$ through

$$\tilde{f} : L^2(\mathcal{G}) \to L^2(\mathcal{G}), F \mapsto [\tilde{f}(F) : \mathcal{G} \to \mathbb{C}, \gamma \mapsto F(\gamma) \cdot f(r(\gamma))].$$

Definition 4.8. *For a groupoid $\mathcal{G}$ the groupoid ring $\mathbb{C}\mathcal{G}$ is the ring of bounded operators on $L^2(\mathcal{G})$ generated by all measurable edges and all elements of $L^\infty(\mathcal{G})$.*

We always denote the elements of $\mathbb{C}\mathcal{G}$ by a linear combination $\sum_{i \in I} \tilde{\Phi}_i \cdot_{\mathbb{C}\mathcal{G}} \tilde{f}_i$ where Φ is a measurable edge, $f \in L^\infty(\mathcal{G}_0)$ and I is a finite set. One can show that each element of $\mathbb{C}\mathcal{G}$ can be (although non-uniquely) represented in such a way.

Example 4.9. *Let G be a group and $\mathcal{G}$ the corresponding groupoid as in Example 4.3. We want to show that the groupoid ring $\mathbb{C}\mathcal{G}$ is isomorphic to the g $\mathbb{C}G$. Because we have only one element in $\mathcal{G}_0$ the only measurable edges we have are the maps from $\bullet$ to a specific group element of G. The elements of $L^\infty(\mathcal{G}_0)$ are just maps from $\bullet$ to a specific element in $\mathbb{C}$. Therefore $L^\infty(\mathcal{G}_0)$ is isomorphic to $\mathbb{C}$. Now let $\Phi_1 : \bullet \mapsto g_1$ and $\Phi_2 : \bullet \mapsto g_2$ be measurable edges. ...*

Remark 4.10. *Let $\mathcal{G}(\rho)$ be an action Groupoid of $\rho : \Gamma \to Bij(X)$. For each element $\gamma \in \Gamma$ we get a measurable edge $\bar{\gamma} : X \to x \times \Gamma$ which maps $x \in X$ to (x, γ).*

4.3 Groupoids of Turing dynamical systems

Definition 4.11. *Let Γ be a discrete, countable group, X a probability measure space and $\rho : \Gamma \to Bij(X)$ a right measure preserving action. The action groupoid $\mathcal{G}(\rho)$ is the groupoid with objects X and morphisms $X \times \Gamma$ such that $s(x,\gamma) = x$ and $r(x,\gamma) = \rho(\gamma)(x)$. The composition of morphisms is defined through $(x,\gamma_1) \cdot_{\mathcal{G}(\rho)} (\rho(\gamma_1)(x), \gamma_2) = (x, \gamma_1 \cdot \gamma_2)$ and the inverse of (x,γ) is $(\rho(\gamma)(x), \gamma^{-1})$.*

Remark 4.12. *If G is a locally compact Hausdorff topological group then there exists a unique normalized left Haar measure on G which is in particular left translation invariant. Let (T_X) be a computing Turing dynamical system. Then the Haar measure is defined on $X = \Pi_{j\in J}\mathbb{Z}/2\mathbb{Z}$. From now on every computing Turing dynamical system will always be equipped with the corresponding normalized Haar measure.*

Lemma 4.13. *Let (T_X) be a computing Turing dynamical system with $X = \Pi_{j\in J}\mathbb{Z}/2\mathbb{Z}$ and equipped with the normalized Haar measure μ. For a set $M = \{(x_j)_{j\in J} \in X \mid x_k = 0 \ \forall k \in I_1, x_h = 1 \ \forall h \in I_2 \ \ I_1, I_2 \subset J\}$ it holds that*

$$\mu(M) = \begin{cases} \frac{1}{2^{|I_1|\cdot|I_2|}} & if \ I_1 \ and \ I_2 \ are \ finite \\ 0 & if \ I_1 \ or \ I_2 \ is \ infinite \end{cases}.$$

Proof. Let $x \in X$ be an element with $\overline{0}$ on every coordinate $i \notin I_1 \cup I_2$. Then xM is disjoint with M and because of the left translation invariance of the measure it holds that $\mu(M) = \mu(xM)$. Because for every $i \in I_1 \cup I_2$ we can set the corresponding coordinate either to $\overline{1}$ or to $\overline{0}$ we get $2^{|I_1|\cdot|I_2|}$ of these sets if I_1 and I_2 are finite and an infinite amount of sets if one of them is infinite. The disjoint union of all of these sets is the whole set X so the sum of these measures equals 1 and the claim follows. □

4.4 Traces of Turing dynamical systems

4.5 Pontryagin duality

We will now create a link between the groupoid ring and the group ring. For this we need the Pontryagin duals. For the rest of this chapter X is a locally compact abelian group, Γ is another group acting on X by continuous group automorphisms. We will call this action $\rho : \Gamma \to Aut(X)$.

Definition 4.14. *A character of X is a homomorphism $\hat{x} : X \to S$, where S is the multiplicative group of complex numbers of absolute value 1.The group of all characters of X is called the character group or Pontryagin dual group of X and is denoted by $\hat{X}$.*

Definition 4.15. *The Pontryagin dual $\hat{\rho} : \Gamma \to Aut(\hat{X})$ of the action ρ is an action on $\hat{X}$ and defined as $\hat{\rho}(\gamma)(f)(x) = f(\rho(\gamma^{-1})(x))$ for $f \in \hat{X}$ and $x \in X$.*

Remark 4.16. *An element of the Pontryagin dual $\hat{x} \in \hat{X}$ can also be seen as an element of $L^\infty(X)$. We call this element $P(x)$*

Definition 4.17. *Let N, H be groups and $\Theta : H \to Aut(N)$ a homomorphism. The semidirect product $N \rtimes_\Theta H$ of H and N is defined on the Set $N \times H$ with multiplication*

$$(n_1, h_1) \cdot_{N \rtimes_\Theta H} (n_2, h_2) = (n_1 \cdot_N \Theta(h_1)(n_2), h_1 \cdot_H h_2).$$

For more information about semidirect products see e.g. [4].

Theorem 4.18. *The map*

$$P \otimes 1 : \mathbb{C}(\hat{X} \rtimes_{\hat{\rho}} \Gamma) \to \mathbb{C}\mathcal{G}(\rho), \sum_I c_i \cdot (\hat{x}_i, \gamma_i) \mapsto \sum_I c_i \cdot \widetilde{P(\hat{x}_i)} \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{\gamma}}_i$$

with $c_i \in \mathbb{C}$, $\hat{(x_i)} \in \hat{X}$ and $\gamma_i \in \Gamma$ is

i) a ringhomomorphism

ii) trace-preserving

Proof. *i)* Because of the definition it is clear that $P \otimes 1$ preserves the addition. So let $c_i \in \mathbb{C}$, $\hat{(x_i)} \in \hat{X}$ and $\gamma_i \in \Gamma$ with $i \in \{1, 2\}$. It follows that

$$\begin{aligned} &P \otimes 1(c_1 \cdot (\hat{x}_1, \gamma_1) \cdot_{C(\hat{X} \rtimes_{\hat{\rho}} \Gamma)} c_2 \cdot (\hat{x}_2, \gamma_2)) \\ &= P \otimes 1(c_1 \cdot c_2 \cdot (\hat{x}_1, \gamma_1) \cdot_{N \rtimes_{\hat{\rho}} H} (\hat{x}_2, \gamma_2)) \\ &= P \otimes 1(c_1 \cdot c_2 \cdot (\hat{x}_1 \cdot_{\hat{X}} \hat{\rho}(\gamma_1)(\hat{x}_2), \gamma_1 \cdot_\Gamma \gamma_2)) \\ &= c_1 \cdot c_2 \cdot P(\widetilde{\hat{x}_1 \cdot_{\hat{X}} \hat{\rho}(\gamma_1)}(\hat{x}_2)) \cdot_{\mathbb{C}\mathcal{G}(\rho)} \overline{\widetilde{\gamma_1 \cdot_\Gamma \gamma_2}}. \end{aligned}$$

From the definition of the groupoid ring it follows directly that the last term equals

$$c_1 \cdot c_2 \cdot \widetilde{P(\hat{x}_1)} \cdot_{\mathbb{C}\mathcal{G}(\rho)} P(\widetilde{\hat{\rho}(\gamma_1)}(\hat{x}_2)) \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{\gamma}}_1 \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{\gamma}}_2.$$

Because elements of $\mathbb{C}$ commute with all elements in the groupoid ring it remains to show that

$$P(\widetilde{\hat{\rho}(\gamma_1)}(\hat{x}_2)) \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{\gamma}}_1 = \tilde{\tilde{\gamma}}_1 \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{x}}_2$$

in the groupoid ring. Let $F \in L^2\mathcal{G}(\rho)$ and $\alpha \in \mathcal{G}$. Then the left side equals

$$P(\widetilde{\hat{\rho}(\gamma_1)}(\hat{x}_2)) \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{\gamma}}_1(F)(\alpha) = \hat{x}_2(\rho(\gamma_1^{-1})(r(\alpha))) \cdot F(\alpha \cdot_{\mathcal{G}(\rho)} \bar{\gamma}_1{}^{-1}(r(\alpha)))$$

and the right side equals

$$\tilde{\tilde{\gamma}}_1 \cdot_{\mathbb{C}\mathcal{G}(\rho)} \tilde{\tilde{x}}_2(F)(\alpha) = \tilde{\tilde{\gamma}}_1(\hat{x}_2(r(\alpha)) \cdot F(\alpha)) = \hat{x}_2(r(\alpha \cdot_{\mathcal{G}(\rho)} \bar{\gamma}_1{}^{-1}(r(\alpha))) \cdot F(\alpha \cdot_{\mathcal{G}(\rho)} \bar{\gamma}_1{}^{-1}(r(\alpha)))$$

which is the same.

ii) □

5 The lamplighter group

Definition 5.1. *The lamplighter group L is defined as*

$$L = (\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes_{\zeta} \mathbb{Z}.$$

where ζ is the shift operation defined in 3.8.

Lets first look at the Turing dynamical system corresponding to a read-only Turing machine as constructed in 3.9. If we can assure that the resulting Turing dynamical system has disjoint accepting chains and stops at any configuration we can assure that it is computing. By using 4.1 we see that $\mu(I) - \Omega_1(T_X)$ is a l^2-Betti number arising from $\hat{X} \rtimes_{\hat{\rho}} \Gamma$. We then get

$$\begin{aligned} \hat{X} \rtimes_{\hat{\rho}} \Gamma &= (\widehat{(\prod_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z})} \times \mathbb{Z}/2\mathbb{Z}^n) \rtimes_{\hat{\rho}} (\mathbb{Z} \times Aut(\mathbb{Z}/2\mathbb{Z}^n)) \\ &= (\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}^n) \rtimes_{\rho} (\mathbb{Z} \times Aut(\mathbb{Z}/2\mathbb{Z}^n)). \end{aligned}$$

Because $\mathbb{Z}$ acts only on the first part and $Aut(\mathbb{Z}/2\mathbb{Z}^n)$ only on the second part of X and $\mathbb{Z}$ acts by shift we get

$$\begin{aligned} \hat{X} \rtimes_{\hat{\rho}} \Gamma &= ((\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \rtimes_{\zeta} \mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}^n \rtimes_{\rho|_{Aut(\mathbb{Z}/2\mathbb{Z}^n)}} Aut(\mathbb{Z}/2\mathbb{Z}^n)) \\ &= L \times (\mathbb{Z}/2\mathbb{Z}^n \rtimes_{\rho|_{Aut(\mathbb{Z}/2\mathbb{Z}^n)}} Aut(\mathbb{Z}/2\mathbb{Z}^n)). \end{aligned}$$

Therefore we see that read-only Turing machines offer a way to compute the l^2-betti number of $L \times H$ for some discrete finite group H. We will use this fact to prove the following theorem.

Theorem 5.2. *Every positive rational number arises from the lamplighter group.*

Lets start by constructing the used Turing dynamical system.

Remark 5.3. *In this chapter we will often describe Turing machines instead of Turing dynamical systems but because of 3.9 we can always construct the corresponding Turing dynamical system.*

Lemma 5.4. *For every $m \in \mathbb{N}$ such that $\frac{1}{m}$ has a finite binary expansion there exists a computing Turing dynamical system T_X with $\frac{\Omega_1(T_X)}{\mu(I)} = \frac{1}{m}$.*

Proof. If $m = 1$ we can just construct a Turing machine which accepts every input. So let $m \neq 1$ Let $0.a_1 a_2 \dots a_k = \frac{1}{m}$ be the binary expansion of $\frac{1}{m}$. We construct a Turing machine T with $k + 3$ states. These states are divided into

- one accepting and one rejecting state called s_A and s_R
- one initial state called s_I

- k states called $s_1, s_2 \dots s_k$.

The transition function is defined through

$$\delta(s_I, x) = (s_1, x, 1) \; \forall x \in \mathbb{Z}/2\mathbb{Z}$$
$$\delta(s_i, \overline{0}) = \begin{cases} (s_{i+1}, \overline{0}, 1) & if\ i \neq k \\ (s_R, \overline{0}, 0) & if\ i = k \end{cases}$$
$$\delta(s_i, \overline{1}) = \begin{cases} (s_A, \overline{1}, 0) & if\ a_i = 1 \\ (s_R, \overline{1}, 0) & if\ a_i = 0 \end{cases}$$

for all $i \in \{1 \dots k\}$. We now see that the Turing machine holds after at most $k+2$ steps independent of the current configuration. Therefore the resulting Turing dynamical system (T_X) stops at any configuration. It is also clear that it does not restart. To ensure that (T_X) has disjoint accepting chains we reduce the initial set I of (T_X) from $[\underline{x}][s_I]$ to $[\underline{\overline{1}}][s_I]$ (with $x \in \mathbb{Z}/2\mathbb{Z}$). Because T holds at the first $\overline{1}$ it reads on the right side of its starting point we see that each configuration in the accepting or rejecting set of (T_X) must be contained in

$$[\overline{1}\ \overline{0}^n\ \underline{\overline{1}}][s] \text{ with } s \in \{s_A, s_R\},\ n \in \mathbb{N} \text{ or } [\overline{1}\ \overline{0}^{k-1}\ \underline{\overline{0}}][s_R].$$

Each of these elements can be traced back to a single element in

$$[\underline{\overline{1}}\ \overline{0}^n\ \overline{1}][s_I] \text{ or } [\underline{\overline{1}}\ \overline{0}^k][s_I].$$

where each entry which is not fixed stays the same. It remains to show that $\frac{\Omega_1(T_X)}{\mu(I)} = \frac{1}{m}$. Let $K \subset \mathbb{N}$ be such that $a_i = 1 \Leftrightarrow i \in K$. It then holds that

$$\mathcal{F}_1(T_X) = \bigcup_{i \in K} [\underline{\overline{1}}\ \overline{0}^{i-1}\overline{1}][s_I].$$

Let $n \in \mathbb{N}$ denote the number such that the space X of (T_X) equals $(\prod_{i\in\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}^n$, i.e. the smallest $n \in \mathbb{N}$ such that $k+3 < 2^n$. Then

$$\mu([\underline{\overline{1}}\ \overline{0}^{i-1}\overline{1}][s_I]) = \frac{1}{2^n \cdot 2^{i+1}}$$

and

$$\mu(I) = \mu([\underline{\overline{1}}][s_I]) = \frac{1}{2^{n+1}}.$$

Therefore

$$\frac{\Omega_1(T_X)}{\mu(I)} = \sum_{i \in K} \frac{1}{2^i}$$

which is exactly $\frac{1}{m}$. $\square$

Lemma 5.5. *For every $m \in \mathbb{N}$ with $m \neq 1$ the binary expansion of $\frac{1}{m}$ is of the form $0.a\overline{b}$ where a is a nonrepeating and b a repeating part. The length of a and the length of b are both finite.*

Lemma 5.6. *For every $m \in \mathbb{N}$ there exists a computing Turing dynamical system T_X with $\frac{\Omega_1(T_X)}{\mu(I)} = \frac{1}{m}$.*

Proof. W.l.o.g. $m \neq 1$. Then with 5.5 follows that $\frac{1}{m} = 0.a_1a_2 \dots a_k b_1 b_2 \dots b_h$ where $a_1a_2 \dots a_k$ is nonrepeating and $b_1b_2 \dots b_h$ is repeating. We then construct the same Turing machine as in 5.4 but add h many states called $r_1, r_2 \dots r_h$. In addition we replace $\delta(s_k, \overline{0}) = (s_R, \overline{0}, 0)$ with $\delta(s_k, \overline{0}) = (r_1, \overline{0}, 1)$ and set

$$\delta(r_i, \overline{0}) = \begin{cases} (r_{i+1}, \overline{0}, 1) & if\ i \neq h \\ (r_1, \overline{0}, 1) & if\ i = h \end{cases}$$

$$\delta(r_i, \overline{1}) = \begin{cases} (s_A, \overline{1}, 0) & if\ b_i = 1 \\ (s_R, \overline{1}, 0) & if\ b_i = 0 \end{cases}$$

which takes care of the repeating part of $\frac{1}{m}$. The rest of the transition function δ stays the same as in 5.4. We can see that this Turing machines does not hold for every input, because if we have only zeros on the right side of our staring point we never arrive at s_A or s_R. But because the set $[\underline{x}\ \overline{0}\ \overline{0} \dots][\sigma]$ (with $x \in \mathbb{Z}/2\mathbb{Z}$ and σ an arbitrary state) has measure zero the resulting Turing dynamical system still stops at any configuration. The rest follows exactly like in the proof of 5.4 because the infinite sum of measures converges. □

We can now start to proof 5.2. Because of 2.21 it suffices to show that for every $m \in \mathbb{N}$ $\frac{1}{m}$ arises from L. Let $m \in \mathbb{N}$ and (T_X) be the Turing dynamical system constructed in the proof of 5.6. First we see that

$$\mu(I) - \Omega_1(X) = \mu(I)(1 - \frac{\Omega_1(T_X)}{\mu(I)}) = \mu(I)(1 - \frac{1}{m})$$

is a l^2-Betti number of $L \times (\mathbb{Z}/2\mathbb{Z}^n \rtimes_{\rho|_{Aut(\mathbb{Z}/2\mathbb{Z}^n)}} Aut(\mathbb{Z}/2\mathbb{Z}^n))$ for some $n \in \mathbb{N}$. Let k denote the number of states of the Turing machine T constructed in 5.6 and $k' \in \mathbb{N}$ be such that $k < k'$ and $k' = 2^h$ for some $h \in \mathbb{N}$. We now change (T_X) by adding "dummy states" until (T_X) has $2^{k'}$ many states. Doing so does not change the value of $\frac{\Omega_1(T_X)}{\mu(I)}$. We now construct a Turing dynamical system $(T_{X'})$ in the same way as in 3.9 but we set $X' = \mathbb{Z}/2\mathbb{Z}^{\mathbb{Z}} \times \mathbb{Z}/2\mathbb{Z}^{k'}$ and identify every state of T with an element of the standard basis of $\mathbb{Z}/2\mathbb{Z}^{k'}$. In addition we set $\Gamma = \mathbb{Z} \times Rot$ where Rot is the subgroup of $Aut(\mathbb{Z}/2\mathbb{Z}^{k'})$ generated by the automorphism which rotates every element one to the right, i.e.

$$\phi : \mathbb{Z}/2\mathbb{Z}^{k'} \to \mathbb{Z}/2\mathbb{Z}^{k'}, \sum_{i=1}^{k'} z_i \cdot e_i \mapsto \sum_{i=2}^{k'} z_{i-1} \cdot e_i + z_{k'} \cdot e_1$$

where e_i denotes the standard basis of $\mathbb{Z}/2\mathbb{Z}^{k'}$ and each $z_i \in \mathbb{Z}/2\mathbb{Z}$. Then Rot acts transitively on the standard basis of $\mathbb{Z}/2\mathbb{Z}^{k'}$ and has exactly $k' = 2^h$ elements. Then $\frac{1}{2^{k'+1}}(1 - \frac{1}{m})$ is a l^2-Betti number of $L \times (\mathbb{Z}/2\mathbb{Z}^{k'} \rtimes_{\rho|_{Rot}} Rot)$. With 2.22 follows that

$$|(\mathbb{Z}/2\mathbb{Z}^{k'} \rtimes_{\rho|_{Rot}} Rot)| \cdot \frac{1}{2^{k'+1}}(1 - \frac{1}{m}) = 2^{k'} \cdot 2^h \cdot \frac{1}{2^{k'+1}}(1 - \frac{1}{m}) = 2^{h-1}(1 - \frac{1}{m})$$

is a l^2-Betti number of L. By changing I to

$$[\overline{1}^{h-1}\overline{\underline{1}}][s_I]$$

we get that $(1 - \frac{1}{m})$ is a l^2-Betti number of L. At last by interchanging s_A and s_R we get that $\frac{1}{m}$ is a l^2-Betti number of L which concludes the proof.

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Erklärung

Hiermit versichere ich, dass ich diese Arbeit selbständig verfasst und keine anderen, als die angegebenen Quellen und Hilfsmittel benutzt, die wörtlich oder inhaltlich übernommenen Stellen als solche kenntlich gemacht und die Satzung des Karlsruher Instituts für Technologie zur Sicherung guter wissenschaftlicher Praxis in der jeweils gültigen Fassung beachtet habe.

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