c) $2^n = (2+x)^n - \binom{n}{1}x^1(2+x)^{n-1}$
$+ \binom{n}{2}x^2(2+x)^{n-2} - \cdots + (-1)^n\binom{n}{n}x^n$

32. Determine x if $\sum_{i=0}^{50}\binom{50}{i}8^i = x^{100}$.

33. a) If a_0, a_1, a_2, a_3 is a list of four real numbers, what is $\sum_{i=1}^{3}(a_i - a_{i-1})$?

b) Given a list—$a_0, a_1, a_2, \ldots, a_n$—of $n+1$ real numbers, where n is a positive integer, determine $\sum_{i=1}^{n}(a_i - a_{i-1})$.

c) Determine the value of $\sum_{i=1}^{100}\left(\frac{1}{i+2} - \frac{1}{i+1}\right)$.

34. a) Write a computer program (or develop an algorithm) that lists all selections of size 2 from the objects 1, 2, 3, 4, 5, 6.

b) Repeat part (a) for selections of size 3.

1.4 Combinations with Repetition

When repetitions are allowed, we have seen that for n distinct objects an arrangement of size r of these objects can be obtained in n^r ways, for an integer $r \geq 0$. We now turn to the comparable problem for combinations and once again obtain a related problem whose solution follows from our previous enumeration principles.

EXAMPLE 1.28

On their way home from track practice, seven high school freshmen stop at a restaurant, where each of them has one of the following: a cheeseburger, a hot dog, a taco, or a fish sandwich. How many different purchases are possible (from the viewpoint of the restaurant)?

Let c, h, t, and f represent cheeseburger, hot dog, taco, and fish sandwich, respectively. Here we are concerned with how many of each item are purchased, not with the order in which they are purchased, so the problem is one of selections, or combinations, with repetition.

In Table 1.6 we list some possible purchases in column (a) and another means of representing each purchase in column (b).

Table 1.6

(a)	(b)
1. c, c, h, h, t, t, f	1. x x | x x | x x | x
2. c, c, c, c, h, t, f	2. x x x x | x | x | x
3. c, c, c, c, c, c, f	3. x x x x x x | | | x
4. h, t, t, f, f, f, f	4. | x | x x | x x x x
5. t, t, t, t, t, f, f	5. | | x x x x x | x x
6. t, t, t, t, t, t, t	6. | | x x x x x x x |
7. f, f, f, f, f, f, f	7. | | | x x x x x x x

For a purchase in column (b) of Table 1.6 we realize that each x to the left of the first bar (|) represents a c, each x between the first and second bars represents an h, the x's between the second and third bars stand for t's, and each x to the right of the third bar stands for an f. The third purchase, for example, has three consecutive bars because no one bought a hot dog or taco; the bar at the start of the fourth purchase indicates that there were no cheeseburgers in that purchase.

Once again a correspondence has been established between two collections of objects, where we know how to count the number in one collection. For the representations in column (b) of Table 1.6, we are enumerating all arrangements of 10 symbols consisting of seven x's and three |'s, so by our correspondence the number of different purchases for column (a) is

$$\frac{10!}{7!\,3!} = \binom{10}{7}.$$

In this example we note that the seven x's (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the $3 + 1 = 4$ possible food items that can be chosen.

When we wish to select, *with repetition*, r of n distinct objects, we find (as in Table 1.6) that we are considering all arrangements of r x's and $n - 1$ |'s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}.$$

Consequently, the number of combinations of n objects taken r at a time, *with repetition*, is $C(n + r - 1, r)$.

(In Example 1.28, $n = 4$, $r = 7$, so it is possible for r to exceed n when repetitions are allowed.)

EXAMPLE 1.29

A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind when we enter the shop, we can select a dozen donuts in $C(20 + 12 - 1, 12) = C(31, 12) = 141{,}120{,}525$ ways. (Here $n = 20$, $r = 12$.)

EXAMPLE 1.30

President Helen has four vice presidents: (1) Betty, (2) Goldie, (3) Mary Lou, and (4) Mona. She wishes to distribute among them \$1000 in Christmas bonus checks, where each check will be written for a multiple of \$100.

a) Allowing the situation in which one or more of the vice presidents get nothing, President Helen is making a selection of size 10 (one for each unit of \$100) from a collection of size 4 (four vice presidents), with repetition. This can be done in $C(4 + 10 - 1, 10) = C(13, 10) = 286$ ways.

b) If there are to be no hard feelings, each vice president should receive at least \$100. With this restriction, President Helen is now faced with making a selection of size 6 (the remaining six units of \$100) from the same collection of size 4, and the choices now number $C(4 + 6 - 1, 6) = C(9, 6) = 84$. [For example, here the selection 2, 3, 3, 4, 4, 4 is interpreted as follows: Betty does not get anything extra—for there is no 1 in the selection. The one 2 in the selection indicates that Goldie gets an additional \$100. Mary Lou receives an additional \$200 (\$100 for each of the two 3's in the selection). Due to the three 4's, Mona's bonus check will total $\$100 + 3(\$100) = \$400$.]

c) If each vice president must get at least \$100 and Mona, as executive vice president, gets at least \$500, then the number of ways President Helen can distribute the bonus checks is

$$\underbrace{C(3+2-1,2)}_{\text{Mona gets exactly \$500}} + \underbrace{C(3+1-1,1)}_{\text{Mona gets exactly \$600}} + \underbrace{C(3+0-1,0)}_{\text{Mona gets exactly \$700}} = 10 = \underbrace{C(4+2-1,2)}_{\text{Using the technique in part (b)}}$$

Having worked examples utilizing combinations with repetition, we now consider two examples involving other counting principles as well.

EXAMPLE 1.31

In how many ways can we distribute seven bananas and six oranges among four children so that each child receives at least one banana?

After giving each child one banana, consider the number of ways the remaining three bananas can be distributed among these four children. Table 1.7 shows four of the distributions we are considering here. For example, the second distribution in part (a) of Table 1.7—namely, 1, 3, 3—indicates that we have given the first child (designated by 1) one additional banana and the third child (designated by 3) two additional bananas. The corresponding arrangement in part (b) of Table 1.7 represents this distribution in terms of three b's and three bars. These six symbols—three of one type (the b's) and three others of a second type (the bars)—can be arranged in $6!/(3!\,3!) = C(6, 3) = C(4+3-1, 3) = 20$ ways. [Here $n = 4$, $r = 3$.] Consequently, there are 20 ways in which we can distribute the three additional bananas among these four children. Table 1.8 provides the comparable situation for distributing the six oranges. In this case we are arranging nine symbols—six of one type (the o's) and three of a second type (the bars). So now we learn that the number of ways we can distribute the six oranges among these four children is $9!/(6!\,3!) = C(9, 6) = C(4+6-1, 6) = 84$ ways. [Here $n = 4$, $r = 6$.] Therefore, by the rule of product, there are $20 \times 84 = 1680$ ways to distribute the fruit under the stated conditions.

Table 1.7

(a)	(b)
1) 1, 2, 3	1) $b \mid b \mid b \mid$
2) 1, 3, 3	2) $b \mid \mid b\,b \mid$
3) 3, 4, 4	3) $\mid \mid b \mid b\,b$
4) 4, 4, 4	4) $\mid \mid \mid b\,b\,b$

Table 1.8

(a)	(b)
1) 1, 2, 2, 3, 3, 4	1) $o \mid o\,o \mid o\,o \mid o$
2) 1, 2, 2, 4, 4, 4	2) $o \mid o\,o \mid \mid o\,o\,o$
3) 2, 2, 2, 3, 3, 3	3) $\mid o\,o\,o \mid o\,o\,o \mid$
4) 4, 4, 4, 4, 4, 4	4) $\mid \mid \mid o\,o\,o\,o\,o\,o$

EXAMPLE 1.32

A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 (blank) spaces between the symbols, with at least three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

There are 12! ways to arrange the 12 different symbols, and for each of these arrangements there are 11 positions between the 12 symbols. Because there must be at least three spaces between successive symbols, we use up 33 of the 45 spaces and must now locate the remaining 12 spaces. This is now a selection, with repetition, of size 12 (the spaces) from a collection of size 11 (the locations), and this can be accomplished in $C(11+12-1, 12) = 646{,}646$ ways.

Consequently, by the rule of product the transmitter can send such messages with the required spacing in $(12!)\binom{22}{12} \doteq 3.097 \times 10^{14}$ ways.

In the next example an idea is introduced that appears to have more to do with number theory than with combinations or arrangements. Nonetheless, the solution of this example will turn out to be equivalent to counting combinations with repetitions.

EXAMPLE 1.33

Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \qquad \text{where } x_i \geq 0 \quad \text{for all } 1 \leq i \leq 4.$$

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$. (This is different from a solution such as $x_1 = 1, x_2 = 0, x_3 = 3, x_4 = 3$, even though the same four integers are being used.) A possible interpretation for the solution $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$ is that we are distributing seven pennies (identical objects) among four children (distinct containers), and here we have given three pennies to each of the first two children, nothing to the third child, and the last penny to the fourth child. Continuing with this interpretation, we see that each nonnegative integer solution of the equation corresponds to a selection, with repetition, of size 7 (the *identical* pennies) from a collection of size 4 (the *distinct* children), so there are $C(4 + 7 - 1, 7) = 120$ solutions.

At this point it is crucial that we recognize the equivalence of the following:

a) The number of integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = r, \qquad x_i \geq 0, \qquad 1 \leq i \leq n.$$

b) The number of selections, with repetition, of size r from a collection of size n.

c) The number of ways r identical objects can be distributed among n distinct containers.

In terms of distributions, part (c) is valid only when the r objects being distributed are identical and the n containers are distinct. When both the r objects and the n containers are distinct, we can select any of the n containers for each one of the objects and get n^r distributions by the rule of product.

When the objects are distinct but the containers are identical, we shall solve the problem using the Stirling numbers of the second kind (Chapter 5). For the final case, in which both objects and containers are identical, the theory of partitions of integers (Chapter 9) will provide some necessary results.

EXAMPLE 1.34

In how many ways can one distribute 10 (identical) white marbles among six distinct containers?

Solving this problem is equivalent to finding the number of nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. That number is the number of selections of size 10, with repetition, from a collection of size 6. Hence the answer is $C(6 + 10 - 1, 10) = 3003$.

We now examine two other examples related to the theme of this section.

EXAMPLE 1.35

From Example 1.34 we know that there are 3003 nonnegative integer solutions to the equation $x_1 + x_2 + \cdots + x_6 = 10$. How many such solutions are there to the inequality $x_1 + x_2 + \cdots + x_6 < 10$?

One approach that may seem feasible in dealing with this inequality is to determine the number of such solutions to $x_1 + x_2 + \cdots + x_6 = k$, where k is an integer and $0 \leq k \leq 9$. Although feasible now, the technique becomes unrealistic if 10 is replaced by a somewhat larger number, say 100. In Example 3.12 of Chapter 3, however, we shall establish a combinatorial identity that will help us obtain an alternative solution to the problem by using this approach.

For the present we transform the problem by noting the correspondence between the nonnegative integer solutions of

$$x_1 + x_2 + \cdots + x_6 < 10 \tag{1}$$

and the integer solutions of

$$x_1 + x_2 + \cdots + x_6 + x_7 = 10, \qquad 0 \leq x_i, \qquad 1 \leq i \leq 6, \qquad 0 < x_7. \tag{2}$$

The number of solutions of Eq. (2) is the same as the number of nonnegative integer solutions of $y_1 + y_2 + \cdots + y_6 + y_7 = 9$, where $y_i = x_i$ for $1 \leq i \leq 6$, and $y_7 = x_7 - 1$. This is $C(7 + 9 - 1, 9) = 5005$.

Our next result takes us back to the binomial and multinomial expansions.

EXAMPLE 1.36

In the binomial expansion for $(x + y)^n$, each term is of the form $\binom{n}{k} x^k y^{n-k}$, so the total number of terms in the expansion is the number of nonnegative integer solutions of $n_1 + n_2 = n$ (n_1 is the exponent for x, n_2 the exponent for y). This number is $C(2 + n - 1, n) = n + 1$.

Perhaps it seems that we have used a rather long-winded argument to get this result. Many of us would probably be willing to believe the result on the basis of our experiences in expanding $(x + y)^n$ for various small values of n.

Although experience is worthwhile in pattern recognition, it is not always enough to find a general principle. Here it would prove of little value if we wanted to know how many terms there are in the expansion of $(w + x + y + z)^{10}$.

Each distinct term here is of the form $\binom{10}{n_1, n_2, n_3, n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$, where $0 \leq n_i$ for $1 \leq i \leq 4$, and $n_1 + n_2 + n_3 + n_4 = 10$. This last equation can be solved in $C(4 + 10 - 1, 10) = 286$ ways, so there are 286 terms in the expansion of $(w + x + y + z)^{10}$.

And now once again the binomial expansion will come into play, as we find ourselves using part (a) of Corollary 1.1

EXAMPLE 1.37

a) Let us determine all the different ways in which we can write the number 4 as a sum of positive integers, where the order of the summands is considered relevant. These representations are called the *compositions* of 4 and may be listed as follows:

1) 4	**5)** $2 + 1 + 1$
2) $3 + 1$	**6)** $1 + 2 + 1$
3) $1 + 3$	**7)** $1 + 1 + 2$
4) $2 + 2$	**8)** $1 + 1 + 1 + 1$

Here we include the sum consisting of only one summand—namely, 4. We find that for the number 4 there are eight compositions in total. (If we do *not* care about the order of the summands, then the representations in (2) and (3) are no longer considered to be different—nor are the representations in (5), (6), and (7). Under these circumstances we find that there are five *partitions* for the number 4—namely, 4; $3+1$; $2+2$; $2+1+1$; and $1+1+1+1$. We shall learn more about partitions of positive integers in Section 9.3.)

b) Now suppose that we wish to *count* the number of compositions for the number 7. Here we do *not* want to list all of the possibilities—which include 7; $6+1$; $1+6$; $5+2$; $1+2+4$; $2+4+1$; and $3+1+2+1$. To count all of these compositions, let us consider the number of possible summands.

i) For one summand there is only one composition—namely, 7.

ii) If there are two (positive) summands, we want to count the number of integer solutions for

$$w_1 + w_2 = 7, \qquad \text{where } w_1, w_2 > 0.$$

This is equal to the number of integer solutions for

$$x_1 + x_2 = 5, \qquad \text{where } x_1, x_2 \geq 0.$$

The number of such solutions is $\binom{2+5-1}{5} = \binom{6}{5}$.

iii) Continuing with our next case, we examine the compositions with three (positive) summands. So now we want to count the number of *positive* integer solutions for

$$y_1 + y_2 + y_3 = 7.$$

This is equal to the number of *nonnegative* integer solutions for

$$z_1 + z_2 + z_3 = 4,$$

and that number is $\binom{3+4-1}{4} = \binom{6}{4}$.

We summarize cases (i), (ii), and (iii), and the other four cases in Table 1.9, where we recall for case (i) that $1 = \binom{6}{6}$.

Table 1.9

n = The Number of Summands in a Composition of 7	The Number of Compositions of 7 with n Summands
(i) $n = 1$	(i) $\binom{6}{6}$
(ii) $n = 2$	(ii) $\binom{6}{5}$
(iii) $n = 3$	(iii) $\binom{6}{4}$
(iv) $n = 4$	(iv) $\binom{6}{3}$
(v) $n = 5$	(v) $\binom{6}{2}$
(vi) $n = 6$	(vi) $\binom{6}{1}$
(vii) $n = 7$	(vii) $\binom{6}{0}$

Consequently, the results from the right-hand side of our table tell us that the (total) number of compositions of 7 is

$$\binom{6}{6}+\binom{6}{5}+\binom{6}{4}+\binom{6}{3}+\binom{6}{2}+\binom{6}{1}+\binom{6}{0}=\sum_{k=0}^{6}\binom{6}{k}.$$

From part (a) of Corollary 1.1 this reduces to 2^6.

In general, one finds that for each positive integer m, there are $\sum_{k=0}^{m-1}\binom{m-1}{k}=2^{m-1}$ compositions.

EXAMPLE 1.38

From Example 1.37 we know that there are $2^{12-1}=2^{11}=2048$ compositions of 12. If our interest is in those compositions where each summand is even, then we consider, for instance, compositions such as

$$2+4+6=2(1+2+3) \qquad 2+8+2=2(1+4+1)$$
$$8+2+2=2(4+1+1) \qquad 6+6=2(3+3).$$

In each of these four examples, the parenthesized expression is a composition of 6. This observation indicates that the number of compositions of 12, where each summand is even, equals the number of (all) compositions of 6, which is $2^{6-1}=2^5=32$.

Our next two examples provide applications from the area of computer science. Furthermore, the second example will lead to an important summation formula that we shall use in many later chapters.

EXAMPLE 1.39

Consider the following program segment, where i, j, and k are integer variables.

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      print (i * j + k)
```

How many times is the **print** statement executed in this program segment?

Among the possible choices for i, j, and k (in the order i–first, j–second, k–third) that will lead to execution of the **print** statement, we list (1) 1, 1, 1; (2) 2, 1, 1; (3) 15, 10, 1; and (4) 15, 10, 7. We note that $i=10$, $j=12$, $k=5$ is not one of the selections to be considered, because $j=12>10=i$; this violates the condition set forth in the second **for** loop. Each of the above four selections where the **print** statement is executed satisfies the condition $1\le k\le j\le i\le 20$. In fact, any selection a, b, c ($a\le b\le c$) of size 3, with repetitions allowed, from the list 1, 2, 3, . . . , 20 results in one of the correct selections: here, $k=a$, $j=b$, $i=c$. Consequently the **print** statement is executed

$$\binom{20+3-1}{3}=\binom{22}{3}=1540 \text{ times.}$$

If there had been r (≥ 1) **for** loops instead of three, the **print** statement would have been executed $\binom{20+r-1}{r}$ times.

EXAMPLE 1.40

Here we use a program segment to derive a summation formula. In this program segment, the variables i, j, n, and *counter* are integer variables. Furthermore, we assume that the value of n has been set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    counter := counter + 1
```

From the results in Example 1.39, after this segment is executed the value of (the variable) *counter* will be $\binom{n+2-1}{2} = \binom{n+1}{2}$. (This is also the number of times that the statement

(*) $\quad$ `counter := counter + 1`

is executed.)

This result can also be obtained as follows: When $i := 1$, then j varies from 1 to 1 and (*) is executed once; when i is assigned the value 2, then j varies from 1 to 2 and (*) is executed twice; j varies from 1 to 3 when i is assigned the value 3, and (*) is executed three times; in general, for $1 \le k \le n$, when $i := k$, then j varies from 1 to k and (*) is executed k times. In total, the variable *counter* is incremented [and the statement (*) is executed] $1 + 2 + 3 + \cdots + n$ times.

Consequently,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

The derivation of this summation formula, obtained by counting the same result in two different ways, constitutes a combinatorial proof.

Our last example for this section introduces the idea of a run, a notion that arises in statistics — in particular, in the detecting of trends in a statistical process.

EXAMPLE 1.41

The counter at Patti and Terri's Bar has 15 bar stools. Upon entering the bar Darrell finds the stools occupied as follows:

O O E O O O O E E E O O O E O,

where O indicates an occupied stool and E an empty one. (Here we are not concerned with the occupants of the stools, just whether or not a stool is occupied.) In this case we say that the occupancy of the 15 stools determines seven runs, as shown:

$$\underbrace{\text{OO}}_{\textbf{Run}}\ \underbrace{\text{E}}_{\textbf{Run}}\ \underbrace{\text{OOOO}}_{\textbf{Run}}\ \underbrace{\text{EEE}}_{\textbf{Run}}\ \underbrace{\text{OOO}}_{\textbf{Run}}\ \underbrace{\text{E}}_{\textbf{Run}}\ \underbrace{\text{O}}_{\textbf{Run}}.$$

In general, a *run* is a consecutive list of identical entries that are preceded and followed by different entries or no entries at all.

A second way in which five E's and 10 O's can be arranged to provide seven runs is

E O O O E E O O E O O O O O E.

We want to find the total number of ways five E's and 10 O's can determine seven runs. If the first run starts with an E, then there must be four runs of E's and three runs of O's. Consequently, the last run must end with an E.

Let x_1 count the number of E's in the first run, x_2 the number of O's in the second run, x_3 the number of E's in the third run, ..., and x_7 the number of E's in the seventh run. We want to find the number of integer solutions for

$$x_1 + x_3 + x_5 + x_7 = 5, \qquad x_1, x_3, x_5, x_7 > 0 \tag{3}$$

and

$$x_2 + x_4 + x_6 = 10, \qquad x_2, x_4, x_6 > 0. \tag{4}$$

The number of integer solutions for Eq. (3) equals the number of integer solutions for

$$y_1 + y_3 + y_5 + y_7 = 1, \qquad y_1, y_3, y_5, y_7 \geq 0.$$

This number is $\binom{4+1-1}{1} = \binom{4}{1} = 4$. Similarly, for Eq. (4), the number of solutions is $\binom{3+7-1}{7} = \binom{9}{7} = 36$. Consequently, by the rule of product there are $4 \cdot 36 = 144$ arrangements of five E's and 10 O's that determine seven runs, the first run starting with E.

The seven runs may also have the first run starting with an O and the last run ending with an O. So now let w_1 count the number of O's in the first run, w_2 the number of E's in the second run, w_3 the number of O's in the third run, . . . , and w_7 the number of O's in the seventh run. Here we want the number of integer solutions for

$$w_1 + w_3 + w_5 + w_7 = 10, \qquad w_1, w_3, w_5, w_7 > 0$$

and

$$w_2 + w_4 + w_6 = 5, \qquad w_2, w_4, w_6 > 0.$$

Arguing as above, we find that the number of ways to arrange five E's and 10 O's, resulting in seven runs where the first run starts with an O, is $\binom{4+6-1}{6}\binom{3+2-1}{2} = \binom{9}{6}\binom{4}{2} = 504$.

Consequently, by the rule of sum, the five E's and 10 O's can be arranged in $144 + 504 = 648$ ways to produce seven runs.

EXERCISES 1.4

1. In how many ways can 10 (identical) dimes be distributed among five children if (a) there are no restrictions? (b) each child gets at least one dime? (c) the oldest child gets at least two dimes?

2. In how many ways can 15 (identical) candy bars be distributed among five children so that the youngest gets only one or two of them?

3. Determine how many ways 20 coins can be selected from four large containers filled with pennies, nickels, dimes, and quarters. (Each container is filled with only one type of coin.)

4. A certain ice cream store has 31 flavors of ice cream available. In how many ways can we order a dozen ice cream cones if (a) we do not want the same flavor more than once? (b) a flavor may be ordered as many as 12 times? (c) a flavor may be ordered no more than 11 times?

5. a) In how many ways can we select five coins from a collection of 10 consisting of one penny, one nickel, one dime, one quarter, one half-dollar, and five (identical) Susan B. Anthony dollars?

b) In how many ways can we select n objects from a collection of size $2n$ that consists of n distinct and n identical objects?

6. Answer Example 1.32, where the 12 symbols being transmitted are four A's, four B's, and four C's.

7. Determine the number of integer solutions of

$$x_1 + x_2 + x_3 + x_4 = 32,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 4$ **b)** $x_i > 0, \quad 1 \leq i \leq 4$

c) $x_1, x_2 \geq 5, \quad x_3, x_4 \geq 7$

d) $x_i \geq 8, \quad 1 \leq i \leq 4$ **e)** $x_i \geq -2, \quad 1 \leq i \leq 4$

f) $x_1, x_2, x_3 > 0, \quad 0 < x_4 \leq 25$

8. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants at least one donut of each kind?

9. Columba has two dozen each of n different colored beads. If she can select 20 beads (with repetitions of colors allowed) in 230,230 ways, what is the value of n?

10. In how many ways can Lisa toss 100 (identical) dice so that at least three of each type of face will be showing?

11. Two n-digit integers (leading zeros allowed) are considered equivalent if one is a rearrangement of the other. (For example, 12033, 20331, and 01332 are considered equivalent five-digit integers.) (a) How many five-digit integers are not equivalent? (b) If the digits 1, 3, and 7 can appear at most once, how many nonequivalent five-digit integers are there?

12. Determine the number of integer solutions for

$$x_1 + x_2 + x_3 + x_4 + x_5 < 40,$$

where

a) $x_i \geq 0, \quad 1 \leq i \leq 5$

b) $x_i \geq -3, \quad 1 \leq i \leq 5$

13. In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?

14. a) Find the coefficient of v^2w^4xz in the expansion of $(3v + 2w + x + y + z)^8$.

b) How many distinct terms arise in the expansion in part (a)?

15. In how many ways can Beth place 24 different books on four shelves so that there is at least one book on each shelf? (For any of these arrangements consider the books on each shelf to be placed one next to the other, with the first book at the left of the shelf.)

16. For which positive integer n will the equations

(1) $x_1 + x_2 + x_3 + \cdots + x_{19} = n$, and

(2) $y_1 + y_2 + y_3 + \cdots + y_{64} = n$

have the same number of positive integer solutions?

17. How many ways are there to place 12 marbles of the same size in five distinct jars if (a) the marbles are all black? (b) each marble is a different color?

18. a) How many nonnegative integer solutions are there to the pair of equations $x_1 + x_2 + x_3 + \cdots + x_7 = 37$, $x_1 + x_2 + x_3 = 6$?

b) How many solutions in part (a) have $x_1, x_2, x_3 > 0$?

19. How many times is the **print** statement executed for the following program segment? (Here, i, j, k, and m are integer variables.)

```
for i := 1 to 20 do
  for j := 1 to i do
    for k := 1 to j do
      for m := 1 to k do
        print (i * j) + (k * m)
```

20. In the following program segment, i, j, k, and *counter* are integer variables. Determine the value that the variable *counter* will have after the segment is executed.

```
counter := 10
for i := 1 to 15 do
  for j := i to 15 do
    for k := j to 15 do
      counter := counter + 1
```

21. Find the value of *sum* after the given program segment is executed. (Here i, j, k, *increment*, and *sum* are integer variables.)

```
increment := 0
sum := 0
for i := 1 to 10 do
  for j := 1 to i do
    for k := 1 to j do
      begin
        increment := increment + 1
        sum := sum + increment
      end
```

22. Consider the following program segment, where i, j, k, n, and *counter* are integer variables and the value of n (a positive integer) is set prior to this segment.

```
counter := 0
for i := 1 to n do
  for j := 1 to i do
    for k := 1 to j do
      counter := counter + 1
```

We shall determine, in two different ways, the number of times the statement

```
counter := counter + 1
```

is executed. (This is also the value of *counter* after execution of the program segment.) From the result in Example 1.39, we know that the statement is executed $\binom{n+3-1}{3} = \binom{n+2}{3}$ times. For a fixed value of i, the **for** loops involving j and k result in $\binom{i+1}{2}$ executions of the counter increment statement. Consequently, $\binom{n+2}{3} = \sum_{i=1}^{n} \binom{i+1}{2}$. Use this result to obtain a summation formula for

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2.$$

23. a) Given positive integers m, n with $m \geq n$, show that the number of ways to distribute m identical objects into n distinct containers with no container left empty is

$$C(m-1, m-n) = C(m-1, n-1).$$

b) Show that the number of distributions in part (a) where each container holds at least r objects ($m \geq nr$) is

$$C(m-1+(1-r)n, n-1).$$

24. Write a computer program (or develop an algorithm) to list the integer solutions for

a) $x_1 + x_2 + x_3 = 10, \quad 0 \leq x_i, \quad 1 \leq i \leq 3$

b) $x_1 + x_2 + x_3 + x_4 = 4, \quad -2 \leq x_i, \quad 1 \leq i \leq 4$

25. Consider the 2^{19} compositions of 20. (a) How many have each summand even? (b) How many have each summand a multiple of 4?

26. Let n, m, k be positive integers with $n = mk$. How many compositions of n have each summand a multiple of k?

27. Frannie tosses a coin 12 times and gets five heads and seven tails. In how many ways can these tosses result in (a) two runs of heads and one run of tails; (b) three runs; (c) four runs;

(d) five runs; (e) six runs; and (f) equal numbers of runs of heads and runs of tails?

28. **a)** For $n \geq 4$, consider the strings made up of n bits—that is, a total of n 0's and 1's. In particular, consider those strings where there are (exactly) two occurrences of 01. For example, if $n = 6$ we want to include strings such as 010010 and 100101, but not 101111 or 010101. How many such strings are there?

b) For $n \geq 6$, how many strings of n 0's and 1's contain (exactly) three occurrences of 01?

c) Provide a combinatorial proof for the following: For $n \geq 1$,

$$2^n = \binom{n+1}{1} + \binom{n+1}{3} + \cdots + \begin{cases} \binom{n+1}{n}, & n \text{ odd} \\ \binom{n+1}{n+1}, & n \text{ even.} \end{cases}$$

1.5 The Catalan Numbers (Optional)

In this section a very prominent sequence of numbers is introduced. This sequence arises in a wide variety of combinatorial situations. We'll begin by examining one specific instance where it is found.

EXAMPLE 1.42

Let us start at the point (0, 0) in the xy-plane and consider two kinds of moves:

$$\text{R: } (x, y) \to (x+1, y) \qquad \text{U: } (x, y) \to (x, y+1).$$

We want to know how we can move from (0, 0) to (5, 5) using such moves—one unit to the right or one unit up. So we'll need five R's and five U's. At this point we have a situation like that in Example 1.14, so we know there are $10!/(5!\,5!) = \binom{10}{5}$ such paths. But now we'll add a twist! In going from (0, 0) to (5, 5) one may touch but *never* rise above the line $y = x$. Consequently, we want to include paths such as those shown in parts (a) and (b) of Fig. 1.9 but not the path shown in part (c).

The first thing that is evident is that each such arrangement of five R's and five U's must start with an R (and end with a U). Then as we move across this type of arrangement—going from left to right—the number of R's at any point must equal or exceed the number of U's. Note how this happens in parts (a) and (b) of Fig. 1.9 but not in part (c). Now we can solve the problem at hand if we can count the paths [like the one in part (c)] that go from (0, 0) to (5, 5) but rise above the line $y = x$. Look again at the path in part (c) of Fig. 1.9. Where does the situation there break down for the first time? After all, we start with the requisite R—then follow it by a U. So far, so good! But then there is a second U and, at this (first) time, the number of U's exceeds the number of R's.

Now let us consider the following transformation:

$$\text{R, U, U, } \mid \text{ U, R, R, R, U, U, R} \leftrightarrow \text{R, U, U, } \mid \text{ R, U, U, U, R, R, U.}$$

What have we done here? For the path on the left-hand side of the transformation, we located the first move (the second U) where the path rose above the line $y = x$. The moves up to and including this move (the second U) remain as is, but the moves that follow are interchanged—each U is replaced by an R and each R by a U. The result is the path on the right-hand side of the transformation—an arrangement of four R's and six U's, as seen in part (d) of Fig. 1.9. Part (e) of that figure provides another path to be avoided; part (f) shows what happens when this path is transformed by the method described above. Now suppose we start with an arrangement of six U's and four R's, say

$$\text{R, U, R, R, U, U, U, } \mid \text{ U, U, R.}$$

21. $\binom{n}{3}$ $\qquad \binom{n}{3} - n - n(n-4),\ n \geq 4$

23. a) $\binom{12}{9}$ b) $\binom{12}{9}(2^3)$ c) $\binom{12}{9}(2^9)(-3)^3$

25. a) $\binom{4}{1,1,2} = 12$ b) 12 c) $\binom{4}{1,1,2}(2)(-1)(-1)^2 = -24$
d) -216 e) $\binom{8}{3,2,1,2}(2^3)(-1)^2(3)(-2)^2 = 161{,}280$

27. a) 2^3 b) 2^{10} c) 3^{10} d) 4^5 e) 4^{10}

29. $$n\binom{m+n}{m} = n\frac{(m+n)!}{m!\,n!} = \frac{(m+n)!}{m!(n-1)!} = (m+1)\frac{(m+n)!}{(m+1)(m!)(n-1)!}$$
$$= (m+1)\frac{(m+n)!}{(m+1)!(n-1)!} = (m+1)\binom{m+n}{m+1}$$

31. Consider the expansions of (a) $[(1+x)-x]^n$; (b) $[(2+x)-(x+1)]^n$; and (c) $[(2+x)-x]^n$.

33. a) $a_3 - a_0$ b) $a_n - a_0$ c) $\frac{1}{102} - \frac{1}{2} = \frac{-25}{51}$

Section 1.4—p. 34

1. a) $\binom{14}{10}$ b) $\binom{9}{5}$ c) $\binom{12}{8}$ 3. $\binom{23}{20}$ 5. a) 2^5 b) 2^n

7. a) $\binom{35}{32}$ b) $\binom{31}{28}$ c) $\binom{11}{8}$ d) 1 e) $\binom{43}{40}$ f) $\binom{31}{28} - \binom{6}{3}$

9. $n = 7$ 11. a) $\binom{14}{5}$ b) $\binom{11}{5} + 3\binom{10}{4} + 3\binom{9}{3} + \binom{8}{2}$

13. a) $\binom{7}{4}$ b) $\sum_{i=0}^{3}\binom{9-2i}{7-2i}$ 15. $\binom{23}{20}(24!)$ 17. a) $\binom{16}{12}$ b) 5^{12}

19. $\binom{23}{4}$ 21. $24{,}310 = \sum_{i=1}^{n} i$ $\left[\text{for } n = \binom{12}{3}\right]$

23. a) Place one of the m identical objects into each of the n distinct containers. This leaves $m-n$ identical objects to be placed into the n distinct containers, resulting in $\binom{n+(m-n)-1}{m-n} = \binom{m-1}{m-n} = \binom{m-1}{n-1}$ distributions.

25. a) 2^9 b) 2^4

27. a) $\binom{2+3-1}{3} = 4$ b) 10 c) 48 d) $\binom{3+4-1}{4}\binom{2+3-1}{3} + \binom{3+2-1}{2}\binom{2+5-1}{5} = 96$
e) 180 f) 420

Section 1.5—p. 40

1. $$\binom{2n}{n} - \binom{2n}{n-1} = \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!(n+1)!} = \frac{(2n)!(n+1)}{(n+1)!\,n!} - \frac{(2n)!\,n}{n!(n+1)!} =$$
$$\frac{(2n)![(n+1)-n]}{(n+1)!\,n!} = \frac{1}{(n+1)}\frac{(2n)!}{n!\,n!} = \left(\frac{1}{n+1}\right)\binom{2n}{n}$$

3. a) $5\ (= b_3)$; $14\ (= b_4)$
b) For $n \geq 0$ there are $b_n \left(= \frac{1}{(n+1)}\binom{2n}{n}\right)$ such paths from $(0, 0)$ to (n, n).
c) For $n \geq 0$ the first move is U and the last is R.

5. Using the results in the third column of Table 1.10 we have:

111000	110010	101010
1 2 3	1 2 5	1 3 5
4 5 6	3 4 6	2 4 6

7. There are $b_5 (= 42)$ ways.

9. (i) When $n = 4$ there are $14\ (= b_4)$ such diagrams.
(ii) For each $n \geq 0$, there are b_n different drawings of n semicircles on and above a horizontal line, with no two semicircles intersecting. Consider, for instance, the diagram in part (f) of Fig. 1.10. Going from left to right, write 1 the first time you encounter a semicircle and write 0 the second time that semicircle is encountered. Here we get the list 110100. The list 110010 corresponds with the drawing in part (g). This correspondence shows that the number of such drawings for n semicircles is the same as the number of lists of n 1's and n 0's where, as the list is read from left to right, the number of 0's never exceeds the number of 1's.

11. $\left(\frac{1}{7}\right)\binom{12}{6}(6!)(6!) = \left(\frac{1}{7}\right)(12!) = 68{,}428{,}800$

Supplementary Exercises—p. 43

1. $\binom{4}{1}\binom{7}{2} + \binom{4}{2}\binom{7}{4} + \binom{4}{3}\binom{7}{6}$

3. Select any four of these twelve points (on the circumference). As seen in the figure, these points determine a pair of chords that intersect. Consequently, the largest number of points of