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Multirate systems include more than one sample rate

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Multirate systems include more than one sample rate

Why bother?:

- May need to change the sample rate
  e.g. Audio sample rates include 32, 44.1, 48, 96 kHz

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  e.g. Audio sample rates include 32, 44.1, 48, 96 kHz
- Can relax analog or digital filter requirements
  e.g. Audio DAC increases sample rate so that the reconstruction filter can have a more gradual cutoff

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- May need to change the sample rate
  e.g. Audio sample rates include 32, 44.1, 48, 96 kHz
- Can relax analog or digital filter requirements
  e.g. Audio DAC increases sample rate so that the reconstruction filter can have a more gradual cutoff
- Reduce computational complexity
  FIR filter length $\propto \frac{f_s}{\Delta f}$ where Δf is width of transition band
  Lower $f_s \Rightarrow$ shorter filter + fewer samples $\Rightarrow$computation $\propto f_s^2$

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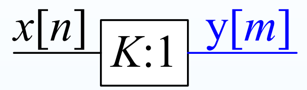
Downsample



$$y[m] = x[Km]$$

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Downsample $x[n] \rightarrow \boxed{K{:}1} \rightarrow y[m]$ $\quad y[m] = x[Km]$

Upsample $u[m] \rightarrow \boxed{1{:}K} \rightarrow v[n]$ $\quad v[n] = \begin{cases} u\left[\frac{n}{K}\right] & K \mid n \\ 0 & \text{else} \end{cases}$

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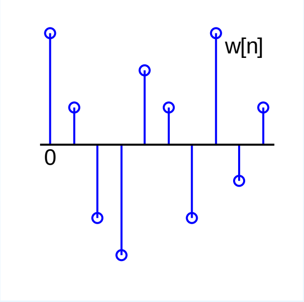
Example:

Downsample by 3 then upsample by 4

$w[n] \to \boxed{3:1} \to x[m] \to \boxed{1:4} \to y[r]$



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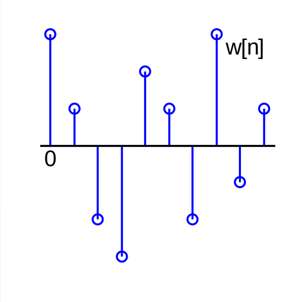

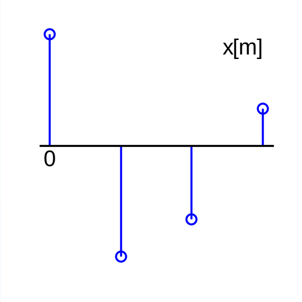

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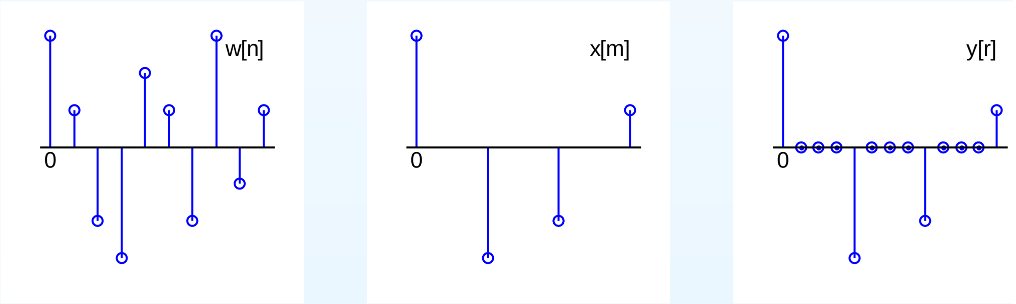

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Building blocks

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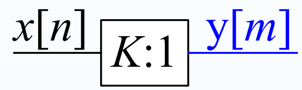
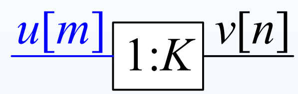


$$y[m] = x[Km]$$

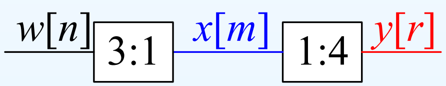
Upsample

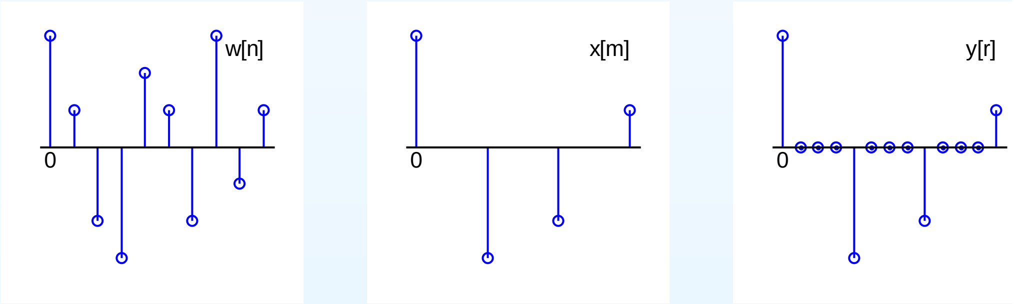


$$v[n] = \begin{cases} u\left[\frac{n}{K}\right] & K \mid n \\ 0 & \text{else} \end{cases}$$

Example:

Downsample by 3 then upsample by 4





- We use different index variables (n, m, r) for different sample rates

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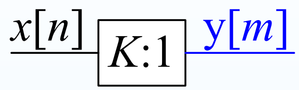
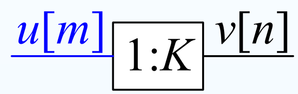


$$y[m] = x[Km]$$

Upsample



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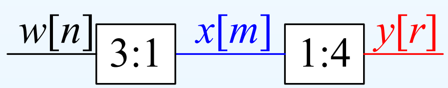

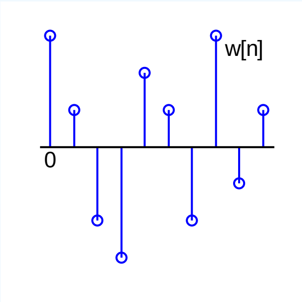

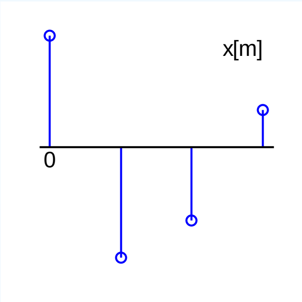

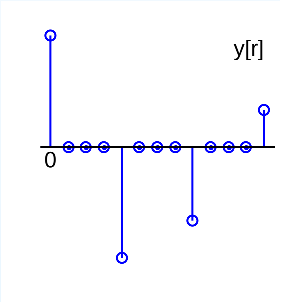

- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)

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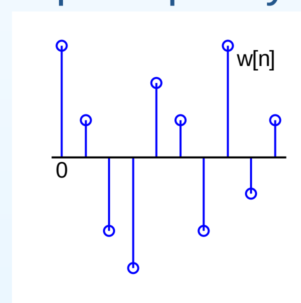

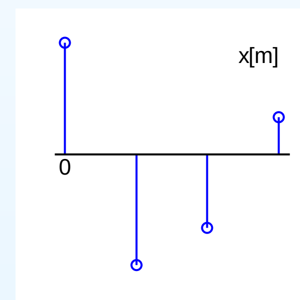

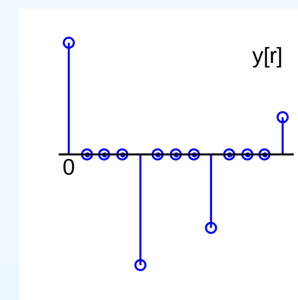

- We use different index variables (n, m, r) for different sample rates
- Use different colours for signals at different rates (sometimes)
- Synchronization: all signals have a sample at $n = 0$.

Resampling Cascades

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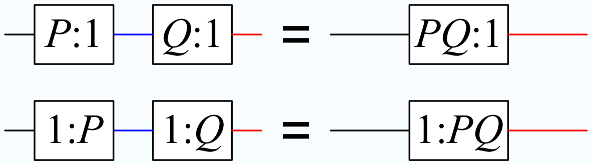
Successive downsamplers or upsamplers can be combined



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Resampling Cascades

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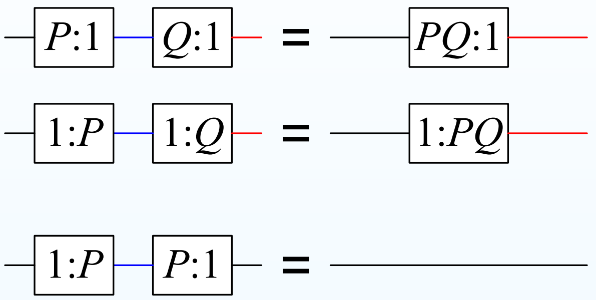
Upsampling can be exactly inverted



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$$-\boxed{1{:}P}-\boxed{1{:}Q}- \;=\; -\boxed{1{:}PQ}-$$

Upsampling can be exactly inverted

$$-\boxed{1{:}P}-\boxed{P{:}1}- \;=\; ———$$

Downsampling destroys information permanently $\Rightarrow$ uninvertible

$$-\boxed{P{:}1}-\boxed{1{:}P}- \;\neq\; ———$$

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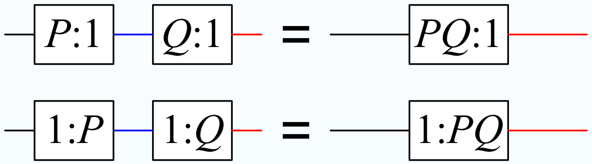

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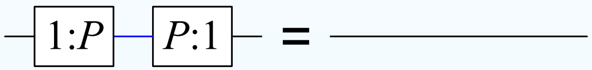
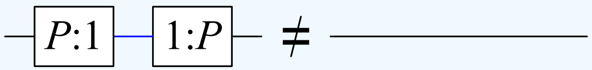
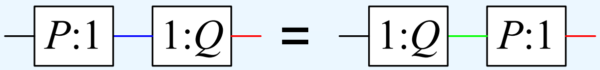


Downsampling destroys information permanently $\Rightarrow$ uninvertible



Resampling can be interchanged iff P and Q are coprime (surprising!)



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Upsampling can be exactly inverted

$$\boxed{1{:}P}\,\boxed{P{:}1} = \text{———}$$

Downsampling destroys information permanently $\Rightarrow$ uninvertible

$$\boxed{P{:}1}\,\boxed{1{:}P} \neq \text{———}$$

Resampling can be interchanged iff P and Q are coprime (surprising!)

$$\boxed{P{:}1}\,\boxed{1{:}Q} = \boxed{1{:}Q}\,\boxed{P{:}1}$$

Proof: Left side: $y[n] = x\left[\frac{P}{Q}n\right]$ if $Q \mid n$ else $y[n] = 0$.

[Note: $a \mid b$ means "b is a multiple of a"]

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Upsampling can be exactly inverted

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Downsampling destroys information permanently $\Rightarrow$ uninvertible

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Right side: $y[n] = x\left[\frac{P}{Q}n\right]$ if $Q \mid Pn$.

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Right side: $y[n] = x\left[\frac{P}{Q}n\right]$ if $Q \mid Pn$.

But $\{Q \mid Pn \;\Rightarrow\; Q \mid n\}$ iff P and Q are coprime.

[Note: $a \mid b$ means "b is a multiple of a"]

Noble Identities

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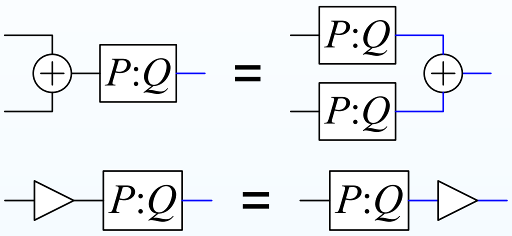
Resamplers commute with addition and multiplication



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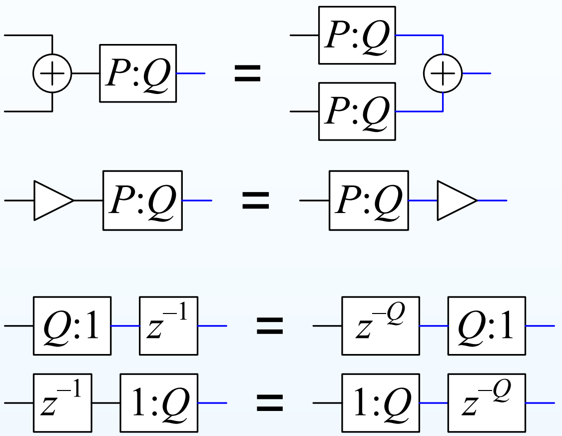
Resamplers commute with addition and multiplication

Delays must be multiplied by the resampling ratio



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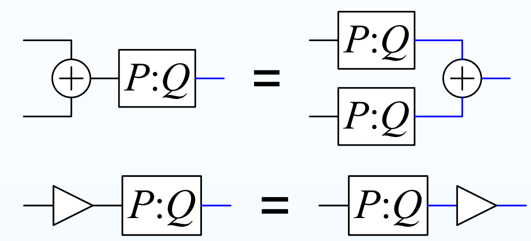

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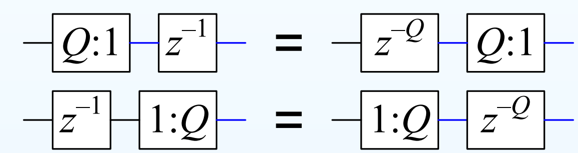
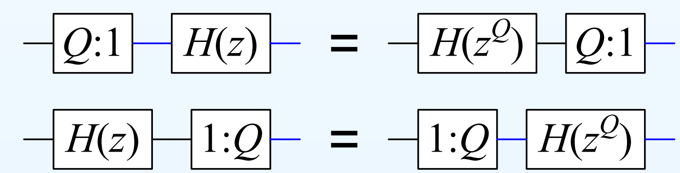


Noble identities:
Exchange resamplers and filters



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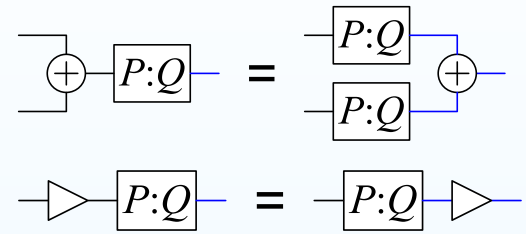
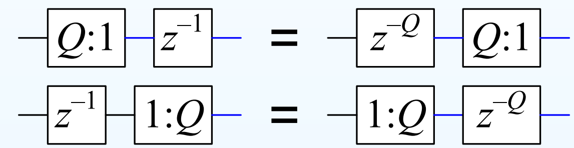
Noble Identities

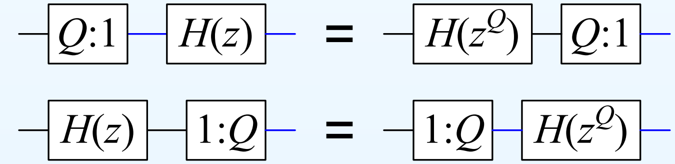
Resamplers commute with addition and multiplication



Delays must be multiplied by the resampling ratio



Noble identities:
Exchange resamplers and filters



Example: $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots$

$$H(z^3) = h[0] + h[1]z^{-3} + h[2]z^{-6} + \cdots$$

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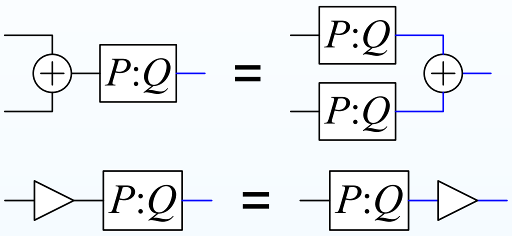

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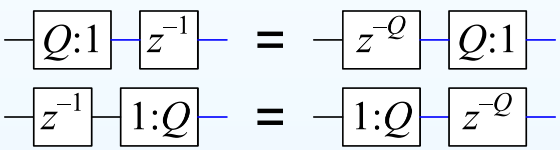

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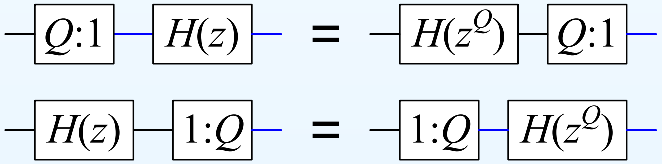
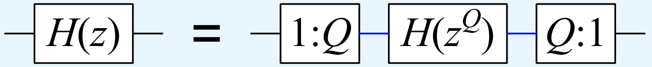


Corrollary



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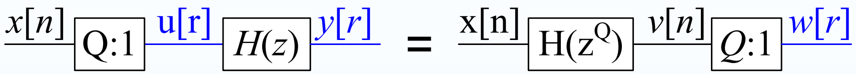
Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.



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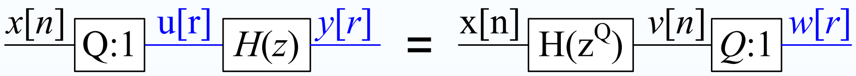

Assume that $h[r]$ is of length $M+1$ so that $h_Q[n]$ is of length $QM+1$.

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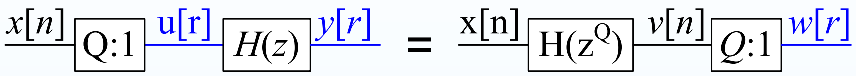

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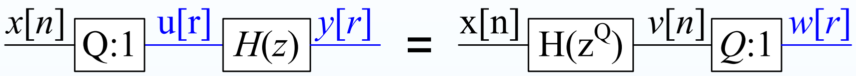

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$$w[r]=v[Qr]=\sum_{s=0}^{QM} h_Q[s]x[Qr-s]$$

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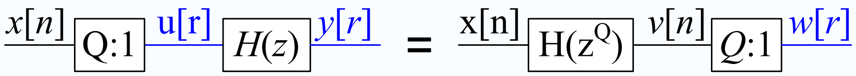

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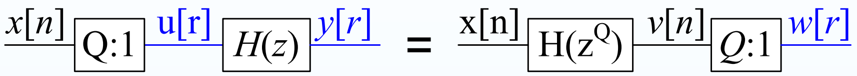

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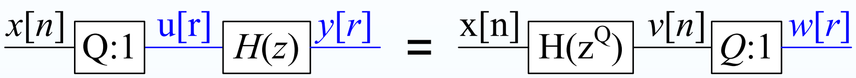

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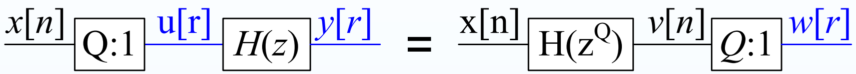

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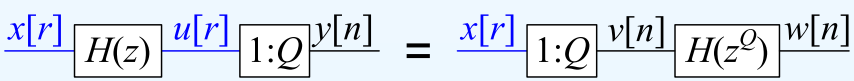

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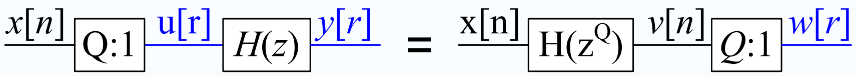

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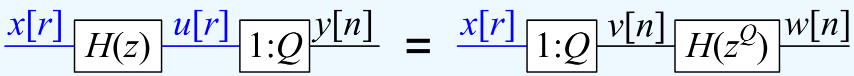

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$$w[n] = \sum_{s=0}^{QM} h_Q[s]v[n-s]$$

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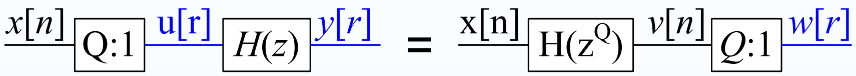

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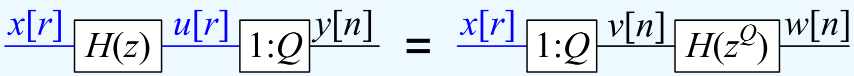

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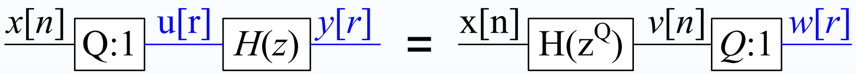

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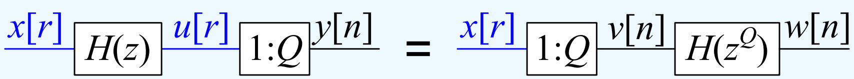

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If $Q \nmid n$, then $v[n-Qm] = 0 \; \forall m$ so $w[n] = 0 = y[n]$

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If $Q \nmid n$, then $v[n-Qm]=0\ \forall m$ so $w[n]=0=y[n]$

If $Q \mid n = Qr$, then $w[Qr] = \sum_{m=0}^{M} h[m]v[Qr-Qm]$

Noble Identities Proof

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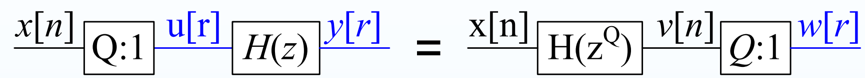
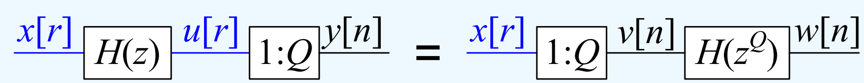
Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.



Assume that $h[r]$ is of length $M+1$ so that $h_Q[n]$ is of length $QM+1$.
We know that $h_Q[n] = 0$ except when $Q \mid n$ and that $h[r] = h_Q[Qr]$.

$$\begin{aligned} w[r] = v[Qr] &= \sum_{s=0}^{QM} h_Q[s]x[Qr-s] \\ &= \sum_{m=0}^{M} h_Q[Qm]x[Qr-Qm] = \sum_{m=0}^{M} h[m]x[Q(r-m)] \\ &= \sum_{m=0}^{M} h[m]u[r-m] = y[r] \quad ☺ \end{aligned}$$

Upsampled Noble Identity:



We know that $v[n] = 0$ except when $Q \mid n$ and that $v[Qr] = x[r]$.

$$\begin{aligned} w[n] &= \sum_{s=0}^{QM} h_Q[s]v[n-s] = \sum_{m=0}^{M} h_Q[Qm]v[n-Qm] \\ &= \sum_{m=0}^{M} h[m]v[n-Qm] \end{aligned}$$

If $Q \nmid n$, then $v[n-Qm] = 0 \; \forall m$ so $w[n] = 0 = y[n]$

If $Q \mid n = Qr$, then $w[Qr] = \sum_{m=0}^{M} h[m]v[Qr-Qm]$
$$= \sum_{m=0}^{M} h[m]x[r-m] = u[r]$$

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Noble Identities Proof

Define $h_Q[n]$ to be the impulse response of $H(z^Q)$.

$x[n] \rightarrow \boxed{Q{:}1} \xrightarrow{u[r]} \boxed{H(z)} \xrightarrow{y[r]} \quad = \quad x[n] \rightarrow \boxed{H(z^Q)} \xrightarrow{v[n]} \boxed{Q{:}1} \xrightarrow{w[r]}$

Assume that $h[r]$ is of length $M+1$ so that $h_Q[n]$ is of length $QM+1$.
We know that $h_Q[n]=0$ except when $Q \mid n$ and that $h[r]=h_Q[Qr]$.

$$\begin{aligned} w[r] = v[Qr] &= \textstyle\sum_{s=0}^{QM} h_Q[s]x[Qr-s] \\ &= \textstyle\sum_{m=0}^{M} h_Q[Qm]x[Qr-Qm] = \sum_{m=0}^{M} h[m]x[Q(r-m)] \\ &= \textstyle\sum_{m=0}^{M} h[m]u[r-m] = y[r] \quad ☺ \end{aligned}$$

Upsampled Noble Identity:

$x[r] \rightarrow \boxed{H(z)} \xrightarrow{u[r]} \boxed{1{:}Q} \xrightarrow{y[n]} \quad = \quad x[r] \rightarrow \boxed{1{:}Q} \xrightarrow{v[n]} \boxed{H(z^Q)} \xrightarrow{w[n]}$

We know that $v[n]=0$ except when $Q \mid n$ and that $v[Qr]=x[r]$.

$$\begin{aligned} w[n] &= \textstyle\sum_{s=0}^{QM} h_Q[s]v[n-s] = \sum_{m=0}^{M} h_Q[Qm]v[n-Qm] \\ &= \textstyle\sum_{m=0}^{M} h[m]v[n-Qm] \end{aligned}$$

If $Q \nmid n$, then $v[n-Qm]=0\ \forall m$ so $w[n]=0=y[n]$

If $Q \mid n = Qr$, then $w[Qr] = \sum_{m=0}^{M} h[m]v[Qr-Qm]$
$$= \textstyle\sum_{m=0}^{M} h[m]x[r-m] = u[r] = y[Qr] \quad ☺$$

Upsampled z-transform

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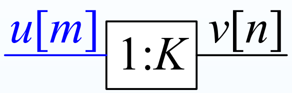
$$V(z) = \sum_n v[n] z^{-n}$$



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$$V(z) = \sum_n v[n] z^{-n} = \sum_{n:K|n} u[\tfrac{n}{K}] z^{-n}$$

$$u[m] \longrightarrow \boxed{1:K} \longrightarrow v[n]$$

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$$V(z) = \sum_n v[n]z^{-n} = \sum_{n:K|n} u[\tfrac{n}{K}]z^{-n}$$
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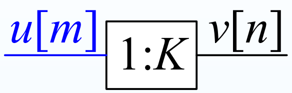

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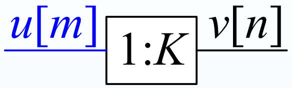
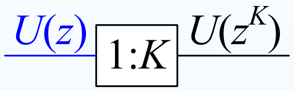
$$V(z) = \sum_n v[n] z^{-n} = \sum_{n:K|n} u[\tfrac{n}{K}] z^{-n}$$
$$= \sum_m u[m] z^{-Km} = U(z^K)$$





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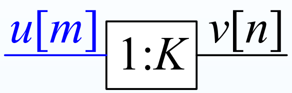

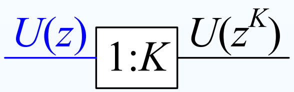

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

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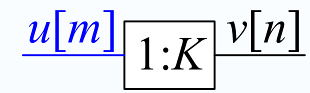

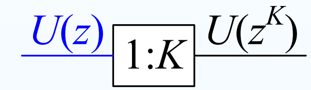

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

Upsampled z-transform

$$V(z) = \sum_n v[n] z^{-n} = \sum_{n:K|n} u[\frac{n}{K}] z^{-n}$$
$$= \sum_m u[m] z^{-Km} = U(z^K)$$

$u[m] \rightarrow \boxed{1{:}K} \rightarrow v[n]$

$U(z) \rightarrow \boxed{1{:}K} \rightarrow U(z^K)$

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

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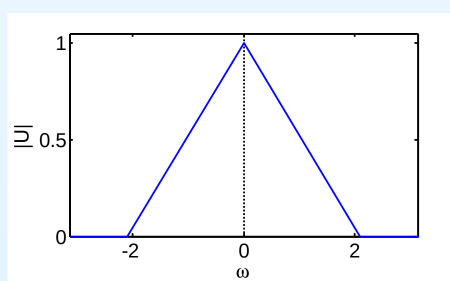
Example:



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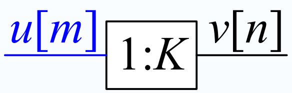

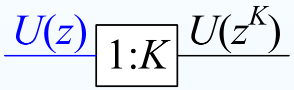

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

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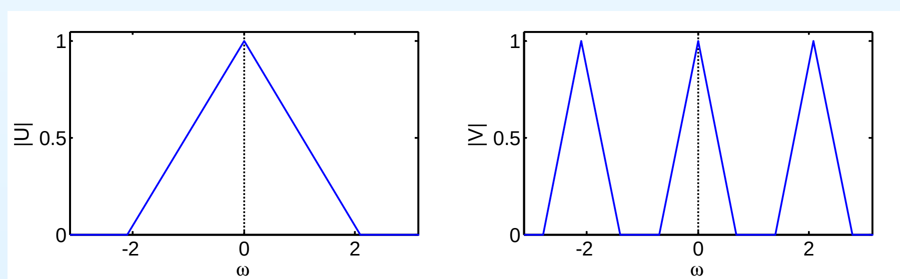
Example:

$K = 3$: three images of the original spectrum in all.



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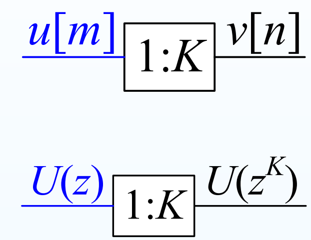

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

Total energy unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

Example:

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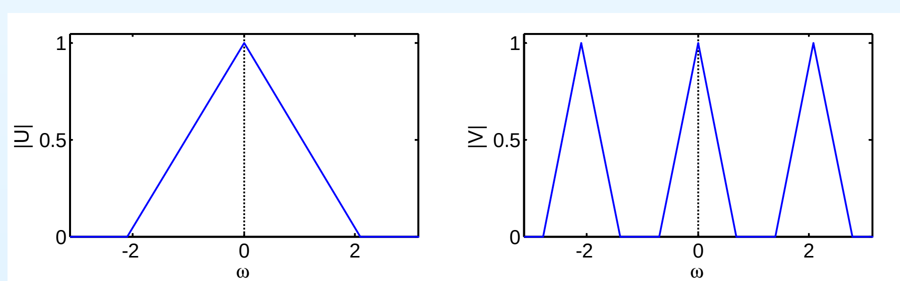

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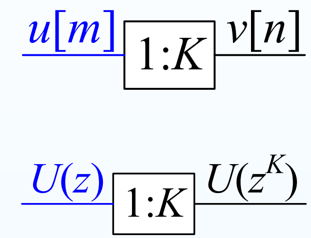


Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

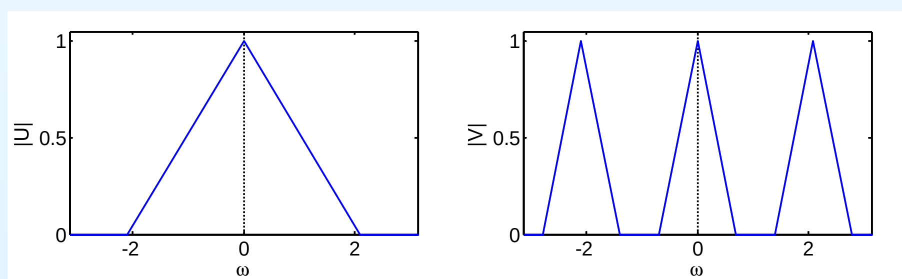
Spectrum is horizontally shrunk and replicated K times.

Total energy unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

Example:

$K = 3$: three images of the original spectrum in all.

Energy unchanged: $\frac{1}{2\pi} \int \left|U(e^{j\omega})\right|^2 d\omega = \frac{1}{2\pi} \int \left|V(e^{j\omega})\right|^2 d\omega$



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$u[m] \to \boxed{1\!:\!K} \to v[n]$

$U(z) \to \boxed{1\!:\!K} \to U(z^K)$

Spectrum: $V(e^{j\omega}) = U(e^{jK\omega})$

Spectrum is horizontally shrunk and replicated K times.

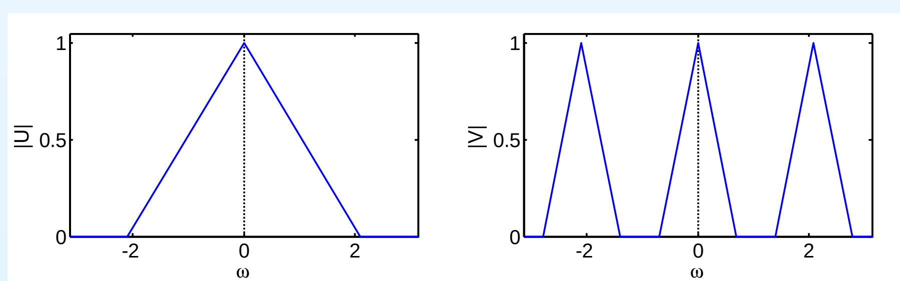
Total energy unchanged; power (= energy/sample) multiplied by $\frac{1}{K}$

Upsampling normally followed by a LP filter to remove images.

Example:

$K = 3$: three images of the original spectrum in all.

Energy unchanged: $\frac{1}{2\pi} \int \left| U(e^{j\omega}) \right|^2 d\omega = \frac{1}{2\pi} \int \left| V(e^{j\omega}) \right|^2 d\omega$



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Define $c_K[n] = \delta_{K|n}[n]$

$$x[n] \rightarrow \boxed{K:1} \xrightarrow{y[m]} \boxed{1:K} \rightarrow x_K[n]$$

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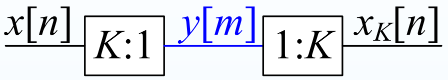
Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$



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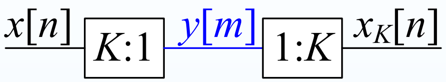

Define $c_K[n] = \delta_{K|n}[n] = \frac{1}{K}\sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}}$

Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases}$

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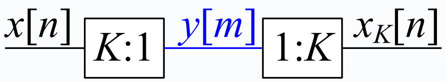

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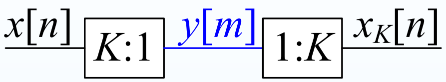

Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$X_K(z) = \sum_n x_K[n]z^{-n}$

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$$x[n] \longrightarrow \boxed{K{:}1} \longrightarrow y[m] \longrightarrow \boxed{1{:}K} \longrightarrow x_K[n]$$

Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$$X_K(z) = \sum_n x_K[n]z^{-n} = \frac{1}{K}\sum_n \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}} x[n]z^{-n}$$

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$$x[n] \to \boxed{K{:}1} \xrightarrow{y[m]} \boxed{1{:}K} \to x_K[n]$$

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Now define $x_K[n] = \begin{cases} x[n] & K \mid n \\ 0 & K \nmid n \end{cases} = c_K[n]x[n]$

$$X_K(z) = \sum_n x_K[n]z^{-n} = \frac{1}{K}\sum_n \sum_{k=0}^{K-1} e^{\frac{j2\pi kn}{K}} x[n]z^{-n}$$

$$= \frac{1}{K}\sum_{k=0}^{K-1}\sum_n x[n]\left(e^{\frac{-j2\pi k}{K}}z\right)^{-n}$$

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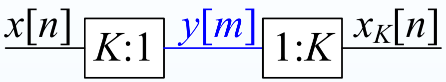
Downsampled z-transform



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From previous slide:

$$X_K(z) = Y(z^K)$$

Downsampled z-transform

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$$X_K(z) = Y(z^K)$$

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Downsampled z-transform

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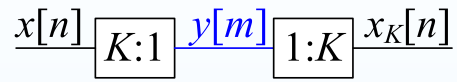

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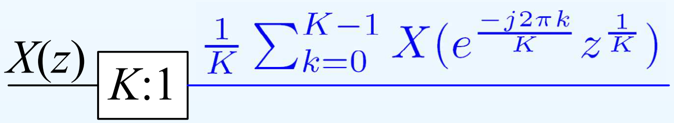

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Frequency Spectrum:

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Downsampled z-transform

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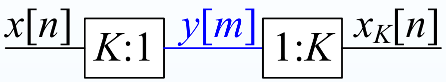

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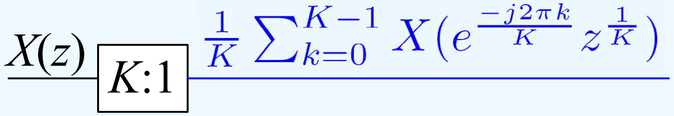

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Average of K aliased versions, each expanded in ω by a factor of K.

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Downsampled z-transform

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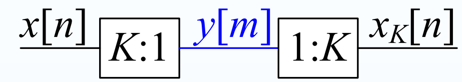

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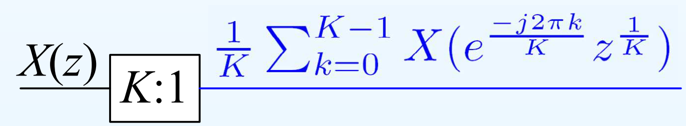

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Downsampling is normally preceded by a LP filter to prevent aliasing.

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Downsampled Spectrum

$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$

$$x[n] \rightarrow \boxed{K:1} \rightarrow y[m]$$

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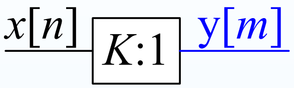
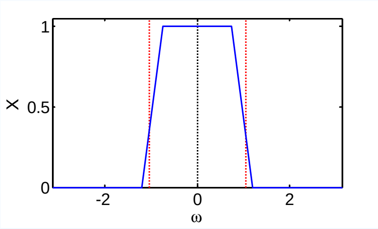


Example 1:

$K = 3$

Not quite limited to $\pm\frac{\pi}{K}$



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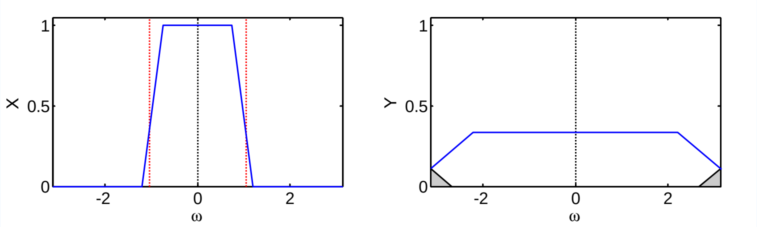

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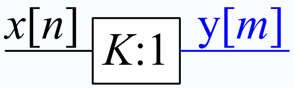

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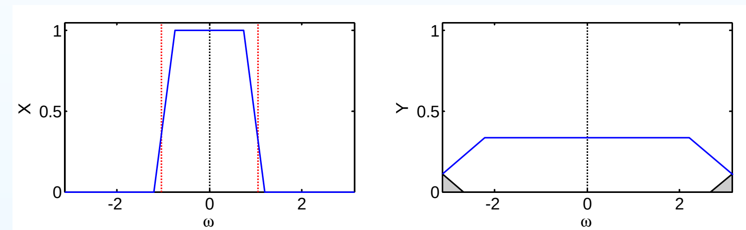

Energy decreases: $\frac{1}{2\pi}\int\left|Y(e^{j\omega})\right|^2 d\omega \approx \frac{1}{2\pi K}\int\left|X(e^{j\omega})\right|^2 d\omega$

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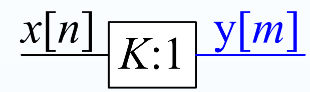

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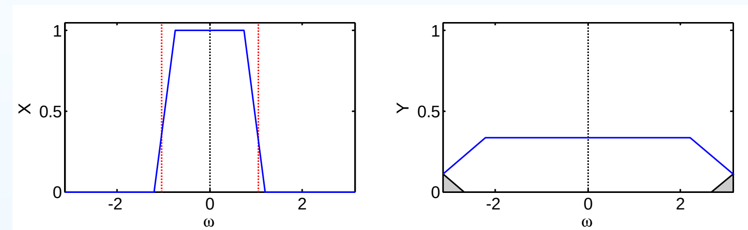

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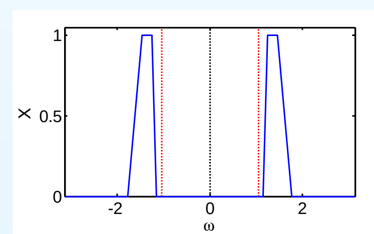
Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$



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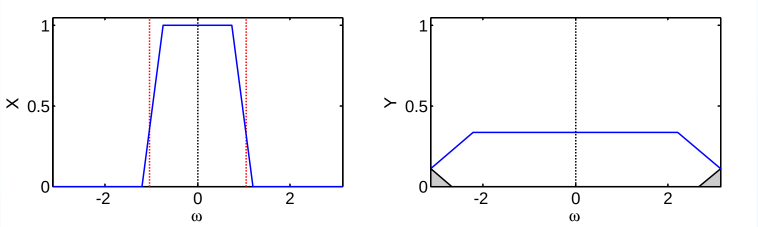

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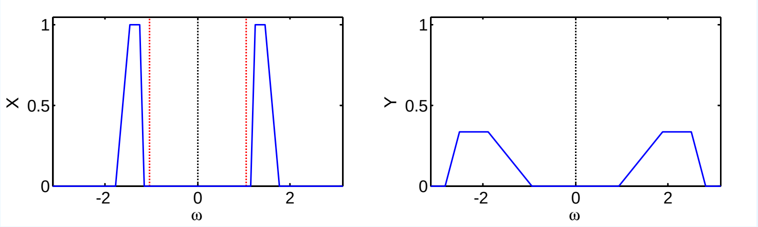

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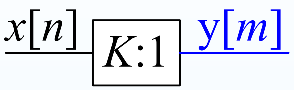

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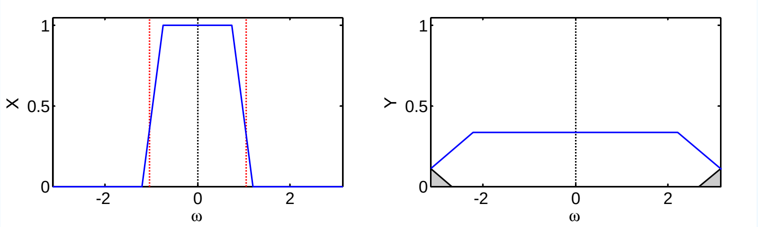

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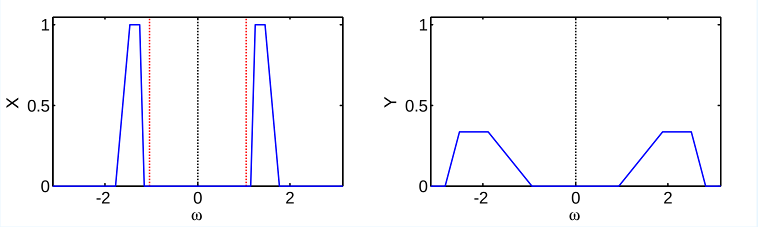

No aliasing: All energy is in $r\frac{\pi}{K} \le |\omega| < (r+1)\frac{\pi}{K}$ for some integer r

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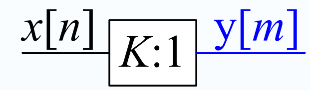

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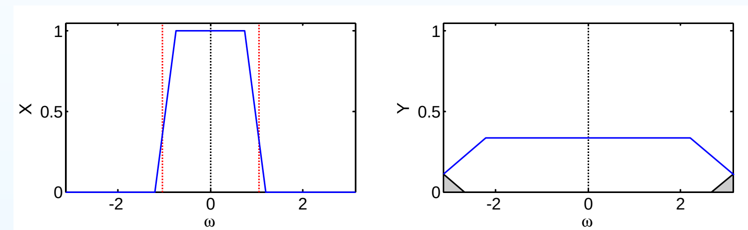

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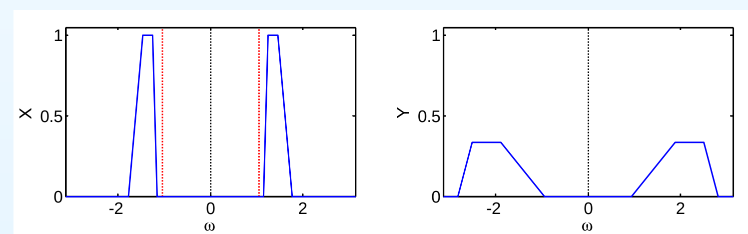
Energy all in $\frac{\pi}{K} \leq |\omega| < 2\frac{\pi}{K}$

No aliasing: ☺



No aliasing: All energy is in $r\frac{\pi}{K} \leq |\omega| < (r+1)\frac{\pi}{K}$ for some integer r

Normal case ($r = 0$): All energy in $0 \leq |\omega| \leq \frac{\pi}{K}$

Downsampled Spectrum

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$$Y(e^{j\omega}) = \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{j(\omega-2\pi k)}{K}})$$

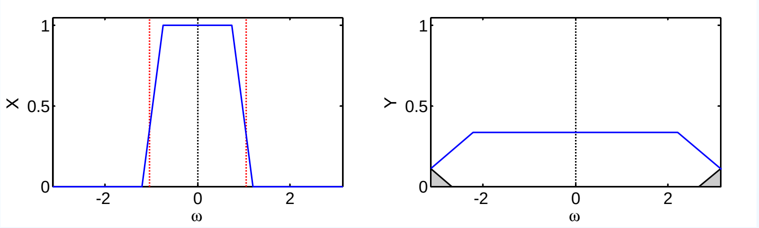
$x[n] \rightarrow \boxed{K{:}1} \rightarrow y[m]$

Example 1:

$K = 3$

Not quite limited to $\pm\frac{\pi}{K}$

Shaded region shows aliasing



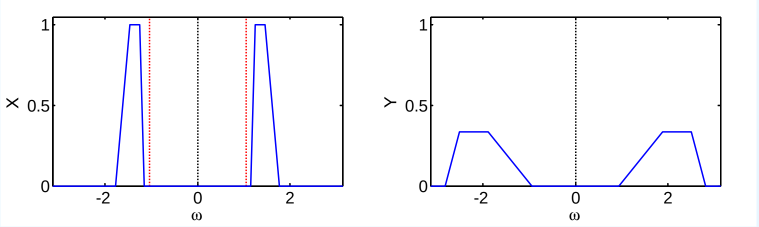
Energy decreases: $\frac{1}{2\pi}\int \left|Y(e^{j\omega})\right|^2 d\omega \approx \frac{1}{2\pi K}\int \left|X(e^{j\omega})\right|^2 d\omega$

Example 2:

$K = 3$

Energy all in $\frac{\pi}{K} \le |\omega| < 2\frac{\pi}{K}$

No aliasing: ☺



No aliasing: All energy is in $r\frac{\pi}{K} \le |\omega| < (r+1)\frac{\pi}{K}$ for some integer r

Normal case ($r = 0$): All energy in $0 \le |\omega| \le \frac{\pi}{K}$

Downsampling: Total energy multiplied by $\frac{1}{K}$ if there is no aliasing

Average power (= energy/sample) ≈unchanged

Power Spectral Density

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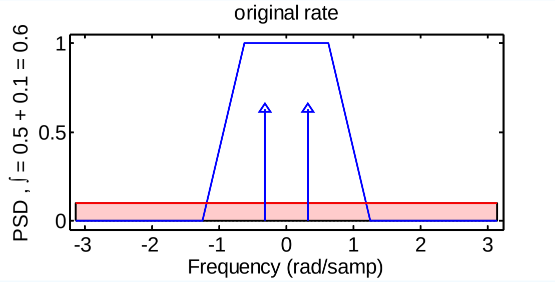
Example: Signal in $\omega \in \pm 0.4\pi$ + Tones @ $\omega = \pm 0.1\pi$ + White noise



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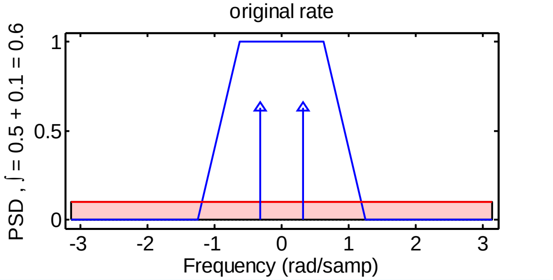
Power $=$ Energy/sample $=$ Average PSD



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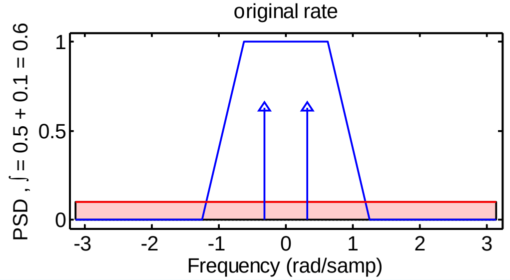
Power Spectral Density

Example: Signal in $\omega \in \pm 0.4\pi$ + Tones @ $\omega = \pm 0.1\pi$ + White noise

Power $=$ Energy/sample $=$ Average PSD

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{PSD}(\omega)d\omega$$



Power Spectral Density

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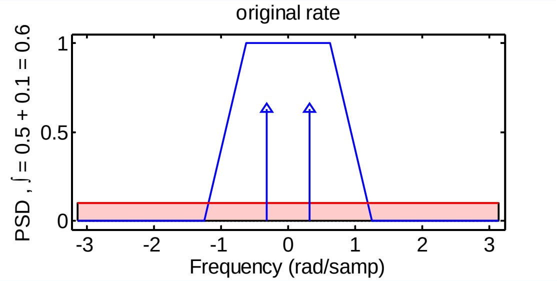

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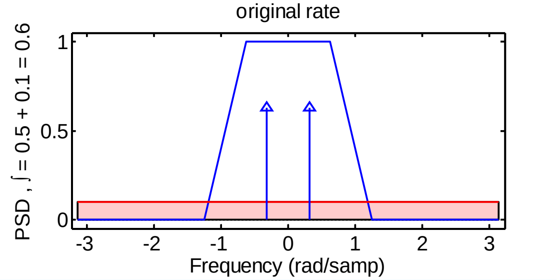
Signal Power = 0.5, Noise Power = 0.1

Total Power = Energy/sample = 0.6



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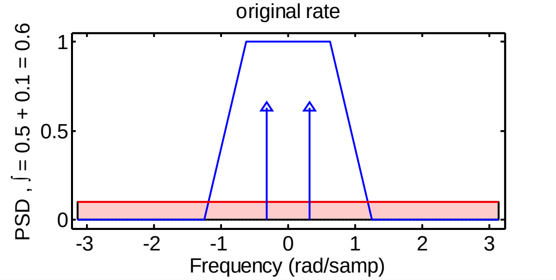
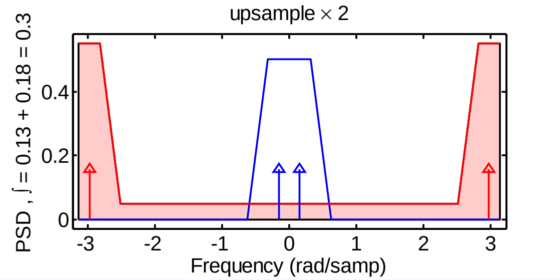


Upsampling:

Same energy per second

$\Rightarrow$ Power is $\div K$



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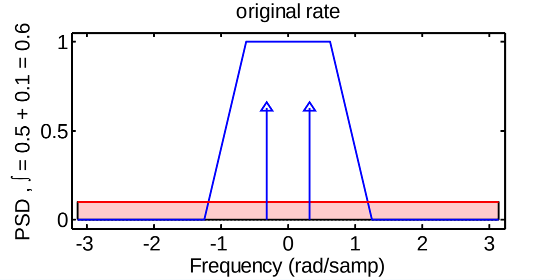

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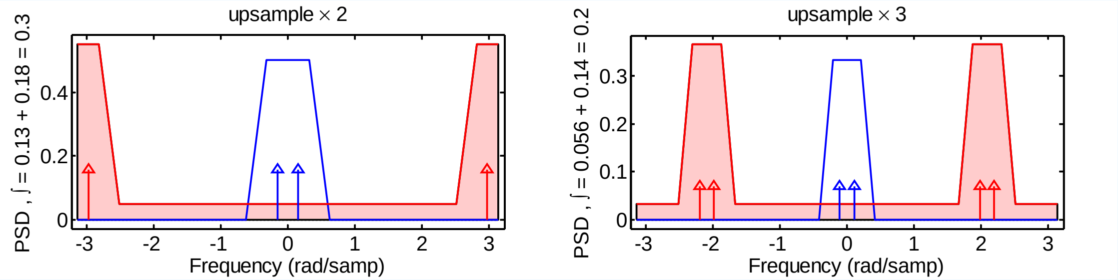

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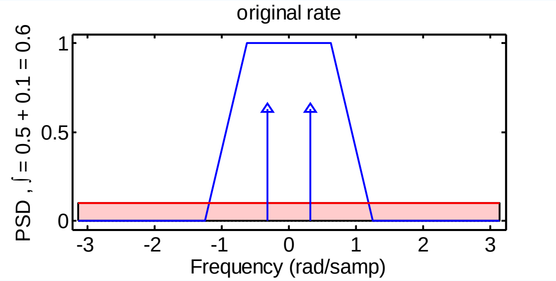

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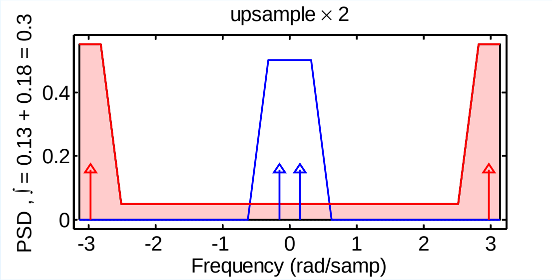
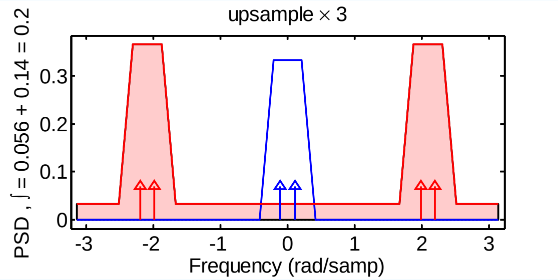
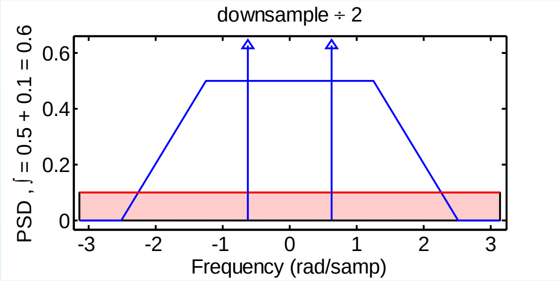




Downsampling:

Average power is unchanged.



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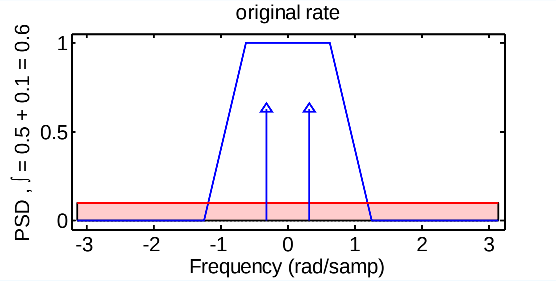
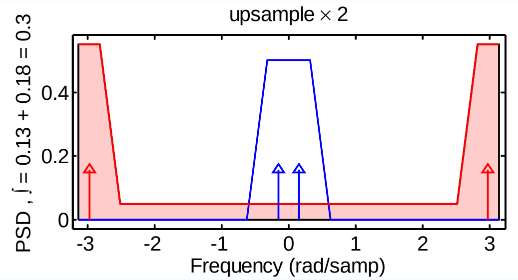
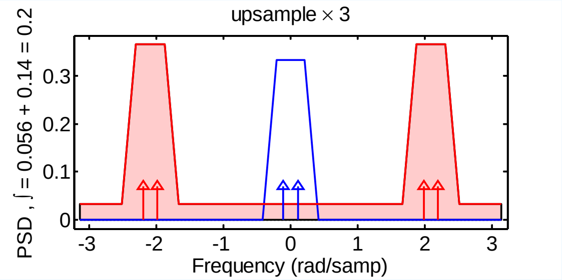


Upsampling:

Same energy per second

$\Rightarrow$ Power is $\div K$

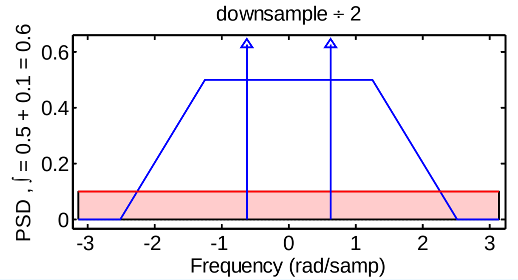


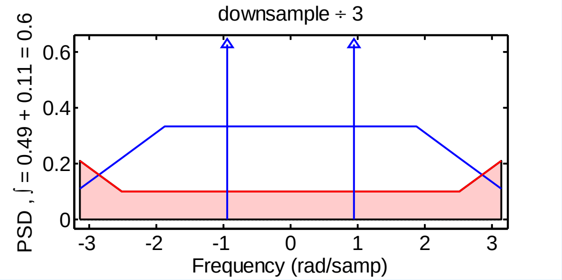


Downsampling:

Average power is unchanged.

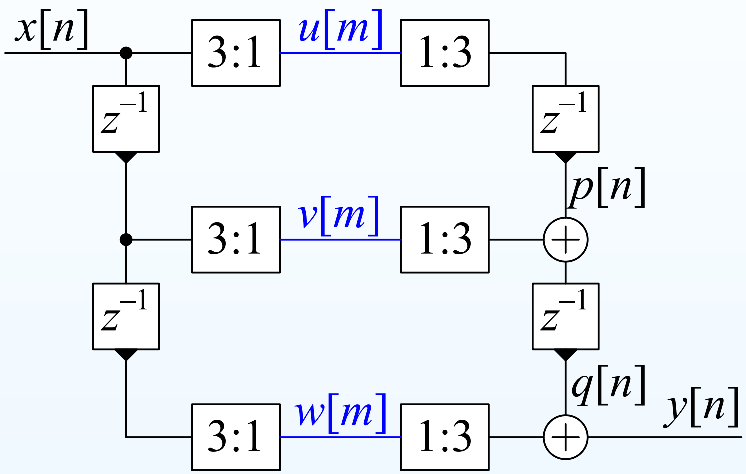
$\exists$ aliasing in the $\div 3$ case.





Perfect Reconstruction

$x[n]$ `cdefghijklmn`



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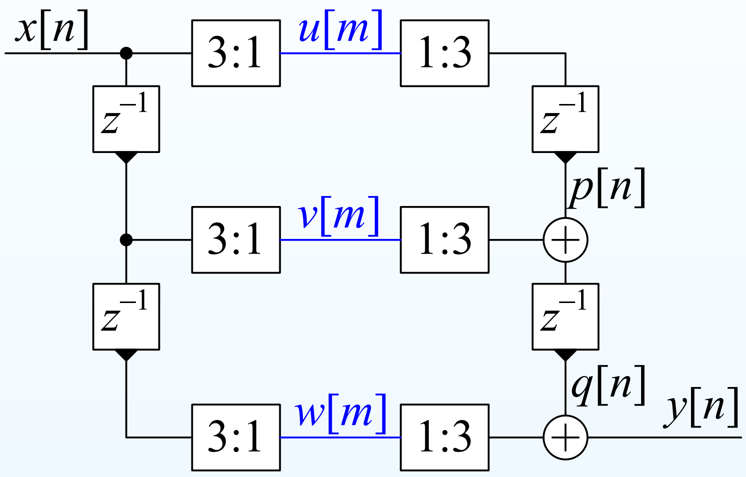
$x[n]$ `cdefghijklmn`
$u[m]$ `c  f  i  l`



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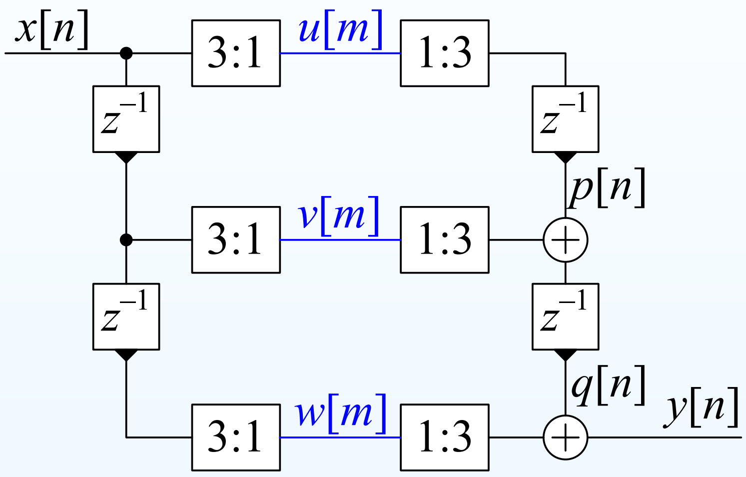
$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$p[n]$	`-c--f--i--l`



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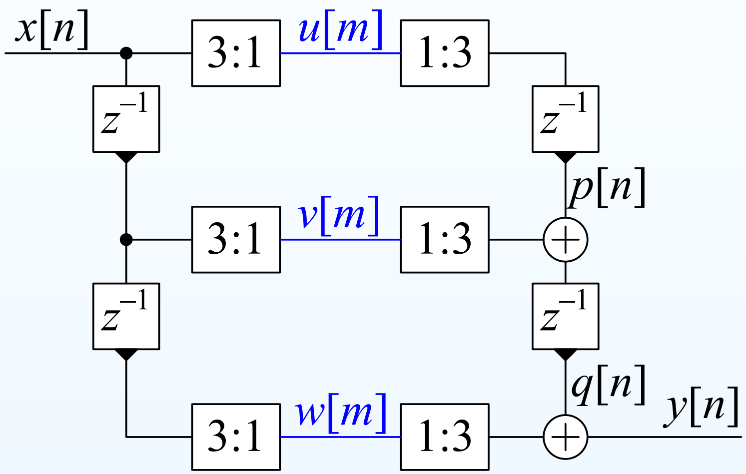

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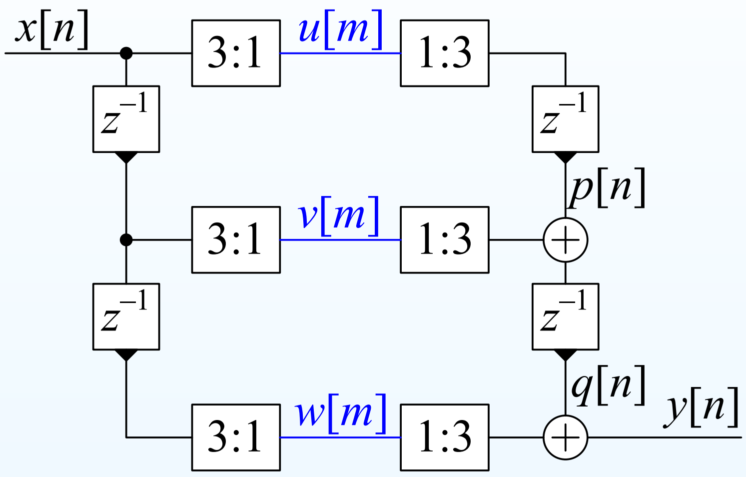

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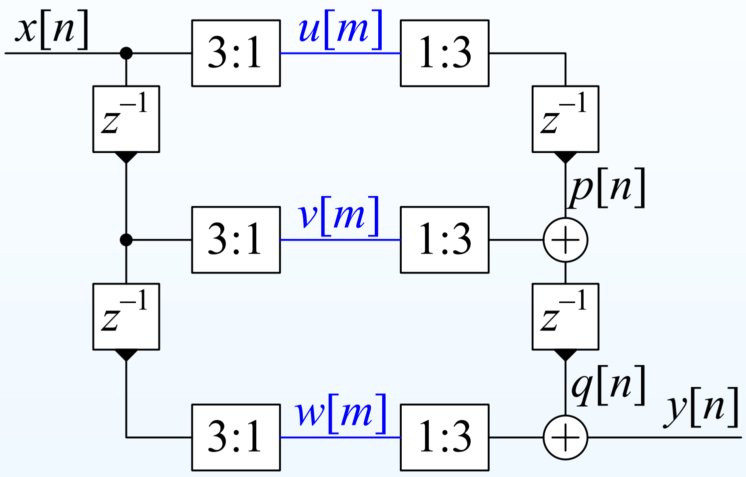

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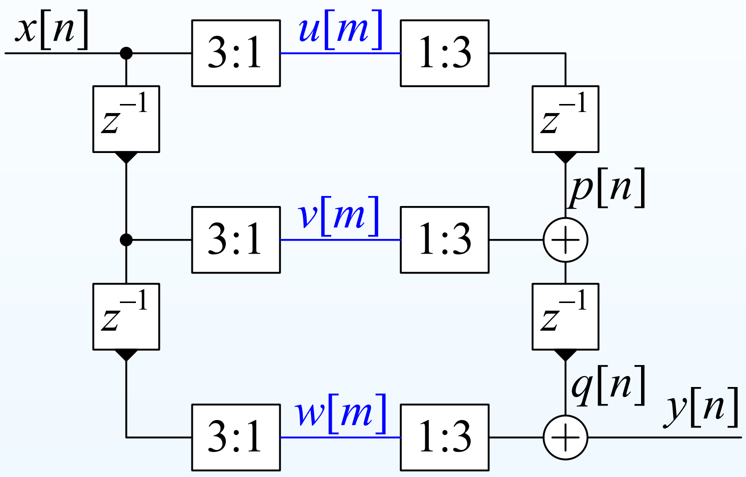

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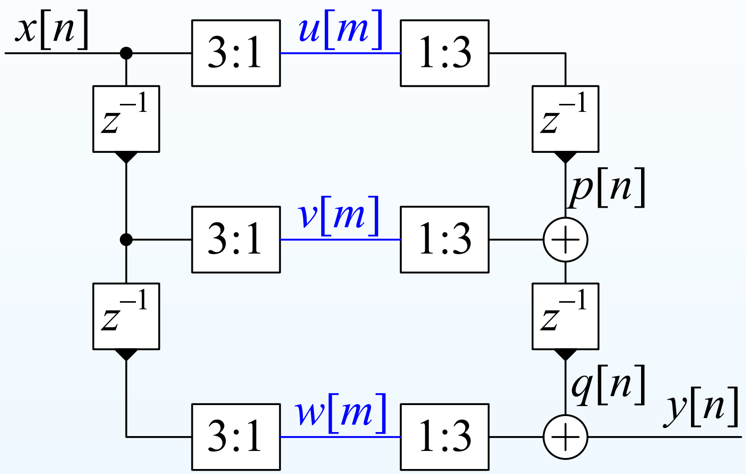

Input sequence $x[n]$ is split into three streams at $\frac{1}{3}$ the sample rate:

$$u[m] = x[3m],\ v[m] = x[3m-1],\ w[m] = x[3m-2]$$

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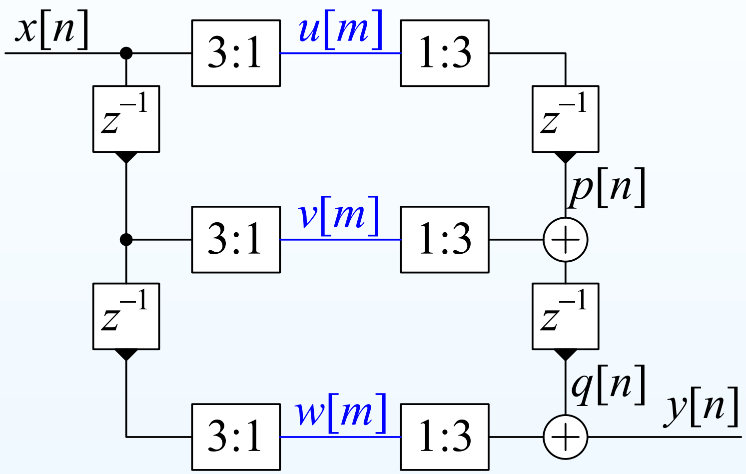

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Following upsampling, the streams are aligned by the delays and then added to give:

$$y[n] = x[n-2]$$

Perfect Reconstruction

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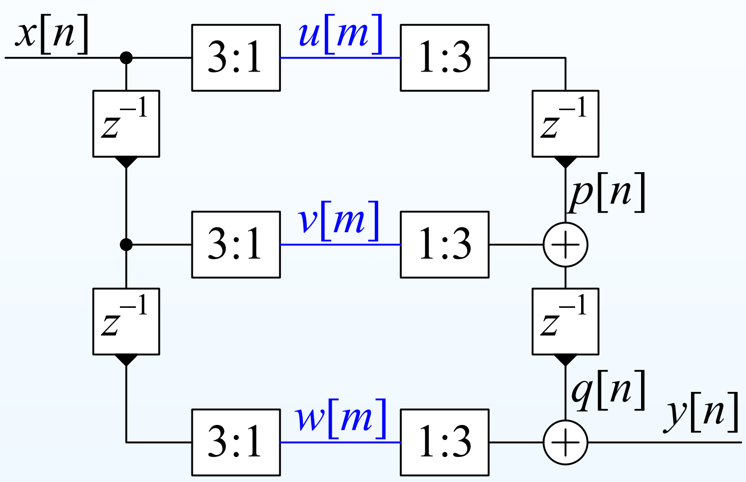

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Perfect Reconstruction: output is a delayed scaled replica of the input

Commutators

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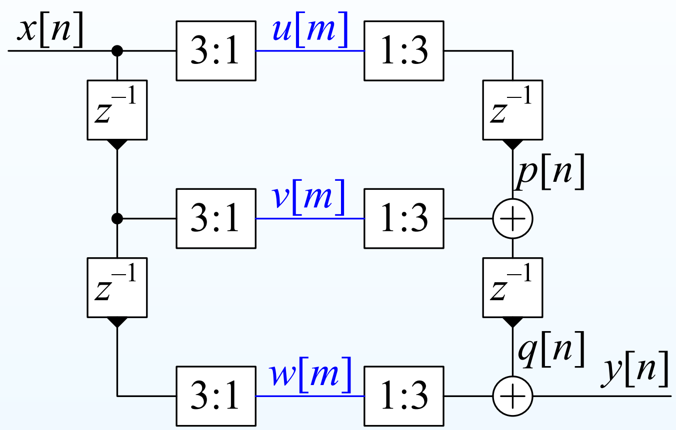

$x[n]$	`cdefghijklmn`			
$u[m]$	`c`	`f`	`i`	`l`
$v[m]$	`b`	`e`	`h`	`k`
$w[m]$	`a`	`d`	`g`	`j`

$y[n]$ `abcdefghijkl`

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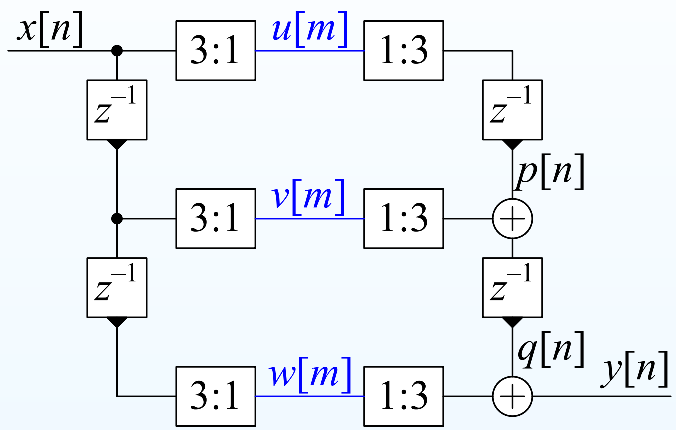

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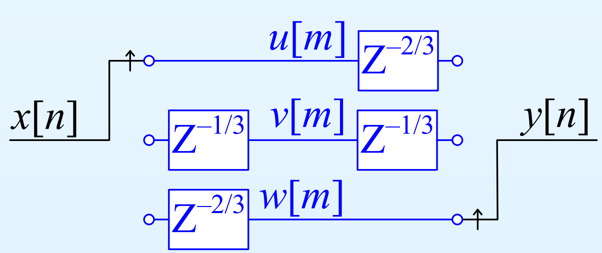
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

The combination of delays and downsamplers can be regarded as a commutator that distributes values in sequence to u, w and v.



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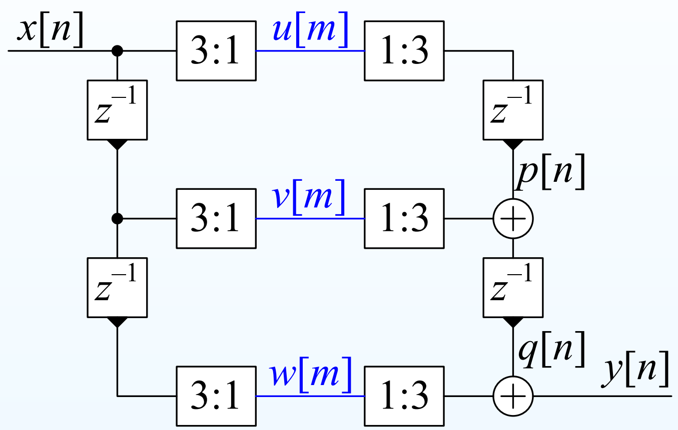

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

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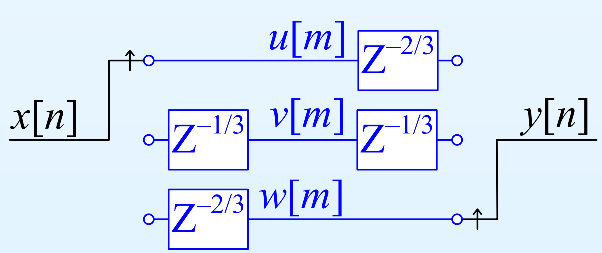

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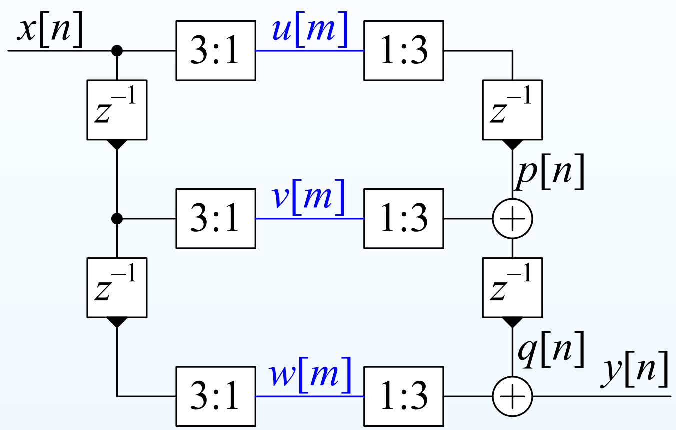

$x[n]$	`cdefghijklmn`
$u[m]$	`c  f  i  l`
$v[m]$	`b  e  h  k`
$w[m]$	`a  d  g  j`
$y[n]$	`abcdefghijkl`

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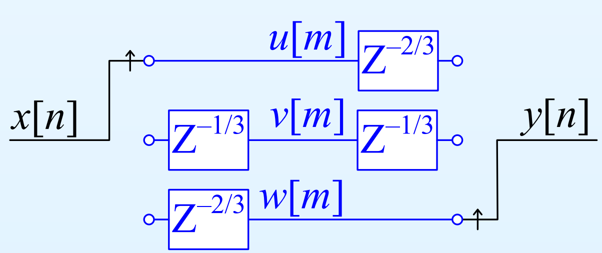

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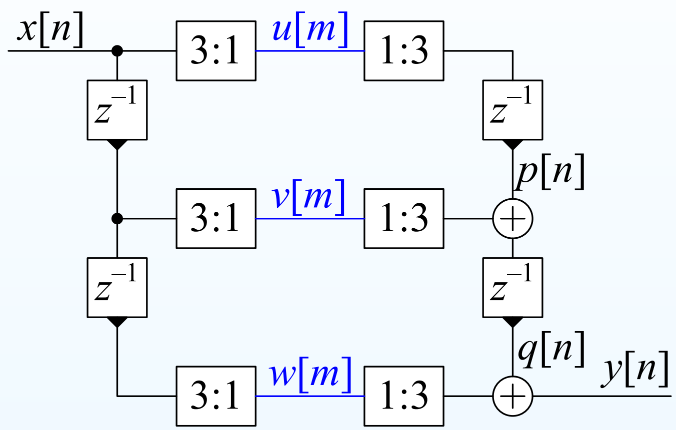

$x[n]$	c	d	e	f	g	h	i	j	k	l	m	n
$u[m]$	c			f			i			l		
$v[m]$	b			e			h			k		
$w[m]$	a			d			g			j		
$v[m+\frac{1}{3}]$			e			h			k			l
$w[m+\frac{2}{3}]$		d			g			j			m	
$y[n]$	a	b	c	d	e	f	g	h	i	j	k	l

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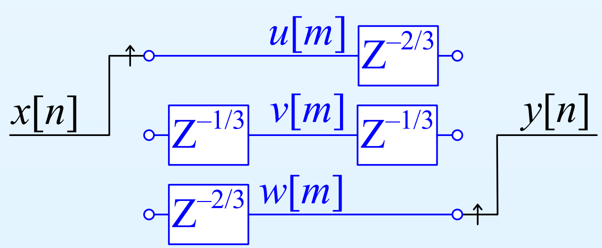

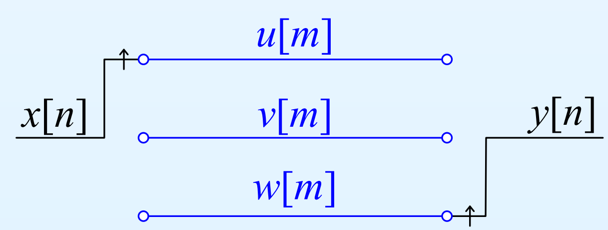

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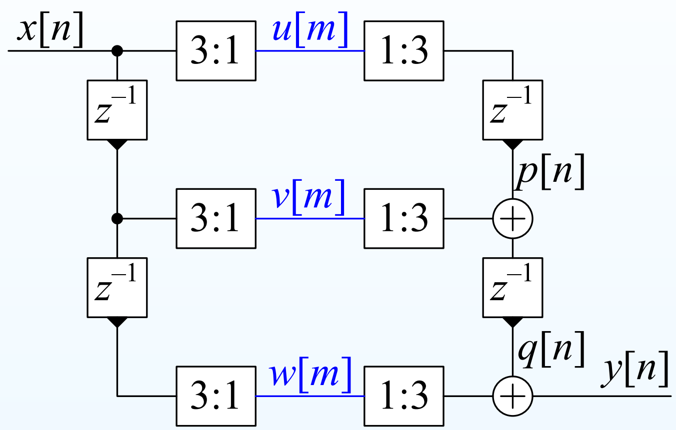

$x[n]$	`cdefghijklmn`
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$w[m]$	`a  d  g  j`
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$w[m+\frac{2}{3}]$	` d  g  j  m`
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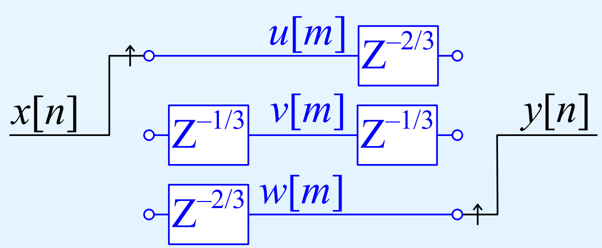

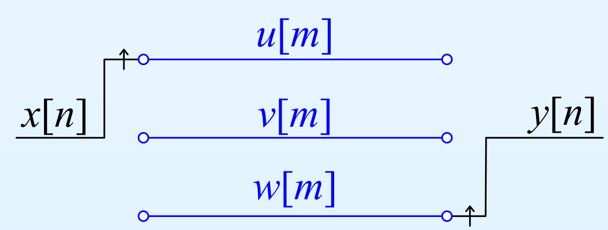

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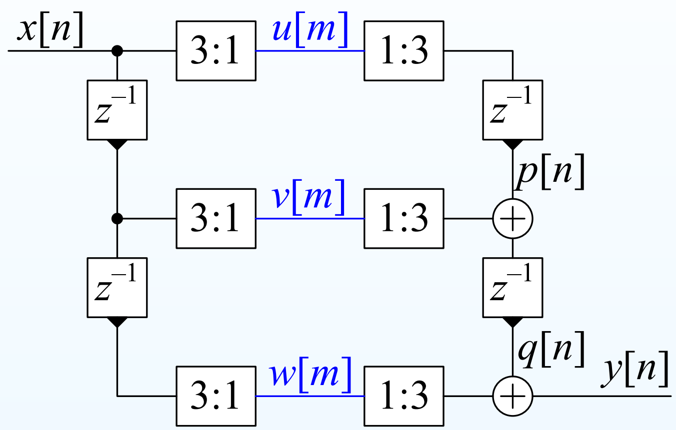

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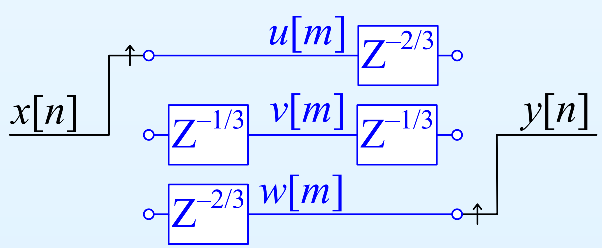

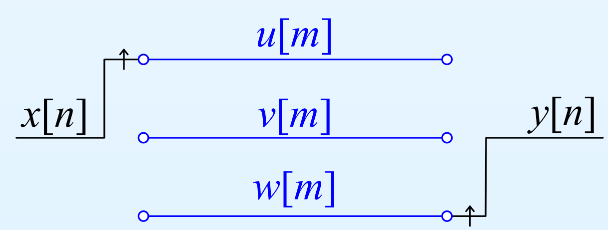

The commutator direction is against the direction of the z^{-1} delays.

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- Multirate Building Blocks
  - Upsample: $X(z) \xrightarrow{1:K} X(z^K)$
    Invertible, Inserts $K-1$ zeros between samples
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    Follow by LP filter to remove images

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  - Downsample: $X(z) \stackrel{K:1}{\rightarrow} \frac{1}{K}\sum_{k=0}^{K-1} X(e^{\frac{-j2\pi k}{K}} z^{\frac{1}{K}})$
    Destroys information and energy, keeps every K^{th} sample
    Expands and aliasses spectrum
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For further details see Mitra: 13.

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resample	change sampling rate