2

Fundamentals of Logic

In the first chapter we derived a summation formula in Example 1.40 (Section 1.4). We obtained this formula by counting the same collection of objects (the statements that were executed in a certain program segment) in two different ways and then equating the results. Consequently, we say that the formula was established by a combinatorial proof. This is one of many different techniques for arriving at a proof.

In this chapter we take a close look at what constitutes a valid argument and a more conventional proof. When a mathematician wishes to provide a proof for a given situation, he or she must use a system of logic. This is also true when a computer scientist develops the algorithms needed for a program or system of programs. The logic of mathematics is applied to decide whether one statement follows from, or is a logical consequence of, one or more other statements.

Some of the rules that govern this process are described in this chapter. We shall use these rules in proofs (provided in the text and required in the exercises) throughout subsequent chapters. However, at no time can we hope to arrive at a point at which we can apply the rules in an automatic fashion. As in applying the counting ideas discussed in Chapter 1, we should always analyze and seek to understand the situation given. This often calls for attributes we cannot learn in a book, such as insight and creativity. Merely trying to apply formulas or invoke rules will not get us very far either in proving results (such as theorems) or in doing enumeration problems.

2.1 Basic Connectives and Truth Tables

In the development of any mathematical theory, assertions are made in the form of sentences. Such verbal or written assertions, called *statements* (or *propositions*), are declarative sentences that are either true or false — but *not* both. For example, the following are statements, and we use the lowercase letters of the alphabet (such as p, q, and r) to represent these statements.

p: Combinatorics is a required course for sophomores.

q: Margaret Mitchell wrote *Gone with the Wind.*

r: $2 + 3 = 5$.

On the other hand, we do not regard sentences such as the exclamation

"What a beautiful evening!"

or the command

"Get up and do your exercises."

as statements since they do not have *truth values* (true or false).

The preceding statements represented by the letters p, q, and r are considered to be *primitive* statements, for there is really no way to break them down into anything simpler. New statements can be obtained from existing ones in two ways.

1) Transform a given statement p into the statement $\neg p$, which denotes its *negation* and is read "Not p."

For the statement p above, $\neg p$ is the statement "Combinatorics is not a required course for sophomores." (We do not consider the negation of a primitive statement to be a primitive statement.)

2) Combine two or more statements into a *compound* statement, using the following *logical connectives.*

a) Conjunction: The *conjunction* of the statements p, q is denoted by $p \wedge q$, which is read "p and q." In our example the compound statement $p \wedge q$ is read "Combinatorics is a required course for sophomores, **and** Margaret Mitchell wrote *Gone with the Wind.*"

b) Disjunction: The expression $p \vee q$ denotes the *disjunction* of the statements p, q and is read "p or q." Hence "Combinatorics is a required course for sophomores, **or** Margaret Mitchell wrote *Gone with the Wind*" is the verbal translation for $p \vee q$, when p, q are as above. We use the word "or" in the *inclusive* sense here. Consequently, $p \vee q$ is true if one or the other of p, q is true or if *both* of the statements p, q are true. In English we sometimes write "and/or" to point this out. The *exclusive* "or" is denoted by $p \veebar q$. The compound statement $p \veebar q$ is true if one or the other of p, q is true but *not both* of the statements p, q are true. One way to express $p \veebar q$ for the example here is "Combinatorics is a required course for sophomores, or Margaret Mitchell wrote *Gone with the Wind*, but not both."

c) Implication: We say that "p implies q" and write $p \rightarrow q$ to designate the statement, which is the *implication* of q by p. Alternatively, we can also say

(i) "If p, then q."
(ii) "p is *sufficient* for q."
(iii) "p is a *sufficient condition* for q."
(iv) "q is *necessary* for p."
(v) "q is a *necessary condition* for p."
(vi) "p only if q."

A verbal translation of $p \rightarrow q$ for our example is "If combinatorics is a required course for sophomores, then Margaret Mitchell wrote *Gone with the Wind.*" The statement p is called the *hypothesis* of the implication; q is called the *conclusion*. When statements are combined in this manner, there need not be any causal relationship between the statements for the implication to be true.

d) Biconditional: Last, the *biconditional* of two statements p, q, is denoted by $p \leftrightarrow q$, which is read "p if and only if q," or "p is necessary and sufficient for q." For our p, q, "Combinatorics is a required course for sophomores if and only if Margaret Mitchell wrote *Gone with the Wind*" conveys the meaning of $p \leftrightarrow q$. We sometimes abbreviate "p if and only if q" as "p iff q."

Throughout our discussion on logic we must realize that a sentence such as

"The number x is an integer."

is *not* a statement because its truth value (true or false) cannot be determined until a numerical value is assigned for x. If x were assigned the value 7, the result would be a true statement. Assigning x a value such as $\frac{1}{2}$, $\sqrt{2}$, or π, however, would make the resulting statement false. (We shall encounter this type of situation again in Sections 2.4 and 2.5 of this chapter.)

In the foregoing discussion, we mentioned the circumstances under which the *compound* statements $p \vee q$, $p \veebar q$ are considered true, on the basis of the truth of their components p, q. This idea of the truth or falsity of a compound statement being dependent only on the truth values of its components is worth further investigation. Tables 2.1 and 2.2 summarize the truth and falsity of the negation and the different kinds of compound statements on the basis of the truth values of their components. In constructing such *truth tables*, we write "0" for false and "1" for true.

Table 2.1

p	$\neg p$
0	1
1	0

Table 2.2

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

The four possible truth assignments for p, q can be listed in any order. For later work, the particular order presented here will prove useful.

We see that the columns of truth values for p and $\neg p$ are the opposite of each other. The statement $p \wedge q$ is true only when both p, q are true, whereas $p \vee q$ is false only when both the component statements p, q are false. As we noted before, $p \veebar q$ is true when exactly one of p, q is true.

For the implication $p \rightarrow q$, the result is true in all cases except where p is true and q is false. We do not want a true statement to lead us into believing something that is false. However, we regard as true a statement such as "If $2 + 3 = 6$, then $2 + 4 = 7$," even though the statements "$2 + 3 = 6$" and "$2 + 4 = 7$" are both false.

Finally, the biconditional $p \leftrightarrow q$ is true when the statements p, q have the same truth value and is false otherwise.

Now that we have been introduced to certain concepts, let us investigate a little further some of these initial ideas about connectives. Our first two examples should prove useful for such an investigation.

EXAMPLE 2.1

Let s, t, and u denote the following primitive statements:

s: Phyllis goes out for a walk.

t: The moon is out.

u: It is snowing.

The following English sentences provide possible translations for the given (symbolic) compound statements.

a) $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

b) $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if it is not snowing Phyllis goes out for a walk. [So $\neg u \rightarrow s$ is understood to mean $(\neg u) \rightarrow s$ as opposed to $\neg(u \rightarrow s)$.]

c) $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Now we will work in reverse order and examine the logical (or symbolic) notation for three given English sentences:

d) "Phyllis will go out walking if and only if the moon is out." Here the words "if and only if" indicate that we are dealing with a biconditional. In symbolic form this becomes $s \leftrightarrow t$.

e) "If it is snowing and the moon is not out, then Phyllis will not go out for a walk." This compound statement is an implication where the hypothesis is also a compound statement. One may express this statement in symbolic form as $(u \wedge \neg t) \rightarrow \neg s$.

f) "It is snowing but Phyllis will still go out for a walk." Now we come across a new connective — namely, *but*. In our study of logic we shall follow the convention that the connectives *but* and *and* convey the same meaning. Consequently, this sentence may be represented as $u \wedge s$.

Now let us return to the results in Table 2.2, particularly the sixth column. For if this is one's first encounter with the truth table for the implication $p \rightarrow q$, then it may be somewhat difficult to accept the stated entries — especially the results in the first two rows (where p has the truth value 0). The following example should help make these truth value assignments easier to grasp.

EXAMPLE 2.2

Consider the following scenario. It is almost the week before Christmas and Penny will be attending several parties that week. Ever conscious of her weight, she plans not to weigh herself until the day after Christmas. Considering what those parties may do to her waistline by then, she makes the following resolution for the December 26 outcome: "If I weigh more than 120 pounds, then I shall enroll in an exercise class."

Here we let p and q denote the (primitive) statements

p: I weigh more than 120 pounds.

q: I shall enroll in an exercise class.

Then Penny's statement (implication) is given by $p \rightarrow q$.

We shall consider the truth values of this particular example of $p \rightarrow q$ for the rows of Table 2.2. Consider first the easier cases in rows 4 and 3.

- Row 4: p and q both have the truth value 1. On December 26 Penny finds that she weighs more than 120 pounds and promptly enrolls in an exercise class, just as she said she would. Here we consider $p \rightarrow q$ to be true and assign it the truth value 1.
- Row 3: p has the truth value 1, q has the truth value 0. Now that December 26 has arrived, Penny finds her weight to be over 120 pounds, but she makes no attempt to enroll in an exercise class. In this case we feel that Penny has broken her resolution — in other words, the implication $p \rightarrow q$ is false (and has the truth value 0).

The cases in rows 1 and 2 may not immediately agree with our intuition, but the example should make these results a little easier to accept.

• Row 1: p and q both have the truth value 0. Here Penny finds that on December 26 her weight is 120 pounds or less and she does not enroll in an exercise class. She has not violated her resolution; we take her statement $p \rightarrow q$ to be true and assign it the truth value 1.

• Row 2: p has the truth value 0, q has the truth value 1. This last case finds Penny weighing 120 pounds or less on December 26 but still enrolling in an exercise class. Perhaps her weight is 119 or 120 pounds and she feels this is still too high. Or maybe she wants to join an exercise class because she thinks it will be good for her health. No matter what the reason, she has not gone against her resolution $p \rightarrow q$. Once again, we accept this compound statement as true, assigning it the truth value 1.

Our next example discusses a related notion: the *decision* (or *selection*) structure in computer programming.

EXAMPLE 2.3

In computer science the **if-then** and **if-then-else** decision structures arise (in various formats) in high-level programming languages such as Java and C++. The hypothesis p is often a relational expression such as $x > 2$. This expression then becomes a (logical) statement that has the truth value 0 or 1, depending on the value of the variable x at that point in the program. The conclusion q is usually an "executable statement." (So q is not one of the logical statements that we have been discussing.) When dealing with "**if** p **then** q," in this context, the computer executes q only on the condition that p is true. For p false, the computer goes to the next instruction in the program sequence. For the decision structure "**if** p **then** q **else** r," q is executed when p is true and r is executed when p is false.

Before continuing, a word of caution: Be careful when using the symbols $\rightarrow$ and $\leftrightarrow$. The implication and the biconditional are not the same, as evidenced by the last two columns of Table 2.2.

In our everyday language, however, we often find situations where an implication is used when the intention actually calls for a biconditional. For example, consider the following implications that a certain parent might direct to his or her child.

$s \rightarrow t$: If you do your homework, then you will get to watch the baseball game.

$t \rightarrow s$: You will get to watch the baseball game only if you do your homework.

• Case 1: The implication $s \rightarrow t$. When the parent says to the child, "If you do your homework, then you will get to watch the baseball game," he or she is trying a positive approach by emphasizing the enjoyment in watching the baseball game.

• Case 2: The implication $t \rightarrow s$. Here we find the negative approach and the parent who warns the child in saying, "You will get to watch the baseball game only if you do your homework." This parent places the emphasis on the punishment (lack of enjoyment) to be incurred.

In either case, the parent probably wants his or her implication — be it $s \rightarrow t$ or $t \rightarrow s$ — to be understood as the biconditional $s \leftrightarrow t$. For in case 1 the parent wants to hint at the punishment while promising the enjoyment; in case 2, where the punishment has been used (perhaps, to threaten), if the child does in fact do the homework, then that child will definitely be given the opportunity to enjoy watching the baseball game.

In scientific writing one must make every effort to be unambiguous — when an implication is given, it ordinarily cannot, and should not, be interpreted as a biconditional. Definitions are a notable exception, which we shall discuss in Section 2.5.

Before we continue let us take a step back. When we summarized the material that gave us Tables 2.1 and 2.2, we may not have stressed enough that the results were for any statements p, q — *not* just primitive statements p, q. Examples 2.4 through 2.6 should help to reinforce this.

EXAMPLE 2.4

Let us examine the truth table for the compound statement "Margaret Mitchell wrote *Gone with the Wind*, and if $2 + 3 \neq 5$, then combinatorics is a required course for sophomores." In symbolic notation this statement is written as $q \wedge (\neg r \rightarrow p)$, where p, q, and r represent the primitive statements introduced at the start of this section. The last column of Table 2.3 contains the truth values for this result. We obtained these truth values by using the fact that the conjunction of any two statements is true if and only if both statements are true. This is what we said earlier in Table 2.2, and now one of our statements — namely, the implication $\neg r \rightarrow p$ — is definitely a compound statement, not a primitive one. Columns 4, 5, and 6 in this table show how we build the truth table up by considering smaller parts of the compound statement and by using the results from Tables 2.1 and 2.2.

Table 2.3

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

EXAMPLE 2.5

In Table 2.4 we develop the truth tables for the compound statements $p \vee (q \wedge r)$ (column 5) and $(p \vee q) \wedge r$ (column 7).

Table 2.4

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Because the truth values in columns 5 and 7 differ (in rows 5 and 7), we must avoid writing a compound statement such as $p \vee q \wedge r$. Without parentheses to indicate which of the connectives $\vee$ and $\wedge$ should be applied first, we have no idea whether we are dealing with $p \vee (q \wedge r)$ or $(p \vee q) \wedge r$.

Our last example for this section illustrates two special types of statements.

EXAMPLE 2.6

The results in columns 4 and 7 of Table 2.5 reveal that the statement $p \rightarrow (p \vee q)$ is true and that the statement $p \wedge (\neg p \wedge q)$ is false for all truth value assignments for the component statements p, q.

Table 2.5

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Definition 2.1

A compound statement is called a *tautology* if it is true for all truth value assignments for its component statements. If a compound statement is false for all such assignments, then it is called a *contradiction*.

Throughout this chapter we shall use the symbol T_0 to denote any tautology and the symbol F_0 to denote any contradiction.

We can use the ideas of tautology and implication to describe what we mean by a valid argument. This will be of primary interest to us in Section 2.3, and it will help us develop needed skills for proving mathematical theorems. In general, an argument starts with a list of *given* statements called *premises* and a statement called the *conclusion* of the argument. We examine these premises, say $p_1, p_2, p_3, \ldots, p_n$, and try to show that the conclusion q follows logically from these given statements — that is, we try to show that if each of $p_1, p_2, p_3, \ldots, p_n$ is a true statement, then the statement q is also true. To do so one way is to examine the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n)^\dagger \rightarrow q,$$

where the hypothesis is the conjunction of the n premises. If any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then no matter what truth value q has, the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is true. Consequently, if we start with the premises $p_1, p_2, p_3, \ldots, p_n$ — each with truth value 1 — and find that under these circumstances q also has the value 1, then the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$$

is a *tautology* and we have a *valid argument*.

† At this point we have dealt only with the conjunction of two statements, so we must point out that the conjunction $p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n$ of n statements is true if and only if each p_i, $1 \leq i \leq n$, is true. We shall deal with this generalized conjunction in detail in Example 4.16 of Section 4.2.

EXERCISES 2.1

1. Determine whether each of the following sentences is a statement.

a) In 2003 George W. Bush was the president of the United States.

b) $x + 3$ is a positive integer.

c) Fifteen is an even number.

d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

e) What time is it?

f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

2. Identify the primitive statements in Exercise 1.

3. Let p, q be primitive statements for which the implication $p \rightarrow q$ is false. Determine the truth values for each of the following.

a) $p \wedge q$ **b)** $\neg p \vee q$ **c)** $q \rightarrow p$ **d)** $\neg q \rightarrow \neg p$

4. Let p, q, r, s denote the following statements:

p: I finish writing my computer program before lunch.
q: I shall play tennis in the afternoon.
r: The sun is shining.
s: The humidity is low.

Write the following in symbolic form.

a) If the sun is shining, I shall play tennis this afternoon.

b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

c) Low humidity and sunshine are sufficient for me to play tennis this afternoon.

5. Let p, q, r denote the following statements about a particular triangle ABC.

p: Triangle ABC is isosceles.
q: Triangle ABC is equilateral.
r: Triangle ABC is equiangular.

Translate each of the following into an English sentence.

a) $q \rightarrow p$ **b)** $\neg p \rightarrow \neg q$

c) $q \leftrightarrow r$ **d)** $p \wedge \neg q$

e) $r \rightarrow p$

6. Determine the truth value of each of the following implications.

a) If $3 + 4 = 12$, then $3 + 2 = 6$.

b) If $3 + 3 = 6$, then $3 + 4 = 9$.

c) If Thomas Jefferson was the third president of the United States, then $2 + 3 = 5$.

7. Rewrite each of the following statements as an implication in the **if-then** form.

a) Practicing her serve daily is a sufficient condition for Darci to have a good chance of winning the tennis tournament.

b) Fix my air conditioner or I won't pay the rent.

c) Mary will be allowed on Larry's motorcycle only if she wears her helmet.

8. Construct a truth table for each of the following compound statements, where p, q, r denote primitive statements.

a) $\neg(p \vee \neg q) \rightarrow \neg p$ **b)** $p \rightarrow (q \rightarrow r)$

c) $(p \rightarrow q) \rightarrow r$ **d)** $(p \rightarrow q) \rightarrow (q \rightarrow p)$

e) $[p \wedge (p \rightarrow q)] \rightarrow q$ **f)** $(p \wedge q) \rightarrow p$

g) $q \leftrightarrow (\neg p \vee \neg q)$

h) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

9. Which of the compound statements in Exercise 8 are tautologies?

10. Verify that $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology.

11. a) How many rows are needed for the truth table of the compound statement $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$, where p, q, r, s, and t are primitive statements?

b) Let $p_1, p_2, \ldots, p_n$ denote n primitive statements. Let p be a compound statement that contains at least one occurrence each of p_i, for $1 \leq i \leq n$—and p contains no other primitive statement. How many rows are needed to construct the truth table for p?

12. Determine all truth value assignments, if any, for the primitive statements p, q, r, s, t that make each of the following compound statements false.

a) $[(p \wedge q) \wedge r] \rightarrow (s \vee t)$

b) $[p \wedge (q \wedge r)] \rightarrow (s \veebar t)$

13. If statement q has the truth value 1, determine all truth value assignments for the primitive statements, p, r, and s for which the truth value of the statement

$$(q \rightarrow [(\neg p \vee r) \wedge \neg s]) \wedge [\neg s \rightarrow (\neg r \wedge q)]$$

is 1.

14. At the start of a program (written in pseudocode) the integer variable n is assigned the value 7. Determine the value of n after each of the following *successive* statements is encountered during the execution of this program. [Here the value of n following the execution of the statement in part (a) becomes the value of n for the statement in part (b), and so on, through the statement in part (d). For positive integers a, b, $\lfloor a/b \rfloor$ returns the integer part of the quotient—for example, $\lfloor 6/2 \rfloor = 3$, $\lfloor 7/2 \rfloor = 3$, $\lfloor 2/5 \rfloor = 0$, and $\lfloor 8/3 \rfloor = 2$.]

a) `if n > 5 then n := n + 2`

b) **if** ((n + 2 = 8) **or** (n - 3 = 6)) **then**
```
    n := 2 * n + 1
```
c) **if** ((n - 3 = 16) **and** (⌊n/6⌋ = 1)) **then**
```
    n := n + 3
```
d) **if** ((n ≠ 21) **and** (n - 7 = 15)) **then**
```
    n := n - 4
```

15. The integer variables m and n are assigned the values 3 and 8, respectively, during the execution of a program (written in pseudocode). Each of the following *successive* statements is then encountered during program execution. [Here the values of m, n following the execution of the statement in part (a) become the values of m, n for the statement in part (b), and so on, through the statement in part (e).] What are the values of m, n after each of these statements is encountered?

a) **if** n - m = 5 **then** n := n - 2

b) **if** ((2 * m = n) **and** (⌊n/4⌋ = 1)) **then**
```
    n := 4 * m - 3
```
c) **if** ((n < 8) **or** (⌊m/2⌋= 2)) **then** n := 2 * m
  else m := 2 * n

d) **if** ((m < 20) **and** (⌊n/6⌋ = 1)) **then**
```
    m := m - n - 5
```
e) **if** ((n = 2 * m) **or** (⌊n/2⌋ = 5)) **then**
```
    m := m + 2
```

16. In the following program segment i, j, m, and n are integer variables. The values of m and n are supplied by the user earlier in the execution of the total program.

```
for i := 1 to m do
  for j := 1 to n do
    if i ≠ j then
      print i + j
```

How many times is the **print** statement in the segment executed when (a) $m = 10$, $n = 10$; (b) $m = 20$, $n = 20$; (c) $m = 10$, $n = 20$; (d) $m = 20$, $n = 10$?

17. After baking a pie for the two nieces and two nephews who are visiting her, Aunt Nellie leaves the pie on her kitchen table to cool. Then she drives to the mall to close her boutique for the day. Upon her return she finds that someone has eaten one-quarter of the pie. Since no one was in her house that day—except for the four visitors—Aunt Nellie questions each niece and nephew about who ate the piece of pie. The four "suspects" tell her the following:

Charles:	Kelly ate the piece of pie.
Dawn:	I did not eat the piece of pie.
Kelly:	Tyler ate the pie.
Tyler:	Kelly lied when she said I ate the pie.

If only one of these four statements is true and only one of the four committed this heinous crime, who is the vile culprit that Aunt Nellie will have to punish severely?

2.2 Logical Equivalence: The Laws of Logic

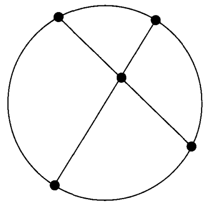
In all areas of mathematics we need to know when the entities we are studying are equal or essentially the same. For example, in arithmetic and algebra we know that two nonzero real numbers are equal when they have the same magnitude and algebraic sign. Hence, for two nonzero real numbers x, y, we have $x = y$ if $|x| = |y|$ and $xy > 0$, and conversely (that is, if $x = y$, then $|x| = |y|$ and $xy > 0$). When we deal with triangles in geometry, the notion of congruence arises. Here triangle ABC and triangle DEF are congruent if, for instance, they have equal corresponding sides—that is, the length of side AB = the length of side DE, the length of side BC = the length of side EF, and the length of side CA = the length of side FD.

Our study of logic is often referred to as the *algebra of propositions* (as opposed to the algebra of real numbers). In this algebra we shall use the truth tables of the statements, or propositions, to develop an idea of when two such entities are essentially the same. We begin with an example.

EXAMPLE 2.7

For primitive statements p and q, Table 2.6 provides the truth tables for the compound statements $\neg p \vee q$ and $p \rightarrow q$. Here we see that the corresponding truth tables for the two statements $\neg p \vee q$ and $p \rightarrow q$ are exactly the same.

intersection for all possible chords is $\binom{12}{4} = 495$.



5. a) 10^{25} **b)** $(10)(11)(12)\cdots(34) = 34!/9!$ **c)** $(25!)\binom{24}{9}$
7. a) $C(12, 8)$ **b)** $P(12, 8)$ **9. a)** 12 **b)** 49
11. $(1/11)[11!/(5!\,3!\,3!)]$
13. a) (i) $\binom{5}{4} + \binom{5}{2}\binom{4}{2} + \binom{4}{4}$ (ii) $\binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4}$ (iii) $\binom{8}{4} + \binom{6}{2}\binom{5}{2} + \binom{7}{4} - 9$
b) (i) $\binom{5}{1}\binom{4}{3} + \binom{5}{3}\binom{4}{1}$ (ii) and (iii) $\binom{5}{1}\binom{6}{3} + \binom{7}{3}\binom{4}{1}$
15. a) $2\binom{9}{4} + \binom{9}{3} = 343$ **b)** $[2\binom{12}{4} - 9] + [\binom{12}{3} - 1] = 1200$
17. a) $(5)(9!)$ **b)** $(3)(8!)$
19. a) $\binom{4}{2}7^5$ **b)** $2[\binom{3}{2}7^4 + \binom{4}{2}7^5]$
21. $0 = (1 + (-1))^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n\binom{n}{n}$, so
$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$
23. a) $P(20, 12) = 20!/8!$ **b)** $\binom{17}{9}(12!)$
25. a) $\binom{9}{1} + \binom{10}{3} + \cdots + \binom{16}{15} + \binom{17}{17} = \sum_{k=0}^{8}\binom{9+k}{1+2k}$ **b)** $\sum_{k=0}^{9}\binom{9+k}{2k}$
c) $n = 2k + 1,\ k \geq 0$: $\sum_{i=0}^{k}\binom{k+1+i}{1+2i}$
$n = 2k,\ k \geq 1$: $\sum_{i=0}^{k}\binom{k+i}{2i}$
27. a) $\binom{r+(n-r)-1}{n-r} = \binom{n-1}{n-r} = \binom{n-1}{r-1}$
b) $\sum_{r=1}^{n}\binom{n-1}{r-1} = \binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1} = 2^{n-1}$
29. a) $11!/(7!\,4!)$ **b)** $[11!/(7!\,4!)] - [4!/(2!\,2!)][4!/(3!\,1!)]$
c) $[11!/(7!\,4!)] + [10!/(6!\,3!\,1!)] + [9!/(5!\,2!\,2!)] + [8!/(4!\,1!\,3!)] + [7!/(3!\,4!)]$ [in part (a)]
$\{[11!/(7!\,4!)] + [10!/(6!\,3!\,1!)] + [9!/(5!\,2!\,2!)] + [8!/(4!\,1!\,3!)] + [7!/(3!\,4!)]\}$
$- [\{[4!/(2!\,2!)] + [3!/(1!\,1!\,1!)] + [2!/2!]\} \times \{[4!/(3!\,1!)] + [3!/(2!\,1!)]\}]$
[in part (b)]
31. $\binom{9}{2}\binom{6}{2} = 540$ **33.** $\binom{6}{4}(12)(11)(10)(9) = 178{,}200$

Chapter 2
Fundamentals of Logic

Section 2.1 – p. 54

1. The sentences in parts (a), (c), (d), and (f) are statements. The other two sentences are not.
3. a) 0 **b)** 0 **c)** 1 **d)** 0
5. a) If triangle ABC is equilateral, then it is isosceles.
b) If triangle ABC is not isosceles, then it is not equilateral.
d) Triangle ABC is isosceles, but it is not equilateral.
7. a) If Darci practices her serve daily then she will have a good chance of winning the tennis tournament.
b) If you do not fix my air conditioner, then I shall not pay the rent.
c) If Mary is to be allowed on Larry's motorcycle, then she must wear her helmet.
9. Statements (a), (e), (f), and (h) are tautologies.
11. a) $2^5 = 32$ **b)** 2^n **13.** p: 0; r: 0; s: 0
15. a) $m = 3, n = 6$ **b)** $m = 3, n = 9$ **c)** $m = 18, n = 9$ **d)** $m = 4, n = 9$
e) $m = 4, n = 9$
17. Dawn