12. Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.

a) If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

b) If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore (sometime today) Lois will mow her lawn.

13. a) Given primitive statements p, q, r, show that the implication

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

is a tautology.

b) The tautology in part (a) provides the rule of inference known as *resolution*, where the conclusion $(q \vee r)$ is called the *resolvent*. This rule was proposed in 1965 by J. A. Robinson and is the basis of many computer programs designed to automate a reasoning system.

In applying resolution each premise (in the hypothesis) and the conclusion are written as *clauses*. A clause is a primitive statement or its negation, or it is the disjunction of terms each of which is a primitive statement or the negation of such a statement. Hence the given rule has the clauses $(p \vee q)$ and $(\neg p \vee r)$ as premises and the clause $(q \vee r)$ as its conclusion (or, resolvent). Should we have the premise $\neg(p \wedge q)$, we replace this by the logically equivalent clause $\neg p \vee \neg q$, by the first of DeMorgan's Laws. The premise $\neg(p \vee q)$ can be replaced by the two clauses $\neg p$, $\neg q$. This is due to the second DeMorgan Law and the Rule of Conjunctive Simplification. For the premise $p \vee (q \wedge r)$, we apply the Distributive Law of $\vee$ over $\wedge$ and the Rule of Conjunctive Simplification to arrive at either of the two clauses $p \vee q$, $p \vee r$. Finally, the premise $p \rightarrow q$ becomes the clause $\neg p \vee q$.

Establish the validity of the following arguments, using resolution (along with the rules of inference and the laws of logic).

(i)
$$\begin{array}{l} p \vee (q \wedge r) \\ \underline{p \rightarrow s} \\ \therefore r \vee s \end{array}$$

(ii)
$$\begin{array}{l} p \\ \underline{p \leftrightarrow q} \\ \therefore q \end{array}$$

(iii)
$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ \underline{r \rightarrow s} \\ \therefore q \vee s \end{array}$$

(iv)
$$\begin{array}{l} \neg p \vee q \vee r \\ \neg q \\ \underline{\neg r} \\ \therefore \neg p \end{array}$$

(v)
$$\begin{array}{l} \neg p \vee s \\ \neg t \vee (s \wedge r) \\ \neg q \vee r \\ \underline{p \vee q \vee t} \\ \therefore r \vee s \end{array}$$

c) Write the following argument in symbolic form, then use resolution (along with the rules of inference and the laws of logic) to establish its validity.

Jonathan does not have his driver's license or his new car is out of gas. Jonathan has his driver's license or he does not like to drive his new car. Jonathan's new car is not out of gas or he does not like to drive his new car. Therefore, Jonathan does not like to drive his new car.

2.4 The Use of Quantifiers

In Section 2.1, we mentioned how sentences that involve a variable, such as x, need not be statements. For example, the sentence "The number $x + 2$ is an even integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by -5, -1, or 3, for instance, the resulting statement is false. In fact, it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, however, the resulting statement is true.

We refer to the sentence "The number $x + 2$ is an even integer" as an *open statement*, which we formally define as follows.

Definition 2.5 A declarative sentence is an *open statement* if

1) it contains one or more variables, and

2) it is not a statement, but
3) it becomes a statement when the variables in it are replaced by certain allowable choices.

When we examine the sentence "The number $x + 2$ is an even integer" in light of this definition, we find it is an open statement that contains the single variable x. With regard to the third element of the definition, in our earlier discussion we restricted the "certain allowable choices" to integers. These allowable choices constitute what is called the *universe* or *universe of discourse* for the open statement. The universe comprises the choices we wish to consider or allow for the variable(s) in the open statement. (The universe is an example of a *set*, a concept we shall examine in some detail in the next chapter.)

In dealing with open statements, we use the following notation:

The open statement "The number $x + 2$ is an even integer" is denoted by $p(x)$ [or $q(x)$, $r(x)$, etc.]. Then $\neg p(x)$ may be read "The number $x + 2$ is *not* an even integer."

We shall use $q(x, y)$ to represent an open statement that contains two variables. For example, consider

$q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 2y$ are even integers.

In the case of $q(x, y)$, there is more than one occurrence of each of the variables x, y. It is understood that when we replace one of the x's by a choice from our universe, we replace the other x by the same choice. Likewise, when a substitution (from the universe) is made for one occurrence of y, that same substitution is made for all other occurrences of the variable y.

With $p(x)$ and $q(x, y)$ as above, and the universe still stipulating the integers as our only allowable choices, we get the following results when we make some replacements for the variables x, y.

$p(5)$: The number $7 (= 5 + 2)$ is an even integer. (FALSE)

$\neg p(7)$: The number 9 is not an even integer. (TRUE)

$q(4, 2)$: The numbers 4, 2, and 8 are even integers. (TRUE)

We also note, for example, that $q(5, 2)$ and $q(4, 7)$ are both false statements, whereas $\neg q(5, 2)$ and $\neg q(4, 7)$ are true.

Consequently, we see that for both $p(x)$ and $q(x, y)$, as already given, some substitutions result in true statements and others in false statements. Therefore we can make the following true statements.

1) For some x, $p(x)$.

2) For some x, y, $q(x, y)$.

Note that in this situation, the statements "For some x, $\neg p(x)$" and "For some x, y, $\neg q(x, y)$" are also true. [Since the statements "For some x, $p(x)$" and "For some x, $\neg p(x)$" are both true, we realize that the second statement is *not* the negation of the first—even though the open statement $\neg p(x)$ is the negation of the open statement $p(x)$. And a similar result is true for the statements involving $q(x, y)$ and $\neg q(x, y)$.]

The phrases "For some x" and "For some x, y" are said to *quantify* the open statements $p(x)$ and $q(x, y)$, respectively. Many postulates, definitions, and theorems in mathematics involve statements that are quantified open statements. These result from the two types of *quantifiers*, which are called the *existential* and the *universal quantifiers*.

Statement (1) uses the *existential quantifier* "For some x," which can also be expressed as "For at least one x" or "There exists an x such that." This quantifier is written in symbolic form as $\exists x$. Hence the statement "For some x, $p(x)$" becomes $\exists x\ p(x)$, in symbolic form.

Statement (2) becomes $\exists x\ \exists y\ q(x, y)$ in symbolic form. The notation $\exists x, y$ can be used to abbreviate $\exists x\ \exists y\ q(x, y)$ to $\exists x, y\ q(x, y)$.

The *universal quantifier* is denoted by $\forall x$ and is read "For all x," "For any x," "For each x," or "For every x." "For all x, y," "For any x, y," "For every x, y," or "For all x and y" is denoted by $\forall x\ \forall y$, which can be abbreviated to $\forall x, y$.

Taking $p(x)$ as defined earlier and using the universal quantifier, we can change the open statement $p(x)$ into the (quantified) statement $\forall x\ p(x)$, a false statement.

If we consider the open statement $r(x)$: "$2x$ is an even integer" with the same universe (of all integers), then the (quantified) statement $\forall x\ r(x)$ is a true statement. When we say that $\forall x\ r(x)$ is true, we mean that no matter which integer (from our universe) is substituted for x in $r(x)$, the resulting statement is true. Also note that the statement $\exists x\ r(x)$ is a true statement, whereas $\forall x\ \neg r(x)$ and $\exists x\ \neg r(x)$ are both false.

The variable x in each of open statements $p(x)$ and $r(x)$ is called a *free variable* (of the open statement). As x varies over the universe for an open statement, the truth value of the statement (that results upon the replacement of each occurrence of x) may vary. For instance, in the case of $p(x)$, we found $p(5)$ to be false—while $p(6)$ turns out to be a true statement. The open statement $r(x)$, however, becomes a true statement for every replacement (for x) taken from the universe of all integers. In contrast to the open statement $p(x)$ the statement $\exists x\ p(x)$ has a fixed truth value—namely, true. And in the symbolic representation $\exists x\ p(x)$ the variable x is said to be a *bound* variable—it is bound by the existential quantifier $\exists$. This is also the case for the statements $\forall x\ r(x)$ and $\forall x\ \neg r(x)$, where in each case the variable x is bound by the universal quantifier $\forall$.

For the open statement $q(x, y)$ we have two free variables, each of which is bound by the quantifier $\exists$ in either of the statements $\exists x\ \exists y\ q(x, y)$ or $\exists x, y\ q(x, y)$.

The following example shows how these new ideas about quantifiers can be used in conjunction with the logical connectives.

EXAMPLE 2.36

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & x \geq 0 & r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & s(x): & x^2 - 3 > 0. \end{array}$$

Then the following statements are true.

1) $$\exists x\ [p(x) \wedge r(x)]$$

This follows because the real number 4, for example, is a member of the universe and is such that both of the statements $p(4)$ and $r(4)$ are true.

2) $$\forall x\ [p(x) \rightarrow q(x)]$$

If we replace x in $p(x)$ by a negative real number a, then $p(a)$ is false, but $p(a) \rightarrow q(a)$ is true regardless of the truth value of $q(a)$. Replacing x in $p(x)$ by a nonnegative real number b, we find that $p(b)$ and $q(b)$ are both true, as is $p(b) \rightarrow q(b)$. Consequently, $p(x) \rightarrow q(x)$ is true for all replacements x taken from the universe of all real numbers, and the (quantified) statement $\forall x\ [p(x) \rightarrow q(x)]$ is true.

This statement may be translated into any of the following:

a) For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a nonnegative square.

c) The square of any nonnegative real number is a nonnegative real number.

d) All nonnegative real numbers have nonnegative squares.

Also, the statement $\exists x\,[p(x) \rightarrow q(x)]$ is true.

The next statements we examine are false.

1′) $$\forall x\,[q(x) \rightarrow s(x)]$$

We want to show that the statement is false, so we need exhibit only one *counterexample*—that is, *one value of* x for which $q(x) \rightarrow s(x)$ is false—rather than prove something for all x as we did for statement (2). Replacing x by 1, we find that $q(1)$ is true and $s(1)$ is false. Therefore $q(1) \rightarrow s(1)$ is false, and consequently the (quantified) statement $\forall x\,[q(x) \rightarrow s(x)]$ is false. [Note that $x = 1$ does not produce the only counterexample: Every real number a between $-\sqrt{3}$ and $\sqrt{3}$ will make $q(a)$ true and $s(a)$ false.]

2′) $$\forall x\,[r(x) \vee s(x)]$$

Here there are many values for x, such as $1, \frac{1}{2}, -\frac{3}{2}$, and 0, that produce counterexamples. Upon changing quantifiers, however, we find that the statement $\exists x\,[r(x) \vee s(x)]$ is true.

3′) $$\forall x\,[r(x) \rightarrow p(x)]$$

The real number -1 is a solution of the equation $x^2 - 3x - 4 = 0$, so $r(-1)$ is true while $p(-1)$ is false. Therefore the choice of -1 provides the unique counterexample we need to show that this (quantified) statement is false.

Statement (3′) may be translated into either of the following:

a) For every real number x, if $x^2 - 3x - 4 = 0$, then $x \geq 0$.

b) For every real number x, if x is a solution of the equation $x^2 - 3x - 4 = 0$, then $x \geq 0$.

Now we make the following observations. Let $p(x)$ denote any open statement (in the variable x) with a prescribed *nonempty* universe (that is, the universe contains at least one member). Then if $\forall x\; p(x)$ is true, so is $\exists x\; p(x)$, or

$$\forall x\; p(x) \Rightarrow \exists x\; p(x).$$

When we write $\forall x\; p(x) \Rightarrow \exists x\; p(x)$ we are saying that the implication $\forall x\; p(x) \rightarrow \exists x\; p(x)$ is a logical implication—that is, $\exists x\; p(x)$ is true whenever $\forall x\; p(x)$ is true. Also, we realize that the hypothesis of this implication is the quantified *statement* $\forall x\; p(x)$, and the conclusion is $\exists x\; p(x)$, another quantified *statement*. On the other hand, it does not follow that if $\exists x\; p(x)$ is true, then $\forall x\; p(x)$ must be true. Hence $\exists x\; p(x)$ does not logically imply $\forall x\; p(x)$, in general.

Our next example brings out the fact that the quantification of an open statement may not be as explicit as we might prefer.

EXAMPLE 2.37

a) Let us consider the universe of all real numbers and examine the sentences:

1) If a number is rational, then it is a real number.

2) If x is rational, then x is real.

We should agree that these sentences convey the same information. But we should also question whether the sentences are statements or open statements. In the case of sentence (2) we at least have the presence of the variable x. But neither sentence contains an expression such as "For all," or "For every," or "For each." Our one and only clue to indicate that we are dealing with universally quantified statements here is the presence of the indefinite article "a" in the first sentence. In situations like these the use of the universal quantifier is *implicit* as opposed to *explicit*.

If we let $p(x)$, $q(x)$ be the open statements

$p(x)$: x is a rational number $\qquad$ $q(x)$: x is a real number,

then we must recognize the fact that both of the given sentences are somewhat informal ways of expressing the quantified statement

$$\forall x\ [p(x) \rightarrow q(x)].$$

b) For the universe of all triangles in the plane, the sentence

"An equilateral triangle has three angles of 60°, and conversely."

provides another instance of implicit quantification. Here the indefinite article "An" is the only indication that we might be able to express this sentence as a statement with a universal quantifier. If the open statements

$e(t)$: Triangle t is equilateral.

$a(t)$: Triangle t has three angles of 60°.

are defined for this universe, then the given sentence can be written in the explicit quantified form

$$\forall t\ [e(t) \leftrightarrow a(t)].$$

c) In the typical trigonometry textbook one often comes across the trigonometric identity

$$\sin^2 x + \cos^2 x = 1.$$

This identify contains no explicit quantification, and the reader must understand or be told that it is defined for all real numbers x. When the universe of all real numbers is specified (or at least understood), then the identity can be expressed by the (explicitly) quantified statement

$$\forall x\ [\sin^2 x + \cos^2 x = 1].$$

d) Finally, consider the universe of all positive integers and the sentence

"The integer 41 is equal to the sum of two perfect squares."

Here we have one more example where the quantification is implicit—but this time the quantification is existential. We may express the result here in a more formal (and symbolic) manner as

$$\exists m\ \exists n\ [41 = m^2 + n^2].$$

The next example demonstrates that the truth value of a quantified statement may depend on the universe prescribed.

EXAMPLE 2.38

Consider the open statement $p(x)$: $x^2 \geq 1$.

1) If the universe consists of all positive integers, then the quantified statement $\forall x\ p(x)$ is true.
2) For the universe of all positive real numbers, however, the same quantified statement $\forall x\ p(x)$ is false. The positive real number $1/2$ provides one of many possible counterexamples.

Yet for either universe, the quantified statement $\exists x\ p(x)$ is true.

One use of quantifiers in a computer science setting is illustrated in the following example.

EXAMPLE 2.39

In the following program segment, n is an integer variable and the variable A is an array $A[1], A[2], \ldots, A[20]$ of 20 integer values.

```
for n := 1 to 20 do
  A[n] := n * n - n
```

The following statements about the array A can be represented in quantified form, where the universe consists of all integers from 1 to 20, inclusive.

1) Every entry in the array is nonnegative:

$$\forall n\ (A[n] \geq 0).$$

2) There exist two consecutive entries in A where the larger entry is twice the smaller:

$$\exists n\ (A[n+1] = 2A[n]).$$

3) The entries in the array are sorted in (strictly) ascending order:

$$\forall n\ [(1 \leq n \leq 19) \rightarrow (A[n] < A[n+1])].$$

Our last statement requires the use of two integer variables m, n.

4) The entries in the array are distinct:

$$\forall m\ \forall n\ [(m \neq n) \rightarrow (A[m] \neq A[n])], \quad \text{or}$$
$$\forall m, n\ [(m < n) \rightarrow (A[m] \neq A[n])].$$

Before continuing, we summarize and somewhat extend, in Table 2.21, what we have learned about quantifiers.

The results in Table 2.21 may appear to involve only one open statement. However, we should realize that the open statement $p(x)$ in the table may stand for a conjunction of open statements, such as $q(x) \wedge r(x)$, or an implication of open statements, such as $s(x) \rightarrow t(x)$. If, for example, we want to know when the statement $\exists x\ [s(x) \rightarrow t(x)]$ is true, then we look at the table for $\exists x\ p(x)$ and use the information provided there. The table tells us that $\exists x\ [s(x) \rightarrow t(x)]$ is true when $s(a) \rightarrow t(a)$ is true for some (at least one) a in the prescribed universe.

We will look further into quantified statements involving more than one open statement. Before doing so, however, we need to examine the following definition. This definition is comparable to Definitions 2.2 and 2.4 where we defined the ideas of logically equivalent statements and logical implication. It settles the same types of questions for open statements.

Table 2.21

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

Definition 2.6

Let $p(x), q(x)$ be open statements defined for a given universe.

The open statements $p(x)$ and $q(x)$ are called (*logically*) *equivalent*, and we write $\forall x\ [p(x) \Longleftrightarrow q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe (that is, $p(a) \Longleftrightarrow q(a)$ for each a in the universe). If the implication $p(a) \to q(a)$ is true for each a in the universe (that is, $p(a) \Rightarrow q(a)$ for each a in the universe), then we write $\forall x\ [p(x) \Rightarrow q(x)]$ and say that $p(x)$ *logically implies* $q(x)$.

For the universe of all triangles in the plane, let $p(x), q(x)$ denote the open statements

$$p(x):\quad x \text{ is equiangular} \qquad\qquad q(x):\quad x \text{ is equilateral.}$$

Then for every particular triangle a (a replacement for x) we know that $p(a) \leftrightarrow q(a)$ is true (that is, $p(a) \Longleftrightarrow q(a)$, for every triangle in the plane). Consequently, $\forall x\ [p(x) \Longleftrightarrow q(x)]$.

Observe that here and, in general, $\forall x\ [p(x) \Longleftrightarrow q(x)]$ if and only if $\forall x\ [p(x) \Rightarrow q(x)]$ and $\forall x\ [q(x) \Rightarrow p(x)]$.

We also realize that a definition similar to Definition 2.6 can be given for two open statements that involve two or more variables.

Now we take another look at the logical equivalence of statements (not open statements) as we examine the converse, inverse, and contrapositive of a statement of the form $\forall x\ [p(x) \to q(x)]$.

Definition 2.7

For open statements $p(x), q(x)$—defined for a prescribed universe—and the universally quantified statement $\forall x\ [p(x) \to q(x)]$, we define:

1) The *contrapositive* of $\forall x\ [p(x) \to q(x)]$ to be $\forall x\ [\neg q(x) \to \neg p(x)]$.
2) The *converse* of $\forall x\ [p(x) \to q(x)]$ to be $\forall x\ [q(x) \to p(x)]$.
3) The *inverse* of $\forall x\ [p(x) \to q(x)]$ to be $\forall x\ [\neg p(x) \to \neg q(x)]$.

The following two examples illustrate Definition 2.7.

EXAMPLE 2.40

For the universe of all quadrilaterals in the plane let $s(x)$ and $e(x)$ denote the open statements

$$s(x): \quad x \text{ is a square} \qquad e(x): \quad x \text{ is equilateral.}$$

a) The statement

$$\forall x\,[s(x) \rightarrow e(x)]$$

is a true statement and is logically equivalent to its contrapositive

$$\forall x\,[\neg e(x) \rightarrow \neg s(x)]$$

because $[s(a) \rightarrow e(a)] \Leftrightarrow [\neg e(a) \rightarrow \neg s(a)]$ for each replacement a. Hence

$$\forall x\,[s(x) \rightarrow e(x)] \Leftrightarrow \forall x\,[\neg e(x) \rightarrow \neg s(x)].$$

b) The statement

$$\forall x\,[e(x) \rightarrow s(x)]$$

is a false statement and is the converse of the true statement

$$\forall x\,[s(x) \rightarrow e(x)].$$

The false statement

$$\forall x\,[\neg s(x) \rightarrow \neg e(x)]$$

is the inverse of the given statement $\forall x\,[s(x) \rightarrow e(x)]$.

Since $[e(a) \rightarrow s(a)] \Leftrightarrow [\neg s(a) \rightarrow \neg e(a)]$ for each specific quadrilateral a, we find that the converse and inverse are logically equivalent—that is,

$$\forall x\,[e(x) \rightarrow s(x)] \Leftrightarrow \forall x\,[\neg s(x) \rightarrow \neg e(x)].$$

EXAMPLE 2.41

Here $p(x)$ and $q(x)$ are the open statements

$$p(x): \quad |x| > 3 \qquad q(x): \quad x > 3$$

and the universe consists of all real numbers.

a) The statement $\forall x\,[p(x) \rightarrow q(x)]$ is a false statement. For example, if $x = -5$, then $p(-5)$ is true while $q(-5)$ is false. Consequently, $p(-5) \rightarrow q(-5)$ is false, and so is $\forall x\,[p(x) \rightarrow q(x)]$.

b) We can express the converse of the given statement [in part (a)] as follows:

Every real number greater than 3 has magnitude
(or, absolute value) greater than 3.

In symbolic form this true statement is written $\forall x\,[q(x) \rightarrow p(x)]$.

c) The inverse of the given statement is also a true statement. In symbolic form we have $\forall x\,[\neg p(x) \rightarrow \neg q(x)]$, which can be expressed in words by

If the magnitude of a real number is less than or equal to 3,
then the number itself is less than or equal to 3.

And this is logically equivalent to the (converse) statement given in part (b).

d) Here the contrapositive of the statement in part (a) is given by $\forall x\,[\neg q(x) \rightarrow \neg p(x)]$. This false statement is logically equivalent to $\forall x\,[p(x) \rightarrow q(x)]$ and can be expressed as follows:

If a real number is less than or equal to 3, then so is its magnitude.

e) Together with $p(x)$ and $q(x)$ as above, consider the open statement

$$r(x): \quad x < -3,$$

which is also defined for the universe of all real numbers. The following four statements are all true:

Statement:	$\forall x\, [p(x) \rightarrow (r(x) \vee q(x))]$
Contrapositive:	$\forall x\, [\neg(r(x) \vee q(x)) \rightarrow \neg p(x)]$
Converse:	$\forall x\, [(r(x) \vee q(x)) \rightarrow p(x)]$
Inverse:	$\forall x\, [\neg p(x) \rightarrow \neg(r(x) \vee q(x))]$

In this case (because the statement and its converse are both true) we find that the statement $\forall x\, [p(x) \leftrightarrow (r(x) \vee q(x))]$ is true.

Now we use the results of Table 2.21 once again as we examine the next example.

EXAMPLE 2.42

Here the universe consists of all the integers, and the open statements $r(x)$, $s(x)$ are given by

$$r(x): \quad 2x + 1 = 5 \qquad s(x): \quad x^2 = 9.$$

We see that the statement $\exists x\, [r(x) \wedge s(x)]$ is false because there is no one integer a such that $2a + 1 = 5$ and $a^2 = 9$. However, there is an integer b $(= 2)$ such that $2b + 1 = 5$, and there is a second integer c $(= 3$ or $-3)$ such that $c^2 = 9$. Therefore the statement $\exists x\, r(x) \wedge \exists x\, s(x)$ is true. Consequently, the existential quantifier $\exists x$ does not distribute over the logical connective $\wedge$. This one counterexample is enough to show that

$$\exists x\, [r(x) \wedge s(x)] \not\Leftrightarrow [\exists x\, r(x) \wedge \exists x\, s(x)],$$

where $\not\Leftrightarrow$ is read "is *not* logically equivalent to." It also demonstrates that

$$[\exists x\, r(x) \wedge \exists x\, s(x)] \not\Rightarrow \exists x\, [r(x) \wedge s(x)],$$

where $\not\Rightarrow$ is read "does *not* logically imply." So the statement

$$[\exists x\, r(x) \wedge \exists x\, s(x)] \rightarrow \exists x\, [r(x) \wedge s(x)]$$

is *not* a tautology.

What, however, can we say about the converse of a quantified statement of this form? At this point we present a general argument for *any* (arbitrary) open statements $p(x)$, $q(x)$ and *any* (arbitrary) prescribed universe.

Examining the statement

$$\exists x\, [p(x) \wedge q(x)] \rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)],$$

we find that when the hypothesis $\exists x\, [p(x) \wedge q(x)]$ is true, there is at least one element c in the universe for which the statement $p(c) \wedge q(c)$ is true. By the Rule of Conjunctive Simplification (see Section 2.3), $[p(c) \wedge q(c)] \Rightarrow p(c)$. From the truth of $p(c)$ we have the true statement $\exists x\, p(x)$. Similarly we obtain $\exists x\, q(x)$, another true statement. So $\exists x\, p(x) \wedge$

$\exists x\, q(x)$ is a true statement. Since $\exists x\, p(x) \wedge \exists x\, q(x)$ is true whenever $\exists x\, [p(x) \wedge q(x)]$ is true, it follows that

$$\exists x\, [p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)].$$

Arguments similar to the one for Example 2.42 provide the logical equivalences and logical implications listed in Table 2.22. In addition to those listed in Table 2.22 many other logical equivalences and logical implications can be derived.

Table 2.22 Logical Equivalences and Logical Implications for Quantified Statements in One Variable

For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x:
$\exists x\, [p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$
$\exists x\, [p(x) \vee q(x)] \Longleftrightarrow [\exists x\, p(x) \vee \exists x\, q(x)]$
$\forall x\, [p(x) \wedge q(x)] \Longleftrightarrow [\forall x\, p(x) \wedge \forall x\, q(x)]$
$[\forall x\, p(x) \vee \forall x\, q(x)] \Rightarrow \forall x\, [p(x) \vee q(x)]$

Our next example lists several of these and demonstrates how two of them are verified.

EXAMPLE 2.43

Let $p(x)$, $q(x)$, and $r(x)$ denote open statements for a given universe. We find the following logical equivalences. (Many more are also possible.)

1) $\forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Longleftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)]$
To show that this statement is a logical equivalence we proceed as follows:
For each a in the universe, consider the *statements* $p(a) \wedge (q(a) \wedge r(a))$ and $(p(a) \wedge q(a)) \wedge r(a)$. By the Associative Law for $\wedge$, we have

$$p(a) \wedge (q(a) \wedge r(a)) \Longleftrightarrow (p(a) \wedge q(a)) \wedge r(a).$$

Consequently, for the *open statements* $p(x) \wedge (q(x) \wedge r(x))$ and $(p(x) \wedge q(x)) \wedge r(x)$, it follows that

$$\forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Longleftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)].$$

2) $\exists x\, [p(x) \rightarrow q(x)] \Longleftrightarrow \exists x\, [\neg p(x) \vee q(x)]$
For each c in the universe, it follows from Example 2.7 that

$$[p(c) \rightarrow q(c)] \Longleftrightarrow [\neg p(c) \vee q(c)].$$

Therefore the statement $\exists x\, [p(x) \rightarrow q(x)]$ is true (respectively, false) if and only if the statement $\exists x\, [\neg p(x) \vee q(x)]$ is true (respectively, false), so

$$\exists x\, [p(x) \rightarrow q(x)] \Longleftrightarrow \exists x\, [\neg p(x) \vee q(x)].$$

3) Other logical equivalences that we shall often find useful include the following.

a) $\forall x\, \neg\neg p(x) \Longleftrightarrow \forall x\, p(x)$
b) $\forall x\, \neg[p(x) \wedge q(x)] \Longleftrightarrow \forall x\, [\neg p(x) \vee \neg q(x)]$
c) $\forall x\, \neg[p(x) \vee q(x)] \Longleftrightarrow \forall x\, [\neg p(x) \wedge \neg q(x)]$

4) The results for the logical equivalences in 3(a), (b), and (c) remain valid when all of the universal quantifiers are replaced by existential quantifiers.

The results of Tables 2.21 and 2.22 and Examples 2.42 and 2.43 will now help us with a very important concept. How do we negate quantified statements that involve a single variable?

Consider the statement $\forall x\ p(x)$. Its negation—namely, $\neg[\forall x\ p(x)]$—can be stated as "It is not the case that for all x, $p(x)$ holds." This is not a very useful remark, so we consider $\neg[\forall x\ p(x)]$ further. When $\neg[\forall x\ p(x)]$ is true, then $\forall x\ p(x)$ is false, and so for some replacement a from the universe $\neg p(a)$ is true and $\exists x\ \neg p(x)$ is true. Conversely, whenever the statement $\exists x\ \neg p(x)$ is true we know that $\neg p(b)$ is true for some member b of the universe. Hence $\forall x\ p(x)$ is false and $\neg[\forall x\ p(x)]$ is true. So the statement $\neg[\forall x\ p(x)]$ is true if and only if the statement $\exists x\ \neg p(x)$ is true. (Similar considerations also tell us that $\neg[\forall x\ p(x)]$ is false if and only if $\exists x\ \neg p(x)$ is false.)

These observations lead to the following rule for negating the statement $\forall x\ p(x)$:

$$\neg[\forall x\ p(x)] \Longleftrightarrow \exists x\ \neg p(x).$$

In a similar way, Table 2.21 shows us that the statement $\exists x\ p(x)$ is true (false) precisely when the statement $\forall x\ \neg p(x)$ is false (true). This observation then motivates a rule for negating the statement $\exists x\ p(x)$:

$$\neg[\exists x\ p(x)] \Longleftrightarrow \forall x\ \neg p(x).$$

These two rules for negation, and two others that follow from them, are given in Table 2.23 for convenient reference.

Table 2.23 Rules for Negating Statements with One Quantifier

$\neg[\forall x\ p(x)] \Longleftrightarrow \exists x\ \neg p(x)$
$\neg[\exists x\ p(x)] \Longleftrightarrow \forall x\ \neg p(x)$
$\neg[\forall x\ \neg p(x)] \Longleftrightarrow \exists x\ \neg\neg p(x) \Longleftrightarrow \exists x\ p(x)$
$\neg[\exists x\ \neg p(x)] \Longleftrightarrow \forall x\ \neg\neg p(x) \Longleftrightarrow \forall x\ p(x)$

We use the rules for negating quantified statements in the following example.

EXAMPLE 2.44

Here we find the negation of two statements, where the universe comprises all of the integers.

1) Let $p(x)$ and $q(x)$ be given by

$$p(x):\quad x \text{ is odd} \qquad q(x):\quad x^2 - 1 \text{ is even}.$$

The statement "If x is odd, then $x^2 - 1$ is even" can be symbolized as $\forall x\ [p(x) \rightarrow q(x)]$. (This is a true statement.)

The negation of this statement is determined as follows:

$$\begin{aligned}&\neg[\forall x\ (p(x) \rightarrow q(x))] \Longleftrightarrow \exists x\ [\neg(p(x) \rightarrow q(x))]\\ &\Longleftrightarrow \exists x\ [\neg(\neg p(x) \vee q(x))] \Longleftrightarrow \exists x\ [\neg\neg p(x) \wedge \neg q(x)]\\ &\Longleftrightarrow \exists x\ [p(x) \wedge \neg q(x)]\end{aligned}$$

In words, the negation says, "There exists an integer x such that x is odd and $x^2 - 1$ is odd (that is, not even)." (This statement is false.)

2) As in Example 2.42, let $r(x)$ and $s(x)$ be the open statements

$$r(x):\quad 2x+1=5 \qquad s(x):\quad x^2=9.$$

The quantified statement $\exists x\,[r(x)\wedge s(x)]$ is false because it asserts the existence of at least one integer a such that $2a+1=5$ ($a=2$) and $a^2=9$ ($a=3$ or -3). Consequently, its negation

$$\neg[\exists x\,(r(x)\wedge s(x))] \Longleftrightarrow \forall x\,[\neg(r(x)\wedge s(x))] \Longleftrightarrow \forall x\,[\neg r(x)\vee\neg s(x)]$$

is true. This negation may be given in words as "For every integer x, $2x+1\neq 5$ or $x^2\neq 9$."

Because a mathematical statement may involve more than one quantifier, we continue this section by offering some examples and making some observations on these types of statements.

EXAMPLE 2.45

Here we have two real variables x, y, so the universe consists of all real numbers. The commutative law for the addition of real numbers may be expressed by

$$\forall x\,\forall y\,(x+y=y+x).$$

This statement may also be given as

$$\forall y\,\forall x\,(x+y=y+x).$$

Likewise, in the case of the multiplication of real numbers, we may write

$$\forall x\,\forall y\,(xy=yx) \quad\text{or}\quad \forall y\,\forall x\,(xy=yx).$$

These two examples suggest the following general result. If $p(x, y)$ is an open statement in the two variables x, y (with either a prescribed universe for both x and y or one prescribed universe for x and a second for y), then the statements $\forall x\,\forall y\,p(x, y)$ and $\forall y\,\forall x\,p(x, y)$ are logically equivalent — that is, the statement $\forall x\,\forall y\,p(x, y)$ is true (respectively, false) if and only if the statement $\forall y\,\forall x\,p(x, y)$ is true (respectively, false). Hence

$$\forall x\,\forall y\,p(x, y) \Longleftrightarrow \forall y\,\forall x\,p(x, y).$$

EXAMPLE 2.46

When dealing with the associative law for the addition of real numbers, we find that for all real numbers x, y, and z,

$$x+(y+z)=(x+y)+z.$$

Using universal quantifiers (with the universe of all real numbers), we may express this by

$$\forall x\,\forall y\,\forall z\,[x+(y+z)=(x+y)+z] \quad\text{or}\quad \forall y\,\forall x\,\forall z\,[x+(y+z)=(x+y)+z].$$

In fact, there are $3! = 6$ ways to order these three universal quantifiers, and all six of these quantified statements are logically equivalent to one another.

This is actually true for all open statements $p(x, y, z)$, and to shorten the notation, one may write, for example,

$$\forall x, y, z\;\; p(x, y, z) \Longleftrightarrow \forall y, x, z\;\; p(x, y, z) \Longleftrightarrow \forall x, z, y\;\; p(x, y, z),$$

describing the logical equivalence for three of the six statements.

In Examples 2.45 and 2.46 we encountered quantified statements with two and three bound variables — each such variable bound by a universal quantifier. Our next example examines a situation in which there are two bound variables — and this time each of these variables is bound by an existential quantifier.

EXAMPLE 2.47

For the universe of all integers, consider the true statement "There exist integers x, y such that $x + y = 6$." We may represent this in symbolic form by

$$\exists x\, \exists y\, (x + y = 6).$$

If we let $p(x, y)$ denote the open statement "$x + y = 6$," then an equivalent statement can be given by $\exists y\, \exists x\, p(x, y)$.

In general, for any open statement $p(x, y)$ and universe(s) prescribed for the variables x, y,

$$\exists x\, \exists y\, p(x, y) \Leftrightarrow \exists y\, \exists x\, p(x, y).$$

Similar results follow for statements involving three or more such quantifiers.

When a statement involves both existential and universal quantifiers, however, we must be careful about the order in which the quantifiers are written. Example 2.48 illustrates this case.

EXAMPLE 2.48

We restrict ourselves here to the universe of all integers and let $p(x, y)$ denote the open statement "$x + y = 17$."

1) The statement

$$\forall x\, \exists y\, p(x, y)$$

says that "For every integer x, there exists an integer y such that $x + y = 17$." (We read the quantifiers from left to right.)

This statement is true; once we select *any* x, the integer $y = 17 - x$ does *exist* and $x + y = x + (17 - x) = 17$. But we realize that each value of x gives rise to a different value of y.

2) Now consider the statement

$$\exists y\, \forall x\, p(x, y).$$

This statement is read "There exists an integer y so that for all integers x, $x + y = 17$." This statement is false. Once *an* integer y is selected, the *only* value that x can have (and still satisfy $x + y = 17$) is $17 - y$.

If the statement $\exists y\, \forall x\, p(x, y)$ were true, then every integer (x) would equal $17 - y$ (for some one fixed y). This says, in effect, that all integers are equal!

Consequently, the statements $\forall x\, \exists y\, p(x, y)$ and $\exists y\, \forall x\, p(x, y)$ are generally not logically equivalent.

Translating mathematical statements — be they postulates, definitions, or theorems — into symbolic form can be helpful for two important reasons.

1) Doing so forces us to be very careful and precise about the meanings of statements, the meanings of phrases such as "For all x" and "There exists an x," and the order in which such phrases appear.

2) After we translate a mathematical statement into symbolic form, the rules we have learned should then apply when we want to determine such related statements as the negation or, if appropriate, the contrapositive, converse, or inverse.

Our last two examples illustrate this, and in so doing, extend the results in Table 2.23.

EXAMPLE 2.49

Let $p(x, y)$, $q(x, y)$, and $r(x, y)$ represent three open statements, with replacements for the variables x, y chosen from some prescribed universe(s). What is the negation of the following statement?

$$\forall x \; \exists y \; [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

We find that

$$\begin{aligned} &\neg[\forall x \; \exists y \; [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]] \\ \Longleftrightarrow\; &\exists x \; [\neg\exists y \; [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]] \\ \Longleftrightarrow\; &\exists x \; \forall y \; \neg[(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)] \\ \Longleftrightarrow\; &\exists x \; \forall y \; \neg[\neg[p(x, y) \wedge q(x, y)] \vee r(x, y)] \\ \Longleftrightarrow\; &\exists x \; \forall y \; [\neg\neg[p(x, y) \wedge q(x, y)] \wedge \neg r(x, y)] \\ \Longleftrightarrow\; &\exists x \; \forall y \; [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)]. \end{aligned}$$

Now suppose that we are trying to establish the validity of an argument (or a mathematical theorem) for which

$$\forall x \; \exists y \; [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

is the conclusion. Should we want to try to prove the result by the method of Proof by Contradiction, we would assume as an additional premise the negation of this conclusion. Consequently, our additional premise would be the statement

$$\exists x \; \forall y \; [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)].$$

Finally, we consider how to negate the definition of *limit*, a fundamental concept in calculus.

EXAMPLE 2.50

In calculus, one studies the properties of real-valued functions of a real variable. (Functions will be examined in Chapter 5 of this text.) Among these properties is the existence of limits, and one finds the following definition: Let I be an open interval[†] containing the real number a and suppose the function f is defined throughout I, except possibly at a. We say that f has the *limit* L as x approaches a, and write $\lim_{x \to a} f(x) = L$, if (and only if) for every $\epsilon > 0$ there exists a $\delta > 0$ so that, for all x in I, $(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)$. This can be expressed in symbolic form as

$$\lim_{x \to a} f(x) = L \Longleftrightarrow \forall \epsilon > 0 \; \exists \delta > 0 \; \forall x \; [(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)].$$

[†]The concept of an open interval is defined at the end of Section 3.1.

[Here the universe comprises the real numbers in the open interval I, except possibly a. Also, the quantifiers $\forall\epsilon > 0$ and $\exists\delta > 0$ now contain some restrictive information.] Then, to negate this definition, we do the following (in which certain steps have been combined):

$$\begin{aligned}
&\lim_{x\to a} f(x) \neq L \\
&\Leftrightarrow \neg[\forall\epsilon > 0 \ \exists\delta > 0 \ \forall x \ [(0 < |x-a| < \delta) \rightarrow (|f(x) - L| < \epsilon)]] \\
&\Leftrightarrow \exists\epsilon > 0 \ \forall\delta > 0 \ \exists x \ \neg[(0 < |x-a| < \delta) \rightarrow (|f(x) - L| < \epsilon)] \\
&\Leftrightarrow \exists\epsilon > 0 \ \forall\delta > 0 \ \exists x \ \neg[\neg(0 < |x-a| < \delta) \vee (|f(x) - L| < \epsilon)] \\
&\Leftrightarrow \exists\epsilon > 0 \ \forall\delta > 0 \ \exists x \ [\neg\neg(0 < |x-a| < \delta) \wedge \neg(|f(x) - L| < \epsilon)] \\
&\Leftrightarrow \exists\epsilon > 0 \ \forall\delta > 0 \ \exists x \ [(0 < |x-a| < \delta) \wedge (|f(x) - L| \geq \epsilon)]
\end{aligned}$$

Translating into words, we find that $\lim_{x\to a} f(x) \neq L$ if (and only if) there exists a positive (real) number ϵ such that for every positive (real) number δ, there is an x in I such that $0 < |x - a| < \delta$ (that is, $x \neq a$ and its distance from a is less than δ) but $|f(x) - L| \geq \epsilon$ [that is, the value of $f(x)$ differs from L by at least ϵ].

EXERCISES 2.4

1. Let $p(x)$, $q(x)$ denote the following open statements.

$p(x)$: $x \leq 3$ $\qquad$ $q(x)$: $x + 1$ is odd

If the universe consists of all integers, what are the truth values of the following statements?

a) $q(1)$ **b)** $\neg p(3)$ **c)** $p(7) \vee q(7)$
d) $p(3) \wedge q(4)$ **e)** $\neg(p(-4) \vee q(-3))$
f) $\neg p(-4) \wedge \neg q(-3)$

2. Let $p(x)$, $q(x)$ be defined as in Exercise 1. Let $r(x)$ be the open statement "$x > 0$." Once again the universe comprises all integers.

a) Determine the truth values of the following statements.
- **i)** $p(3) \vee [q(3) \vee \neg r(3)]$
- **ii)** $p(2) \rightarrow [q(2) \rightarrow r(2)]$
- **iii)** $[p(2) \wedge q(2)] \rightarrow r(2)$
- **iv)** $p(0) \rightarrow [\neg q(-1) \leftrightarrow r(1)]$

b) Determine all values of x for which $[p(x) \wedge q(x)] \wedge r(x)$ results in a true statement.

3. Let $p(x)$ be the open statement "$x^2 = 2x$," where the universe comprises all integers. Determine whether each of the following statements is true or false.

a) $p(0)$ **b)** $p(1)$ **c)** $p(2)$
d) $p(-2)$ **e)** $\exists x\ p(x)$ **f)** $\forall x\ p(x)$

4. Consider the universe of all polygons with three or four sides, and define the following open statements for this universe.

$a(x)$: all interior angles of x are equal
$e(x)$: x is an equilateral triangle
$h(x)$: all sides of x are equal
$i(x)$: x is an isosceles triangle
$p(x)$: x has an interior angle that exceeds 180°
$q(x)$: x is a quadrilateral
$r(x)$: x is a rectangle
$s(x)$: x is a square
$t(x)$: x is a triangle

Translate each of the following statements into an English sentence, and determine whether the statement is true or false.

a) $\forall x\ [q(x) \veebar t(x)]$ **b)** $\forall x\ [i(x) \rightarrow e(x)]$
c) $\exists x\ [t(x) \wedge p(x)]$ **d)** $\forall x\ [(a(x) \wedge t(x)) \leftrightarrow e(x)]$
e) $\exists x\ [q(x) \wedge \neg r(x)]$ **f)** $\exists x\ [r(x) \wedge \neg s(x)]$
g) $\forall x\ [h(x) \rightarrow e(x)]$ **h)** $\forall x\ [t(x) \rightarrow \neg p(x)]$
i) $\forall x\ [s(x) \leftrightarrow (a(x) \wedge h(x))]$
j) $\forall x\ [t(x) \rightarrow (a(x) \leftrightarrow h(x))]$

5. Professor Carlson's class in mechanics is comprised of 29 students of which exactly

1) three physics majors are juniors;
2) two electrical engineering majors are juniors;
3) four mathematics majors are juniors;
4) twelve physics majors are seniors;
5) four electrical engineering majors are seniors;
6) two electrical engineering majors are graduate students; and
7) two mathematics majors are graduate students.

Consider the following open statements.

$c(x)$: Student x is in the class (that is, Professor Carlson's mechanics class as already described).

$j(x)$: Student x is a junior.

$s(x)$: Student x is a senior.

$g(x)$: Student x is a graduate student.

$p(x)$: Student x is a physics major.

$e(x)$: Student x is an electrical engineering major.

$m(x)$: Student x is a mathematics major.

Write each of the following statements in terms of quantifiers and the open statements $c(x)$, $j(x)$, $s(x)$, $g(x)$, $p(x)$, $e(x)$, and $m(x)$, and determine whether the given statement is true or false. Here the universe comprises all of the 12,500 students enrolled at the university where Professor Carlson teaches. Furthermore, at this university each student has only one major.

a) There is a mathematics major in the class who is a junior.

b) There is a senior in the class who is not a mathematics major.

c) Every student in the class is majoring in mathematics or physics.

d) No graduate student in the class is a physics major.

e) Every senior in the class is majoring in either physics or electrical engineering.

6. Let $p(x, y)$, $q(x, y)$ denote the following open statements.

$$p(x, y):\quad x^2 \geq y \qquad q(x, y):\quad x + 2 < y$$

If the universe for each of x, y consists of all real numbers, determine the truth value for each of the following statements.

a) $p(2, 4)$ **b)** $q(1, \pi)$

c) $p(-3, 8) \wedge q(1, 3)$ **d)** $p\left(\frac{1}{2}, \frac{1}{3}\right) \vee \neg q(-2, -3)$

e) $p(2, 2) \rightarrow q(1, 1)$ **f)** $p(1, 2) \leftrightarrow \neg q(1, 2)$

7. For the universe of all integers, let $p(x)$, $q(x)$, $r(x)$, $s(x)$, and $t(x)$ be the following open statements.

$p(x)$: $x > 0$

$q(x)$: x is even

$r(x)$: x is a perfect square

$s(x)$: x is (exactly) divisible by 4

$t(x)$: x is (exactly) divisible by 5

a) Write the following statements in symbolic form.

i) At least one integer is even.
ii) There exists a positive integer that is even.
iii) If x is even, then x is not divisible by 5.
iv) No even integer is divisible by 5.
v) There exists an even integer divisible by 5.
vi) If x is even and x is a perfect square, then x is divisible by 4.

b) Determine whether each of the six statements in part (a) is true or false. For each false statement, provide a counterexample.

c) Express each of the following symbolic representations in words.

i) $\forall x\, [r(x) \rightarrow p(x)]$ **ii)** $\forall x\, [s(x) \rightarrow q(x)]$
iii) $\forall x\, [s(x) \rightarrow \neg t(x)]$ **iv)** $\exists x\, [s(x) \wedge \neg r(x)]$

d) Provide a counterexample for each false statement in part (c).

8. Let $p(x)$, $q(x)$, and $r(x)$ denote the following open statements.

$p(x)$: $x^2 - 8x + 15 = 0$

$q(x)$: x is odd

$r(x)$: $x > 0$

For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.

a) $\forall x\, [p(x) \rightarrow q(x)]$ **b)** $\forall x\, [q(x) \rightarrow p(x)]$

c) $\exists x\, [p(x) \rightarrow q(x)]$ **d)** $\exists x\, [q(x) \rightarrow p(x)]$

e) $\exists x\, [r(x) \rightarrow p(x)]$ **f)** $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$

g) $\exists x\, [p(x) \rightarrow (q(x) \wedge r(x))]$

h) $\forall x\, [(p(x) \vee q(x)) \rightarrow r(x)]$

9. Let $p(x)$, $q(x)$, and $r(x)$ be the following open statements.

$p(x)$: $x^2 - 7x + 10 = 0$

$q(x)$: $x^2 - 2x - 3 = 0$

$r(x)$: $x < 0$

a) Determine the truth or falsity of the following statements, where the universe is all integers. If a statement is false, provide a counterexample or explanation.

i) $\forall x\, [p(x) \rightarrow \neg r(x)]$ **ii)** $\forall x\, [q(x) \rightarrow r(x)]$
iii) $\exists x\, [q(x) \rightarrow r(x)]$ **iv)** $\exists x\, [p(x) \rightarrow r(x)]$

b) Find the answers to part (a) when the universe consists of all positive integers.

c) Find the answers to part (a) when the universe contains only the integers 2 and 5.

10. For the following program segment, m and n are integer variables. The variable A is a two-dimensional array $A[1, 1]$, $A[1, 2]$, . . . , $A[1, 20]$, . . . , $A[10, 1]$, . . . , $A[10, 20]$, with 10 rows (indexed from 1 to 10) and 20 columns (indexed from 1 to 20).

```
for m := 1 to 10 do
   for n := 1 to 20 do
      A[m,n] := m + 3 * n
```

Write the following statements in symbolic form. (The universe for the variable m contains only the integers from 1 to 10 inclusive; for n the universe consists of the integers from 1 to 20 inclusive.)

a) All entries of A are positive.

b) All entries of A are positive and less than or equal to 70.

c) Some of the entries of A exceed 60.

d) The entries in each row of A are sorted into (strictly) ascending order.

e) The entries in each column of A are sorted into (strictly) ascending order.

f) The entries in the first three rows of A are distinct.

11. Identify the bound variables and the free variables in each of the following expressions (or statements). In both cases the universe comprises all real numbers.

a) $\forall y\, \exists z\, [\cos(x+y) = \sin(z-x)]$

b) $\exists x\, \exists y\, [x^2 - y^2 = z]$

12. a) Let $p(x, y)$ denote the open statement "x divides y," where the universe for each of the variables x, y comprises all integers. (In this context "divides" means "exactly divides" or "divides evenly.") Determine the truth value of each of the following statements; if a quantified statement is false, provide an explanation or a counterexample.

i) $p(3, 7)$ ii) $p(3, 27)$
iii) $\forall y\, p(1, y)$ iv) $\forall x\, p(x, 0)$
v) $\forall x\, p(x, x)$ vi) $\forall y\, \exists x\, p(x, y)$
vii) $\exists y\, \forall x\, p(x, y)$
viii) $\forall x\, \forall y\, [(p(x, y) \wedge p(y, x)) \rightarrow (x = y)]$

b) Determine which of the eight statements in part (a) will change in truth value if the universe for each of the variables x, y were restricted to just the positive integers.

c) Determine the truth value of each of the following statements. If the statement is false, provide an explanation or a counterexample. [The universe for each of x, y is as in part (b).]

i) $\forall x\, \exists y\, p(x, y)$ ii) $\forall y\, \exists x\, p(x, y)$
iii) $\exists x\, \forall y\, p(x, y)$ iv) $\exists y\, \forall x\, p(x, y)$

13. Suppose that $p(x, y)$ is an open statement where the universe for each of x, y consists of only three integers: 2, 3, 5. Then the quantified statement $\exists y\, p(2, y)$ is logically equivalent to $p(2, 2) \vee p(2, 3) \vee p(2, 5)$. The quantified statement $\exists x\, \forall y\, p(x, y)$ is logically equivalent to $[p(2, 2) \wedge p(2, 3) \wedge p(2, 5)] \vee [p(3, 2) \wedge p(3, 3) \wedge p(3, 5)] \vee [p(5, 2) \wedge p(5, 3) \wedge p(5, 5)]$. Use conjunctions and/or disjunctions to express the following statements without quantifiers.

a) $\forall x\, p(x, 3)$ b) $\exists x\, \exists y\, p(x, y)$ c) $\forall y\, \exists x\, p(x, y)$

14. Let $p(n), q(n)$ represent the open statements

$p(n)$: n is odd $\qquad q(n)$: n^2 is odd

for the universe of all integers. Which of the following statements are logically equivalent to each other?

a) If the square of an integer is odd, then the integer is odd.

b) $\forall n\, [p(n)$ is necessary for $q(n]$

c) The square of an odd integer is odd.

d) There are some integers whose squares are odd.

e) Given an integer whose square is odd, that integer is likewise odd.

f) $\forall n\, [\neg p(x) \rightarrow \neg q(n)]$

g) $\forall n\, [p(n)$ is sufficient for $q(n)]$

15. For each of the following pairs of statements determine whether the proposed negation is correct. If correct, determine which is true: the original statement or the proposed negation. If the proposed negation is wrong, write a correct version of the negation and then determine whether the original statement or your corrected version of the negation is true.

a) Statement: For all real numbers x, y, if $x^2 > y^2$, then $x > y$.
Proposed negation: There exist real numbers x, y such that $x^2 > y^2$ but $x \leq y$.

b) Statement: There exist real numbers x, y such that x and y are rational but $x + y$ is irrational.
Proposed negation: For all real numbers x, y, if $x + y$ is rational, then each of x, y is rational.

c) Statement: For all real numbers x, if x is not 0, then x has a multiplicative inverse.
Proposed negation: There exists a nonzero real number that does not have a multiplicative inverse.

d) Statement: There exist odd integers whose product is odd.
Proposed negation: The product of any two odd integers is odd.

16. Write the negation of each of the following statements as an English sentence—without symbolic notation. (Here the universe consists of all the students at the university where Professor Lenhart teaches.)

a) Every student in Professor Lenhart's C++ class is majoring in computer science or mathematics.

b) At least one student in Professor Lenhart's C++ class is a history major.

17. Write the negation of each of the following true statements. For parts (a) and (b) the universe consists of all integers; for parts (c) and (d) the universe comprises all real numbers.

a) For all integers n, if n is not (exactly) divisible by 2, then n is odd.

b) If k, m, n are any integers where $k - m$ and $m - n$ are odd, then $k - n$ is even.

c) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

d) For all real numbers x, if $|x - 3| < 7$, then $-4 < x < 10$.

18. Negate and simplify each of the following.

a) $\exists x\, [p(x) \vee q(x)]$ b) $\forall x\, [p(x) \wedge \neg q(x)]$

c) $\forall x\, [p(x) \rightarrow q(x)]$

d) $\exists x\, [(p(x) \vee q(x)) \rightarrow r(x)]$

19. For each of the following statements state the converse, inverse, and contrapositive. Also determine the truth value for each given statement, as well as the truth values for its converse, inverse, and contrapositive. (Here "divides" means "exactly divides.")

a) [The universe comprises all positive integers.]
If $m > n$, then $m^2 > n^2$.

b) [The universe comprises all integers.]
If $a > b$, then $a^2 > b^2$.

c) [The universe comprises all integers.]
If m divides n and n divides p, then m divides p.

d) [The universe consists of all real numbers.]
$\forall x\ [(x > 3) \rightarrow (x^2 > 9)]$

e) [The universe consists of all real numbers.]
For all real numbers x, if $x^2 + 4x - 21 > 0$, then $x > 3$ or $x < -7$.

20. Rewrite each of the following statements in the *if-then* form. Then write the converse, inverse, and contrapositive of your implication. For each result in parts (a) and (c) give the truth value for the implication and the truth values for its converse, inverse, and contrapositive. [In part (a) "divisibility" requires a remainder of 0.]

a) [The universe comprises all positive integers.]
Divisibility by 21 is a sufficient condition for divisibility by 7.

b) [The universe comprises all snakes presently slithering about the jungles of Asia.]
Being a cobra is a sufficient condition for a snake to be dangerous.

c) [The universe consists of all complex numbers.]
For every complex number z, z being real is necessary for z^2 to be real.

21. For the following statements the universe comprises all nonzero integers. Determine the truth value of each statement.

a) $\exists x\ \exists y\ [xy = 1]$ b) $\exists x\ \forall y\ [xy = 1]$

c) $\forall x\ \exists y\ [xy = 1]$

d) $\exists x\ \exists y\ [(2x + y = 5) \wedge (x - 3y = -8)]$

e) $\exists x\ \exists y\ [(3x - y = 7) \wedge (2x + 4y = 3)]$

22. Answer Exercise 21 for the universe of all nonzero real numbers.

23. In the arithmetic of real numbers, there is a real number, namely 0, called the identity of addition because $a + 0 = 0 + a = a$ for every real number a. This may be expressed in symbolic form by

$$\exists z\ \forall a\ [a + z = z + a = a].$$

(We agree that the universe comprises all real numbers.)

a) In conjunction with the existence of an additive identity is the existence of additive inverses. Write a quantified statement that expresses "Every real number has an additive inverse." (Do not use the minus sign anywhere in your statement.)

b) Write a quantified statement dealing with the existence of a multiplicative identity for the arithmetic of real numbers.

c) Write a quantified statement covering the existence of multiplicative inverses for the nonzero real numbers. (Do not use the exponent -1 anywhere in your statement.)

d) Do the results in parts (b) and (c) change in any way when the universe is restricted to the integers?

24. Consider the quantified statement $\forall x\ \exists y\ [x + y = 17]$. Determine whether this statement is true or false for each of the following universes: (a) the integers; (b) the positive integers; (c) the integers for x, the positive integers for y; (d) the positive integers for x, the integers for y.

25. Let the universe for the variables in the following statements consist of all real numbers. In each case negate and simplify the given statement.

a) $\forall x\ \forall y\ [(x > y) \rightarrow (x - y > 0)]$

b) $\forall x\ \forall y\ [(x < y) \rightarrow \exists z\ (x < z < y)]$

c) $\forall x\ \forall y\ [(|x| = |y|) \rightarrow (y = \pm x)]$

26. In calculus the definition of the limit L of a sequence of real numbers $r_1, r_2, r_3, \ldots$ can be given as

$$\lim_{n\to\infty} r_n = L$$

if (and only if) for every $\epsilon > 0$ there exists a positive integer k so that for all integers n, if $n > k$ then $|r_n - L| < \epsilon$.

In symbolic form this can be expressed as

$$\lim_{n\to\infty} r_n = L \Longleftrightarrow \forall\epsilon > 0\ \exists k > 0\ \forall n\ [(n > k) \rightarrow |r_n - L| < \epsilon].$$

Express $\lim_{n\to\infty} r_n \neq L$ in symbolic form.

2.5 Quantifiers, Definitions, and the Proofs of Theorems

In this section we shall combine some of the ideas we have already studied in the prior two sections. Although Section 2.3 introduced rules and methods for establishing the validity of an argument, unfortunately the arguments presented there seemed to have little to do with anything mathematical. [The rare exceptions are in Example 2.23 and the erroneous

c) Consider the following assignments.

p: Jonathan has his driver's license.
q: Jonathan's new car is out of gas.
r: Jonathan likes to drive his new car.

Then the given argument can be written in symbolic form as

$$\begin{array}{l} \neg p \vee q \\ p \vee \neg r \\ \underline{\neg q \vee \neg r} \\ \therefore \neg r \end{array}$$

Steps	Reasons
1) $\neg p \vee q$	Premise
2) $p \vee \neg r$	Premise
3) $(p \vee \neg r) \wedge (\neg p \vee q)$	Steps (2), (1), and the Rule of Conjunction
4) $\neg r \vee q$	Step (3) and Resolution
5) $q \vee \neg r$	Step (4) and the Commutative Law of $\vee$
6) $\neg q \vee \neg r$	Premise
7) $(q \vee \neg r) \wedge (\neg q \vee \neg r)$	Steps (5), (6), and the Rule of Conjunction
8) $\neg r \vee \neg r$	Step (7) and Resolution
9) $\therefore \neg r$	Step (8) and the Idempotent Law of $\vee$

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1. a) False **b)** False **c)** False **d)** True **e)** False **f)** False

3. Statements (a), (c), and (e) are true, and statements (b), (d), and (f) are false.

5. a) $\exists x\,[m(x) \wedge c(x) \wedge j(x)]$ True
b) $\exists x\,[s(x) \wedge c(x) \wedge \neg m(x)]$ True
c) $\forall x\,[c(x) \rightarrow (m(x) \veebar p(x))]$ False
d) $\forall x\,[(g(x) \wedge c(x)) \rightarrow \neg p(x)]$, or True
$\forall x\,[(p(x) \wedge c(x)) \rightarrow \neg g(x)]$, or
$\forall x\,[(g(x) \wedge p(x)) \rightarrow \neg c(x)]$
e) $\forall x\,[(c(x) \wedge s(x)) \rightarrow (p(x) \veebar e(x))]$ True

7. a) (i) $\exists x\, q(x)$
(ii) $\exists x\,[p(x) \wedge q(x)]$
(iii) $\forall x\,[q(x) \rightarrow \neg t(x)]$
(iv) $\forall x\,[q(x) \rightarrow \neg t(x)]$
(v) $\exists x\,[q(x) \wedge t(x)]$
(vi) $\forall x\,[(q(x) \wedge r(x)) \rightarrow s(x)]$

b) Statements (i), (ii), (v), and (vi) are true. Statements (iii) and (iv) are false; $x = 10$ provides a counterexample for either statement.

c) (i) If x is a perfect square, then $x > 0$.
(ii) If x is divisible by 4, then x is even.
(iii) If x is divisible by 4, then x is not divisible by 5.
(iv) There exists an integer that is divisible by 4, but it is not a perfect square.

d) (i) Let $x = 0$. (iii) Let $x = 20$.

9. a) (i) True (ii) False Consider $x = 3$.
(iii) True (iv) True

c) (i) True (ii) True
(iii) True (iv) False For $x = 2$ or 5, the truth value of $p(x)$ is 1 while that of $r(x)$ is 0.

11. a) In this case the variable x is free, while the variables y, z are bound.
b) Here the variables x, y are bound; the variable z is free.

13. a) $p(2, 3) \wedge p(3, 3) \wedge p(5, 3)$
b) $[p(2, 2) \vee p(2, 3) \vee p(2, 5)] \vee [p(3, 2) \vee p(3, 3) \vee p(3, 5)] \vee [p(5, 2) \vee p(5, 3) \vee p(5, 5)]$

15. a) The proposed negation is correct and is a true statement.
b) The proposed negation is wrong. A correct version of the negation is: For all rational numbers x, y, the sum $x + y$ is rational. This correct version of the negation is a true statement.
d) The proposed negation is wrong. A correct version of the negation is: For all integers x, y, if x, y are both odd, then xy is even. The (original) statement is true.
17. a) There exists an integer n such that n is not divisible by 2 but n is even (that is, not odd).
b) There exist integers k, m, n such that $k - m$ and $m - n$ are odd, and $k - n$ is odd.
d) There exists a real number x such that $|x - 3| < 7$ and either $x \leq -4$ or $x \geq 10$.
19. a) *Statement*: For all positive integers m, n, if $m > n$, then $m^2 > n^2$. (TRUE)
Converse: For all positive integers m, n, if $m^2 > n^2$, then $m > n$. (TRUE)
Inverse: For all positive integers m, n, if $m \leq n$, then $m^2 \leq n^2$. (TRUE)
Contrapositive: For all positive integers m, n, if $m^2 \leq n^2$, then $m \leq n$. (TRUE)
b) *Statement*: For all integers a, b, if $a > b$, then $a^2 > b^2$. (FALSE—let $a = 1$ and $b = -2$.)
Converse: For all integers a, b, if $a^2 > b^2$, then $a > b$. (FALSE—let $a = -5$ and $b = 3$.)
Inverse: For all integers a, b, if $a \leq b$, then $a^2 \leq b^2$. (FALSE—let $a = -5$ and $b = 3$.)
Contrapositive: For all integers a, b, if $a^2 \leq b^2$, then $a \leq b$. (FALSE—let $a = 1$ and $b = -2$.)
c) *Statement*: For all integers m, n, and p, if m divides n and n divides p, then m divides p. (TRUE)
Converse: For all integers m and p, if m divides p, then for each integer n it follows that m divides n and n divides p. (FALSE—let $m = 1$, $n = 2$, and $p = 3$.)
Inverse: For all integers m, n, and p, if m does not divide n or n does not divide p, then m does not divide p. (FALSE—let $m = 1$, $n = 2$, and $p = 3$.)
Contrapositive: For all integers m and p, if m does not divide p, then for each integer n it follows that m does not divide n or n does not divide p. (TRUE)
e) *Statement*: $\forall x\, [(x^2 + 4x - 21 > 0) \rightarrow [(x > 3) \vee (x < -7)]]$ (TRUE)
Converse: $\forall x\, [[(x > 3) \vee (x < -7)] \rightarrow (x^2 + 4x - 21 > 0)]$ (TRUE)
Inverse: $\forall x\, [(x^2 + 4x - 21 \leq 0) \rightarrow [(x \leq 3) \wedge (x \geq -7)]]$, or $\forall x\, [(x^2 + 4x - 21 \leq 0) \rightarrow (-7 \leq x \leq 3)]$ (TRUE)
Contrapositive: $\forall x\, [[(x \leq 3) \wedge (x \geq -7)] \rightarrow (x^2 + 4x - 21 \leq 0)]$, or $\forall x\, [(-7 \leq x \leq 3) \rightarrow (x^2 + 4x - 21 \leq 0)]$ (TRUE)
21. a) True b) False c) False d) True e) False
23. a) $\forall a\, \exists b\, [a + b = b + a = 0]$ b) $\exists u\, \forall a\, [au = ua = a]$ c) $\forall a \neq 0\, \exists b\, [ab = ba = 1]$
d) The statement in part (b) remains true, but the statement in part (c) is no longer true for this new universe.
25. a) $\exists x\, \exists y\, [(x > y) \wedge (x - y \leq 0)]$ b) $\exists x\, \exists y\, [(x < y) \wedge \forall z\, [x \geq z \vee z \geq y]]$

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1. Although we may write $28 = 25 + 1 + 1 + 1 = 16 + 4 + 4 + 4$, there is no way to express 28 as the sum of at most three perfect squares.
3.

$30 = 25 + 4 + 1$	$40 = 36 + 4$	$50 = 25 + 25$
$32 = 16 + 16$	$42 = 25 + 16 + 1$	$52 = 36 + 16$
$34 = 25 + 9$	$44 = 36 + 4 + 4$	$54 = 25 + 25 + 4$
$36 = 36$	$46 = 36 + 9 + 1$	$56 = 36 + 16 + 4$
$38 = 36 + 1 + 1$	$48 = 16 + 16 + 16$	$58 = 49 + 9$

5. a) The real number π is not an integer.
c) All administrative directors know how to delegate authority.
d) Quadrilateral $MNPQ$ is not equiangular.
7. a) When the statement $\exists x\, [p(x) \vee q(x)]$ is true, there is at least one element c in the prescribed universe where $p(c) \vee q(c)$ is true. Hence at least one of the statements $p(c)$, $q(c)$ has the truth value 1, so at least one of the statements $\exists x\; p(x)$ and $\exists x\; q(x)$ is true. Therefore, it follows that $\exists x\; p(x) \vee \exists x\; q(x)$ is true, and $\exists x\, [p(x) \vee q(x)] \Rightarrow \exists x\; p(x) \vee \exists x\; q(x)$.
Conversely, if $\exists x\; p(x) \vee \exists x\; q(x)$ is true, then at least one of $p(a)$, $q(b)$ has the truth value 1,