inverse, and contrapositive. (Here "divides" means "exactly divides.")

a) [The universe comprises all positive integers.]
If $m > n$, then $m^2 > n^2$.

b) [The universe comprises all integers.]
If $a > b$, then $a^2 > b^2$.

c) [The universe comprises all integers.]
If m divides n and n divides p, then m divides p.

d) [The universe consists of all real numbers.]
$\forall x\ [(x > 3) \to (x^2 > 9)]$

e) [The universe consists of all real numbers.]
For all real numbers x, if $x^2 + 4x - 21 > 0$, then $x > 3$ or $x < -7$.

20. Rewrite each of the following statements in the *if-then* form. Then write the converse, inverse, and contrapositive of your implication. For each result in parts (a) and (c) give the truth value for the implication and the truth values for its converse, inverse, and contrapositive. [In part (a) "divisibility" requires a remainder of 0.]

a) [The universe comprises all positive integers.]
Divisibility by 21 is a sufficient condition for divisibility by 7.

b) [The universe comprises all snakes presently slithering about the jungles of Asia.]
Being a cobra is a sufficient condition for a snake to be dangerous.

c) [The universe consists of all complex numbers.]
For every complex number z, z being real is necessary for z^2 to be real.

21. For the following statements the universe comprises all nonzero integers. Determine the truth value of each statement.

a) $\exists x\ \exists y\ [xy = 1]$ **b)** $\exists x\ \forall y\ [xy = 1]$

c) $\forall x\ \exists y\ [xy = 1]$

d) $\exists x\ \exists y\ [(2x + y = 5) \wedge (x - 3y = -8)]$

e) $\exists x\ \exists y\ [(3x - y = 7) \wedge (2x + 4y = 3)]$

22. Answer Exercise 21 for the universe of all nonzero real numbers.

23. In the arithmetic of real numbers, there is a real number, namely 0, called the identity of addition because $a + 0 = 0 + a = a$ for every real number a. This may be expressed in symbolic form by

$$\exists z\ \forall a\ [a + z = z + a = a].$$

(We agree that the universe comprises all real numbers.)

a) In conjunction with the existence of an additive identity is the existence of additive inverses. Write a quantified statement that expresses "Every real number has an additive inverse." (Do not use the minus sign anywhere in your statement.)

b) Write a quantified statement dealing with the existence of a multiplicative identity for the arithmetic of real numbers.

c) Write a quantified statement covering the existence of multiplicative inverses for the nonzero real numbers. (Do not use the exponent -1 anywhere in your statement.)

d) Do the results in parts (b) and (c) change in any way when the universe is restricted to the integers?

24. Consider the quantified statement $\forall x\ \exists y\ [x + y = 17]$. Determine whether this statement is true or false for each of the following universes: (a) the integers; (b) the positive integers; (c) the integers for x, the positive integers for y; (d) the positive integers for x, the integers for y.

25. Let the universe for the variables in the following statements consist of all real numbers. In each case negate and simplify the given statement.

a) $\forall x\ \forall y\ [(x > y) \to (x - y > 0)]$

b) $\forall x\ \forall y\ [(x < y) \to \exists z\ (x < z < y)]$

c) $\forall x\ \forall y\ [(|x| = |y|) \to (y = \pm x)]$

26. In calculus the definition of the limit L of a sequence of real numbers $r_1, r_2, r_3, \ldots$ can be given as

$$\lim_{n\to\infty} r_n = L$$

if (and only if) for every $\epsilon > 0$ there exists a positive integer k so that for all integers n, if $n > k$ then $|r_n - L| < \epsilon$.

In symbolic form this can be expressed as

$$\lim_{n\to\infty} r_n = L \iff \forall \epsilon > 0\ \exists k > 0\ \forall n\ [(n > k) \to |r_n - L| < \epsilon].$$

Express $\lim_{n\to\infty} r_n \neq L$ in symbolic form.

2.5 Quantifiers, Definitions, and the Proofs of Theorems

In this section we shall combine some of the ideas we have already studied in the prior two sections. Although Section 2.3 introduced rules and methods for establishing the validity of an argument, unfortunately the arguments presented there seemed to have little to do with anything mathematical. [The rare exceptions are in Example 2.23 and the erroneous

argument in part (b) of the material preceding Example 2.26.] Most of the arguments dealt with certain individuals and predicaments they were either in or about to face.

But now that we have learned some of the properties of quantifiers and quantified statements, we are better equipped to handle arguments that will help us to prove mathematical theorems. Before dealing with theorems, however, we shall consider how mathematical definitions are traditionally presented in scientific writing.

Following Example 2.3 in Section 2.1, the discussion concerned how an implication might be used in place of a biconditional in everyday conversation. But in scientific writing, it was noted, we should avoid any and all situations where an ambiguous interpretation might come about — in particular, an implication should not be used when a biconditional is intended. However, there is one major exception to that rule and it concerns the way that mathematical definitions are traditionally presented in mathematics textbooks and other scientific literature. Example 2.51 demonstrates this exception.

EXAMPLE 2.51

a) Let us start with the universe of all quadrilaterals in the plane and try to identify those that are called rectangles.

One person might say that

"If a quadrilateral is a rectangle then it has four equal angles."

Another individual might identify these special quadrilaterals by observing that

"If a quadrilateral has four equal angles, then it is a rectangle."

(Here both people are making implicitly quantified statements, where the quantifier is universal.)

Given the open statements

$p(x)$: x is a rectangle $\qquad$ $q(x)$: x has four equal angles,

we can express what the first person says as

$$\forall x\ [p(x) \to q(x)],$$

while for the second person we would write

$$\forall x\ [q(x) \to p(x)].$$

So which of the preceding (quantified) statements identifies or defines a rectangle? Perhaps we feel that they both do. But how can that be, since one statement is the converse of the other and, in general, the converse of an implication is *not* logically equivalent to the implication.

Here the reader must consider what is intended — not just what each of the two people has said, or the symbolic expressions we have written to represent these statements. In this situation each person is using an implication with the meaning of a biconditional. They are both intending (though not stating)

$$\forall x\ [p(x) \leftrightarrow q(x)],$$

— that is, each is really telling us that

"A quadrilateral is a rectangle *if and only if* it has four equal angles."

b) Within the universe of all integers we can distinguish the even integers by means of a certain property and so we may define them as follows:

For every integer n we call n even if it is divisible by 2.

(By the expression "divisible by 2" we mean "exactly divisible by 2" — that is, there is no remainder upon division of the dividend n by the divisor 2.)

If we consider the open statements

$$p(n): \quad n \text{ is an even integer} \qquad\qquad q(n): \quad n \text{ is divisible by } 2,$$

then it *appears* that the preceding definition may be written symbolically as

$$\forall n\ [q(n) \to p(n)].$$

After all, the given quantified statement (in the preceding definition) is an implication. However, the situation here is similar to that given in part (a). What appears to be stated is not what is intended. The intention is for the reader to interpret the given definition as

$$\forall n\ [q(n) \leftrightarrow p(n)],$$

that is,

"For every integer n, we call n even *if and only if* n is divisible by 2."

(Note that the open statement "n is divisible by 2" can also be expressed by the *open* statement "$n = 2k$, for some integer k." Don't be misled here by the presence of the quantifier "for some integer k" — for the expression $\exists k\ [n = 2k]$ is still an open statement because n remains a free variable.)

So now we see how quantifiers may enter into the way we state mathematical definitions — and that the traditional way in which such a definition appears is as an implication. But beware and remember: It is *only in definitions* that an implication can be (mis)read and correctly interpreted as a biconditional.

Note how we defined the limit concept in Example 2.50. There we wrote "if (and only if)" since we wanted to let the reader know our intention. Now we are free to replace "if (and only if)" by simply "if."

Having settled our discussion on the nature of mathematical definitions, we continue now with an investigation of arguments involving quantified statements.

EXAMPLE 2.52

Suppose that we start with the universe that comprises only the 13 integers 2, 4, 6, 8, . . . , 24, 26. Then we can establish the statement:

For all n (meaning $n = 2, 4, 6, \ldots, 26$),

we can write n as the sum of at most three perfect squares.

The results in Table 2.24 provide a case-by-case verification showing the given (quantified) statement to be true. (We might call this statement a theorem.)

Table 2.24

$2 = 1 + 1$	$10 = 9 + 1$	$20 = 16 + 4$
$4 = 4$	$12 = 4 + 4 + 4$	$22 = 9 + 9 + 4$
$6 = 4 + 1 + 1$	$14 = 9 + 4 + 1$	$24 = 16 + 4 + 4$
$8 = 4 + 4$	$16 = 16$	$26 = 25 + 1$
	$18 = 16 + 1 + 1$	

This exhaustive listing is an example of a proof using the technique we call, rather appropriately, the *method of exhaustion*. This method is reasonable when we are dealing with a fairly small universe. If we are confronted with a situation in which the universe is larger but within the range of a computer that is available to us, then we might write a program to check all of the individual cases.

(Note that for certain cases in Table 2.24 more than one answer may be possible. For example, we could have written $18 = 9 + 9$ and $26 = 16 + 9 + 1$. But this is all right. We were told that each positive even integer less than or equal to 26 could be written as the sum of one, two, or three perfect squares. We were *not* told that each such representation had to be unique, so more than one possibility could occur. What we had to check in each case was that there was at least one possibility.)

In the previous example we mentioned the word *theorem*. We also found this term used in Chapter 1 — for example, in results like the binomial theorem and the multinomial theorem where we were introduced to certain types of enumeration problems. Without getting overly technical, we shall consider *theorems* to be statements of mathematical interest, statements that are known to be true. Sometimes the term *theorem* is used only to describe major results that have many and varied consequences. Certain of these consequences that follow rather immediately from a theorem are termed *corollaries* (as in the case of Corollary 1.1 in Section 1.3). In this text, however, we shall not be so particular in our use of the word theorem.

Example 2.52 is a nice starting point to examine the proof of a quantified statement. Unfortunately, a great number of mathematical statements and theorems often deal with universes that do not lend themselves to the method of exhaustion. When faced with establishing or proving a result for all integers, for example, or for all real numbers, then we cannot use a case-by-case method like the one in Example 2.52. So what can we do?

We start by considering the following rule.

The Rule of Universal Specification: If an open statement becomes true for *all* replacements by the members in a given universe, then that open statement is true for *each specific* individual member in that universe. (A bit more symbolically — if $p(x)$ is an open statement for a given universe, and if $\forall x\ p(x)$ is true, then $p(a)$ is true for each a in the universe.)

This rule indicates that the truth of an open statement in one particular instance follows (as a special case) from the more general (for the entire universe) truth of that universally quantified open statement. The following examples will show us how to apply this idea.

EXAMPLE 2.53

a) For the universe of all people, consider the open statements

$m(x)$: x is a mathematics professor $\qquad$ $c(x)$: x has studied calculus.

Now consider the following argument.

All mathematics professors have studied calculus.

Leona is a mathematics professor.

Therefore Leona has studied calculus.

If we let l represent this particular woman (in our universe) named Leona, then we can rewrite this argument in symbolic form as

$$\begin{array}{l} \forall x\ [m(x) \to c(x)] \\ m(l) \\ \hline \therefore c(l) \end{array}$$

Here the two statements above the line are the premises of the argument, and the statement $c(l)$ below the line is its conclusion. This is comparable to what we saw in Section 2.3, except now we have a premise that is a universally quantified statement. As was the case in Section 2.3, the premises are all assumed to be true and we must try to establish that the conclusion is also true under these circumstances. Now, to establish the validity of the given argument, we proceed as follows.

Steps	**Reasons**
1) $\forall x\ [m(x) \to c(x)]$	Premise
2) $m(l)$	Premise
3) $m(l) \to c(l)$	Step (1) and the Rule of Universal Specification
4) $\therefore c(l)$	Steps (2) and (3) and the Rule of Detachment

Note that the statements in steps (2) and (3) are *not* quantified statements. They are the types of statements we studied earlier in the chapter. In particular, we can apply the rules of inference we learned in Section 2.3 to these two statements to deduce the conclusion in step (4).

We see here that the Rule of Universal Specification enables us to take a universally quantified premise and deduce from it an ordinary statement (that is, one that is not quantified). This (ordinary) statement — namely, $m(l) \to c(l)$ — is one specific true instance of the universally quantified true premise $\forall x\ [m(x) \to c(x)]$.

b) For an example of a more mathematical nature let us consider the universe of all triangles in the plane in conjunction with the open statements

$p(t)$: t has two sides of equal length.

$q(t)$: t is an isosceles triangle.

$r(t)$: t has two angles of equal measure.

Let us also focus our attention on one specific triangle with no two angles of equal measure. This triangle will be called triangle XYZ and will be designated by c. Then we find that the argument

In triangle XYZ there is no pair of angles of equal measure.	$\neg r(c)$
If a triangle has two sides of equal length, then it is isosceles.	$\forall t\ [p(t) \to q(t)]$
If a triangle is isosceles, then it has two angles of equal measure.	$\forall t\ [q(t) \to r(t)]$
Therefore triangle XYZ has no two sides of equal length.	$\therefore \neg p(c)$

is a valid one — as evidenced by the following.

Steps	Reasons
1) $\forall t\,[p(t) \to q(t)]$	Premise
2) $p(c) \to q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall t\,[q(t) \to r(t)]$	Premise
4) $q(c) \to r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \to r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\neg r(c)$	Premise
7) $\therefore \neg p(c)$	Steps (5) and (6) and Modus Tollens

Once again we see how the Rule of Universal Specification helps us. Here it has taken the universally quantified statements at steps (1) and (3) and has provided us with the (ordinary) statements at steps (2) and (4), respectively. Then at this point we were able to apply the rules of inference we learned in Section 2.3 (namely, the Law of the Syllogism and Modus Tollens) to derive the conclusion $\neg p(c)$ in step (7).

c) Now for one last argument to drive the point home! Here we'll consider the universe to be made up of the entire student body at a particular college. One specific student, Mary Gusberti, will be designated by m.

For this universe and the open statements

$j(x)$: x is a junior $\qquad$ $s(x)$: x is a senior

$p(x)$: x is enrolled in a physical education class

we consider the argument:

No junior or senior is enrolled in a physical education class.

Mary Gusberti is enrolled in a physical education class.

Therefore Mary Gusberti is not a senior.

In symbolic form this argument becomes

$$\begin{array}{l} \forall x\,[(j(x) \vee s(x)) \to \neg p(x)] \\ p(m) \\ \hline \therefore \neg s(m) \end{array}$$

Now the following steps (and reasons) establish the validity of this argument.

Steps	Reasons
1) $\forall x\,[(j(x) \vee s(x)) \to \neg p(x)]$	Premise
2) $p(m)$	Premise
3) $(j(m) \vee s(m)) \to \neg p(m)$	Step (1) and the Rule of Universal Specification
4) $p(m) \to \neg(j(m) \vee s(m))$	Step (3), $(q \to t) \Leftrightarrow (\neg t \to \neg q)$, and the Law of Double Negation
5) $p(m) \to (\neg j(m) \wedge \neg s(m))$	Step (4) and DeMorgan's Law
6) $\neg j(m) \wedge \neg s(m)$	Steps (2) and (5) and the Rule of Detachment (or Modus Ponens)
7) $\therefore \neg s(m)$	Step (6) and the Rule of Conjunctive Simplification

In Example 2.53 we have had our first opportunity to apply the Rule of Universal Specification. Using the rule in conjunction with Modus Ponens (or the Rule of Detachment) and

Modus Tollens, we are able to state the following corresponding analogs, each of which involves a universally quantified premise. In either case we consider a fixed universe that includes a specific member c and make use of the open statements $p(x)$, $q(x)$ defined for this universe.

$$(1)\quad \begin{array}{l} \forall x\,[p(x) \to q(x)] \\ p(c) \\ \hline \therefore q(c) \end{array} \qquad (2)\quad \begin{array}{l} \forall x\,[p(x) \to q(x)] \\ \neg q(c) \\ \hline \therefore \neg p(c) \end{array}$$

These two valid arguments are presented here for the same reason we presented them for the rules of inference — Modus Ponens and Modus Tollens — in Section 2.3 (in the discussion between Examples 2.25 and 2.26). We want to examine some possible errors that may arise when the results in (1) and (2) are not used correctly.

Let us start with the universe of all polygons in the plane. Within this universe we shall let c denote one specific polygon — the quadrilateral $EFGH$, where the measure of angle E is $91°$. For the open statements

$$p(x):\quad x \text{ is a square} \qquad\qquad q(x):\quad x \text{ has four sides,}$$

the following argument is *invalid.*

(1′) All squares have four sides.
Quadrilateral $EFGH$ has four sides.
Therefore quadrilateral $EFGH$ is a square.

In symbolic form this argument translates into

(1″)
$$\begin{array}{l} \forall x\,[p(x) \to q(x)] \\ q(c) \\ \hline \therefore p(c) \end{array}$$

Unfortunately, although the premises are true, the conclusion is false. (For a square has no angle of measure $91°$.) We admit that there might be some confusion between this argument and the valid one in (1) above. For when we apply the Rule of Universal Specification to the quantified premise in (1″), in this instance we arrive at the *invalid* argument

$$\begin{array}{l} p(c) \to q(c) \\ q(c) \\ \hline \therefore p(c) \end{array}$$

And here, as in Section 2.3, the error in reasoning lies in our attempt to argue by the converse.

A second invalid argument — from the misuse of argument (2) above — can also be given, as shown in the following.

(2′) All squares have four sides.
Quadrilateral $EFGH$ is not a square.
Therefore quadrilateral $EFGH$ does not have four sides.

Translating (2′) into symbolic form results in

(2″)
$$\begin{array}{l} \forall x\,[p(x) \to q(x)] \\ \neg p(c) \\ \hline \therefore \neg q(c) \end{array}$$

This time the Rule of Universal Specification leads us to

$$\begin{array}{l} p(c) \to q(c) \\ \neg p(c) \\ \hline \therefore \neg q(c) \end{array}$$

where the fallacy arises because we are trying to argue by the inverse.

And now let us look back at the three parts of Example 2.53. Although the arguments presented there involved premises that were universally quantified statements, there was never any instance where a universally quantified statement appeared in the conclusion. We now want to remedy this situation, since many theorems in mathematics have the form of a universally quantified statement. To do so we need the following considerations.

Start with a given universe and the open statement $p(x)$. To establish the truth of the statement $\forall x\ p(x)$, we must establish the truth of $p(c)$ for each member c in the given universe. But if the universe has many members or, for example, contains all the positive integers, then this exhaustive, if not exhausting, task of validating the truth of each $p(c)$ becomes difficult, if not impossible. To get around this situation we shall prove that $p(c)$ is true — but now we do it for the case where c denotes a *specific but arbitrarily chosen* member from the prescribed universe.

Should the preceding open statement $p(x)$ have the form $q(x) \to r(x)$, for open statements $q(x)$ and $r(x)$, then we shall assume the truth of $q(c)$ as an additional premise and try to deduce the truth of $r(c)$ — by using definitions, axioms, previously proven theorems, and the logical principles we have studied. For when $q(c)$ is false, the implication $q(c) \to r(c)$ is true, regardless of the truth value of $r(c)$.

The reason that the element c must be arbitrary (or generic) is to make sure that what we do and prove about c is applicable for *all* the other elements in the universe. If we are dealing with the universe of all integers, for example, we cannot choose c in an arbitrary manner by selecting c as 4, or by selecting c as an even integer. In general, we cannot make any assumptions about our choice for c unless these assumptions are valid for *all* the other elements of the universe. The word *generic* is applied to the element c here because it indicates that our choice (for c) must share all of the common characteristics of the elements for the given universe.

The principle we have described in the preceding three paragraphs is named and summarized as follows.

The Rule of Universal Generalization: If an open statement $p(x)$ is proved to be true when x is replaced by any *arbitrarily chosen* element c from our universe, then the universally quantified statement $\forall x\ p(x)$ is true. Furthermore, the rule extends beyond a single variable. So if, for example, we have an open statement $q(x, y)$ that is proved to be true when x and y are replaced by *arbitrarily chosen* elements from the same universe, or their own respective universes, then the universally quantified statement $\forall x\ \forall y\ q(x, y)$ [or, $\forall x, y\ q(x, y)$] is true. Similar results hold for the cases of three or more variables.

Before we demonstrate the use of this rule in any examples, we wish to look back at part (1) of Example 2.43 in Section 2.4. It turns out that the explanation given there to establish that

$$\forall x\ [p(x) \wedge (q(x) \wedge r(x))] \Longleftrightarrow \forall x\ [(p(x) \wedge q(x)) \wedge r(x)]$$

anticipated what we have now described in detail as the Rules of Universal Specification and Universal Generalization.

Now we'll turn to an example which is strictly symbolic. This example provides an opportunity to apply the Rule of Universal Generalization.

EXAMPLE 2.54

Let $p(x)$, $q(x)$, and $r(x)$ be open statements that are defined for a given universe. We show that the argument

$$\begin{array}{c} \forall x\ [p(x) \to q(x)] \\ \forall x\ [q(x) \to r(x)] \\ \hline \therefore \forall x\ [p(x) \to r(x)] \end{array}$$

is valid by considering the following.

Steps	**Reasons**
1) $\forall x\ [(p(x) \to q(x)]$	Premise
2) $p(c) \to q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x\ [q(x) \to r(x)]$	Premise
4) $q(c) \to r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \to r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\therefore \forall x\ [p(x) \to r(x)]$	Step (5) and the Rule of Universal Generalization

Here the element c introduced in steps (2) and (4) is the same *specific* but *arbitrarily chosen* element from the universe. Since this element c has no *special or distinguishing properties* but does share all of the common features of every other element in this universe, we can use the Rule of Universal Generalization to go from step (5) to step (6).

And so at last we have dealt with a valid argument where a universally quantified statement appears as the conclusion, as well as among the premises.

The question that now may be at the back of the reader's mind is one of practicality. Namely, when would we ever need to use the argument that we validated in Example 2.54? We may find that we have already used it (perhaps, unknowingly) in earlier algebra and geometry courses, as we demonstrate in the following example.

EXAMPLE 2.55

a) For the universe of all real numbers, consider the open statements

$$p(x):\quad 3x - 7 = 20 \qquad q(x):\quad 3x = 27 \qquad r(x):\quad x = 9.$$

The following solution of an algebraic equation parallels the valid argument from Example 2.54.

1) If $3x - 7 = 20$, then $3x = 27$.
2) If $3x = 27$, then $x = 9$.
3) Therefore, if $3x - 7 = 20$, then $x = 9$.

$$\begin{array}{c} \forall x\ [p(x) \to q(x)] \\ \forall x\ [q(x) \to r(x)] \\ \hline \therefore \forall x\ [p(x) \to r(x)] \end{array}$$

b) When we dealt with the universe of all quadrilaterals in plane geometry, we may have found ourselves relating something like this:

"Since every square is a rectangle, and every rectangle is a parallelogram, it follows that every square is a parallelogram."

In this case we are using the argument in Example 2.54 for the open statements

$p(x)$: x is a square $\qquad q(x)$: x is a rectangle $\qquad r(x)$: x is a parallelogram.

Now we continue with one more argument to validate.

EXAMPLE 2.56

The steps and reasons needed to establish the validity of the argument

$$\begin{array}{l} \forall x\ [p(x) \vee q(x)] \\ \forall x\ [(\neg p(x) \wedge q(x)) \rightarrow r(x)] \\ \hline \therefore \forall x\ [\neg r(x) \rightarrow p(x)] \end{array}$$

are given as follows. [Here the element c is in the universe assigned for the argument. Also, since the conclusion is a universally quantified implication, we can assume $\neg r(c)$ as an additional premise — as was mentioned earlier when the Rule of Universal Generalization was first introduced.]

Steps	**Reasons**
1) $\forall x\ [p(x) \vee q(x)]$	Premise
2) $p(c) \vee q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x\ [(\neg p(x) \wedge q(x)) \rightarrow r(x)]$	Premise
4) $[\neg p(c) \wedge q(c)] \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $\neg r(c) \rightarrow \neg[\neg p(c) \wedge q(c)]$	Step (4) and $s \rightarrow t \Leftrightarrow \neg t \rightarrow \neg s$
6) $\neg r(c) \rightarrow [p(c) \vee \neg q(c)]$	Step (5), DeMorgan's Law, and the Law of Double Negation
7) $\neg r(c)$	Premise (assumed)
8) $p(c) \vee \neg q(c)$	Steps (7) and (6) and Modus Ponens
9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$	Steps (2) and (8) and the Rule of Conjunction
10) $p(c) \vee [q(c) \wedge \neg q(c)]$	Step (9) and the Distributive Law of $\vee$ over $\wedge$
11) $p(c)$	Step (10), $q(c) \wedge \neg q(c) \Leftrightarrow F_0$, and $p(c) \vee F_0 \Leftrightarrow p(c)$
12) $\therefore \forall x\ [\neg r(x) \rightarrow p(x)]$	Steps (7) and (11) and the Rule of Universal Generalization

Before going on we want to point out a convention that the reader may not like but will have to get used to. It concerns our coverage of the Rules of Universal Specification and Universal Generalization. In the first case we started with the statement $\forall x\ p(x)$ and then dealt with $p(c)$ for some specific element c in our universe. For the Rule of Universal Generalization we obtained the truth of $\forall x\ p(x)$ from that of $p(c)$, where c was arbitrarily selected from the universe. Unfortunately, we'll often find ourselves using the letter x instead of c to denote the element — but as long as we understand what is happening we shall soon find the convention easy enough to work with.

The results of Example 2.54 and especially Example 2.56 lead us to believe that we can use universally quantified statements and the rules of inference — including the Rules of Universal Specification and Universal Generalization — to formalize and prove a variety of arguments and, hopefully, theorems. When we do so it appears that the validation of some rather short arguments requires quite a number of steps, because we have been very meticulous and included all the steps and reasons — we left little, if anything, to the imagination. The reader should rest assured that when we start to prove mathematical theorems, we shall present the proofs in the more conventional paragraph style. We shall no longer mention

each and every application of the laws of logic and the other tautologies or the rules of inference. On occasion we may single out a certain rule of inference, but our attention will be primarily directed to the use of definitions, mathematical axioms and principles (other than those we have found in our study of logic), and other (earlier) theorems we have been able to prove. Why then have we been learning all of this material on validating arguments? Because it will provide us with a framework to fall back on whenever we doubt whether a given attempt at a proof really does the job. If in doubt, we have our study of logic to supply us with a somewhat mechanical but strictly objective means to help us decide.

And now we present paragraph-style proofs for some results about the integers. (These results may be considered rather obvious to us — in fact, we may find we have already seen and used some of them. But they provide an excellent setting for writing some simple proofs.) The proofs we shall presently introduce use the following ideas, which we now formally define. [The first idea was mentioned earlier in part (b) of Example 2.51.]

Definition 2.8 Let n be an integer. We call n *even* if n is divisible by 2 — that is, if there exists an integer r so that $n = 2r$. If n is not even, then we call n *odd* and find for this case that there exists an integer s where $n = 2s + 1$.

THEOREM 2.2 For all integers k and l, if k, l are both odd, then $k + l$ is even.

Proof: In this proof we shall number the steps so that we may refer to them for some later remarks. After this we shall no longer number the steps.

1) Since k and l are odd, we may write $k = 2a + 1$ and $l = 2b + 1$, for some integers a, b. This is due to Definition 2.8.
2) Then

$$k + l = (2a + 1) + (2b + 1) = 2(a + b + 1),$$

by virtue of the Commutative and Associative Laws of Addition and the Distributive Law of Multiplication over Addition — all of which hold for integers.

3) Since a, b are integers, $a + b + 1 = c$ is an integer; with $k + l = 2c$, it follows from Definition 2.8 that $k + l$ is even.

Remarks

1) In step (1) of the preceding proof k and l were chosen in an arbitrary manner, so we know by the Rule of Universal Generalization that the result obtained is true for all odd integers.
2) Although we may not realize it, we are using the Rule of Universal Specification (twice) in step (1). The first argument implicit in this step reads as follows.
   i) If n is an odd integer, then $n = 2r + 1$ for some integer r.
   ii) The integer k is a specific (but arbitrarily chosen) odd integer.
   iii) Therefore we may write $k = 2a + 1$ for some (specific) integer a.
3) In step (1) we do not have $k = 2a + 1$ and $l = 2a + 1$. Since k, l are arbitrarily chosen, it *may* be the case that $k = l$ — and when this happens we have $2a + 1 = k = l = 2b + 1$, from which it follows that $a = b$. [Since k may not equal l, it follows

that $(k-1)/2 = a$ may not equal $b = (l-1)/2$. Thus we should use the different variables a and b.]

Before we proceed with another theorem — written in the more conventional manner — let us examine the following.

EXAMPLE 2.57

Consider the following statement for the universe of integers.

If n is an integer, then $n^2 = n$ — or, $\forall n\ [n^2 = n]$.

Now for $n = 0$ it is true that $n^2 = 0^2 = 0 = n$. And if $n = 1$, it is also true that $n^2 = 1^2 = 1 = n$. However, we *cannot* conclude $n^2 = n$ for every integer n. The Rule of Universal Generalization does *not* apply here, for we cannot consider the choice of 0 (or 1) as an arbitrarily chosen integer. If $n = 2$, we have $n^2 = 4 \neq 2 = n$, and this one counterexample is enough to tell us that the given statement is false. However, either replacement — namely, $n = 0$ or $n - 1$ — is enough to establish the truth of the statement:

For some integer n, $n^2 = n$ — or, $\exists n\ [n^2 = n]$.

We close — at last — with three results to demonstrate how we shall write proofs throughout the remainder of the text.

THEOREM 2.3

For all integers k and l, if k and l are both odd, then their product kl is also odd.

Proof: Since k and l are both odd, we may write $k = 2a + 1$ and $l = 2b + 1$, for some integers a and b — because of Definition 2.8. Then the product $kl = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$, where $2ab + a + b$ is an integer. Therefore, by Definition 2.8 once again, it follows that kl is odd.

The preceding proof is an example of a direct proof. In our next example we shall prove a result in three ways: first by a direct argument (or proof), then by the contrapositive method, and finally by the method of proof by contradiction. [For the (method of) proof by contradiction we put in some extra details, since this is our first opportunity to use this technique.] The reader should not assume, however, that every theorem can be so readily proved in a variety of ways.

THEOREM 2.4

If m is an even integer, then $m + 7$ is odd.

Proof:

1) Since m is even, we have $m = 2a$ for some integer a. Then $m + 7 = 2a + 7 = 2a + 6 + 1 = 2(a+3) + 1$. Since $a + 3$ is an integer, we know that $m + 7$ is odd.
2) Suppose that $m + 7$ is not odd, hence even. Then $m + 7 = 2b$ for some integer b and $m = 2b - 7 = 2b - 8 + 1 = 2(b-4) + 1$, where $b - 4$ is an integer. Hence m is odd. [The result follows because the statements $\forall m\ [p(m) \to q(m)]$ and $\forall m[\neg q(m) \to \neg p(m)]$ are logically equivalent.]

3) Now assume that m is even and that $m + 7$ is also even. (This assumption is the negation of what we want to prove.) Then $m + 7$ even implies that $m + 7 = 2c$ for some integer c. And, consequently, $m = 2c - 7 = 2c - 8 + 1 = 2(c - 4) + 1$ with $c - 4$ an integer, so m is odd. Now we have our contradiction. We started with m even and deduced m odd — an impossible situation, since no integer can be both even and odd. How did we arrive at this dilemma? Simple — we made a mistake! This mistake is the false assumption — namely, $m + 7$ is even — that we wanted to believe at the start of the proof. Since the assumption is false, its negation is true, and so we now have $m + 7$ odd.

The second and third proofs for Theorem 2.4 appear to be somewhat similar. This is because the contradiction we derived in the third proof arises from the hypothesis of the theorem and its negation. We shall see as we progress (as early as the next chapter) that a contradiction may also be obtained by deriving the negation of a known fact — a fact that is *not* the hypothesis of the theorem we are attempting to prove. For now, however, let us think about this similarity a little more. Suppose we start with the open statements $p(m)$ and $q(m)$ — for a prescribed universe — and consider a theorem of the form $\forall m\ [p(m) \to q(m)]$. If we try to prove this result by the contrapositive method, then we shall actually prove the logically equivalent statement $\forall m\ [\neg q(m) \to \neg p(m)]$. To do so we assume the truth of $\neg q(m)$ (for any specific but arbitrarily chosen m in the universe) and show that this leads to the truth of $\neg p(m)$. On the other hand, if we wish to prove the theorem $\forall m\ [p(m) \to q(m)]$ by the method of proof by contradiction, then we assume that the statement $\forall m\ [p(m) \to q(m)]$ is false. This amounts to the fact that $p(m) \to q(m)$ is false for at least one replacement for m from the universe — that is, there is some element m in the universe for which $p(m)$ is true and $q(m)$ is false [or $\neg q(m)$ is true]. We then use the truth of $p(m)$ and $\neg q(m)$ to derive a contradiction. [In the third proof of Theorem 2.4 we obtained $p(m) \wedge \neg p(m)$.] These two methods can be compared symbolically in the following — where m is specific but arbitrarily chosen for the method of contraposition.

	Assumption	**Result Derived**
Contraposition	$\neg q(m)$	$\neg p(m)$
Contradiction	$p(m)$ and $\neg q(m)$	F_0

In general, when we are able to establish a theorem by either a direct proof or an indirect proof, the direct approach is less cumbersome than an indirect approach. (This certainly appears to be the case for the three proofs presented for Theorem 2.4.) When we do not have any prescribed directions given for attempting the proof of a certain theorem, we might start with a direct approach. If we succeed, then all is well. If not, then we might consider trying to find a counterexample to what we thought was a theorem. Should our search for a counterexample fail, then we might consider an indirect approach. We might prove the contrapositive of the theorem, or obtain a contradiction, as we did in the third proof of Theorem 2.4, by assuming the truth of the hypothesis and the truth of the negation of the conclusion (for some element m in the universe) in the given theorem.

We close this section with one more indirect proof by the method of contraposition.

THEOREM 2.5 For all positive real numbers x and y, if the product xy exceeds 25, then $x > 5$ or $y > 5$.

Proof: Consider the negation of the conclusion — that is, suppose that $0 < x \leq 5$ and $0 < y \leq 5$. Under these circumstances we find that $0 = 0 \cdot 0 < x \cdot y \leq 5 \cdot 5 = 25$, so the product

xy does *not* exceed 25. (This indirect method of proof now establishes the given statement, since we know that an implication is logically equivalent to its contrapositive.)

EXERCISES 2.5

1. In Example 2.52 why did we stop at 26 and not at 28?

2. In Example 2.52 why didn't we include the odd integers between 2 and 26?

3. Use the method of exhaustion to show that every even integer between 30 and 58 (including 30 and 58) can be written as a sum of at most three perfect squares.

4. Let n be a positive integer greater than 1. We call n *prime* if the only positive integers that (exactly) divide n are 1 and n itself. For example, the first seven primes are 2, 3, 5, 7, 11, 13, and 17. (We shall learn more about primes in Chapter 4.) Use the method of exhaustion to show that every integer in the universe 4, 6, 8, . . . , 36, 38 can be written as the sum of two primes.

5. For each of the following (universes and) pairs of statements, use the Rule of Universal Specification, in conjunction with Modus Ponens and Modus Tollens, in order to fill in the blank line so that a valid argument results.

a) [The universe comprises all real numbers.]
All integers are rational numbers.
The real number π is not a rational number.
$\therefore$ _____

b) [The universe comprises the present population of the United States.]
All librarians know the Library of Congress Classification System.

_____

$\therefore$ Margaret knows the Library of Congress Classification System.

c) [The same universe as in part (b).]

_____

Sondra is an administrative director.
$\therefore$ Sondra knows how to delegate authority.

d) [The universe consists of all quadrilaterals in the plane.]
All rectangles are equiangular.

_____

$\therefore$ Quadrilateral $MNPQ$ is not a rectangle.

6. Determine which of the following arguments are valid and which are invalid. Provide an explanation for each answer. (Let the universe consist of all people presently residing in the United States.)

a) All mail carriers carry a can of mace.
Mrs. Bacon is a mail carrier.
Therefore Mrs. Bacon carries a can of mace.

b) All law-abiding citizens pay their taxes.
Mr. Pelosi pays his taxes.
Therefore Mr. Pelosi is a law-abiding citizen.

c) All people who are concerned about the environment recycle their plastic containers.
Margarita is not concerned about the environment.
Therefore Margarita does not recycle her plastic containers.

7. For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x, prove that

a) $\exists x\ [p(x) \vee q(x)] \Longleftrightarrow \exists x\ p(x) \vee \exists x\ q(x)$

b) $\forall x\ [p(x) \wedge q(x)] \Longleftrightarrow \forall x\ p(x) \wedge \forall x\ q(x)$

8. a) Let $p(x), q(x)$ be open statements in the variable x, with a given universe. Prove that

$$\forall x\ p(x) \vee \forall x\ q(x) \Rightarrow \forall x\ [p(x) \vee q(x)].$$

[That is, prove that when the statement $\forall x\ p(x) \vee \forall x\ q(x)$ is true, then the statement $\forall x\ [p(x) \vee q(x)]$ is true.]

b) Find a counterexample for the converse in part (a). That is, find open statements $p(x), q(x)$ and a universe such that $\forall x\ [p(x) \vee q(x)]$ is true, while $\forall x\ p(x) \vee \forall x\ q(x)$ is false.

9. Provide the reasons for the steps verifying the following argument. (Here a denotes a specific but arbitrarily chosen element from the given universe.)

$$\begin{array}{l} \forall x\ [p(x) \rightarrow (q(x) \wedge r(x))] \\ \forall x\ [p(x) \wedge s(x)] \\ \hline \therefore \forall x\ [r(x) \wedge s(x)] \end{array}$$

Steps	Reasons
1) $\forall x\ [p(x) \rightarrow (q(x) \wedge r(x))]$	
2) $\forall x\ [p(x) \wedge s(x)]$	
3) $p(a) \rightarrow (q(a) \wedge r(a))$	
4) $p(a) \wedge s(a)$	
5) $p(a)$	
6) $q(a) \wedge r(a)$	
7) $r(a)$	
8) $s(a)$	
9) $r(a) \wedge s(a)$	
10) $\therefore \forall x\ [r(x) \wedge s(x)]$	

10. Provide the missing reasons for the steps verifying the following argument:

$$\begin{array}{l} \forall x\ [p(x) \vee q(x)] \\ \exists x\ \neg p(x) \\ \forall x\ [\neg q(x) \vee r(x)] \\ \forall x\ [s(x) \rightarrow \neg r(x)] \\ \hline \therefore \exists x\ \neg s(x) \end{array}$$

Steps	Reasons
1) $\forall x\,[p(x) \vee q(x)]$	Premise
2) $\exists x\,\neg p(x)$	Premise
3) $\neg p(a)$	Step (2) and the definition of the truth for $\exists x\,\neg p(x)$. [Here a is an element (replacement) from the universe for which $\neg p(x)$ is true.] The reason for this step is also referred to as the *Rule of Existential Specification.*
4) $p(a) \vee q(a)$	
5) $q(a)$	
6) $\forall x\,[\neg q(x) \vee r(x)]$	
7) $\neg q(a) \vee r(a)$	
8) $q(a) \rightarrow r(a)$	
9) $r(a)$	
10) $\forall x\,[s(x) \rightarrow \neg r(x)]$	
11) $s(a) \rightarrow \neg r(a)$	
12) $r(a) \rightarrow \neg s(a)$	
13) $\neg s(a)$	
14) $\therefore \exists x\,\neg s(x)$	Step (13) and the definition of the truth for $\exists x\,\neg s(x)$. The reason for this step is also referred to as the *Rule of Existential Generalization.*

11. Write the following argument in symbolic form. Then either verify the validity of the argument or explain why it is invalid. [Assume here that the universe comprises all adults (18 or over) who are presently residing in the city of Las Cruces (in New Mexico). Two of these individuals are Roxe and Imogene.]

All credit union employees must know COBOL. All credit union employees who write loan applications must know Excel.† Roxe works for the credit union, but she doesn't know Excel. Imogene knows Excel but doesn't know COBOL. Therefore Roxe doesn't write loan applications and Imogene doesn't work for the credit union.

12. Give a direct proof (as in Theorem 2.3) for each of the following.

a) For all integers k and l, if k, l are both even, then $k + l$ is even.

b) For all integers k and l, if k, l are both even, then kl is even.

13. For each of the following statements provide an indirect proof [as in part (2) of Theorem 2.4] by stating and proving the contrapositive of the given statement.

a) For all integers k and l, if kl is odd, then k, l are both odd.

b) For all integers k and l, if $k + l$ is even, then k and l are both even or both odd.

14. Prove that for every integer n, if n is odd, then n^2 is odd.

15. Provide a proof by contradiction for the following: For every integer n, if n^2 is odd, then n is odd.

16. Prove that for every integer n, n^2 is even if and only if n is even.

17. Prove the following result in three ways (as in Theorem 2.4): If n is an odd integer, then $n + 11$ is even.

18. Let m, n be two positive integers. Prove that if m, n are perfect squares, then the product mn is also a perfect square.

19. Prove or disprove: If m, n are positive integers and m, n are perfect squares, then $m + n$ is a perfect square.

20. Prove or disprove: There exist positive integers m, n, where m, n, and $m + n$ are all perfect squares.

21. Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

22. Prove that for every integer n, $4n + 7$ is odd.

23. Let n be an integer. Prove that n is odd if and only if $7n + 8$ is odd.

24. Let n be an integer. Prove that n is even if and only if $31n + 12$ is even.

2.6 Summary and Historical Review

This second chapter has introduced some of the fundamentals of logic — in particular, some of the rules of inference and methods of proof necessary for establishing mathematical theorems.

The first systematic study of logical reasoning is found in the work of the Greek philosopher Aristotle (384–322 B.C.). In his treatises on logic Aristotle presented a collection of principles for deductive reasoning. These principles were designed to provide a foundation

†The Excel spreadsheet is a product of Microsoft, Inc.

15. **a)** The proposed negation is correct and is a true statement.
b) The proposed negation is wrong. A correct version of the negation is: For all rational numbers x, y, the sum $x + y$ is rational. This correct version of the negation is a true statement.
d) The proposed negation is wrong. A correct version of the negation is: For all integers x, y, if x, y are both odd, then xy is even. The (original) statement is true.

17. **a)** There exists an integer n such that n is not divisible by 2 but n is even (that is, not odd).
b) There exist integers k, m, n such that $k - m$ and $m - n$ are odd, and $k - n$ is odd.
d) There exists a real number x such that $|x - 3| < 7$ and either $x \leq -4$ or $x \geq 10$.

19. **a)** *Statement*: For all positive integers m, n, if $m > n$, then $m^2 > n^2$. (TRUE)
Converse: For all positive integers m, n, if $m^2 > n^2$, then $m > n$. (TRUE)
Inverse: For all positive integers m, n, if $m \leq n$, then $m^2 \leq n^2$. (TRUE)
Contrapositive: For all positive integers m, n, if $m^2 \leq n^2$, then $m \leq n$. (TRUE)
b) *Statement*: For all integers a, b, if $a > b$, then $a^2 > b^2$. (FALSE—let $a = 1$ and $b = -2$.)
Converse: For all integers a, b, if $a^2 > b^2$, then $a > b$. (FALSE—let $a = -5$ and $b = 3$.)
Inverse: For all integers a, b, if $a \leq b$, then $a^2 \leq b^2$. (FALSE—let $a = -5$ and $b = 3$.)
Contrapositive: For all integers a, b, if $a^2 \leq b^2$, then $a \leq b$. (FALSE—let $a = 1$ and $b = -2$.)
c) *Statement*: For all integers m, n, and p, if m divides n and n divides p, then m divides p. (TRUE)
Converse: For all integers m and p, if m divides p, then for each integer n it follows that m divides n and n divides p. (FALSE—let $m = 1$, $n = 2$, and $p = 3$.)
Inverse: For all integers m, n, and p, if m does not divide n or n does not divide p, then m does not divide p. (FALSE—let $m = 1$, $n = 2$, and $p = 3$.)
Contrapositive: For all integers m and p, if m does not divide p, then for each integer n it follows that m does not divide n or n does not divide p. (TRUE)
e) *Statement*: $\forall x\ [(x^2 + 4x - 21 > 0) \rightarrow [(x > 3) \vee (x < -7)]]$ (TRUE)
Converse: $\forall x\ [[(x > 3) \vee (x < -7)] \rightarrow (x^2 + 4x - 21 > 0)]$ (TRUE)
Inverse: $\forall x\ [(x^2 + 4x - 21 \leq 0) \rightarrow [(x \leq 3) \wedge (x \geq -7)]]$, or $\forall x\ [(x^2 + 4x - 21 \leq 0) \rightarrow (-7 \leq x \leq 3)]$ (TRUE)
Contrapositive: $\forall x\ [[(x \leq 3) \wedge (x \geq -7)] \rightarrow (x^2 + 4x - 21 \leq 0)]$, or $\forall x\ [(-7 \leq x \leq 3) \rightarrow (x^2 + 4x - 21 \leq 0)]$ (TRUE)

21. **a)** True **b)** False **c)** False **d)** True **e)** False

23. **a)** $\forall a\ \exists b\ [a + b = b + a = 0]$ **b)** $\exists u\ \forall a\ [au = ua = a]$ **c)** $\forall a \neq 0\ \exists b\ [ab = ba = 1]$
d) The statement in part (b) remains true, but the statement in part (c) is no longer true for this new universe.

25. **a)** $\exists x\ \exists y\ [(x > y) \wedge (x - y \leq 0)]$ **b)** $\exists x\ \exists y\ [(x < y) \wedge \forall z\ [x \geq z \vee z \geq y]]$

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1. Although we may write $28 = 25 + 1 + 1 + 1 = 16 + 4 + 4 + 4$, there is no way to express 28 as the sum of at most three perfect squares.

3.

$30 = 25 + 4 + 1$	$40 = 36 + 4$	$50 = 25 + 25$
$32 = 16 + 16$	$42 = 25 + 16 + 1$	$52 = 36 + 16$
$34 = 25 + 9$	$44 = 36 + 4 + 4$	$54 = 25 + 25 + 4$
$36 = 36$	$46 = 36 + 9 + 1$	$56 = 36 + 16 + 4$
$38 = 36 + 1 + 1$	$48 = 16 + 16 + 16$	$58 = 49 + 9$

5. **a)** The real number π is not an integer.
c) All administrative directors know how to delegate authority.
d) Quadrilateral $MNPQ$ is not equiangular.

7. **a)** When the statement $\exists x\ [p(x) \vee q(x)]$ is true, there is at least one element c in the prescribed universe where $p(c) \vee q(c)$ is true. Hence at least one of the statements $p(c)$, $q(c)$ has the truth value 1, so at least one of the statements $\exists x\ p(x)$ and $\exists x\ q(x)$ is true. Therefore, it follows that $\exists x\ p(x) \vee \exists x\ q(x)$ is true, and $\exists x\ [p(x) \vee q(x)] \Rightarrow \exists x\ p(x) \vee \exists x\ q(x)$. Conversely, if $\exists x\ p(x) \vee \exists x\ q(x)$ is true, then at least one of $p(a)$, $q(b)$ has the truth value 1,

for some a, b in the prescribed universe. Assume without loss of generality that it is $p(a)$. Then $p(a) \vee q(a)$ has truth value 1 so $\exists x\ [p(x) \vee q(x)]$ is a true statement, and $\exists x\ p(x) \vee \exists x\ q(x) \Rightarrow \exists x\ [p(x) \vee q(x)]$.

b) First consider when the statement $\forall x\ [p(x) \wedge q(x)]$ is true. This occurs when $p(a) \wedge q(a)$ is true for each a in the prescribed universe. Then $p(a)$ is true [as is $q(a)$] for all a in the universe, so the statements $\forall x\ p(x)$ and $\forall x\ q(x)$ are true. Therefore, the statement $\forall x\ p(x) \wedge \forall x\ q(x)$ is true and $\forall x\ [p(x) \wedge q(x)] \Rightarrow \forall x\ p(x) \wedge \forall x\ q(x)$. Conversely, suppose that $\forall x\ p(x) \wedge \forall x\ q(x)$ is a true statement. Then $\forall x\ p(x)$, $\forall x\ q(x)$ are both true. So now let c be any element in the prescribed universe. Then $p(c)$, $q(c)$, and $p(c) \wedge q(c)$ are all true. And since c was chosen arbitrarily, it follows that the statement $\forall x\ [p(x) \wedge q(x)]$ is true, and $\forall x\ p(x) \wedge \forall x\ q(x) \Rightarrow \forall x\ [p(x) \wedge q(x)]$.

9. **1)** Premise
2) Premise
3) Step (1) and the Rule of Universal Specification
4) Step (2) and the Rule of Universal Specification
5) Step (4) and the Rule of Conjunctive Simplification
6) Steps (5) and (3) and Modus Ponens
7) Step (6) and the Rule of Conjunctive Simplification
8) Step (4) and the Rule of Conjunctive Simplification
9) Steps (7) and (8) and the Rule of Conjunction
10) Step (9) and the Rule of Universal Generalization

11. Consider the open statements

$w(x)$: x works for the credit union
$\ell(x)$: x writes loan applications
$c(x)$: x knows COBOL
$q(x)$: x knows Excel

and let r represent Roxe and i represent Imogene.

In symbolic form the given argument is as follows:

$$\begin{array}{l} \forall x\ [w(x) \rightarrow c(x)] \\ \forall x\ [(w(x) \wedge \ell(x)) \rightarrow q(x)] \\ w(r) \wedge \neg q(r) \\ q(i) \wedge \neg c(i) \\ \hline \therefore \neg \ell(r) \wedge \neg w(i) \end{array}$$

The steps (and reasons) needed to verify this argument can now be presented.

Steps	**Reasons**
1) $\forall x\ [w(x) \rightarrow c(x)]$	Premise
2) $q(i) \wedge \neg c(i)$	Premise
3) $\neg c(i)$	Step (2) and the Rule of Conjunctive Simplification
4) $w(i) \rightarrow c(i)$	Step (1) and the Rule of Universal Specification
5) $\neg w(i)$	Steps (3) and (4) and Modus Tollens
6) $\forall x\ [(w(x) \wedge \ell(x)) \rightarrow q(x)]$	Premise
7) $w(r) \wedge \neg q(r)$	Premise
8) $\neg q(r)$	Step (7) and the Rule of Conjunctive Simplification
9) $(w(r) \wedge \ell(r)) \rightarrow q(r)$	Step (6) and the Rule of Universal Specification
10) $\neg(w(r) \wedge \ell(r))$	Steps (8) and (9) and Modus Tollens
11) $w(r)$	Step (7) and the Rule of Conjunctive Simplification
12) $\neg w(r) \vee \neg \ell(r)$	Step (10) and DeMorgan's Law
13) $\neg \ell(r)$	Steps (11) and (12) and the Rule of Disjunctive Syllogism
14) $\therefore \neg \ell(r) \wedge \neg w(i)$	Steps (13) and (5) and the Rule of Conjunction

13. **a)** *Contrapositive*: For all integers k and ℓ, if k, ℓ are not both odd, then $k\ell$ is not odd — OR, For all integers k and ℓ, if at least one of k, ℓ is even, then $k\ell$ is even.
Proof: Let us assume (without loss of generality) that k is even. Then $k = 2c$ for some integer c — because of Definition 2.8. Then $k\ell = (2c)\ell = 2(c\ell)$, by the associative law of multiplication for integers — and $c\ell$ is an integer. Consequently, $k\ell$ is even — once again, by Definition 2.8. (Note that this result does not require anything about the integer ℓ.)

15. *Proof*: Assume that for some integer n, n^2 is odd while n is not odd. Then n is even and we may write $n = 2a$, for some integer a — by Definition 2.8. Consequently, $n^2 = (2a)^2 = (2a)(2a) = (2 \cdot 2)(a \cdot a)$, by the commutative and associative laws of multiplication for integers. Hence, we may write $n^2 = 2(2a^2)$, with $2a^2$ an integer — and this means that n^2 is even. Thus we have arrived at a contradiction, since we now have n^2 both odd (at the start) and even. This contradiction came about from the false assumption that n is not odd. Therefore, for every integer n, it follows that n^2 odd $\Rightarrow n$ odd.

17. *Proof*:
(1) Since n is odd, we have $n = 2a + 1$ for some integer a. Then $n + 11 = (2a + 1) + 11 = 2a + 12 = 2(a + 6)$, where $a + 6$ is an integer. So by Definition 2.8 it follows that $n + 11$ is even.
(2) If $n + 11$ is not even, then it is odd and we have $n + 11 = 2b + 1$, for some integer b. So $n = (2b + 1) - 11 = 2b - 10 = 2(b - 5)$, where $b - 5$ is an integer, and it follows from Definition 2.8 that n is even — that is, not odd.
(3) In this case we stay with the hypothesis — that n is odd — and also assume that $n + 11$ is not even — hence, odd. So we may write $n + 11 = 2b + 1$, for some integer b. This then implies that $n = 2(b - 5)$, for the integer $b - 5$. So by Definition 2.8 it follows that n is even. But with n both even (as shown) and odd (as in the hypothesis), we have arrived at a contradiction. So our assumption was wrong, and it now follows that $n + 11$ is even for every odd integer n.

19. This result is not true, in general. For example, $m = 4 = 2^2$ and $n = 1 = 1^2$ are two positive integers that are perfect squares, but $m + n = 2^2 + 1^2 = 5$ is not a perfect square.

21. *Proof*:
We shall prove the given result by establishing the truth of its (logically equivalent) contrapositive.

Let us consider the negation of the conclusion — that is, $x < 50$ and $y < 50$. Then with $x < 50$ and $y < 50$ it follows that $x + y < 50 + 50 = 100$, and we have the negation of the hypothesis. The given result now follows by this indirect method of proof (by the contrapositive).

23. *Proof*: If n is odd, then $n = 2k + 1$ for some (particular) integer k. Then $7n + 8 = 7(2k + 1) + 8 = 14k + 7 + 8 = 14k + 15 = 14k + 14 + 1 = 2(7k + 7) + 1$. It then follows from Definition 2.8 that $7n + 8$ is odd.

To establish the converse, suppose that n is not odd. Then n is even, so we can write $n = 2t$, for some (particular) integer t. But then $7n + 8 = 7(2t) + 8 = 14t + 8 = 2(7t + 4)$, so it follows from Definition 2.8 that $7n + 8$ is even — that is, $7n + 8$ is not odd. Consequently, the converse follows by contraposition.

Supplementary Exercises – p. 120

1.

p	q	r	s	$q \wedge r$	$\neg(s \vee r)$	$\overbrace{[(q \wedge r) \rightarrow \neg(s \vee r)]}^{t}$	$p \leftrightarrow t$
0	0	0	0	0	1	1	0
0	0	0	1	0	0	1	0
0	0	1	0	0	0	1	0
0	0	1	1	0	0	1	0
0	1	0	0	0	1	1	0
0	1	0	1	0	0	1	0
0	1	1	0	1	0	0	1
0	1	1	1	1	0	0	1