EE518 Digital Signal Processing **University of Washington**
Autumn 2001 **Dept. of Electrical Engineering**

Lecture 9: Upsampling and Downsampling

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9.1 Review

$$x_c(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_c(j\Omega)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t-nT) \stackrel{\mathcal{F}}{\longleftrightarrow} X_s(j\Omega) = \frac{1}{2\pi}X_c(j\Omega) * S(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s)) \quad (\Omega_s = \frac{2\pi}{T})$$

$x_c(t)$ is recoverable from $x_s(t)$ if $2\Omega_N < \Omega_s$, otherwise aliasing occurs. Also

$$\begin{aligned}
X_s(j\Omega) &= \int_{-\infty}^{\infty} x_s(t) e^{-j\Omega t} dt \\
&= \sum_{n=-\infty}^{\infty} x_c(nT) e^{-j\Omega T n} \\
&= \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega T n} \\
&= X(e^{-j\omega})|_{\omega = \Omega T = \Omega / f_s} \qquad f_s = \text{ sampling frequency}
\end{aligned}$$

so ω is a sampling frequency normalized frequency variable.

Also $n = t/T$ is "like" a time normalized variable.

Therefore

$$X(e^{-j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\omega/T - 2\pi k/T))$$

No aliasing if $2\Omega_N < \Omega_s$ or $2f_N < f_s$.

Time Domain View of Aliasing

Consider the set of frequencies

$$A_{\Omega_0} = \{\Omega : \cos(\Omega_0 nT) = \cos(\Omega nT), \forall n, 0 \le \Omega_0 \le \frac{\pi}{T}\}$$

where A_{Ω_0} is the set of all frequencies of sinusoids that will be an alias with $\cos(\Omega_0 nT)$ when sampled with sampling period T.

Theorem 9.1.

$$\Omega \in A_{\Omega_0} \qquad \textit{but } |\Omega| \notin \{0 \le \Omega_0 \le \frac{\pi}{T}\}$$

unless

$$|\Omega| = \Omega_0$$

Proof. See text for frequency domain proof.

Continuous-Time Processing of Discrete-Time Signals

This is not often done and is included here for completeness.

Basic idea as shown in Fig. 9.1

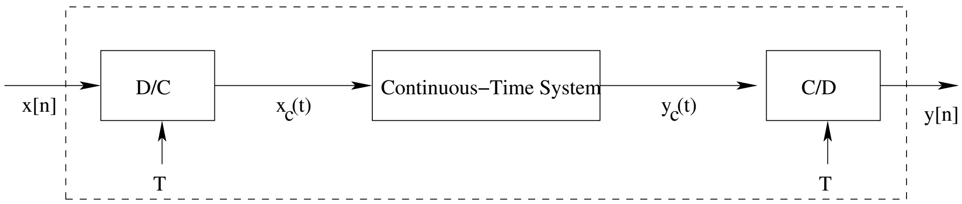

Figure 9.1: Continuous-time processing of discrete-time signals.

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t-nT) = \sum_{n=-\infty}^{\infty} x[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$$(h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} \overset{\mathcal{F}}{\longleftrightarrow} H_r(j\Omega) \text{ is lowpass filter with cutoff frequency at } \pi/T)$$

$$y_c(t) = \sum_{n=-\infty}^{\infty} y[n]\frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

So

$$X_c(j\Omega) = TX(e^{j\omega}) \qquad |\Omega| < \frac{\pi}{T}$$

$$Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega) \qquad |\Omega| < \frac{\pi}{T}$$

$$Y(e^{j\omega}) = \frac{1}{T}Y_c(j\frac{\omega}{T}) = \frac{1}{T}H_c(j\omega/T)TX(e^{j\omega}) = H_c(j\omega/T)X(e^{j\omega}) \qquad |\omega| < \pi$$

i.e.,

$$H(e^{j\omega}) = H_c(j\omega/T) \qquad |\omega| < \pi$$

or

$$H_c(j\Omega) = H(e^{j\Omega T}) \qquad |\Omega| < \frac{\pi}{T} \tag{9.1}$$

Ex Non-integer delay.

$$H(e^{j\omega}) = e^{-j\omega\Delta}$$

When Δ is an integer, $y[n] = x[n-\Delta]$, which is an ideal integer delay.

When Δ is not an integer, this should shift by a non-integer delay (i.e., just a linear phase adjustment.)

Changing the sampling rate in discrete time

$$x[n] = x_c(nT)$$

to

$$x'[n] = x_c(nT')$$

Note, if $\Omega_s > 2\Omega_N$, then this should be doable. All information is contained in $x[n]$, this is just re-sampling that same information.

9.2 Reduction (Downsampling)

$$x_d[n] = x[nM] = x_c(nMT)$$

Notation:

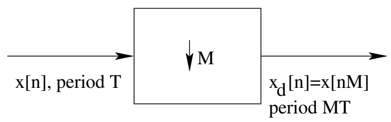

Figure 9.2: Downsampling notation.

$$T' = MT$$

If $X_c(j\Omega)$ is bandlimited, i.e., $X_c(j\Omega) = 0$ for $\Omega \geq \Omega_N$, $x_d[n]$ is exact if

$$\Omega_s' = \frac{2\pi}{T'} > 2\Omega_N$$

So the original sampling rate must be

$$\Omega_s = \frac{2\pi}{T} > M \cdot 2\Omega_N$$

Note

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T})) \tag{9.2}$$

So

$$X_d(e^{j\omega}) = \frac{1}{TM} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{MT} - \frac{2\pi k}{MT}))$$

Change the above to two sum using

$$r = i + kM \qquad 0 \leq i \leq M-1 \quad \forall k$$

then

$$\begin{aligned} X_d(e^{j\omega}) =& \frac{1}{M} \sum_{i=0}^{M-1} \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{MT} - \frac{2\pi k}{MT} - \frac{2\pi i}{MT})) \\ =& \frac{1}{M} \sum_{i=0}^{M-1} X(e^{j(\frac{\omega}{M} - \frac{2\pi i}{M})}) \end{aligned} \tag{9.3}$$

Compare with Eq. (9.3) and Eq. (9.2)

I. shifted copies of both version

Eq. (9.2): infinite copies

Eq. (9.3): only M shifted copies

II. scale factor $1/T$ versus $1/M$

III.
Eq. (9.2): normalized frequency $\Omega = \omega/T$
Eq. (9.3): re-normalized frequency $\omega' = \omega/M$ (so stretches out the ω-axis)

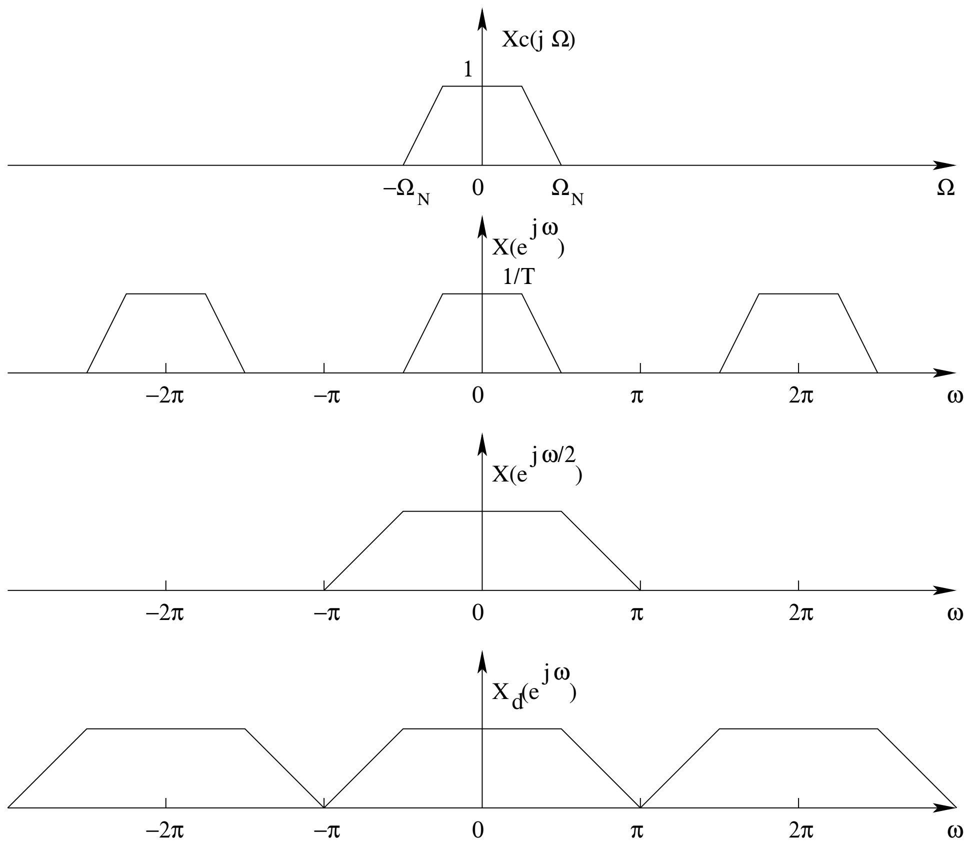

Figure 9.3: Samples at frequency $\Omega_s = \frac{2\pi}{T} = 4\Omega_N$. $T = \frac{\pi}{2\Omega_N}$ $\omega = \Omega T = \frac{\Omega}{2}\frac{\pi}{\Omega_N}$. $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T}))$.$\Omega = \Omega_N$ when $\omega = \pi/2$. Now downsample at $M = 2$. $X_d(e^{j\omega}) = \frac{1}{2}X(e^{j\omega/2}) + \frac{1}{2}X(e^{j(\omega-2\pi)/2})$

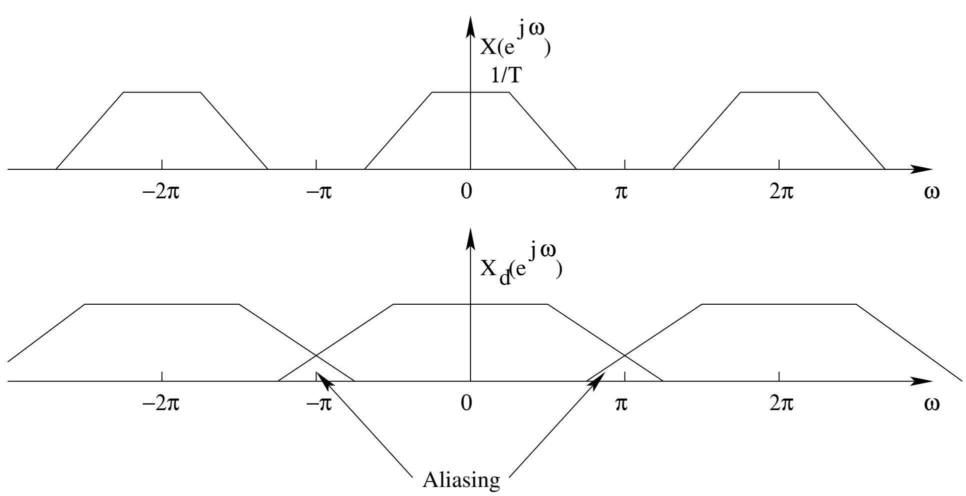

Figure 9.4: $X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T}))$. Now downsample at $M = 2$. Aliasing occurs.

Ex As illustrated in Fig. 9.3 and Fig. 9.4. In Fig. 9.4, aliasing occurred.

What if aliasing occurs? Suppose $\Omega_s = \frac{2\pi}{T} = 3\Omega_N$, $\Omega = \Omega_N$ occurs when $\omega = 2\pi/3$.

So to avoid aliasing, need

$$\Omega_s' = \frac{2\pi}{T'} > 2\Omega_N \tag{9.4}$$

or

$$M < \frac{\pi}{T\Omega_N} \tag{9.5}$$

or

$$\omega_N M < \pi \tag{9.6}$$

where ω_N is discrete time analog of Nyquist frequency.

We can lowpass filter the signal at first to assume no aliasing, as shown in Fig. 9.5. This is called a decimator system.

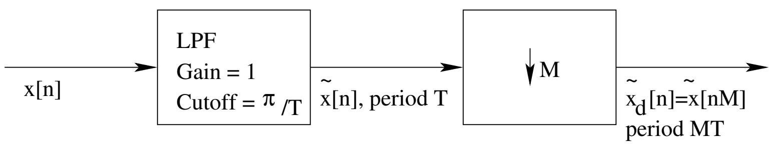

Figure 9.5: A decimator system: lowpass filtering before downsampling

Note $\tilde{x}[n]$ is bandlimited within π/M.

9.3 Upsampling

Obtain $x_i[n] = x_c(nT')$ from $x[n] = x_c(nT)$ when $T' < T$, i.e. $T' = T/L$.

Question: Is this possible?

Note:

$$x_i[n] = x[n/L] = x_c(nT/L)$$

when $n = kL$ and k is an integer. How to do this in discrete time?

$$x_e[n] = \begin{cases} x[n/L] & n = kL \\ 0 & \text{else} \end{cases} = \sum_{k=-infty}^{\infty} x[k]\delta[n-kL] \tag{9.7}$$

Notation for upsampling is shown in Fig. 9.6 Note there are $L-1$ zeros between each sample.

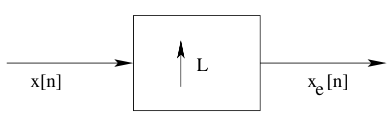

Figure 9.6: Upsampling

$$X_e(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-infty}^{\infty} x[k]\delta[n-kL] \right) e^{-j\omega n} = \sum_{k=-infty}^{\infty} x[k]e^{-j\omega kL} = X(e^{j\omega L}) \tag{9.8}$$

So this scales (shrinking here) frequency axis by a factor of L, since $\omega' = \omega L$, or $\omega = \Omega T'$ or $T' = T/L$.

If we then lowpass filter the output $x_e[n]$, we get general expander, as shown in Fig. 9.7. Note: need gain L in the

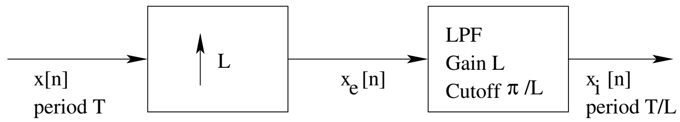

Figure 9.7: An expander.

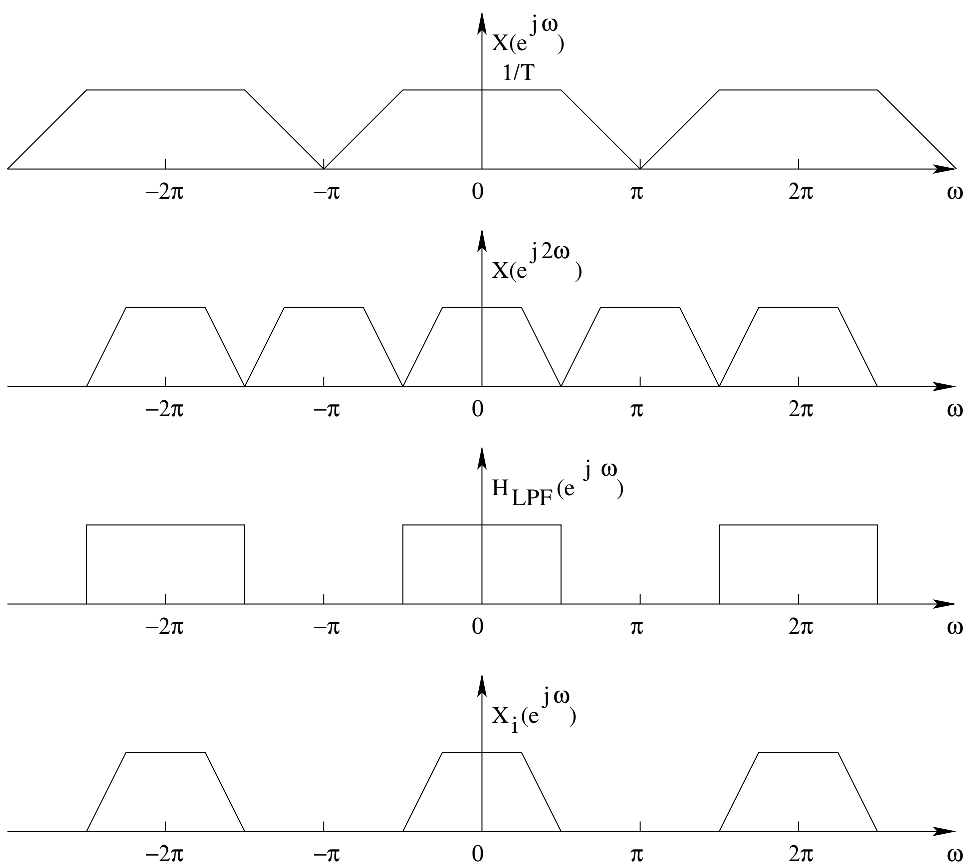

Figure 9.8: Upsampling of $x[n]$ at $L = 2$.

lowpass filter if sampling period had been $T' = T/L$. Originally, gain would be L/T so need to "correct" for this to be consistent.

Ex (frequency domain) Shown in Fig. 9.8 Note: using more samples to encode the same amount of information, we would expect the spectrum to have some "gapa" where we could place other signals. Note the spectrum of $H_i(e^{j\omega})$. We are using a system with more capacity than necessary to encode $X(e^{j\omega})$.

Also if $T' = T/L$, and T matches Nyquist criterion, then $\Omega_s > L \cdot 2\Omega_N$, again higher than necessary.

Why might we do this? This allows simpler LPF design "oversampling". We only need a "cheap" LPF to get back to continuous time.

9.4 Rational Sample Rate Change

Consider the system shown in Fig. 9.9 versus the system shown in Fig. 9.10, the output signal both have a sampling

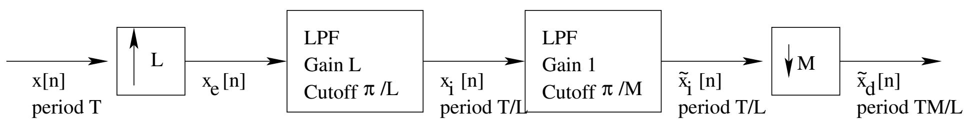

Figure 9.9: Rational sample rate change 1.

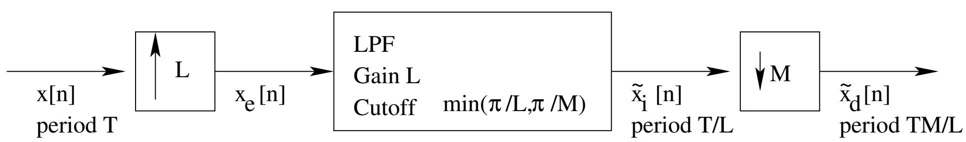

Figure 9.10: Rational sample rate change 2.

period of $T' = TM/L$. Hence the sampling rate can have any rational change.

Ex: $M = 22$ and $L = 7$ produces a change in rate by about 2π.

Note: we might want to reduce sample rate to data storage, etc.