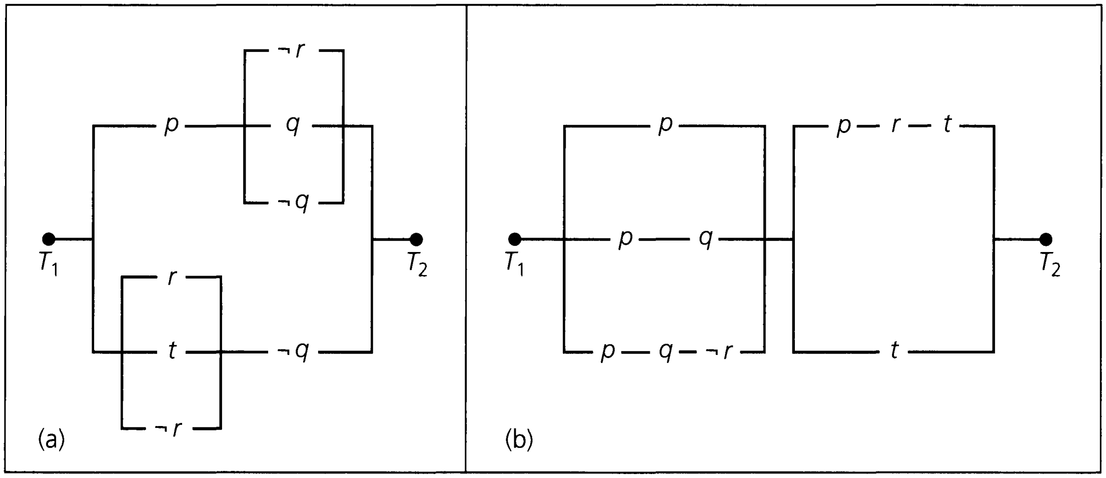

Figure 2.3

b) $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$ **Reasons**
$\Leftrightarrow (p \rightarrow q) \wedge \neg q$
$\Leftrightarrow (\neg p \vee q) \wedge \neg q$
$\Leftrightarrow \neg q \wedge (\neg p \vee q)$
$\Leftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$
$\Leftrightarrow (\neg q \wedge \neg p) \vee F_0$
$\Leftrightarrow \neg q \wedge \neg p$
$\Leftrightarrow \neg(q \vee p)$

19. Provide the steps and reasons, as in Exercise 18, to establish the following logical equivalences.

a) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$

b) $p \vee q \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow p \vee q \vee r$

c) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

20. Simplify each of the networks shown in Fig. 2.3.

2.3 Logical Implication: Rules of Inference

At the end of Section 2.1 we mentioned the notion of a valid argument. Now we will begin a formal study of what we shall mean by an argument and when such an argument is valid. This in turn will help us when we investigate how to prove theorems throughout the text.

We start by considering the general form of an argument, one we wish to show is valid. So let us consider the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q.$$

Here n is a positive integer, the statements $p_1, p_2, p_3, \ldots, p_n$ are called the *premises* of the argument, and the statement q is the *conclusion* for the argument.

The preceding argument is called *valid* if whenever each of the premises $p_1, p_2, p_3, \ldots, p_n$ is true, then the conclusion q is likewise true. [Note that if any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then the hypothesis $p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n$ is false and the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is automatically true, regardless of the truth value of q.] Consequently, one way to establish the validity of a given argument is to show that the statement $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology.

The following examples illustrate this particular approach.

EXAMPLE 2.19

Let p, q, r denote the primitive statements given as

p: Roger studies.

q: Roger plays racketball.

r: Roger passes discrete mathematics.

Now let p_1, p_2, p_3 denote the premises

p_1: If Roger studies, then he will pass discrete mathematics.

p_2: If Roger doesn't play racketball, then he'll study.

p_3: Roger failed discrete mathematics.

We want to determine whether the argument

$$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$$

is valid. To do so, we rewrite p_1, p_2, p_3 as

$$p_1:\quad p \rightarrow r \qquad p_2:\quad \neg q \rightarrow p \qquad p_3:\quad \neg r$$

and examine the truth table for the implication

$$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$$

given in Table 2.14. Because the final column in Table 2.14 contains all 1's, the implication is a tautology. Hence we can say that $(p_1 \wedge p_2 \wedge p_3) \rightarrow q$ is a valid argument.

Table 2.14

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \wedge (\neg q \rightarrow p) \wedge \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

EXAMPLE 2.20

Let us now consider the truth table in Table 2.15. The results in the last column of this table show that for any primitive statements p, r, and s, the implication

$$[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$$

Table 2.15

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[(p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

is a tautology. Consequently, for premises

$$p_1: \quad p \qquad p_2: \quad (p \wedge r) \to s$$

and conclusion q: $(r \to s)$, we know that $(p_1 \wedge p_2) \to q$ is a valid argument, and we may say that the truth of the conclusion q is *deduced* or *inferred* from the truth of the premises p_1, p_2.

The idea presented in the preceding two examples leads to the following.

Definition 2.4 If p, q are arbitrary statements such that $p \to q$ is a tautology, then we say that p *logically implies* q and we write $p \Rightarrow q$ to denote this situation.

When p, q are statements and $p \Rightarrow q$, the implication $p \to q$ is a tautology and we refer to $p \to q$ as a *logical implication.* Note that we can avoid dealing with the idea of a tautology here by saying that $p \Rightarrow q$ (that is, p logically implies q) if q is true whenever p is true.

In Example 2.6 we found that for primitive statements p, q, the implication $p \to (p \vee q)$ is a tautology. In this case, therefore, we can say that p logically implies $p \vee q$ and write $p \Rightarrow (p \vee q)$. Furthermore, because of the first substitution rule, we also find that $p \Rightarrow (p \vee q)$ for any statements p, q—that is, $p \to (p \vee q)$ is a tautology for any statements p, q, whether or not they are primitive statements.

Let p, q be arbitrary statements.

1) If $p \Longleftrightarrow q$, then the statement $p \leftrightarrow q$ is a tautology, so the statements p, q have the same (corresponding) truth values. Under these conditions the statements $p \to q$, $q \to p$ are tautologies, and we have $p \Rightarrow q$ and $q \Rightarrow p$.

2) Conversely, suppose that $p \Rightarrow q$ and $q \Rightarrow p$. The logical implication $p \to q$ tells us that we never have statement p with the truth value 1 and statement q with the truth value 0. But could we have q with the truth value 1 and p with the truth value 0? If this occurred, we could not have the logical implication $q \to p$. Therefore, when $p \Rightarrow q$ and $q \Rightarrow p$, the statements p, q have the same (corresponding) truth values and $p \Longleftrightarrow q$.

Finally, the notation $p \not\Rightarrow q$ is used to indicate that $p \to q$ is *not* a tautology—so the given implication (namely, $p \to q$) is *not* a logical implication.

EXAMPLE 2.21 From the results in Example 2.8 (Table 2.9) and the first substitution rule, we know that for statements p, q,

$$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q.$$

Consequently,

$$\neg(p \wedge q) \Rightarrow (\neg p \vee \neg q) \quad \text{and} \quad (\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$$

for all statements p, q. Alternatively, because each of the implications

$$\neg(p \wedge q) \to (\neg p \vee \neg q) \quad \text{and} \quad (\neg p \vee \neg q) \to \neg(p \wedge q)$$

is a tautology, we may also write

$$[\neg(p \wedge q) \to (\neg p \vee \neg q)] \Longleftrightarrow T_0 \quad \text{and} \quad [(\neg p \vee \neg q) \to \neg(p \wedge q)] \Longleftrightarrow T_0.$$

Returning now to our study of techniques for establishing the validity of an argument, we must take a careful look at the size of Tables 2.14 and 2.15. Each table has eight rows. For Table 2.14 we were able to express the three premises p_1, p_2, and p_3, and the conclusion q, in terms of the three primitive statements p, q, and r. A similar situation arose for the argument we analyzed in Table 2.15, where we had only two premises. But if we were confronted, for example, with establishing whether

$$[(p \rightarrow r) \land (r \rightarrow s) \land (t \lor \neg s) \land (\neg t \lor u) \land \neg u] \rightarrow \neg p$$

is a logical implication (or presents a valid argument), the needed table would require $2^5 = 32$ rows. As the number of premises gets larger and our truth tables grow to 64, 128, 256, or more rows, this first technique for establishing the validity of an argument rapidly loses its appeal.

Furthermore, looking at Table 2.14 once again, we realize that in order to establish whether

$$[(p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r] \rightarrow q$$

is a valid argument, we need to consider only those rows of the table where each of the three premises $p \rightarrow r$, $\neg q \rightarrow p$, and $\neg r$ has the truth value 1. (Remember that if the hypothesis — consisting of the conjunction of all of the premises — is false, then the implication is true regardless of the truth value of the conclusion.) This happens only in the third row, so a good deal of Table 2.14 is not really necessary. (It is not always the case that only one row has all of the premises true. Note that in Table 2.15 we would be concerned with the results in rows 5, 6, and 8.)

Consequently, what these observations are telling us is that we can possibly eliminate a great deal of the effort put into constructing the truth tables in Table 2.14 and Table 2.15. And since we want to avoid even larger tables, we are persuaded to develop a list of techniques called *rules of inference* that will help us as follows:

1) Using these techniques will enable us to consider only the cases wherein all the premises are true. Hence we consider the conclusion only for those rows of a truth table wherein each premise has the truth value 1 — and we do *not* construct the truth table.
2) The rules of inference are fundamental in the development of a step-by-step validation of how the conclusion q logically follows from the premises $p_1, p_2, p_3, \ldots, p_n$ in an implication of the form

$$(p_1 \land p_2 \land p_3 \land \cdots \land p_n) \rightarrow q.$$

Such a development will establish the validity of the given argument, for it will show how the truth of the conclusion can be deduced from the truth of the premises.

Each rule of inference arises from a logical implication. In some cases, the logical implication is stated without proof. (However, several of these proofs will be dealt with in the Section Exercises.)

Many rules of inference arise in the study of logic. We concentrate on those that we need to help us validate the arguments that arise in our study of logic. These rules will also help us later when we turn to methods for proving theorems throughout the remainder of the text. Table 2.19 (on p. 78) summarizes the rules we shall now start to investigate.

EXAMPLE 2.22

For a first example we consider the rule of inference called *Modus Ponens,* or the *Rule of Detachment.* (*Modus Ponens* comes from Latin and may be translated as "the method of affirming.") In symbolic form this rule is expressed by the logical implication

$$[p \wedge (p \rightarrow q)] \rightarrow q,$$

which is verified in Table 2.16, where we find that the fourth row is the only one where both of the premises p and $p \rightarrow q$ (and the conclusion q) are true.

Table 2.16

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

The actual rule will be written in the tabular form

$$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore q \end{array}$$

where the three dots ($\therefore$) stand for the word "therefore," indicating that q is the conclusion for the premises p and $p \rightarrow q$, which appear above the horizontal line.

This rule arises when we argue that *if* (1) p is true, *and* (2) $p \rightarrow q$ is true (or $p \Rightarrow q$), *then* the conclusion q must also be true. (After all, if q were false and p were true, then we could not have $p \rightarrow q$ true.)

The following valid arguments show us how to apply the Rule of Detachment.

a) 1) Lydia wins a ten-million-dollar lottery. p
2) If Lydia wins a ten-million-dollar lottery, then Kay will quit her job. $p \rightarrow q$
3) Therefore Kay will quit her job. $\therefore q$

b) 1) If Allison vacations in Paris, then she will have to win a scholarship. $p \rightarrow q$
2) Allison is vacationing in Paris. p
3) Therefore Allison won a scholarship. $\therefore q$

Before closing the discussion on our first rule of inference let us make one final observation. The two examples in (a) and (b) might suggest that the valid argument $[p \wedge (p \rightarrow q)] \rightarrow q$ is appropriate only for primitive statements p, q. However, since $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology for primitive statements p, q, it follows from the first substitution rule that (all occurrences of) p or q may be replaced by compound statements — and the resulting implication will also be a tautology. Consequently, if r, s, t, and u are primitive statements, then

$$\begin{array}{l} \quad r \vee s \\ \quad (r \vee s) \rightarrow (\neg t \wedge u) \\ \hline \therefore \neg t \wedge u \end{array}$$

is a valid argument, by the Rule of Detachment — just as $[(r \vee s) \wedge [(r \vee s) \rightarrow (\neg t \wedge u)]] \rightarrow (\neg t \wedge u)$ is a tautology.

A similar situation — in which we can apply the first substitution rule — occurs for each of the rules of inference we shall study. However, we shall not mention this so explicitly with these other rules of inference.

EXAMPLE 2.23

A second rule of inference is given by the logical implication

$$[(p \to q) \land (q \to r)] \to (p \to r),$$

where p, q, and r are any statements. In tabular form it is written

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

This rule, which is referred to as the *Law of the Syllogism,* arises in many arguments. For example, we may use it as follows:

1) If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. $p \to q$

2) If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. $\underline{q \to r}$

3) Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. $\therefore p \to r$

The next example involves a slightly longer argument that uses the rules of inference developed in Examples 2.22 and 2.23. In fact, we find here that there may be more than one way to establish the validity of an argument.

EXAMPLE 2.24

Consider the following argument.

1) Rita is baking a cake.

2) If Rita is baking a cake, then she is not practicing her flute.

3) If Rita is not practicing her flute, then her father will not buy her a car.

4) Therefore Rita's father will not buy her a car.

Concentrating on the forms of the statements in the preceding argument, we may write the argument as

$$\begin{array}{l} p \\ p \to \neg q \\ \underline{\neg q \to \neg r} \\ \therefore \neg r \end{array} \qquad (*)$$

Now we need no longer worry about what the statements actually stand for. Our objective is to use the two rules of inference that we have studied so far in order to deduce the truth of the statement $\neg r$ from the truth of the three premises p, $p \to \neg q$, and $\neg q \to \neg r$.

We establish the validity of the argument as follows:

Steps	**Reasons**
1) $p \rightarrow \neg q$	Premise
2) $\neg q \rightarrow \neg r$	Premise
3) $p \rightarrow \neg r$	This follows from steps (1) and (2) and the Law of the Syllogism
4) p	Premise
5) $\therefore \neg r$	This follows from steps (4) and (3) and the Rule of Detachment

Before continuing with a third rule of inference we shall show that the argument presented at (*) can be validated in a second way. Here our "reasons" will be shortened to the form we shall use for the rest of the section. However, we shall always list whatever is needed to demonstrate how each step in an argument comes about, or follows, from prior steps.

A second way to validate the argument follows.

Steps	**Reasons**
1) p	Premise
2) $p \rightarrow \neg q$	Premise
3) $\neg q$	Steps (1) and (2) and the Rule of Detachment
4) $\neg q \rightarrow \neg r$	Premise
5) $\therefore \neg r$	Steps (3) and (4) and the Rule of Detachment

EXAMPLE 2.25

The rule of inference called *Modus Tollens* is given by

$$\begin{array}{r} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

This follows from the logical implication $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$. *Modus Tollens* comes from Latin and can be translated as "method of denying." This is appropriate because we deny the conclusion, q, so as to prove $\neg p$. (Note that we can also obtain this rule from the one for Modus Ponens by using the fact that $p \rightarrow q \Longleftrightarrow \neg q \rightarrow \neg p$.)

The following exemplifies the use of Modus Tollens is making a valid inference:

1) If Connie is elected president of Phi Delta sorority, then Helen will pledge that sorority. $p \rightarrow q$

2) Helen did not pledge Phi Delta sorority. $\neg q$

3) Therefore Connie was not elected president of Phi Delta sorority. $\therefore \neg p$

And now we shall use Modus Tollens to show that the following argument is valid (for primitive statements p, r, s, t, and u).

$$\begin{array}{c} p \rightarrow r \\ r \rightarrow s \\ t \vee \neg s \\ \neg t \vee u \\ \neg u \\ \hline \therefore \neg p \end{array}$$

Both Modus Tollens and the Law of the Syllogism come into play, along with the logical equivalence we developed in Example 2.7.

Steps	**Reasons**
1) $p \to r, r \to s$	Premises
2) $p \to s$	Step (1) and the Law of the Syllogism
3) $t \vee \neg s$	Premise
4) $\neg s \vee t$	Step (3) and the Commutative Law of $\vee$
5) $s \to t$	Step (4) and the fact that $\neg s \vee t \Longleftrightarrow s \to t$
6) $p \to t$	Steps (2) and (5) and the Law of the Syllogism
7) $\neg t \vee u$	Premise
8) $t \to u$	Step (7) and the fact that $\neg t \vee u \Longleftrightarrow t \to u$
9) $p \to u$	Steps (6) and (8) and the Law of the Syllogism
10) $\neg u$	Premise
11) $\therefore \neg p$	Steps (9) and (10) and Modus Tollens

Before continuing with another rule of inference let us summarize what we have just accomplished (and *not* accomplished). The preceding argument shows that

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \Rightarrow \neg p.$$

We have *not* used the laws of logic, as in Section 2.2, to express the statement

$$(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u$$

as a simpler logically equivalent statement. Note that

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \not\Leftrightarrow \neg p.$$

For when p has the truth value 0 and u has the truth value 1, the truth value of $\neg p$ is 1 while that of $\neg u$ and $(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u$ is 0.

Let us once more examine a tabular form for each of the two related rules of inference, Modus Ponens and Modus Tollens.

$$\text{Modus Ponens:}\quad \begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array} \qquad\qquad \text{Modus Tollens:}\quad \begin{array}{l} p \to q \\ \underline{\neg q \qquad} \\ \therefore \neg p \end{array}$$

The reason we wish to do this is that there are other tabular forms that may arise — and these are similar in appearance but present *invalid* arguments — where each of the premises is true but the conclusion is false.

a) Consider the following argument:

1) If Margaret Thatcher is the president of the United States, then she is at least 35 years old. $\qquad p \to q$

2) Margaret Thatcher is at least 35 years old. $\qquad \underline{q \qquad}$

3) Therefore Margaret Thatcher is the president of the United States. $\qquad \therefore p$

Here we find that $[(p \to q) \wedge q] \to p$ is *not* a tautology. For if we consider the truth value assignments p: 0 and q: 1, then each of the premises $p \to q$ and q is true while the conclusion p is false. This *invalid* argument results from the *fallacy* (error in reasoning) where we try to argue by the converse — that is, while $[(p \to q) \wedge p] \Rightarrow q$, it is *not the case* that $[(p \to q) \wedge q] \Rightarrow p$.

b) A second argument where the conclusion doesn't necessarily follow from the premises may be given by:

1) If $2 + 3 = 6$, then $2 + 4 = 6$.	$p \to q$
2) $2 + 3 \neq 6$.	$\neg p$
3) Therefore $2 + 4 \neq 6$.	$\therefore \neg q$

In this case we find that $[(p \to q) \land \neg p] \to \neg q$ is *not* a tautology. Once again the truth value assignments p: 0 and q: 1 show us that the premises $p \to q$ and $\neg p$ can both be true while the conclusion $\neg q$ is false. The fallacy behind this invalid argument arises from our attempt to argue by the inverse—for although $[(p \to q) \land \neg q] \Rightarrow \neg p$, it does *not* follow that $[(p \to q) \land \neg p] \Rightarrow \neg q$.

Before proceeding further we now mention a rather simple but important rule of inference.

EXAMPLE 2.26

The following rule of inference arises from the observation that if p, q are true statements, then $p \land q$ is a true statement.

Now suppose that statements p, q occur in the development of an argument. These statements may be (given) premises or results that are derived from premises and/or from results developed earlier in the argument. Then under these circumstances the two statements p, q can be combined into their conjunction $p \land q$, and this new statement can be used in later steps as the argument continues.

We call this rule the *Rule of Conjunction* and write it in tabular form as

$$\begin{array}{l} p \\ \underline{q \quad\quad} \\ \therefore p \land q \end{array}$$

As we proceed further with our study of rules of inference, we find another fairly simple but important rule.

EXAMPLE 2.27

The following rule of inference—one we may feel just illustrates good old common sense—is called the *Rule of Disjunctive Syllogism*. This rule comes about from the logical implication

$$[(p \lor q) \land \neg p] \to q,$$

which we can derive from Modus Ponens by observing that $p \lor q \Leftrightarrow \neg p \to q$.

In tabular form we write

$$\begin{array}{l} p \lor q \\ \underline{\neg p \quad\quad} \\ \therefore q \end{array}$$

This rule of inference arises when there are exactly two possibilities to consider and we are able to eliminate one of them as being true. Then the other possibility has to be true. The following illustrates one such application of this rule.

1) Bart's wallet is in his back pocket or it is on his desk.	$p \lor q$
2) Bart's wallet is not in his back pocket.	$\neg p$
3) Therefore Bart's wallet is on his desk.	$\therefore q$

At this point we have examined five rules of inference. But before we try to validate any more arguments like the one (with 11 steps) in Example 2.25, we shall look at one more of these rules. This one underlies a method of proof that is sometimes confused with the contrapositive method (or proof) given in Modus Tollens. The confusion arises because both methods involve the negation of a statement. However, we will soon realize that these are two distinct methods. (Toward the end of Section 2.5 we shall compare and contrast these two methods once again.)

EXAMPLE 2.28

Let p denote an arbitrary statement, and F_0 a contradiction. The results in column 5 of Table 2.17 show that the implication $(\neg p \rightarrow F_0) \rightarrow p$ is a tautology, and this provides us with the rule of inference called the *Rule of Contradiction*. In tabular form this rule is written as

$$\begin{array}{l} \neg p \rightarrow F_0 \\ \hline \therefore p \end{array}$$

Table 2.17

p	$\neg p$	F_0	$\neg p \rightarrow F_0$	$(\neg p \rightarrow F_0) \rightarrow p$
1	0	0	1	1
0	1	0	0	1

This rule tells us that if p is a statement and $\neg p \rightarrow F_0$ is true, then $\neg p$ must be false because F_0 is false. So then we have p true.

The Rule of Contradiction is the basis of a method for establishing the validity of an argument — namely, the method of *Proof by Contradiction*, or *Reductio ad Absurdum*. The idea behind the method of Proof by Contradiction is to establish a statement (namely, the conclusion of an argument) by showing that, if this statement were false, then we would be able to deduce an impossible consequence. The use of this method arises in certain arguments which we shall now describe.

In general, when we want to establish the validity of the argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q,$$

we can establish the validity of the logically equivalent argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q) \rightarrow F_0.$$

[This follows from the tautology in column 7 of Table 2.18 and the first substitution rule — where we replace the primitive statement p by the statement $(p_1 \wedge p_2 \wedge \cdots \wedge p_n)^{\dagger}$.]

Table 2.18

p	q	F_0	$p \wedge \neg q$	$(p \wedge \neg q) \rightarrow F_0$	$p \rightarrow q$	$(p \rightarrow q) \leftrightarrow [(p \wedge \neg q) \rightarrow F_0]$
0	0	0	0	1	1	1
0	1	0	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

†In Section 4.2 we shall provide the reason why we know that for any statements $p_1, p_2, \ldots, p_n$, and q, it follows that $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \wedge \neg q \Longleftrightarrow p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q$.

When we apply the method of Proof by Contradiction, we first assume that what we are trying to validate (or prove) is actually false. Then we use this assumption as an additional premise in order to produce a contradiction (or impossible situation) of the form $s \wedge \neg s$, for some statement s. Once we have derived this contradiction we may then conclude that the statement we were given was in fact true — and this validates the argument (or completes the proof).

We shall turn to the method of Proof by Contradiction when it is (or appears to be) easier to use $\neg q$ *in conjunction with* the premises $p_1, p_2, \ldots, p_n$ in order to deduce a contradiction than it is to deduce the conclusion q directly from the premises $p_1, p_2, \ldots, p_n$. The method of Proof by Contradiction will be used in some of the later examples for this section — namely, Examples 2.32 and 2.35. We shall also find it frequently reappearing in other chapters in the text.

Now that we have examined six rules of inference, we summarize these rules and introduce several others in Table 2.19 (on the following page).

The next five examples will present valid arguments. In so doing, these examples will show us how to apply the rules listed in Table 2.19 in conjunction with other results, such as the laws of logic.

EXAMPLE 2.29

Our first example demonstrates the validity of the argument

$$\begin{array}{l} p \rightarrow r \\ \neg p \rightarrow q \\ \underline{q \rightarrow s} \\ \therefore \neg r \rightarrow s \end{array}$$

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $\neg r \rightarrow \neg p$	Step (1) and $p \rightarrow r \Leftrightarrow \neg r \rightarrow \neg p$
3) $\neg p \rightarrow q$	Premise
4) $\neg r \rightarrow q$	Steps (2) and (3) and the Law of the Syllogism
5) $q \rightarrow s$	Premise
6) $\therefore \neg r \rightarrow s$	Steps (4) and (5) and the Law of the Syllogism

A second way to validate the given argument proceeds as follows.

Steps	**Reasons**
1) $p \rightarrow r$	Premise
2) $q \rightarrow s$	Premise
3) $\neg p \rightarrow q$	Premise
4) $p \vee q$	Step (3) and $(\neg p \rightarrow q) \Leftrightarrow (\neg\neg p \vee q) \Leftrightarrow (p \vee q)$, where the second logical equivalence follows by the Law of Double Negation
5) $r \vee s$	Steps (1), (2), and (4) and the Rule of the Constructive Dilemma
6) $\therefore \neg r \rightarrow s$	Step (5) and $(r \vee s) \Leftrightarrow (\neg\neg r \vee s) \Leftrightarrow (\neg r \rightarrow s)$, where the Law of Double Negation is used in the first logical equivalence

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The next example is somewhat more involved.

Table 2.19

	Rule of Inference	Related Logical Implication	Name of Rule
1)	$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$	$[p \land (p \to q)] \to q$	Rule of Detachment (Modus Ponens)
2)	$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$	$[(p \to q) \land (q \to r)] \to (p \to r)$	Law of the Syllogism
3)	$\begin{array}{l} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$	$[(p \to q) \land \neg q] \to \neg p$	Modus Tollens
4)	$\begin{array}{l} p \\ q \\ \hline \therefore p \land q \end{array}$		Rule of Conjunction
5)	$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore q \end{array}$	$[(p \lor q) \land \neg p] \to q$	Rule of Disjunctive Syllogism
6)	$\begin{array}{l} \neg p \to F_0 \\ \hline \therefore p \end{array}$	$(\neg p \to F_0) \to p$	Rule of Contradiction
7)	$\begin{array}{l} p \land q \\ \hline \therefore p \end{array}$	$(p \land q) \to p$	Rule of Conjunctive Simplification
8)	$\begin{array}{l} p \\ \hline \therefore p \lor q \end{array}$	$p \to p \lor q$	Rule of Disjunctive Amplification
9)	$\begin{array}{l} p \land q \\ p \to (q \to r) \\ \hline \therefore r \end{array}$	$[(p \land q) \land [p \to (q \to r)]] \to r$	Rule of Conditional Proof
10)	$\begin{array}{l} p \to r \\ q \to r \\ \hline \therefore (p \lor q) \to r \end{array}$	$[(p \to r) \land (q \to r)] \to [(p \lor q) \to r]$	Rule for Proof by Cases
11)	$\begin{array}{l} p \to q \\ r \to s \\ p \lor r \\ \hline \therefore q \lor s \end{array}$	$[(p \to q) \land (r \to s) \land (p \lor r)] \to (q \lor s)$	Rule of the Constructive Dilemma
12)	$\begin{array}{l} p \to q \\ r \to s \\ \neg q \lor \neg s \\ \hline \therefore \neg p \lor \neg r \end{array}$	$[(p \to q) \land (r \to s) \land (\neg q \lor \neg s)] \to (\neg p \lor \neg r)$	Rule of the Destructive Dilemma

EXAMPLE 2.30

Establish the validity of the argument

$$\begin{array}{l} p \to q \\ q \to (r \land s) \\ \neg r \lor (\neg t \lor u) \\ p \land t \\ \hline \therefore u \end{array}$$

Steps	Reasons
1) $p \rightarrow q$	Premise
2) $q \rightarrow (r \wedge s)$	Premise
3) $p \rightarrow (r \wedge s)$	Steps (1) and (2) and the Law of the Syllogism
4) $p \wedge t$	Premise
5) p	Step (4) and the Rule of Conjunctive Simplification
6) $r \wedge s$	Steps (5) and (3) and the Rule of Detachment
7) r	Step (6) and the Rule of Conjunctive Simplification
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg(r \wedge t) \vee u$	Step (8), the Associative Law of $\vee$, and DeMorgan's Laws
10) t	Step (4) and the Rule of Conjunctive Simplification
11) $r \wedge t$	Steps (7) and (10) and the Rule of Conjunction
12) $\therefore u$	Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism

EXAMPLE 2.31

This example will provide a way to show that the following argument is valid.

> If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were canceled, then refunds would have had to be made. No refunds were made.
>
> Therefore the band could play rock music.

First we convert the given argument into symbolic form by using the following statement assignments:

p: The band could play rock music.
q: The refreshments were delivered on time.
r: The New Year's party was canceled.
s: Alicia was angry.
t: Refunds had to be made.

The argument above now becomes

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \neg t \\ \hline \therefore p \end{array}$$

We can establish the validity of this argument as follows.

Steps	Reasons
1) $r \rightarrow t$	Premise
2) $\neg t$	Premise
3) $\neg r$	Steps (1) and (2) and Modus Tollens
4) $\neg r \vee \neg s$	Step (3) and the Rule of Disjunctive Amplification
5) $\neg(r \wedge s)$	Step (4) and DeMorgan's Laws
6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
7) $\neg(\neg p \vee \neg q)$	Steps (6) and (5) and Modus Tollens
8) $p \wedge q$	Step (7), DeMorgan's Laws, and the Law of Double Negation
9) $\therefore p$	Step (8) and the Rule of Conjunctive Simplification

EXAMPLE 2.32

In this instance we shall use the method of Proof by Contradiction. Consider the argument

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \underline{\neg r \quad} \\ \therefore p \end{array}$$

To establish the validity for this argument, we assume the negation $\neg p$ of the conclusion p as another premise. The objective now is to use these four premises to derive a contradiction F_0. Our derivation follows.

Steps	**Reasons**
1) $\neg p \leftrightarrow q$	Premise
2) $(\neg p \rightarrow q) \land (q \rightarrow \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Longleftrightarrow [(\neg p \rightarrow q) \land (q \rightarrow \neg p)]$
3) $\neg p \rightarrow q$	Step (2) and the Rule of Conjunctive Simplification
4) $q \rightarrow r$	Premise
5) $\neg p \rightarrow r$	Steps (3) and (4) and the Law of the Syllogism
6) $\neg p$	Premise (the one assumed)
7) r	Steps (5) and (6) and the Rule of Detachment
8) $\neg r$	Premise
9) $r \land \neg r (\Longleftrightarrow F_0)$	Steps (7) and (8) and the Rule of Conjunction
10) $\therefore p$	Steps (6) and (9) and the method of Proof by Contradiction

If we examine further what has happened here, we find that

$$[(\neg p \leftrightarrow q) \land (q \rightarrow r) \land \neg r \land \neg p] \Rightarrow F_0.$$

This requires the truth value of $[(\neg p \leftrightarrow q) \land (q \rightarrow r) \land \neg r \land \neg p]$ to be 0. Because $\neg p \leftrightarrow q$, $q \rightarrow r$, and $\neg r$ are the given premises, each of these statements has the truth value 1. Consequently, for $[(\neg p \leftrightarrow q) \land (q \rightarrow r) \land \neg r \land \neg p]$ to have the truth value 0, the statement $\neg p$ must have the truth value 0. Therefore p has the truth value 1, and the conclusion p of the argument is true.

Before we consider our next example, we need to examine columns 5 and 7 of Table 2.20. These identical columns tell us that for primitive statements p, q, and r,

$$[p \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p \land q) \rightarrow r].$$

Using the first substitution rule, let us replace each occurrence of p by the compound statement $(p_1 \land p_2 \land \cdots \land p_n)$. Then we obtain the new result

$$[(p_1 \land p_2 \land \cdots \land p_n) \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p_1 \land p_2 \land \cdots \land p_n \land q)^{\dagger} \rightarrow r].$$

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†In Section 4.2 we shall present a formal proof of why

$$(p_1 \land p_2 \land \cdots \land p_n) \land q \Longleftrightarrow p_1 \land p_2 \land \cdots \land p_n \land q.$$

Table 2.20

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

This result tells us that if we wish to establish the validity of the argument (*) we may be able to do so by establishing the validity of the corresponding argument (**).

$$(*) \quad \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \rightarrow r \end{array} \qquad\qquad (**) \quad \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ q \\ \hline \therefore r \end{array}$$

After all, suppose we want to show that $q \rightarrow r$ has the truth value 1, when each of $p_1, p_2, \ldots, p_n$ does. If the truth value for q is 0, then there is nothing left to do, since the truth value for $q \rightarrow r$ is 1. Hence the real problem is to show that $q \rightarrow r$ has truth value 1, when each of $p_1, p_2, \ldots, p_n$, *and* q does—that is, we need to show that when $p_1, p_2, \ldots, p_n, q$ each have truth value 1, then the truth value of r is 1.

We demonstrate this principle in the next example.

EXAMPLE 2.33

In order to establish the validity of the argument

$$(*) \qquad \begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ \hline \therefore q \rightarrow p \end{array}$$

we consider the corresponding argument

$$(**) \qquad \begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}$$

[Note that q is the hypothesis of the conclusion $q \rightarrow p$ for argument (*) and that it becomes another premise for argument (**) where the conclusion is p.]

To validate the argument (**) we proceed as follows.

Steps	**Reasons**
1) q	Premise
2) $q \rightarrow (u \wedge s)$	Premise
3) $u \wedge s$	Steps (1) and (2) and the Rule of Detachment
4) u	Step (3) and the Rule of Conjunctive Simplification
5) $u \rightarrow r$	Premise
6) r	Steps (4) and (5) and the Rule of Detachment
7) s	Step (3) and the Rule of Conjunctive Simplification
8) $r \wedge s$	Steps (6) and (7) and the Rule of Conjunction
9) $(r \wedge s) \rightarrow (p \vee t)$	Premise
10) $p \vee t$	Steps (8) and (9) and the Rule of Detachment
11) $\neg t$	Premise
12) $\therefore p$	Steps (10) and (11) and the Rule of Disjunctive Syllogism

We now know that for argument (**)

$$[(u \rightarrow r) \wedge [(r \wedge s) \rightarrow (p \vee t)] \wedge [q \rightarrow (u \wedge s)] \wedge \neg t \wedge q] \Rightarrow p,$$

and for argument (*) it follows that

$$[(u \rightarrow r) \wedge [(r \wedge s) \rightarrow (p \vee t)] \wedge [q \rightarrow (u \wedge s)] \wedge \neg t] \Rightarrow (q \rightarrow p).$$

Examples 2.29 through 2.33 have given us some idea of how to establish the validity of an argument. Following Example 2.25 we discussed two situations indicating when an argument is invalid — namely, when we try to argue by the converse or the inverse. So now it is time for us to learn a little more about how to determine when an argument is invalid.

Given an argument

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \underline{p_n} \\ \therefore q \end{array}$$

we say that the argument is invalid if it is possible for each of the premises $p_1, p_2, p_3, \ldots, p_n$ to be true (with truth value 1), while the conclusion q is false (with truth value 0).

The next example illustrates an indirect method whereby we may be able to show that an argument we *feel* is invalid (perhaps because we cannot find a way to show that it is valid) actually *is* invalid.

EXAMPLE 2.34

Consider the primitive statements p, q, r, s, and t and the argument

$$\begin{array}{l} p \\ p \vee q \\ q \rightarrow (r \rightarrow s) \\ \underline{t \rightarrow r} \\ \therefore \neg s \rightarrow \neg t \end{array}$$

To show that this is an invalid argument, we need *one* assignment of truth values for each of the statements p, q, r, s, and t such that the conclusion $\neg s \rightarrow \neg t$ is false (has the truth value 0) while the four premises are all true (have the truth value 1). The only time the conclusion $\neg s \rightarrow \neg t$ is false is when $\neg s$ is true and $\neg t$ is false. This implies that the truth value for s is 0 and that the truth value for t is 1.

Because p is one of the premises, its truth value must be 1. For the premise $p \vee q$ to have the truth value 1, q may be either true (1) or false (0). So let us consider the premise $t \rightarrow r$ where we know that t is true. If $t \rightarrow r$ is to be true, then r must be true (have the truth value 1). Now with r true (1) and s false (0), it follows that $r \rightarrow s$ is false (0), and that the truth value of the premise $q \rightarrow (r \rightarrow s)$ will be 1 only when q is false (0).

Consequently, under the truth value assignments

$$p\colon\ 1 \qquad q\colon\ 0 \qquad r\colon\ 1 \qquad s\colon\ 0 \qquad t\colon\ 1,$$

the four premises

$$p \qquad p \vee q \qquad q \rightarrow (r \rightarrow s) \qquad t \rightarrow r$$

all have the truth value 1, while the conclusion

$$\neg s \rightarrow \neg t$$

has the truth value 0. In this case we have shown the given argument to be invalid.

The truth value assignments p: 1, q: 0, r: 1, s: 0, and t: 1 of Example 2.34 provide one case that *disproves* what we thought might have been a valid argument. We should now start to realize that in trying to show that an implication of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$$

presents a valid argument, we need to consider *all* cases where the premises $p_1, p_2, p_3, \ldots, p_n$ are true. [Each such case is an assignment of truth values for the primitive statements (that make up the premises) where $p_1, p_2, p_3, \ldots, p_n$ are true.] In order to do so—namely, to cover the cases without writing out the truth table—we have been using the rules of inference together with the laws of logic and other logical equivalences. To cover all the necessary cases, we cannot use one specific example (or case) as a means of establishing the validity of the argument (for all possible cases). However, whenever we wish to show that an implication (of the preceding form) is not a tautology, all we need to find is one case for which the implication is false—that is, one case in which all the premises are true but the conclusion is false. This *one* case provides a *counterexample* for the argument and shows it to be invalid.

Let us consider a second example wherein we try the indirect approach of Example 2.34.

EXAMPLE 2.35

What can we say about the validity or invalidity of the following argument? Here p, q, r, and s denote primitive statements.)

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \underline{\neg p \veebar r} \\ \therefore \neg p \end{array}$$

Can the conclusion $\neg p$ be false while the four premises are all true? The conclusion $\neg p$ is false when p has the truth value 1. So for the premise $p \rightarrow q$ to be true, the truth value of q must be 1. From the truth of the premise $q \rightarrow s$, the truth of q forces the truth of s. Consequently, at this point we have statements p, q, and s all with the truth value 1.

Continuing with the premise $r \rightarrow \neg s$, we find that because s has the truth value 1, the truth value of r must be 0. Hence r is false. But with $\neg p$ false and the premise $\neg p \veebar r$ true, we also have r true. Therefore we find that $p \Rightarrow (\neg r \wedge r)$.

We have failed in our attempt to find a counterexample to the validity of the given argument. However, this failure has shown us that the given argument is valid — and the validity follows by using the method of Proof by Contradiction.

This introduction to the rules of inference has been far from exhaustive. Several of the books cited among the references listed near the end of this chapter offer additional material for the reader who wishes to pursue this topic further. In Section 2.5 we shall apply the ideas developed in this section to statements of a more mathematical nature. For we shall want to learn how to develop a proof for a theorem. And then in Chapter 4 another very important proof technique called *mathematical induction* will be added to our arsenal of weapons for proving mathematical theorems. First, however, the reader should carefully complete the exercises for this section.

EXERCISES 2.3

1. The following are three valid arguments. Establish the validity of each by means of a truth table. In each case, determine which rows of the table are crucial for assessing the validity of the argument and which rows can be ignored.

a) $[p \wedge (p \rightarrow q) \wedge r] \rightarrow [(p \vee q) \rightarrow r]$

b) $[[(p \wedge q) \rightarrow r] \wedge \neg q \wedge (p \rightarrow \neg r)] \rightarrow (\neg p \vee \neg q)$

c) $[[p \vee (q \vee r)] \wedge \neg q] \rightarrow (p \vee r)$

2. Use truth tables to verify that each of the following is a logical implication.

a) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

b) $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$

c) $[(p \vee q) \wedge \neg p] \rightarrow q$

d) $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$

3. Verify that each of the following is a logical implication by showing that it is impossible for the conclusion to have the truth value 0 while the hypothesis has the truth value 1.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $[(p \vee q) \wedge \neg p] \rightarrow q$

d) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$

e) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$

4. For each of the following pairs of statements, use Modus Ponens or Modus Tollens to fill in the blank line so that a valid argument is presented.

a) If Janice has trouble starting her car, then her daughter Angela will check Janice's spark plugs.
Janice had trouble starting her car.
$\therefore$ ____________________

b) If Brady solved the first problem correctly, then the answer he obtained is 137.
Brady's answer to the first problem is not 137.
$\therefore$ ____________________

c) If this is a **repeat-until** loop, then the body of this loop is executed at least once.
____
$\therefore$ The body of the loop is executed at least once.

d) If Tim plays basketball in the afternoon, then he will not watch television in the evening.
____
$\therefore$ Tim didn't play basketball in the afternoon.

5. Consider each of the following arguments. If the argument is valid, identify the rule of inference that establishes its validity. If not, indicate whether the error is due to an attempt to argue by the converse or by the inverse.

a) Andrea can program in C++, and she can program in Java.
Therefore Andrea can program in C++.

b) A sufficient condition for Bubbles to win the golf tournament is that her opponent Meg not sink a birdie on the last hole.
Bubbles won the golf tournament.
Therefore Bubbles' opponent Meg did not sink a birdie on the last hole.

c) If Ron's computer program is correct, then he'll be able to complete his computer science assignment in at most two hours.
It takes Ron over two hours to complete his computer science assignment.
Therefore Ron's computer program is not correct.

d) Eileen's car keys are in her purse, or they are on the kitchen table.

Eileen's car keys are not on the kitchen table.
Therefore Eileen's car keys are in her purse.

e) If interest rates fall, then the stock market will rise.
Interest rates are not falling.
Therefore the stock market will not rise.

6. For primitive statements p, q, and r, let P denote the statement

$$[p \wedge (q \wedge r)] \vee \neg[p \vee (q \wedge r)],$$

while P_1 denotes the statement

$$[p \wedge (q \vee r)] \vee \neg[p \vee (q \vee r)].$$

a) Use the rules of inference to show that

$$q \wedge r \Rightarrow q \vee r.$$

b) Is it true that $P \Rightarrow P_1$?

7. Give the reason(s) for each step needed to show that the following argument is valid.

$$[p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)] \rightarrow (s \vee t)$$

Steps	**Reasons**
1) p	
2) $p \rightarrow q$	
3) q	
4) $r \rightarrow \neg q$	
5) $q \rightarrow \neg r$	
6) $\neg r$	
7) $s \vee r$	
8) s	
9) $\therefore s \vee t$	

8. Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \underline{\neg u \rightarrow \neg t} \\ \therefore p \end{array}$$

Steps	**Reasons**
1) $\neg s \wedge \neg u$	
2) $\neg u$	
3) $\neg u \rightarrow \neg t$	
4) $\neg t$	
5) $\neg s$	
6) $\neg s \wedge \neg t$	
7) $r \rightarrow (s \vee t)$	
8) $\neg(s \vee t) \rightarrow \neg r$	
9) $(\neg s \wedge \neg t) \rightarrow \neg r$	
10) $\neg r$	
11) $(\neg p \vee q) \rightarrow r$	
12) $\neg r \rightarrow \neg(\neg p \vee q)$	
13) $\neg r \rightarrow (p \wedge \neg q)$	
14) $p \wedge \neg q$	
15) $\therefore p$	

9. a) Give the reasons for the steps given to validate the argument

$$[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)] \rightarrow (\neg q \rightarrow s).$$

Steps	**Reasons**
1) $\neg(\neg q \rightarrow s)$	
2) $\neg q \wedge \neg s$	
3) $\neg s$	
4) $\neg r \vee s$	
5) $\neg r$	
6) $p \rightarrow q$	
7) $\neg q$	
8) $\neg p$	
9) $p \vee r$	
10) r	
11) $\neg r \wedge r$	
12) $\therefore \neg q \rightarrow s$	

b) Give a direct proof for the result in part (a).

c) Give a direct proof for the result in Example 2.32.

10. Establish the validity of the following arguments.

a) $[(p \wedge \neg q) \wedge r] \rightarrow [(p \wedge r) \vee q]$

b) $[p \wedge (p \rightarrow q) \wedge (\neg q \vee r)] \rightarrow r$

c) $$\begin{array}{l} p \rightarrow q \\ \neg q \\ \underline{\neg r} \\ \therefore \neg(p \vee r) \end{array}$$

d) $$\begin{array}{l} p \rightarrow q \\ r \rightarrow \neg q \\ \underline{r} \\ \therefore \neg p \end{array}$$

e) $$\begin{array}{l} p \rightarrow (q \rightarrow r) \\ \neg q \rightarrow \neg p \\ \underline{p} \\ \therefore r \end{array}$$

f) $$\begin{array}{l} p \wedge q \\ p \rightarrow (r \wedge q) \\ r \rightarrow (s \vee t) \\ \underline{\neg s} \\ \therefore t \end{array}$$

g) $$\begin{array}{l} p \rightarrow (q \rightarrow r) \\ p \vee s \\ t \rightarrow q \\ \underline{\neg s} \\ \therefore \neg r \rightarrow \neg t \end{array}$$

h) $$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \underline{\neg r} \\ \therefore q \end{array}$$

11. Show that each of the following arguments is invalid by providing a counterexample—that is, an assignment of truth values for the given primitive statements p, q, r, and s such that all premises are true (have the truth value 1) while the conclusion is false (has the truth value 0).

a) $[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow \neg r$

b) $[[(p \wedge q) \rightarrow r] \wedge (\neg q \vee r)] \rightarrow p$

c) $$\begin{array}{l} p \leftrightarrow q \\ q \rightarrow r \\ r \vee \neg s \\ \underline{\neg s \rightarrow q} \\ \therefore s \end{array}$$

d) $$\begin{array}{l} p \\ p \rightarrow r \\ p \rightarrow (q \vee \neg r) \\ \underline{\neg q \vee \neg s} \\ \therefore s \end{array}$$

12. Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.

a) If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

b) If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore (sometime today) Lois will mow her lawn.

13. a) Given primitive statements p, q, r, show that the implication

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

is a tautology.

b) The tautology in part (a) provides the rule of inference known as *resolution*, where the conclusion $(q \vee r)$ is called the *resolvent*. This rule was proposed in 1965 by J. A. Robinson and is the basis of many computer programs designed to automate a reasoning system.

In applying resolution each premise (in the hypothesis) and the conclusion are written as *clauses*. A clause is a primitive statement or its negation, or it is the disjunction of terms each of which is a primitive statement or the negation of such a statement. Hence the given rule has the clauses $(p \vee q)$ and $(\neg p \vee r)$ as premises and the clause $(q \vee r)$ as its conclusion (or, resolvent). Should we have the premise $\neg(p \wedge q)$, we replace this by the logically equivalent clause $\neg p \vee \neg q$, by the first of DeMorgan's Laws. The premise $\neg(p \vee q)$ can be replaced by the two clauses $\neg p$, $\neg q$. This is due to the second DeMorgan Law and the Rule of Conjunctive Simplification. For the premise $p \vee (q \wedge r)$, we apply the Distributive Law of $\vee$ over $\wedge$ and the Rule of Conjunctive Simplification to arrive at either of the two clauses $p \vee q$, $p \vee r$. Finally, the premise $p \rightarrow q$ becomes the clause $\neg p \vee q$.

Establish the validity of the following arguments, using resolution (along with the rules of inference and the laws of logic).

(i)
$$\begin{array}{l} p \vee (q \wedge r) \\ \underline{p \rightarrow s} \\ \therefore r \vee s \end{array}$$

(ii)
$$\begin{array}{l} p \\ \underline{p \leftrightarrow q} \\ \therefore q \end{array}$$

(iii)
$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ \underline{r \rightarrow s} \\ \therefore q \vee s \end{array}$$

(iv)
$$\begin{array}{l} \neg p \vee q \vee r \\ \neg q \\ \underline{\neg r} \\ \therefore \neg p \end{array}$$

(v)
$$\begin{array}{l} \neg p \vee s \\ \neg t \vee (s \wedge r) \\ \neg q \vee r \\ \underline{p \vee q \vee t} \\ \therefore r \vee s \end{array}$$

c) Write the following argument in symbolic form, then use resolution (along with the rules of inference and the laws of logic) to establish its validity.

Jonathan does not have his driver's license or his new car is out of gas. Jonathan has his driver's license or he does not like to drive his new car. Jonathan's new car is not out of gas or he does not like to drive his new car. Therefore, Jonathan does not like to drive his new car.

2.4 The Use of Quantifiers

In Section 2.1, we mentioned how sentences that involve a variable, such as x, need not be statements. For example, the sentence "The number $x + 2$ is an even integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by -5, -1, or 3, for instance, the resulting statement is false. In fact, it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, however, the resulting statement is true.

We refer to the sentence "The number $x + 2$ is an even integer" as an *open statement*, which we formally define as follows.

Definition 2.5 A declarative sentence is an *open statement* if

1) it contains one or more variables, and

b) If $-1 < 3$ and $3+7=10$, then $\sin\left(\frac{3\pi}{2}\right) = -1$. (TRUE)
Converse: If $\sin\left(\frac{3\pi}{2}\right) = -1$, then $-1 < 3$ and $3+7=10$. (TRUE)
Inverse: If $-1 \geq 3$ or $3+7 \neq 10$, then $\sin\left(\frac{3\pi}{2}\right) \neq -1$. (TRUE)
Contrapositive: If $\sin\left(\frac{3\pi}{2}\right) \neq -1$, then $-1 \geq 3$ or $3+7 \neq 10$. (TRUE)

11. a) $(q \rightarrow r) \vee \neg p$ **b)** $(\neg q \vee r) \vee \neg p$

13.

p	q	r	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$	$[(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$
0	0	0	1	1
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

15. a) $(p \uparrow p)$ **b)** $(p \uparrow p) \uparrow (q \uparrow q)$ **c)** $(p \uparrow q) \uparrow (p \uparrow q)$ **d)** $p \uparrow (q \uparrow q)$
e) $(r \uparrow s) \uparrow (r \uparrow s)$, where r stands for $p \uparrow (q \uparrow q)$ and s for $q \uparrow (p \uparrow p)$

17.

p	q	$\neg(p \downarrow q)$	$(\neg p \uparrow \neg q)$	$\neg(p \uparrow q)$	$(\neg p \downarrow \neg q)$
0	0	0	0	0	0
0	1	1	1	0	0
1	0	1	1	0	0
1	1	1	1	1	1

19. a)

	Reasons
$p \vee [p \wedge (p \vee q)]$	
$\Leftrightarrow p \vee p$	Absorption Law
$\Leftrightarrow p$	Idempotent Law of $\vee$

c)

	Reasons
$[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)]$	
$\Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$	$s \rightarrow t \Leftrightarrow \neg s \vee t$
$\Leftrightarrow (\neg\neg p \wedge \neg\neg q) \vee (p \wedge q \wedge r)$	DeMorgan's Laws
$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$	Law of Double Negation
$\Leftrightarrow p \wedge q$	Absorption Law

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1. a)

p	q	r	$p \rightarrow q$	$(p \vee q)$	$(p \vee q) \rightarrow r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

The validity of the argument follows from the results in the last row. (The first seven rows may be ignored.)

c)

p	q	r	$q \vee r$	$p \vee (q \vee r)$	$\neg q$	$p \vee r$
0	0	0	0	0	1	0
0	0	1	1	1	1	1
0	1	0	1	1	0	0
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	1	1
1	1	0	1	1	0	1
1	1	1	1	1	0	1

The results in rows 2, 5, and 6 establish the validity of the given argument. (The results in the other five rows of the table may be disregarded.)

3. a) If p has the truth value 0, then so does $p \wedge q$.

b) When $p \vee q$ has the truth value 0, then the truth value of p (and that of q) is 0.

c) If q has truth value 0, then the truth value of $[(p \vee q) \wedge \neg p]$ is 0, regardless of the truth value of p.

d) The statement $q \vee s$ has truth value 0 only when each of q, s has truth value 0. Then $(p \rightarrow q)$ has truth value 1 when p has truth value 0; $(r \rightarrow s)$ has truth value 1 when r has truth value 0. But then $(p \vee r)$ must have truth value 0, not 1.

5. a) Rule of Conjunctive Simplification

b) Invalid — attempt to argue by the converse

c) Modus Tollens

d) Rule of Disjunctive Syllogism

e) Invalid — attempt to argue by the inverse

7. 1) and 2) Premise

3) Steps (1) and (2) and the Rule of Detachment

4) Premise

5) Step (4) and $(r \rightarrow \neg q) \Leftrightarrow (\neg\neg q \rightarrow \neg r) \Leftrightarrow (q \rightarrow \neg r)$

6) Steps (3) and (5) and the Rule of Detachment

7) Premise

8) Steps (6) and (7) and the Rule of Disjunctive Syllogism

9) Step (8) and the Rule of Disjunctive Amplification

9. a)

1) Premise (The Negation of the Conclusion)

2) Step (1) and $\neg(\neg q \rightarrow s) \Leftrightarrow \neg(\neg\neg q \vee s) \Leftrightarrow \neg(q \vee s) \Leftrightarrow \neg q \wedge \neg s$

3) Step (2) and the Rule of Conjunctive Simplification

4) Premise

5) Steps (3) and (4) and the Rule of Disjunctive Syllogism

6) Premise

7) Step (2) and the Rule of Conjunctive Simplification

8) Steps (6) and (7) and Modus Tollens

9) Premise

10) Steps (8) and (9) and the Rule of Disjunctive Syllogism

11) Steps (5) and (10) and the Rule of Conjunction

12) Step (11) and the Method of Proof by Contradiction

b)

1)	$p \rightarrow q$	Premise
2)	$\neg q \rightarrow \neg p$	Step (1) and $(p \rightarrow q) \Leftrightarrow (\neg q \rightarrow \neg p)$
3)	$p \vee r$	Premise
4)	$\neg p \rightarrow r$	Step (3) and $(p \vee r) \Leftrightarrow (\neg p \rightarrow r)$
5)	$\neg q \rightarrow r$	Steps (2) and (4) and the Law of the Syllogism
6)	$\neg r \vee s$	Premise
7)	$r \rightarrow s$	Step (6) and $(\neg r \vee s) \Leftrightarrow (r \rightarrow s)$
8)	$\therefore \neg q \rightarrow s$	Steps (5) and (7) and the Law of the Syllogism

11. **a)** $p: 1 \quad q: 0 \quad r: 1$
b) $p: 0 \quad q: 0 \quad r: 0$ or 1
$p: 0 \quad q: 1 \quad r: 1$
c) $p, q, r: 1 \quad s: 0$
d) $p, q, r: 1 \quad s: 0$

13. **a)**

p	q	r	$p \vee q$	$\neg p \vee r$	$(p \vee q) \wedge (\neg p \vee r)$	$q \vee r$	$[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$
0	0	0	0	1	0	0	1
0	0	1	0	1	0	1	1
0	1	0	1	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	1	0	0	0	1
1	0	1	1	1	1	1	1
1	1	0	1	0	0	1	1
1	1	1	1	1	1	1	1

From the last column of the truth table it follows that $[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$ is a tautology.

b) (i)

Steps	**Reasons**
1) $p \vee (q \wedge r)$	Premise
2) $(p \vee q) \wedge (p \vee r)$	Step (1) and the Distributive Law of $\vee$ over $\wedge$
3) $p \vee r$	Step (2) and the Rule of Conjunctive Simplification
4) $p \to s$	Premise
5) $\neg p \vee s$	Step (4), $p \to s \Leftrightarrow \neg p \vee s$
6) $\therefore r \vee s$	Steps (3), (5), the Rule of Conjunction, and Resolution

(iii)

Steps	**Reasons**
1) $p \vee q$	Premise
2) $p \to r$	Premise
3) $\neg p \vee r$	Step (2), $p \to r \Leftrightarrow \neg p \vee r$
4) $[(p \vee q) \wedge (\neg p \vee r)]$	Steps (1), (3), and the Rule of Conjunction
5) $q \vee r$	Step (4) and Resolution
6) $r \to s$	Premise
7) $\neg r \vee s$	Step (6), $r \to s \Leftrightarrow \neg r \vee s$
8) $[(r \vee q) \wedge (\neg r \vee s)]$	Steps (5), (7), the Commutative Law of $\vee$, and the Rule of Conjunction
9) $\therefore q \vee s$	Step (8) and Resolution

(iv)

Steps	**Reasons**
1) $\neg p \vee q \vee r$	Premise
2) $q \vee (\neg p \vee r)$	Step (1) and the Commutative and Associative Laws of $\vee$
3) $\neg q$	Premise
4) $\neg q \vee (\neg p \vee r)$	Step (3) and the Rule of Disjunctive Amplification
5) $[[q \vee (\neg p \vee r)] \wedge [\neg q \vee (\neg p \vee r)]]$	Steps (2), (4), and the Rule of Conjunction
6) $(\neg p \vee r)$	Step (5), Resolution, and the Idempotent Law of $\wedge$
7) $\neg r$	Premise
8) $\neg r \vee \neg p$	Step (7) and the Rule of Disjunctive Amplification
9) $[(r \vee \neg p) \wedge (\neg r \vee \neg p)]$	Steps (6), (8), the Commutative Law of $\vee$, and the Rule of Conjunction
10) $\therefore \neg p$	Step (9), Resolution, and the Idempotent Law of $\vee$

c) Consider the following assignments.

p: Jonathan has his driver's license.
q: Jonathan's new car is out of gas.
r: Jonathan likes to drive his new car.

Then the given argument can be written in symbolic form as

$$\begin{array}{l} \neg p \vee q \\ p \vee \neg r \\ \underline{\neg q \vee \neg r} \\ \therefore \neg r \end{array}$$

Steps	Reasons
1) $\neg p \vee q$	Premise
2) $p \vee \neg r$	Premise
3) $(p \vee \neg r) \wedge (\neg p \vee q)$	Steps (2), (1), and the Rule of Conjunction
4) $\neg r \vee q$	Step (3) and Resolution
5) $q \vee \neg r$	Step (4) and the Commutative Law of $\vee$
6) $\neg q \vee \neg r$	Premise
7) $(q \vee \neg r) \wedge (\neg q \vee \neg r)$	Steps (5), (6), and the Rule of Conjunction
8) $\neg r \vee \neg r$	Step (7) and Resolution
9) $\therefore \neg r$	Step (8) and the Idempotent Law of $\vee$

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1. a) False **b)** False **c)** False **d)** True **e)** False **f)** False

3. Statements (a), (c), and (e) are true, and statements (b), (d), and (f) are false.

5. a) $\exists x\, [m(x) \wedge c(x) \wedge j(x)]$ True
b) $\exists x\, [s(x) \wedge c(x) \wedge \neg m(x)]$ True
c) $\forall x\, [c(x) \rightarrow (m(x) \veebar p(x))]$ False
d) $\forall x\, [(g(x) \wedge c(x)) \rightarrow \neg p(x)]$, or True
$\forall x\, [(p(x) \wedge c(x)) \rightarrow \neg g(x)]$, or
$\forall x\, [(g(x) \wedge p(x)) \rightarrow \neg c(x)]$
e) $\forall x\, [(c(x) \wedge s(x)) \rightarrow (p(x) \veebar e(x))]$ True

7. a) (i) $\exists x\, q(x)$
(ii) $\exists x\, [p(x) \wedge q(x)]$
(iii) $\forall x\, [q(x) \rightarrow \neg t(x)]$
(iv) $\forall x\, [q(x) \rightarrow \neg t(x)]$
(v) $\exists x\, [q(x) \wedge t(x)]$
(vi) $\forall x\, [(q(x) \wedge r(x)) \rightarrow s(x)]$
b) Statements (i), (ii), (v), and (vi) are true. Statements (iii) and (iv) are false; $x = 10$ provides a counterexample for either statement.
c) (i) If x is a perfect square, then $x > 0$.
(ii) If x is divisible by 4, then x is even.
(iii) If x is divisible by 4, then x is not divisible by 5.
(iv) There exists an integer that is divisible by 4, but it is not a perfect square.
d) (i) Let $x = 0$. (iii) Let $x = 20$.

9. a) (i) True (ii) False Consider $x = 3$.
(iii) True (iv) True
c) (i) True (ii) True
(iii) True (iv) False For $x = 2$ or 5, the truth value of $p(x)$ is 1 while that of $r(x)$ is 0.

11. a) In this case the variable x is free, while the variables y, z are bound.
b) Here the variables x, y are bound; the variable z is free.

13. a) $p(2, 3) \wedge p(3, 3) \wedge p(5, 3)$
b) $[p(2, 2) \vee p(2, 3) \vee p(2, 5)] \vee [p(3, 2) \vee p(3, 3) \vee p(3, 5)] \vee [p(5, 2) \vee p(5, 3) \vee p(5, 5)]$