b) **if** $((n + 2 = 8)$ **or** $(n - 3 = 6))$ **then**
    $n := 2 * n + 1$

c) **if** $((n - 3 = 16)$ **and** $(\lfloor n/6 \rfloor = 1))$ **then**
    $n := n + 3$

d) **if** $((n \neq 21)$ **and** $(n - 7 = 15))$ **then**
    $n := n - 4$

15. The integer variables m and n are assigned the values 3 and 8, respectively, during the execution of a program (written in pseudocode). Each of the following *successive* statements is then encountered during program execution. [Here the values of m, n following the execution of the statement in part (a) become the values of m, n for the statement in part (b), and so on, through the statement in part (e).] What are the values of m, n after each of these statements is encountered?

a) **if** $n - m = 5$ **then** $n := n - 2$

b) **if** $((2 * m = n)$ **and** $(\lfloor n/4 \rfloor = 1))$ **then**
    $n := 4 * m - 3$

c) **if** $((n < 8)$ **or** $(\lfloor m/2 \rfloor = 2))$ **then** $n := 2 * m$
  else $m := 2 * n$

d) **if** $((m < 20)$ **and** $(\lfloor n/6 \rfloor = 1))$ **then**
    $m := m - n - 5$

e) **if** $((n = 2 * m)$ **or** $(\lfloor n/2 \rfloor = 5))$ **then**
    $m := m + 2$

16. In the following program segment i, j, m, and n are integer variables. The values of m and n are supplied by the user earlier in the execution of the total program.

```
for i := 1 to m do
  for j := 1 to n do
    if i ≠ j then
      print i + j
```

How many times is the **print** statement in the segment executed when (a) $m = 10$, $n = 10$; (b) $m = 20$, $n = 20$; (c) $m = 10$, $n = 20$; (d) $m = 20$, $n = 10$?

17. After baking a pie for the two nieces and two nephews who are visiting her, Aunt Nellie leaves the pie on her kitchen table to cool. Then she drives to the mall to close her boutique for the day. Upon her return she finds that someone has eaten one-quarter of the pie. Since no one was in her house that day—except for the four visitors—Aunt Nellie questions each niece and nephew about who ate the piece of pie. The four "suspects" tell her the following:

Charles:	Kelly ate the piece of pie.
Dawn:	I did not eat the piece of pie.
Kelly:	Tyler ate the pie.
Tyler:	Kelly lied when she said I ate the pie.

If only one of these four statements is true and only one of the four committed this heinous crime, who is the vile culprit that Aunt Nellie will have to punish severely?

2.2 Logical Equivalence: The Laws of Logic

In all areas of mathematics we need to know when the entities we are studying are equal or essentially the same. For example, in arithmetic and algebra we know that two nonzero real numbers are equal when they have the same magnitude and algebraic sign. Hence, for two nonzero real numbers x, y, we have $x = y$ if $|x| = |y|$ and $xy > 0$, and conversely (that is, if $x = y$, then $|x| = |y|$ and $xy > 0$). When we deal with triangles in geometry, the notion of congruence arises. Here triangle ABC and triangle DEF are congruent if, for instance, they have equal corresponding sides—that is, the length of side AB = the length of side DE, the length of side BC = the length of side EF, and the length of side CA = the length of side FD.

Our study of logic is often referred to as the *algebra of propositions* (as opposed to the algebra of real numbers). In this algebra we shall use the truth tables of the statements, or propositions, to develop an idea of when two such entities are essentially the same. We begin with an example.

EXAMPLE 2.7

For primitive statements p and q, Table 2.6 provides the truth tables for the compound statements $\neg p \vee q$ and $p \rightarrow q$. Here we see that the corresponding truth tables for the two statements $\neg p \vee q$ and $p \rightarrow q$ are exactly the same.

Table 2.6

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

This situation leads us to the following idea.

Definition 2.2 Two statements s_1, s_2 are said to be *logically equivalent*, and we write $s_1 \Longleftrightarrow s_2$, when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

Note that when $s_1 \Longleftrightarrow s_2$ the statements s_1 and s_2 provide the same truth tables because s_1, s_2 have the same truth values for *all* choices of truth values for their primitive components.

As a result of this concept we see that we can express the connective for the implication (of primitive statements) in terms of negation and disjunction—that is, $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$. In the same manner, from the result in Table 2.7 we have $(p \leftrightarrow q) \Longleftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, and this helps validate the use of the term *biconditional.* Using the logical equivalence from Table 2.6, we find that we can also write $(p \leftrightarrow q) \Longleftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$. Consequently, if we so choose, we can eliminate the connectives $\rightarrow$ and $\leftrightarrow$ from compound statements.

Table 2.7

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Examining Table 2.8, we find that negation, along with the connectives $\wedge$ and $\vee$, are all we need to replace the *exclusive or* connective, $\veebar$. In fact, we may even eliminate either $\wedge$ or $\vee$. However, for the related applications we want to study later in the text, we shall need both $\wedge$ and $\vee$ as well as negation.

Table 2.8

p	q	$p \veebar q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

We now use the idea of logical equivalence to examine some of the important properties that hold for the algebra of propositions.

For all real numbers a, b, we know that $-(a+b) = (-a) + (-b)$. Is there a comparable result for primitive statements p, q?

EXAMPLE 2.8

In Table 2.9 we have constructed the truth tables for the statements $\neg(p \wedge q)$, $\neg p \vee \neg q$, $\neg(p \vee q)$, and $\neg p \wedge \neg q$, where p, q are primitive statements. Columns 4 and 7 reveal that $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$; columns 9 and 10 reveal that $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$. These results are known as *DeMorgan's Laws*. They are similar to the familiar law for real numbers,

$$-(a+b) = (-a) + (-b),$$

already noted, which shows the negative of a sum to be equal to the sum of the negatives. Here, however, a crucial difference emerges: The negation of the *conjunction* of two primitive statements p, q results in the *disjunction* of their negations $\neg p$, $\neg q$, whereas the negation of the *disjunction* of these same statements p, q is logically equivalent to the *conjunction* of their negations $\neg p$, $\neg q$.

Table 2.9

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

Although p, q were primitive statements in the preceding example we shall soon learn that DeMorgan's Laws hold for any two arbitrary statements.

In the arithmetic of real numbers, the operations of addition and multiplication are both involved in the principle called the Distributive Law of Multiplication over Addition: For all real numbers a, b, c,

$$a \times (b+c) = (a \times b) + (a \times c).$$

The next example shows that there is a similar law for primitive statements. There is also a second related law (for primitive statements) that has no counterpart in the arithmetic of real numbers.

EXAMPLE 2.9

Table 2.10 contains the truth tables for the statements $p \wedge (q \vee r)$, $(p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r)$, and $(p \vee q) \wedge (p \vee r)$. From the table it follows that for all primitive statements p, q, and r,

$$p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r) \qquad \text{The Distributive Law of } \wedge \text{ over } \vee$$
$$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r) \qquad \text{The Distributive Law of } \vee \text{ over } \wedge$$

The second distributive law has no counterpart in the arithmetic of real numbers. That is, it is not true for all real numbers a, b, and c that the following holds: $a + (b \times c) = (a+b) \times (a+c)$. For $a = 2$, $b = 3$, and $c = 5$, for instance, $a + (b \times c) = 17$ but $(a+b) \times (a+c) = 35$.

Table 2.10

p	q	r	$p \land (q \lor r)$	$(p \land q) \lor (p \land r)$	$p \lor (q \land r)$	$(p \lor q) \land (p \lor r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Before going any further, we note that, in general, if s_1, s_2 are statements and $s_1 \leftrightarrow s_2$ is a tautology, then s_1, s_2 must have the same corresponding truth values (that is, for each assignment of truth values to the primitive statements in s_1 and s_2, s_1 is true if and only if s_2 is true and s_1 is false if and only if s_2 is false) and $s_1 \Longleftrightarrow s_2$. When s_1 and s_2 are logically equivalent statements (that is, $s_1 \Longleftrightarrow s_2$), then the compound statement $s_1 \leftrightarrow s_2$ is a tautology. Under these circumstances it is also true that $\neg s_1 \Longleftrightarrow \neg s_2$, and $\neg s_1 \leftrightarrow \neg s_2$ is a tautology.

If s_1, s_2, and s_3 are statements where $s_1 \Longleftrightarrow s_2$ and $s_2 \Longleftrightarrow s_3$ then $s_1 \Longleftrightarrow s_3$. When two statements s_1 and s_2 are not logically equivalent, we may write $s_1 \not\Longleftrightarrow s_2$ to designate this situation.

Using the concepts of logical equivalence, tautology, and contradiction, we state the following list of laws for the algebra of propositions.

The Laws of Logic

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0,

1) $\neg\neg p \Longleftrightarrow p$	**Law of *Double Negation***
2) $\neg(p \lor q) \Longleftrightarrow \neg p \land \neg q$ $\neg(p \land q) \Longleftrightarrow \neg p \lor \neg q$	***DeMorgan's* Laws**
3) $p \lor q \Longleftrightarrow q \lor p$ $p \land q \Longleftrightarrow q \land p$	***Commutative* Laws**
4) $p \lor (q \lor r) \Longleftrightarrow (p \lor q) \lor r$[†] $p \land (q \land r) \Longleftrightarrow (p \land q) \land r$	***Associative* Laws**
5) $p \lor (q \land r) \Longleftrightarrow (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \Longleftrightarrow (p \land q) \lor (p \land r)$	***Distributive* Laws**
6) $p \lor p \Longleftrightarrow p$ $p \land p \Longleftrightarrow p$	***Idempotent* Laws**
7) $p \lor F_0 \Longleftrightarrow p$ $p \land T_0 \Longleftrightarrow p$	***Identity* Laws**

[†]We note that because of the Associative Laws, there is no ambiguity in statements of the form $p \lor q \lor r$ or $p \land q \land r$.

8) $p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	*Inverse* Laws
9) $p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	*Domination* Laws
10) $p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	*Absorption* Laws

We now turn our attention to proving all of these properties. In so doing we realize that we could simply construct the truth tables and compare the results for the corresponding truth values in each case — as we did in Examples 2.8 and 2.9. However, before we start writing, let us take one more look at this list of 19 laws, which, aside from the Law of Double Negation, fall naturally into pairs. This pairing idea will help us after we examine the following concept.

Definition 2.3 Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the *dual* of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

If p is any primitive statement, then p^d is the same as p — that is, the dual of a primitive statement is simply the same primitive statement. And $(\neg p)^d$ is the same as $\neg p$. The statements $p \vee \neg p$ and $p \wedge \neg p$ are duals of each other whenever p is primitive — and so are the statements $p \vee T_0$ and $p \wedge F_0$.

Given the primitive statements p, q, r and the compound statement

$$s: \quad (p \wedge \neg q) \vee (r \wedge T_0),$$

we find that the dual of s is

$$s^d: \quad (p \vee \neg q) \wedge (r \vee F_0).$$

(Note that $\neg q$ is unchanged as we go from s to s^d.)

We now state and use a theorem without proving it. However, in Chapter 15 we shall justify the result that appears here.

THEOREM 2.1 *The Principle of Duality.* Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

As a result, laws 2 through 10 in our list can be established by proving one of the laws in each pair and then invoking this principle.

We also find that it is possible to derive many other logical equivalences. For example, if q, r, s are primitive statements, the results in columns 5 and 7 of Table 2.11 show us that

$$(r \wedge s) \rightarrow q \Leftrightarrow \neg(r \wedge s) \vee q$$

or that $[(r \wedge s) \rightarrow q] \leftrightarrow [\neg(r \wedge s) \vee q]$ is a tautology. However, instead of always constructing more (and, unfortunately, larger) truth tables it might be a good idea to recall from Example 2.7 that for primitive statements p, q, the compound statement

$$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$$

Table 2.11

q	r	s	$r \wedge s$	$(r \wedge s) \rightarrow q$	$\neg(r \wedge s)$	$\neg(r \wedge s) \vee q$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	1	1
0	1	1	1	0	0	0
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

is a tautology. If we were to *replace* each occurrence of this primitive statement p by the compound statement $r \wedge s$, then we would obtain the earlier tautology

$$[(r \wedge s) \rightarrow q] \leftrightarrow [\neg(r \wedge s) \vee q].$$

What has happened here illustrates the first of the following two *substitution rules*:

1) Suppose that the compound statement P is a tautology. If p is a *primitive* statement that appears in P and we replace *each* occurrence of p by the *same* statement q, then the resulting compound statement P_1 is also a tautology.
2) Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Longleftrightarrow p$. Suppose that in P we replace one or more occurrences of p by q. Then this replacement yields the compound statement P_1. Under these circumstances $P_1 \Longleftrightarrow P$.

These rules are further illustrated in the following two examples.

EXAMPLE 2.10

a) From the first of DeMorgan's Laws we know that for all primitive statements p, q, the compound statement

$$P: \quad \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

is a tautology. When we replace each occurrence of p by $r \wedge s$, it follows from the first substitution rule that

$$P_1: \quad \neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$$

is also a tautology. Extending this result one step further, we may replace each occurrence of q by $t \rightarrow u$. The same substitution rule now yields the tautology

$$P_2: \quad \neg[(r \wedge s) \vee (t \rightarrow u)] \leftrightarrow [\neg(r \wedge s) \wedge \neg(t \rightarrow u)],$$

and hence, by the remarks following shortly after Example 2.9, the logical equivalence

$$\neg[(r \wedge s) \vee (t \rightarrow u)] \Longleftrightarrow [\neg(r \wedge s) \wedge \neg(t \rightarrow u)].$$

b) For primitive statements p, q, we learn from the last column of Table 2.12 that the compound statement $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology. Consequently, if r, s, t, u are any statements, then by the first substitution rule we obtain the new tautology

$$[(r \rightarrow s) \wedge [(r \rightarrow s) \rightarrow (\neg t \vee u)]] \rightarrow (\neg t \vee u)$$

when we replace each occurrence of p by $r \rightarrow s$ and each occurrence of q by $\neg t \vee u$.

Table 2.12

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

EXAMPLE 2.11

a) For an application of the second substitution rule, let P denote the compound statement $(p \to q) \to r$. Because $(p \to q) \iff \neg p \vee q$ (as shown in Example 2.7 and Table 2.6), if P_1 denotes the compound statement $(\neg p \vee q) \to r$, then $P_1 \iff P$. (We also find that $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.)

b) Now let P represent the compound statement (actually a tautology) $p \to (p \vee q)$. Since $\neg\neg p \iff p$, the compound statement P_1: $p \to (\neg\neg p \vee q)$ is derived from P by replacing *only the second occurrence* (but *not* the first occurrence) of p by $\neg\neg p$. The second substitution rule still implies that $P_1 \iff P$. [Note that P_2: $\neg\neg p \to (\neg\neg p \vee q)$, derived by replacing *both* occurrences of p by $\neg\neg p$, is also logically equivalent to P.]

Our next example demonstrates how we can use the idea of logical equivalence together with the laws of logic and the substitution rules.

EXAMPLE 2.12

Negate and simplify the compound statement $(p \vee q) \to r$.

We organize our explanation as follows:

1) $(p \vee q) \to r \iff \neg(p \vee q) \vee r$ [by the first substitution rule because $(s \to t) \leftrightarrow (\neg s \vee t)$ is a tautology for primitive statements s, t].

2) Negating the statements in step (1), we have $\neg[(p \vee q) \to r] \iff \neg[\neg(p \vee q) \vee r]$.

3) From the first of DeMorgan's Laws and the first substitution rule, $\neg[\neg(p \vee q) \vee r] \iff \neg\neg(p \vee q) \wedge \neg r$.

4) The Law of Double Negation and the second substitution rule now gives us $\neg\neg(p \vee q) \wedge \neg r \iff (p \vee q) \wedge \neg r$.

From steps (1) through (4) we have $\neg[(p \vee q) \to r] \iff (p \vee q) \wedge \neg r$.

When we wanted to write the negation of an implication, as in Example 2.12, we found that the concept of logical equivalence played a key role — in conjunction with the laws of logic and the substitution rules. This idea is important enough to warrant a second look.

EXAMPLE 2.13

Let p, q denote the primitive statements

p: Joan goes to Lake George. $\quad$ q: Mary pays for Joan's shopping spree.

and consider the implication

$p \to q$: If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.

Here we want to write the negation of $p \rightarrow q$ in a way other than simply $\neg(p \rightarrow q)$. We want to avoid writing the negation as "It is not the case that if Joan goes to Lake George, then Mary will pay for Joan's shopping spree."

To accomplish this we consider the following. Since $p \rightarrow q \Longleftrightarrow \neg p \vee q$, it follows that $\neg(p \rightarrow q) \Longleftrightarrow \neg(\neg p \vee q)$. Then by DeMorgan's Law we have $\neg(\neg p \vee q) \Longleftrightarrow \neg\neg p \wedge \neg q$, and from the Law of Double Negation and the second substitution rule it follows that $\neg\neg p \wedge \neg q \Longleftrightarrow p \wedge \neg q$. Consequently,

$$\neg(p \rightarrow q) \Longleftrightarrow \neg(\neg p \vee q) \Longleftrightarrow \neg\neg p \wedge \neg q \Longleftrightarrow p \wedge \neg q,$$

and we may write the negation of $p \rightarrow q$ in this case as

$\neg(p \rightarrow q)$: Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

(*Note*: The negation of an if-then statement does *not* begin with the word *if.* It is *not* another *implication.*)

EXAMPLE 2.14

In Definition 2.3 the dual s^d of a statement s was defined only for statements involving negation and the basic connectives $\wedge$ and $\vee$. How does one determine the dual of a statement such as s: $p \rightarrow q$, where p, q are primitive?

Because $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$, s^d is logically equivalent to the statement $(\neg p \vee q)^d$, which is $\neg p \wedge q$.

The implication $p \rightarrow q$ and certain statements related to it are now examined in the following example.

EXAMPLE 2.15

Table 2.13 gives the truth tables for the statements $p \rightarrow q$, $\neg q \rightarrow \neg p$, $q \rightarrow p$, and $\neg p \rightarrow \neg q$. The third and fourth columns of the table reveal that

$$(p \rightarrow q) \Longleftrightarrow (\neg q \rightarrow \neg p).$$

Table 2.13

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$q \rightarrow p$	$\neg p \rightarrow \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

The statement $\neg q \rightarrow \neg p$ is called the *contrapositive* of the implication $p \rightarrow q$. Columns 5 and 6 of the table show that

$$(q \rightarrow p) \Longleftrightarrow (\neg p \rightarrow \neg q).$$

The statement $q \rightarrow p$ is called the *converse* of $p \rightarrow q$; $\neg p \rightarrow \neg q$ is called the *inverse* of $p \rightarrow q$. We also see from Table 2.13 that

$$(p \rightarrow q) \not\Longleftrightarrow (q \rightarrow p) \quad \text{and} \quad (\neg p \rightarrow \neg q) \not\Longleftrightarrow (\neg q \rightarrow \neg p).$$

Consequently, we must keep the implication and its converse straight. The fact that a certain implication $p \rightarrow q$ is true (in particular, as in row 2 of the table) does *not* require that the converse $q \rightarrow p$ also be true. However, it does necessitate the truth of the contrapositive $\neg q \rightarrow \neg p$.

Let us consider a specific example where p, q represent the statements

p: Jeff is concerned about his cholesterol (HDL and LDL) levels.

q: Jeff walks at least two miles three times a week.

Then we obtain

- (The implication: $p \rightarrow q$). If Jeff is concerned about his cholesterol levels, then he will walk at least two miles three times a week.
- (The contrapositive: $\neg q \rightarrow \neg p$). If Jeff does not walk at least two miles three times a week, then he is not concerned about his cholesterol levels.
- (The converse: $q \rightarrow p$). If Jeff walks at least two miles three times a week, then he is concerned about his cholesterol levels.
- (The inverse: $\neg p \rightarrow \neg q$). If Jeff is not concerned about his cholesterol levels, then he will not walk at least two miles three times a week.

If p is true and q is false, then the implication $p \rightarrow q$ and the contrapositive $\neg q \rightarrow \neg p$ are false, while the converse $q \rightarrow p$ and the inverse $\neg p \rightarrow \neg q$ are true. For the case where p is false and q is true, the implication $p \rightarrow q$ and the contrapositive $\neg q \rightarrow \neg p$ are now true, while the converse $q \rightarrow p$ and the inverse $\neg p \rightarrow \neg q$ are false. When p, q are both true or both false, then the implication is true, as are the contrapositive, converse, and inverse.

We turn now to two examples involving the simplification of compound statements. For simplicity, we shall list the major laws of logic being used, but we shall not mention any applications of our two substitution rules.

EXAMPLE 2.16

For primitive statements p, q, is there any simpler way to express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$ — that is, can we find a simpler statement that is logically equivalent to the one given?

Here one finds that

$(p \vee q) \wedge \neg(\neg p \wedge q)$	**Reasons**
$\Longleftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	DeMorgan's Law
$\Longleftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Law of Double Negation
$\Longleftrightarrow (p \vee (q \wedge \neg q)$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee F_0$	Inverse Law
$\Longleftrightarrow p$	Identity Law

Consequently, we see that

$$(p \vee q) \wedge \neg(\neg p \wedge q) \Longleftrightarrow p,$$

so we can express the given compound statement by the simpler logically equivalent statement p.

EXAMPLE 2.17

Consider the compound statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

where p, q, r are primitive statements. This statement contains four occurrences of primitive statements, three negation symbols, and three connectives.

From the laws of logic it follows that

$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$	**Reasons**
$\Longleftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q$	DeMorgan's Law
$\Longleftrightarrow [(p \vee q) \wedge r] \wedge q$	Law of Double Negation
$\Longleftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative Law of $\wedge$
$\Longleftrightarrow (p \vee q) \wedge (q \wedge r)$	Commutative Law of $\wedge$
$\Longleftrightarrow [(p \vee q) \wedge q] \wedge r$	Associative Law of $\wedge$
$\Longleftrightarrow q \wedge r$	Absorption Law (as well as the Commutative Laws for $\wedge$ and $\vee$)

Consequently, the original statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$$

is logically equivalent to the much simpler statement

$$q \wedge r,$$

where we find only two primitive statements, no negation symbols, and only one connective.

Note further that from Example 2.7 we have

$$\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Longleftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

so it follows that

$$\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Longleftrightarrow q \wedge r.$$

We close this section with an application on how the ideas in Examples 2.16 and 2.17 can be used in simplifying switching networks.

EXAMPLE 2.18

A switching network is made up of wires and switches connecting two terminals T_1 and T_2. In such a network, each switch is either open (0), so that no current flows through it, or closed (1), so that current does flow through it.

In Fig. 2.1(a) we have a network with one switch. Each of parts (b) and (c) contains two (independent) switches.

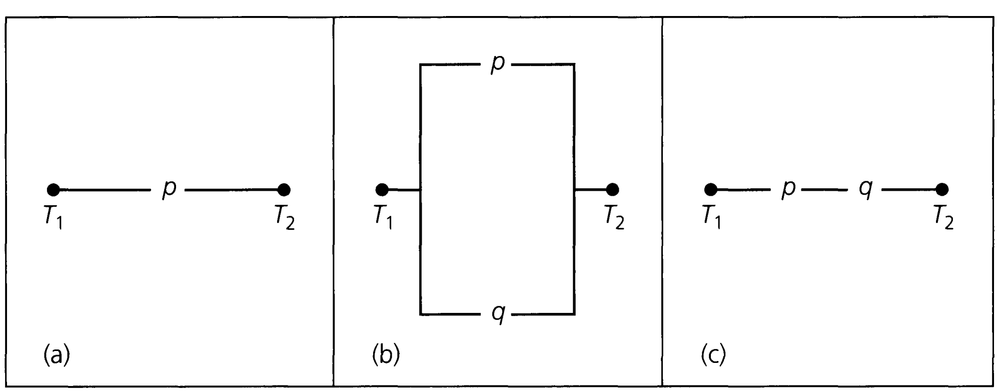

Figure 2.1

For the network in part (b), current flows from T_1 to T_2 if either of the switches p, q is closed. We call this a *parallel* network and represent it by $p \vee q$. The network in part (c) requires that each of the switches p, q be closed in order for current to flow from T_1 to T_2. Here the switches are in *series*; this network is represented by $p \wedge q$.

The switches in a network need not act independently of each other. Consider the network shown in Fig. 2.2(a). Here the switches labeled t and $\neg t$ are not independent. We have coupled these two switches so that t is open (closed) if and only if $\neg t$ is simultaneously closed (open). The same is true for the switches at q, $\neg q$. (Also, for example, the three switches labeled p are not independent.)

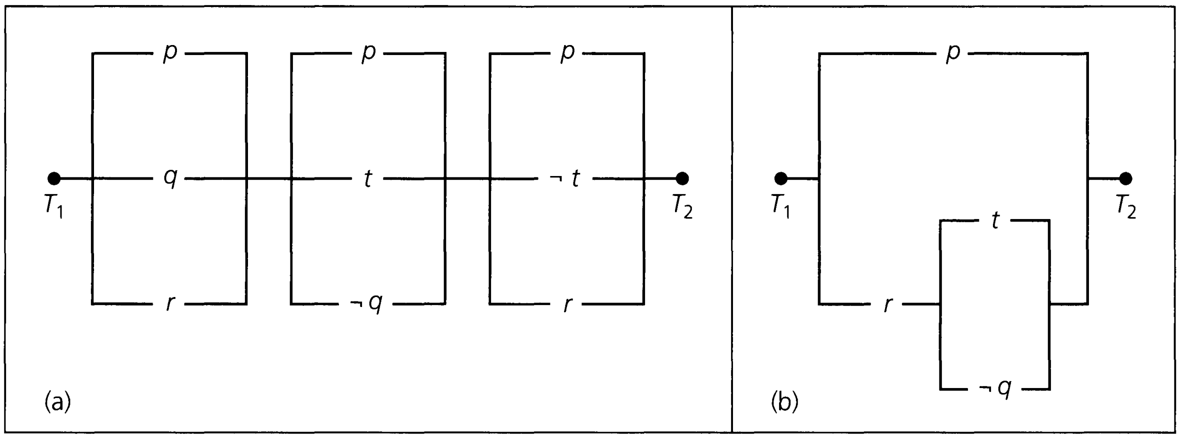

Figure 2.2

This network is represented by the statement $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$. Using the laws of logic, we may simplify this statement as follows.

	Reasons
$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$	
$\Longleftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$	Commutative Law of $\wedge$
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$	Law of Double Negation
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$	DeMorgan's Law
$\Longleftrightarrow p \vee [\neg(\neg t \wedge q) \wedge ((\neg t \wedge q) \vee r)]$	Commutative Law of $\wedge$ (twice)
$\Longleftrightarrow p \vee [(\neg(\neg t \wedge q) \wedge (\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$	Distributive Law of $\wedge$ over $\vee$
$\Longleftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$	$\neg s \wedge s \Longleftrightarrow F_0$, for any statement s
$\Longleftrightarrow p \vee [(\neg(\neg t \wedge q)) \wedge r]$	F_0 is the identity for $\vee$
$\Longleftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$	Commutative Law of $\wedge$
$\Longleftrightarrow p \vee [r \wedge (t \vee \neg q)]$	DeMorgan's Law and the Law of Double Negation

Hence $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \Longleftrightarrow p \vee [r \wedge (t \vee \neg q)]$, and the network shown in Fig. 2.2(b) is equivalent to the original network in the sense that current flows from T_1 to T_2 in network (a) exactly when it does so in network (b). But network (b) has only four switches, five fewer than network (a).

EXERCISES 2.2

1. Let p, q, r denote primitive statements.

a) Use truth tables to verify the following logical equivalences.

i) $p \rightarrow (q \wedge r) \Longleftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$
ii) $[(p \vee q) \rightarrow r] \Longleftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$
iii) $[p \rightarrow (q \vee r)] \Longleftrightarrow [\neg r \rightarrow (p \rightarrow q)]$

b) Use the substitution rules to show that

$$[p \rightarrow (q \vee r)] \Longleftrightarrow [(p \wedge \neg q) \rightarrow r].$$

2. Verify the first Absorption Law by means of a truth table.

3. Use the substitution rules to verify that each of the following is a tautology. (Here p, q, and r are primitive statements.)

a) $[p \vee (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$

b) $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$

4. For primitive statements p, q, r, and s, simplify the compound statement

$$[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \rightarrow s.$$

5. Negate and express each of the following statements in smooth English.

a) Kelsey will get a good education if she puts her studies before her interest in cheerleading.

b) Norma is doing her homework, and Karen is practicing her piano lessons.

c) If Harold passes his C++ course and finishes his data structures project, then he will graduate at the end of the semester.

6. Negate each of the following and simplify the resulting statement.

a) $p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)$

b) $(p \wedge q) \rightarrow r$

c) $p \rightarrow (\neg q \wedge r)$

d) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$

7. a) If p, q are primitive statements, prove that

$$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Longleftrightarrow (p \wedge q).$$

b) Write the dual of the logical equivalence in part (a).

8. Write the dual for (a) $q \rightarrow p$, (b) $p \rightarrow (q \wedge r)$, (c) $p \leftrightarrow q$, and (d) $p \veebar q$, where p, q, and r are primitive statements.

9. Write the converse, inverse, and contrapositive of each of the following implications. For each implication, determine its truth value as well as the truth values of its corresponding converse, inverse, and contrapositive.

a) If $0 + 0 = 0$, then $1 + 1 = 1$.

b) If $-1 < 3$ and $3 + 7 = 10$, then $\sin\left(\frac{3\pi}{2}\right) = -1$.

10. Determine whether each of the following is true or false. Here p, q are arbitrary statements.

a) An equivalent way to express the converse of "p is sufficient for q" is "p is necessary for q."

b) An equivalent way to express the inverse of "p is necessary for q" is "$\neg q$ is sufficient for $\neg p$."

c) An equivalent way to express the contrapositive of "p is necessary for q" is "$\neg q$ is necessary for $\neg p$."

11. Let p, q, and r denote primitive statements. Find a form of the contrapositive of $p \rightarrow (q \rightarrow r)$ with (a) only one occurrence of the connective $\rightarrow$; (b) no occurrences of the connective $\rightarrow$.

12. Show that for primitive statements p, q,

$$p \veebar q \Longleftrightarrow [(p \wedge \neg q) \vee (\neg p \wedge q)] \Longleftrightarrow \neg(p \leftrightarrow q).$$

13. Verify that $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Longleftrightarrow [(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$, for primitive statements p, q, and r.

14. For primitive statements p, q,

a) verify that $p \rightarrow [q \rightarrow (p \wedge q)]$ is a tautology.

b) verify that $(p \vee q) \rightarrow [q \rightarrow q]$ is a tautology by using the result from part (a) along with the substitution rules and the laws of logic.

c) is $(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$ a tautology?

15. Define the connective "Nand" or "Not ... and ..." by $(p \uparrow q) \Longleftrightarrow \neg(p \wedge q)$, for any statements p, q. Represent the following using only this connective.

a) $\neg p$ **b)** $p \vee q$ **c)** $p \wedge q$
d) $p \rightarrow q$ **e)** $p \leftrightarrow q$

16. The connective "Nor" or "Not ... or ..." is defined for any statements p, q by $(p \downarrow q) \Longleftrightarrow \neg(p \vee q)$. Represent the statements in parts (a) through (e) of Exercise 15, using only this connective.

17. For any statements p, q, prove that

a) $\neg(p \downarrow q) \Longleftrightarrow (\neg p \uparrow \neg q)$

b) $\neg(p \uparrow q) \Longleftrightarrow (\neg p \downarrow \neg q)$

18. Give the reasons for each step in the following simplifications of compound statements.

a)

	Reasons
$[(p \vee q) \wedge (p \vee \neg q)] \vee q$	
$\Longleftrightarrow [p \vee (q \wedge \neg q)] \vee q$	
$\Longleftrightarrow (p \vee F_0) \vee q$	
$\Longleftrightarrow p \vee q$	

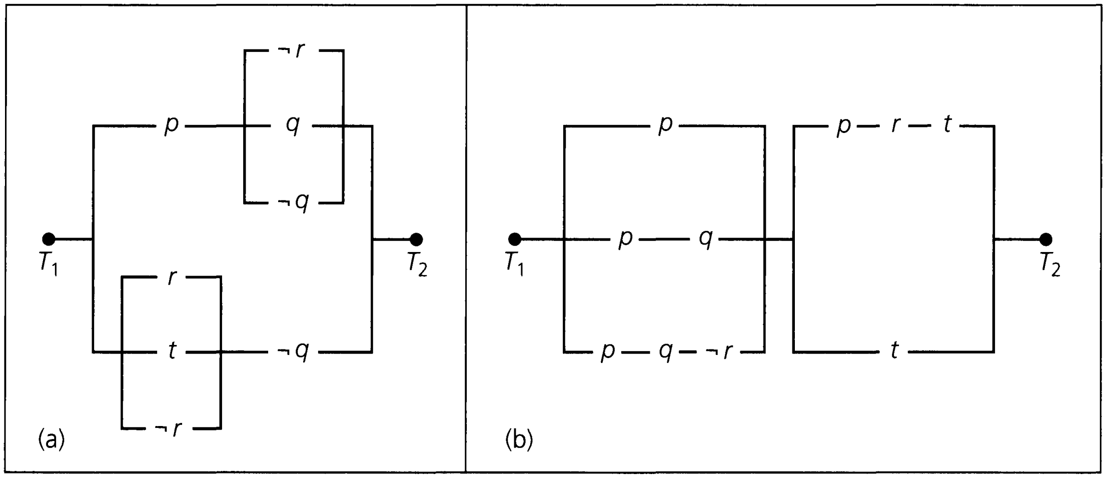

Figure 2.3

b) $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)]$ **Reasons**

$\Longleftrightarrow (p \to q) \wedge \neg q$

$\Longleftrightarrow (\neg p \vee q) \wedge \neg q$

$\Longleftrightarrow \neg q \wedge (\neg p \vee q)$

$\Longleftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$

$\Longleftrightarrow (\neg q \wedge \neg p) \vee F_0$

$\Longleftrightarrow \neg q \wedge \neg p$

$\Longleftrightarrow \neg(q \vee p)$

19. Provide the steps and reasons, as in Exercise 18, to establish the following logical equivalences.

a) $p \vee [p \wedge (p \vee q)] \Longleftrightarrow p$

b) $p \vee q \vee (\neg p \wedge \neg q \wedge r) \Longleftrightarrow p \vee q \vee r$

c) $[(\neg p \vee \neg q) \to (p \wedge q \wedge r)] \Longleftrightarrow p \wedge q$

20. Simplify each of the networks shown in Fig. 2.3.

2.3 Logical Implication: Rules of Inference

At the end of Section 2.1 we mentioned the notion of a valid argument. Now we will begin a formal study of what we shall mean by an argument and when such an argument is valid. This in turn will help us when we investigate how to prove theorems throughout the text.

We start by considering the general form of an argument, one we wish to show is valid. So let us consider the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \to q.$$

Here n is a positive integer, the statements $p_1, p_2, p_3, \ldots, p_n$ are called the *premises* of the argument, and the statement q is the *conclusion* for the argument.

The preceding argument is called *valid* if whenever each of the premises $p_1, p_2, p_3, \ldots, p_n$ is true, then the conclusion q is likewise true. [Note that if any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then the hypothesis $p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n$ is false and the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \to q$ is automatically true, regardless of the truth value of q.] Consequently, one way to establish the validity of a given argument is to show that the statement $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \to q$ is a tautology.

The following examples illustrate this particular approach.

EXAMPLE 2.19

Let p, q, r denote the primitive statements given as

p: Roger studies.

q: Roger plays racketball.

r: Roger passes discrete mathematics.

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1. a) (i)

p	q	r	$q \wedge r$	$p \rightarrow (q \wedge r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \wedge (p \rightarrow r)$
0	0	0	0	1	1	1	1
0	0	1	0	1	1	1	1
0	1	0	0	1	1	1	1
0	1	1	1	1	1	1	1
1	0	0	0	0	0	0	0
1	0	1	0	0	0	1	0
1	1	0	0	0	1	0	0
1	1	1	1	1	1	1	1

(iii)

p	q	r	$q \vee r$	$p \rightarrow (q \vee r)$	$p \rightarrow q$	$\neg r \rightarrow (p \rightarrow q)$
0	0	0	0	1	1	1
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	0	0	0
1	0	1	1	1	0	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

b) $[p \rightarrow (q \vee r)] \Longleftrightarrow [\neg r \rightarrow (p \rightarrow q)]$ From part (iii) of part (a)

$\Longleftrightarrow [\neg r \rightarrow (\neg p \vee q)]$ By the 2nd Substitution Rule, and $(p \rightarrow q) \Longleftrightarrow (\neg p \vee q)$

$\Longleftrightarrow [\neg(\neg p \vee q) \rightarrow \neg\neg r]$ By the 1st Substitution Rule, and $(s \rightarrow t) \Longleftrightarrow (\neg t \rightarrow \neg s)$ for any primitive statements s, t

$\Longleftrightarrow [(\neg\neg p \wedge \neg q) \rightarrow r]$ By DeMorgan's Law, Double Negation, and the 2nd Substitution Rule

$\Longleftrightarrow [(p \wedge \neg q) \rightarrow r]$ By Double Negation and the 2nd Substitution Rule

3. a) For any primitive statement s, $s \vee \neg s \Longleftrightarrow T_0$. Replace each occurrence of s by $p \vee (q \wedge r)$, and the result follows by the 1st Substitution Rule.

b) For any primitive statements s, t, we have $(s \rightarrow t) \Longleftrightarrow (\neg t \rightarrow \neg s)$. Replace each occurrence of s by $p \vee q$, and each occurrence of t by r, and the result is a consequence of the 1st Substitution Rule.

5. a) Kelsey placed her studies before her interest in cheerleading, but she (still) did not get a good education.

b) Norma is not doing her mathematics homework or Karen is not practicing her piano lesson.

c) Harold did pass his C++ course and he did finish his data structures project, but he did not graduate at the end of the semester.

7. a)

p	q	$(\neg p \vee q) \wedge (p \wedge (p \wedge q))$	$p \wedge q$
0	0	0	0
0	1	0	0
1	0	0	0
1	1	1	1

b) $(\neg p \wedge q) \vee (p \vee (p \vee q)) \Longleftrightarrow p \vee q$

9. a) If $0 + 0 = 0$, then $1 + 1 = 1$. (FALSE)

Contrapositive: If $1 + 1 \neq 1$, then $0 + 0 \neq 0$. (FALSE)

Converse: If $1 + 1 = 1$, then $0 + 0 = 0$. (TRUE)

Inverse: If $0 + 0 \neq 0$, then $1 + 1 \neq 1$. (TRUE)

b) If $-1 < 3$ and $3 + 7 = 10$, then $\sin\left(\frac{3\pi}{2}\right) = -1$. (TRUE)
Converse: If $\sin\left(\frac{3\pi}{2}\right) = -1$, then $-1 < 3$ and $3 + 7 = 10$. (TRUE)
Inverse: If $-1 \geq 3$ or $3 + 7 \neq 10$, then $\sin\left(\frac{3\pi}{2}\right) \neq -1$. (TRUE)
Contrapositive: If $\sin\left(\frac{3\pi}{2}\right) \neq -1$, then $-1 \geq 3$ or $3 + 7 \neq 10$. (TRUE)

11. **a)** $(q \rightarrow r) \vee \neg p$ **b)** $(\neg q \vee r) \vee \neg p$

13.

p	q	r	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$	$[(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$
0	0	0	1	1
0	0	1	0	0
0	1	0	0	0
0	1	1	0	0
1	0	0	0	0
1	0	1	0	0
1	1	0	0	0
1	1	1	1	1

15. **a)** $(p \uparrow p)$ **b)** $(p \uparrow p) \uparrow (q \uparrow q)$ **c)** $(p \uparrow q) \uparrow (p \uparrow q)$ **d)** $p \uparrow (q \uparrow q)$
e) $(r \uparrow s) \uparrow (r \uparrow s)$, where r stands for $p \uparrow (q \uparrow q)$ and s for $q \uparrow (p \uparrow p)$

17.

p	q	$\neg(p \downarrow q)$	$(\neg p \uparrow \neg q)$	$\neg(p \uparrow q)$	$(\neg p \downarrow \neg q)$
0	0	0	0	0	0
0	1	1	1	0	0
1	0	1	1	0	0
1	1	1	1	1	1

19. **a)**

	Reasons
$p \vee [p \wedge (p \vee q)]$	
$\Leftrightarrow p \vee p$	Absorption Law
$\Leftrightarrow p$	Idempotent Law of $\vee$

c)

	Reasons
$[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)]$	
$\Leftrightarrow \neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)$	$s \rightarrow t \Leftrightarrow \neg s \vee t$
$\Leftrightarrow (\neg\neg p \wedge \neg\neg q) \vee (p \wedge q \wedge r)$	DeMorgan's Laws
$\Leftrightarrow (p \wedge q) \vee (p \wedge q \wedge r)$	Law of Double Negation
$\Leftrightarrow p \wedge q$	Absorption Law

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1. **a)**

p	q	r	$p \rightarrow q$	$(p \vee q)$	$(p \vee q) \rightarrow r$
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	1	1	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	1	1	1

The validity of the argument follows from the results in the last row. (The first seven rows may be ignored.)