

(d) five runs; (e) six runs; and (f) equal numbers of runs of heads and runs of tails?

28. a) For $n \geq 4$, consider the strings made up of n bits — that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?

b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?

c) Provide a combinatorial proof for the following:
For $n \geq 1$,

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

1.5

The Catalan Numbers (Optional)

In this section a very prominent sequence of numbers is introduced. This sequence arises in a wide variety of combinatorial situations. We'll begin by examining one specific instance where it is found.

EXAMPLE 1.42

Let us start at the point $(0, 0)$ in the xy -plane and consider two kinds of moves:

$$R: (x, y) \rightarrow (x + 1, y) \quad U: (x, y) \rightarrow (x, y + 1).$$

We want to know how we can move from $(0, 0)$ to $(5, 5)$ using such moves — one unit to the right or one unit up. So we'll need five R's and five U's. At this point we have a situation like that in Example 1.14, so we know there are $10!/(5! 5!) = \binom{10}{5}$ such paths. But now we'll add a twist! In going from $(0, 0)$ to $(5, 5)$ one may touch but *never* rise above the line $y = x$. Consequently, we want to include paths such as those shown in parts (a) and (b) of Fig. 1.9 but not the path shown in part (c).

The first thing that is evident is that each such arrangement of five R's and five U's must start with an R (and end with a U). Then as we move across this type of arrangement — going from left to right — the number of R's at any point must equal or exceed the number of U's. Note how this happens in parts (a) and (b) of Fig. 1.9 but not in part (c). Now we can solve the problem at hand if we can count the paths [like the one in part (c)] that go from $(0, 0)$ to $(5, 5)$ but rise above the line $y = x$. Look again at the path in part (c) of Fig. 1.9. Where does the situation there break down for the first time? After all, we start with the requisite R — then follow it by a U. So far, so good! But then there is a second U and, at this (first) time, the number of U's exceeds the number of R's.

Now let us consider the following transformation:

$$R, U, U, \downarrow U, R, R, R, U, U, R \leftrightarrow R, U, U, \downarrow R, U, U, U, R, R, U.$$

What have we done here? For the path on the left-hand side of the transformation, we located the first move (the second U) where the path rose above the line $y = x$. The moves up to and including this move (the second U) remain as is, but the moves that follow are interchanged — each U is replaced by an R and each R by a U. The result is the path on the right-hand side of the transformation — an arrangement of four R's and six U's, as seen in part (d) of Fig. 1.9. Part (e) of that figure provides another path to be avoided; part (f) shows what happens when this path is transformed by the method described above. Now suppose we start with an arrangement of six U's and four R's, say

$$R, U, R, R, U, U, U, \downarrow U, U, R.$$

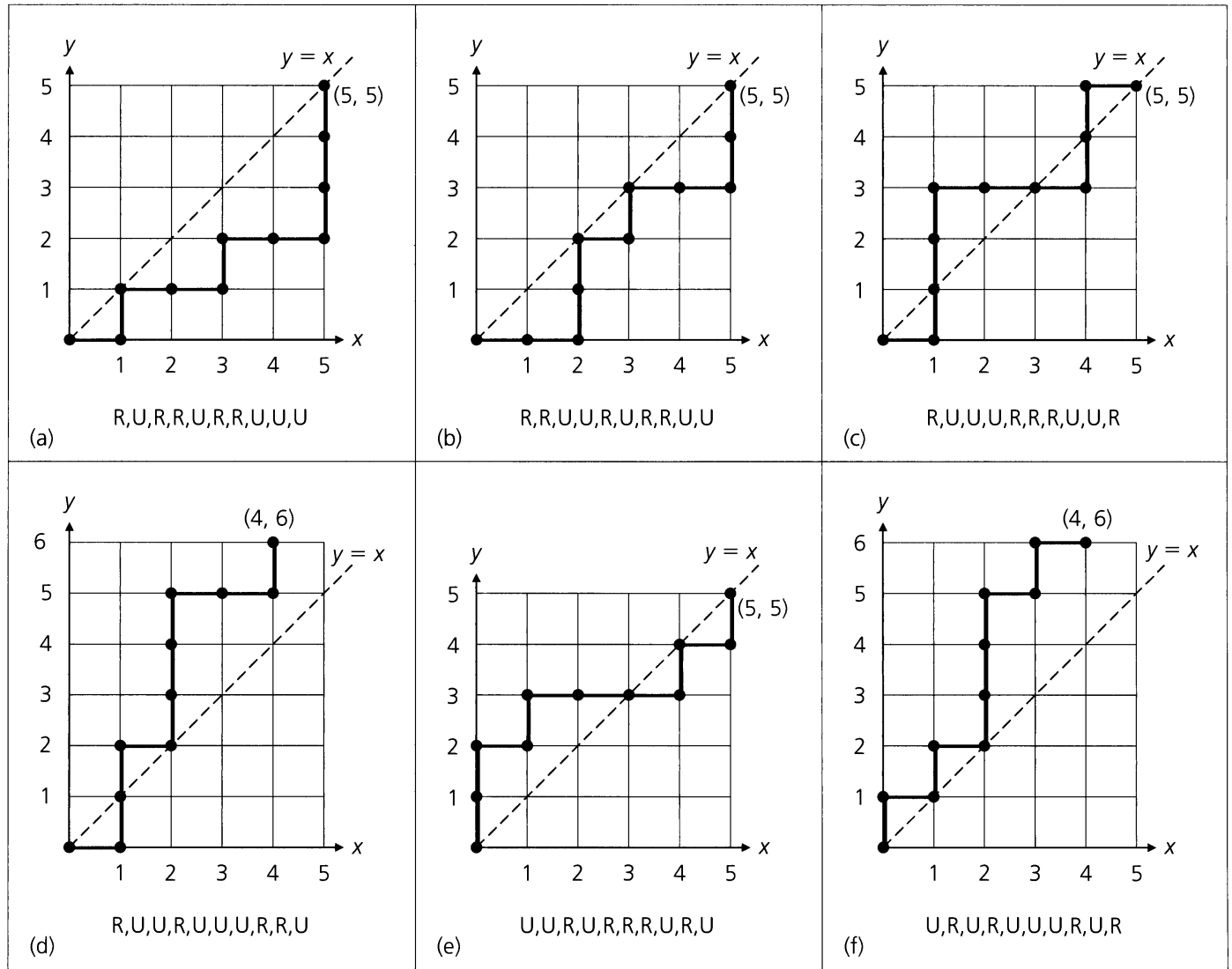


Figure 1.9

Focus on the first place where the number of U's exceeds the number of R's. Here it is in the seventh position, the location of the fourth U. This arrangement is now transformed as follows: The moves up to and including the fourth U remain as they are; the last three moves are interchanged—each U is replaced by an R, each R by a U. This results in the arrangement

$$R, U, R, R, U, U, U, \quad \vdots \quad R, R, U.$$

—one of the *bad* arrangements (of five R's and five U's) we wish to avoid as we go from (0, 0) to (5, 5). The correspondence established by these transformations gives us a way to count the number of bad arrangements. We alternatively count the number of ways to arrange four R's and six U's—this is $10!/(4! 6!) = \binom{10}{4}$. Consequently, the number of ways to go from (0, 0) to (5, 5) without rising above the line $y = x$ is

$$\begin{aligned} \binom{10}{5} - \binom{10}{4} &= \frac{10!}{5! 5!} - \frac{10!}{4! 6!} = \frac{6(10)! - 5(10)!}{6! 5!} \\ &= \left(\frac{1}{6}\right) \left(\frac{10!}{5! 5!}\right) = \frac{1}{(5+1)} \binom{10}{5} = \frac{1}{(5+1)} \binom{2 \cdot 5}{5} = 42. \end{aligned}$$

The above result generalizes as follows. For any integer $n \geq 0$, the number of paths (made up of n R's and n U's) going from $(0, 0)$ to (n, n) , without rising above the line $y = x$, is

$$b_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 1, \quad b_0 = 1.$$

The numbers b_0, b_1, b_2, \dots are called the *Catalan numbers*, after the Belgian mathematician Eugène Charles Catalan (1814–1894), who used them in determining the number of ways to parenthesize the product $x_1 x_2 x_3 x_4 \cdots x_n$. For instance, the five ($= b_3$) ways to parenthesize $x_1 x_2 x_3 x_4$ are:

$$((x_1 x_2) x_3) x_4 \quad ((x_1 (x_2 x_3)) x_4) \quad ((x_1 x_2) (x_3 x_4)) \quad (x_1 ((x_2 x_3) x_4)) \quad (x_1 (x_2 (x_3 x_4))).$$

The first seven Catalan numbers are $b_0 = 1, b_1 = 1, b_2 = 2, b_3 = 5, b_4 = 14, b_5 = 42$, and $b_6 = 132$.

EXAMPLE 1.43

Here are some other situations where the Catalan numbers arise. Some of these examples are very much like the result in Example 1.42. A change in vocabulary is often the only difference.

- a) In how many ways can one arrange three 1's and three -1 's so that all six partial sums (starting with the first summand) are nonnegative? There are five ($= b_3$) such arrangements:

$$\begin{array}{lll} 1, 1, 1, -1, -1, -1 & 1, 1, -1, -1, 1, -1 & 1, -1, 1, 1, -1, -1 \\ & 1, 1, -1, 1, -1, -1 & 1, -1, 1, -1, 1, -1 \end{array}$$

In general, for $n \geq 0$, one can arrange n 1's and n -1 's, with all $2n$ partial sums nonnegative, in b_n ways.

- b) Given four 1's and four 0's, there are 14 ($= b_4$) ways to list these eight symbols so that in each list the number of 0's never exceeds the number of 1's (as a list is read from left to right). The following shows these 14 lists:

$$\begin{array}{lll} 10101010 & 11001010 & 11100010 \\ 10101100 & 11001100 & 11100100 \\ 10110010 & 11010010 & 11101000 \\ 10110100 & 11010100 & \\ 10111000 & 11011000 & 11110000 \end{array}$$

For $n \geq 0$, there are b_n such lists of n 1's and n 0's.

c)

Table 1.10

$((ab)c)d$	$((abc$	111000
$((a(bc))d$	$((a(bc$	110100
$((ab)(cd))$	$((ab(c$	110010
$(a((bc)d))$	$(a((bc$	101100
$(a(b(cd)))$	$(a(b(c$	101010

Consider the first column in Table 1.10. Here we find five ways to parenthesize the product $abcd$. The first of these is $((ab)c)d$. Reading left to right, we list the three occurrences of the left parenthesis “(” and the letters a, b, c — maintaining the order in which these six symbols occur. This results in $((abc$, the first expression in col-

umn 2 of Table 1.10. Likewise, $((a(bc))d)$ in column 1 corresponds to $((a(bc$ in column 2 — and so on, for the other three entries in each of columns 1 and 2. Now one can also go backward, from column 2 to column 1. Take an expression in column 2 and append “ d ” to the right end. For instance, $((ab(c$ becomes $((ab(cd)$. Reading this new expression from left to right, we now insert a right parenthesis “ $)$ ” whenever a product of two results arises. So, for example, $((ab(cd)$ becomes

$$\begin{array}{ccc}
 & & ((ab)(cd)) \\
 & \nearrow & \nearrow \\
 \text{For the} & & \text{For the} \\
 \text{product of} & & \text{product of} \\
 a \text{ and } b & & (ab) \text{ and } (cd)
 \end{array}$$

The correspondence between the entries in columns 2 and 3 is more immediate. For an entry in column 2 replace each “(” by a “1” and each letter by a “0”. Reversing this process, we replace each “1” by a “(”, the first 0 by a , the second by b , and the third by c . This takes us from the entries in column 3 to those in column 2.

Now consider the correspondence between columns 1 and 3. (This correspondence arises from the correspondence between columns 1 and 2 and the one between columns 2 and 3.) It shows us that the number of ways to parenthesize the product $abcd$ equals the number of ways to list three 1’s and three 0’s so that, as such a list is read from left to right, the number of 1’s always equals or exceeds the number of 0’s. The number of ways here is 5 ($= b_3$).

In general, one can parenthesize the product $x_1x_2x_3 \cdots x_n$ in b_{n-1} ways.

- d) Let us arrange the integers 1, 2, 3, 4, 5, 6 in two rows of three so that (1) the integers increase in value as each row is read, from left to right, and (2) in any column the smaller integer is on top. For example, one way to do this is

$$\begin{array}{ccc}
 1 & 2 & 4 \\
 3 & 5 & 6
 \end{array}$$

Now consider three 1’s and three 0’s. Arrange these six symbols in a list so that the 1’s are in positions 1, 2, 4 (the top row) and the 0’s are in positions 3, 5, 6 (the bottom row). The result is 110100. Reversing the process, start with another list, say 101100 (where the number of 0’s never exceeds the number of 1’s, as the list is read from left to right). The 1’s are in positions 1, 3, 4 and the 0’s are in positions 2, 5, 6. This corresponds to the arrangement

$$\begin{array}{ccc}
 1 & 3 & 4 \\
 2 & 5 & 6
 \end{array}$$

which satisfies conditions (1) and (2), as stated above. From this correspondence we learn that the number of ways to arrange 1, 2, 3, 4, 5, 6, so that conditions (1) and (2) are satisfied, is the number of ways to arrange three 1’s and three 0’s in a list so that as the six symbols are read, from left to right, the number of 0’s never exceeds the number of 1’s. Consequently, one can arrange 1, 2, 3, 4, 5, 6 and satisfy conditions (1) and (2) in $b_3 (= 5)$ ways.

In closing let us mention that the Catalan numbers will come up in other sections — in particular, Section 5 of Chapter 10. Further examples can be found in reference [3] by M. Gardner. For even more results about these numbers one should consult the references for Chapter 10.

EXERCISES 1.5

1. Verify that for each integer $n \geq 1$,

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1} \binom{2n}{n}.$$

2. Determine the value of b_7 , b_8 , b_9 , and b_{10} .

3. a) In how many ways can one travel in the xy -plane from $(0, 0)$ to $(3, 3)$ using the moves R: $(x, y) \rightarrow (x + 1, y)$ and U: $(x, y) \rightarrow (x, y + 1)$, if the path taken may touch but *never* fall below the line $y = x$? In how many ways from $(0, 0)$ to $(4, 4)$?

b) Generalize the results in part (a).

c) What can one say about the first and last moves of the paths in parts (a) and (b)?

4. Consider the moves

$$R: (x, y) \rightarrow (x + 1, y) \quad \text{and} \quad U: (x, y) \rightarrow (x, y + 1),$$

as in Example 1.42. In how many ways can one go

a) from $(0, 0)$ to $(6, 6)$ and not rise above the line $y = x$?

b) from $(2, 1)$ to $(7, 6)$ and not rise above the line $y = x - 1$?

c) from $(3, 8)$ to $(10, 15)$ and not rise above the line $y = x + 5$?

5. Find the other three ways to arrange 1, 2, 3, 4, 5, 6 in two rows of three so that the conditions in part (d) of Example 1.43 are satisfied.

6. There are $b_4 (= 14)$ ways to arrange 1, 2, 3, ..., 8 in two rows of four so that (1) the integers increase in value as each row is read, from left to right, and (2) in any column the smaller integer is on top. Find, as in part (d) of Example 1.43,

a) the arrangements that correspond to each of the following.

i) 10110010 ii) 11001010 iii) 11101000

b) the lists of four 1's and four 0's that correspond to each of these arrangements of 1, 2, 3, ..., 8.

i) 1 3 4 5 ii) 1 2 3 7 iii) 1 2 4 5
2 6 7 8 4 5 6 8 3 6 7 8

7. In how many ways can one parenthesize the product $abcdef$?

8. There are 132 ways in which one can parenthesize the product $abcdefg$.

a) Determine, as in part (c) of Example 1.43, the list of five 1's and five 0's that corresponds to each of the following.

i) $((ab)c)(d(ef)))$

ii) $(a(b(c(d(ef))))))$

iii) $((((ab)(cd))e)f)$

- b) Find, as in Example 1.43, the way to parenthesize $abcdef$ that corresponds to each given list of five 1's and five 0's.

i) 1110010100

ii) 1100110010

iii) 1011100100

9. Consider drawing n semicircles on and above a horizontal line, with no two semicircles intersecting. In parts (a) and (b) of Fig. 1.10 we find the two ways this can be done for $n = 2$; the results for $n = 3$ are shown in parts (c)–(g).

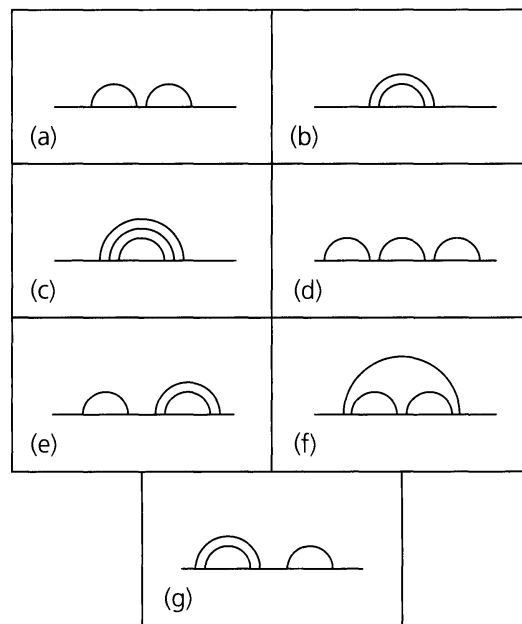


Figure 1.10

- i) How many different drawings are there for four semicircles?

- ii) How many for any $n \geq 0$? Explain why.

10. a) In how many ways can one go from $(0, 0)$ to $(7, 3)$ if the only moves permitted are R: $(x, y) \rightarrow (x + 1, y)$ and U: $(x, y) \rightarrow (x, y + 1)$, and the number of U's may never exceed the number of R's along the path taken?

b) Let m, n be positive integers with $m > n$. Answer the question posed in part (a), upon replacing 7 by m and 3 by n .

11. Twelve patrons, six each with a \$5 bill and the other six each with a \$10 bill, are the first to arrive at a movie theater, where the price of admission is five dollars. In how many ways can these 12 individuals (all loners) line up so that the number with a \$5 bill is never exceeded by the number with a \$10 bill (and, as a result, the ticket seller is always able to make any necessary change from the bills taken in from the first 11 of these 12 patrons)?

21. $\binom{n}{3} - \binom{n}{3} - n - n(n-4), n \geq 4$
23. a) $\binom{12}{9}$ b) $\binom{12}{9}(2^3)$ c) $\binom{12}{9}(2^9)(-3)^3$
25. a) $\binom{4}{1,1,2} = 12$ b) 12 c) $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$
 d) -216 e) $\binom{8}{3,2,1,2}(2^3)(-1)^2(3)(-2)^2 = 161,280$
27. a) 2^3 b) 2^{10} c) 3^{10} d) 4^5 e) 4^{10}
29. $n \binom{m+n}{m} = n \frac{(m+n)!}{m!n!} = \frac{(m+n)!}{m!(n-1)!} = (m+1) \frac{(m+n)!}{(m+1)(m!)(n-1)!}$
 $= (m+1) \frac{(m+n)!}{(m+1)!(n-1)!} = (m+1) \binom{m+n}{m+1}$
31. Consider the expansions of (a) $[(1+x) - x]^n$; (b) $[(2+x) - (x+1)]^n$; and (c) $[(2+x) - x]^n$.
33. a) $a_3 - a_0$ b) $a_n - a_0$ c) $\frac{1}{102} - \frac{1}{2} = \frac{-25}{51}$

Section 1.4—p. 34

1. a) $\binom{14}{10}$ b) $\binom{9}{5}$ c) $\binom{12}{8}$ 3. $\binom{23}{20}$ 5. a) 2^5 b) 2^n
7. a) $\binom{35}{32}$ b) $\binom{31}{28}$ c) $\binom{11}{8}$ d) 1 e) $\binom{43}{40}$ f) $\binom{31}{28} - \binom{6}{3}$
9. $n = 7$ 11. a) $\binom{14}{5}$ b) $\binom{11}{5} + 3\binom{10}{4} + 3\binom{9}{3} + \binom{8}{2}$
13. a) $\binom{7}{4}$ b) $\sum_{i=0}^3 \binom{9-2i}{7-2i}$ 15. $\binom{23}{20}(24!)$ 17. a) $\binom{16}{12}$ b) 5^{12}
19. $\binom{23}{4}$ 21. $24,310 = \sum_{i=1}^n i$ [for $n = \binom{12}{3}$]
23. a) Place one of the m identical objects into each of the n distinct containers. This leaves $m-n$ identical objects to be placed into the n distinct containers, resulting in $\binom{n+(m-n)-1}{m-n} = \binom{m-1}{m-n} = \binom{m-1}{n-1}$ distributions.
25. a) 2^9 b) 2^4
27. a) $\binom{2+3-1}{3} = 4$ b) 10 c) 48 d) $\binom{3+4-1}{4}\binom{2+3-1}{3} + \binom{3+2-1}{2}\binom{2+5-1}{5} = 96$
 e) 180 f) 420

Section 1.5—p. 40

1. $\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!(n+1)}{(n+1)!n!} - \frac{(2n)!n}{n!(n+1)!} = \frac{(2n)![(n+1)-n]}{(n+1)!n!} = \frac{1}{(n+1)} \frac{(2n)!}{n!n!} = \left(\frac{1}{n+1}\right) \binom{2n}{n}$
3. a) $5 (= b_3); 14 (= b_4)$
 b) For $n \geq 0$ there are $b_n \left(= \frac{1}{(n+1)} \binom{2n}{n} \right)$ such paths from $(0, 0)$ to (n, n) .
 c) For $n \geq 0$ the first move is U and the last is R.
5. Using the results in the third column of Table 1.10 we have:

111000	110010	101010
1 2 3	1 2 5	1 3 5
4 5 6	3 4 6	2 4 6

7. There are $b_5 (= 42)$ ways.
9. (i) When $n = 4$ there are $14 (= b_4)$ such diagrams.
 (ii) For each $n \geq 0$, there are b_n different drawings of n semicircles on and above a horizontal line, with no two semicircles intersecting. Consider, for instance, the diagram in part (f) of Fig. 1.10. Going from left to right, write 1 the first time you encounter a semicircle and write 0 the second time that semicircle is encountered. Here we get the list 110100. The list 110010 corresponds with the drawing in part (g). This correspondence shows that the number of such drawings for n semicircles is the same as the number of lists of n 1's and n 0's where, as the list is read from left to right, the number of 0's never exceeds the number of 1's.
11. $\left(\frac{1}{7}\right) \binom{12}{6} (6!)(6!) = \left(\frac{1}{7}\right) (12!) = 68,428,800$

Supplementary
Exercises—p. 43

1. $\binom{4}{1}\binom{7}{2} + \binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{6}$
3. Select any four of these twelve points (on the circumference). As seen in the figure, these points determine a pair of chords that intersect. Consequently, the largest number of points of