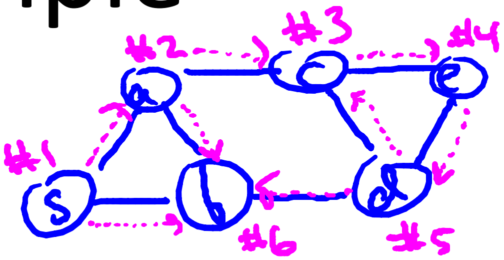
Graph Primitives

Depth-First Search

Design and Analysis of Algorithms I

Overview and Example

Depth-First Search (DFS) : explore aggressively, only backtrack when necessary.



-- also computes a topological ordering of a directed acyclic graph
-- and strongly connected components of directed graphs

Run Time : O(m+n)

The Code

Exercise : mimic BFS code, use a stack instead of a queue [+ some other minor modifications]

Recursive version : DFS(graph G, start vertex s)

-- mark s as explored

-- for every edge (s,v) :

    -- if v unexplored

        -- DFS(G,v)

Basic DFS Properties

Claim #1 : at the end of the algorithm, v marked as explored <==> there exists a path from s to v in G.

Reason : particular instantiation of generic search procedure

Claim #2 : running time is $O(n_s+m_s)$,

where n_s = # of nodes reachable from s

m_s = # of edges reachable from s

Reason : looks at each node in the connected component of s at most once, each edge at most twice.

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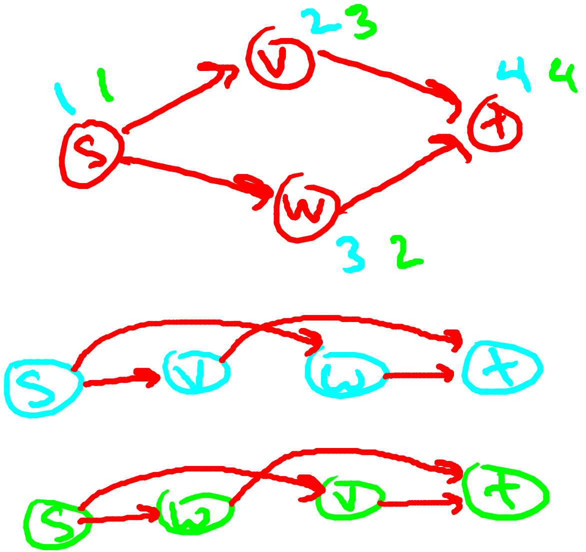
Application: Topological Sort

Definition : A topological ordering of a directed graph G is a labeling f of G's nodes such that:

1. The f(v)'s are the set {1,2,..,n}
2. $(u,v) \in G => f(u) < f(v)$

The condition asserts that all of G's (directed) edges should travel forward in the ordering



Motivation : sequence tasks while respecting all precedence constraints.

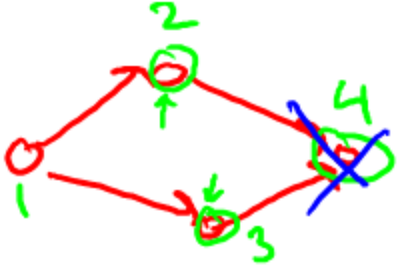
Note : G has directed cycle => no topological ordering

Theorem : no directed cycle => can compute topological ordering in O(m+n) time.

Straightforward Solution

Note : every directed acyclic graph has a sink vertex.

It doesn't have outgoing edge



Reason : if not, can keep following outgoing arcs to produce a directed cycle.

To compute topological ordering :

-- let v be a sink vertex of G

-- set f(v) = n

-- recurse on G-{v}

Why does it work? : when v is assigned to position i, all outgoing arcs already deleted => all lead to later vertices in ordering.

Topological Sort via DFS (Slick)

DFS-Loop (graph G)

-- mark all nodes unexplored

-- current-label = n [to keep track of ordering]

-- for each vertex

    -- if v not yet explored [in previous DFS call]

        -- DFS(G,v)

DFS(graph G, start vertex s)

-- for every edge (s,v)

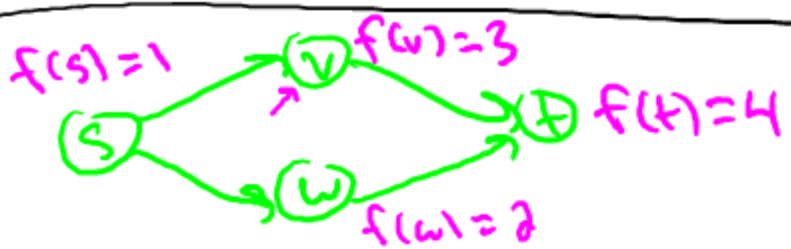
    -- if v not yet explored

        -- mark v explored

        -- DFS(G,v)

-- set f(s) = current_label

-- current_label = current_label-1



Topological Sort via DFS (con’d)

Running Time : O(m+n).

Reason : O(1) time per node, O(1) time per edge.

Correctness : need to show that if (u,v) is an edge, then f(u) < f(v)

(since no directed cycles)

Case 1 : u visited by DFS before v => recursive call corresponding to v finishes before that of u (since DFS).
$\Rightarrow$f(v) > f(u)

Case 2 : v visited before u => v’s recursive call finishes before u’s even starts. => f(v) > f(u)

Q.E.D.