4

Properties of the Integers: Mathematical Induction

Having known about the integers since our first encounters with arithmetic, in this chapter we examine a special property exhibited by the subset of positive integers. This property will enable us to establish certain mathematical formulas and theorems by using a technique called *mathematical induction*. This method of proof will play a key role in many of the results we shall obtain in the later chapters of this text. Furthermore, this chapter will provide us with an introduction to five sets of numbers that are very important in the study of discrete mathematics and combinatorics — namely, the triangular numbers, the harmonic numbers, the Fibonacci numbers, the Lucas numbers, and the Eulerian numbers.

When $x, y \in \mathbf{Z}$, we know that $x + y, xy, x - y \in \mathbf{Z}$. Thus we say that the set $\mathbf{Z}$ is *closed* under (the binary operations of) addition, multiplication, and subtraction. Turning to division, however, we find, for example, that $2, 3 \in \mathbf{Z}$ but that the rational number $\frac{2}{3}$ is *not* a member of $\mathbf{Z}$. So the set $\mathbf{Z}$ of all integers is *not closed* under the binary operation of *nonzero* division. To cope with this situation, we shall introduce a somewhat restricted form of division for $\mathbf{Z}$ and shall concentrate on special elements of $\mathbf{Z}^+$ called *primes*. These primes turn out to be the "building blocks" of the integers, and they provide our first example of a representation theorem — in this case the Fundamental Theorem of Arithmetic.

4.1 The Well-Ordering Principle: Mathematical Induction

Given any two distinct integers x, y, we know that we must have either $x < y$ or $y < x$. However, this is also true if, instead of being integers, x and y are rational numbers or real numbers. What makes $\mathbf{Z}$ special in this situation?

Suppose we try to express the subset $\mathbf{Z}^+$ of $\mathbf{Z}$, using the inequality symbols $>$ and $\geq$. We find that we can define the set of positive elements of $\mathbf{Z}$ as

$$\mathbf{Z}^+ = \{x \in \mathbf{Z} | x > 0\} = \{x \in \mathbf{Z} | x \geq 1\}.$$

When we try to do likewise for the rational and real numbers, however, we find that

$$\mathbf{Q}^+ = \{x \in \mathbf{Q} | x > 0\} \qquad \text{and} \qquad \mathbf{R}^+ = \{x \in \mathbf{R} | x > 0\},$$

but we cannot represent $\mathbf{Q}^+$ or $\mathbf{R}^+$ using $\geq$ as we did for $\mathbf{Z}^+$.

The set $\mathbf{Z}^+$ is different from the sets $\mathbf{Q}^+$ and $\mathbf{R}^+$ in that *every* nonempty subset X of $\mathbf{Z}^+$ contains an integer a such that $a \leq x$, for all $x \in X$—that is, X contains a *least* (or *smallest*) element. This is not so for either $\mathbf{Q}^+$ or $\mathbf{R}^+$. The sets themselves do not contain least elements. There is no smallest positive rational number or smallest positive real number. If q is a positive rational number, then since $0 < q/2 < q$, we would have the smaller positive rational number $q/2$.

These observations lead us to the following property of the set $\mathbf{Z}^+ \subset \mathbf{Z}$.

The Well-Ordering Principle: Every *nonempty* subset of $\mathbf{Z}^+$ contains a smallest element. (We often express this by saying that $\mathbf{Z}^+$ is *well ordered.*)

This principle serves to distinguish $\mathbf{Z}^+$ from $\mathbf{Q}^+$ and $\mathbf{R}^+$. But does it lead anywhere that is mathematically interesting or useful? The answer is a resounding "Yes!" It is the basis of a proof technique known as mathematical induction. This technique will often help us to prove a general mathematical statement involving positive integers when certain instances of that statement suggest a general pattern.

We now establish the basis for this induction technique.

THEOREM 4.1 *The Principle of Mathematical Induction.* Let $S(n)$ denote an open mathematical statement (or set of such open statements) that involves one or more occurrences of the variable n, which represents a positive integer.

a) If $S(1)$ is true; and

b) If whenever $S(k)$ is true (for some particular, but arbitrarily chosen, $k \in \mathbf{Z}^+$), then $S(k + 1)$ is true;

then $S(n)$ is true for all $n \in \mathbf{Z}^+$.

Proof: Let $S(n)$ be such an open statement satisfying conditions (a) and (b), and let $F = \{t \in \mathbf{Z}^+ | S(t) \text{ is false}\}$. We wish to prove that $F = \emptyset$, so to obtain a contradiction we assume that $F \neq \emptyset$. Then by the Well-Ordering Principle, F has a least element m. Since $S(1)$ is true, it follows that $m \neq 1$, so $m > 1$, and consequently $m - 1 \in \mathbf{Z}^+$. With $m - 1 \notin F$, we have $S(m - 1)$ true. So by condition (b) it follows that $S((m - 1) + 1) = S(m)$ is true, contradicting $m \in F$. This contradiction arose from the assumption that $F \neq \emptyset$. Consequently, $F = \emptyset$.

We have now seen how the Well-Ordering Principle is used in the proof of the Principle of Mathematical Induction. It is also true that the Principle of Mathematical Induction is useful if one wants to prove the Well-Ordering Principle. However, we shall not concern ourselves with that fact right now. In this section our major goal will center on understanding and using the Principle of Mathematical Induction. (But in the exercises for Section 4.2 we shall examine how the Principle of Mathematical Induction is used to prove the Well-Ordering Principle.)

In the statement of Theorem 4.1 the condition in part (a) is referred to as the *basis step*, while that in part (b) is called the *inductive step*.

The choice of 1 in the first condition of Theorem 4.1 is not mandatory. All that is needed is for the open statement $S(n)$ to be true for some *first* element $n_0 \in \mathbf{Z}$ so that the induction process has a starting place. We need the truth of $S(n_0)$ for our basis step. The integer n_0 could be 5 just as well as 1. It could even be zero or negative because the set $\mathbf{Z}^+$ in union with $\{0\}$ or any *finite* set of negative integers is well ordered. (When we do an induction proof and start with $n_0 < 0$, we are considering the set of all *consecutive* negative integers $\geq n_0$ in union with $\{0\}$ and $\mathbf{Z}^+$.)

Under these circumstances, we may express the Principle of Mathematical Induction, using quantifiers, as

$$[S(n_0) \wedge [\forall k \geq n_0 \; [S(k) \Rightarrow S(k+1)]]] \Rightarrow \forall n \geq n_0 \; S(n).$$

We may get a somewhat better understanding of why this method of proof is valid by using our intuition in conjunction with the situation presented in Fig. 4.1.

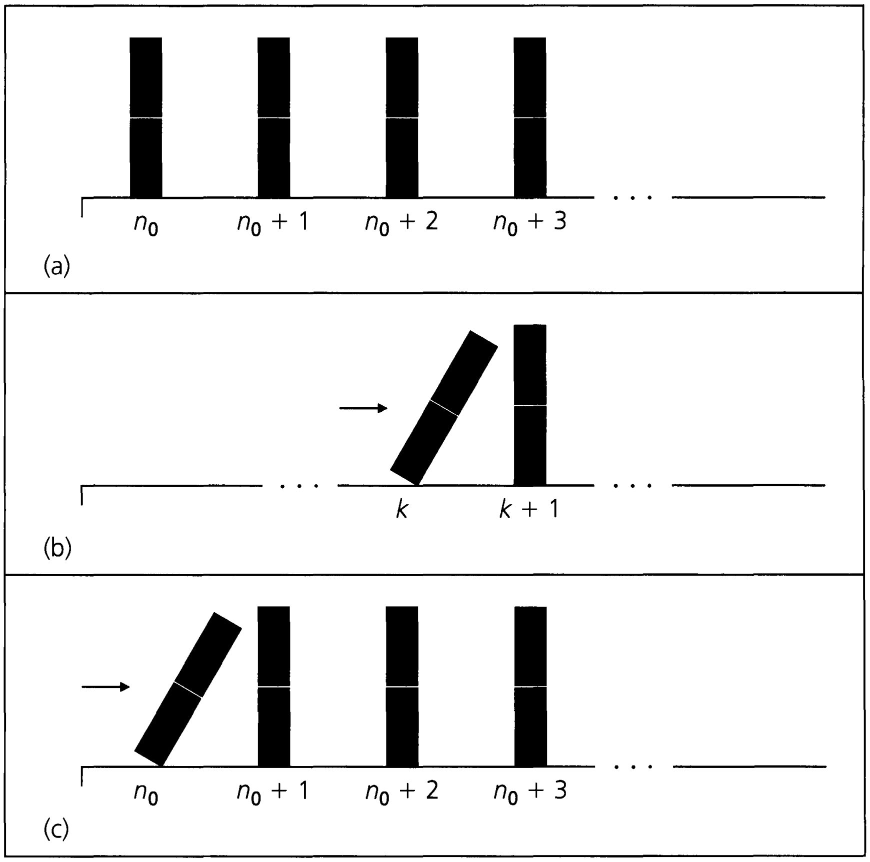

Figure 4.1

In part (a) of the figure we see the first four of an infinite (ordered) arrangement of dominos, each standing on end. The spacing between any two consecutive dominos is always the same, and it is such that if any one domino (say the kth) is pushed over to the right, then it will knock over the next ($(k+1)$st) domino. This process is suggested in Fig. 4.1(b). Our intuition leads us to feel that this process will continue, the $(k+1)$st domino toppling and knocking over (to the right) the $(k+2)$nd domino, and so on. Part (c) of the figure indicates how the truth of $S(n_0)$ provides the push (to the right) to the first domino (at n_0). This provides the basis step and sets the process in motion. The truth of $S(k)$

forcing the truth of $S(k+1)$ gives us the inductive step and continues the toppling process. We then infer the fact that $S(n)$ is true for all $n \geq n_0$ as we imagine *all* the successive dominos toppling (to the right.)

We shall now demonstrate several results that call for the use of Theorem 4.1.

EXAMPLE 4.1

For all $n \in \mathbf{Z}^+$, $\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$.

Proof: For $n = 1$ the open statement

$$S(n): \quad \sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

becomes $S(1)$: $\sum_{i=1}^{1} i = 1 = (1)(1+1)/2$. So $S(1)$ is true and we have our *basis step*— and a starting point from which to begin the induction. Assuming the result true for $n = k$ (for some $k \in \mathbf{Z}^+$), we want to establish our *inductive step* by showing how the truth of $S(k)$ "forces" us to accept the truth of $S(k+1)$. [The assumption of the truth of $S(k)$ is our *induction hypothesis*.] To establish the truth of $S(k+1)$, we need to show that

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}.$$

We proceed as follows.

$$\sum_{i=1}^{k+1} i = 1+2+\cdots+k+(k+1) = \left(\sum_{i=1}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1),$$

for we are assuming the truth of $S(k)$. But

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$

establishing the inductive step [condition (b)] of the theorem.

Consequently, by the Principle of Mathematical Induction, $S(n)$ is true for all $n \in \mathbf{Z}^+$.

Now that we have obtained the summation formula for $\sum_{i=1}^{n} i$ in two ways (see Example 1.40), we shall digress from our main topic and consider two examples that use this summation formula.

EXAMPLE 4.2

A wheel of fortune has the numbers from 1 to 36 painted on it in a random manner. Show that regardless of how the numbers are situated, there are three consecutive (on the wheel) numbers whose total is 55 or more.

Let x_1 be any number on the wheel. Counting clockwise from x_1, label the other numbers $x_2, x_3, \ldots, x_{36}$. For the result to be false, we must have $x_1 + x_2 + x_3 < 55$, $x_2 + x_3 + x_4 < 55, \ldots, x_{34} + x_{35} + x_{36} < 55$, $x_{35} + x_{36} + x_1 < 55$, and $x_{36} + x_1 + x_2 < 55$. In these 36 inequalities, each of the terms $x_1, x_2, \ldots, x_{36}$ appears (exactly) three times, so each of the integers $1, 2, \ldots, 36$ appears (exactly) three times. Adding all 36 inequalities, we find that $3\sum_{i=1}^{36} x_i = 3\sum_{i=1}^{36} i < 36(55) = 1980$. But $\sum_{i=1}^{36} i = (36)(37)/2 = 666$, and this gives us the contradiction that $1998 = 3(666) < 1980$.

EXAMPLE 4.3

Among the 900 three-digit integers (from 100 to 999) those such as 131, 222, 303, 717, 848, and 969, where the integer is the same whether it is read from left to right or from right to left, are called *palindromes*. Without actually determining all of these three-digit palindromes, we would like to determine their sum.

The typical palindrome under study here has the form $aba = 100a + 10b + a = 101a + 10b$, where $1 \le a \le 9$ and $0 \le b \le 9$. With nine choices for a and ten for b, it follows from the rule of product that there are 90 such three-digit palindromes. Their sum is

$$\begin{aligned}
\sum_{a=1}^{9}\left(\sum_{b=0}^{9} aba\right) &= \sum_{a=1}^{9}\sum_{b=0}^{9} aba = \sum_{a=1}^{9}\sum_{b=0}^{9}(101a + 10b) \\
&= \sum_{a=1}^{9}\left[10(101a) + 10\sum_{b=0}^{9} b\right] = \sum_{a=1}^{9}\left[10(101a) + 10\sum_{b=1}^{9} b\right] \\
&= \sum_{a=1}^{9}\left[1010a + \frac{10(9\cdot 10)}{2}\right] = \sum_{a=1}^{9}(1010a + 450) \\
&= 1010\sum_{a=1}^{9} a + 9(450) \\
&= \frac{1010(9\cdot 10)}{2} + 4050 = 49{,}500.
\end{aligned}$$

The next summation formula takes us from first powers to squares.

EXAMPLE 4.4

Prove that for each $n \in \mathbf{Z}^+$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof: Here we are dealing with the open statement

$$S(n):\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Basis Step: We start with the statement $S(1)$ and find that

$$\sum_{i=1}^{1} i^2 = 1^2 = \frac{1(1+1)(2(1)+1)}{6}.$$

so $S(1)$ is true.

Inductive Step: Now we assume the truth of $S(k)$, for some (particular) $k \in \mathbf{Z}^+$—that is, we assume that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

is a true statement (when n is replaced by k). From this assumption we want to deduce the truth of

$$\begin{aligned}
S(k+1):\quad \sum_{i=1}^{k+1} i^2 &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

Using the induction hypothesis $S(k)$, we find that

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 \\ &= \left[\frac{k(k+1)(2k+1)}{6}\right] + (k+1)^2 \\ &= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] = (k+1)\left[\frac{2k^2+7k+6}{6}\right] \\ &= \frac{(k+1)(k+2)(2k+3)}{6},\end{aligned}$$

and the general result follows by the Principle of Mathematical Induction.

The formulas from Examples 4.1 and 4.4 prove handy in deriving our next result.

EXAMPLE 4.5

Figure 4.2 provides the first four entries of the sequence of *triangular* numbers. We see that $t_1 = 1$, $t_2 = 3$, $t_3 = 6$, $t_4 = 10$, and, in general, $t_i = 1 + 2 + \cdots + i = i(i+1)/2$, for each $i \in \mathbf{Z}^+$. For a fixed $n \in \mathbf{Z}^+$ we want a formula for the sum of the first n triangular numbers—that is, $t_1 + t_2 + \cdots + t_n = \sum_{i=1}^{n} t_i$. When $n = 2$ we have $t_1 + t_2 = 4$. For $n = 3$ the sum is 10. Considering n fixed (but arbitrary) we find that

$$\begin{aligned}\sum_{i=1}^{n} t_i &= \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\sum_{i=1}^{n}(i^2+i) = \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{1}{2}\sum_{i=1}^{n} i \\ &= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{1}{2}\left[\frac{n(n+1)}{2}\right] = n(n+1)\left[\frac{2n+1}{12} + \frac{1}{4}\right] \\ &= \frac{n(n+1)(n+2)}{6}.\end{aligned}$$

Consequently, if we wish to know the sum of the first 100 triangular numbers, we have

$$t_1 + t_2 + \cdots + t_{100} = \frac{100(101)(102)}{6} = 171{,}700.$$

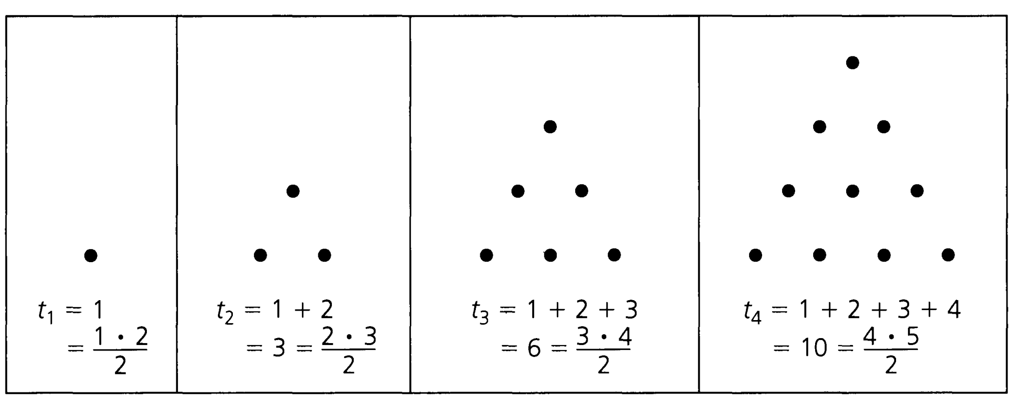

Figure 4.2

Before we present any more results, let us note how we started the proofs in Examples 4.1 and 4.4. In both cases we simply replaced the variable n by 1 and verified the truth of some rather easy equalities. Considering how the inductive step in each of these proofs was definitely more complicated to establish, we might question the need for bothering with these basis steps. So let us examine the following example.

EXAMPLE 4.6

If $n \in \mathbf{Z}^+$, establish the validity of the open statement

$$S(n):\quad \sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n^2+n+2}{2}.$$

This time we shall go directly to the inductive step. Assuming the truth of the statement

$$S(k):\quad \sum_{i=1}^{k} i = 1+2+3+\cdots+k = \frac{k^2+k+2}{2}$$

for some (particular) $k \in \mathbf{Z}^+$, we want to infer the truth of the statement

$$S(k+1):\quad \sum_{i=1}^{k+1} i = 1+2+3+\cdots+k+(k+1) = \frac{(k+1)^2+(k+1)+2}{2} = \frac{k^2+3k+4}{2}.$$

As we did previously, we use the induction hypothesis and calculate as follows:

$$\begin{aligned}\sum_{i=1}^{k+1} i &= 1+2+3+\cdots+k+(k+1) = \left(\sum_{i=1}^{k} i\right) + (k+1)\\ &= \frac{k^2+k+2}{2} + (k+1)\\ &= \frac{k^2+k+2}{2} + \frac{2k+2}{2} = \frac{k^2+3k+4}{2}.\end{aligned}$$

Hence, for each $k \in \mathbf{Z}^+$, it follows that $S(k) \Rightarrow S(k+1)$. But before we decide to accept the statement $\forall n\; S(n)$ as a true statement, let us reconsider Example 4.1. From that example we learned that $\sum_{i=1}^{n} i = n(n+1)/2$, for all $n \in \mathbf{Z}^+$. Therefore, we can use these two results (from Example 4.1 and the one already "established" here) to conclude that for all $n \in \mathbf{Z}^+$,

$$\frac{n(n+1)}{2} = \sum_{i=1}^{n} i = \frac{n^2+n+2}{2},$$

which implies that $n(n+1) = n^2+n+2$ and $0 = 2$. (Something is wrong somewhere!)

If $n = 1$, then $\sum_{i=1}^{1} 1 = 1$, but $(n^2+n+2)/2 = (1+1+2)/2 = 2$. So $S(1)$ is not true. But we may feel that this result just indicates that we have the wrong starting point. Perhaps $S(n)$ is true for all $n \geq 7$, or all $n \geq 137$. Using the preceding argument, however, we know that for any starting point $n_0 \in \mathbf{Z}^+$, if $S(n_0)$ were true, then

$$\frac{n_0^2+n_0+2}{2} = \sum_{i=1}^{n_0} i = 1+2+3+\cdots+n_0.$$

From the result in Example 4.1 we have $\sum_{i=1}^{n_0} i = n_0(n_0+1)/2$, so it follows once again that $0 = 2$, and we have no possible starting point.

This example should indicate to the reader the need to establish the basis step — no matter how easy it may be to verify it.

Now consider the following pseudocode procedures. The procedure in Fig. 4.3 uses a **for** loop to accumulate the sum of the squares. The second procedure (Fig. 4.4) demonstrates how the result of Example 4.4 can be used in place of such a loop. In both procedures the input is a positive integer n and the output is $\sum_{i=1}^{n} i^2$. However, whereas the pseudocode within the **for** loop of the procedure in Fig. 4.3 entails a total of n additions and n multiplications (not to mention the $n - 1$ additions for incrementing the counter variable i), the procedure in Fig. 4.4 requires only two additions, three multiplications, and one (integer) division. And this total number of additions, multiplications, and (integer) divisions is still 6 as the value of n increases. Consequently, the procedure in Fig. 4.4 is considered more efficient. (This idea of a *more efficient* procedure will be examined further in Sections 5.7 and 5.8.)

```
procedure SumOfSquares1 (n: positive integer)
begin
  sum := 0
  for i := 1 to n do
    sum := sum + i²
end
```

Figure 4.3

```
procedure SumOfSquares2 (n: positive integer)
begin
  sum := n * (n + 1) * (2 * n + 1)/6
end
```

Figure 4.4

Looking back at our first two applications of mathematical induction (in Examples 4.1 and 4.4), we might wonder whether this principle applies only to the verification of *known* summation formulas. The next seven examples show that mathematical induction is a vital tool in many other circumstances as well.

EXAMPLE 4.7

Let us consider the sums of consecutive odd positive integers.

1) $1 \qquad = 1 \qquad (= 1^2)$

2) $1 + 3 \qquad = 4 \qquad (= 2^2)$

3) $1 + 3 + 5 \qquad = 9 \qquad (= 3^2)$

4) $1 + 3 + 5 + 7 \qquad = 16 \qquad (= 4^2)$

From these first four cases we *conjecture* the following result: The sum of the first n consecutive odd positive integers is n^2; that is, for all $n \in \mathbf{Z}^+$,

$$S(n): \quad \sum_{i=1}^{n} (2i - 1) = n^2.$$

Now that we have developed what we feel is a true summation formula, we use the Principle of Mathematical Induction to verify its truth for *all* $n \geq 1$.

From the preceding calculations, we see that $S(1)$ is true [as are $S(2)$, $S(3)$, and $S(4)$], and so we have our basis step. For the inductive step we assume the truth of $S(k)$ for some k (≥ 1) and have

$$\sum_{i=1}^{k}(2i-1) = k^2.$$

We now deduce the truth of $S(k+1)$: $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$. Since we have assumed the truth of $S(k)$, our induction hypothesis, we may now write

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + [2(k+1)-1] = k^2 + [2(k+1)-1]$$
$$= k^2 + 2k + 1 = (k+1)^2.$$

Consequently, the result $S(n)$ is true for all $n \geq 1$, by the Principle of Mathematical Induction.

Now it is time to investigate some results that are not summation formulas.

EXAMPLE 4.8

In Table 4.1, we have listed in adjacent columns the values of $4n$ and $n^2 - 7$ for the positive integers n, where $1 \leq n \leq 8$. From the table, we see that $(n^2 - 7) < 4n$ for $n = 1, 2, 3, 4, 5$; but when $n = 6, 7, 8$, we have $4n < (n^2 - 7)$. These last three observations lead us to conjecture: For all $n \geq 6$, $4n < (n^2 - 7)$.

Table 4.1

n	$4n$	$n^2 - 7$	n	$4n$	$n^2 - 7$
1	4	−6	5	20	18
2	8	−3	6	24	29
3	12	2	7	28	42
4	16	9	8	32	57

Once again, the Principle of Mathematical Induction is the proof technique we need to verify our conjecture. Let $S(n)$ denote the open statement: $4n < (n^2 - 7)$. Then Table 4.1 confirms that $S(6)$ is true [as are $S(7)$ and $S(8)$], and we have our basis step. (At last we have an example wherein the starting point is an integer $n_0 \neq 1$.)

In this example, the induction hypothesis is $S(k)$: $4k < (k^2 - 7)$, where $k \in \mathbf{Z}^+$ and $k \geq 6$. In order to establish the inductive step, we need to obtain the truth of $S(k+1)$ from that of $S(k)$. That is, from $4k < (k^2 - 7)$ we must conclude that $4(k+1) < [(k+1)^2 - 7]$. Here are the necessary steps:

$$4k < (k^2 - 7) \Rightarrow 4k + 4 < (k^2 - 7) + 4 < (k^2 - 7) + (2k + 1)$$

(because for $k \geq 6$, we find $2k + 1 \geq 13 > 4$), and

$$4k + 4 < (k^2 - 7) + (2k + 1) \Rightarrow 4(k+1) < (k^2 + 2k + 1) - 7 = (k+1)^2 - 7.$$

Therefore, by the Principle of Mathematical Induction, $S(n)$ is true for all $n \geq 6$.

EXAMPLE 4.9

Among the many interesting sequences of numbers encountered in discrete mathematics and combinatorics, one finds the *harmonic numbers* $H_1, H_2, H_3, \ldots,$ where

$$\begin{aligned} H_1 &= 1 \\ H_2 &= 1 + \frac{1}{2} \\ H_3 &= 1 + \frac{1}{2} + \frac{1}{3}, \\ &\ldots, \end{aligned}$$

and, in general, $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, for each $n \in \mathbf{Z}^+$.

The following property of the harmonic numbers provides one more opportunity for us to apply the Principle of Mathematical Induction.

$$\text{For all } n \in \mathbf{Z}^+, \sum_{j=1}^{n} H_j = (n+1)H_n - n.$$

Proof: As we have done in the earlier examples (that is, Examples 4.1, 4.4, and 4.7), we verify the basis step at $n = 1$ for the open statement $S(n)$: $\sum_{j=1}^{n} H_j = (n+1)H_n - n$. This result follows readily from

$$\sum_{j=1}^{1} H_j = H_1 = 1 = 2 \cdot 1 - 1 = (1+1)H_1 - 1.$$

To verify the inductive step, we assume the truth of $S(k)$, that is,

$$\sum_{j=1}^{k} H_j = (k+1)H_k - k.$$

This assumption then leads us to the following:

$$\begin{aligned} \sum_{j=1}^{k+1} H_j = \sum_{j=1}^{k} H_j + H_{k+1} &= [(k+1)H_k - k] + H_{k+1} \\ &= (k+1)H_k - k + H_{k+1} \\ &= (k+1)[H_{k+1} - (1/(k+1))] - k + H_{k+1} \\ &= (k+2)H_{k+1} - 1 - k \\ &= (k+2)H_{k+1} - (k+1). \end{aligned}$$

Consequently, we now know from the Principle of Mathematical Induction that $S(n)$ is true for all positive integers n.

EXAMPLE 4.10

For all $n \geq 0$ let $A_n \subset \mathbf{R}$, where $|A_n| = 2^n$ and the elements of A_n are listed in ascending order. If $r \in \mathbf{R}$, prove that in order to determine whether $r \in A_n$ (by the procedure developed below), we must compare r with no more than $n + 1$ elements in A_n.

When $n = 0$, $A_0 = \{a\}$ and only one comparison is needed. So the result is true for $n = 0$ (and we have our basis step). For $n = 1$, $A_1 = \{a_1, a_2\}$ with $a_1 < a_2$. In order to determine whether $r \in A_1$, at most two comparisons must be made. Hence the result follows when $n = 1$. Now if $n = 2$, we write $A_2 = \{b_1, b_2, c_1, c_2\} = B_1 \cup C_1$, where $b_1 < b_2 < c_1 < c_2$, $B_1 = \{b_1, b_2\}$, and $C_1 = \{c_1, c_2\}$. Comparing r with b_2, we determine which of the two possibilities—(i) $r \in B_1$; or (ii) $r \in C_1$—can occur. Since $|B_1| = |C_1| = 2$, either one of the two possibilities requires at most two more comparisons (from the prior case where $n = 1$). Consequently, we can determine whether $r \in A_2$ by making no more than $2 + 1 = n + 1$ comparisons.

We now argue in general. Assume the result true for some $k \geq 0$ and consider the case for A_{k+1}, where $|A_{k+1}| = 2^{k+1}$. In order to establish our inductive step, let $A_{k+1} = B_k \cup C_k$, where $|B_k| = |C_k| = 2^k$, and the elements of B_k, C_k are in ascending order with the largest element x in B_k smaller than the least element in C_k. Let $r \in \mathbf{R}$. To determine whether $r \in A_{k+1}$, we consider whether $r \in B_k$ or $r \in C_k$.

a) First we compare r and x. (One comparison)

b) If $r \leq x$, then because $|B_k| = 2^k$, it follows by the induction hypothesis that we can determine whether $r \in B_k$ by making no more than $k + 1$ additional comparisons.

c) If $r > x$, we do likewise with the elements in C_k. We make at most $k + 1$ additional comparisons to see whether $r \in C_k$.

In any event, at most $(k + 1) + 1$ comparisons are made.

The general result now follows by the Principle of Mathematical Induction.

EXAMPLE 4.11

One of our first concerns when we evaluate the quality of a computer program is whether the program does what it is supposed to do. Just as we cannot prove a theorem by checking specific cases, so we cannot establish the correctness of a program simply by testing various sets of data. (Furthermore, doing this would be quite difficult if our program were to become a part of a larger software package wherein, perhaps, a data set is internally generated.) Since software development places a great deal of emphasis on structured programming, this has brought about the need for *program verification*. Here the programmer or the programming team must prove that the program being developed is correct *regardless* of the data set supplied. The effort invested at this stage considerably reduces the time that must be spent in debugging the program (or software package). One of the methods that can play a major role in such program verification is mathematical induction. Let us see how.

The pseudocode program segment shown in Fig. 4.5 is supposed to produce the answer $x(y^n)$ for real variables x, y with n a nonnegative integer. (The values for these three variables are assigned earlier in the program.) We shall verify the correctness of this program segment by mathematical induction for the open statement.

$S(n)$: For *all* $x, y \in \mathbf{R}$, if the program reaches the top of the **while** loop with $n \in \mathbf{N}$, after the loop is bypassed (for $n = 0$) or the two loop instructions are executed n (> 0) times, then the value of the real variable *answer* is $x(y^n)$.

```
while n ≠ 0 do
  begin
    x := x * y
    n := n - 1
  end
answer := x
```

Figure 4.5

The flowchart for this program segment is shown in Fig. 4.6. Referring to it will help us as we develop our proof.

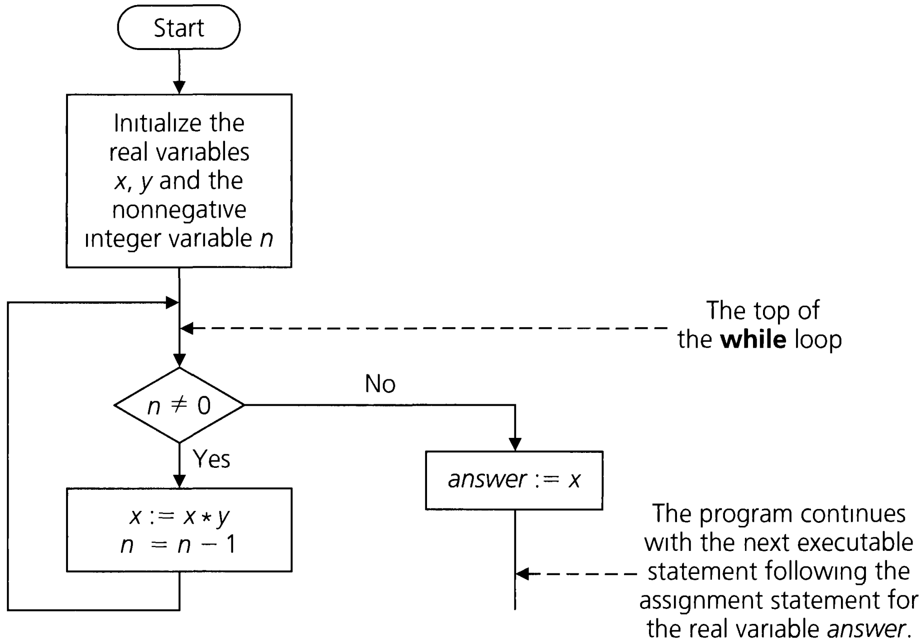

Figure 4.6

First consider $S(0)$, the statement for the case where $n = 0$. Here the program reaches the top of the **while** loop, but since $n = 0$, it follows the No branch in the flowchart and assigns the value $x = x(1) = x(y^0)$ to the real variable *answer*. Consequently, the statement $S(0)$ is true and the basis step of our induction argument is established.

Now we assume the truth of $S(k)$, for some nonnegative integer k. This provides us with the induction hypothesis.

$S(k)$: For *all* $x, y \in \mathbf{R}$, if the program reaches the top of the **while** loop with $k \in \mathbf{N}$, after the loop is bypassed (for $k = 0$) or the two loop instructions are executed k (> 0) times, then the value of the real variable *answer* is $x(y^k)$.

Continuing with the inductive step of the proof, when dealing with the statement $S(k+1)$, we note that because $k + 1 \geq 1$, the program will not simply follow the No branch and bypass the instructions in the **while** loop. Those two instructions (in the **while** loop) will be executed at least once. When the program reaches the top of the **while** loop for the first time, $n = k + 1 > 0$, so the loop instructions are executed and the program returns to the top of the **while** loop where now we find that

- The value of y is unchanged.
- The value of x is $x_1 = x(y^1) = xy$.
- The value of n is $(k + 1) - 1 = k$.

But now, by our induction hypothesis (applied to the real numbers x_1, y), we know that after the **while** loop for x_1, y and $n = k$ is bypassed (for $k = 0$) or the two loop instructions are executed k (> 0) times, then the value assigned to the real variable *answer* is

$$x_1(y^k) = (xy)(y^k) = x(y^{k+1}).$$

So by the Principle of Mathematical Induction, $S(n)$ is true for all $n \geq 0$ and the correctness of the program segment is established.

EXAMPLE 4.12

Recall (from Examples 1.37 and 3.11) that for a given $n \in \mathbf{Z}^+$, a *composition* of n is an *ordered* sum of positive-integer summands summing to n. In Fig. 4.7 we find the compositions of 1, 2, 3, and 4. We see that

a) 1 has $1 = 2^0 = 2^{1-1}$ composition, 2 has $2 = 2^1 = 2^{2-1}$ compositions, 3 has $4 = 2^2 = 2^{3-1}$ compositions, and 4 has $8 = 2^3 = 2^{4-1}$ compositions; and

b) the eight compositions of 4 arise from the four compositions of 3 in two ways: (i) Compositions (1′)–(4′) result by increasing the last summand (in each corresponding composition of 3) by 1; (ii) Each of compositions (1″)–(4″) is obtained by appending "+1" to the corresponding composition of 3.

$(n = 1)$	1		$(n = 4)$	(1′)	4
				(2′)	$1 + 3$
$(n = 2)$	2			(3′)	$2 + 2$
	$1 + 1$			(4′)	$1 + 1 + 2$
$(n = 3)$	(1)	3		(1″)	$3 + 1$
	(2)	$1 + 2$		(2″)	$1 + 2 + 1$
	(3)	$2 + 1$		(3″)	$2 + 1 + 1$
	(4)	$1 + 1 + 1$		(4″)	$1 + 1 + 1 + 1$

Figure 4.7

The observations in part (a) suggest that for all $n \in \mathbf{Z}^+$, $S(n)$: n has 2^{n-1} compositions. The result [in part (a)] for $n = 1$ provides our basis step, $S(1)$. So now let us assume the result true for some (fixed) $k \in \mathbf{Z}^+$ — namely, $S(k)$: k has 2^{k-1} compositions. At this point consider $S(k + 1)$. One can develop the compositions of $k + 1$ from those of k as in part (b) above (where $k = 3$). For $k \geq 1$, we find that the compositions of $k + 1$ fall into two distinct cases:

1) The compositions of $k + 1$, where the last summand is an integer $t > 1$: Here this last summand t is replaced by $t - 1$, and this type of replacement provides a correspondence between all of the compositions of k and all those compositions of $k + 1$, where the last summand exceeds 1.

2) The compositions of $k + 1$, where the last summand is 1: In this case we delete "+1" from the right side of this type of composition of $k + 1$. Once again we get a correspondence between all the compositions of k and all those compositions of $k + 1$, where the last summand is 1.

Therefore, the number of compositions of $k + 1$ is twice the number for k. Consequently, it follows from the induction hypothesis that the number of compositions of $k + 1$ is $2(2^{k-1}) = 2^k$. The Principle of Mathematical Induction now tells us that for all $n \in \mathbf{Z}^+$, $S(n)$: n has 2^{n-1} compositions (as we learned earlier in Examples 1.37 and 3.11).

EXAMPLE 4.13

We learn from the equation $14 = 3 + 3 + 8$ that we can express 14 using only 3's and 8's as summands. But what may prove to be surprising is that for all $n \geq 14$,

$S(n)$: $\quad n$ can be written as a sum of 3's and/or 8's (with no regard to order).

As we start to verify $S(n)$ for all $n \geq 14$, we realize that the given introductory sentence shows us that the basis step $S(14)$ is true. For the inductive step we assume the truth of $S(k)$ for some $k \in \mathbf{Z}^+$, where $k \geq 14$, and then consider what can happen for $S(k+1)$. If there is at least one 8 in the sum (of 3's and/or 8's) that equals k, then we can replace this 8 by three 3's and obtain $k+1$ as a sum of 3's and/or 8's. But suppose that no 8 appears as a summand of k. Then the only summand used is a 3, and, since $k \geq 14$, we must have at least five 3's as summands. And now if we replace five of these 3's by two 8's, we obtain the sum $k+1$, where the only summands are 3's and/or 8's. Consequently, we have shown how $S(k) \Rightarrow S(k+1)$ and so the result follows for all $n \geq 14$ by the Principle of Mathematical Induction.

Now that we have seen several applications of the Principle of Mathematical Induction, we shall close this section by introducing another form of mathematical induction. This second form is sometimes referred to as the *Alternative Form of the Principle of Mathematical Induction* or the *Principle of Strong Mathematical Induction.*

Once again we shall consider a statement of the form $\forall\, n \geq n_0\; S(n)$, where $n_0 \in \mathbf{Z}^+$, and we shall establish both a basis step and an inductive step. However, this time the basis step may require proving more than just the first case — where $n = n_0$. And in the inductive step we shall assume the truth of all the statements $S(n_0)$, $S(n_0+1)$, $\ldots$, $S(k-1)$, and $S(k)$, in order to establish the truth of the statement $S(k+1)$. We formally present this second Principle of Mathematical Induction in the following theorem.

THEOREM 4.2 *The Principle of Mathematical Induction — Alternative Form.* Let $S(n)$ denote an open mathematical statement (or set of such open statements) that involves one or more occurrences of the variable n, which represents a positive integer. Also let $n_0, n_1 \in \mathbf{Z}^+$ with $n_0 \leq n_1$.

a) If $S(n_0)$, $S(n_0+1)$, $S(n_0+2)$, $\ldots$, $S(n_1-1)$, and $S(n_1)$ are true; and

b) If whenever $S(n_0)$, $S(n_0+1)$, $\ldots$, $S(k-1)$, and $S(k)$ are true for some (particular but arbitrarily chosen) $k \in \mathbf{Z}^+$, where $k \geq n_1$, then the statement $S(k+1)$ is also true;

then $S(n)$ is true for all $n \geq n_0$.

As in Theorem 4.1, condition (a) is called the *basis step* and condition (b) is called the *inductive step.*

The proof of Theorem 4.2 is similar to that of Theorem 4.1 and will be requested in the Section Exercises. We shall also learn in the exercises for Section 4.2 that the two forms of mathematical induction (given in Theorems 4.1 and 4.2) are equivalent, for each can be shown to be a valid proof technique when we assume the truth of the other.

Before we give any examples where Theorem 4.2 is applied, let us mention, as we did for Theorem 4.1, that n_0 need not actually be a positive integer — it may, in reality, be 0 or even possibly a negative integer. And now that we have taken care of that point once again, let us see how we might apply this new proof technique.

Our first example should be familiar. We shall simply apply Theorem 4.2 in order to obtain the result in Example 4.13 in a second way.

EXAMPLE 4.14

The following calculations indicate that it is possible to write (without regard to order) the integers 14, 15, 16 using only 3's and/or 8's as summands:

$$14 = 3 + 3 + 8 \qquad 15 = 3 + 3 + 3 + 3 + 3 \qquad 16 = 8 + 8$$

On the basis of these three results, we make the conjecture

For every $n \in \mathbf{Z}^+$ where $n \geq 14$,

$$S(n): \quad n \text{ can be written as a sum of 3's and/or 8's.}$$

Proof: It is apparent that the statements $S(14)$, $S(15)$, and $S(16)$ are true—and this establishes our basis step. (Here $n_0 = 14$ and $n_1 = 16$.)

For the inductive step we assume the truth of the statements

$$S(14),\ S(15),\ \ldots,\ S(k-2),\ S(k-1),\ \text{and } S(k)$$

for some $k \in \mathbf{Z}^+$, where $k \geq 16$. [The assumption of the truth of these $(k - 14) + 1$ statements constitutes our induction hypothesis.] And now if $n = k + 1$, then $n \geq 17$ and $k + 1 = (k - 2) + 3$. But since $14 \leq k - 2 \leq k$, from the truth of $S(k - 2)$ we know that $(k - 2)$ can be written as a sum of 3's and/or 8's; so $(k + 1) = (k - 2) + 3$ can also be written in this form. Consequently, $S(n)$ is true for all $n \geq 14$ by the alternative form of the Principle of Mathematical Induction.

In Example 4.14 we saw how the truth of $S(k + 1)$ was deduced by using the truth of the one prior result $S(k - 2)$. Our last example presents a situation wherein the truth of more than one prior result is needed.

EXAMPLE 4.15

Let us consider the integer sequence $a_0, a_1, a_2, a_3, \ldots,$ where

$$a_0 = 1,\ a_1 = 2,\ a_2 = 3, \qquad \text{and}$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \qquad \text{for all } n \in \mathbf{Z}^+ \text{ where } n \geq 3.$$

(Then, for instance, we find that $a_3 = a_2 + a_1 + a_0 = 3 + 2 + 1 = 6$; $a_4 = a_3 + a_2 + a_1 = 6 + 3 + 2 = 11$; and $a_5 = a_4 + a_3 + a_2 = 11 + 6 + 3 = 20$.)

We claim that the entries in this sequence are such that $a_n \leq 3^n$ for all $n \in \mathbf{N}$—that is, $\forall n \in \mathbf{N}\ S'(n)$, where $S'(n)$ is the open statement: $a_n \leq 3^n$.

For the basis step, we observe that

i) $a_0 = 1 = 3^0 \leq 3^0$;
ii) $a_1 = 2 \leq 3 = 3^1$; and
iii) $a_2 = 3 \leq 9 = 3^2$.

Consequently, we know that $S'(0)$, $S'(1)$, and $S'(2)$ are true statements.

So now we turn our attention to the inductive step where we assume the truth of the statements $S'(0), S'(1), S'(2), \ldots, S'(k-1), S'(k)$, for some $k \in \mathbf{Z}^+$ where $k \geq 2$. For the case where $n = k + 1 \geq 3$ we see that

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} + a_{k-2} \\ &\leq 3^k + 3^{k-1} + 3^{k-2} \\ &\leq 3^k + 3^k + 3^k = 3(3^k) = 3^{k+1}, \end{aligned}$$

so $[S'(k-2) \wedge S'(k-1) \wedge S'(k)] \Rightarrow S'(k+1)$.

Therefore it follows from the alternative form of the Principle of Mathematical Induction that $a_n \leq 3^n$ for all $n \in \mathbf{N}$.

Before we close this section, let us take a second look at the preceding two results. In both Example 4.14 and Example 4.15 we established the basis step by verifying the truth of three statements: $S(14)$, $S(15)$, and $S(16)$ in Example 4.14; and, $S'(0)$, $S'(1)$, and $S'(2)$ in Example 4.15. However, to obtain the truth of $S(k+1)$ in Example 4.14, we actually used only one of the $(k-14)+1$ statements in the induction hypothesis — namely, the statement $S(k-2)$. For Example 4.15 we used three of the $k+1$ statements in the induction hypothesis — in this case, the statements $S'(k-2)$, $S'(k-1)$, and $S'(k)$.

EXERCISES 4.1

1. Prove each of the following for all $n \geq 1$ by the Principle of Mathematical Induction.

a) $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

b) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$

c) $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

d) $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \left(\sum_{i=1}^{n} i\right)^2$

2. Establish each of the following for all $n \geq 1$ by the Principle of Mathematical Induction.

a) $\displaystyle\sum_{i=1}^{n} 2^{i-1} = \sum_{i=0}^{n-1} 2^i = 2^n - 1$

b) $\displaystyle\sum_{i=1}^{n} i(2^i) = 2 + (n-1)2^{n+1}$

c) $\displaystyle\sum_{i=1}^{n} (i)(i!) = (n+1)! - 1$

3. a) Note how $\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=0}^{n} (i+1)^3 = \sum_{i=0}^{n} (i^3 + 3i^2 + 3i + 1)$. Use this result to obtain a formula for $\sum_{i=1}^{n} i^2$. (Compare with the formula given in Example 4.4.)

b) Use the idea presented in part (a) to find a formula for $\sum_{i=1}^{n} i^3$ and one for $\sum_{i=1}^{n} i^4$. [Compare the result for $\sum_{i=1}^{n} i^3$ with the formula in part (d) of Exercise 1 for this section.]

4. A wheel of fortune has the integers from 1 to 25 placed on it in a random manner. Show that regardless of how the numbers are positioned on the wheel, there are three adjacent numbers whose sum is at least 39.

5. Consider the following program segment (written in pseudocode):

```
for i := 1 to 123 do
  for j := 1 to i do
    print i * j
```

a) How many times is the print statement of the third line executed?

b) Replace i in the second line by i^2, and answer the question in part (a).

6. a) For the four-digit integers (from 1000 to 9999) how many are palindromes and what is their sum?

b) Write a computer program to check the answer for the sum in part (a).

7. A lumberjack has $4n + 110$ logs in a pile consisting of n layers. Each layer has two more logs than the layer directly above it. If the top layer has six logs, how many layers are there?

8. Determine the positive integer n for which

$$\sum_{i=1}^{2n} i = \sum_{i=1}^{n} i^2.$$

9. Evaluate each of the following:

a) $\sum_{i=11}^{33} i$ **b)** $\sum_{i=11}^{33} i^2$.

10. Determine $\sum_{i=51}^{100} t_i$, where t_i denotes the ith triangular number, for $51 \leq i \leq 100$.

11. a) Derive a formula for $\sum_{i=1}^{n} t_{2i}$, where t_{2i} denotes the $2i$th triangular number for $1 \leq i \leq n$.

b) Determine $\sum_{i=1}^{100} t_{2i}$.

c) Write a computer program to check the result in part (b).

12. a) Prove that $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$, where $i \in \mathbf{C}$ and $i^2 = -1$.

b) Using induction, prove that for all $n \in \mathbf{Z}^+$,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

(This result is known as *DeMoivre's Theorem.*)

c) Verify that $1 + i = \sqrt{2}(\cos 45° + i\sin 45°)$, and compute $(1+i)^{100}$.

13. a) Consider an 8×8 chessboard. It contains sixty-four 1×1 squares and one 8×8 square. How many 2×2

squares does it contain? How many 3×3 squares? How many squares in total?

b) Now consider an $n \times n$ chessboard for some fixed $n \in \mathbf{Z}^+$. For $1 \leq k \leq n$, how many $k \times k$ squares are contained in this chessboard? How many squares in total?

14. Prove that for all $n \in \mathbf{Z}^+$, $n > 3 \Rightarrow 2^n < n!$

15. Prove that for all $n \in \mathbf{Z}^+$, $n > 4 \Rightarrow n^2 < 2^n$.

16. a) For $n = 3$ let $X_3 = \{1, 2, 3\}$. Now consider the sum

$$s_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3} = \sum_{\emptyset \neq A \subseteq X_3} \frac{1}{p_A},$$

where p_A denotes the product of all elements in a nonempty subset A of X_3. Note that the sum is taken over all the nonempty subsets of X_3. Evaluate this sum.

b) Repeat the calculation in part (a) for s_2 (where $n = 2$ and $X_2 = \{1, 2\}$) and s_4 (where $n = 4$ and $X_4 = \{1, 2, 3, 4\}$).

c) Conjecture the general result suggested by the calculations from parts (a) and (b). Prove your conjecture using the Principle of Mathematical Induction.

17. For $n \in \mathbf{Z}^+$, let H_n denote the nth harmonic number (as defined in Example 4.9).

a) For all $n \in \mathbf{N}$ prove that $1 + \left(\frac{n}{2}\right) \leq H_{2^n}$.

b) Prove that for all $n \in \mathbf{Z}^+$,

$$\sum_{j=1}^{n} jH_j = \left[\frac{n(n+1)}{2}\right] H_{n+1} - \left[\frac{n(n+1)}{4}\right].$$

18. Consider the following four equations:

1) $1 = 1$
2) $2 + 3 + 4 = 1 + 8$
3) $5 + 6 + 7 + 8 + 9 = 8 + 27$
4) $10 + 11 + 12 + 13 + 14 + 15 + 16 = 27 + 64$

Conjecture the general formula suggested by these four equations, and prove your conjecture.

19. For $n \in \mathbf{Z}^+$, let $S(n)$ be the open statement

$$\sum_{i=1}^{n} i = \frac{(n + (1/2))^2}{2}.$$

Show that the truth of $S(k)$ implies the truth of $S(k+1)$ for all $k \in \mathbf{Z}^+$. Is $S(n)$ true for all $n \in \mathbf{Z}^+$?

20. Let S_1 and S_2 be two sets where $|S_1| = m$, $|S_2| = r$, for $m, r \in \mathbf{Z}^+$, and the elements in each of S_1, S_2 are in ascending order. It can be shown that the elements in S_1 and S_2 can be merged into ascending order by making no more than $m + r - 1$ comparisons. (See Lemma 12.1.) Use this result to establish the following.

For $n \geq 0$, let S be a set with $|S| = 2^n$. Prove that the number of comparisons needed to place the elements of S in ascending order is bounded above by $n \cdot 2^n$.

21. During the execution of a certain program segment (written in pseudocode), the user assigns to the integer variables x and n any (possibly different) positive integers. The segment shown in Fig. 4.8 immediately follows these assignments. If the program reaches the top of the **while** loop, state and prove (by mathematical induction) what the value assigned to *answer* will be after the two loop instructions are executed n (> 0) times.

```
while n ≠ 0 do
  begin
    x := x * n
    n := n - 1
  end
answer := x
```

Figure 4.8

22. In the program segment shown in Fig. 4.9, x, y, and *answer* are real variables, and n is an integer variable. Prior to execution of this **while** loop, the user supplies real values for x and y and a nonnegative integer value for n. Prove (by mathematical induction) that for all $x, y \in \mathbf{R}$, if the program reaches the top of the **while** loop with $n \in \mathbf{N}$, after the loop is bypassed (for $n = 0$) or the two loop instructions are executed n (> 0) times, then the value assigned to *answer* is $x + ny$.

```
while n ≠ 0 do
  begin
    x := x + y
    n := n - 1
  end
answer := x
```

Figure 4.9

23. a) Let $n \in \mathbf{Z}^+$, where $n \neq 1, 3$. Prove that n can be expressed as a sum of 2's and/or 5's.

b) For all $n \in \mathbf{Z}^+$ show that if $n \geq 24$, then n can be written as a sum of 5's and/or 7's.

24. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_1 = 1 \qquad a_2 = 2 \qquad a_n = a_{n-1} + a_{n-2},\ n \geq 3.$$

a) Determine the values of a_3, a_4, a_5, a_6, and a_7.

b) Prove that for all $n \geq 1$, $a_n < (7/4)^n$.

25. For a fixed $n \in \mathbf{Z}^+$, let X be the random variable where $Pr(X = x) = \frac{1}{n}$, $x = 1, 2, 3, \ldots, n$. (Here X is called a *uniform discrete* random variable.) Determine $E(X)$ and $\mathrm{Var}(X)$.

26. Let a_0 be a fixed constant and, for $n \geq 1$, let $a_n = \sum_{i=0}^{n-1} \binom{n-1}{i} a_i a_{(n-1)-i}$.

a) Show that $a_1 = a_0^2$ and that $a_2 = 2a_0^3$.

b) Determine a_3 and a_4 in terms of a_0.

c) Conjecture a formula for a_n in terms of a_0 when $n \geq 0$. Prove your conjecture using the Principle of Mathematical Induction.

27. Verify Theorem 4.2.

28. **a)** Of the $2^{5-1} = 2^4 = 16$ compositions of 5, determine how many start with (i) 1; (ii) 2; (iii) 3; (iv) 4; and (v) 5.

b) Provide a combinatorial proof for the result in part (a) of Exercise 2.

4.2 Recursive Definitions

Let us start this section by considering the integer sequence $b_0, b_1, b_2, b_3, \ldots$, where $b_n = 2n$ for all $n \in \mathbf{N}$. Here we find that $b_0 = 2 \cdot 0 = 0$, $b_1 = 2 \cdot 1 = 2$, $b_2 = 2 \cdot 2 = 4$, and $b_3 = 2 \cdot 3 = 6$. If, for instance, we need to determine b_6, we simply calculate $b_6 = 2 \cdot 6 = 12$—without the need to calculate the value of b_n for any other $n \in \mathbf{N}$. We can perform such calculations because we have an *explicit* formula—namely, $b_n = 2n$—that tells us how b_n is determined from n (alone).

In Example 4.15 of the preceding section, however, we considered the integer sequence $a_0, a_1, a_2, a_3, \ldots$, where

$$a_0 = 1, a_1 = 2, a_2 = 3, \qquad \text{and}$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \qquad \text{for all } n \in \mathbf{Z}^+ \text{ where } n \geq 3.$$

Here we do *not* have an *explicit* formula that defines each a_n in terms of n for all $n \in \mathbf{N}$. If we want the value of a_6, for example, we need to know the values of a_5, a_4, and a_3. And these values (of a_5, a_4, and a_3) require that we also know the values of a_2, a_1, and a_0. Unlike the rather easy situation where we determined $b_6 = 2 \cdot 6 = 12$, in order to calculate a_6, here we might find ourselves writing

$$\begin{aligned}
a_6 &= a_5 + a_4 + a_3 \\
&= (a_4 + a_3 + a_2) + (a_3 + a_2 + a_1) + (a_2 + a_1 + a_0) \\
&= [(a_3 + a_2 + a_1) + (a_2 + a_1 + a_0) + a_2] \\
&\quad + [(a_2 + a_1 + a_0) + a_2 + a_1] + (a_2 + a_1 + a_0) \\
&= [[(a_2 + a_1 + a_0) + a_2 + a_1] + (a_2 + a_1 + a_0) + a_2] \\
&\quad + [(a_2 + a_1 + a_0) + a_2 + a_1] + (a_2 + a_1 + a_0) \\
&= [[(3 + 2 + 1) + 3 + 2] + (3 + 2 + 1) + 3] \\
&\quad + [(3 + 2 + 1) + 3 + 2] + (3 + 2 + 1) \\
&= 37.
\end{aligned}$$

Or, in a somewhat easier manner, we could have gone in the opposite direction with these considerations:

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 = 3 + 2 + 1 = 6 \\
a_4 &= a_3 + a_2 + a_1 = 6 + 3 + 2 = 11 \\
a_5 &= a_4 + a_3 + a_2 = 11 + 6 + 3 = 20 \\
a_6 &= a_5 + a_4 + a_3 = 20 + 11 + 6 = 37.
\end{aligned}$$

No matter how we arrive at a_6, we realize that the two integer sequences—$b_0, b_1, b_2, b_3, \ldots$, and $a_0, a_1, a_2, a_3, \ldots$—are more than just numerically different. The integers $b_0, b_1, b_2, b_3, \ldots$, can be very readily listed as $0, 2, 4, 6, \ldots$, and for any $n \in \mathbf{N}$ we have

d)

A	B	C	$A \bigtriangleup B$	$A \bigtriangleup C$	$B \bigtriangleup C$
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	0	1
0	1	1	1	1	0
1	0	0	1	1	0
1	0	1	1	0	1
1	1	0	0	1	1
1	1	1	0	0	0

When $A \bigtriangleup B = C$, we consider rows 1, 4, 6, and 7. In these cases, $A \bigtriangleup C = B$ and $B \bigtriangleup C = A$.

15. a) $\binom{r+1}{m}$ $(m \leq r+1)$ **b)** $\binom{n-k+1}{k}$ $(2k \leq n+1)$

17. a) 23 **b)** 8 **19.** $7^{15} - 3(3^{15}) + 3$ **21.** $\binom{12}{4}\binom{10}{3}/\binom{22}{7} = 0.3483$

23. a) $\sum_{i=0}^{8} \binom{2i}{i}\binom{i+8}{8-i} = \sum_{i=0}^{8} \frac{(i+8)!}{i!\,i!(8-i)!}$

b) (i) $\binom{12}{6}\binom{14}{2}/\left[\sum_{i=0}^{8}\binom{2i}{i}\binom{i+8}{8-i}\right]$ (ii) $\binom{12}{6}\binom{13}{1}/\left[\sum_{i=0}^{8}\binom{2i}{i}\binom{i+8}{8-i}\right]$

(iii) $\left[\binom{16}{8} + \binom{12}{6}\binom{14}{2} + \binom{8}{4}\binom{12}{4} + \binom{4}{2}\binom{10}{6} + \binom{0}{0}\binom{8}{8}\right] / \left[\sum_{i=0}^{8}\binom{2i}{i}\binom{i+8}{8-i}\right]$

25. $A \cup B = [-2, 4]$, $A \cap B = \{3\}$ **27.** $135/512 \doteq 0.263672$

29. $Pr(A \cap (B \cup C)) = Pr((A \cap B) \cup (A \cap C)) =$ $Pr(A \cap B) + Pr(A \cap C) - Pr((A \cap B) \cap (A \cap C))$. Since A, B, C are independent and $(A \cap B) \cap (A \cap C) = (A \cap A) \cap (B \cap C) = A \cap B \cap C$, $Pr(A \cap (B \cup C)) = Pr(A)Pr(B) +$ $Pr(A)Pr(C) - Pr(A)Pr(B)Pr(C) = Pr(A)[Pr(B) + Pr(C) - Pr(B)Pr(C)] =$ $Pr(A)[Pr(B) + Pr(C) - Pr(B \cap C)] = Pr(A)Pr(B \cup C)$, so A and $B \cup C$ are independent.

31. a) 0.99 **b)** $(0.99)^3 = 0.970299$ **33.** $3/5$

35. $\binom{5}{3}(0.8)^3(0.2)^2 + \binom{5}{4}(0.8)^4(0.2) + \binom{5}{5}(0.8)^5 \doteq 0.94208$

37. $675/2048$ **39. a)** $c = 1/50$ **b)** 0.82 **c)** 13/41 **d)** 2.8 **e)** 1.64

41. a) $3/\binom{47}{2}$ **b)** $\left[\binom{10}{2} - 3\right]/\binom{47}{2}$ **c)** $\left[3\binom{4}{1}\binom{4}{1} - 3\right]/\binom{47}{2}$

43. $2/[m(m+1)]$

45. a) $Pr(X = 1) = 7/16$; $Pr(X = 2) = 3/8$; $Pr(X = 3) = 3/16$

b) $7/4$ **c)** $\sigma_X = 3/4$

Chapter 4
Properties of the Integers: Mathematical Induction

Section 4.1 – p. 208

1. b) Since $1 \cdot 3 = (1)(2)(9)/6$, the result is true for $n = 1$. Assume the result is true for $n = k\ (\geq 1)$: $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + k(k+2) = k(k+1)(2k+7)/6$. Then consider the case for $n = k+1$: $[1 \cdot 3 + 2 \cdot 4 + \cdots + k(k+2)] + (k+1)(k+3) = [k(k+1)(2k+7)/6] +$ $(k+1)(k+3) = [(k+1)/6][k(2k+7) + 6(k+3)] = (k+1)(2k^2 + 13k + 18)/6 =$ $(k+1)(k+2)(2k+9)/6$. Hence the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

c) $S(n)$: $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$

$S(1)$: $\displaystyle\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(2)} = \frac{1}{1+1}$, so $S(1)$ is true.

Assume $S(k)$: $\displaystyle\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. Consider $S(k+1)$.

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= [k(k+2)+1]/[(k+1)(k+2)] = (k+1)/(k+2),$$

so $S(k) \Rightarrow S(k+1)$ and the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

3. a) From $\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=0}^{n}(i^3+3i^2+3i+1) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, we have $(n+1)^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1} i + (n+1)$. Consequently,

$$\begin{aligned}3\sum_{i=1}^{n} i^2 &= (n^3+3n^2+3n+1) - 3[(n)(n+1)/2] - n - 1\\ &= n^3 + (3/2)n^2 + (1/2)n\\ &= (1/2)[2n^3+3n^2+n] = (1/2)n(2n^2+3n+1)\\ &= (1/2)n(n+1)(2n+1), \text{ so}\end{aligned}$$

$\sum_{i=1}^{n} i^2 = (1/6)n(n+1)(2n+1)$ (as shown in Example 4.4).

b) From $\sum_{i=1}^{n} i^4 + (n+1)^4 = \sum_{i=0}^{n}(i+1)^4 = \sum_{i=0}^{n}(i^4+4i^3+6i^2+4i+1) = \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, it follows that $(n+1)^4 = 4\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$. Consequently,

$$\begin{aligned}4\sum_{i=1}^{n} i^3 &= (n+1)^4 - 6[n(n+1)(2n+1)/6] - 4[n(n+1)/2] - (n+1)\\ &= n^4+4n^3+6n^2+4n+1 - (2n^3+3n^2+n) - (2n^2+2n) - (n+1)\\ &= n^4+2n^3+n^2 = n^2(n^2+2n+1) = n^2(n+1)^2.\end{aligned}$$

So $\sum_{i=1}^{n} i^3 = (1/4)n^2(n+1)^2$ [as shown in part (d) of Exercise 1 for this section].

From $\sum_{i=1}^{n} i^5 + (n+1)^5 = \sum_{i=0}^{n}(i+1)^5 = \sum_{i=0}^{n}(i^5+5i^4+10i^3+10i^2+5i+1) = \sum_{i=1}^{n} i^5 + 5\sum_{i=1}^{n} i^4 + 10\sum_{i=1}^{n} i^3 + 10\sum_{i=1}^{n} i^2 + 5\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, we have $5\sum_{i=1}^{n} i^4 = (n+1)^5 - (10/4)n^2(n+1)^2 - (10/6)n(n+1)(2n+1) - (5/2)n(n+1) - (n+1)$. So

$$\begin{aligned}5\sum_{i=1}^{n} i^4 &= n^5+5n^4+10n^3+10n^2+5n+1-(5/2)n^4\\ &\quad - 5n^3 - (5/2)n^2 - (10/3)n^3 - 5n^2 - (5/3)n - (5/2)n^2 - (5/2)n - n - 1\\ &= n^5 + (5/2)n^4 + (5/3)n^3 - (1/6)n.\end{aligned}$$

Consequently, $\sum_{i=1}^{n} i^4 = (1/30)n(n+1)(6n^3+9n^2+n-1)$.

5. a) 7626 **b)** 627,874 **7.** $n = 10$ **9. a)** 506 **b)** 12,144

11. a) $\sum_{i=1}^{n} t_{2i} = \sum_{i=1}^{n} \frac{(2i)(2i+1)}{2} = \sum_{i=1}^{n}(2i^2+i) = 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = 2[(n)(n+1)(2n+1)/6] + [n(n+1)/2] = [n(n+1)(2n+1)/3] + [n(n+1)/2] = n(n+1)[\frac{2n+1}{3} + \frac{1}{2}] = n(n+1)[\frac{4n+5}{6}] = n(n+1)(4n+5)/6$.

b) $\sum_{i=1}^{100} t_{2i} = 100(101)(405)/6 = 681{,}750$.

c)
```
begin
  sum := 0
  for i := 1 to 100 do
    sum := sum + (2 * i) * (2 * i + 1)/2
  print sum
end
```

13. a) There are 49 $(= 7^2)$ 2×2 squares and 36 $(= 6^2)$ 3×3 squares. In total there are $1^2 + 2^2 + 3^2 + \cdots + 8^2 = (8)(8+1)(2\cdot 8+1)/6 = (8)(9)(17)/6 = 204$ squares.

b) For each $1 \le k \le n$ the $n \times n$ chessboard contains $(n-k+1)^2$ $k \times k$ squares. In total there are $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ squares.

15. For $n = 5$, $2^5 = 32 > 25 = 5^2$. Assume the result for $n = k$ (≥ 5): $2^k > k^2$. For $k > 3$, $k(k-2) > 1$, or $k^2 > 2k+1$. $2^k > k^2 \Rightarrow 2^k + 2^k > k^2 + k^2 \Rightarrow 2^{k+1} > k^2 + k^2 > k^2 + (2k+1) = (k+1)^2$. Hence the result is true for $n \ge 5$ by the Principle of Mathematical Induction.

17. b) Starting with $n = 1$ we find that

$$\sum_{j=1}^{1} jH_j = H_1 = 1 = [(2)(1)/2](3/2) - [(2)(1)/4] = [(2)(1)/2]H_2 - [(2)(1)/4].$$

Assuming the truth of the given (open) statement for $n = k$, we have

$$\sum_{j=1}^{k} jH_j = [(k+1)(k)/2]H_{k+1} - [(k+1)(k)/4].$$

For $n = k + 1$ we now find that

$$\begin{aligned}\sum_{j=1}^{k+1} jH_j &= \sum_{j=1}^{k} jH_j + (k+1)H_{k+1}\\ &= [(k+1)(k)/2]H_{k+1} - [(k+1)(k)/4] + (k+1)H_{k+1}\\ &= (k+1)[1+(k/2)]H_{k+1} - [(k+1)(k)/4]\\ &= (k+1)[1+(k/2)][H_{k+2} - (1/(k+2))] - [(k+1)(k)/4]\\ &= [(k+2)(k+1)/2]H_{k+2} - [(k+1)(k+2)]/[2(k+2)] - [(k+1)(k)/4]\\ &= [(k+2)(k+1)/2]H_{k+2} - [(1/4)[2(k+1) + k(k+1)]]\\ &= [(k+2)(k+1)/2]H_{k+2} - [(k+2)(k+1)/4].\end{aligned}$$

Consequently, by the Principle of Mathematical Induction, it follows that the given (open) statement is true for all $n \in \mathbf{Z}^+$.

19. Assume $S(k)$. For $S(k+1)$, we find that $\sum_{i=1}^{k+1} i = ([k + (1/2)]^2/2) + (k+1) = (k^2 + k + (1/4) + 2k + 2)/2 = [(k+1)^2 + (k+1) + (1/4)]/2 = [(k+1) + (1/2)]^2/2$. So $S(k) \Rightarrow S(k+1)$. However, we have no first value of k where $S(k)$ is true: for all $k \geq 1$, $\sum_{i=1}^{k} i = (k)(k+1)/2$ and $(k)(k+1)/2 = [k + (1/2)]^2/2 \Rightarrow 0 = 1/4$.

21. Let $S(n)$ denote the following (open) statement: For $x, n \in \mathbf{Z}^+$, if the program reaches the top of the **while** loop, after the two loop instructions are executed n (> 0) times, then the value of the integer variable *answer* is $x(n!)$.

First consider $S(1)$, the statement for the case where $n = 1$. Here the program (if it reaches the top of the **while** loop) will result in one execution of the **while** loop: x will be assigned the value $x \cdot 1 = x(1!)$, and the value of n will be decreased to 0. With the value of n equal to 0, the loop is not processed again and the value of the variable *answer* is $x(1!)$. Hence $S(1)$ is true.

Now assume the truth for $n = k$ (≥ 1): For $x, k \in \mathbf{Z}^+$, if the program reaches the top of the **while** loop, then upon exiting the loop, the value of the variable *answer* is $x(k!)$. To establish the truth of $S(k+1)$, if the program reaches the top of the **while** loop, then the following occur during the first execution:

The value assigned to the variable x is $x(k+1)$.

The value of n is decreased to $(k+1) - 1 = k$.

But then we can apply the induction hypothesis to the integers $x(k+1)$ and k, and upon exiting the **while** loop for these values, the value of the variable *answer* is $(x(k+1))(k!) = x(k+1)!$

Consequently, $S(n)$ is true for all $n \geq 1$, and we have verified the correctness of this program segment by using the Principle of Mathematical Induction.

23. b) $24 = 5+5+7+7$ $\quad$ $25 = 5+5+5+5+5$ $\quad$ $26 = 5+7+7+7$
$27 = 5+5+5+5+7$ $\quad$ $28 = 7+7+7+7$

Hence the result is true for all $24 \leq n \leq 28$. Assume the result true for 24, 25, 26, 27, 28, $\ldots$, k, and consider $n = k+1$. Since $k + 1 \geq 29$, we may write $k + 1 = [(k+1) - 5] + 5 = (k-4) + 5$, where $k - 4$ can be expressed as a sum of 5's and 7's. Hence $k + 1$ can be expressed as such a sum and the result follows for all $n \geq 24$ by the alternative form of the Principle of Mathematical Induction.

25. $$E(X) = \sum_x x\,Pr(X = x) = \sum_{x=1}^{n} x\left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)\sum_{x=1}^{n} x = \left(\frac{1}{n}\right)\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}$$
$$E(X^2) = \sum_x x^2\,Pr(X = x) = \sum_{x=1}^{n} x^2\left(\frac{1}{n}\right) = \left(\frac{1}{n}\right)\sum_{x=1}^{n} x^2 = \left(\frac{1}{n}\right)\left[\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \frac{(n+1)(2n+1)}{6}$$
$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = (n+1)\left[\frac{2n+1}{6} - \frac{n+1}{4}\right]$$
$$= (n+1)\left[\frac{4n+2-(3n+3)}{12}\right] = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}.$$

27. Let $T = \{n \in \mathbf{Z}^+ | n \geq n_0 \text{ and } S(n) \text{ is false }\}$. Since $S(n_0)$, $S(n_0+1)$, $S(n_0+2)$, $\ldots$, $S(n_1)$ are true, we know that $n_0, n_0+1, n_0+2, \ldots, n_1 \notin T$. If $T \neq \emptyset$, then T has a least element r, because $T \subseteq \mathbf{Z}^+$. However, since $S(n_0)$, $S(n_0+1)$, $\ldots$, $S(r-1)$ are true, it follows that $S(r)$ is true. Hence $T = \emptyset$ and the result follows.

Section 4.2—p. 219

1. a) $c_1 = 7$; $c_{n+1} = c_n + 7$, for $n \geq 1$. **b)** $c_1 = 7$; $c_{n+1} = 7c_n$, for $n \geq 1$.
c) $c_1 = 10$; $c_{n+1} = c_n + 3$, for $n \geq 1$. **d)** $c_1 = 7$; $c_{n+1} = c_n$, for $n \geq 1$.

3. Let $T(n)$ denote the following statement: For $n \in \mathbf{Z}^+$, $n \geq 2$, and the statements $p, q_1, q_2, \ldots, q_n$,
$$p \vee (q_1 \wedge q_2 \wedge \cdots \wedge q_n) \Longleftrightarrow (p \vee q_1) \wedge (p \vee q_2) \wedge \cdots \wedge (p \vee q_n).$$

The statement $T(2)$ is true by virtue of the Distributive Law of $\vee$ over $\wedge$. Assuming $T(k)$, for some $k \geq 2$, we now examine the situation for the statements $p, q_1, q_2, \ldots, q_k, q_{k+1}$. We find that $p \vee (q_1 \wedge q_2 \wedge \cdots \wedge q_k \wedge q_{k+1})$
$$\Longleftrightarrow p \vee [(q_1 \wedge q_2 \wedge \cdots \wedge q_k) \wedge q_{k+1}]$$
$$\Longleftrightarrow [p \vee (q_1 \wedge q_2 \wedge \cdots \wedge q_k)] \wedge (p \vee q_{k+1})$$
$$\Longleftrightarrow [(p \vee q_1) \wedge (p \vee q_2) \wedge \cdots \wedge (p \vee q_k)] \wedge (p \vee q_{k+1})$$
$$\Longleftrightarrow (p \vee q_1) \wedge (p \vee q_2) \wedge \cdots \wedge (p \vee q_k) \wedge (p \vee q_{k+1}).$$

It then follows by the Principle of Mathematical Induction that the statement $T(n)$ is true for all $n \geq 2$.

5. a) (i) The intersection of A_1, A_2 is $A_1 \cap A_2$.
(ii) The intersection of $A_1, A_2, \ldots, A_n, A_{n+1}$ is given by $A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1} = (A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1}$, the intersection of the *two* sets $A_1 \cap A_2 \cap \cdots \cap A_n$ and A_{n+1}.
b) Let $S(n)$ denote the given (open) statement. Then the truth of $S(3)$ follows from the Associative Law of $\cap$. Assuming $S(k)$ true for some $k \geq 3$, consider the case for $k+1$ sets.
(1) If $r = k$, then
$$(A_1 \cap A_2 \cap \cdots \cap A_k) \cap A_{k+1} = A_1 \cap A_2 \cap \cdots \cap A_k \cap A_{k+1},$$
from the recursive definition given in part (a).
(2) For $1 \leq r < k$, we have
$$(A_1 \cap A_2 \cap \cdots \cap A_r) \cap (A_{r+1} \cap \cdots \cap A_k \cap A_{k+1})$$
$$= (A_1 \cap A_2 \cap \cdots \cap A_r) \cap [(A_{r+1} \cap \cdots \cap A_k) \cap A_{k+1}]$$
$$= [(A_1 \cap A_2 \cap \cdots \cap A_r) \cap (A_{r+1} \cap \cdots \cap A_k)] \cap A_{k+1}$$
$$= (A_1 \cap A_2 \cap \cdots \cap A_r \cap A_{r+1} \cap \cdots \cap A_k) \cap A_{k+1}$$
$$= A_1 \cap A_2 \cap \cdots \cap A_r \cap A_{r+1} \cap \cdots \cap A_k \cap A_{k+1},$$
and by the Principle of Mathematical Induction, $S(n)$ is true for all $n \geq 3$ and all $1 \leq r < n$.

CHAPTER 4
PROPERTIES OF THE INTEGERS: MATHEMATICAL INDUCTION

Section 4.1

1. (a) $S(n): 1^2+3^2+5^2+\ldots+(2n-1)^2 = (n)(2n-1)(2n+1)/3$.
$S(1): 1^2 = (1)(1)(3)/3$. This is true.
Assume $S(k): 1^2+3^2+\ldots+(2k-1)^2 = (k)(2k-1)(2k+1)/3$, for some $k \geq 1$.
Consider $S(k+1)$. $[1^2+3^2+\ldots+(2k-1)^2]+(2k+1)^2 = [(k)(2k-1)(2k+1)/3]+(2k+1)^2 = [(2k+1)/3][k(2k-1)+3(2k+1)] = [(2k+1)/3][2k^2+5k+3] = (k+1)(2k+1)(2k+3)/3$, so $S(k) \Longrightarrow S(k+1)$ and the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(c) $S(n): \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$
$S(1):\ \sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(2)} = \frac{1}{1+1}$, so $S(1)$ is true.
Assume $S(k):\ \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. Consider $S(k+1)$.
$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)} = [k(k+2)+1]/[(k+1)(k+2)] = (k+1)/(k+2)$, so $S(k) \Longrightarrow S(k+1)$ and the result follows for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.

The proofs of the remaining parts are similar.

2. (a) $S(n): \sum_{i=1}^{n} 2^{i-1} = 2^n - 1$
$S(1):\ \sum_{i=1}^{1} 2^{i-1} = 2^{1-1} = 2^1 - 1$, so $S(1)$ is true.
Assume $S(k): \sum_{i=1}^{k} 2^{i-1} = 2^k - 1$. Consider $S(k+1)$.
$\sum_{i=1}^{k+1} 2^{i-1} = \sum_{i=1}^{k} 2^{i-1} + 2^k = 2^k - 1 + 2^k = 2^{k+1} - 1$, so $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(b) For $n = 1, \sum_{i=1}^{1} i(2^i) = 2 = 2 + (1-1)2^{1+1}$, so the statement $S(1)$ is true. Assume $S(k)$ true – that is, $\sum_{i=1}^{k} i(2^i) = 2 + (k-1)2^{k+1}$. For $n = k+1, \sum_{i=1}^{k+1} i(2^i) = \sum_{i=1}^{k} i(2^i) + (k+1)2^{k+1} = 2 + (k-1)2^{k+1} + (k+1)2^{k+1} = 2 + (2k)2^{k+1} = 2 + k \cdot 2^{k+2}$, so $S(n)$ is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(c) For $n = 1$, we find that $\sum_{i=1}^{1} (i)(i!) = 1 = (1+1)! - 1$, so $S(1)$ is true. We assume the truth of $S(k)$ – that is, $\sum_{i=1}^{k} i(i!) = (k+1)! - 1$. Now for the case where $n = k+1$ we have $\sum_{i=1}^{k+1} i(i!) = \sum_{i=1}^{k} i(i!) + (k+1)(k+1)! = (k+1)! - 1 + (k+1)(k+1)! = [1+(k+1)](k+1)! - 1 = (k+2)(k+1)! - 1 = (k+2)! - 1$. Hence $S(k) \Longrightarrow S(k+1)$, and since $S(1)$ is true it follows that the statement is true for all $n \geq 1$, by the Principle of Mathematical Induction.

3. (a) From $\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=0}^{n} (i^3+3i^2+3i+1) = \sum_{i=1}^{n} i^3 + 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=0}^{n} 1$, we have $(n+1)^3 = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1} i + (n+1)$. Consequently,

$$\begin{aligned}3\sum_{i=1}^{n} i^2 &= (n^3+3n^2+3n+1)-3[(n)(n+1)/2]-n-1\\ &= n^3+(3/2)n^2+(1/2)n\\ &= (1/2)[2n^3+3n^2+n]=(1/2)n(2n^2+3n+1)\\ &= (1/2)n(n+1)(2n+1), \text{ so}\end{aligned}$$

$\sum_{i=1}^{n} i^2 = (1/6)n(n+1)(2n+1)$ (as shown in Example 4.4).
(b) From $\sum_{i=1}^{n} i^4+(n+1)^4 = \sum_{i=0}^{n}(i+1)^4 = \sum_{i=0}^{n}(i^4+4i^3+6i^2+4i+1) = \sum_{i=1}^{n} i^4+4\sum_{i=1}^{n} i^3+6\sum_{i=1}^{n} i^2+4\sum_{i=1}^{n} i+\sum_{i=0}^{n} 1$, it follows that $(n+1)^4 = 4\sum_{i=1}^{n} i^3+6\sum_{i=1}^{n} i^2+4\sum_{i=1}^{n} i+\sum_{i=0}^{n} 1$. Consequently,
$4\sum_{i=1}^{n} i^3 = (n+1)^4-6[n(n+1)(2n+1)/6]-4[n(n+1)/2]-(n+1) = n^4+4n^3+6n^2+4n+1-(2n^3+3n^2+n)-(2n^2+2n)-(n+1) = n^4+2n^3+n^2 = n^2(n^2+2n+1) = n^2(n+1)^2$.
So $\sum_{i=1}^{n} i^3 = (1/4)n^2(n+1)^2$ [as shown in part (d) of Exercise 1 for this section].

From $\sum_{i=1}^{n} i^5 + (n+1)^5 = \sum_{i=0}^{n}(i+1)^5 = \sum_{i=0}^{n}(i^5+5i^4+10i^3+10i^2+5i+1) = \sum_{i=1}^{n} i^5+5\sum_{i=1}^{n} i^4+10\sum_{i=1}^{n} i^3+10\sum_{i=1}^{n} i^2+5\sum_{i=1}^{n} i+\sum_{i=0}^{n} 1$, we have $5\sum_{i=1}^{n} i^4 = (n+1)^5-(10/4)n^2(n+1)^2-(10/6)n(n+1)(2n+1)-(5/2)n(n+1)-(n+1)$. So

$$\begin{aligned}5\sum_{i=1}^{n} i^4 &= n^5+5n^4+10n^3+10n^2+5n+1-(5/2)n^4\\ &\quad -5n^3-(5/2)n^2-(10/3)n^3-5n^2-(5/3)n-(5/2)n^2-(5/2)n-n-1\\ &= n^5+(5/2)n^4+(5/3)n^3-(1/6)n.\end{aligned}$$

Consequently, $\sum_{i=1}^{n} i^4 = (1/30)n(n+1)(6n^3+9n^2+n-1)$.

4. Let $x_1, x_2, \ldots, x_{25}$ denote the numbers (in their order on the wheel), and assume that $x_1+x_2+x_3<39, x_2+x_3+x_4<39, \ldots, x_{24}+x_{25}+x_1<39$, and $x_{25}+x_1+x_2<39$. Then $\sum_{i=1}^{25} 3x_i < 25(39)$. But $\sum_{i=1}^{25} 3x_i = 3\sum_{i=1}^{25} i = (3)(25)(26)/2 = (39)(25)$.

5. (a) 7626 (b) 627,874

6. a) The typical palindrome under study here has the form $abba$ where $1 \le a \le 9$ and $0 \le b \le 9$. Consequently there are $9\cdot 10 = 90$ such palindromes, by the rule of product. Their sum is $\sum_{a=1}^{9}(\sum_{b=0}^{9} abba) = \sum_{a=1}^{9}\sum_{b=0}^{9}(1001a+110b) = \sum_{a=1}^{9}[10(1001a)+110\sum_{b=0}^{9} b] = \sum_{a=1}^{9}(10010a+110(9\cdot 10/2)) = 10010\sum_{a=1}^{9} a+\sum_{a=1}^{9} 4950 = 10010(9\cdot 10/2)+9(4950) = 450450+44550 = 495000$.

b)
```
begin
    sum := 0
    for a := 1 to 9 do
        for b := 0 to 9 do
            sum := sum + 1001 * a + 110 * b
    print sum
end
```

7.

$$\begin{aligned} 4n + 110 &= 6 + 8 + 10 + \cdots + [6 + (n-1)2] \\ &= 6n + [0 + 2 + 4 + \cdots + (n-1)2] \\ &= 6n + 2[1 + 2 + \cdots + (n-1)] \\ &= 6n + 2[(n-1)(n)/2] \\ &= 6n + (n-1)(n) = n^2 + 5n \\ n^2 + n - 110 &= (n+11)(n-10) = 0, \end{aligned}$$

so $n = 10$ – the number of layers.

8. Here we have $\sum_{i=1}^{n} i^2 = (n)(n+1)(2n+1)/6 = (2n)(2n+1)/2 = \sum_{i=1}^{2n} i$,

and $(n)(n+1)(2n+1)/6 = (2n)(2n+1)/2 \Rightarrow (n)(n+1)/6 = (2n)/2 \Rightarrow$

$(n+1)/6 = 1 \Rightarrow n+1 = 6 \Rightarrow n = 5$.

9. (a) $\sum_{i=11}^{33} i = \sum_{i=1}^{33} i - \sum_{i=1}^{10} i = [(33)(34)/2] - [(10)(11)/2] = 561 - 55 = 506$
(b) $\sum_{i=11}^{33} i^2 = \sum_{i=1}^{33} i^2 - \sum_{i=1}^{10} i^2 = [(33)(34)(67)/6] - [(10)(11)(21)/6] = 12144$

10. $\sum_{i=51}^{100} t_i = \sum_{i=1}^{100} t_i - \sum_{i=1}^{50} t_i = (100)(101)(102)/6 - (50)(51)(52)/6 = 171,700 - 22,100 = 149,600$.

11. a) $\sum_{i=1}^{n} t_{2i} = \sum_{i=1}^{n} \frac{(2i)(2i+1)}{2} = \sum_{i=1}^{n}(2i^2+i) = 2\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = 2[(n)(n+1)(2n+1)/6] + [n(n+1)/2] = [n(n+1)(2n+1)/3] + [n(n+1)/2] = n(n+1)[\frac{2n+1}{3} + \frac{1}{2}] = n(n+1)[\frac{4n+5}{6}] = n(n+1)(4n+5)/6$.

b) $\sum_{i=1}^{100} t_{2i} = 100(101)(405)/6 = 681,750$.

c)
```
begin
    sum := 0
    for i := 1 to 100 do
        sum := sum + (2 * i) * (2 * i + 1)/2
    print sum
end
```

12. (a) $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta) = \cos 2\theta + i\sin 2\theta$.
(b) $S(n) : (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. $S(1)$ is true, so assume $S(k) : (\cos\theta + i\sin\theta)^k = (\cos k\theta + i\sin k\theta)$. Consider $S(k+1) : (\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta) = (\cos k\theta + i\sin k\theta)\cdot(\cos\theta + i\sin\theta) = (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin\theta\cos k\theta + \sin k\theta\cos\theta) = \cos(k+1)\theta + i\sin(k+1)\theta$. So $S(k) \Longrightarrow S(k+1)$ and the result is true for all $n \in \mathbf{Z}^+$ by the Principle of Mathematical Induction.
(c) $(1+i)^{100} = 2^{50}(\cos 4500° + i\sin 4500°) = 2^{50}(\cos 180° + i\sin 180°) = -(2^{50})$.

13. (a) There are $49(=7^2)$ 2×2 squares and $36(=6^2)$ 3×3 squares. In total there are $1^2+2^2+3^2+\cdots+8^2=(8)(8+1)(2\cdot 8+1)/6=(8)(9)(17)/6=204$ squares.

(b) For each $1 \leq k \leq n$ the $n \times n$ chessboard contains $(n-k+1)^2$ $k \times k$ squares. In total there are $1^2+2^2+3^2+\cdots+n^2=n(n+1)(2n+1)/6$ squares.

14. For $n=4$ we have $2^4=16<24=4!$, so the statement $S(4)$ is true. Assume the truth of $S(k)$ – that is, $2^k<k!$. For $k \geq 4, 2<k+1$, and $[(2^k<k!) \wedge (2<k+1)] \Longrightarrow (2^k)(2)<(k!)(k+1)$, or $2^{k+1}<(k+1)!$ Hence $S(n)$ is true for all $n \geq 4$ by the Principle of Mathematical Induction.

15. For $n=5, 2^5=32>25=5^2$. Assume the result for $n=k(\geq 5): 2^k>k^2$. For $k>2, k(k-2)>1$, or $k^2>2k+1$. But $2^k>k^2 \Longrightarrow 2^k+2^k>k^2+k^2 \Longrightarrow 2^{k+1}>k^2+k^2>k^2+(2k+1)=(k+1)^2$. Hence the result is true for $n \geq 5$ by the Principle of Mathematical Induction.

16. (a) 3 (b) $s_2=2$; $s_4=4$

(c) For $n \geq 1$, $s_n=\sum_{\emptyset \neq A \subseteq X_n} \frac{1}{p_A}=n$.

Proof: For $n=1$, $s_1=\frac{1}{1}=1$, so this first case is true and establishes the basis step. Now, for the inductive step, assume the result true for $n=k(\geq 1)$. That is, $s_k=\sum_{\emptyset \neq A \subseteq X_k} \frac{1}{p_A}=k$. For $n=k+1$ we find that $s_{k+1}=\sum_{\emptyset \neq A \subseteq X_{k+1}} \frac{1}{p_A}=\sum_{\emptyset \neq B \subseteq X_k} \frac{1}{p_B}+\sum_{\{k+1\} \subseteq C \subseteq X_{k+1}} \frac{1}{p_C}$, where the first sum is taken over all nonempty subsets B of X_k and the second sum over all subsets C of X_{k+1} that contain $k+1$. Then $s_{k+1}=s_k+[(\frac{1}{k+1})+(\frac{1}{k+1})s_k]=k+(\frac{1}{k+1})+(\frac{1}{k+1})k=k+(\frac{1}{k+1})(1+k)=k+1$. Consequently, we have deduced the truth for $n=k+1$ from that of $n=k$. The result now follows for all $n \geq 1$ by the Principle of Mathematical Induction.

17. (a) Once again we start at $n=0$. Here we find that $1=1+(0/2) \leq H_1=H_{2^0}$, so this first case is true. Assuming the truth for $n=k(\in \mathbf{N})$ we obtain the induction hypothesis

$$1+(k/2) \leq H_{2^k}.$$

Turning now to the case where $n=k+1$ we find $H_{2^{k+1}}=H_{2^k}+[1/(2^k+1)]+[1/(2^k+2)]+\ldots+[1/(2^k+2^k)] \geq H_{2^k}+[1/(2^k+2^k)]+[1/(2^k+2^k)]+\ldots+[1/(2^k+2^k)]=H_{2^k}+2^k[1/2^{k+1}]=H_{2^k}+(1/2) \geq 1+(k/2)+(1/2)=1+(k+1)/2$.

The result now follows for all $n \geq 0$ by the Principle of Mathematical Induction.

(b) Starting with $n=1$ we find that

$$\sum_{j=1}^{1} jH_j=H_1=1=[(2)(1)/2](3/2)-[(2)(1)/4]=[(2)(1)/2]H_2-[(2)(1)/4].$$

Assuming the truth of the given statement for $n = k$, we have

$$\sum_{j=1}^{k} jH_j = [(k+1)(k)/2]H_{k+1} - [(k+1)(k)/4].$$

For $n = k+1$ we now find that

$$\sum_{j=1}^{k+1} jH_j = \sum_{j=1}^{k} jH_j + (k+1)H_{k+1}$$

$$= [(k+1)(k)/2]H_{k+1} - [(k+1)(k)/4] + (k+1)H_{k+1}$$

$$= (k+1)[1+(k/2)]H_{k+1} - [(k+1)(k)/4]$$

$$= (k+1)[1+(k/2)][H_{k+2} - (1/(k+2))] - [(k+1)(k)/4]$$

$$= [(k+2)(k+1)/2]H_{k+2} - [(k+1)(k+2)]/[2(k+2)] - [(k+1)(k)/4]$$

$$= [(k+2)(k+1)/2]H_{k+2} - [(1/4)[2(k+1) + k(k+1)]]$$

$$= [(k+2)(k+1)/2]H_{k+2} - [(k+2)(k+1)/4].$$

Consequently, by the Principle of Mathematical Induction, it follows that the given statement is true for all $n \in \mathbf{Z}^+$.

18. Conjecture: For all $n \in \mathbf{N}$, $(n^2+1)+(n^2+2)+(n^2+3)+\ldots+(n+1)^2 = \sum_{i=1}^{2n+1}(n^2+i) = n^3+(n+1)^3$.
Proof: $\sum_{i=1}^{2n+1}(n^2+i) = n^2\sum_{i=1}^{2n+1} 1 + \sum_{i=1}^{2n+1} i = n^2(2n+1)+(2n+1)(2n+2)/2 =$
$2n^3+n^2+(2n+1)(n+1) = 2n^3+n^2+2n^2+3n+1 = n^3+[n^3+3n^2+3n+1] =$
$n^3+(n+1)^3$.

19. Assume $S(k)$ true for some $k \geq 1$. For $S(k+1), \sum_{i=1}^{k+1} i = [k+(1/2)]^2/2+(k+1) =$
$((k^2+k)+(1/4)+2k+2)/2 = [(k+1)^2+(k+1)+(1/4)]/2 = [(k+1)+(1/2)]^2/2$.
S $S(k) \Longrightarrow S(k+1)$. However, we have no first value of k where $S(k)$ is true:
For each $k \geq 1, \sum_{i=1}^{k} i = (k)(k+1)/2$ and $(k)(k+1)/2 = [k+(1/2)]^2/2 \Longrightarrow 0 = 1/4$.

20. For $n = 0, S = \{a_1\}$ and 0 comparisons are required. Since $0 = 0 \cdot 2^0$, the result is true when $n = 0$. Assume the result for $n = k (\geq 0)$ and consider the case $n = k+1$. If $|S| = 2^{k+1}$ then $S = S_1 \cup S_2$ where $|S_1| = |S_2| = 2^k$. By the induction hypothesis the number of comparisons needed to place the elements in each of S_1, S_2 in ascending order is bounded by $k \cdot 2^k$. Therefore, by the given information, the elements in S can be placed in ascending order by making at most a total of $(k \cdot 2^k)+(k \cdot 2^k)+(2^k+2^k-1) = (k+1)2^{k+1}-1 \leq (k+1)2^{k+1}$ comparisons.

21. For $x, n \in \mathbf{Z}^+$, let $S(n)$ denote the statement: If the program reaches the top of the **while** loop, after the two loop instructions are executed $n(> 0)$ times, then the value of the integer variable *answer* is $x(n!)$.
First consider $S(1)$, the statement for the case where $n = 1$. Here the program (if it reaches the top of the **while** loop) will result in one execution of the **while** loop: x will be assigned the value $x \cdot 1 = x(1!)$, and the value of n will be decreased to 0. With the value of n equal to 0 the loop is not processed again and the value of the variable *answer* is $x(1!)$. Hence $S(1)$ is true.
Now assume the truth for $n = k$: For $x, k \in \mathbf{Z}^+$, if the program reaches the top of the **while** loop, then upon exiting the loop, the value of the variable *answer* is $x(k!)$. To establish $S(k+1)$, if the program reaches the top of the **while** loop, then the following occur during the first execution:

The value assigned to the variable x is $x(k+1)$.

The value of n is decreased to $(k+1) - 1 = k$.

But then we can apply the induction hypothesis to the integers $x(k+1)$ and k, and after we exit the **while** loop for these values, the value of the variable *answer* is $(x(k+1))(k!) = x(k+1)!$
Consequently, $S(n)$ is true for all $n \geq 1$, and we have verified the correctness of this program segment by using the Principle of Mathematical Induction.

22. If $n = 0$, then the statement '$n \neq 0$' is false so the **while** loop is bypassed and the value assigned to *answer* is $x = x + 0 \cdot y$. So the result is true in the first case.
Now assume the result true for $n = k$ – that is, for $x, y \in \mathbf{R}$, if the program reaches the top of the **while** loop with $k \in \mathbf{Z}$, $k \geq 0$, then upon bypassing the loop when $k = 0$, or executing the two loop instructions $k(> 0)$ times, then the value assigned to *answer* is $x + ny$. To establish the result for $n = k+1$, suppose the program reaches the top of the **while** loop. Since $k \geq 0, n = k+1 > 0$, so the loop is not bypassed. During the first pass through the **while** loop we find that

The value assigned to x is $x + y$; and

The value of n is decreased to $(k+1) - 1 = k$.

Now we apply the induction hypothesis to the real numbers $x+y$ and y and the nonnegative integer $n - 1 = k$, and upon bypassing the loop when $k = 0$, or executing the two loop instructions $k(> 0)$ times, then the value assigned to *answer* is

$$(x + y) + ky = x + (k+1)y.$$

The result now follows for all $n \in \mathbf{N}$ by the Principle of Mathematical Induction.

23. (a) The result is true for $n = 2, 4, 5, 6$. Assume the result is true for all $n = 2, 4, 5, \ldots, k-1, k$, where $k \geq 6$. If $n = k+1$, then $n = 2 + (k-1)$, and since the result is true for $k - 1$, it follows by induction that it is true for $k + 1$. Consequently, by the Alternative Form of the Principle of Mathematical Induction, every $n \in \mathbf{Z}^+, n \neq 1, 3$, can be written as a sum of 2's and 5's.

(b) $24 = 5 + 5 + 7 + 7$ $\quad$ $25 = 5 + 5 + 5 + 5 + 5$ $\quad$ $26 = 5 + 7 + 7 + 7$

$27 = 5+5+5+5+7$ $\qquad\qquad$ $28 = 7+7+7+7$

Hence the result is true for all $24 \le n \le 28$. Assume the result true for $24 \le n \le 28 \le k$, and consider $n = k+1$. Since $k+1 \ge 29$, we may write $k+1 = [(k+1)-5]+5 = (k-4)+5$, where $k-4$ can be expressed as a sum of 5's and 7's. Hence $k+1$ can be expressed as such a sum and the result follows for all $n \ge 24$ by the Alternative Form of the Principle of Mathematical Induction.

24. (a) $a_3 = 3 \qquad a_4 = 5 \qquad a_5 = 8 \qquad a_6 = 13 \qquad a_7 = 21$

(b) $a_1 = 1 < (7/4)^1$, so the result is true for $n = 1$. Likewise, $a_2 = 2 < \frac{49}{16} = (7/4)^2$ and the result holds for $n = 2$.

Assume the result true for all $1 \le n \le k$, where $k \ge 2$. Now for $n = k+1$ we have $a_{k+1} = a_k + a_{k-1} < (7/4)^k + (7/4)^{k-1} = (7/4)^{k-1}[(7/4)+1] = (7/4)^{k-1}(11/4) = (7/4)^{k-1}(44/16) < (7/4)^{k-1}(49/16) = (7/4)^{k-1}(7/4)^2 = (7/4)^{k+1}$. So by the Alternative Form of the Principle of Mathematical Induction it follows that $a_n < (7/4)^n$ for all $n \ge 1$.

25.

$$\begin{aligned}
E(X) &= \textstyle\sum_x xPr(X=x) = \sum_{x=1}^n x(\frac{1}{n}) = (\frac{1}{n})\sum_{x=1}^n x = (\frac{1}{n})[\frac{n(n+1)}{2}] = \frac{n+1}{2} \\
E(X^2) &= \textstyle\sum_x x^2Pr(X=x) = \sum_{x=1}^n x^2(\frac{1}{n}) = (\frac{1}{n})\sum_{x=1}^n x^2 = (\frac{1}{n})[\frac{n(n+1)(2n+1)}{6}] = \frac{(n+1)(2n+1)}{6} \\
\mathrm{Var}(X) &= E(X^2) - E(X)^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = (n+1)[\frac{2n+1}{6} - \frac{n+1}{4}] \\
&= (n+1)[\frac{4n+2-(3n+3)}{12}] = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}.
\end{aligned}$$

26.

$$\begin{aligned}
\text{a)}\quad a_1 &= \textstyle\sum_{i=0}^{1-1}\binom{0}{i}a_i a_{(1-1)-i} = \binom{0}{0}a_0a_0 = a_0^2 \\
a_2 &= \textstyle\sum_{i=0}^{2-1}\binom{1}{i}a_i a_{(2-1)-i} = \binom{1}{0}a_0a_1 + \binom{1}{1}a_1a_0 = 2a_0^3
\end{aligned}$$

$$\begin{aligned}
\text{b)}\quad a_3 &= \textstyle\sum_{i=0}^{3-1}\binom{3-1}{i}a_i a_{(3-1)-i} = \sum_{i=0}^{2}\binom{2}{i}a_i a_{2-i} \\
&= \binom{2}{0}a_0a_2 + \binom{2}{1}a_1a_1 + \binom{2}{2}a_2a_0 \\
&= a_0(2a_0^3) + 2(a_0^2)(a_0^2) + (2a_0^3)a_0 = 6a_0^4
\end{aligned}$$

$$\begin{aligned}
\text{c)}\quad a_4 &= \textstyle\sum_{i=0}^{4-1}\binom{4-1}{i}a_i a_{(4-1)-i} = \sum_{i=0}^{3}\binom{3}{i}a_i a_{3-i} \\
&= \binom{3}{0}a_0a_3 + \binom{3}{1}a_1a_2 + \binom{3}{2}a_2a_1 + \binom{3}{3}a_3a_0 \\
&= a_0(6a_0^4) + 3(a_0^2)(2a_0^3) + 3(2a_0^3)(a_0^2) + (6a_0^4)(a_0) \\
&= 24a_0^5
\end{aligned}$$

c) For $n \ge 0$, $a_n = (n!)a_0^{n+1}$.

Proof: (By the Alternative Form of the Principle of Mathematical Induction)
The result is true for $n = 0$ and this establishes the basis step. [In fact, the calculations in parts (a) and (b) show the result is also true for $n = 1, 2, 3$, and 4.] Assuming the result true for $n = 0, 1, 2, 3, \ldots, k(\geq 0)$ – that is, that $a_n = (n!)a_0^{n+1}$ for $n = 0, 1, 2, 3, \ldots, k(\geq 0)$ – we find that

$$\begin{aligned} a_{k+1} &= \sum_{i=0}^{k} \binom{k}{i} a_i a_{k-i} \\ &= \sum_{i=0}^{k} \binom{k}{i} (i!)(a_0^{i+1})(k-i)!(a_0^{k-i+1}) \\ &= \sum_{i=0}^{k} \binom{k}{i} (i!)(k-i)! a_0^{k+2} \\ &= \sum_{i=0}^{k} k! a_0^{k+2} \\ &= (k+1)[k! a_0^{k+2}] = (k+1)! a_0^{k+2}. \end{aligned}$$

So the truth of the result for $n = 0, 1, 2, \ldots, k(\geq 0)$ implies the truth of the result for $n = k+1$. Consequently, for all $n \geq 0$, $a_n = (n!)a_0^{n+1}$ by the Alternative Form of the Principle of Mathematical Induction.

27. Let $T = \{n \in \mathbf{Z}^+ | n \geq n_0$ and $S(n)$ is false$\}$. Since $S(n_0)$, $S(n_0+1)$, $S(n_0+2), \ldots, S(n_1)$ are true, we know that $n_0, n_0+1, n_0+2, \ldots, n_1 \notin T$. If $T \neq \emptyset$, then by the Well-Ordering Principle T has a least element r, because $T \subseteq \mathbf{Z}^+$. However, since $S(n_0)$, $S(n_0+1), \ldots, S(r-1)$ are true, it follows that $S(r)$ is true. Hence $T = \emptyset$ and the result follows.

28. (a) (i) The number of compositions of 5 that start with 1 is the number of compositions of 4, which is $2^{4-1} = 2^3 = 8$.

(ii) $2^{3-1} = 2^2 = 4$ (iii) $2^{2-1} = 2^1 = 2$
(iv) $2^{1-1} = 2^0 = 1$ (v) 1

(b) In total, there are $2^{(n+1)-1} = 2^n$ compositions for the fixed positive integer $n+1$. For $1 \leq i \leq n$, there are $2^{(n+1-i)-1} = 2^{n-i}$ compositions of $n+1$ that start with i. In addition, there is the composition consisting of only one summand – namely, $(n+1)$. So we have counted the same collection of objects – that is, the compositions of $n+1$ – in two ways. This gives us $2^n = \sum_{i=1}^{n} 2^{n-i} + 1 = (2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 2^0) + 1 = (2^0 + 2^1 + 2^2 + \cdots + 2^{n-3} + 2^{n-2} + 2^{n-1}) + 1 = \sum_{i=0}^{n-1} 2^i + 1$. Consequently, $\sum_{i=0}^{n-1} 2^i = 2^n - 1$.

Section 4.2

1.

(a) $c_1 = 7$; and $c_{n+1} = c_n + 7$, for $n \geq 1$.
(b) $c_1 = 7$; and $c_{n+1} = 7c_n$, for $n \geq 1$.
(c) $c_1 = 10$; and $c_{n+1} = c_n + 3$, for $n \geq 1$.
(d) $c_1 = 7$; and $c_{n+1} = c_n$, for $n \geq 1$.
(e) $c_1 = 1$; and $c_{n+1} = c_n + 2n + 1$, for $n \geq 1$.
(f) $c_1 = 3, c_2 = 1$; and $c_{n+2} = c_n$, for $n \geq 1$.

2. (a) For any statements $p_1, p_2, \ldots, p_n, p_{n+1}$, we define