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Fundamental Principles of Counting

Enumeration, or counting, may strike one as an obvious process that a student learns when first studying arithmetic. But then, it seems, very little attention is paid to further development in counting as the student turns to "more difficult" areas in mathematics, such as algebra, geometry, trigonometry, and calculus. Consequently, this first chapter should provide some warning about the seriousness and difficulty of "mere" counting.

Enumeration does not end with arithmetic. It also has applications in such areas as coding theory, probability and statistics, and in the analysis of algorithms. Later chapters will offer some specific examples of these applications.

As we enter this fascinating field of mathematics, we shall come upon many problems that are very simple to state but somewhat "sticky" to solve. Thus, be sure to learn and understand the basic formulas — but do *not* rely on them too heavily. For without an analysis of each problem, a mere knowledge of formulas is next to useless. Instead, welcome the challenge to solve unusual problems or those that are different from problems you have encountered in the past. Seek solutions based on your own scrutiny, regardless of whether it reproduces what the author provides. There are often several ways to solve a given problem.

1.1 The Rules of Sum and Product

Our study of discrete and combinatorial mathematics begins with two basic principles of counting: the rules of sum and product. The statements and initial applications of these rules appear quite simple. In analyzing more complicated problems, one is often able to break down such problems into parts that can be solved using these basic principles. We want to develop the ability to "decompose" such problems and piece together our partial solutions in order to arrive at the final answer. A good way to do this is to analyze and solve many diverse enumeration problems, taking note of the principles being used. This is the approach we shall follow here.

Our first principle of counting can be stated as follows:

The Rule of Sum: If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be accomplished in any one of $m + n$ ways.

Note that when we say that a particular occurrence, such as a first task, can come about in m ways, these m ways are assumed to be distinct, unless a statement is made to the contrary. This will be true throughout the entire text.

EXAMPLE 1.1

A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology. By the rule of sum, a student at this college can select among $40 + 50 = 90$ textbooks in order to learn more about one or the other of these two subjects.

EXAMPLE 1.2

The rule can be extended beyond two tasks as long as no pair of tasks can occur simultaneously. For instance, a computer science instructor who has, say, seven different introductory books each on C++, Java, and Perl can recommend any one of these 21 books to a student who is interested in learning a first programming language.

EXAMPLE 1.3

The computer science instructor of Example 1.2 has two colleagues. One of these colleagues has three textbooks on the analysis of algorithms, and the other has five such textbooks. If n denotes the maximum number of different books on this topic that this instructor can borrow from them, then $5 \leq n \leq 8$, for here both colleagues *may* own copies of the same textbook(s).

The following example introduces our second principle of counting.

EXAMPLE 1.4

In trying to reach a decision on plant expansion, an administrator assigns 12 of her employees to two committees. Committee A consists of five members and is to investigate possible favorable results from such an expansion. The other seven employees, committee B, will scrutinize possible unfavorable repercussions. Should the administrator decide to speak to just one committee member before making her decision, then by the rule of sum there are 12 employees she can call upon for input. However, to be a bit more unbiased, she decides to speak with a member of committee A on Monday, and then with a member of committee B on Tuesday, before reaching a decision. Using the following principle, we find that she can select two such employees to speak with in $5 \times 7 = 35$ ways.

The Rule of Product: If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage and if, for each of these outcomes, there are n possible outcomes for the second stage, then the total procedure can be carried out, in the designated order, in mn ways.

EXAMPLE 1.5

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in $6 \times 8 = 48$ ways.

EXAMPLE 1.6

Here various extensions of the rule are illustrated by considering the manufacture of license plates consisting of two letters followed by four digits.

a) If no letter or digit can be repeated, there are $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3{,}276{,}000$ different possible plates.

b) With repetitions of letters and digits allowed, $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6{,}760{,}000$ different license plates are possible.

c) If repetitions are allowed, as in part (b), how many of the plates have only vowels (A, E, I, O, U) and even digits? (0 is an even integer.)

EXAMPLE 1.7

In order to store data, a computer's main memory contains a large collection of circuits, each of which is capable of storing a *bit* — that is, one of the *bi*nary dig*its* 0 or 1. These storage circuits are arranged in units called (memory) cells. To identify the cells in a computer's main memory, each is assigned a unique name called its *address*. For some computers, such as embedded microcontrollers (as found in the ignition system for an automobile), an address is represented by an ordered list of eight bits, collectively referred to as a *byte*. Using the rule of product, there are $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$ such bytes. So we have 256 addresses that may be used for cells where certain information may be stored.

A kitchen appliance, such as a microwave oven, incorporates an embedded microcontroller. These "small computers" (such as the PICmicro microcontroller) contain thousands of memory cells and use two-byte addresses to identify these cells in their main memory. Such addresses are made up of two consecutive bytes, or 16 consecutive bits. Thus there are $256 \times 256 = 2^8 \times 2^8 = 2^{16} = 65{,}536$ available addresses that could be used to identify cells in the main memory. Other computers use addressing systems of four bytes. This 32-bit architecture is presently used in the Pentium† processor, where there are as many as $2^8 \times 2^8 \times 2^8 \times 2^8 = 2^{32} = 4{,}294{,}967{,}296$ addresses for use in identifying the cells in main memory. When a programmer deals with the UltraSPARC‡ or Itanium§ processors, he or she considers memory cells with eight-byte addresses. Each of these addresses comprises $8 \times 8 = 64$ bits, and there are $2^{64} = 18{,}446{,}744{,}073{,}709{,}551{,}616$ possible addresses for this architecture. (Of course, not all of these possibilities are actually used.)

EXAMPLE 1.8

At times it is necessary to combine several different counting principles in the solution of one problem. Here we find that the rules of both sum and product are needed to attain the answer.

At the AWL corporation Mrs. Foster operates the Quick Snack Coffee Shop. The menu at her shop is limited: six kinds of muffins, eight kinds of sandwiches, and five beverages (hot coffee, hot tea, iced tea, cola, and orange juice). Ms. Dodd, an editor at AWL, sends her assistant Carl to the shop to get her lunch — either a muffin and a hot beverage or a sandwich and a cold beverage.

By the rule of product, there are $6 \times 2 = 12$ ways in which Carl can purchase a muffin and hot beverage. A second application of this rule shows that there are $8 \times 3 = 24$ possibilities for a sandwich and cold beverage. So by the rule of sum, there are $12 + 24 = 36$ ways in which Carl can purchase Ms. Dodd's lunch.

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† Pentium (R) is a registered trademark of the Intel Corporation.

‡ The UltraSPARC processor is manufactured by Sun (R) Microsystems, Inc.

§ Itanium (TM) is a trademark of the Intel Corporation.

1.2 Permutations

Continuing to examine applications of the rule of product, we turn now to counting linear arrangements of objects. These arrangements are often called *permutations* when the objects are distinct. We shall develop some systematic methods for dealing with linear arrangements, starting with a typical example.

EXAMPLE 1.9

In a class of 10 students, five are to be chosen and seated in a row for a picture. How many such linear arrangements are possible?

The key word here is *arrangement*, which designates the importance of *order*. If A, B, C, . . . , I, J denote the 10 students, then BCEFI, CEFIB, and ABCFG are three such different arrangements, even though the first two involve the same five students.

To answer this question, we consider the positions and possible numbers of students we can choose from in order to fill each position. The filling of a position is a stage of our procedure.

10	$\times$	9	$\times$	8	$\times$	7	$\times$	6
1st position		**2nd position**		**3rd position**		**4th position**		**5th position**

Each of the 10 students can occupy the 1st position in the row. Because repetitions are not possible here, we can select only one of the nine remaining students to fill the 2nd position. Continuing in this way, we find only six students to select from in order to fill the 5th and final position. This yields a total of 30,240 possible arrangements of five students selected from the class of 10.

Exactly the same answer is obtained if the positions are filled from right to left—namely, $6 \times 7 \times 8 \times 9 \times 10$. If the 3rd position is filled first, the 1st position second, the 4th position third, the 5th position fourth, and the 2nd position fifth, then the answer is $9 \times 6 \times 10 \times 8 \times 7$, still the same value, 30,240.

As in Example 1.9, the product of certain consecutive positive integers often comes into play in enumeration problems. Consequently, the following notation proves to be quite useful when we are dealing with such counting problems. It will frequently allow us to express our answers in a more convenient form.

Definition 1.1

For an integer $n \geq 0$, *n factorial* (denoted $n!$) is defined by

$$0! = 1,$$
$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1), \quad \text{for} \quad n \geq 1.$$

One finds that $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $5! = 120$. In addition, for each $n \geq 0$, $(n+1)! = (n+1)(n!)$.

Before we proceed any further, let us try to get a somewhat better appreciation for how fast $n!$ grows. We can calculate that $10! = 3{,}628{,}800$, and it just so happens that this is exactly the number of *seconds* in six *weeks*. Consequently, $11!$ exceeds the number of seconds in one *year*, $12!$ exceeds the number in 12 years, and $13!$ surpasses the number of seconds in a *century*.

If we make use of the factorial notation, the answer in Example 1.9 can be expressed in the following more compact form:

$$10 \times 9 \times 8 \times 7 \times 6 = 10 \times 9 \times 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{5!}.$$

Definition 1.2 Given a collection of n distinct objects, any (linear) arrangement of these objects is called a *permutation* of the collection.

Starting with the letters a, b, c, there are six ways to arrange, or permute, all of the letters: abc, acb, bac, bca, cab, cba. If we are interested in arranging only two of the letters at a time, there are six such size-2 permutations: ab, ba, ac, ca, bc, cb.

If there are n distinct objects and r is an integer, with $1 \leq r \leq n$, then by the rule of product, the number of permutations of size r for the n objects is

$$\begin{aligned} P(n, r) &= \underset{\substack{\text{1st}\\\text{position}}}{n} \times \underset{\substack{\text{2nd}\\\text{position}}}{(n-1)} \times \underset{\substack{\text{3rd}\\\text{position}}}{(n-2)} \times \cdots \times \underset{\substack{r\text{th}\\\text{position}}}{(n-r+1)} \\ &= (n)(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)} \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

For $r = 0$, $P(n, 0) = 1 = n!/(n-0)!$, so $P(n, r) = n!/(n-r)!$ holds for all $0 \leq r \leq n$. A special case of this result is Example 1.9, where $n = 10$, $r = 5$, and $P(10, 5) = 30{,}240$. When permuting all of the n objects in the collection, we have $r = n$ and find that $P(n, n) = n!/0! = n!$.

Note, for example, that if $n \geq 2$, then $P(n, 2) = n!/(n-2)! = n(n-1)$. When $n > 3$ one finds that $P(n, n-3) = n!/[n-(n-3)]! = n!/3! = (n)(n-1)(n-2)\cdots(5)(4)$.

The number of permutations of size r, where $0 \leq r \leq n$, from a collection of n objects, is $P(n, r) = n!/(n-r)!$. (Remember that $P(n, r)$ counts (linear) arrangements in which the objects can*not* be repeated.) However, if repetitions are allowed, then by the rule of product there are n^r possible arrangements, with $r \geq 0$.

EXAMPLE 1.10 The number of permutations of the letters in the word COMPUTER is $8!$. If only five of the letters are used, the number of permutations (of size 5) is $P(8, 5) = 8!/(8-5)! = 8!/3! = 6720$. If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} \doteq 6.872 \times 10^{10}$.†

EXAMPLE 1.11 Unlike Example 1.10, the number of (linear) arrangements of the four letters in BALL is 12, not $4!$ $(= 24)$. The reason is that we do not have four distinct letters to arrange. To get the 12 arrangements, we can list them as in Table 1.1(a).

†The symbol "$\doteq$" is read "is approximately equal to."

Table 1.1

(a)				(b)							
A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

If the two L's are distinguished as L_1, L_2, then we can use our previous ideas on permutations of distinct objects; with the four distinct symbols B, A, L_1, L_2, we have $4! = 24$ permutations. These are listed in Table 1.1(b). Table 1.1 reveals that for each arrangement in which the L's are indistinguishable there corresponds a *pair* of permutations with distinct L's. Consequently,

$$2 \times (\text{Number of arrangements of the letters B, A, L, L}) \\ = (\text{Number of permutations of the symbols B, A, } L_1, L_2),$$

and the answer to the original problem of finding all the arrangements of the four letters in BALL is $4!/2 = 12$.

EXAMPLE 1.12

Using the idea developed in Example 1.11, we now consider the arrangements of all nine letters in DATABASES.

There are $3! = 6$ arrangements with the A's distinguished for each arrangement in which the A's are not distinguished. For example, $DA_1TA_2BA_3SES$, $DA_1TA_3BA_2SES$, $DA_2TA_1BA_3SES$, $DA_2TA_3BA_1SES$, $DA_3TA_1BA_2SES$, and $DA_3TA_2BA_1SES$ all correspond to DATABASES, when we remove the subscripts on the A's. In addition, to the arrangement $DA_1TA_2BA_3SES$ there corresponds the pair of permutations $DA_1TA_2BA_3S_1ES_2$ and $DA_1TA_2BA_3S_2ES_1$, when the S's are distinguished. Consequently,

$$(2!)(3!)(\text{Number of arrangements of the letters in DATABASES}) \\ = (\text{Number of permutations of the symbols D, } A_1, \text{T}, A_2, \text{B}, A_3, S_1, \text{E}, S_2),$$

so the number of arrangements of the nine letters in DATABASES is $9!/(2!\,3!) = 30{,}240$.

Before stating a general principle for arrangements with repeated symbols, note that in our prior two examples we solved a new type of problem by relating it to previous enumeration principles. This practice is common in mathematics in general, and often occurs in the derivations of discrete and combinatorial formulas.

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, ..., and n_r indistinguishable objects of an rth type, where $n_1 + n_2 + \cdots + n_r = n$, then there are $\dfrac{n!}{n_1!\, n_2! \cdots n_r!}$ (linear) arrangements of the given n objects.

EXAMPLE 1.13

The MASSASAUGA is a brown and white venomous snake indigenous to North America. Arranging all of the letters in MASSASAUGA, we find that there are

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

possible arrangements. Among these are

$$\frac{7!}{3!\,1!\,1!\,1!\,1!} = 840$$

in which all four A's are together. To get this last result, we considered all arrangements of the seven symbols AAAA (one symbol), S, S, S, M, U, G.

EXAMPLE 1.14

Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U). The blue lines in Fig. 1.1 show two of these paths.

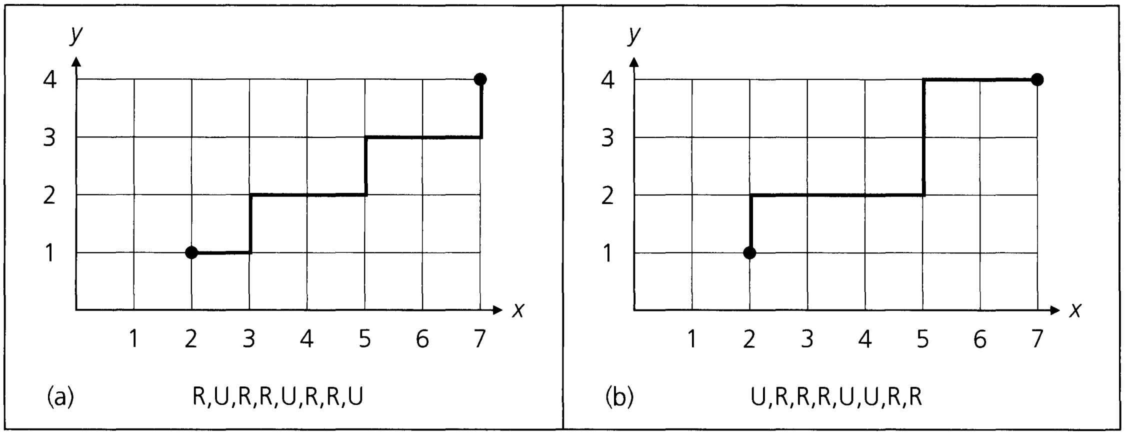

Figure 1.1

Beneath each path in Fig. 1.1 we have listed the individual steps. For example, in part (a) the list R, U, R, R, U, R, R, U indicates that starting at the point (2, 1), we first move one unit to the right [to (3, 1)], then one unit upward [to (3, 2)], followed by two units to the right [to (5, 2)], and so on, until we reach the point (7, 4). The path consists of five R's for moves to the right and three U's for moves upward.

The path in part (b) of the figure is also made up of five R's and three U's. In general, the overall trip from (2, 1) to (7, 4) requires $7 - 2 = 5$ horizontal moves to the right and $4 - 1 = 3$ vertical moves upward. Consequently, each path corresponds to a list of five R's and three U's, and the solution for the number of paths emerges as the number of arrangements of the five R's and three U's, which is $8!/(5!\,3!) = 56$.

EXAMPLE 1.15

We now do something a bit more abstract and prove that if n and k are positive integers with $n = 2k$, then $n!/2^k$ is an integer. Because our argument relies on counting, it is an example of a *combinatorial proof.*

Consider the n symbols $x_1, x_1, x_2, x_2, \ldots, x_k, x_k$. The number of ways in which we can arrange all of these $n = 2k$ symbols is an integer that equals

$$\frac{n!}{\underbrace{2!\,2!\cdots 2!}_{k \text{ factors of } 2!}} = \frac{n!}{2^k}.$$

Finally, we will apply what has been developed so far to a situation in which the arrangements are no longer linear.

EXAMPLE 1.16

If six people, designated as A, B, . . . , F, are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotation? [In Fig. 1.2, arrangements (a) and (b) are considered identical, whereas (b), (c), and (d) are three distinct arrangements.]

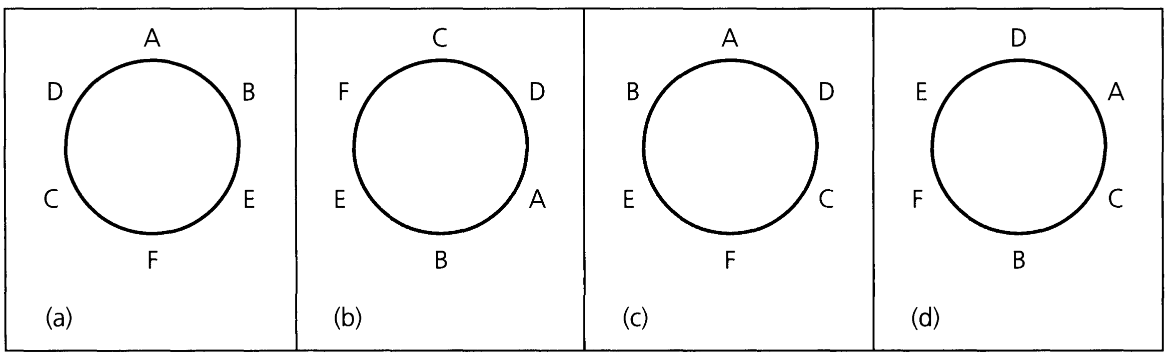

Figure 1.2

We shall try to relate this problem to previous ones we have already encountered. Consider Figs. 1.2(a) and (b). Starting at the top of the circle and moving clockwise, we list the distinct linear arrangements ABEFCD and CDABEF, which correspond to the same circular arrangement. In addition to these two, four other linear arrangements — BEFCDA, DABEFC, EFCDAB, and FCDABE — are found to correspond to the same circular arrangement as in (a) or (b). So inasmuch as each circular arrangement corresponds to six linear arrangements, we have $6 \times$ (Number of circular arrangements of A, B, . . . , F) = (Number of linear arrangements of A, B, . . . , F) = $6!$.

Consequently, there are $6!/6 = 5! = 120$ arrangements of A, B, . . . , F around the circular table.

EXAMPLE 1.17

Suppose now that the six people of Example 1.16 are three married couples and that A, B, and C are the females. We want to arrange the six people around the table so that the sexes alternate. (Once again, arrangements are considered identical if one can be obtained from the other by rotation.)

Before we solve this problem, let us solve Example 1.16 by an alternative method, which will assist us in solving our present problem. If we place A at the table as shown in Fig. 1.3(a), five locations (clockwise from A) remain to be filled. Using B, C, . . . , F to fill

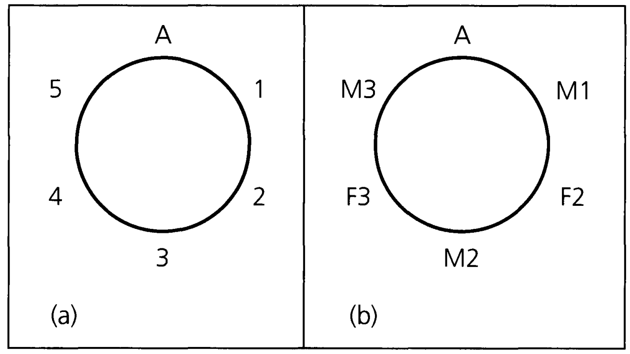

Figure 1.3

these five positions is the problem of permuting B, C, . . . , F in a linear manner, and this can be done in $5! = 120$ ways.

To solve the new problem of alternating the sexes, consider the method shown in Fig. 1.3(b). A (a female) is placed as before. The next position, clockwise from A, is marked M1 (Male 1) and can be filled in three ways. Continuing clockwise from A, position F2 (Female 2) can be filled in two ways. Proceeding in this manner, by the rule of product, there are $3 \times 2 \times 2 \times 1 \times 1 = 12$ ways in which these six people can be arranged with no two men or women seated next to each other.

EXERCISES 1.1 AND 1.2

1. During a local campaign, eight Republican and five Democratic candidates are nominated for president of the school board.

a) If the president is to be one of these candidates, how many possibilities are there for the eventual winner?

b) How many possibilities exist for a pair of candidates (one from each party) to oppose each other for the eventual election?

c) Which counting principle is used in part (a)? in part (b)?

2. Answer part (c) of Example 1.6.

3. Buick automobiles come in four models, 12 colors, three engine sizes, and two transmission types. (a) How many distinct Buicks can be manufactured? (b) If one of the available colors is blue, how many different blue Buicks can be manufactured?

4. The board of directors of a pharmaceutical corporation has 10 members. An upcoming stockholders' meeting is scheduled to approve a new slate of company officers (chosen from the 10 board members).

a) How many different slates consisting of a president, vice president, secretary, and treasurer can the board present to the stockholders for their approval?

b) Three members of the board of directors are physicians. How many slates from part (a) have (i) a physician nominated for the presidency? (ii) exactly one physician appearing on the slate? (iii) at least one physician appearing on the slate?

5. While on a Saturday shopping spree Jennifer and Tiffany witnessed two men driving away from the front of a jewelry shop, just before a burglar alarm started to sound. Although everything happened rather quickly, when the two young ladies were questioned they were able to give the police the following information about the license plate (which consisted of two letters followed by four digits) on the get-away car. Tiffany was sure that the second letter on the plate was either an O or a Q and the last digit was either a 3 or an 8. Jennifer told the investigator that the first letter on the plate was either a C or a G and that the first digit was definitely a 7. How many different license plates will the police have to check out?

6. To raise money for a new municipal pool, the chamber of commerce in a certain city sponsors a race. Each participant pays a $5 entrance fee and has a chance to win one of the different-sized trophies that are to be awarded to the first eight runners who finish.

a) If 30 people enter the race, in how many ways will it be possible to award the trophies?

b) If Roberta and Candice are two participants in the race, in how many ways can the trophies be awarded with these two runners among the top three?

7. A certain "Burger Joint" advertises that a customer can have his or her hamburger with or without any or all of the following: catsup, mustard, mayonnaise, lettuce, tomato, onion, pickle, cheese, or mushrooms. How many different kinds of hamburger orders are possible?

8. Matthew works as a computer operator at a small university. One evening he finds that 12 computer programs have been submitted earlier that day for batch processing. In how many ways can Matthew order the processing of these programs if (a) there are no restrictions? (b) he considers four of the programs higher in priority than the other eight and wants to process those four first? (c) he first separates the programs into four of top priority, five of lesser priority, and three of least priority, and he wishes to process the 12 programs in such a way that the top-priority programs are processed first and the three programs of least priority are processed last?

9. Patter's Pastry Parlor offers eight different kinds of pastry and six different kinds of muffins. In addition to bakery items one can purchase small, medium, or large containers of the following beverages: coffee (black, with cream, with sugar, or with cream and sugar), tea (plain, with cream, with sugar, with cream and sugar, with lemon, or with lemon and sugar), hot cocoa, and orange juice. When Carol comes to Patter's, in how many ways can she order

a) one bakery item and one medium-sized beverage for herself?

b) one bakery item and one container of coffee for herself and one muffin and one container of tea for her boss, Ms. Didio?

c) one piece of pastry and one container of tea for herself, one muffin and a container of orange juice for Ms. Didio, and one bakery item and one container of coffee for each of her two assistants, Mr. Talbot and Mrs. Gillis?

10. Pamela has 15 different books. In how many ways can she place her books on two shelves so that there is at least one book on each shelf? (Consider the books in each arrangement to be stacked one next to the other, with the first book on each shelf at the left of the shelf.)

11. Three small towns, designated by A, B, and C, are interconnected by a system of two-way roads, as shown in Fig. 1.4.

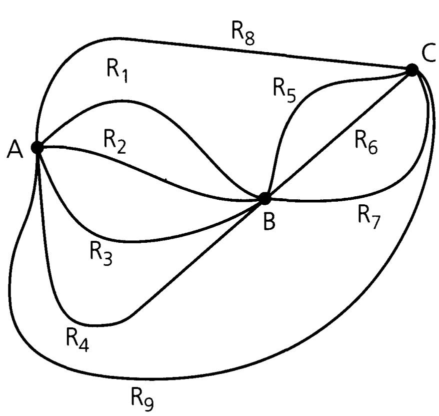

Figure 1.4

a) In how many ways can Linda travel from town A to town C?

b) How many different round trips can Linda travel from town A to town C and back to town A?

c) How many of the round trips in part (b) are such that the return trip (from town C to town A) is at least partially different from the route Linda takes from town A to town C? (For example, if Linda travels from town A to town C along roads R_1 and R_6, then on her return she might take roads R_6 and R_3, or roads R_7 and R_2, or road R_9, among other possibilities, but she does *not* travel on roads R_6 *and* R_1.)

12. List all the permutations for the letters a, c, t.

13. a) How many permutations are there for the eight letters a, c, f, g, i, t, w, x?

b) Consider the permutations in part (a). How many start with the letter t? How many start with the letter t and end with the letter c?

14. Evaluate each of the following.

a) $P(7, 2)$ **b)** $P(8, 4)$ **c)** $P(10, 7)$ **d)** $P(12, 3)$

15. In how many ways can the symbols a, b, c, d, e, e, e, e, e be arranged so that no e is adjacent to another e?

16. An alphabet of 40 symbols is used for transmitting messages in a communication system. How many distinct messages (lists of symbols) of 25 symbols can the transmitter generate if symbols can be repeated in the message? How many if 10 of the 40 symbols can appear only as the first and/or last symbols of the message, the other 30 symbols can appear anywhere, and repetitions of all symbols are allowed?

17. In the Internet each network interface of a computer is assigned one, or more, Internet addresses. The nature of these Internet addresses is dependent on network size. For the Internet Standard regarding reserved network numbers (STD 2), each address is a 32-bit string which falls into one of the following three classes: (1) A class A address, used for the largest networks, begins with a 0 which is then followed by a seven-bit *network number*, and then a 24-bit *local address*. However, one is restricted from using the network numbers of all 0's or all 1's and the local addresses of all 0's or all 1's. (2) The class B address is meant for an intermediate-sized network. This address starts with the two-bit string 10, which is followed by a 14-bit network number and then a 16-bit local address. But the local addresses of all 0's or all 1's are not permitted. (3) Class C addresses are used for the smallest networks. These addresses consist of the three-bit string 110, followed by a 21-bit network number, and then an eight-bit local address. Once again the local addresses of all 0's or all 1's are excluded. How many different addresses of each class are available on the Internet, for this Internet Standard?

18. Morgan is considering the purchase of a low-end computer system. After some careful investigating, she finds that there are seven basic systems (each consisting of a monitor, CPU, keyboard, and mouse) that meet her requirements. Furthermore, she also plans to buy one of four modems, one of three CD ROM drives, and one of six printers. (Here each peripheral device of a given type, such as the modem, is compatible with all seven basic systems.) In how many ways can Morgan configure her low-end computer system?

19. A computer science professor has seven different programming books on a bookshelf. Three of the books deal with C++, the other four with Java. In how many ways can the professor arrange these books on the shelf (a) if there are no restrictions? (b) if the languages should alternate? (c) if all the C++ books must be next to each other? (d) if all the C++ books must be next to each other and all the Java books must be next to each other?

20. Over the Internet, data are transmitted in structured blocks of bits called *datagrams*.

a) In how many ways can the letters in DATAGRAM be arranged?

b) For the arrangements of part (a), how many have all three A's together?

21. **a)** How many arrangements are there of all the letters in SOCIOLOGICAL?

b) In how many of the arrangements in part (a) are A and G adjacent?

c) In how many of the arrangements in part (a) are all the vowels adjacent?

22. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

23. Twelve clay targets (identical in shape) are arranged in four hanging columns, as shown in Fig. 1.5. There are four red targets in the first column, three white ones in the second column, two green targets in the third column, and three blue ones in the fourth column. To join her college drill team, Deborah must break all 12 of these targets (using her pistol and only 12 bullets) and in so doing must always break the existing target at the bottom of a column. Under these conditions, in how many different orders can Deborah shoot down (and break) the 12 targets?

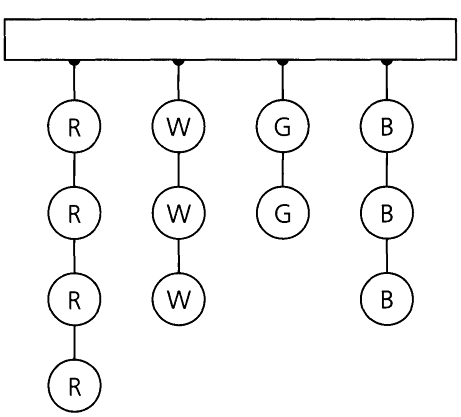

Figure 1.5

24. Show that for all integers $n, r \geq 0$, if $n + 1 > r$, then

$$P(n+1, r) = \left(\frac{n+1}{n+1-r}\right) P(n, r).$$

25. Find the value(s) of n in each of the following: (a) $P(n, 2) = 90$, (b) $P(n, 3) = 3P(n, 2)$, and (c) $2P(n, 2) + 50 = P(2n, 2)$.

26. How many different paths in the xy-plane are there from (0, 0) to (7, 7) if a path proceeds one step at a time by going either one space to the right (R) or one space upward (U)? How many such paths are there from (2, 7) to (9, 14)? Can any general statement be made that incorporates these two results?

27. **a)** How many distinct paths are there from $(-1, 2, 0)$ to $(1, 3, 7)$ in Euclidean three-space if each move is one of the following types?

$$\text{(H): } (x, y, z) \rightarrow (x+1, y, z);$$
$$\text{(V): } (x, y, z) \rightarrow (x, y+1, z);$$
$$\text{(A): } (x, y, z) \rightarrow (x, y, z+1)$$

b) How many such paths are there from $(1, 0, 5)$ to $(8, 1, 7)$?

c) Generalize the results in parts (a) and (b).

28. **a)** Determine the value of the integer variable *counter* after execution of the following program segment. (Here i, j, and k are integer variables.)

```
counter := 0
for i := 1 to 12 do
   counter := counter + 1
for j := 5 to 10 do
   counter := counter + 2
for k := 15 downto 8 do
   counter := counter + 3
```

b) Which counting principle is at play in part (a)?

29. Consider the following program segment where i, j, and k are integer variables.

```
for i := 1 to 12 do
   for j := 5 to 10 do
      for k := 15 downto 8 do
         print (i - j)*k
```

a) How many times is the **print** statement executed?

b) Which counting principle is used in part (a)?

30. A sequence of letters of the form abcba, where the expression is unchanged upon reversing order, is an example of a *palindrome* (of five letters). (a) If a letter may appear more than twice, how many palindromes of five letters are there? of six letters? (b) Repeat part (a) under the condition that no letter appears more than twice.

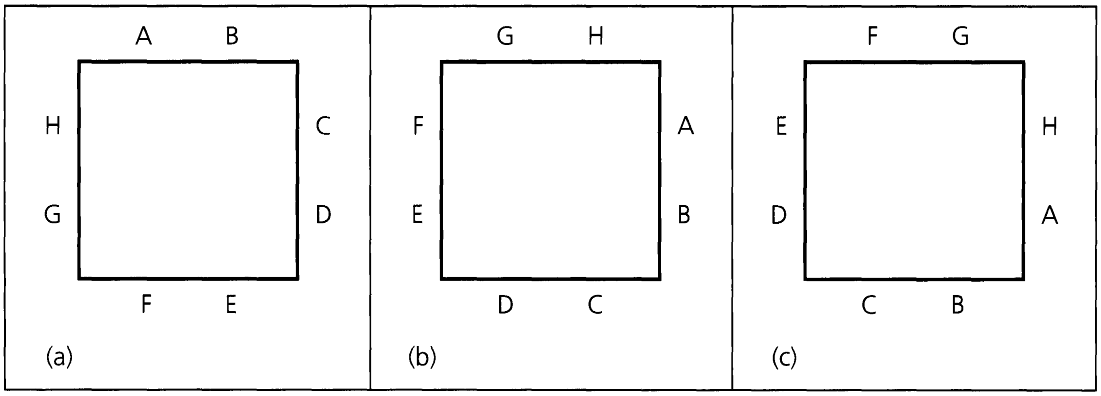

Figure 1.6

31. Determine the number of six-digit integers (no leading zeros) in which (a) no digit may be repeated; (b) digits may be repeated. Answer parts (a) and (b) with the extra condition that the six-digit integer is (i) even; (ii) divisible by 5; (iii) divisible by 4.

32. a) Provide a combinatorial argument to show that if n and k are positive integers with $n = 3k$, then $n!/(3!)^k$ is an integer.

b) Generalize the result of part (a).

33. a) In how many possible ways could a student answer a 10-question true-false test?

b) In how many ways can the student answer the test in part (a) if it is possible to leave a question unanswered in order to avoid an extra penalty for a wrong answer?

34. How many distinct four-digit integers can one make from the digits 1, 3, 3, 7, 7, and 8?

35. a) In how many ways can seven people be arranged about a circular table?

b) If two of the people insist on sitting next to each other, how many arrangements are possible?

36. a) In how many ways can eight people, denoted A, B, . . . , H be seated about the square table shown in Fig. 1.6, where Figs. 1.6(a) and 1.6(b) are considered the same but are distinct from Fig. 1.6(c)?

b) If two of the eight people, say A and B, do not get along well, how many different seatings are possible with A and B not sitting next to each other?

37. Sixteen people are to be seated at two circular tables, one of which seats 10 while the other seats six. How many different seating arrangements are possible?

38. A committee of 15—nine women and six men—is to be seated at a circular table (with 15 seats). In how many ways can the seats be assigned so that no two men are seated next to each other?

39. Write a computer program (or develop an algorithm) to determine whether there is a three-digit integer abc $(= 100a + 10b + c)$ where $abc = a! + b! + c!$.

1.3 Combinations: The Binomial Theorem

The standard deck of playing cards consists of 52 cards comprising four suits: clubs, diamonds, hearts, and spades. Each suit has 13 cards: ace, 2, 3, . . . , 9, 10, jack, queen, king. If we are asked to draw three cards from a standard deck, in succession and without replacement, then by the rule of product there are

$$52 \times 51 \times 50 = \frac{52!}{49!} = P(52, 3)$$

possibilities, one of which is AH (ace of hearts), 9C (nine of clubs), KD (king of diamonds). If instead we simply select three cards at one time from the deck so that the order of selection of the cards is no longer important, then the six permutations AH–9C–KD, AH–KD–9C, 9C–AH–KD, 9C–KD–AH, KD–9C–AH, and KD–AH–9C all correspond to just one (unordered) selection. Consequently, each selection, or combination, of three cards, *with no reference to order*, corresponds to 3! permutations of three cards. In equation form

Solutions

Chapter 1
Fundamental Principles of Counting

Sections 1.1 and 1.2–p. 11

1. a) 13 **b)** 40 **c)** The rule of sum in part (a); the rule of product in part (b)
3. a) 288 **b)** 24
5. $2 \times 2 \times 1 \times 10 \times 10 \times 2 = 800$ different license plates
7. 2^9 **9. a)** $(14)(12) = 168$ **b)** $(14)(12)(6)(18) = 18{,}144$ **c)** 73,156,608
11. a) $12 + 2 = 14$ **b)** $14 \times 14 = 196$ **c)** 182
13. a) $P(8, 8) = 8!$ **b)** $7!$ $6!$ **15.** $4! = 24$
17. Class A: $(2^7 - 2)(2^{24} - 2) = 2{,}113{,}928{,}964$
Class B: $2^{14}(2^{16} - 2) = 1{,}073{,}709{,}056$
Class C: $2^{12}(2^8 - 2) = 1{,}040{,}384$
19. a) $7! = 5040$ **b)** $(4!)(3!) = 144$ **c)** $(5!)(3!) = 720$ **d)** 288
21. a) $12!/(3!\,2!\,2!\,2!)$ **b)** $2[11!/(3!\,2!\,2!\,2!)]$ **c)** $[7!/(2!\,2!)][6!/(3!\,2!)]$
23. $12!/(4!\,3!\,2!\,3!) = 277{,}200$ **25. a)** $n = 10$ **b)** $n = 5$ **c)** $n = 5$
27. a) $(10!)/(2!\,7!) = 360$ **b)** 360
c) Let x, y, and z be any real numbers and let m, n, and p be any nonnegative integers. The number of paths from (x, y, z) to $(x + m, y + n, z + p)$, as described in part (a), is $(m + n + p)!/(m!\,n!\,p!)$.
29. a) 576 **b)** The rule of product
31. a) $9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136{,}080$ **b)** 9×10^5
(i) (a) 68,880 (b) 450,000
(ii) (a) 28,560 (b) 180,000
(iii) (a) 33,600 (b) 225,000
33. a) 2^{10} **b)** 3^{10} **35. a)** $6!$ **b)** $2(5!) = 240$
37. $\binom{16}{10}9!\,5! = 348{,}713{,}164{,}800$

Section 1.3–p. 24

1. $\binom{6}{2} = 6!/(2!\,4!) = 15$. The selections of size 2 are *ab*, *ac*, *ad*, *ae*, *af*, *bc*, *bd*, *be*, *bf*, *cd*, *ce*, *cf*, *de*, *df*, and *ef*.
3. a) $C(10, 4) = 10!/(4!\,6!) = 210$ **b)** $\binom{12}{7} = 12!/(7!\,5!) = 792$
c) $C(14, 12) = 91$ **d)** $\binom{15}{10} = 3003$
5. a) $P(5, 3) = 60$
b) a, f, m a, f, r a, f, t a, m, r a, m, t
a, r, t f, m, r f, m, t f, r, t m, r, t
7. a) $\binom{20}{12} = 125{,}970$ **b)** $\binom{10}{6}\binom{10}{6} = 44{,}100$ **c)** $\sum_{i=1}^{5}\binom{10}{12-2i}\binom{10}{2i}$
d) $\sum_{i=7}^{10}\binom{10}{i}\binom{10}{12-i}$ **e)** $\sum_{i=8}^{10}\binom{10}{i}\binom{10}{12-i}$
9. a) $\binom{8}{2} = 28$ **b)** 70 **c)** $\binom{8}{6} = 28$ **d)** 37
11. a) 120 **b)** 56 **c)** 100
13. $\binom{8}{4}\left(\frac{7!}{4!\,2!}\right) = 7350$
15. a) $\binom{15}{2} = 105$ **b)** $\binom{25}{3} = 2300$; $\binom{25}{3}$; $\binom{25}{4} = 12{,}650$
17. a) $\sum_{k=2}^{n}\frac{1}{k!}$ **c)** $\sum_{j=1}^{7}(-1)^{j-1}j^3 = \sum_{k=1}^{7}(-1)^{k+1}k^3$ **d)** $\sum_{i=0}^{n}\frac{i+1}{n+i}$
19. $\binom{10}{3} + \binom{10}{1}\binom{9}{1} + \binom{10}{1} = 220$ $\binom{10}{4} + \binom{10}{2} + \binom{10}{1}\binom{9}{2} + \binom{10}{1}\binom{9}{1} = 705$
$2^{10}\left(\sum_{i=0}^{5}\binom{10}{2i}\right)$