Optimisation and Statistical Data Analysis

Exercise Set 11 Solutions

Set the size of figures.

```
figure; set(gcf,'position',[0 0 500 300])
```

Problem 1

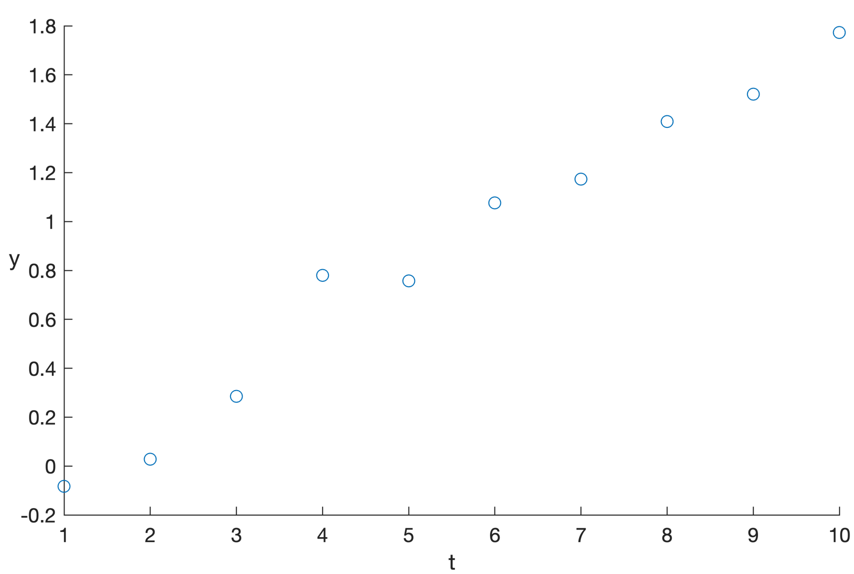
part a

Enter the data.

```
t=(1:10)';
y=[-0.083 0.028 0.285 0.780 0.757 1.076 1.173 1.409 1.521 1.773]';
```

Always look at the data first!

```
plot(t,y,'o')
xlabel('t'), ylabel('y','rot',0), box off
```



This plot indicates that it is reasonable to fit a line to this data.

Compute the posterior distribution's parameters $\widehat{\beta}$, C, $\widehat{\sigma}$ as on slide 14.

```
n=length(y);
X=[t(:) ones(n,1)];  % design matrix
betahat=X\y          % least squares
```

```
betahat = 2×1
    0.2077
   -0.2705
```

```
C=inv(X'*X);          % covariance factor
u=length(betahat);
sigmahat=norm(y-X*betahat)/sqrt(n-u);
```

The marginal posterior distribution of a parameter is (from slide 13):

$$\beta_i \mid y_{1:n} = \widehat{\beta}_i + \widehat{\sigma}\sqrt{c_{ii}}\, z, \quad z \sim \mathrm{t}_{n-2}.$$

The 95% confidence intervals for the parameters:

```
CI95=tinv(0.975,n-u);   % 95 percent CI of t_{n-u} dist. is [-1,1]*CI95
for i=1:u
    betahat(i) + [-1 1]*sqrt(C(i,i))*sigmahat*CI95
end
```

```
ans = 1×2
    0.1816    0.2338
ans = 1×2
   -0.4326   -0.1084
```

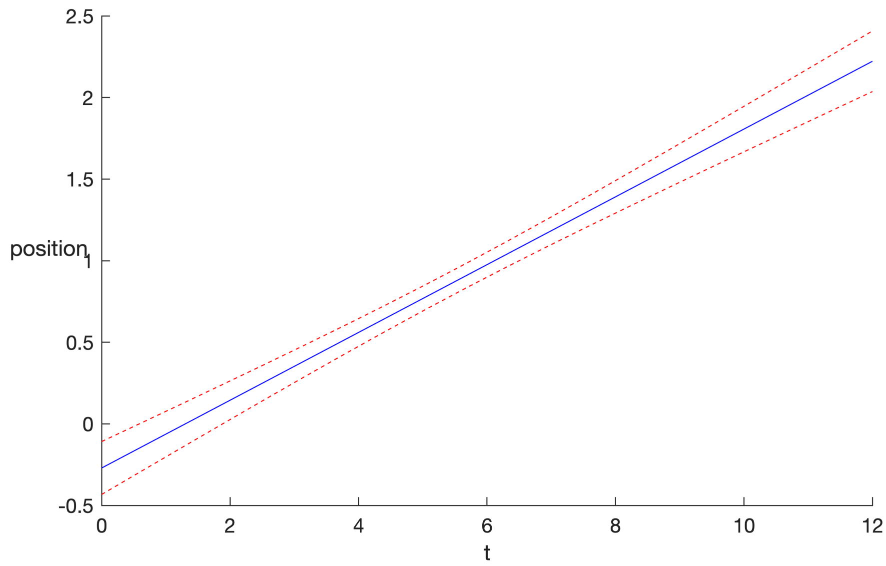
part b

The fitted line (model value) at a given time ξ is (from slide 13):

$$\beta_1\xi + \beta_2 \mid y_{1:n} = [\xi\ 1]\beta \mid y_{1:n} = [\xi\ 1]\widehat{\beta} + \widehat{\sigma}\sqrt{[\xi\ 1]\, C\, [\xi\ 1]^T}\, z, \quad z \sim \mathrm{t}_{n-2}$$

Plot the mean and 95% confidence interval of the fitted line:

```
xx=linspace(0,12,20)';       % dense mesh of times
XX=[xx ones(size(xx))];      % design matrix for dense mesh
mvxxmean=XX*betahat;                          % model value mean
mvxxscale=sigmahat*sqrt(diag(XX*C*XX'));   % model value scale
plot(xx,mvxxmean,'b-',...
    xx,mvxxmean+CI95*mvxxscale,'r--',...
    xx,mvxxmean-CI95*mvxxscale,'r--')
box off, xlabel('t'), ylabel('position ','rot',0)
```



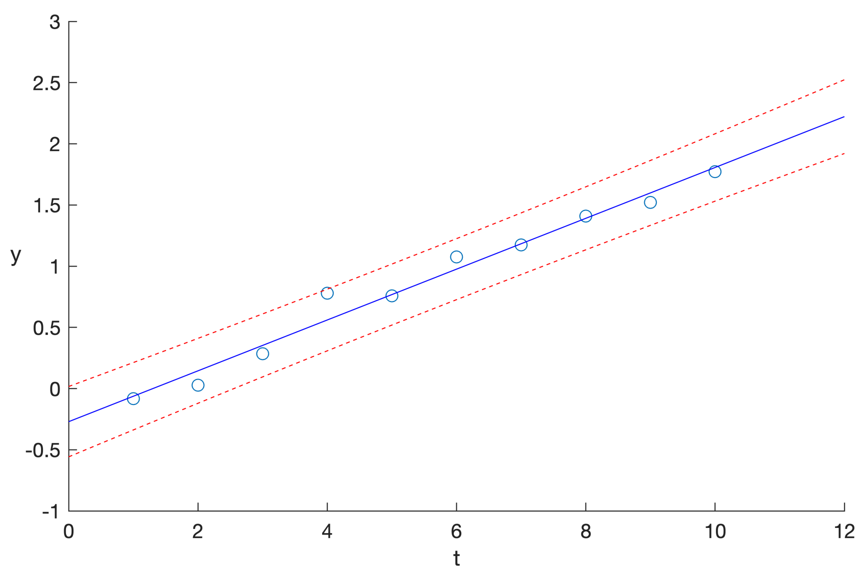
part c

The posterior predicted measurement at a given ξ is (from slide 15):

$$\tilde{y} \mid y_{1:n} = [\xi\ 1]\hat{\beta} + \hat{\sigma}\sqrt{1 + [\xi\ 1]\, C\, [\xi\ 1]^T}\, z, \quad z \sim \mathrm{t}_{n-2}$$

Plot the mean and 95% CI, together with the actual measurements:

```
ypredscale=sigmahat*sqrt(1+diag(XX*C*XX'));
plot(t,y,'o',xx,mvxxmean,'b-',...
    xx,mvxxmean+CI95*ypredscale,'r--',...
    xx,mvxxmean-CI95*ypredscale,'r--')
box off, xlabel('t'), ylabel('y ','rot',0)
```



The model fit is good: the distribution of observations agrees with the predictive distribution, and there are no evident trends (e.g. curvature) in the sequence of residuals.

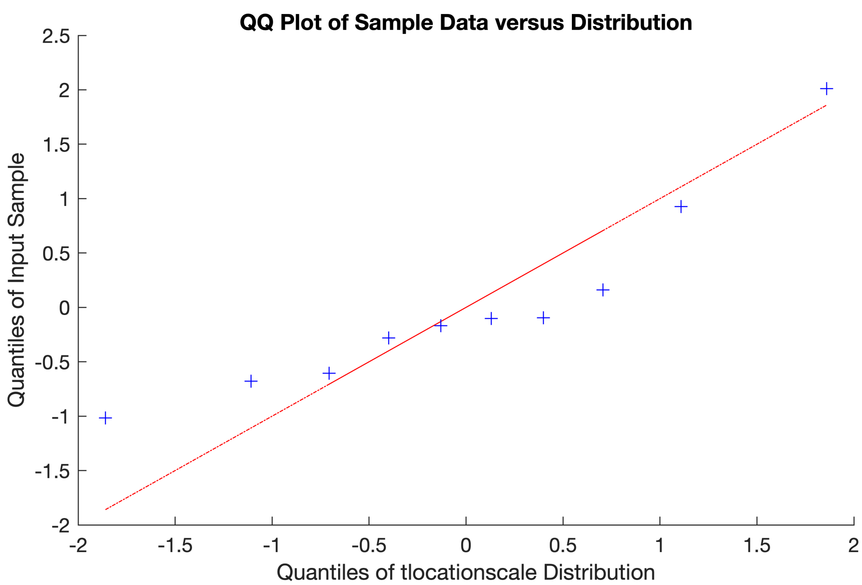
part d

The scaled residual of observation y_i is (as in slide 15)

$$\frac{y_i - [x_i\ 1]\hat{\beta}}{\hat{\sigma}\sqrt{1 + [x_i\ 1]\, C\, [x_i\ 1]^T}}$$

The QQ plot of scaled residuals vs t_{n-2} distribution:

```
scaledres=(y-X*betahat)./(sigmahat*sqrt(1+diag(X*C*X')));
h=qqplot(scaledres,makedist('tlocationscale','nu',n-u));
h(2).YData=h(2).XData;    % make a 45-degree reference line
h(3).YData=h(3).XData;
```



There's a bit of right-skewness (the points form a "smile") but the fit is reasonably good.

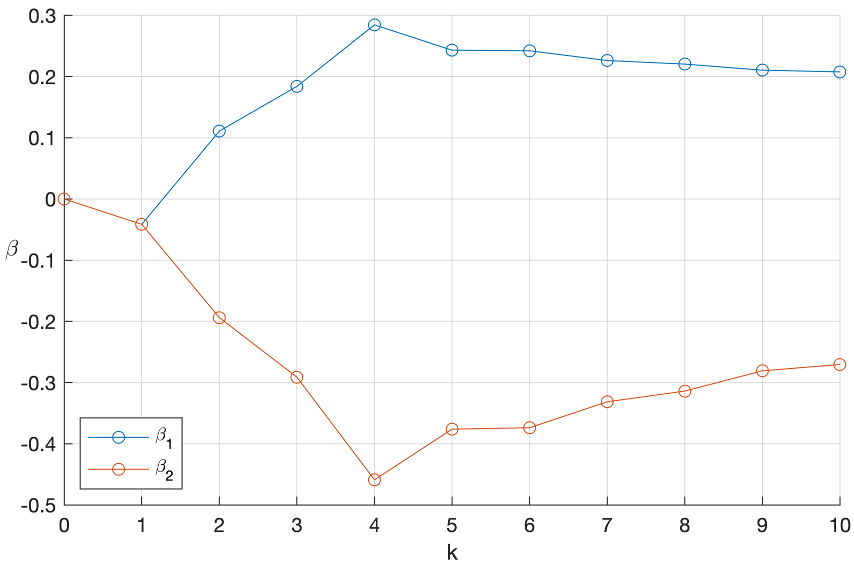
part e

As in part a, but using the "known" σ in place of $\hat{\sigma}$, and the normal distribution in place of Student-t.

Update the parameter distributions one measurement at a time, using the formula on slide 24.

```
sigma=0.1;
betahats=nan(u,n);          % preallocate with NaN
b0=[0;0]; P=100*eye(2);    % prior mean and covariance
R=sigma^2;                  % measurement noise variance
betahat=b0;
for k=1:n
    H=[t(k) 1];
    S=H*P*H'+R;
```

```
    K=P*H'/S;          % backslash is preferable to inv( )
    P=P-K*S*K';
    betahat=betahat+K*(y(k)-H*betahat);
    betahats(:,k)=betahat;
end
plot(0:n,[b0 betahats],'o-')
box off, grid on, xlabel('k'), ylabel('\beta ','rot',0)
legend('\beta_1','\beta_2','location','southwest')
```



The posterior distribution, given all measurements, is $\beta \mid y_{1:n} \sim \mathrm{MVN}(\widehat{\beta}_n, P_n)$ with

```
betahat, P
```

```
betahat = 2×1
    0.2077
   -0.2705
P = 2×2
    0.0001   -0.0007
   -0.0007    0.0047
```

The marginal distributions are $\beta_i \mid y_{1:n} \sim \mathrm{norm}(\widehat{\beta}_n(i), \sqrt{P_n(i,i)})$. Their 95% confidence intervals are

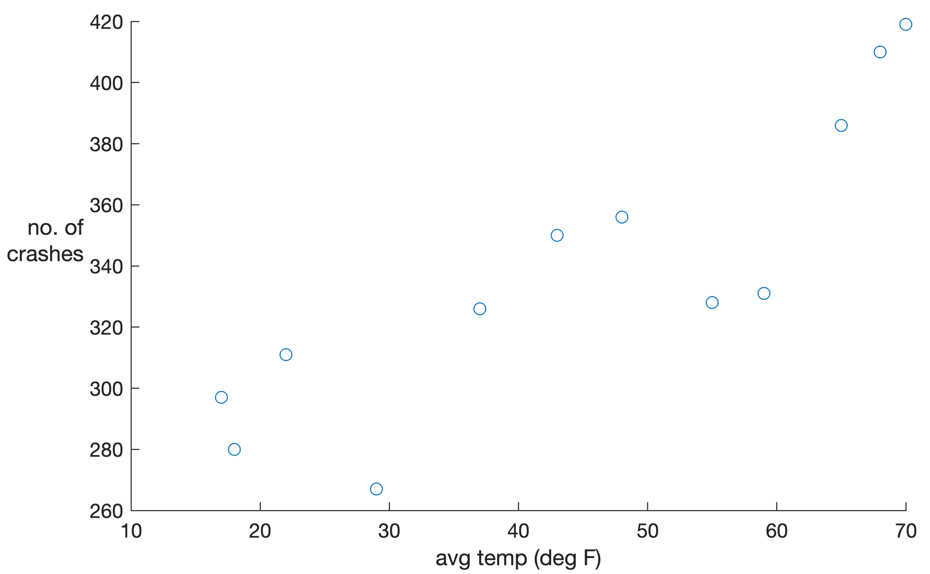
```
CI95=norminv(0.975);   % 95 percent CI of standard normal dist. is [-1,1]*CI95
for i=1:u
    betahat(i) + [-1 1]*sqrt(P(i,i))*CI95
end
```

```
ans = 1×2
    0.1861    0.2293
ans = 1×2
   -0.4044   -0.1366
```

Problem 2

Enter the data and plot it

```
t=[  17  18  29  43  55  65  70  68  59  48  37  22]';
y=[ 297 280 267 350 328 386 419 410 331 356 326 311]';
plot(t,y,'o')
xlabel('avg temp (deg F)'), box off
ylabel({'no. of';'crashes'},'rot',0,'hor','right')
```



This plot indicates that it is reasonable to fit a line to this data.

Compute the posterior distribution's parameters $\widehat{\beta}$, C, $\widehat{\sigma}$ as on slide 14.

```
n=length(y);
X=[t(:) ones(n,1)];   % design matrix
betahat=X\y           % least squares
```

```
betahat = 2×1
    2.1440
  243.5465
```

```
C=inv(X'*X);          % covariance factor
u=length(betahat);
sigmahat=norm(y-X*betahat)/sqrt(n-u);
```

The marginal posterior distribution of the slope parameter is (from slide 13):

$$\beta_1 \mid y_{1:n} = \widehat{\beta}_1 + \widehat{\sigma}\sqrt{c_{11}}\, z, \quad z \sim \mathrm{t}_{n-2}.$$

Thus $\mathrm{Prob}(\beta_1 \mid y_{1:12} > 0) = \mathrm{Prob}(z > -\dfrac{\widehat{\beta}_1}{\widehat{\sigma}\sqrt{c_{11}}}) = 1 - \mathrm{Prob}(z \le -\dfrac{\widehat{\beta}_1}{\widehat{\sigma}\sqrt{c_{11}}})$

```
format long   % show more digits
Prob=1-tcdf(-betahat(1)/sigmahat/sqrt(C(1,1)),n-2)
```

```
Prob =
   0.999882490125677
```

The news headline illustrates a well-known mistake known as the *third-cause fallacy* or *ignoring a common cause.* In this data there are other relevant factors, such as *number of journeys made* and *average speed,* that should be included in a probabilistic model of the number of fatal road accidents on any given day.