Chapter 1

Symmetry in Geometry

You've probably seen a few types of symmetry already. This chapter will look at some examples in geometry that you've likely seen, and a few you might not have seen, and look at them in a way that you perhaps haven't thought of them before.

1.1 Rotational Symmetry

Definitions

The first type of symmetry that we'll look at is **rotational symmetry**. We'll define that as follows: a shape has rotational symmetry if there's an angle that you can rotate it through (and a point, called the **centre** to rotate it around) and get the same shape back.

Example 1.1

A square has rotational symmetry, because you can rotate it through 90 degrees (or 180 degrees, or 270 degrees) and get back to a square.

You might notice that these positions are evenly spaced. This will always be the case, and we'll see why later.

Combining Rotations

We could easily do two rotations to the same shape. We might also want to think about the single symmetry that you get by combining them, so let's have a theorem to pin it down:

Theorem 1.1

Rotating a shape through an angle of a then b around the same centre is the same as rotating through an angle of $a+b$.

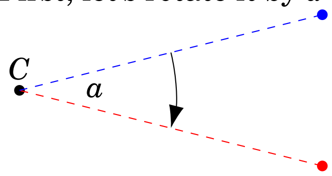
Note

This might seem too obvious to be worth proving (and the proof will turn out to be quite easy), but this is exactly the sort of thing that's important to check, and we'll see examples of similar things that are **not** true later.

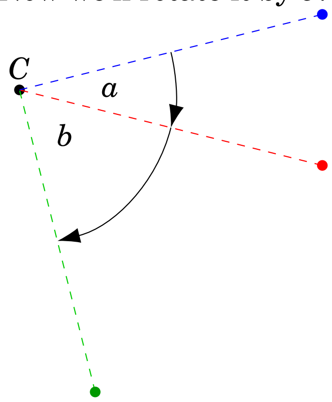
Proof

To show that two transformations are the same, we need to show that they take every point to the same place as each other. So, let's do that. Let's pick a point, and we'll mark in our centre of rotation as well (I've labelled it C for Centre):

First, let's rotate it by a (I've also added in some useful lines):



Now we'll rotate it by b:



Now, we're done: notice that the angle between the blue line and the green line is $a+b$, and that angle is at C, so we've just rotated by $a+b$ around C. Since we didn't use anything special about the point that we picked, this is true for every point.

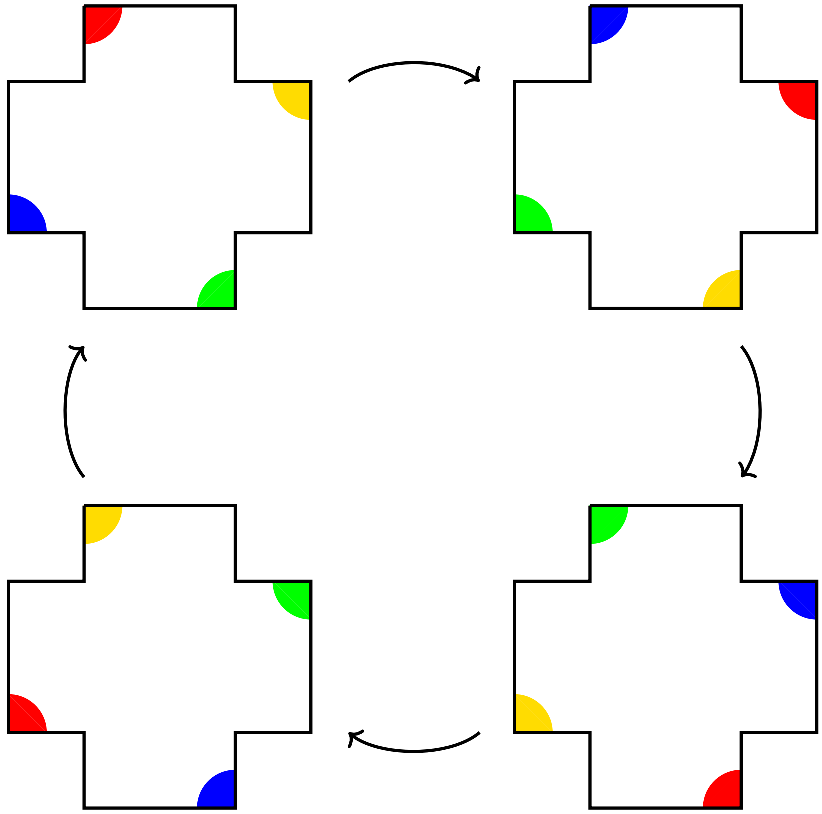
Definition 1.1

If we repeat a rotation lots of times, we might get back to where we started. If we do, the smallest number of rotations that we can use to do this is called the **order** of the rotation.

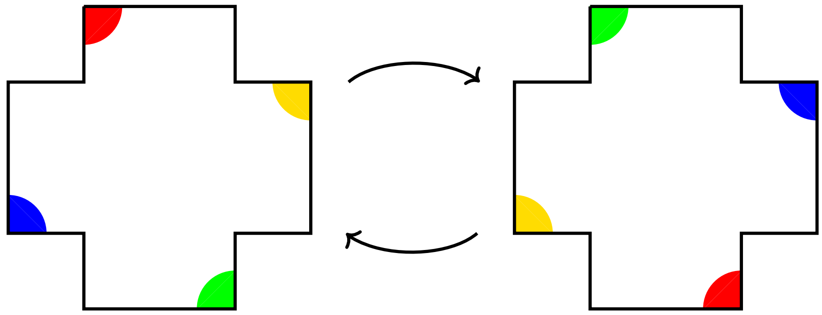
The rotational symmetry **order** of a shape is the biggest order of any of its rotational symmetries (if there is one - more on that later).

Example 1.2

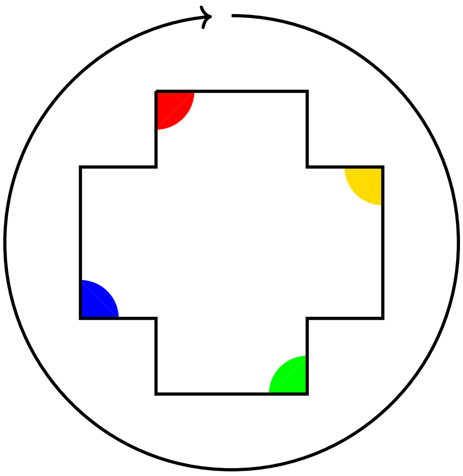
Some of the corners of this shape have been coloured so that you can see the rotation.



As you can see, it takes four rotations to get back to where we started, so this symmetry has **order** 4. The shape also has a rotational symmetry of order 2:



It also, if we're being slightly silly, has a rotational symmetry of order 1:

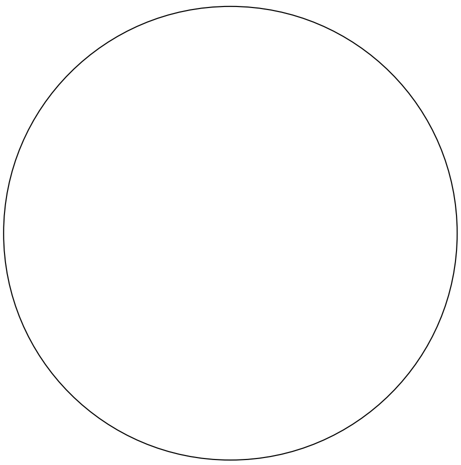


where we rotate it the whole way around and back to where we started. Since 4 is the largest order or any of its rotational symmetries, this shape has rotational symmetry of order 4.

We'll also quickly note that you can have rotational symmetries of infinite order (and so shapes with infinite rotational symmetries). There's not many easy-to-draw examples, and this is the simplest:

Example 1.3

This circle has a lot of rotational symmetry:



In fact, if we rotate it by **any** angle, we will get the same circle back. In fact, if you pick any whole number bigger than 0, and divide 360° by that number, you'll find a rotation with order that number. In fact, it also has rotations that will **never** get back to where you started, no matter how

many times you repeat it. We'll prove that by just finding one. First, we'll find something that's true for every rotation that does eventually get you back to where you started, then we'll find a rotational that it isn't true for.
For the first part, take a rotation by an angle $a°$ that has order n. That is: rotating by $a°$ again and again n times gets you back to where you started, so, by Theorem 1.1, we have $a + \cdots + a = 360$, with n copies of a being added, so $n \times a = 360$, and (dividing by n) $a = \dfrac{360}{n}$.
That is: every angle that we can repeatedly rotate by to get back to where we started can be written as a fraction, so all we need is a number that can't be written as a fraction. We'll not prove that here, since it's a bit off-topic, but you might want to give it a go. The proof will be in Appendix B:

Theorem 1.2 Proof on page 92

$\sqrt{2}$ cannot be written as a fraction.

Definition 1.2

The **inverse** of a symmetry is the symmetry that gets you back to the start when you do it after the symmetry.

Example 1.4

For example, the inverse of a rotation by a is a rotation by $360° - a$, because rotating by a then rotating by $360° - a$ is the same as rotating by $360°$ by Theorem 1.1, which is the same as doing nothing.

Note

We've quietly assumed that there's only one inverse to any symmetry in this definition. This is true, but we'll prove it later. You might want to think about why it has to be true.
You might also notice that rotating by $360° - a$ then rotating by a is also the same as doing nothing. That is: rotating by a is the inverse of its own inverse. This is true for all symmetries. Again, we'll prove it later.
If we put this together with our definition of order, we see that if a symmetry has order n, its inverse must be the thing you get by doing it $n - 1$ times: for reflections, that is $(n-1)a = 360° - a$, which is equivalent to $a = 360°/n$, which we already knew.

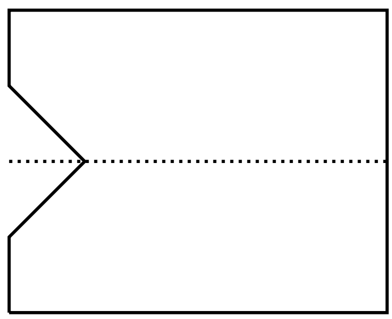
1.2 Reflective Symmetry

Definitions

The second type of symmetry that we will look at is **reflective symmetry**. You've probably defined this as something like this: a shape has reflective symmetry if there's a line where you could draw it on a piece of paper and fold it in half

Example 1.5

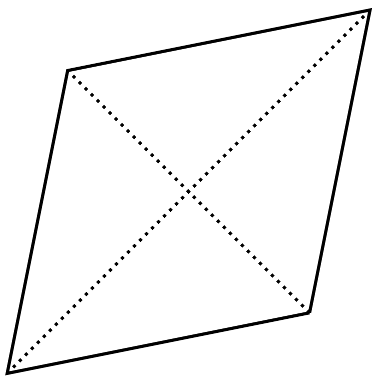
This shape has one **reflective symmetry**, which is drawn with a dotted line.



You can also have multiple reflective symmetries.

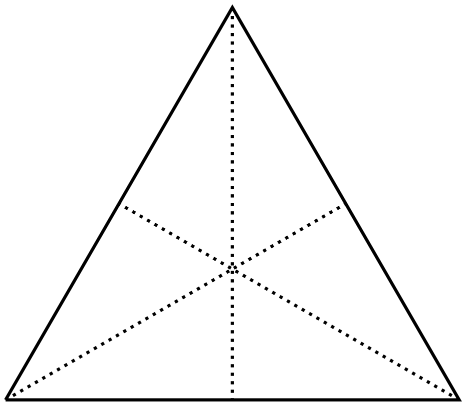
Example 1.6

This **rhombus** has two **reflective symmetries**, drawn with dotted lines



Example 1.7

This **equilateral triangle** has three **reflective symmetries**, drawn with dotted lines



Investigation 1.1 Solution on page 63

All of these shapes so far have the lines of symmetry evenly spaced: the rhombus has them at right angles, and the equilateral triangle has them all at 60 degree angles to each other. Are there any shapes where they aren't evenly spaced?

Note

You might be better placed to answer this question after reading Section 1.2: Combining Reflections, which is also where you'll find the solution.

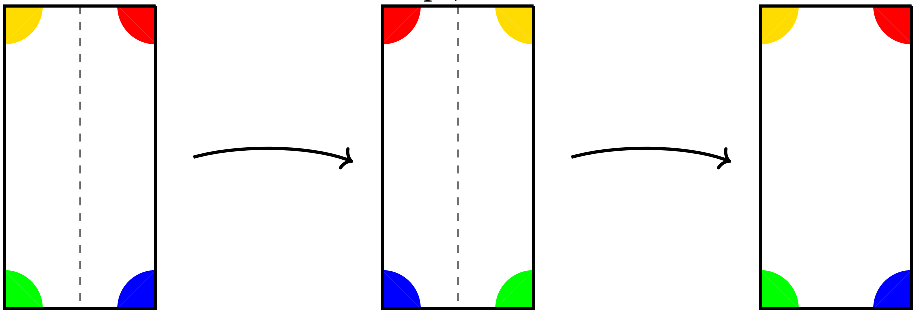
Hint if you want to try it now: Every shape we've seen with more than one line of symmetry also has rotational symmetry. Is that always the case?

Combining Reflections

Just like rotational symmetry, we can talk about the **order** of a reflective symmetry. This just doesn't tend to happen very often, because it's not that interesting on its own: every reflective symmetry has **order** 2: if you reflect a shape, then reflect it back, you get right back to where you started.

Example 1.8

Here, we reflect the rectangle in the vertical axis. Notice that after two steps, we're back where we started.



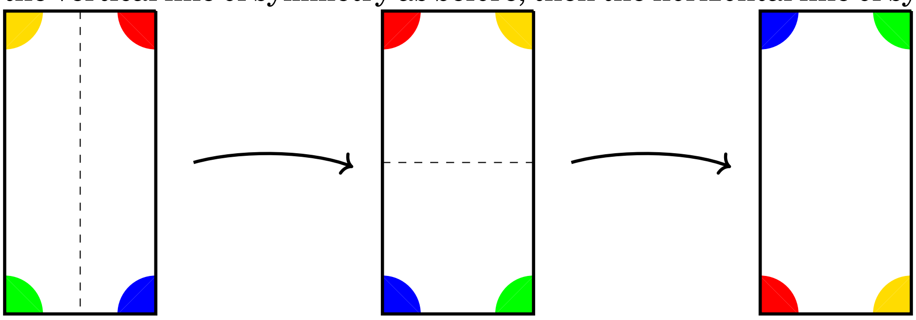
Note

You might notice that having order 2 is exactly the same thing as being your own inverse, so every reflection is its own inverse.

But what if we were to combine different reflections? With rotations around the same point, this was easy: you just add the angles. With reflections, something a little more interesting happens:

Example 1.9

Let's reflect that same rectangle again, but this time, we'll reflect it first in the vertical line of symmetry as before, then the horizontal line of symmetry.



That's not what we started with. It's also not a position we can get to from the starting position with any reflection (try proving this!). Instead, it's a rotation: we've rotated the original shape through 180 degrees (this is twice the angle between the two lines).

In fact, this is always true:

Theorem 1.3

Reflecting a shape twice is the same as rotating it by twice the angle between the lines of reflection.

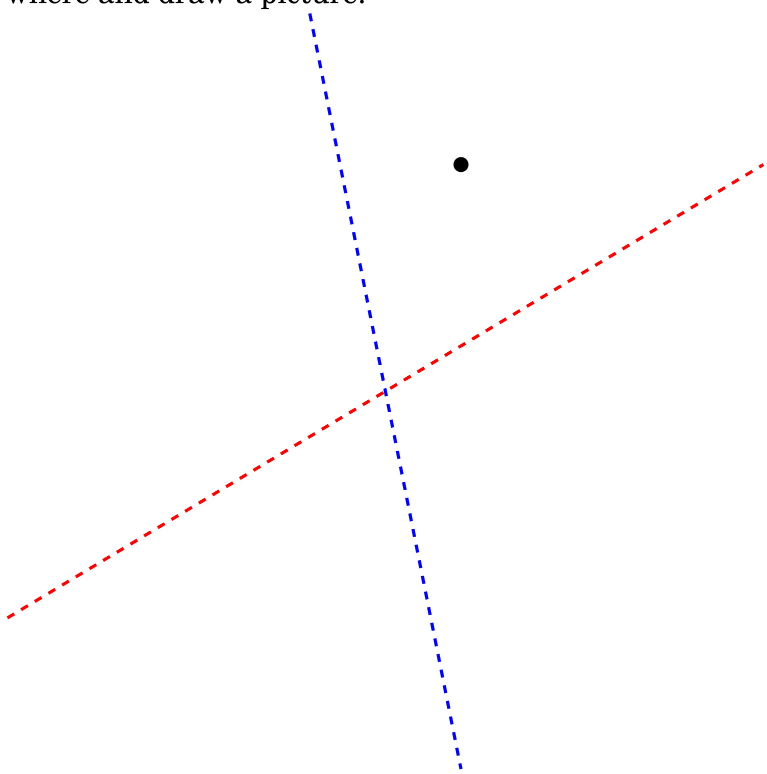
Note

You might notice that this includes the previous example, with the angle being either 0° or 180°.

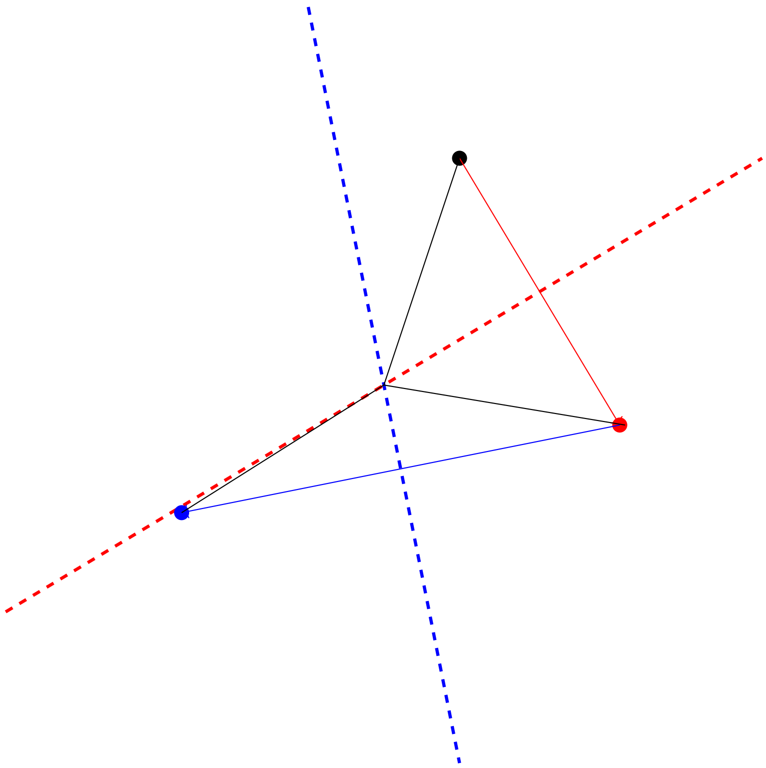
Proof

Our strategy here will be a common one: we'll pick one point and check that it's moved to the right place, but we'll do it without using anything special about the point, so that our checking will work exactly the same for every other point.

So, let's do that: if we've got two lines of symmetry, let's pick a point somewhere and draw a picture:



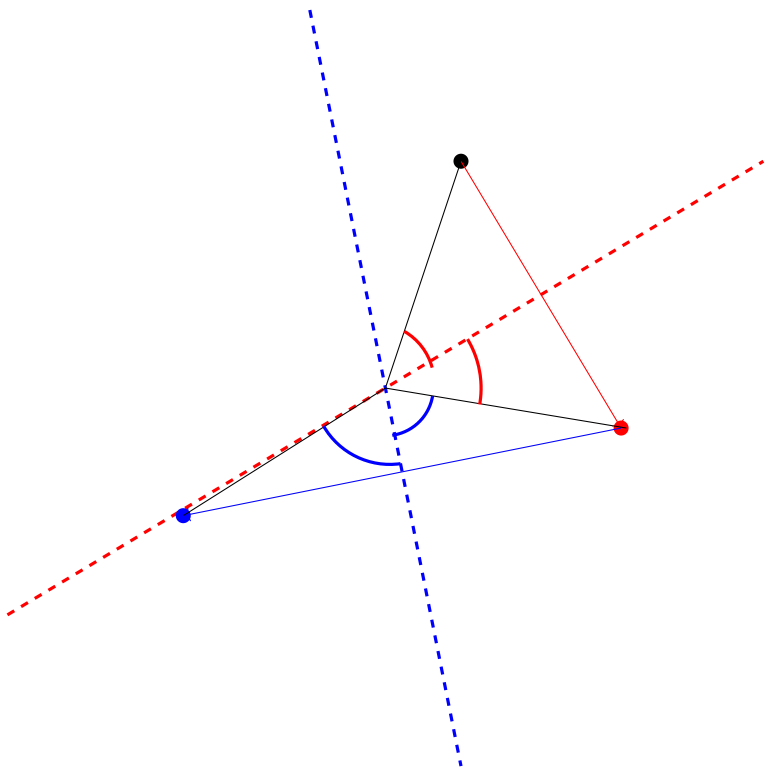
We'll reflect that point in the red line, then in the blue line, and put the points in. We'll also draw in some lines and angles to make things easier:



The solid and dashed red lines cross at a right angle, and the two parts of the solid red line are the same length (this is the definition of a reflection), and the line from the place where the dashed lines cross to where the two red lines cross is the same length as itself (obviously!). That means that everything to do with the two triangles touching the dashed red line is the same: the two angles between the solid red and black lines are the same, the two black lines are the same, and the two angles touching the point where the dashed lines cross.

All of that is exactly the same for the two blue lines. That means that all three black lines are the same length. In other words, the point we're looking up stays the same distance from the crossing point in the middle.

Let's mark the angles that we know are the same with the same colour:



If you stare at that picture, you might notice that the angle between the two dotted line is made of one red angle and one blue angle, and the angle between where our point starts and where it finishes is made of two red angles and two blue angles, so is twice the angle between the two lines.

Putting those two together, that means that our point gets moved to exactly the same place that a rotation by twice the angle between the two lines would take it to. Because we didn't use anything special about the point, the same thing is true for **every** point, so the whole shape is moved to exactly where that rotation would take it to, and so we've actually done that rotation.

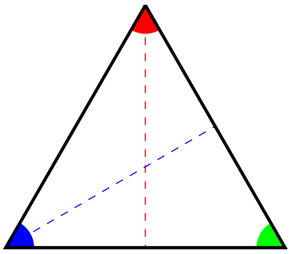
Note

This answers the question in the hint to Investigation 1.1 on page 10, so you now have what you need to solve that investigation, if you haven't already.

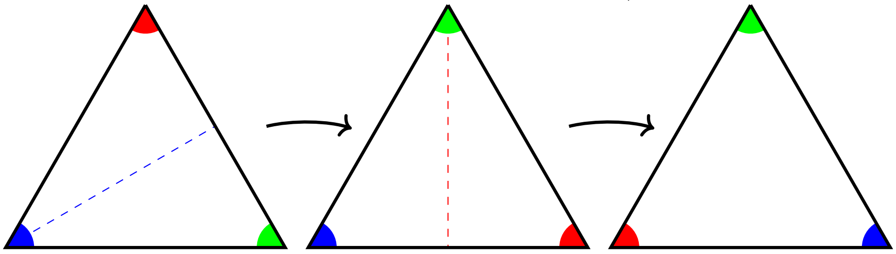
This isn't even the end of the interesting bits: let's try two very similar examples, and see what happens.

Example 1.10

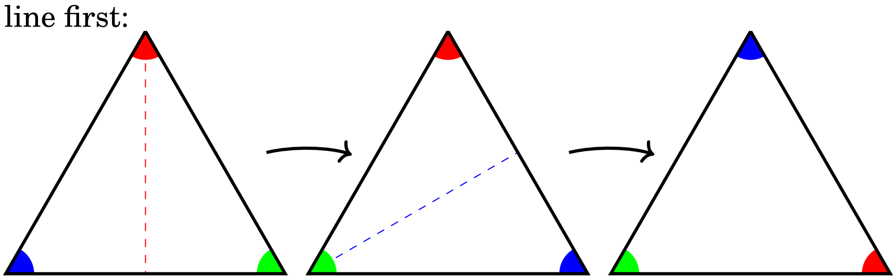
Let's look at the following shape, and the two marked lines of symmetry:



We'll reflect it in the blue line, then the red line:



Now, let's do the same thing again, but this time, we'll reflect it in the red line first:



You'll notice that our results are different: despite using the same two reflections on the same shape, we've ended up in different positions, just by changing the order. This is our first example of something **noncommutative**: that just means that the order matters (the opposite, **commutative** means that the order doesn't matter - for example, rotations are **commutative**, or we might say that rotations **commute** with each other).

Investigation 1.2 Solution on page 64

The above two statements might seem contradictory. We've shown that:

1. Any two reflections make a rotation by the angle between the lines of reflection.

2. Combining the same two reflections in a different order can give a different outcome.

How come?

You might also see that we can do this backwards: if we want to make any rotation, we can make it out of reflections, in exactly the same way: if we want to rotate by an angle a, we can reflect in a vertical line, then in a line at an angle of a to that vertical line (we could also start with any other line instead of a vertical line, of course, and get the same rotation).

Note

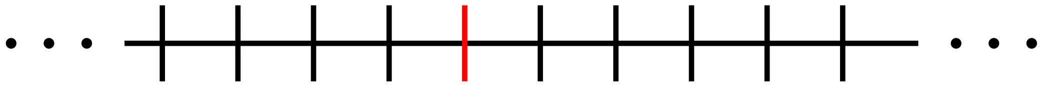
Notice that this works both ways around: any rotation can be written as two reflections (in lots of different ways: you can pick any line through the centre of the rotation to be one of your lines of symmetry).

1.3 Translational Symmetry

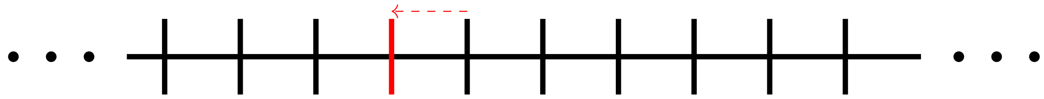
This is something you've certainly seen, but probably haven't thought of as a symmetry:

Example 1.11

Take a look at this picture:



The dots mean that it continues forever in each direction. I've drawn one of the lines red, just to keep track of where it is, but really, they're all the same, just like the coloured corners in previous examples.
If we move this sideways by the distance between the lines (or any multiple of that distance), we'll get the same picture again, just moved over sideways:



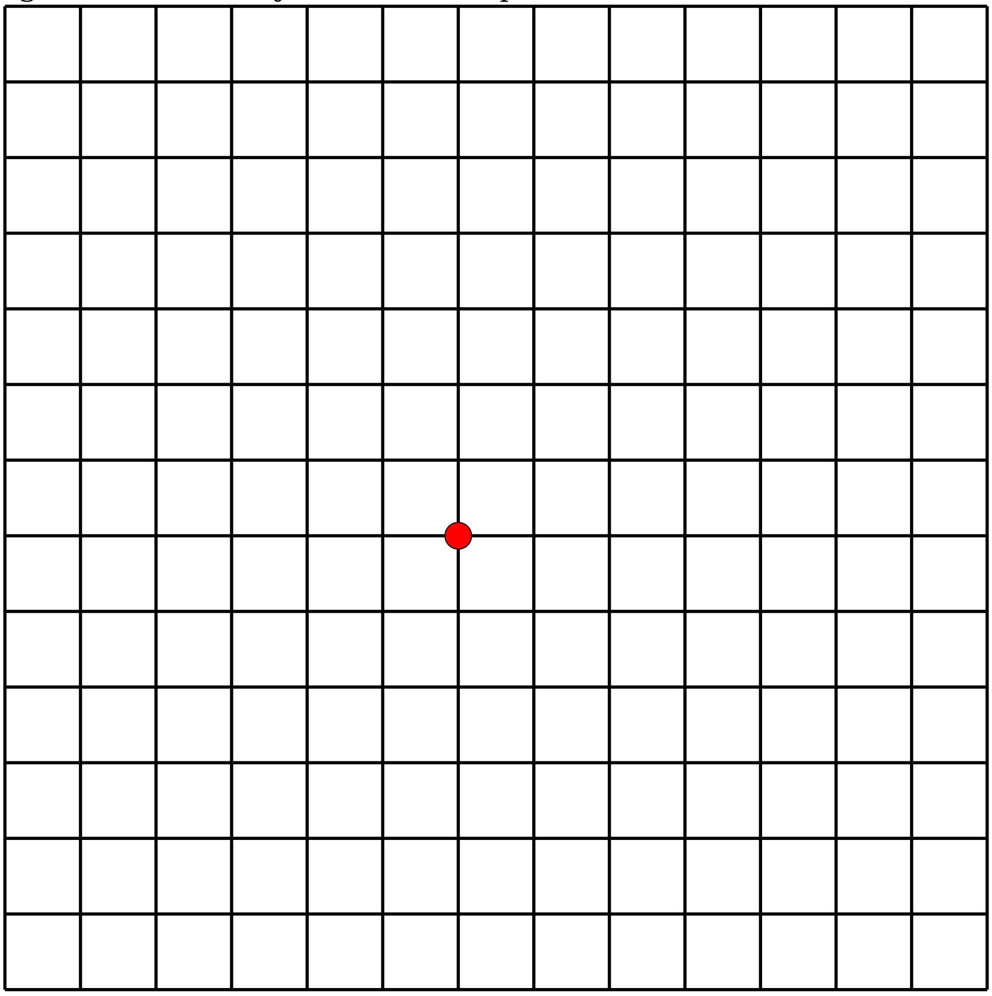
You could also think of this differently: instead of the line moving, you could imagine the line staying still, and us moving backwards and forwards along it. We'll use both approaches, as it doesn't actually matter, and call both of them the same symmetry.

This is another type of symmetry, called **translation**. The most famous example of this is a number line: if you write a whole number below each line, moving right is counting, and moving left is counting backwards.

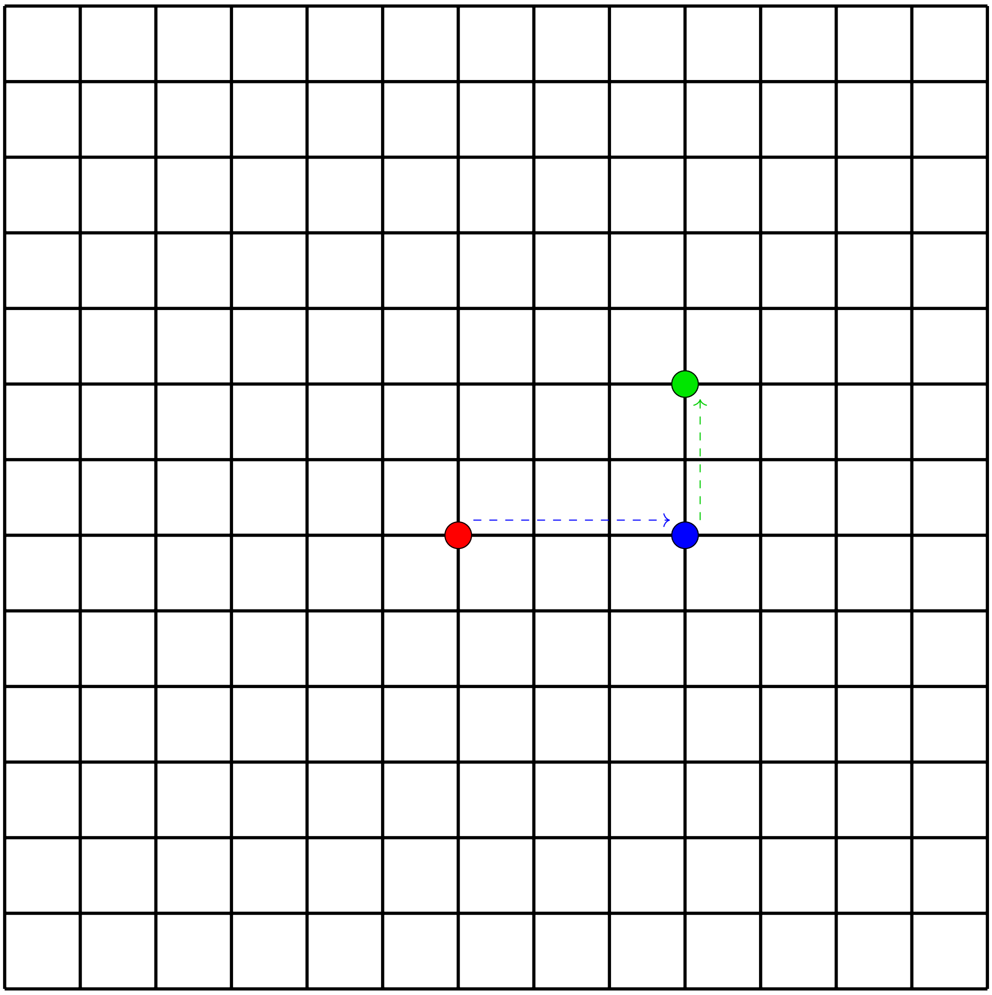
You might notice that this is basically one dimensional (the little vertical lines don't matter, and could just as easily be dots on the main line), unlike our examples above. We could also do it in two dimensions:

Example 1.12

Again, imagine that the pattern carries on forever in all directions, and once again the red dot is just there to keep track of where it is.



This time, we can move the grid either horizontally or vertically, or any combination of those: we could move it left 3 then down 2, for example.



In this picture, I've coloured the marks red, then blue, then green.
Again, we could instead think of keeping the grid still and moving the dot around: notice that this is what it looks like happened anyway, which is part of why a lot of people prefer this version.

We can now do the one case of combining reflections we didn't do earlier:

Theorem 1.4

Reflecting in one line, then in another parallel line is the same as translating by twice the distance from the first line to the second, in that direction.

Proof

Like with the other cases, we'll track one arbitrary point and make sure it ends up in the right place. In the picture, we'll reflect in the red line first, then the blue one. To simplify things, I'll draw the lines vertically with the red line on the left, but everything will work fine if we had them at any other angle (just rotate the page you're reading this on). The advantage of doing it this way is that the points will only move horizontally, not vertically, so we only actually need to keep track of one dimension. Because of that, we can just label it with numbers: we'll put the first line at 0, the second line at a and the point we're starting with at b.

Now, because we put our first line at 0, reflecting in it is easy: it takes b to $-b$. Then, reflecting in the second line takes $-b$ to the position of the second line plus the distance from $-b$ to that second line, which is $a+(a-(-b)) = 2a+b$, so our point moved from b to $2a+b$, ie it translated by $2a$, which is exactly what we wanted.

Note

Unlike our previous proofs, which were geometric, with lots of pictures, this one is basically algebraic: we turn the geometry problem into an algebra problem to make it easier (if you don't believe me that it's easier, try doing this geometrically - it's much more complicated!). This is useful in lots of cases, as is the opposite (turning algebra problems into geometry problems to make them easier).

Note

We've now done every possible combination of two reflections: either the lines cross, in which case we get a rotation (see Theorem 1.2 on page 10), or they don't, in which case we get a translation. If you think about where the point of intersection moves as the lines get closer to being parallel, this is another way in which translations are like rotations around a point that's "infinitely far away".

Note

Like with rotations, this works backwards: you can make any translation by combining two reflections in parallel lines half the distance apart and both perpendicular to the direction of the translation.

Now, let's combine translations together, with perhaps the most obvious theorem in this chapter.

Theorem 1.5 Proof on page 95

Translating by x units horizontally and y units vertically, then by z units horizontally and w units vertically is the same as translating by $x+z$ units horizontally and $y+w$ units vertically.

I'll leave this one as an exercise for you to prove: check the proof afterwards.

You may notice that (apart from the 'do nothing' translation by $(0,0)$), all translations have infinite order: no matter how many times we move in a direction, we'll never get back to where we started.

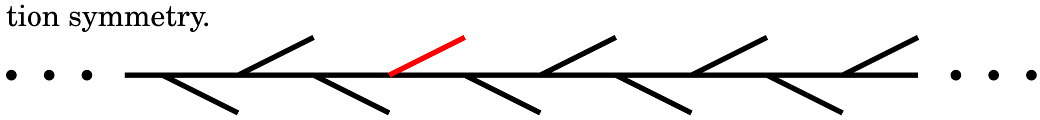
The inverse of a translation by p is just a translation by $-p$.

1.4 Glide Reflective Symmetry

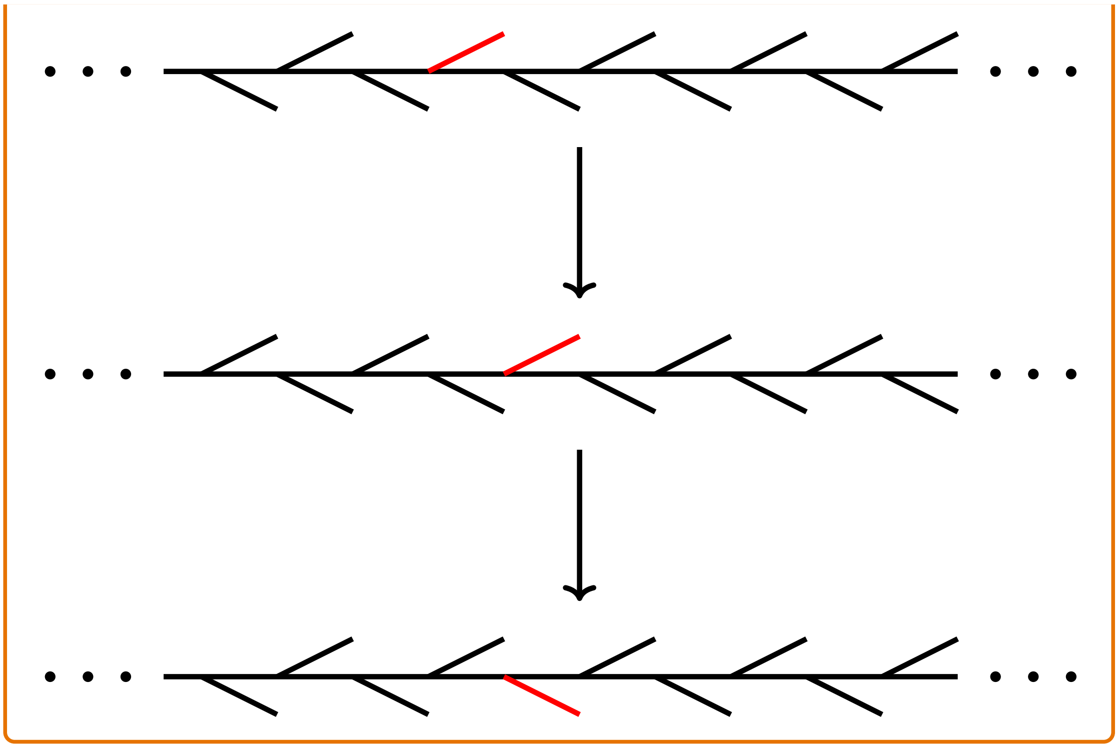
There's one last type of symmetry I want to talk about here: **Glide Reflective Symmetry**. This is what you get when you combine a reflection with a translation along the line:

Example 1.13

This picture (again, extending forever in both directions) has a glide reflection symmetry.



We can see this by sliding it along so that the red line matches up with the line going down next to it, then reflecting it vertically:



As with other symmetries, we can combine these together and get a nice simple result. Why don't you see if you can find out what?

Investigation 1.3 Solution on page 64

What happens when we combine two glide reflections along the same line?

Note

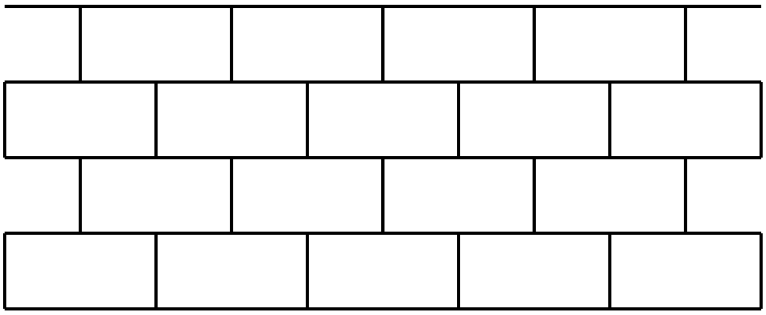
Since a glide reflection is a translation then a reflection, and we already know that translations can be made from a pair of parallel reflections, any glide reflection is made by three reflections: two parallel, then one at right angles to them.

Just like translations, glide reflections all have infinite order. The inverse of a glide reflection by v in L is a glide reflection by $-v$ in L - the two reflections cancel out, as do the translations.

1.5 A Return to Rotation

So far, we've only looked at rotations around one point. Can a shape have rotational symmetry around two or more different points? Have a think (hint: it's slightly a trick question), then turn the page to find out.

As promised, the answer is a bit of a trick: it depends what you mean by “shape". Nothing like any of the shapes we’ve looked at so far can have more than one, and in fact anything with more than one will have to have infinitely many (we’ll see why later), so our “shapes" are actually going to have to be infinite repeating patterns. Here’s one with a bunch of rotational symmetries:



Where the pattern rotates forever in all directions. If you pick any one of the corners and rotate through 180 degrees, you’ll get the same pattern again, and the same if you pick the centre of any of the rectangles. You might notice that this pattern has loads of other symmetries, too, we’ll come back to that later.

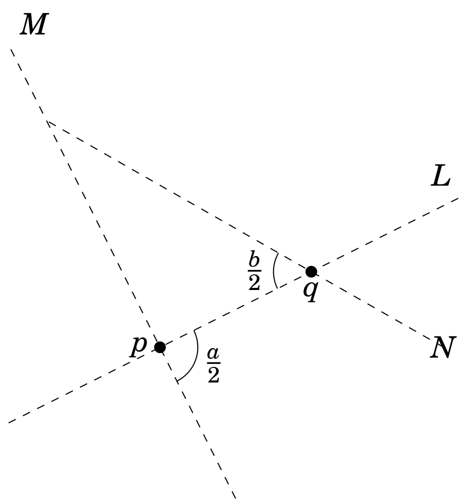
So, let’s ask our same question again: what happens if we combine two rotations, but this time, with different centres?

Theorem 1.6

If we rotate around a point p by an angle a then another point q by an angle b, that’s the same as rotating around a third point (which we’ll find later) by an angle of $|b-a|$, unless $b = 360° - a$, in which case it is a translation.

Proof

First, we’ll deal with the case where $b \neq ° - a$.
We’re going to use Theorem 1.2 from page 12, as well as the note after it: we’ll write our rotations as pairs of reflections, picking our lines cleverly: we’ll choose the second line of symmetry for our first rotation and the first line of symmetry for our second line to be the same line of symmetry (the line from p to q, which we’ll call L): that makes the first line of symmetry for the first rotation the line through p at an angle of $2a$ anticlockwise from L (let’s call it M, and the second line of symmetry for the second rotation to be the line through q at an angle of $2b$ clockwise from L (let’s call it N, so we get this picture (where M and N have to cross because $b \neq 360° - a$):



Now, as usual, let's just pick a point and see what happens: first, we'll reflect it in M, then in L (so that does the first rotation), and then we'll reflect it in L, and then in N (so that does the second rotation. But if you just look at that last sentence again, you'll notice that in the middle we reflect in L twice without doing anything else in between, which is the same as just doing nothing, so actually, we can just skip that, and what's actually happening to our point is it's getting reflected in M, then in N - but we know exactly what that is from Theorem 1.2 on page 10: it's just a rotation around the point where M and N cross by twice the angle between them. Working out exactly where they cross and what the angle is requires some trigonometry and is rather ugly, so we'll relegate it to an appendix. You can find it on page 93

Finally, we'll deal with what happens when $b = 360° - a$, so M and N don't cross at all. In this case, after cancelling out the two reflections in L, we have reflections in two parallel lines, so by 1.3 on page 19, we have a translation by twice the distance between them.

Note

You might notice that as M and N above get closer to being parallel, the place where they cross gets further and further away, and what's happening around our points looks closer and closer to a translation - this is another way in which translations are like rotations around a point that's infinitely far way.

Note

This “turn everything into reflections" strategy is really powerful, and we’ll be using it for nearly everything going forwards.

1.6 Combining Symmetry Types

So far, we’ve combined everything with things like it - rotations with other rotations, reflections with other reflections, translations with other translations, and glide reflections with other glide reflections. Now, let’s start combining different types.

Theorem 1.7

Rotating through an angle of a around a point p then reflecting in a line L that passes through p is the same as reflecting in the line through p at an angle of $360° - \frac{a}{2}$ from L.

Proof

We could do this by seeing where a point goes, but instead, let’s use our “write everything as reflections" trick: we can write the rotation as a reflection in L after a reflection in a line through p at an angle of $360° - \frac{a}{2}$ to L by 1.2 on page 12. The two reflections in L are now happening with nothing in between, so they do nothing and we can skip them, just leaving our other reflection.
Isn’t that easier?

We’ll leave the other way of combining these things to you:

Investigation 1.4 Solution on page 64

What single transformation is reflecting in a line L then rotating through an angle a around a point p on L is the same as?

Theorem 1.8

Rotating by an angle a around a point p then translating by $v = (x, y)$ is the same as rotating by a around a different point.

Proof

Let's write our translation as two reflections in lines separated by v, with the first one passing through p, and our rotation as two reflections, the second in a line perpendicular to v and the first at an angle of $360° - \frac{a}{2}$ to the second. Combining the rotation and transformation, we put two reflections in the same line together, so those cancel out, leaving us with reflections in two lines with the second at an angle of $\frac{a}{2}$ from the first, crossing at some new point, so we have a rotation by the same angle around a different point.

The calculation of which point is given on page 96 (it's short and simple, but needs a little trigonometry).

The proof for the other direction (with the translation first) is very similar, have a go yourself. If you've done some trigonometry, have a go at finding which point, as well.

Investigation 1.5 Solution on page 96

Translating by $v = (x, y)$ then rotating by an angle a around a point p is the same as rotating by a around a different point.

Theorem 1.9 Proof on page 65

Reflecting in a line L then translating by v parallel to L is the same as a glide reflection in L by v.

Note

You might notice that this is equivalent to showing that reflections commute with translations parallel to the line of reflection.
The proof is simple - have a go yourself before you have a look.

Theorem 1.10

Translating by a vector v then reflecting in L perpendicular to v is the same as reflecting in the line M parallel to L but translated by $-v/2$.

Proof

Let's do this one with a bit of algebra. We'll do this by picking some co-ordinates that make it easy. Instead of the usual x and y with x going horizontally and y going vertically, we'll use s and t, with s going along v

and t at right angles (so along L), with $s = 0$ on L, so that $p = (a, b)$, $v = (c, 0)$ in (s, t) coordinates for some a, b, and c, and $s = -c/2$ on M.
Now, translating by v takes p to $(a + c, b)$, and reflecting that in L gives $q = (-(a + c), b)$. Reflecting that in M leaves the t coordinate the same, and takes the s coordinate to $-c - s$, so sends q to $(-(a + c) + 2(a + c - c/2), b) = (-a - c + 2a + 2c - c) = (a, b) = p$. Since reflections have order 2, this means than reflecting in M takes p to q.

Investigation 1.6 Solution on page 65

Prove the following theorem. Try doing it similarly to the above, or differently:

Theorem 1.11

Reflecting in a line L then translating by a vector v perpendicular to L is the same as reflecting in the line M parallel to L but translated by $v/2$.

Theorem 1.12

Translating by $v = (x, y)$ then reflecting in a line L at an angle a to v is the same as a glide reflection in a line parallel to L.

Proof

First, we're going to split our translation into two different translations: one parallel to L, which we'll call $v \| L$, and one perpendicular to L, which we'll call $V \perp L$. Let M be the line parallel to L and offset by $-v \perp L$. By Theorem 1.6, translating by $v \perp L$ then reflecting in L is the same as reflecting in M. By Theorem 1.9, translating by $v \| L$ then reflecting in M is the same as a glide reflection by $v \perp L$ in M.

Note

This trick, of proving some easier cases first, then putting them together to prove the main thing, is very often useful.

Investigation 1.7 Solution on page 66

Prove the following very similar theorem:

Theorem 1.13

Reflecting in a line L then translating by a vector v at an angle a to L is the same as a glide reflection in a line parallel to L.

Theorem 1.14

Rotating through an angle of a around a point p then reflecting in a line L that misses p by a distance of x is a glide reflection.

Proof

Let's use some of the theorems we've already proved. Let's write the rotation as two reflections: the first in a line M at an angle of $360° - a/2$ to L, the second parallel to L. The second can then be combined with the reflection in L to give a translation perpendicular to L of length x. Then, we're just combining a reflection in M with a translation at an angle of $a/2 + 90°$ to M, so by Investigation 1.7, we have a glide reflection along a line parallel to M.

Investigation 1.8 Solution on page 66

Prove the result for the other order:
Reflecting in a line L then rotating through an angle a around a point p not on L is a glide reflection.

Combinations Including Glide Reflections

We've now done every combination that doesn't include glide reflections (we've left those for last as they're made of a translation and reflection, so we should be able to do nearly everything with things we've already done, and also because they're the most complicated to do by themselves). Now let's do the rest. They're mostly going to go very similarly: we'll split the glide reflection into a translation and a reflection, then use what we've already done.

Theorem 1.15

Rotating through an angle of a around a point p then glide reflecting by a vector v in a line L is a glide reflection.

Proof

Again, we'll split things into loads of reflections: we'll split our rotation into a reflection perpendicular to L after one at an angle of $180° - a/2$ to that, and our glide reflection into two reflections perpendicular to L separated by v, with the first passing through p, followed by a reflection parallel to L. We can than combine them differently: we again have two of the same reflection together, so those cancel out, leaving the reflection at an angle of $180° - a/2$, then a reflection perpendicular to L at a distance of $|v|$ along L from p, then a reflection in L. Combining the last two reflections gives a rotation, leaving us with a reflection followed by a rotation, which is a glide reflection by Investigation 1.8.

Not much changes if we do the glide reflection first, so have a go at that yourself:

Investigation 1.9 Solution on page 66

Glide reflecting by a vector v in a line L then rotating through an angle of a around a point p is a glide reflection

Investigation 1.10 Solution on page 66

Show that reflecting in a line L then glide reflecting by a vector v in L is the same as translating by v.

The proof here is easy: have a go yourself.

Investigation 1.11 Solution on page 66

What is reflecting in a line L then glide reflecting by a vector v in a line M at an angle of a to L the same as?

Note

You might want to split this one up: what happens if M and L are parallel?

The other way around is similar, have a go:

Investigation 1.12 Solution on page 67

What if the glide reflection comes before the reflection?

Investigation 1.13 Solution on page 67

What is translating by a vector v then glide reflecting by a vector w in a line L the same as?

Note

This one might be worth splitting into parts that you can then put together: what happens if v and w are parallel? What if they're perpendicular? Then put those two together to get the full answer.

The other direction, again, is very similar:

Investigation 1.14 Solution on page 68

What happens if we translate after the glide reflection above, instead of before?

Finally, we'll combine glide reflections by other glide reflections. We already did the case where they're in the same line (see Investigation 1.3 on page 22, so we'll move on to the other two cases:

Theorem 1.16

Glide reflecting in L by v followed by glide reflecting by w in M parallel to L and separated from it by u is the same as translating by $v+w+2u$.

Proof

Again, split the glide reflections into reflections and translations. That gives us, in order: a translation by v, a reflection in L, a translation by w, and a reflection in M. Combining the reflection in L with the translation by w gives a glide reflection by w in L. Combining that with the translation by v gives a glide reflection by $v+w$ in L. Combining that with the reflection in M gives a translation by $v+w+2u$.

Our final case is quite similar:

Investigation 1.15 Solution on page 69

Glide reflecting in L by v followed by glide reflecting by w in M intersecting L at a point p and angle a is the same as a rotation by a.

Throughout this chapter, we've effectively filled in this table:

Combinations of Symmetries in 2-dimensional Euclidean Space

First symmetry		
Second Symmetry	Result	Page
Rotation through angle a around point p		
Rotation by b around p	Rotation by $a+b$ around p	4
Rotation by $-a$ around q	Translation	24
Rotation by b around q	Rotation around third point	24
Reflection in line through p at angle b	Reflection in line through p at angle $360° - \frac{a}{2}$	26
Reflection in line not through p	Glide Reflection	29
Translation	Rotation by a around a new point	26
Glide reflection by v in line L	Glide reflection	29
Reflection in line L		
Rotation around p on L by angle a	Reflection in line through p at angle $a/2$ from L	26
Rotation around p not on L by angle a	Reflection	29
Reflection in M parallel to L	Translation by $2(L-M)$	19
Reflection in M intersecting L at p at angle a	Rotation by $2a$ about p	10
Translation by v parallel to L	Glide reflection in L by v	27
Translation by v perpendicular to L	Reflection in $L+v$	28
Translation by v	Glide reflection in $L+v\perp L$ by $v\|L$	29
Glide reflection by v in L	Translation by v	30
Glide reflection by v in M parallel to L	Translation by $(v+2(L-M))$	30
Glide reflection by v in M not parallel to L	Rotation	30
Translation by v		
Rotation by a about p	Rotation	27
Reflection in L parallel to v	Glide reflection in L by $\lvert v\rvert$	21

Reflection in L perpendicular to v	Reflection in $L-v$	27
Reflection in L at angle a to v	Glide reflection parallel to L	28
Translation by w	Translation by $v+w$	21
Glide reflection by w in L	Glide Reflection by $w+v\|w$ in $L-v\perp w$	31
Glide Reflection in L by v		
Rotation by a around p	Glide Reflection	30
Reflection in L	Translation by v	30
Reflection in M parallel to L	Translation by $v+2(M-L)$	30
Reflection in M intersecting L at p at angle a	Rotation	30
Translation by w	Glide Reflection parallel to L	31
Glide reflection in L by w	Translation by $v+w$	22
Glide reflection in M parallel to L by w	Translation	31
Glide reflection in M intersecting L at p by w	Rotation	31

Now, there's just one last thing to do to make sure we've not missed any possible symmetries. This sequence of investigations will let you prove this theorem:

Theorem 1.17

Any symmetry (in 2 dimensions) is either a rotation, a reflection, a translation, or a glide reflection.

Investigation 1.16 Solution on page 69

Take any four points a, b, c, and d with a and c the same distance apart as b and d. How many symmetries can there be that send a to c and b to d?

Investigation 1.17 Solution on page 70

Show that any symmetry sending a to c and b to d can be written as at most three reflections.

Investigation 1.18 Solution on page 71

Show that any symmetry at all can be written as at most three reflections.

Note

Hint: this is really easy.

Investigation 1.19 Solution on page 71

Prove Theorem 1.6 (just try all of the combinations of reflections).

We've now confirmed that our table above is complete: we have every possible 2-dimensional symmetry in the above table.

Chapter 2

Classifying Symmetry Groups

In this section, we'll tackle a slightly different question: what combinations of symmetries can we have in an object? First, we'll look at rotations:

2.1 Rotations Only

Firstly, having a rotational symmetry of order n means that our shape has to have at least n rotational symmetries. This is, in fact, basically the only available option:

Theorem 2.1

Every shape with n rotational symmetries around the same point has a rotational symmetry of order n.

If this seems obvious to you, think about it for a bit: why couldn't you have a rotation of order 3 and a rotation of order 2 that combine together to make 6 (or 5?) different rotations? After a while, you might find that it stops feeling so obvious. It's still true, though:

Proof

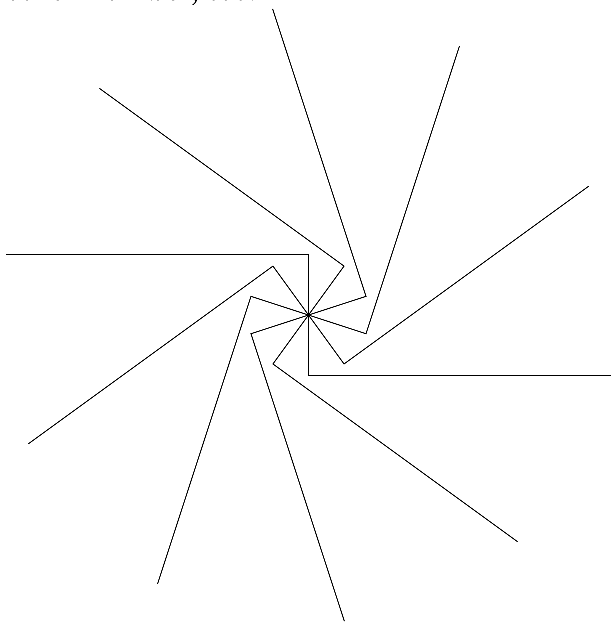
Put our n rotational symmetries in order of the angle that they rotate by. Look at the one with the smallest angle (other than 0°). Call that angle a. Take any other rotation by an angle b (again, no 0°), and we'll show that b is a multiple of a. We must have $b \geq a$ (since a is the smallest), so there's some whole number $k \geq 1$ so that $ka \leq b < (k+1)a$. Let $c = b - ka$, which must be at least 0 (since $b \geq ka$) and smaller than a (since $(k+1)a > b$). But rotating by c is the same as rotating by b then rotating anticlockwise k times by a

each time, and since all of those are symmetries of our shape, so is c. If c is not 0, this is a problem: we've found a symmetry with a smaller positive angle than a, but we picked a to be the smallest. So c must by 0, which means $b = ka$, so b is the same as rotating k times by a, and so every one of our rotations must come from rotating by a some number of times. Since there are exactly n of them, that means a must have order n.

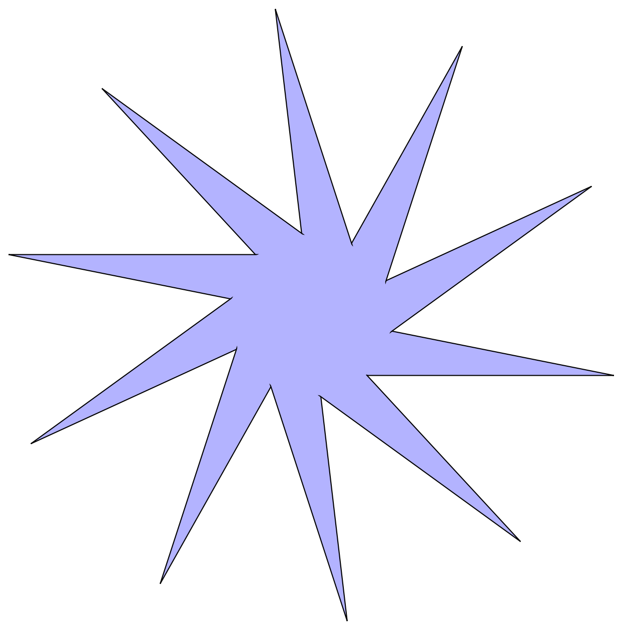
Example 2.1

Finally, we can have a shape with n rotational symmetries (and no other symmetries) for any $n > 0$: start with any asymmetric shape, say this tick shape:

If you put n of them together with one of the ends touching , you'll get a shape with n symmetries. I'll draw $n = 10$ here, but it will work with any other number, too:



If you prefer your shapes to have some width rather than just being edges, just add a few extra lines:



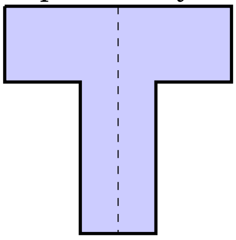
Dealing with the case where there are infinite symmetries is somewhat more complicated, and we'll do it later.

This kind of symmetry, where we have one base symmetry and all other symmetries are repeats of that is called **Cyclic** symmetry. We'll see lots of other examples of this type of symmetry. If that base symmetry has order n, we'd way we're dealing with symmetry of **group type** C_n, where the C stands for "cyclic".

2.2 Reflections Only

This one is very simple: if a shape has two different lines of reflective symmetry, it must also have another symmetry: if they overlap, it has a rotational symmetry around the point where they cross. If not, it has a translational symmetry of length twice the distance between them. Thus, the only shapes with only reflective symmetry are the ones with only one reflective symmetry. We can actually have this:

Example 2.2

This shape has only the marked line of reflection, and no rotational symmetries. 

Notice that this symmetry has group type C_2.

2.3 Translations Only

The 1-dimensional part of this is actually very similar to rotations only (this is yet another case where rotations and translations act similarly).

Theorem 2.2

If a shape has translational symmetry in a direction, and there is a translation with the smallest positive length ℓ of any translation in that direction, then all translations in that direction are by lengths equal to a multiple of ℓ.

The proof is basically the same as for rotations, so I'll leave it up to you:

Investigation 2.1 Solution on page 72

Prove the theorem above.

Again, the case without a smallest positive translation is a bit more complicated, so we'll do it later.

If you'll allow a little bit of cheating about what n can be in our C_n notation, we'll call this symmetry of **group type C_∞**.

If we have translation in two different directions, they just work separately:

Theorem 2.3

If a shape has translational symmetry in two different directions, and doesn't have arbitrarily short translational symmetries, then there are two vectors v and w such that

1. Translating by v is a symmetry of the shape.
2. Translating by w is a symmetry of the shape.
3. Every translational symmetry of the shape is by $kv + mw$ for some whole numbers k and m.

Proof

Since we don't have arbitrarily short translational symmetries, we can pick v to be the shortest translational symmetry it has, and let w be the shortest translational symmetry not in the same/opposite direction to v. All translations can be written in the form $kv + mw$ for some numbers k and m by Theorem 1.5, so all we need to do is show that k and m are whole numbers. Now, suppose there is some translational symmetry with vector $kv + mw$ with k and m not whole numbers. Choose the shortest with k and v positive. Then translating by $-v-w$ then $kv+mw$ gives a symmetry $(k-1)v+(m-1)w$. Since our first symmetry was the shortest, this new one must be at least as long. But that means that $(k-1)^2 + (m-1)^2 \geq k^2 + m^2$, so $-2m - 2k + 2 \geq 0$, so $k + m \leq 1$. But then the length of $kv + mw$ is $k|v| + m|w| \leq k|w| + m|w| \leq (k+m)|w| \leq |w|$. Since w was the shortest translation that isn't in the v direction, we must have $|kv + mw| = |w|$, so $k + m = 1$.
But now $m = 1 - k$, and our symmetry vector is $kv + (1-k)w$. But now, translating by $-v$ then by $kv + (1-k)w$ gives a symmetry translating by $(k-1)v+(1-k)w$. This has to be at least as long as $kv+(1-k)w$, but $|(k-1)v+(1-k)w|^2 = (k-1)^2((a-c)^2 + k^2(b-d)^2)$ and $|kv+(1-k)w|^2 = k^2((a-c)^2+(b-d)^2)+c^2+d^2-2k(c^2-ac+d^2-bd)$, so $(k-1)^2((a-c)^2+k^2(b-d)^2) \geq k^2((a-c)^2+(b-d)^2)+c^2+d^2-2k(c^2-ac+d^2-bd)$, so $1-k \geq k$, and $k \leq \frac{1}{2}$. But similarly, translating by $-w$ then by $kv + (1-k)w$ and doing the same argument (you should check this!) tells us that $k \geq \frac{1}{2}$, so $k = \frac{1}{2}$, and our translation is by $\frac{1}{2}v + \frac{1}{2}w$. But then $|w| \leq |\frac{1}{2}v + \frac{1}{2}w| = \frac{1}{2}|v+w| \leq \frac{1}{2}(|v|+|w|) \leq \frac{1}{2}(|w|+|w|) = |w|$, so all of those $\leq$ signs must actually be $=$ signs, so in particular $|v+w| = |v|+|w|$. But the only possible arrangement of v and w that makes that true is for them to point in opposite directions, which contradicts our definition of w.

Again, the case where there's no shortest symmetries is more complicated, and will be done later [TODO].

This is our first symmetry group type that isn't just C_n. Since we've basically got two copies of C_n that don't interact (one in each direction), we'll call it C_∞^2 or $C_\infty \times C_\infty$.

2.4 Glide Reflections Only

This one is even easier: if we have a glide reflection symmetry, then we have the result of composing it with itself, so we have a translation symmetry. Thus, there are no shapes with only glide reflection symmetry.

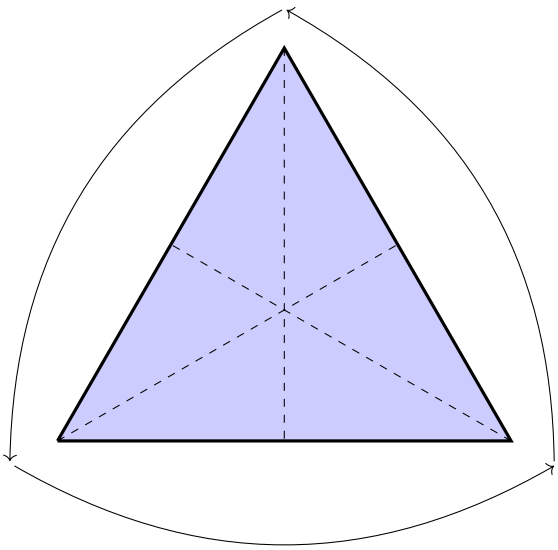
2.5 Point Fixing Groups

Next, we'll tackle the collections of symmetries that keep a point still. These can't include translations or glide reflections, as those move everything. We also can't have rotations around more than one point, as each rotation would move the fixed point of the other, or reflections around lines that don't cross (composing them would give a translation), or rotations around a point not on all of our lines of reflection (composing them would give a glide reflection). That last means that all of our lines of reflection have to cross in the same place (otherwise, composing two pairs that cross in different places would give rotations around different points). Thus, all we can have is a bunch of rotations around a single point, and a bunch of reflections in lines that go through that point.

We've already seen shapes with no symmetry apart from "do nothing" (Example 2.1 with $n = 1$), with a single reflection symmetry and nothing else (Example 2.2), and with rotations only, of any number (Example 2.1).

We can also have both rotations and reflections. The simplest example is the humble equilateral triangle:

Example 2.3



You can see that it has rotation and reflection symmetries around the centre. Notice that combining any two reflections gives one of the rotations, and that we have exactly three reflections and three rotations.

The simplest examples for other numbers of rotations/reflections are very similar: they're just the regular polygons with other numbers of sides. This is a new symmetry group type (it's the first one where the order where we do things matters). We'll call it **Dihedral**, and say that the symmetry group type of a shape with n sides is D_n (eg the triangle above has symmetry group type D_3.

Theorem 2.4

The only finite collections of symmetries that fix a point p in the plane are:

1. No symmetry at all.
2. A single reflection in a line through p, plus the “do nothing" symmetry.
3. A rotation by an angle $a = \dfrac{360°}{n}$ around p with n a whole number, together with all n compositions of that rotation by itself.
4. A collection of n rotations as in (3), together with n reflections through p, each at an angle of $a/2$ from the next, with n at least 3.

Proof

First, notice that the examples above mean that all of the above do actually exist. Now, we'll check that there are no others. Items (1), (2) and (3) above deal with the cases where there are only reflections or only rotations, which we have already looked at, so we'll assume that we have both exactly n rotations (with $n > 1$) and a reflection. Composing each nth rotation with one reflection gives a second reflection at an angle of $na/2$ to the first, so we must have at least n reflections. If we have any extra reflections, say at an angle of b to the nearest other rotation, then by Theorem 1.2, we also have a rotation by $\frac{b}{2}$, contradicting the definition of n. Thus, we must be in case (4) above.

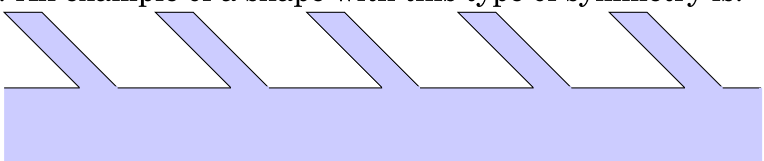
The remaining cases will all involve translations. To see this, notice that if we have a glide reflection, translation, or reflections in parallel lines, we're done, so the only remaining possibilities are rotations around more than one point or a rotation and a reflection in a line it is not on. In the latter case, we can get a reflection in a crossing line by composing the two, then a rotation around a second point by composing the two reflections. In the former, composing the two gives a rotation around some third point, so we only need to worry about the case where we have two rotations by angles a and b around points p and q (with $q \neq q$, or we'd be in the first case we looked at, and $a \neq b$, or we get a translation by composing the first with the inverse of the second).

2.6 The Frieze Groups

We'll now look at the cases with translations, but only in one direction.

Again, we'll assume there's a single shortest translation for now. We'll say it's a translation by v. By Theorem 2.3, every other translation has to be by a multiple of v. There are quite a few possibilities here.

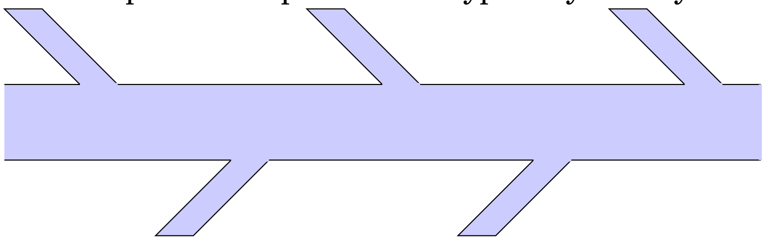
Firstly, if we only have translations, we're in the case we dealt with in Theorem 2.3. An example of a shape with this type of symmetry is:



Secondly, if we have a shortest glide reflection by w in L and no symmetries other than glide reflections and translations, we must have a translation by $2w$ (by doing the glide reflection twice), so $2w$ has to be a multiple of v (or v wouldn't be the shortest translation). We're going to show that in fact $2w = v$. If $2w = 2v$, then composing our glide reflection with the inverse of our translation gives a reflection, which is an extra symmetry we said we didn't have. If $2w > 2v$, then that same composition gives a shorter glide reflection that we said we didn't have.

So in fact every symmetry is a multiple of our glide reflection, and our symmetry group type is again C_∞.

An example of a shape with this type of symmetry is:



The third case I'll leave for you:

Investigation 2.2 Solution on page 72

If we only have translations and reflections, then:

1. What angle(s) can the reflections be at to the translations?
2. How many can there be, and how can they be spaced out?
3. Is this symmetry group type the same as any other we've seen?
4. Give an example of a shape with this type of symmetry.

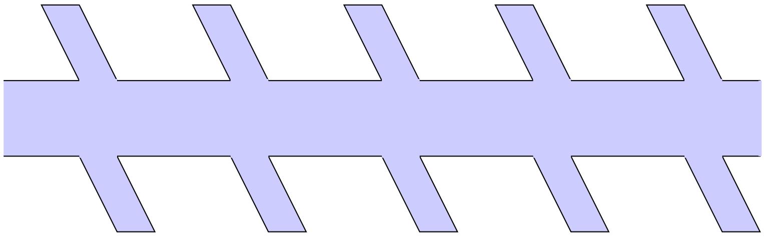
Have a go at the next one with a bit less structure:

Investigation 2.3 Solution on page 73

Describe what happens when we have all three of translations, reflections, and glide reflections.

Now we're just left with the cases involving rotations.

If we only have translations and rotations, the rotation must be by 180° (otherwise we will get translations in another direction). We must have infinitely many of them, with the centres of rotation all in a straight line spaced out by v: there must be at least that many, as translating by v has to take any centre of rotation to another centre of rotation. If there was an extra one on the line, we'd get a shorter translation by combining it with a rotation whose centre is closer to it than v. If there were one off the line, we'd get a translation in another direction. You might notice that this is very similar to what we had in the case with only reflections and translations. In fact, the symmetry group type we get is "the same", for a meaning of "the same" that we'll explain properly in a later chapter, and we'll again call it D_∞. This shape has these symmetries:



Otherwise, we have translations, rotations, and something else. The rotations still have to be by 180°, with centres spaced by v as above.

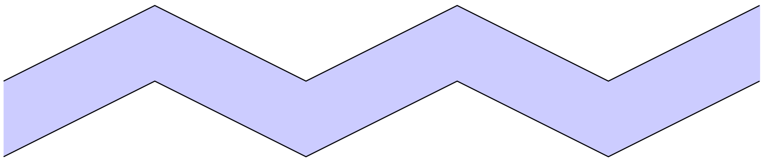
If we have a glide reflection, then it must be in a line parallel to v and by a multiple of v, as before, and combining it with our rotation gives a perpendicular reflection.

If we have a reflection, it's either parallel or perpendicular to v. If it's parallel, then combining with our rotation gives a perpendicular reflection as well.

If we have a perpendicular reflection, combining it with a rotation around a point not on its line gives a glide reflection.

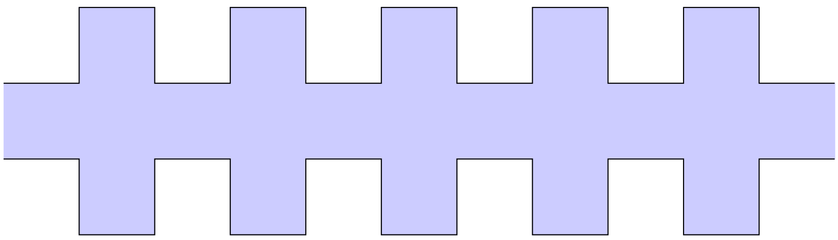
So in all remaining cases, we have at least translations, rotations, glide reflections, and perpendicular reflections. The perpendicular reflections are restricted as before: they must be through the centres of rotation. The shortest glide reflection must be by either v or $v/2$, as before. If it is by v, then combining it with the inverse of our translation gives a parallel reflection in the same line (which is the only possible parallel reflection). If it is by $v/2$, then we do not have that parallel reflection.

Both of these remaining options are possible. This shape has all symmetries apart from parallel reflections:



It's harder to see, but this one again has the D_∞ symmetry group type. We'll see why properly in a later chapter.

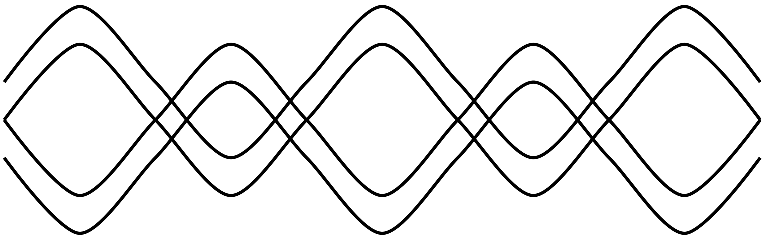
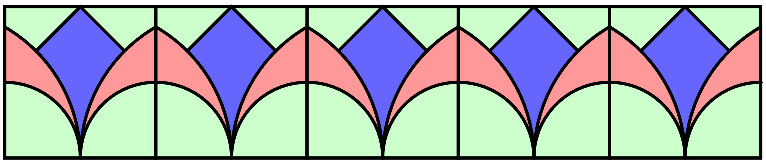
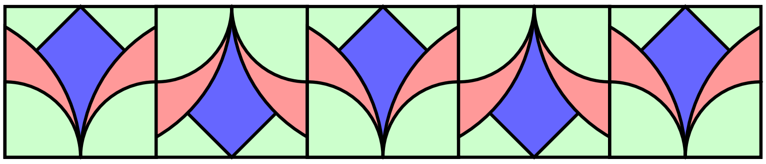
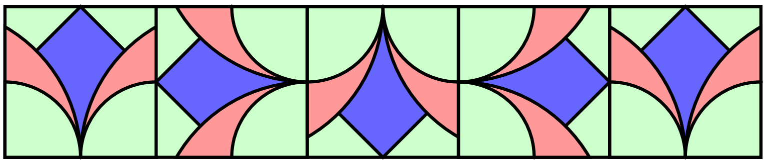
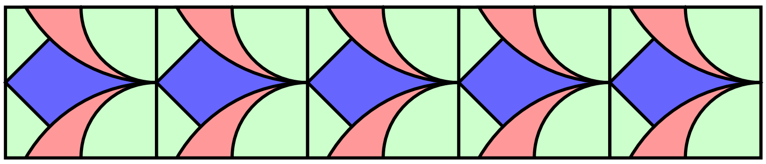
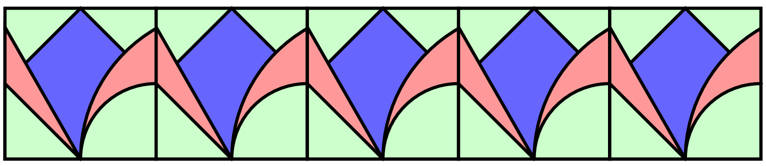
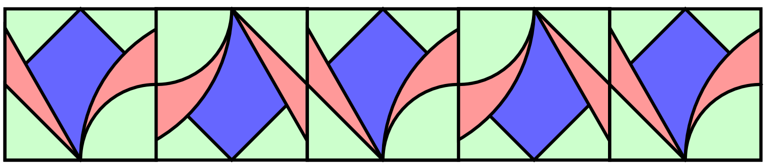
This has all possible symmetries:



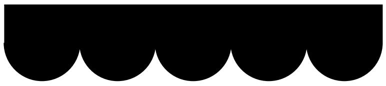
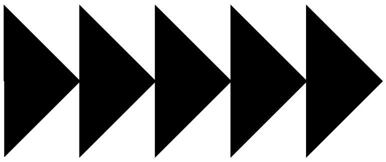
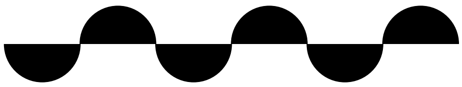
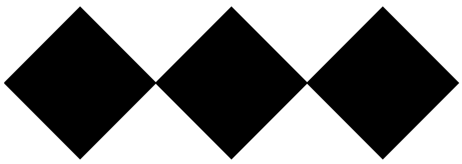
This one is a new symmetry group type: if we ignore the rotations, it's D_∞ as before. Adding those rotations back in gives an extra symmetry of order 2 that is separate to the rest, so we'll call this $D_\infty \times C_2$.

Investigation 2.4 Solution on page 73

These shapes and symmetry types we've seen with translations in one direction are called **frieze groups**, named after the symmetrical strips of decorative patterns you often see around buildings. Look at these examples (or any others you might see while walking around), and work out their symmetry group types.

1. 
2. 
3. 
4. 
5. 
6. 
7. 

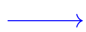
For a reminder, the seven types of frieze groups are:

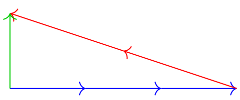
Translations only	C_∞	
Glide reflections & translations	C_∞	
Reflections & translations	D_∞	
Reflections, glide reflections, & translations	$C_2 \times C_\infty$	
Rotations & translations	D_∞	
Rotations, glide reflections, perpendicular reflections, & translations	D_∞	
Rotations, glide reflections, perpendicular & parallel reflections, & translations	$D_\infty \times C_2$	

2.7 The Wallpaper Groups

We're now going to do the same kind of classification as we did above, but with translations in more than one direction. This time, you're going to do it, but this is a lot more complicated than previous investigations, so I'll help by giving you some structure.

Initial Setup

If we have translations in more than one direction, we can only have two "independent" ones, plus combinations of them. To see this, notice that for any third translation, we can make it out of the first two by repeating one until we're "lined up" along the second, then repeating the second to complete it. For example, in this picture, we do a translation by  using translations by  and :



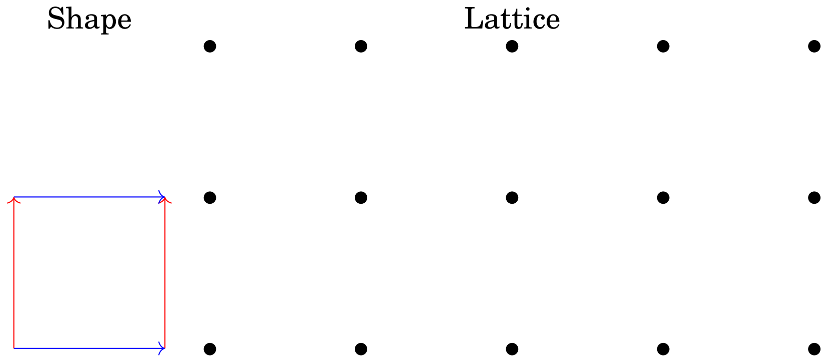
This is always possible if you allow fractions of a translation (you should think about why, and convince yourself of it). Since we're still sticking to cases with a smallest translation, we can always pick two translations that do it with whole numbers of translations (try proving this: if not, take the two shortest, and make a shorter one).

For all of the rest of this section, we'll call our two vectors that we're building our translations out of v and w. There will be lots of choices of them (in the example above, we could also have built red out of -3 copies of blue and one copy of green, for example), but we'll leave our choice of which one we're picking for later. The rule that we'll stick to is that they always have to be the *shortest* vector in that direction. Otherwise, we'll miss symmetries.

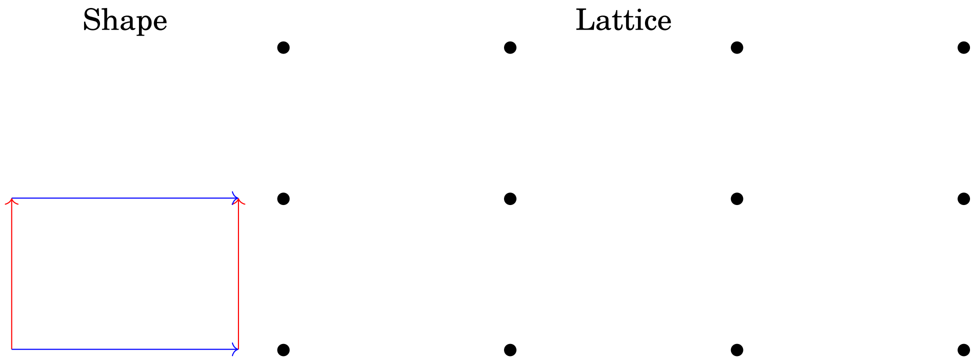
These two vectors actually determine a lot about what the pattern looks like, but don't change the group at all (see [TODO: linear algebra appendix not yet written]).

We'll look at two things about those vectors: whether they're the same length, or different lengths, and what the angle between them is. These two things determine the shape made when you put them together in both orders, which determines what's called the **Lattice Type**: that's the pattern that you get when you pick one point and mark all of the points the translations take it to.

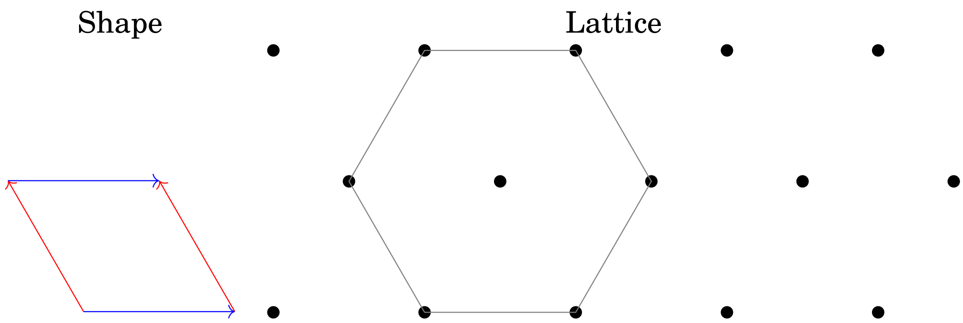
If we can choose v and w so that they are both the same length and at right angles to each other, that shape is a square, which gives us a **square lattice**:



If they can be at right angles, but different lengths, you get a **rectangular** lattice:

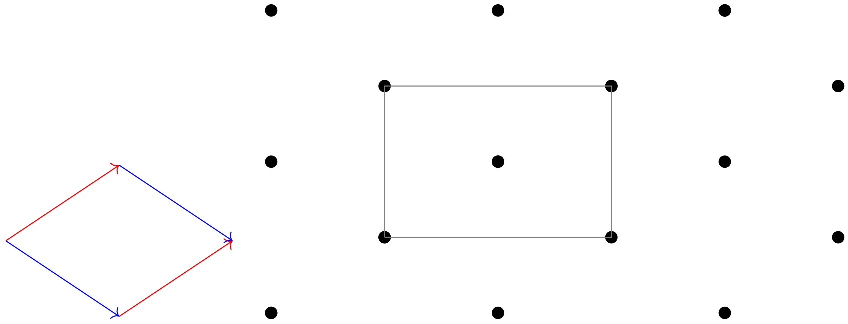


If they can be chosen to be at an angle of 120° (or 60°) and the same length, you get a rhombus made of two equilateral triangles stuck together, and the lattice is called **hexagonal** (which will make more sense if you look at the picture, which I've drawn one of the hexagons on):



If they are the same length at any other angle, the shape is a rhombus, and you get a **centered rectangular** lattice (again, I've drawn a rectangle - notice the extra dot in the centre).

Shape Lattice

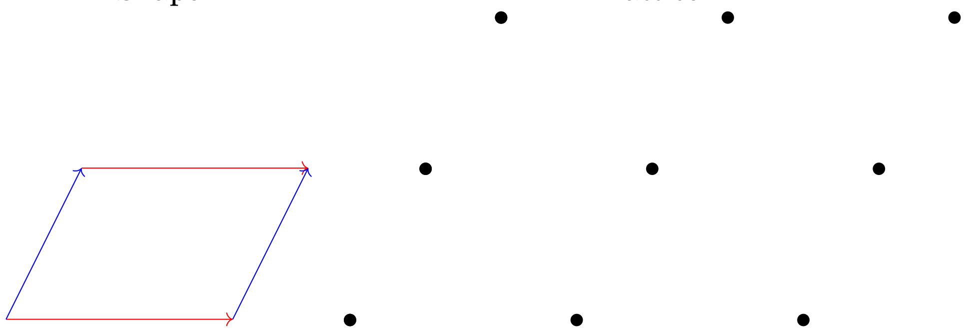


Investigation 2.5 Solution on page 74

You might want to think about why those particular angles (60°, 90°, 120°) are so different to every other angle – or are they actually different?

Finally, if they are different lengths and not at right angles to each other, we get a parallelogram, and an **oblique lattice**:

Shape Lattice



You might want to think about the symmetry group types of the shapes above – it will be important later.

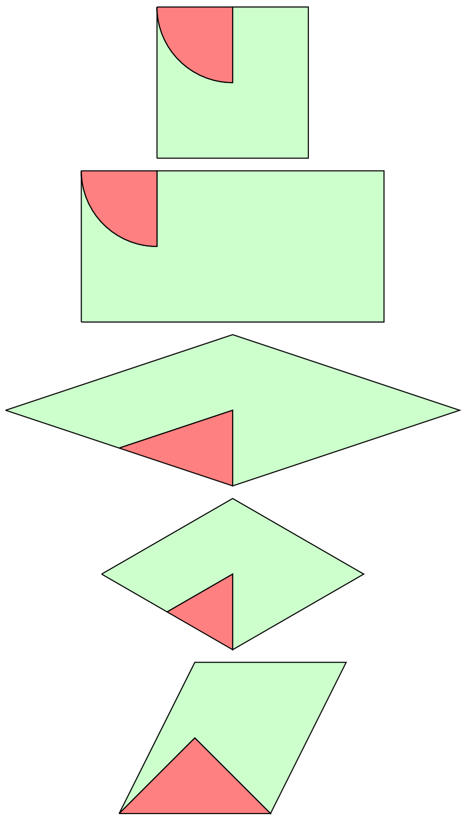
Translations Only

Just like before, we'll start with the simplest case. A few questions to help you think it through:

Investigation 2.6 Solution on page 74

1. Does it matter what order we do translations in?
2. Does it matter what angle the translations are at to each other or how long they are?
3. What would the symmetry group type be if we picked just one translation and ignored the other?
4. What do you think the symmetry group type will be?

This type can happen on any of our lattice types. Try drawing one for each, then check the list on the next page:



Square

Rectangular

Centred Rectangular

Hexagonal

Oblique

Translations and Glide Reflections

Investigation 2.7 Solution on page 74

If we have glide reflections that aren't parallel, is it possible that we only have glide reflections and translations?

Investigation 2.8 Solution on page 74

Given two parallel glide translation symmetries, both by a vector x in lines L and M separated by a perpendicular vector y (going forward, we'll assume that there are no other glide reflection lines between L and M, and that x is the shortest possible), what other symmetries do we get?

Investigation 2.9 Solution on page 75

Can there be any translations other than the ones guaranteed above?

We're now nearly done: we know what all of our symmetries are: since our translation by $2x$ is the result of combining a glide reflection with itself, while the translation by $2y$ is independent, our symmetry group type is, again, $C_\infty \times C_\infty$. We're also forced into either a square or rectangular lattice, for much the same reasons as before. Again, both are possible: try drawing some (turning one into the other is usually easy).

Translations, Glide Reflections, and Reflections

This is the first case that slightly more complicated, in that we will have multiple possible cases with different lattice types.

As in Investigation 2.7, we can still only have parallel glide reflections, or we'd get rotations, and they still need to be evenly spaced. We still get all of the same symmetries from Investigation 2.8. The only thing has changed is with Investigation 2.9.

Investigation 2.10 Solution on page 75

What extra symmetries can we have that we couldn't in the previous section?

Notice that our reflection lines are always glide lines, because we can compose our reflection with a parallel translation to get a glide with the same line.

We'll now look at the two cases separately: whether we have the translations by x, or not. First, let's assume that we do.

Investigation 2.11 Solution on page 75

If we do have a translation by x, what lattice shapes could we have?

We can't have a translation by y, or we'd have extra reflections by composing it with one of our reflection, so we're done: our group consists of reflections in parallel lines spaced out by y, glide reflections in those same lines, translations by sums and multiples of x and $2y$. Parallel to the reflections, we just have translations, so our symmetry group type is C_∞. Perpendicular to them, we have translations and perpendicular reflections, so our symmetry group type is D_∞. Combining those, the overall symmetry group type is $C_\infty \times D_\infty$.

Investigation 2.12 Solution on page 75

Draw a pattern with this symmetry type.

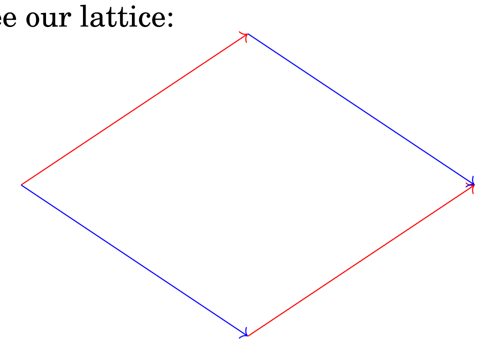
Now, if we don't have translations by x, then our original glide lines are *not* reflection lines, so we have reflection lines spaced out by y, with glide lines half way between each pair.

Combining a glide reflection with an adjacent reflection gives a translation by $x+y$. Similarly, there is a translation by $x-y$, given by combining the glide reflection with the adjacent reflection on the other side.

Investigation 2.13 Solution on page 76

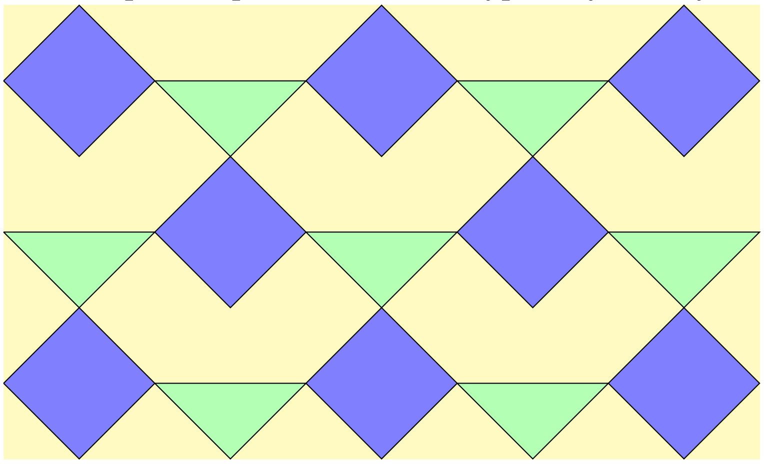
Prove that all translational symmetries of this pattern are given by combining our translations by $x+y$ and $x-y$.

Finally, if we draw $x+y$ and $x-y$, as well as translated copies of them, we'll see our lattice:



So we must have a rhombic lattice (or a square one if $x=y$).

An example of a pattern with this type of symmetry is:



The Crystallographic Restriction Theorem

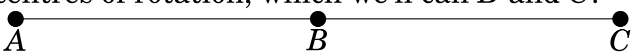
Before we go on to rotations, we need to pin down what rotations are even possible. This is the rather grandly named Crystallographic Restriction Theorem

Theorem 2.1: The Crystallographic Restriction Theorem

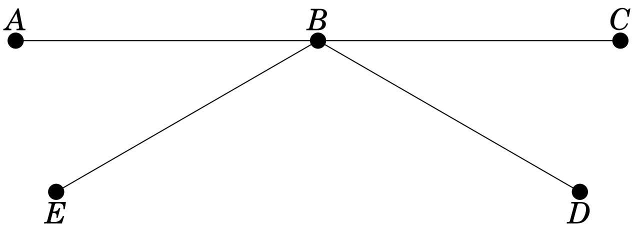
For any shape with translational symmetry in two directions, the only possible rotational symmetries are of order 1, 2, 3, 4, and 6 (so rotations by 360°, 180°, 120°, 90°, and 60° and multiples of those).

Proof

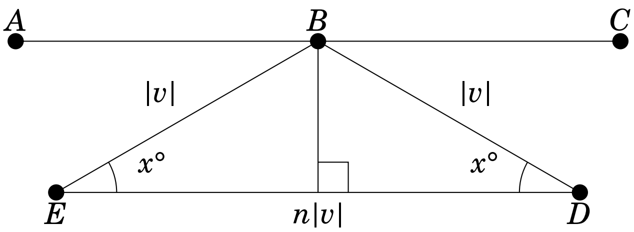
Call our two translations vectors v and w. Pick any centre of rotation in our shape. Call it A. Applying our translation by v to A twice gives two more centres of rotation, which we'll call B and C.



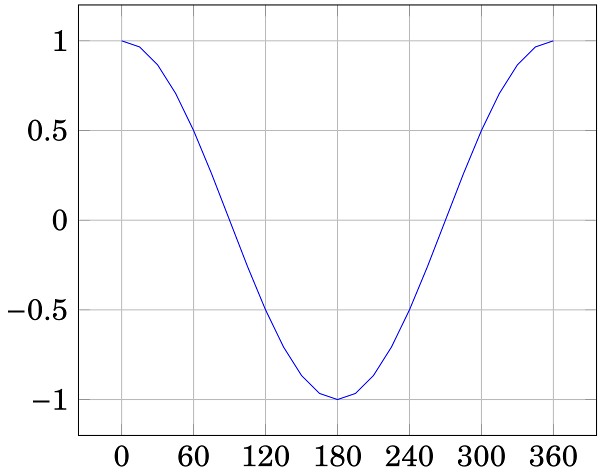
If we have a rotational symmetry by $x°$, we can rotate C around B by $x°$ and A around B by $-x°$, getting two more centres of rotation, which we'll call D and E.



Applying our translation by w some number of times moves the line of points through A, B, and C to the line of points through D and E, and D and E must be separated by some multiple of v, say nv. That gives us this picture:



If we look at one of those two right-angled triangles, we can see that $\frac{n}{2}|v| = |v|\cos(x°)$, so $\frac{n}{2} = \cos(x°)$. Since n has to be an integer, $\cos(x°)$ has to be a multiple of $\frac{1}{2}$. Since $\cos(360°) = 1$, $\cos(180°) = -1$, $\cos(120°) = -\frac{1}{2}$, $\cos(90°) = 0$, and $\cos(60°) = \frac{1}{2}$, we at least don't rule out any of our angles. If we look at a graph of $\cos(x)$:



You can see that it only hits a multiple of $\frac{1}{2}$ at $x° = 0°, 60°, 90°, 120°, 180°, 240°, 270°, 300°$ and $360°$. Notice that all of those have order 1 (for $x° = 0°, 360°$), 2 (for $x° = 180°$), 3 (for $x° = 120°, 240°$), 4 (for $x = 90°, 270°$), or 6 (for $x° = 60°, 300°$), so we're done.

Now, we'll just try all of those orders (except order 1, since that's just doing nothing). Of course, having a rotation of one order gives us rotations with order all of its factors (by just repeating it). Also, if we have rotations of order 4 and 6 (ie rotations by 90° and 60°, we would obtain a rotation by 30° around some point by doing one followed by the inverse of the other, so we can't have those. So the only thing we need to think about is our highest order rotation.

Notice also that if we pick two adjacent centres of rotation and rotate one around the other repeatedly and join the dots, we get a regular polygon whose number of sides is the order of the rotation, so the order of rotation tells us about our lattice type: if our highest order rotation is 2, we could have any lattice type. If it is order 3 or 6, our lattice must be hexagonal. If it is order 4, our lattice must be square.

Translations and Rotations of Order 2 Only

This is perhaps the most flexible case: we just need a regular grid of rotation centres (and no reflection or glide reflection symmetry). If you stare at it, you'll notice that this is just the same as the case with translations and rotations only from when we had translations in only one direction, plus translations in some other direction. Putting those together, that tells us that our symmetry group type is $C_\infty \times C_\infty \times C_2$. To make these things shorter to write, we'll abbreviate that as $C_\infty^2 \times C_2$.

Investigation 2.14 Solution on page 76

Find examples of patterns with this symmetry of each lattice type.

Translations, Glide Reflections, and Rotations of Order 2 Only

Investigation 2.15 Solution on page 77

This one isn't too hard: try doing it yourself. Describe all of the symmetries we get in this case and find out what lattice types we can have.

The symmetry group type here is different to anything we've seen so far: the glide reflections don't commute with the rotations, or the glide reflections in the other direction, so we have three separate collections of symmetries that all don't commute with each other, so we can't split it up into copies of C_n and D_n groups, as we have with everything else so far. Instead, we'll just give it a new name: for lack of anything else to call it, I'll call it B for now.

Translations, Reflections, Glide Reflections, and Rotations of Order 2 Only

This is the most complicated case, so we'll split it up a bit.

First, like before, we have to have glide reflections in two perpendicular directions. We don't, however, need to have reflections in two directions. Let's start by looking at what happens if we only have reflections in one direction.

Investigation 2.16 Solution on page 77

Describe the symmetries and possible lattice types in this case.

Again, this is a new symmetry group type that doesn't split up nicely in the way we've done so far. We'll call it E for now, lacking a better name.

That just leaves the case with reflections in two different directions. They have to be at right angles to each other (because of our rotations), but this time, there's nothing stopping us having centres of rotation off our lines of reflection (as well as the ones that we must have everywhere our lines of rotation cross), or anything forcing us to. Again, we'll split into two cases. First, let's look at what happens if we don't have any centres of rotation that aren't on our reflection axes:

Investigation 2.17 Solution on page 78

Describe the symmetries and possible lattice types in this case.

That only leaves the case where we have extra centres of rotation not on our reflection lines:

Investigation 2.18 Solution on page 78

Describe the symmetries and possible lattice types in this case.

Once again, this is a new symmetry group type. We'll call it F for now.

You might be glad to know that this is by far the hardest case (we're actually already past half way through this section): as we've already said, the rest only have one possible lattice type each (hexagonal for order 3 or 6, square for order 4), and there are only three possibilities for each rotation order.

Translations and Rotations of Order 3 Only

Investigation 2.19 Solution on page 81

This one is somewhat different to the previous examples, in that it's the first one where we can't get all of the transformations by composing the ones we assume that we have. See if you can figure out which extra symmetries we'll have before you read ahead.

Investigation 2.20 Solution on page 82

Try to find an example of a pattern with this symmetry group type (hint: you might need more than one triangle).

Translations, Glide Reflections, and Rotations of Order 3 Only

This isn't possible. As usual, our glide reflection must be a multiple of half of the translation between two points linked by translations. If it takes one such point to another, then taking a glide reflection followed by some translations gives a reflection. Otherwise, after some translations, we have a glide reflection of length exactly half of the distance between two such points. Rotating it gives glide reflections of the same length in two other directions forming an equilateral triangle. But now combining these three glide reflections gives a reflection.

Translations, Reflections, Glide Reflections, and Rotations of Order 3 Only

This is another case where there are two possible groups.

As before, we can split the plane into equilateral triangles, with our shortest translations along the edges. We have to have a reflection line somewhere. Rotating that line around a centre of rotation gives another reflection line at 60° to the first. Combining those reflections gives a rotation around the point where they cross by 120° (of order 3). So whatever happens, we have a grid of equilateral triangles whose edges are along reflection lines and have our shortest translations along them, with centres of rotation at each corner.

Investigation 2.21	Solution on page 82
Are there any extra symmetries that we have to have?	

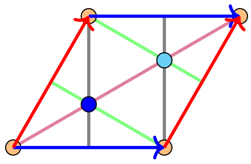
The extra rotation centres is where our two groups come from: those extra rotations can either be on reflection lines, or not. As before, we'll look at them separately.

If those extra rotation centres are not on reflection lines, then we've described all of our possible symmetries: if we had any extra rotation centres, they wouldn't move our triangles to matching triangles. If we had any extra reflections, they'd have to be parallel to the existing ones (to avoid making extra rotations). But they'd also have to be evenly spaced between our current ones, which would make shorter translations. If we had extra glide reflections, they would similarly combine with our existing glide reflections to make new translations.

Investigation 2.22	Solution on page 83
Find some examples of a pattern with this symmetry.	

This is yet another symmetry group type that can't be broken up as we have so far. We'll call it H for now.

If, instead, all of our rotation centres *are* on reflection lines, things are somewhat different. As before, our extra rotation centres must be in the middle of our equilateral triangles of the original rotation centres. Our reflection lines are somewhat different, though. They still have to form a grid of equilateral triangles with centres of rotation in the corners, but this time, they have to go through the centres of our original triangles. That means that they can't go along the edges of those triangles, or we'd have triangles with 30° angles made of symmetry lines, which isn't possible. That means that we have the symmetries in this picture (where circles are centres of rotation, arrows are translations, and lines are reflections)



Notice, in particular, that there are no reflections parallel to either translation.

Investigation 2.23	Solution on page 84
What other symmetries are there?	

Translations and Rotations of order 2, 3 and 6 Only

As already mentioned, we're forced to have a hexagonal lattice of centres of rotation of order 6 with translations along the edges.

Investigation 2.24	Solution on page 85
Describe the other rotations in this case.	

Investigation 2.25	Solution on page 85
Describe the possible patterns with this type of symmetry.	

Again, this is a new group we can't easily write in terms of ones we've already seen. We'll call it I.

Investigation 2.26	Solution on page 86
Find some examples of shapes with this type of symmetry.	

Translations and Rotations of Order 2 and 4 Only

As we already mentioned, our rotations of order 4 force us to have a square lattice, because it's our only lattice shape with rotational symmetry of order 4.

Taking any centre of rotation of order 4 and applying our translations then gives a square grid with a centre of rotation at each corner. Again, there are extra centres of rotation.

Investigation 2.27 Solution on page 86

Prove that there is a centre of rotation of order 4 in the centre of each square, and a rotation of order 2 in the middle of each side of a square.

Now, there can't be any more rotations: combining a rotation of order 2 anywhere else with its closest existing rotation of order 2 (which might be one of the rotations of order 4 repeated twice) would give a shorter translation.

So we're done. Again, we'll just give this one a name for now: it's J.

Investigation 2.28 Solution on page 87

Find some examples of patterns with this symmetry.

Translations, Glide Reflections, and Rotations of Order 2, 3 and 6 Only

This, again, is not possible, for exactly the same reason as in the order 3 case: combining an order 3 rotation (from repeating an order 6 rotation twice) with our glide reflection in either order gives an equilateral triangle of glide reflection lines. Combining all three gives a reflection.

Translations, Reflections, Glide Reflections, and Rotations of Order 2, 3 and 6 Only

Finally, we have the most symmetrical case.

Investigation 2.29 Solution on page 88

1. Which lattice type(s) could we have?
2. What rotation centres do we have to have?
3. Where can we have reflection lines?
4. Do we have to have all of these, or can we have just some of them?
5. What glide reflections do we have?
6. Describe what our patterns have to look like, and draw one of each main possibility.

2.8 Beyond Two Dimensions

It's worth mentioning that you can do all of this classification work in three dimensions (or any other number you like), but it quickly gets somewhat out of hand: in three dimensions, instead of the 17 Wallpaper Groups, there are 230 Crystallographic Groups. Beyond that, it gets even more silly (there are 4,894 in dimension 4, 222,097 in 5 dimensions, and nobody even knows how many there are in dimension 6, but it's definitely more than 28,927,915).

Appendix A

Solutions

Chapter 1

Solution to Investigation 1.1 on page 10

They're all evenly spaced. Suppose we have two lines of symmetry. Then reflecting in one, then the other will, by Theorem 1.2, give a rotation by twice the angle between them. Since both lines are lines of symmetry, this rotation is also a symmetry of our shape, so our shape has rotational symmetry with that angle. Rotating it by that angle also rotates the lines of symmetry by that angle, and gives us two new lines of symmetry, with the same space between them as the original pair, and (because it's exactly twice the angle that we rotate by) the same angle between the two pairs. If one of those lines is a line we already had, then we have evenly spaced lines of symmetry all the way around, and we're done. If not, we now have four lines of symmetry that are evenly spaced. Repeat this process with the new lines of symmetry, and keep doing this. If we ever get a line that we've had before, we're done. If not, the shape is symmetrical under *every* rotation, so the lines of symmetry are everywhere, and so are evenly spaced.

Note

Note that the second part of this argument also proves the claim on page 4 that rotational positions are also evenly spaced.

Solution to Investigation 1.2 on page 16

There are two angles between the two lines of symmetry. The angle of rotation is twice the angle from the line of the first reflection to the line of the second rotation, in that direction.

Solution to Investigation 1.3 on page 22

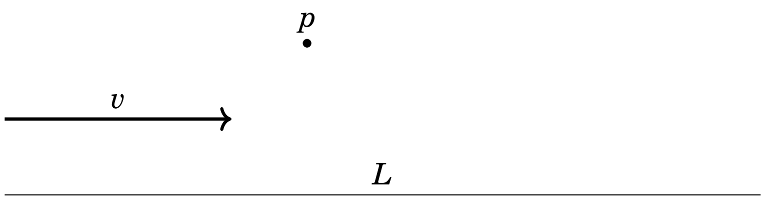
The result is a translation by the sum of the glide lengths. To see this, notice that the reflections and translations **commute** (that is: we get the same thing whether we translate first or reflect first - note that this only works because the reflection is perpendicular to the translation, it won't work in other directions - we'll prove this in more detail later), so combining them is the same as reflecting twice in the same line (which does nothing), then translating by one length than the other (which is the same as translating by the sum).

Solution to Investigation 1.4 on page 26

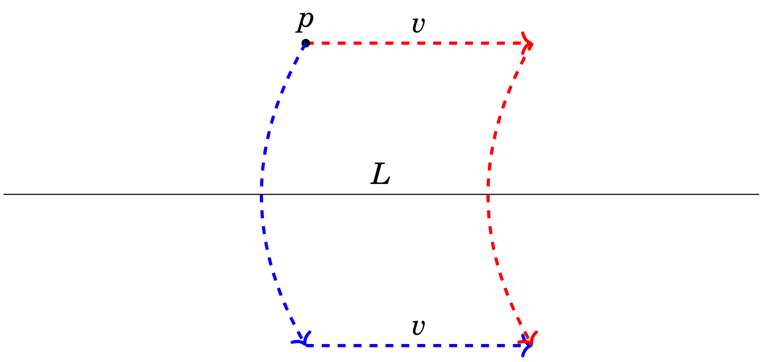
Like before, let's split into reflections: we can write the rotation as a reflection in L followed by a reflection in a line at an angle of $a/2$ from L, so the two reflections in L cancel out, and we're left with just a reflection in a line at an angle of $a/2$ from L.

Proof of Theorem 1.9 on page 27

Take some point p. Let's draw a picture:



Let's draw what happens if we do our translation and reflection in either order. I'll do the translation first in red, the reflection first in blue:



As you can see, they both end up in the same place: both move p by $|v|$ parallel to L and by twice the distance from p to L perpendicular to L.

Solution to Investigation 1.6 on page 28

Here, we'll do it without picking the clever coordinates, so you can see how it helps. Let's say L is the line $y = mx + c$, $p = (a, b)$, and $v = (d, e)$.
Then the line from L to p at right angles to L is just some scaled version of v - say it's $fv = (fd, fe)$. Thus, reflecting in L takes p to $(a-2fd, b-2fe)$. Translating by v then takes that to $q = (a+(1-2f)d, b+(1-2f)e)$.
If instead we translate L by $v/2$, it will go to $y - e/2 = m(x - d/2) + c$, which is the same as $y = mx + (c + e/2 - md/2)$. The line from M to p at right angles to M is also at right angles to L, so is again a scaled version of v, and we've scaled it by $(1/2 - f)$, so reflecting p in M takes it from (a, b) to $(a, b) + 2(1/2 - f)(d, e) = (a + (1-2f)d, a + (1-2f)e)$, which is the same as we got the first time.

Solution to Investigation 1.7 on page 29

Split the translation into $v \perp L$ and $v \| L$ as before. Let M be the line parallel to L and offset by $(v \perp L)/2$. Use Investigation 1.6 to convert the $v \| L$ translation and the reflection in L into a reflection in M, then use the definition of glide reflection to combine the $v \perp L$ translation with the reflection in M to give a glide reflection by $v \perp L$ in M.

Solution to Investigation 1.8 on page 29

Write the rotation as a reflection in a line parallel to L followed by a reflection in a line at an angle of $a/2$ to L. Combine the first with the reflection in L to get a translation perpendicular to L. Combining that with the remaining reflection gives a glide reflection by Theorem 1.6.

Solution to Investigation 1.9 on page 30

Again, split everything into reflections: make the rotation into a reflection in a line perpendicular to L followed by a reflection in a line at an angle of $a/2$ to that. Split the glide reflection into two reflections perpendicular to L separated by v, with the second passing through p, followed by a reflection parallel to L. Again, we have two reflections in the line perpendicular to L passing through p together, which cancel. The first two reflections again combine to give a rotation, leaving us with a rotation followed by a reflection, which is a glide reflection by Theorem 1.6.

Solution to Investigation 1.10 on page 30

As already shown, we can split our glide reflection into a reflection and a translation in either order, If we put the reflection first, it will cancel with the other reflection, leaving just the translation by v.

Solution to Investigation 1.11 on page 30

As the hint suggests, we'll do it in stages. If M and L are parallel, then splitting the glide reflection as in A leaves us with two parallel reflections separated by the perpendicular vector u between M and L followed by a translation by v. The parallel reflections give a translation by $2u$ by 1.3, so by 1.5 we get an overall translation by $2u + v$.

If M and L are not parallel, however, then splitting the glide reflection as above instead leaves us with two crossing reflections followed by a translation. The two crossing reflections give a rotation around the point where they cross by 1.2, so we get a rotation around a new point by 1.6.

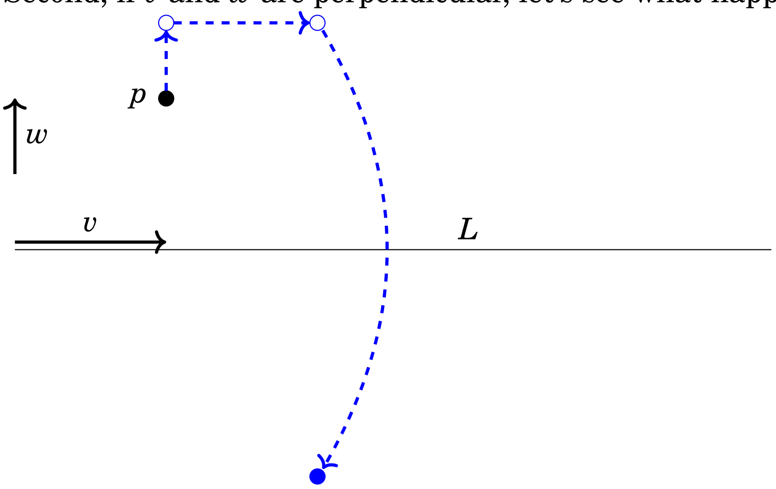
Solution to Investigation 1.12 on page 30

Everything is quite similar:
If M and L are parallel, we can combine the two parallel reflections to get a perpendicular translation, then combine the two translations to get a translation.
If M and L intersect at p, we can combine the two reflections to get a rotation around p, then combine that rotation with the translation to get a rotation.

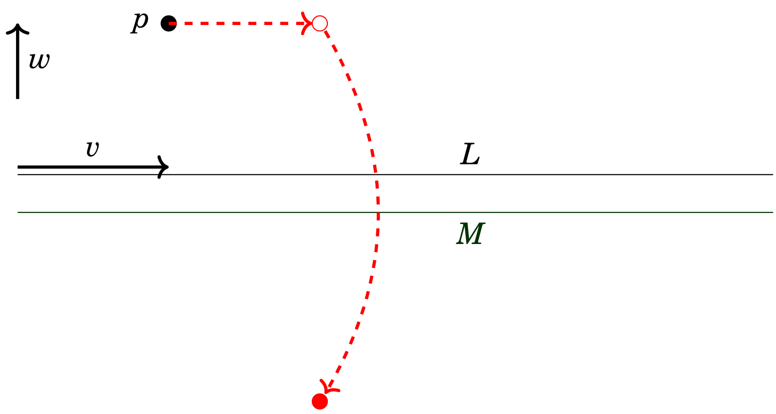
Solution to Investigation 1.13 on page 31

First, if v and w are parallel, we clearly just get a glide reflection by $v+w$ in L.

Second, if v and w are perpendicular, let's see what happens to a point p:



Here, p gets sent to the point that is v away along L, and $w+u$ away from L on the opposite side, where u is the vector from p to L.
If instead we glide reflect p by v in the line M that is $|w/2|$ away from L in the opposite direction to p, we get this picture:

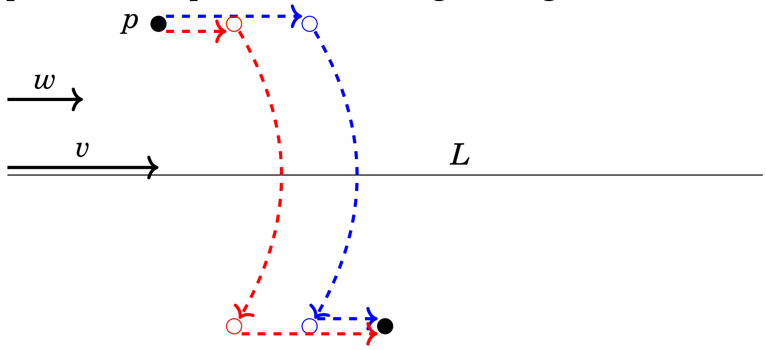


Again, p has been sent to the point that is v away along L and $w+u$ away from L on the opposite side. As p was arbitrary, this is true for any point p, so these are the same transformation.

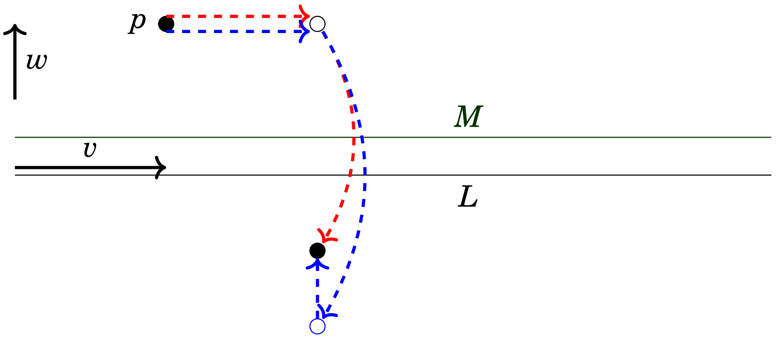
Finally, if v and w are not perpendicular, we can split w into $w\|v$ and $w\perp v$, where $w\|v$ is parallel to v with length $\cos(a)|w|$, with a the angle between v and w, and $w\perp v = w - w\|v$, as in other exercises. Now, we can split out translation into a translation by $w\perp v$ followed by a translation by $w\|v$. Combining that second translation with our glide reflection gives a glide reflection in L by $v+w\|v$ by the first part of this solution. Combining that with the first translation gives a glide reflection by $v+w\|v$ in the line M that is $-(w\perp v)/2$ from L.

Solution to Investigation 1.14 on page 31

If v and w are parallel, it's easy to see by following what happens to a point that we get a glide reflection in L by $v+w$.



If v and w are perpendicular, you can follow a point to see that we get a glide reflection by v in the line M that's separated from L by $w/2$:



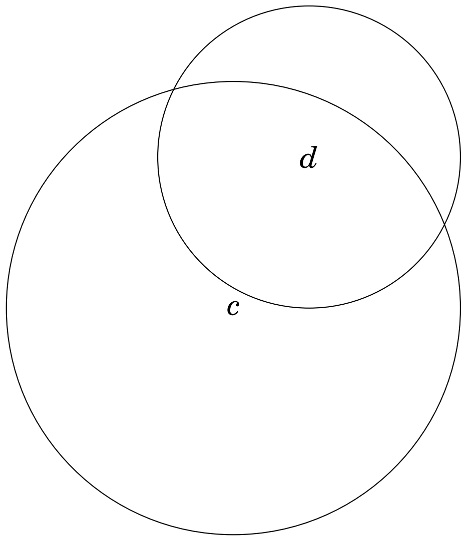
Finally, in all other cases, split w into $w \perp v$ and $w \| v$ as before, and apply the above: first combine the translation by $w \| v$ with the glide reflection to get a glide reflection by $v + (w \| v)$ in L, then combine with the translation by $w \perp v$ to get a glide reflection by $v + (w \| v)$ in the line that is $w \perp v$ away from L.

Solution to Investigation 1.15 on page 31

Again, split the glide reflections into reflections and translations, giving us a translation by v, a reflection in L, a translation by w, and a reflection in M. Combining the middle two gives a glide reflection in the line N that is parallel to L and separated from it by $w \perp v$ by $w \| v$. Combining that with the translation by v gives a glide reflection in N by $(w \| v) + v$. Finally, combining that with the reflection in M gives a rotation by a.

Solution to Investigation 1.16 on page 33

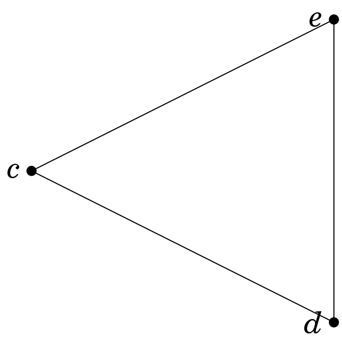
There are at most two: any symmetry has to send any other point e to some point f that's the same distance from c as e is from a, and the same distance from d as e is from b. There's only two such points, one on either side of the line through c and d (or just one, if it's on that line:



The only thing left to check is that we don't mix up what sides of the line from c to d we send this to. This, though, is fairly easy: if two points start on the same side of the line and finish on different sides, the line between them has gone from not crossing the line to crossing it, so our "symmetry" has changed the shape, so isn't a symmetry. Similarly, if two points start on different sides and end up on the same side, the line between them has gone from crossing the line to not crossing it, so again we've changed the shape and don't have a symmetry.

Solution to Investigation 1.17 on page 33

First, let's get a to go to c: we can do this by reflecting in a line perpendicular to the line from a to c and half way along it. That sends b to some other point e (that's the same distance from c as d is). Next, we need to send e to d without moving c. The latter part means we need a reflection in a line that goes through c. Let's draw a picture:



This is an isosceles triangle, so if we reflect in the line through c perpendicular to the opposite side, e will be sent to d.
The only thing left to do is to choose which side of the line from b to d we're sending things to: we can switch between them (without moving b or d) by reflecting in the line through b and d.
That's three reflections, so we're done.

Solution to Investigation 1.18 on page 33

Pick any two points. They have to be sent to some other two points that are the same distance apart as they are, so we're done by Investigation 1.17.

Solution to Investigation 1.19 on page 34

Let's just go through every possible combination of 0, 1, 2, or 3 reflections:

Zero Reflections This is just doing nothing: it's either a rotation by $0°$ or a translation by $(0,0)$, whichever you prefer.

One Reflection This is just a reflection.

Two Reflections Either the two lines of symmetry cross, or they don't. If they cross, we get a rotation around the point where they cross by twice the angle between them by Theorem 1.2. If not, we get a translation by twice the distance between them by Theorem 1.3.

Three Reflections This is where there are a lot of possibilities:

1. If none of the lines of symmetry cross, we can combine the first two to get a translation perpendicular to the lines of symmetry by Theorem 1.3, which leaves us with a translation of length twice the distance between them, then a perpendicular reflection, which gives us a reflection in a line parallel to the third line of symmetry at a distance of the distance between the fist two by Theorem 1.6.

2. If the first two lines of symmetry don't cross, but the third crosses them, we can again combine the first two to make a translation by Theorem 1.3, then combine with the last reflection to get a reflection by Theorem 1.6.

3. If the first two lines of symmetry cross, we can combine them to make

a rotation around the point where they cross by twice the angle between them by Theorem 1.2. This gives a reflection when combined with any reflection by Theorem 1.6 and Theorem 1.6.

That covers all of the possibilities, so we're done.

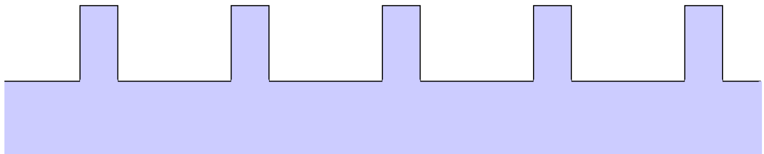
Chapter 2

Solution to Investigation 2.1 on page 38

Take any other translation by m. There are whole numbers k and c such that $m = k\ell + c$ with $0 \le c < \ell$, Translating by c is the same as translating by m, then translating k times by ℓ in the opposite direction, so is also a symmetry of our object. But if $c > 0$, then it's a positive length of a translation in the same direction by less than ℓ, which isn't possible, so we must have $c = 0$, so $m = k\ell$ is a multiple of ℓ.

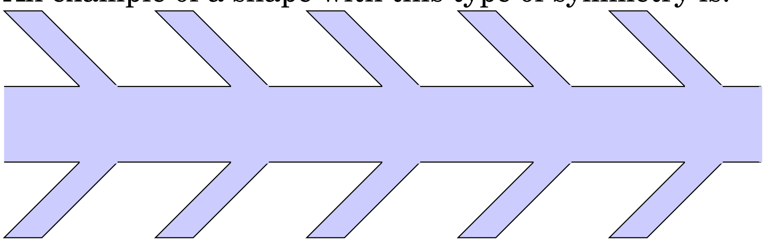
Solution to Investigation 2.2 on page 43

1. The reflections must be at right angles to the translations: otherwise, we'd get glide reflections.
2. There must be infinitely many, spaced apart by v: there can't be less, because every translation has to take each line of symmetry to the next one. There can't be more, or reflecting in a pair that are closer together than v would give a shorter translation than v.
3. This symmetry group is different to all of the others we've seen: it's infinite and not cyclic. It's quite similar to the dihedral examples, though, just with the rotations replaced by translations, so we'll call it the **infinite dihedral** symmetry group type D_∞.
4. An example of a shape with this type of symmetry is:



Solution to Investigation 2.3 on page 43

Our glide reflections must be parallel to v - say the shortest is by w in L. Our reflection must be either horizontal or vertical or we'd get rotations as before. If it were vertical, we'd get a 180° rotation by combining with our glide reflection, so we can only have horizontal reflections. The horizontal reflection must be in L or we'd get a translation in a different direction by combining with our glide reflection, so we have only the one reflection.
Now v and w must be the same: if w were longer, we'd have a shorter glide reflection by combining our translation and reflection. If v were longer, it would have to be a multiple of w, or we'd get a shorter glide reflection or translation by combining them. Since doing our glide reflection twice gives a translation by $2w$, we would have to have $v = 2w$. Combining it with our reflection gives a glide reflection of length v, and combining the two glide reflections gives a translation by $3w$. Combining one of these translations with the inverse of the other gives a translation by $w < v$, which we can't have. So we must have $v = w$. Thus, we don't really need our glide reflection: it's just what you get from putting our translation and reflection again.
This is another new symmetry group type: it doesn't matter what order we do our reflection and translations in, so it isn't D_∞, and it isn't cyclic because we can never get either the translations from the reflection or the reflection from the translations, so it isn't C_∞. It's something new instead. If we look at just the reflection, it looks like C_2, and if we look at just the translations, it looks like C_∞, so we'll call it $C_2 \times C_\infty$, and say that has **bicyclic** symmetry group type.
An example of a shape with this type of symmetry is:



Solution to Investigation 2.4 on page 46

1. This has all possible symmetries, so has symmetry group type $D_\infty \times C_2$.

2. This has translation and vertical reflection symmetries, so has symmetry group type D_∞.

3. This is only missing vertical reflection symmetries, so has symmetry group type D_∞.
4. If you think of the repeating segments as being four tiles wide, it's easier to see that this has translation and vertical reflection symmetries, so again has symmetry group type D_∞.
5. This has horizontal reflection, glide reflection, and translation symmetries, so has symmetry group type $C_2 \times C_\infty$.
6. This has only translation symmetry, so has symmetry group type C_∞.
7. This has translation and rotation symmetries, so has symmetry group type D_∞.

Solution to Investigation 2.5 on page 50

They're not actually different: they're all just oblique lattices that happen to have the right answers, as all of these repeating tiles are special cases of parallelograms.

Solution to Investigation 2.6 on page 51

1. No (this is Theorem 1.5 on page 21).
2. No: because the order doesn't matter, they're independent and don't interact.
3. C_∞ (see Section 2.6 on page 21).
4. Just like other times when we've had two symmetries not interacting, it's $C_\infty \times C_\infty$.

Solution to Investigation 2.7 on page 52

No: if we had two glide reflections that aren't parallel, then their lines cross, and combining them gives a rotation by Investigation 1.15 on page 31.

Solution to Investigation 2.8 on page 52

We get a translation by $2x$ by combining one glide reflection with itself, and a translation by $2y$ by combining the two different glide reflections. Then, of course, we get translations by the sums and multiples of those two vectors, a parallel set of evenly spaced lines of glide reflection separated by y, and glide reflections by all of the multiples of x.

Solution to Investigation 2.9 on page 52

No: a translation by x would give a reflection, a translation by any other vector in the x direction shorter than $2x$ would combine with our glide reflection (and possibly some translations by $2x$) to give a shorter glide reflection. A translation by any other vector would combine with a glide reflection to give a pair of glide reflections closer together than M and L.

Solution to Investigation 2.10 on page 53

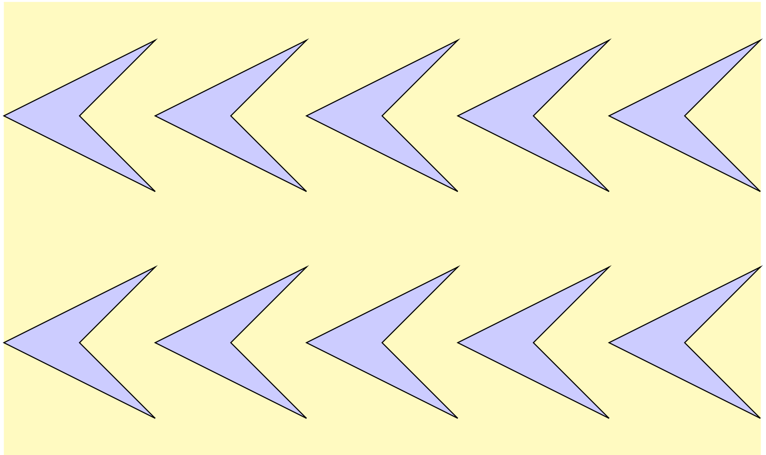
If you look at A on page 75, you'll notice that we only ruled out translations by x because it would give a reflection, but that's now fine, so we can (but don't necessarily have to) have translations by x, as well as reflections.

Solution to Investigation 2.11 on page 53

We can compose a translation by $-x$ with a glide reflection to get a reflection in every glide axis, so our glide axes and reflection axes are the same. Composing two adjacent ones gives a translation by $2y$. We have two translations at right angles to each other, so we must have a square or rectangular lattice.

Solution to Investigation 2.12 on page 53

Here's one possibility:

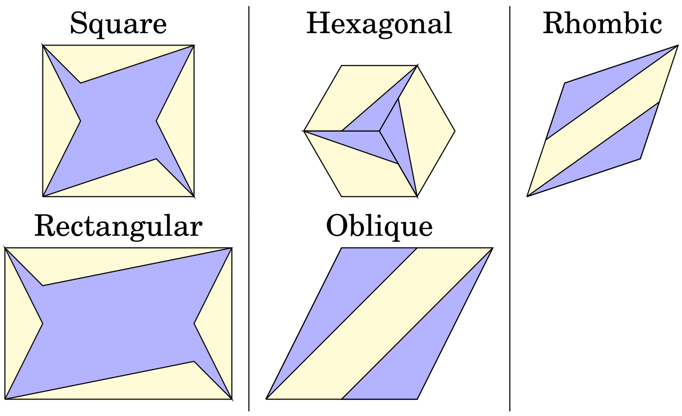


Solution to Investigation 2.13 on page 54

We can obtain translations by $2x$ by combining them as they are, and $2y$ by combining the first with the inverse of the second. If we had a translation by x, then we would have a reflection in the glide axis, which we don't. If we had a translation by y, we could combine it with our translation by $x-y$ to get a translation by x. If we have a translation by any $ax+by$, we can combine it with b copies of our translation by $x-y$ to get a translation by $(a+b)x$, then with copies of our translation by $2x$ to get a translation by $2x$, if $a+b$ is even, or x, if $a+b$ is odd. Since we can't have the second, $a+b$ must be even, so we can get our translation by $(a+b)x$ with $\frac{a+b}{2}$ copies of each of our base translations, and our translation by $ax+by$ with that, followed by $-b$ copies of our translation by $x-y$, so that $ax+by = \frac{a+b}{2}(x+y)+\frac{a-b}{2}(x-y)$.

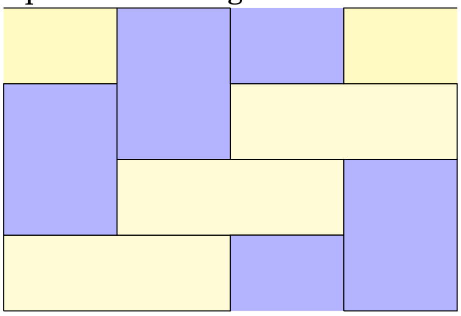
Solution to Investigation 2.14 on page 57

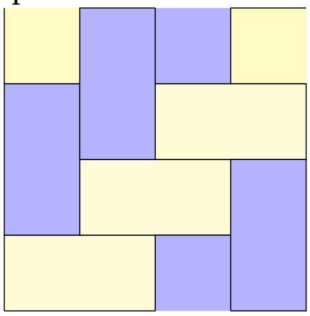
For example:



Solution to Investigation 2.15 on page 57

As always, our glide reflections must be by some multiple of half of the shortest translation in their direction, and in fact it must be exactly half, or we'd have a reflection (just like in the case with translations in only one direction). If there was a rotation on the line of a glide reflection, we'd have a perpendicular reflection by combining it with a rotation, so our rotations must not be on our glide reflection lines. Since rotating must take any glide reflection line to another one, the centres of rotation must be half way between the glide reflection lines. Now, combining our rotation with a glide reflection gives a glide reflection at right angles, so we must have two sets of glide reflection lines at right angles to each other. This also limits us to a square or rectangular lattice. Both are possible:





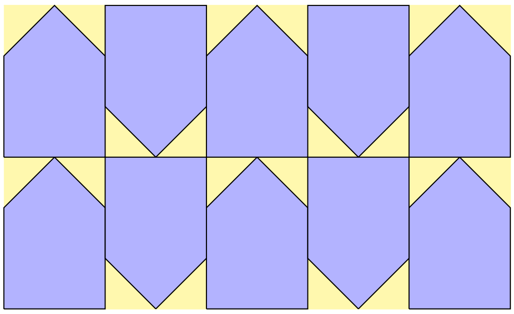
Solution to Investigation 2.16 on page 57

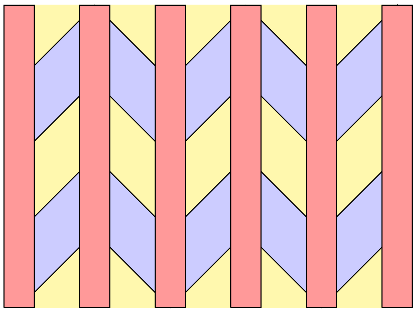
Like before, our centres of rotation can't be on the reflection lines, or we'd have a perpendicular reflection. They must be half way between them instead, so that rotating around them takes the reflection lines to other reflection lines.

However, combining a rotation with a reflection gives a perpendicular glide reflection on a line running through the centre of rotation, so all of our rotations have to sit on our lines of glide reflection.

This gives us a rectangular or square grid of rotation centres (alternating between two different centres that can't be sent to each other), which forces us to have rectangular or square lattice.

Both of these are possible:

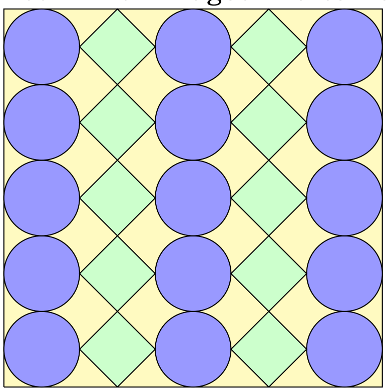


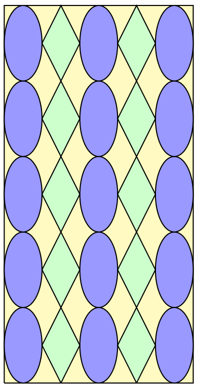


Solution to Investigation 2.17 on page 58

We've actually already listed all of our symmetries: we've assumed that there are no more rotations, any extra reflections at another angle would give rotations by smaller angles, as would any extra glide reflections at different angles, while extra glide reflections at the same angle would give smaller translations.

The reflections give a rectangular grid of centres of rotation, so a square or rectangular lattice. As usual, both are possible (any other grid of squares/rectangles which are symmetrical but different to each other and not mirror images in alternating rows would also work):

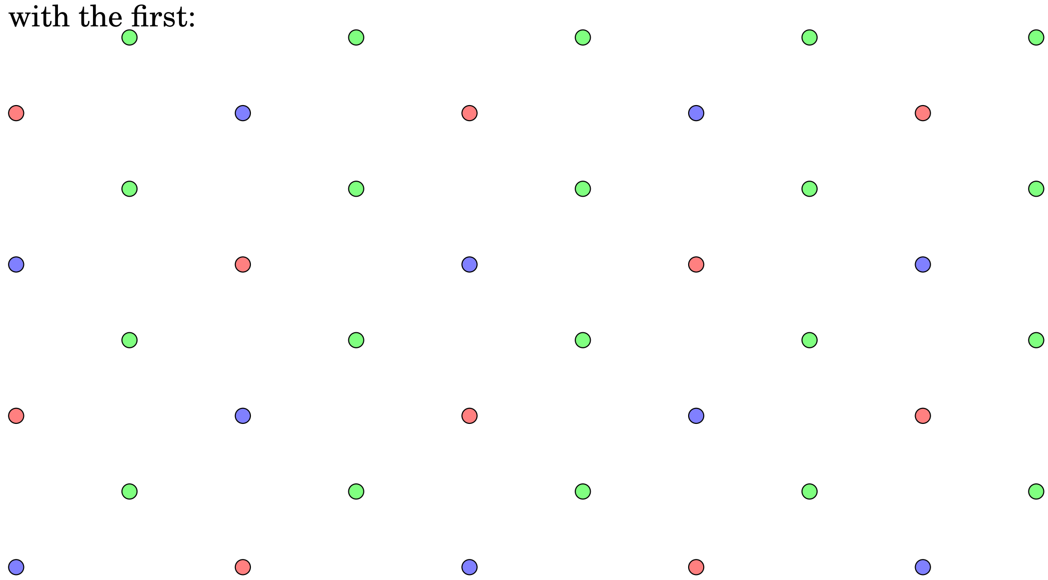




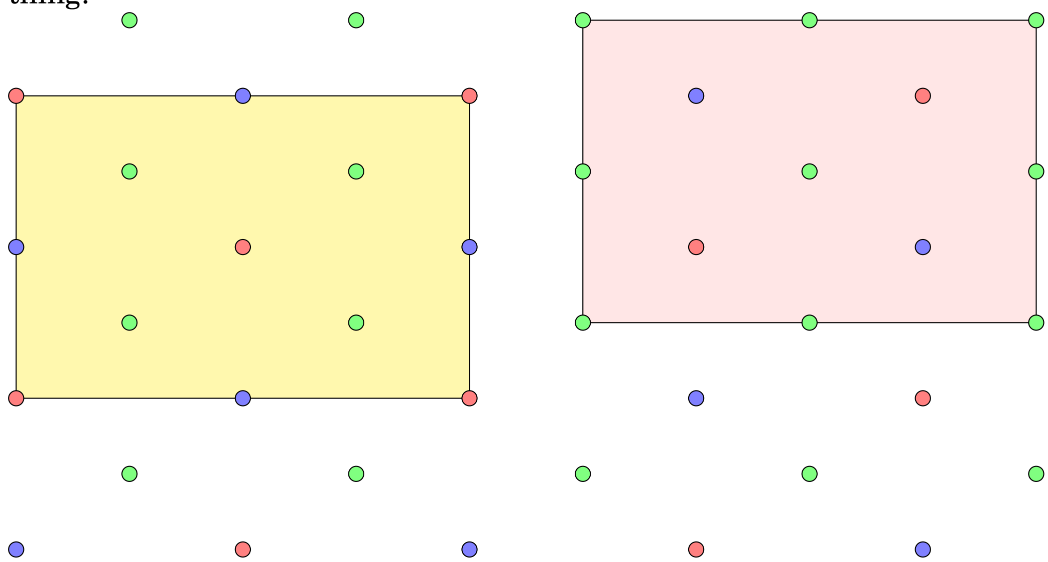
Solution to Investigation 2.18 on page 58

The rotation centres that are not on reflection lines have to be half way between them in both directions, and again, the reflections must be at right angles. Combining that rotation with each of our reflections gives glide reflections parallel to the reflections by the same distance as the gaps between the reflection lines. Combining that rotation with our translations gives rotations around the points where the mirror lines cross.

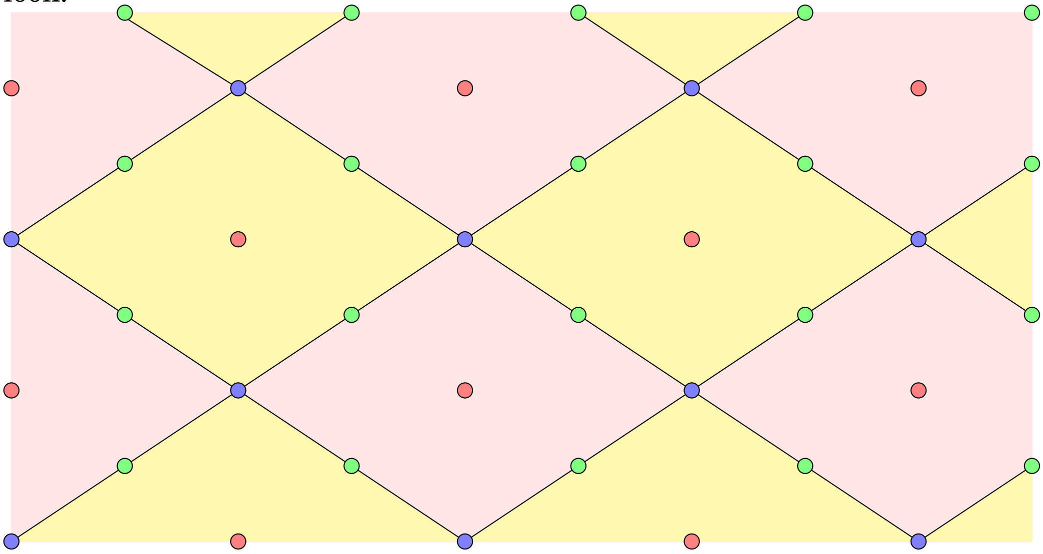
For the lattice types, we need to be slightly careful this time: you might think we have to have a square or rectangular lattice, like in the previous examples, but in fact, this isn't the case. As before, if we take a rotation centre, we get a square or rectangular grid of such centres using our translations. This time, though, there are actually three separate grids. If we take two adjacent centres on reflection lines, they make a square/rectangular grid, as before. If we instead take one of the centres that are not on the reflection lines, we get a second square/rectangular grid that isn't aligned with the first:



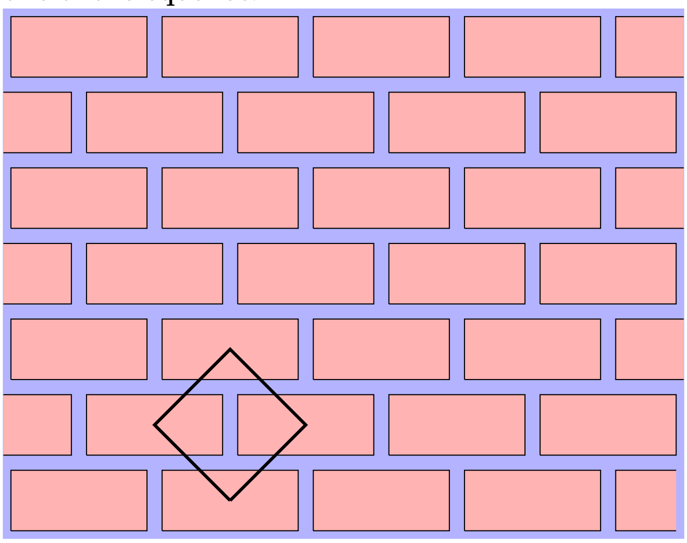
Because of this, our rectangles would have to be quite large to cover everything:



But there is a smaller repeating region - see if you can find it before you look.

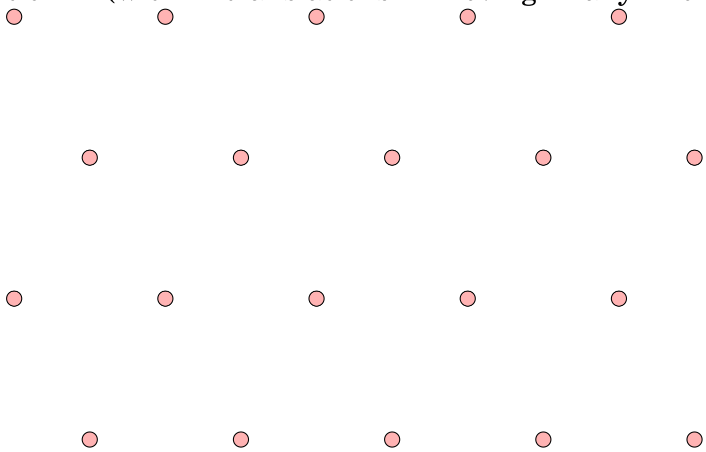


This really is as good as we can do, so we get a rhombic lattice, unless the ratio happens to be right so that it is a square lattice. These are, as always, both possible (the picture shown is square, but stretching it will not change the symmetry. Any other filling of the plane by rhombuses with two lines of symmetry and 180° rotational symmetry will also work). I've marked on one of the squares.

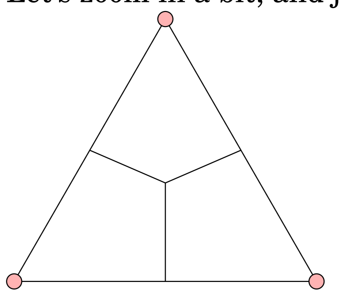


Solution to Investigation 2.19 on page 58

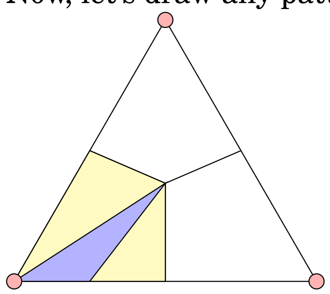
We start with this arrangement of centres of rotation (with translations moving any one to any other):



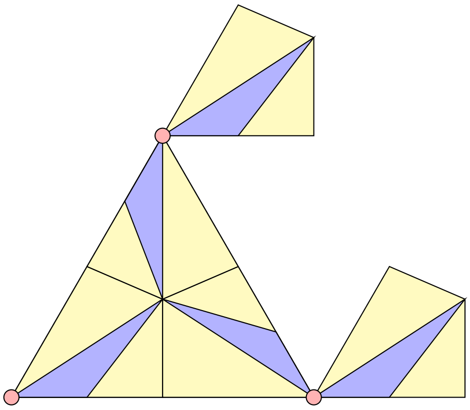
Let's zoom in a bit, and just add some lines to split this triangle up a bit:



Now, let's draw any pattern we like in one of those sections:



Let's apply some of our symmetries and see what happens:

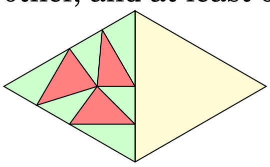


We've got an extra centre of rotation in the middle of each triangle. This will happen in every triangle, and every one will keep the rest of the pattern still as well, which gives us two more sets of rotation centres: one in the middle of the up-pointing triangles, and one in the middle of the down-pointing triangles. (notice that we can't turn one into the other, because that would need a rotation of order 6).

Now we have our rotation centres (we can't have any more, because they wouldn't match up the triangles), we're done. Again, this is a new symmetry group type, which we'll call G for now.

Solution to Investigation 2.20 on page 58

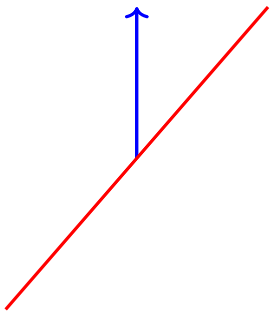
Filling (or "tiling") the plane with copies of this (or any other shape made of two same-sized equilateral triangles stuck together where both have rotational symmetry of order 3, they are different and not mirror images of each other, and at least one has no lines of symmetry) will work:



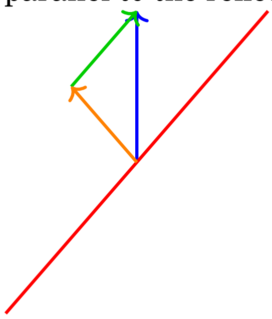
Solution to Investigation 2.21 on page 59

Just like before, there also have to be centres of rotation in the middle of each triangle.

We also have to have some extra glide reflections: to see this, let's look at one translation (blue) and one line of reflection (red) that isn't parallel to it:



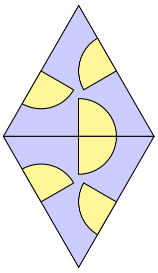
We can write that translation as the composition of two translations, one parallel to the reflection (green) and one at right angles (orange):



Now, if we combine these with the perpendicular translation first, then the parallel translation, then the reflection, we can first put the last two together to get a glide reflection in the same line as the reflection, then combine that with the parallel translation to get a glide reflection in a parallel line offset by the length of the orange arrow. But the triangle in the picture above has an angle of 30° between the orange and blue arrows (since the reflections are at 60 degrees to the translations that they aren't parallel to), so the length of the orange arrow is the length of the blue arrow multiplied by $\sin(30°) = \frac{1}{2}$, ie the extra glide reflections are half way between the reflections.
[TODO: I don't like this, find a better proof]

Solution to Investigation 2.22 on page 59

Tiling with this pattern (or any other pattern that fits in two attached equilateral triangles that are mirror images in the line between them and don't have any other mirror symmetries) will work: [TODO: add these clarifications to all of them]



Solution to Investigation 2.23 on page 60

- There can't be any more translations for the same reason as usual (you'd end up with shorter translations than our shortest ones).
- There can't be any more reflections (they'd have to be parallel to the existing ones to avoid making triangles with the wrong angles, so would combine with the existing ones to give shorter translations.
- There can't be any more rotations: to map the reflections to each other, they would have to be in the centre of each of the small triangles above, and be by 120°, but then combining one with the rotation around one of the corners of its small triangle gives a shorter translation.
- There **are** glide reflections, albeit less obvious ones than before (they're not parallel to either of our shortest translations):

  Combining our two translations gives a translation that is parallel to the purple lines in the above diagram, so combines with those reflections to give glide reflections in those lines. Similarly, there are glide reflections in the green and grey lines (can you find the combinations of translations for them?).

  But this isn't even all of the glide reflections: combining a reflection in one of the gray lines with the red translation, we'll get a glide reflection in a line parallel to the gray one but offset from it by half the horizontal length of the red arrow, so a glide reflection in a line halfway between the two gray lines, with a length of $\sqrt(3)/2$ lots of our shortest (red/blue) translations. Similarly, there are glide reflections of the same length half way between pairs of purple/green lines.

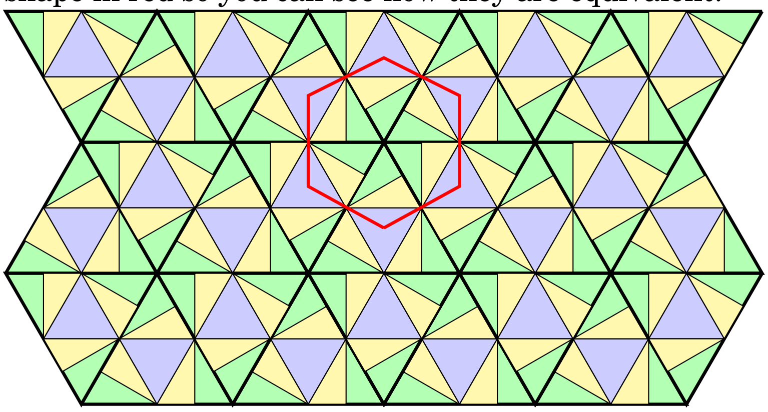
  This, though, is all of the glide reflections: any extras would have to be parallel to the existing ones, and combining them would give new translations, just like with the reflections.

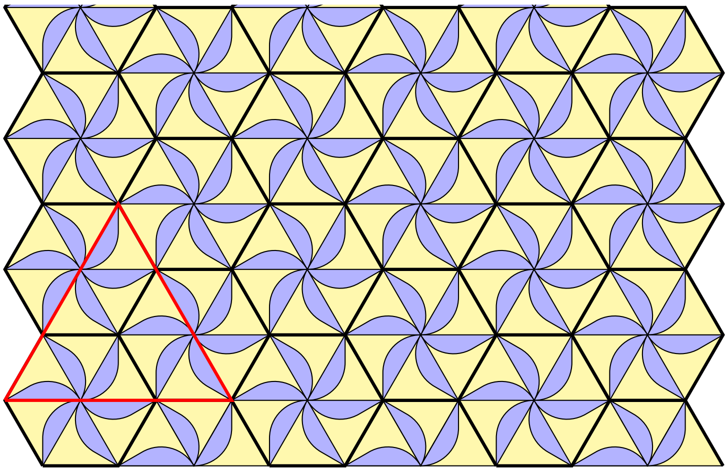
Solution to Investigation 2.24 on page 60

In each equilateral triangle of our lattice, there are also rotations of order 3 rotations in the centres (and only order 3: we order 3 in the corners (by doubling the order 6 rotation), so just like in the order 3 case, we also have can't have order 6 rotations, because the equilateral triangle itself only has rotational symmetry of order 3).
Similarly, if you take one of the triangles, rotate it 180° around one of its corners, then translate it so that it ends up sharing an edge with the original triangle, we'll have done a rotation by 180° around the middle of that edge.
There are no other possible symmetries: these are the only rotations that send all of our lattice points to lattice points.

Solution to Investigation 2.25 on page 60

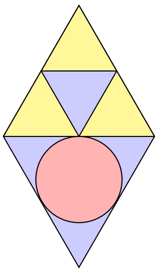
There are two equivalent descriptions: either fill the plane with identical triangles that have rotational symmetry of order 3 but no reflection symmetry, or with identical hexagons that have rotational symmetry of order 6 but no reflection symmetry. In the diagrams below, I've marked one of the other shape in red so you can see how they are equivalent:





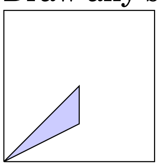
Solution to Investigation 2.26 on page 60

Tilings of this pattern, or any other pattern made of two equilateral triangles connected together, with both having symmetry type D_3 (ie rotational symmetry of order 3 and three lines of mirror symmetry), where they aren't the same or mirror images of each other) will work:

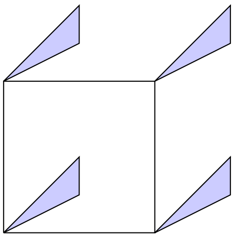


Solution to Investigation 2.27 on page 61

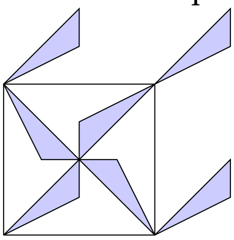
Draw any shape you like in one corner of the square:



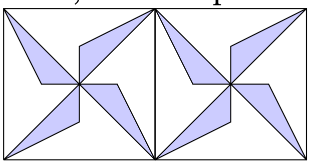
Translate it to each corner of the square:



Finally, rotate each one around the corner it is now touching so that it finishes in the square:

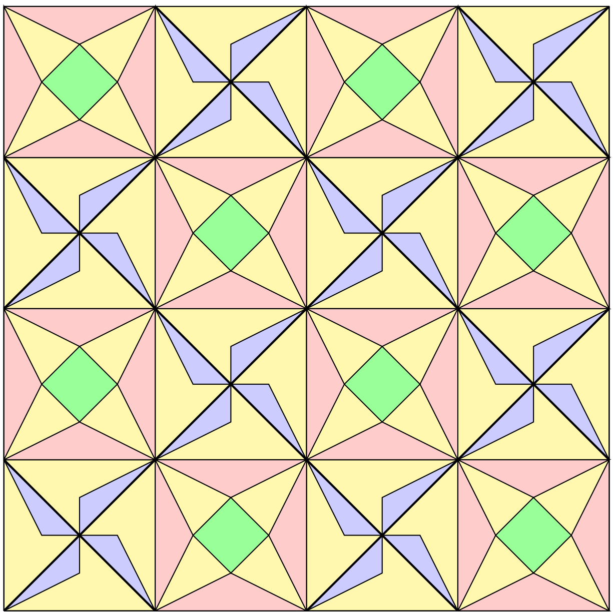


You can see that whatever shape we start with, we get four copies of it positioned rotated around the centre of the square, so we have a centre of rotation of order 4 there.
If we then look at two adjacent squares, you can see that if we rotate aroudn the middle of the edge between them, the two squares will be sent to each other, with the patterns matching perfectly:



Solution to Investigation 2.28 on page 61

There are actually two common ways to do this: you can either, like in the picture above, take any square tile with 4-way rotational symmetry and no reflective symmetry and make an infinite grid of it, or you can take two different tiles like that and make an infinite checkerboard pattern of them. These are actually the same - see if you can see why in the example below.



Solution to Investigation 2.29 on page 61

1. As before, we can only have a hexagonal lattice.

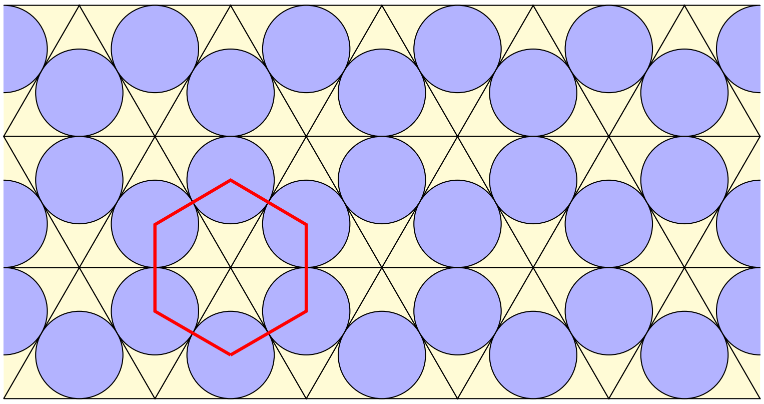
2. Just like in Investigation 2.24, we must have a centre of order 3 in the centre of each equilateral triangle of our lattice, and a centre of order 2 on each edge of each triangle, and we can't have any others.

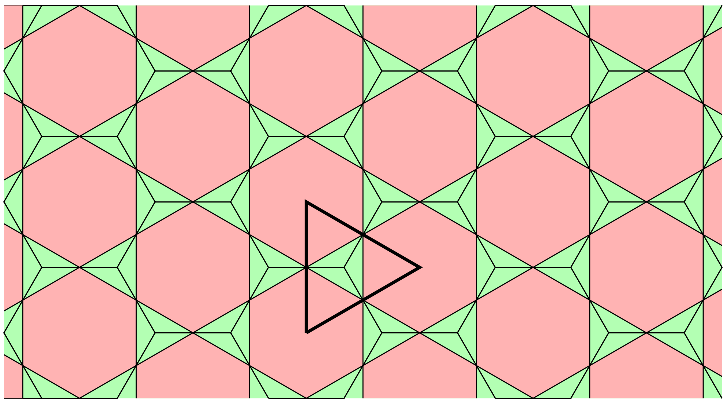
3. Any reflection has to send all of those rotation centres to rotation centres of the same order. That means that they have to either be along the edges of the triangles (through the centres of order 6 and 2), or through the centres of the triangles.

4. We have to have all of them: first, suppose we have one reflection through the centre of a triangles. Applying our rotations and translations give all of the other reflections through the centres. Similarly, if we have one of the edge reflections, then our rotations and translations give all of the others. To show that one forces the other, notice that rotating by 60° then reflecting in one of the edges gives a reflection through one of the centres (because it rotates the line of reflection by 30°), and similarly rotating by 60° then reflecting through one of the centres gives a reflection in an edge.

5. Of course, we have all of the glide reflections in the edges of the triangles given by just combining the reflections in those edges with parallel translations. As is often the case, we're also going to need to have extra glide reflections half way in between each. First, notice that these are the only possible ones (by arguments we've seen a lot, if we had any others, we'd get new translations by combining them with the closest reflection if parallel to them, or, new rotations by combining them with existing reflections if not). Second, they all have to exist: we can get any one by combining one of our reflections with one of our shortest translations that is neither parallel nor perpendicular.

6. There are two main options: it's either a tiling of symmetric triangles, or of symmetric hexagons. As before, I've marked the other shape:





Appendix B

Proofs of Results Used Without Proof

Chapter 1

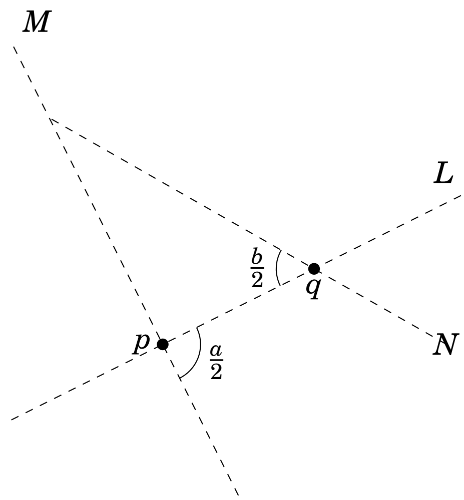
Proof of Theorem 1.2 on page 8

$\sqrt{2}$ cannot be written as a fraction.

We'll tackle this one by what's called **contradiction**: we'll assume it's not true, and prove that something impossible happens, so it must be true.
So, if it's not true, then $\sqrt{2}$ must be able to be written as a fraction. Let's call the numerator of that fraction b and the denominator c, so that $\sqrt{2} = \frac{b}{c}$.
There will be lots of ways to write $\sqrt{2}$ as a fraction (we also have $\sqrt{2} = \frac{2b}{2c} = \frac{3b}{3c} = \cdots$), so to make sure we don't cause any problems, we'll pick the one that makes b (and so also c as small as possible).
But now, let's multiply both sides of that by c, so that $b = \sqrt{2}c$. Looking at that, I can see that $\sqrt{2}$ is the hardest part to think about, so we'll get rid of it by squaring both sides to get $b^2 = 2 \times c^2$.
That tells us that b^2 is even: we just wrote it as 2 multiplied by a whole number. But that means that b is even (because multiplying two odd numbers together gives an odd number). Since b is even, we can write it as 2 multiplied by a whole number. Let's say $b = 2 \times d$, where d is that whole number.
But now we can replace b with $2 \times d$ in the equation above, since they're the same, giving $(2 \times d)^2 = 2 \times c^2$. If we expand out those brackets, that gives us $4 \times d^2 = 2 \times c^2$, and we can divide both sides by 2 to get $2 \times d^2 = c^2$. But just like before, that means that c^2 is even, so c is even, so we can write $c = 2e$ with e a whole number. But then $\sqrt{2} = \frac{b}{c} = \frac{2d}{2e} = \frac{d}{e}$, but we just wrote $\sqrt{2}$ as a fraction with the numbers smaller than b and c, which we said was impossible right at the start.
Now we've found something impossible, our assumption at the start must be false, which is exactly what we were trying to prove.

We shall now work out the details of 1.5 on page 10

Proof



We have this picture:
We've already shown that this is a rotation around the point where M and N cross by twice the angle between them. What actually need to do is work out where the lines cross and at what angle.
First, though, there's one slight issue: I've drawn my picture so that the lines cross above L, but they could just as easily cross below L, and this will change things slightly: we're now going to move our known angles inside the triangle, and one will end up staying the same, the other will be subtracted from 180°, and which one gets changed will be different depending on where the lines cross. It won't end up making much difference though, so I'll just do the version I've drawn, and leave the other one up to you to try.
So, let's move the angle $\frac{a}{2}$ inside the triangle. If we just focus on that bit of the diagram, we've got two angles on a straight line where we know one of them is $\frac{a}{2}$, so the other must be $180° - \frac{a}{2}$. Then, we've got two of the three angles inside the triangles as $180° - \frac{a}{2}$ and $\frac{b}{2}$, so the third angle must be $180° - (180° - \frac{a}{2}) - \frac{b}{2} = \frac{a-b}{2}$. Notice that this is exactly the angle between the two lines we're reflecting in, so our rotation is by an angle of $a - b$ (if the lines crossed on the other side, we'd instead get $b - a$: notice that this happens exactly when $b > a$, so we just always get whichever of $a - b$ and $b - a$ is positive: we'll call this $|a - b|$).
So now we know all three angles of our triangle, as well as one of the sides (we know where p and q are, so we know how long the line between them is - for now, we'll call that length x), so we should be able to figure out the other sides as well. We'll use a thing called the **sine rule** for this: see Theorem B for the details.
This tells us that, if we call the length of the line from p to our new point y

and the length of the line from q to our new point z, we have

$$y = \frac{x \sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$$

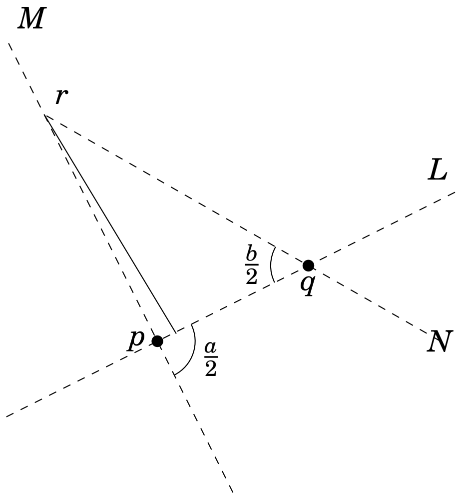
and

$$z = \frac{x \sin\left(180° - \frac{a}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$$

We can simplify that last one slightly: $\sin(180° - \theta) = \sin(\theta)$ for any angle θ, so we can drop the "180°−" part:

$$z = \frac{x \sin\left(\frac{a}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$$

Now, let's be a little more careful and work out exactly where M and N cross, which I'll call r: first, let's add a line from r to L at right angles.



This gives us two right-angled triangles, but we only really need one, so I'll pick the one with q in it. In this triangle, we know that the angle at q is $\frac{b}{2}$, and the length from q to r is z, so the length of the bit of L that's part of this triangle is $z \cos\left(\frac{b}{2}\right)$ and the length of the other shorter side is $z \sin\left(\frac{b}{2}\right)$.

That's actually everything we need to know to work out exactly where r is: first, we can find where it is in the "L" dimension: it's at

$$q - z\sin\left(\frac{b}{2}\right)\left(\frac{1}{x}(p-q)\right) = q - \frac{\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}(p-q),$$

and secondly, we can find how far away from L it is: it's $z\sin\left(\frac{b}{2}\right)$ away.

Now, let's just abbreviate some things: let's define $\operatorname{Par}(a,b) = \dfrac{\sin\left(\frac{a}{2}\right)\cos\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$ and $\operatorname{Per}(a,b) = \dfrac{\sin\left(\frac{a}{2}\right)\sin\left(\frac{b}{2}\right)}{\sin\left(\frac{a-b}{2}\right)}$. I'm also going to define, for any vector $v = (g,h)$, $v^\perp := (-h,g)$ (the funny upside-down T thing is a "perpendicular" sign - feel free to check that the line from $(0,0)$ to $v^\perp$ is actually at right angles to the line from $(0,0)$ to v, which is where the name comes from).

This then makes our formulas look nicer: we now know that $r = q - \operatorname{Par}(a,b)(p-q) + \operatorname{Per}(a,b)(p-q)^\perp$ (if the crossing is on the other side, you'll get a $-$ sign where I have a $+$), but since $\sin(-x) = -\sin(x)$, that's just the same as swapping whether we're using $a-b$ or $b-a$.

Proof of Theorem 1.5 on page 21

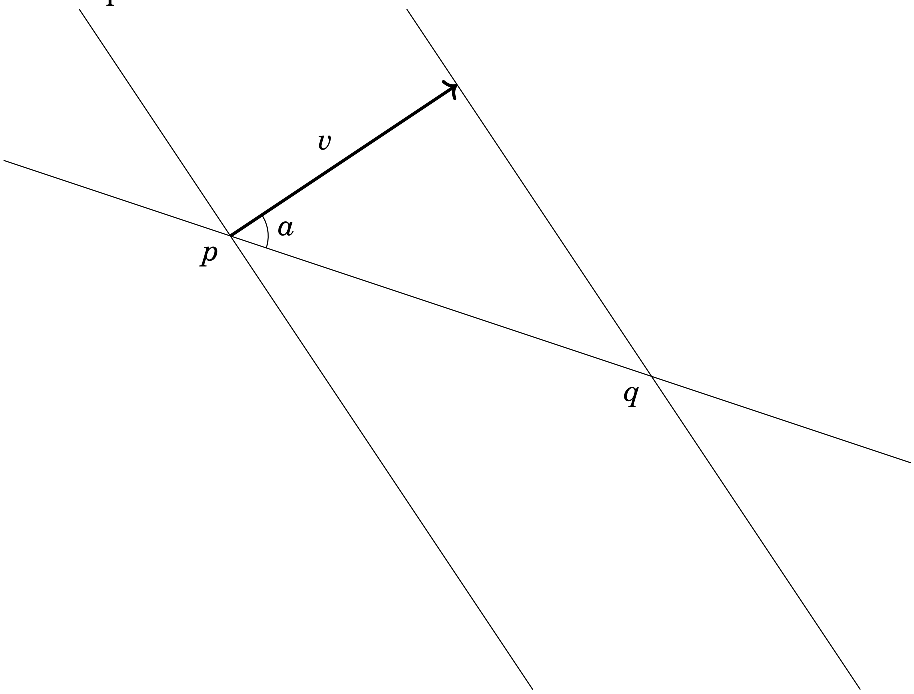
Take any point: say it's (a,b). Then our first translation moves it to $(a+x, b+y)$, and the second moves that to $(a+x+z, b+y+w)$, which is exactly where translating by $x+z$ horizontally and $y+w$ vertically would move it to. Since that works for every point, they must be the same.

Theorem B.1

Rotating by an angle a around a point p then translating by $v=(x,y)$ is the same as rotating by a around q, where q is the point at a distance of $\frac{|v|}{\sin(a)}$ from p at an angle of a from v.

Proof

Everything except the calculation of q is given on page 1.6. To find q, we'll draw a picture:

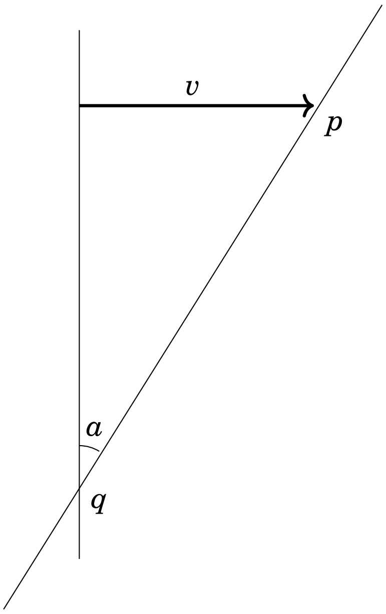


Since the two parallel mirror lines are both at right angles to v, we have a right-angled triangle. Applying the definition of cosine gives the result.

Solution to Investigation 1.5 on page 27

Write the translation as two reflections in lines perpendicular to v, with the first $-v$ away from p, the second through p. Write the rotation as two reflections through p, with the first perpendicular to v, the second at an angle of $a/2$ to the first. The two reflections through p perpendicular to v cancel, leaving two reflections in lines at an angle of $a/2$ to each other, which give a rotation by a.

To find the point we rotate around, let's draw a picture:



This is a right-angled triangle, so the length from p to q is $\frac{|v|}{\sin(a)}$, and the angle from p to q is $a + 180°$, which tells us exactly where q is.

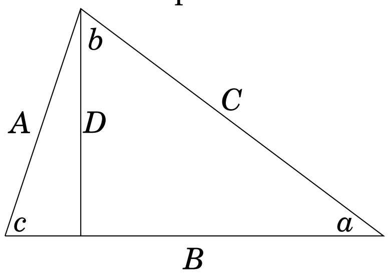
Theorem B.1: The Sine Rule

For any triangle with angles a, b, and c opposite sides of length A, B, and C, we have

$$\frac{\sin(a)}{A} = \frac{\sin(b)}{B} = \frac{\sin(c)}{C}.$$

Proof

Let's draw a picture:



You'll notice that we've added a line splitting our triangle into two right-angled triangles. Now, let's use the definition of sine in the right-hand triangle: that gives us

$$\sin(a) = \frac{D}{C}.$$

Doing the same on the left-hand triangle gives us

$$\sin(c) = \frac{D}{A}.$$

Rearranging those two gives $C\sin(a) = D = A\sin(c)$, and rearranging those gives us

$$\frac{\sin(a)}{A} = \frac{\sin(c)}{C}.$$

The same argument with the triangle rotated will give

$$\frac{\sin(b)}{B} = \frac{\sin(a)}{A},$$

so we're done.