

The new SDPRX

1 Likelihood

$$\begin{aligned}\widehat{\beta}_1 \mid \eta, \beta_1 &\sim N(R_1 \eta \beta_1, R_1/N_1 + aI) \\ \widehat{\beta}_2 \mid \eta, \beta_2 &\sim N(R_2 \eta \beta_2, R_2/N_2 + aI) \\ \widehat{\beta}_3 \mid \eta, \beta_3 &\sim N(R_3 \eta \beta_3, R_3/N_3 + aI)\end{aligned}$$

2 Priors

$$\begin{aligned}\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} &\sim \theta \begin{pmatrix} \delta_0 \\ \delta_0 \\ \delta_0 \end{pmatrix} + (1 - \theta) \sum_k^K \pi_k N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma_k^2 \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix} \right) \\ \theta &\sim \text{unif}(0, 1) \text{ (Also Beta}(1, 1)) \\ V_k &\sim \text{Beta}(1, \alpha) \\ \pi_1 &= V_1 \\ \pi_k &= \prod_{m=1}^{k-1} (1 - V_m) V_k \\ \sigma_k^2 &\sim IG(.5, .5) \\ \alpha &\sim \text{Gamma}(0.1, 0.1) \\ \eta &\sim (0, 10^6)\end{aligned}$$

We will have correlation as

1. ρ_1 : correlation between population 1 and 2
2. ρ_2 : correlation between population 1 and 3
3. ρ_3 : correlation between population 2 and 3

let

$$\begin{aligned}\beta &= (\beta_1, \beta_2, \beta_3)^T \\ \beta_1 &= (\beta_{11}, \beta_{12}, \beta_{13}, \dots, \beta_{1n})^T\end{aligned}$$

and

$$\beta_j = (\beta_{1j}, \beta_{2j}, \beta_{3j})^T$$

that represents the individual i and note that R_1 , R_2 and R_3 are symmetric

3 Posterior

with

$$S_0 = \sigma_k^2 \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}$$

and we note

$$|S_0| = \sigma_k^6 s_0 = \sigma_k^6 (-\rho_1^2 - \rho_2^2 - \rho_3^2 + 2\rho_1\rho_2\rho_3 + 1)$$

and

$$s_0 = -\rho_1^2 - \rho_2^2 - \rho_3^2 + 2\rho_1\rho_2\rho_3 + 1$$

Also

$$S_0^{-1} = \frac{1}{s_0\sigma_k^2} \begin{pmatrix} 1 - \rho_3^2 & \rho_2\rho_3 - \rho_1 & \rho_1\rho_3 - \rho_2 \\ \rho_2\rho_3 - \rho_1 & 1 - \rho_2^2 & \rho_1\rho_2 - \rho_3 \\ \rho_1\rho_3 - \rho_2 & \rho_1\rho_2 - \rho_3 & 1 - \rho_1^2 \end{pmatrix} = \frac{S}{\sigma_k^2}$$

We will have

$$\beta^T S \beta = ((1 - \rho_3^2)\beta_1^2 + (1 - \rho_2^2)\beta_2^2 + (1 - \rho_1^2)\beta_3^2 + 2(\rho_2\rho_3 - \rho_1)\beta_1\beta_2 + 2(\rho_1\rho_3 - \rho_2)\beta_1\beta_3 + 2(\rho_1\rho_2 - \rho_3)\beta_2\beta_3) / s_0$$

3.1 Assignment

$$A_1 = (R_1 + N_1 a I)^{-1} R_1, A_2 = (R_2 + N_2 a I)^{-1} R_2, A_3 = (R_3 + N_3 a I)^{-1} R_3$$

$$B_1 = R_1 A_1, B_2 = R_2 A_2, B_3 = R_3 A_3$$

We will still have the assignment posterior z as:

$$P(z = (k, 1) \mid .)$$

Where $(k, 1)$ means the k th cluster in the non-null population. So we have

$$\begin{aligned} P(z = (k, 1) \mid .) &= \int \int \int P(\hat{\beta}_1 \mid \eta\beta_1) P(\hat{\beta}_2 \mid \eta\beta_2) P(\hat{\beta}_3 \mid \eta\beta_3) P(\beta_1, \beta_2, \beta_3 \mid z = (k, 1), \sigma_k^2) d\beta_1 d\beta_2 d\beta_3 \\ &\quad \times P(z = (k, 1)) \end{aligned}$$

It is

$$\begin{aligned} &\int \int \int \exp\left\{-\frac{1}{2}(\hat{\beta}_1 - \eta R_1 \beta_1)^T (R_1/N_1 + aI)^{-1} (\hat{\beta}_1 - \eta R_1 \beta_1)\right\} \\ &\exp\left\{-\frac{1}{2}(\hat{\beta}_2 - \eta R_2 \beta_2)^T (R_2/N_2 + aI)^{-1} (\hat{\beta}_2 - \eta R_2 \beta_2)\right\} \\ &\exp\left\{-\frac{1}{2}(\hat{\beta}_3 - \eta R_3 \beta_3)^T (R_3/N_3 + aI)^{-1} (\hat{\beta}_3 - \eta R_3 \beta_3)\right\} \\ &\exp\left\{-\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2} \beta_j\right\} \frac{1}{\sqrt{(2\pi)^3 \sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1 - \theta) \pi_k \end{aligned}$$

For the term

$$-\frac{1}{2}(\hat{\beta}_1 - \eta R_1 \beta_1)^T (R_1/N_1 + aI)^{-1} (\hat{\beta}_1 - \eta R_1 \beta_1)$$

We can assume that $M = (R_1/N_1 + aI)^{-1}$ and then it is

$$-\frac{1}{2}(\hat{\beta}_1^T M \hat{\beta}_1 - 2\hat{\beta}_1^T M \eta R_1 \beta_1 + \eta^2 \beta_1^T R_1 M R_1 \beta_1)$$

where $A_1 = (M R_1)/N_1$ and $B_1 = (R_1 M R_1)/N_1$. Retaining terms only related to variable $\beta_{1j}, \beta_{2j}, \beta_{3j}$ we have

$$\begin{aligned} & \int \int \int \exp\left\{-\frac{N_1}{2}\eta^2 B_{1,jj}\beta_{1j}^2 - N_1\eta^2 \sum_{i \neq j} B_{1,ij}\beta_{1i}\beta_{1j} + N_1\eta \sum_i A_{1,ij}\hat{\beta}_{1i}\beta_{1j}\right\} \\ & \exp\left\{-\frac{N_2}{2}\eta^2 B_{2,jj}\beta_{2j}^2 - N_2\eta^2 \sum_{i \neq j} B_{2,ij}\beta_{2i}\beta_{2j} + N_2\eta \sum_i A_{2,ij}\hat{\beta}_{2i}\beta_{2j}\right\} \\ & \exp\left\{-\frac{N_3}{2}\eta^2 B_{3,jj}\beta_{3j}^2 - N_3\eta^2 \sum_{i \neq j} B_{3,ij}\beta_{3i}\beta_{3j} + N_3\eta \sum_i A_{3,ij}\hat{\beta}_{3i}\beta_{3j}\right\} \\ & \exp\left\{-\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2} \beta_j\right\} \frac{1}{\sqrt{(2\pi)^3 \sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1 - \theta)\pi_k \end{aligned}$$

Following the steps given in SDPRX, we then have

$$\begin{aligned} b_{1j} &= \eta \sum_i A_{1,ij}\hat{\beta}_{1i} - \eta^2 \sum_{i \neq j} B_{1,ij}\beta_{1i} \\ b_{2j} &= \eta \sum_i A_{2,ij}\hat{\beta}_{2i} - \eta^2 \sum_{i \neq j} B_{2,ij}\beta_{2i} \\ b_{3j} &= \eta \sum_i A_{3,ij}\hat{\beta}_{3i} - \eta^2 \sum_{i \neq j} B_{3,ij}\beta_{3i} \end{aligned}$$

which makes the formula

$$\begin{aligned} & \int \int \int \exp\left\{-\frac{N_1}{2}\eta^2 B_{1,jj}\beta_{1j}^2 + N_1 b_{1j}\beta_{1j}\right\} \\ & \exp\left\{-\frac{N_2}{2}\eta^2 B_{2,jj}\beta_{2j}^2 + N_2 b_{2j}\beta_{2j}\right\} \\ & \exp\left\{-\frac{N_3}{2}\eta^2 B_{3,jj}\beta_{3j}^2 + N_3 b_{3j}\beta_{3j}\right\} \\ & \exp\left\{\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2} \beta_j\right\} \frac{1}{\sqrt{(2\pi)^3 \sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1 - \theta)\pi_k \\ & = \int \int \int \exp\left\{-\frac{N_1}{2}\eta^2 B_{1,jj}\beta_{1j}^2 + N_1 b_{1j}\beta_{1j}\right\} \\ & \exp\left\{-\frac{N_2}{2}\eta^2 B_{2,jj}\beta_{2j}^2 + N_2 b_{2j}\beta_{2j}\right\} \\ & \exp\left\{-\frac{N_3}{2}\eta^2 B_{3,jj}\beta_{3j}^2 + N_3 b_{3j}\beta_{3j}\right\} \\ & \exp\left\{-\frac{(1 - \rho_3^2)\beta_{1j}^2 + (1 - \rho_2^2)\beta_{2j}^2 + (1 - \rho_1^2)\beta_{3j}^2 + 2(\rho_2\rho_3 - \rho_1)\beta_{1j}\beta_{2j} + 2(\rho_1\rho_3 - \rho_2)\beta_{1j}\beta_{3j} + 2(\rho_1\rho_2 - \rho_3)\beta_{2j}\beta_{3j}}{2s_0\sigma_k^2}\right\} \\ & \frac{1}{\sqrt{(2\pi)^3 \sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1 - \theta)\pi_k \end{aligned}$$

We then denote

$$\begin{aligned}
a_{jk1} &= N_1 \eta^2 B_{1,jj} + \frac{1 - \rho_3^2}{s_0 \sigma_k^2} \\
a_{jk2} &= N_2 \eta^2 B_{2,jj} + \frac{1 - \rho_2^2}{s_0 \sigma_k^2} \\
a_{jk3} &= N_3 \eta^2 B_{3,jj} + \frac{1 - \rho_1^2}{s_0 \sigma_k^2} \\
c_{k1} &= \frac{\rho_2 \rho_3 - \rho_1}{s_0 \sigma_k^2} \\
c_{k2} &= \frac{\rho_1 \rho_3 - \rho_2}{s_0 \sigma_k^2} \\
c_{k3} &= \frac{\rho_1 \rho_2 - \rho_3}{s_0 \sigma_k^2}
\end{aligned}$$

this makes the formula

$$\begin{aligned}
&\int \int \int \exp\left\{-\frac{1}{2}a_{jk1}\beta_{1j}^2 + N_1 b_{1j}\beta_{1j} - \frac{1}{2}a_{jk2}\beta_{2j}^2 + N_2 b_{2j}\beta_{2j} - \frac{1}{2}a_{jk3}\beta_{3j}^2 + N_3 b_{3j}\beta_{3j}\right\} \\
&\quad \exp\{-c_{k1}\beta_{1j}\beta_{2j} - c_{k2}\beta_{1j}\beta_{3j} - c_{k3}\beta_{2j}\beta_{3j}\} \frac{1}{\sqrt{(2\pi)^3 \sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1 - \theta) \pi_k
\end{aligned}$$

To solve the integration we will use the formula for Gaussian Integration

$$\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}x^\top A x + b^\top x + c\right) d^n x = \sqrt{\det(2\pi A^{-1})} e^{\frac{1}{2}b^\top A^{-1}b + c}$$

where A is supposed to be a symmetric matrix. In this case, we will have

$$x = (\beta_{1j}, \beta_{2j}, \beta_{3j})^T, b = (N_1 b_{1j}, N_2 b_{2j}, N_3 b_{3j})^T$$

and also

$$A = \begin{pmatrix} a_{jk1} & c_{k1} & c_{k2} \\ c_{k1} & a_{jk2} & c_{k3} \\ c_{k2} & c_{k3} & a_{jk3} \end{pmatrix}$$

where

$$|A| = a_{jk1}a_{jk2}a_{jk3} + 2c_{k1}c_{k2}c_{k3} - a_{jk1}c_{k3}^2 - a_{jk3}c_{k1}^2 - a_{jk2}c_{k2}^2 = s_1$$

and

$$A^{-1} = \frac{1}{s_1} \begin{pmatrix} a_{jk2}a_{jk3} - c_{k3}^2 & c_{k2}c_{k3} - c_{k1}a_{jk3} & c_{k1}c_{k3} - c_{k2}a_{jk2} \\ c_{k2}c_{k3} - c_{k1}a_{jk3} & a_{jk1}a_{jk3} - c_{k2}^2 & c_{k1}c_{k2} - c_{k3}a_{jk1} \\ c_{k1}c_{k3} - c_{k2}a_{jk2} & c_{k1}c_{k2} - c_{k3}a_{jk1} & a_{jk1}a_{jk2} - c_{k1}^2 \end{pmatrix} = L$$

Here we have

$$\sqrt{\det(2\pi A^{-1})} = \sqrt{\frac{(2\pi)^3}{s_1}}$$

and

$$b^\top A^{-1}b = L_{11}N_1^2 b_{1j}^2 + L_{22}N_2^2 b_{2j}^2 + L_{33}N_3^2 b_{3j}^2 + 2L_{12}N_1 b_{1j}N_2 b_{2j} + 2L_{13}N_1 b_{1j}N_3 b_{3j} + 2L_{23}N_2 b_{2j}N_3 b_{3j}$$

where

$$\begin{aligned}
L_{11} &= \frac{a_{jk2}a_{jk3} - c_{k3}^2}{s_1} \\
L_{22} &= \frac{a_{jk1}a_{jk3} - c_{k2}^2}{s_1} \\
L_{33} &= \frac{a_{jk1}a_{jk2} - c_{k1}^2}{s_1} \\
L_{12} &= \frac{c_{k2}c_{k3} - c_{k1}a_{jk3}}{s_1} \\
L_{13} &= \frac{c_{k1}c_{k3} - c_{k2}a_{jk2}}{s_1} \\
L_{23} &= \frac{c_{k1}c_{k2} - c_{k3}a_{jk1}}{s_1}
\end{aligned}$$

We finally have the results that

$$\begin{aligned}
&P(z_j = (1, k)|\cdot) \propto \\
&\frac{\pi_k(1 - \theta)}{\sqrt{s_1 s_0 \sigma_k^6}} \exp\left\{\frac{1}{2}(L_{11}N_1^2 b_{1j}^2 + L_{22}N_2^2 b_{2j}^2 + L_{33}N_3^2 b_{3j}^2 + 2L_{12}N_1 b_{1j} N_2 b_{2j} + 2L_{13}N_1 b_{1j} N_3 b_{3j} + 2L_{23}N_2 b_{2j} N_3 b_{3j})\right\}
\end{aligned}$$

3.2 beta

The posterior of β should be similar to SDPRX, we select those with non-zero value as $\gamma_1, \gamma_2, \gamma_3$, we will have

$$\begin{aligned}
& P(\beta_{1,\gamma_1}, \beta_{2,\gamma_2}, \beta_{3,\gamma_3} | \cdot) \\
& \propto \exp\left\{-\frac{N_1}{2}\eta^2\beta_1^T B_1\beta_1 + \eta N_1\hat{\beta}_1^T A_1\beta_1\right\} \\
& \exp\left\{-\frac{N_2}{2}\eta^2\beta_2^T B_2\beta_2 + \eta N_2\hat{\beta}_2^T A_2\beta_2\right\} \\
& \exp\left\{-\frac{N_3}{2}\eta^2\beta_3^T B_3\beta_3 + \eta N_3\hat{\beta}_3^T A_3\beta_3\right\} \\
& \exp\left\{-\frac{1}{2}\beta^T \Sigma_0^{-1}\beta\right\} \\
& \propto \exp\left\{\frac{1}{2}\eta^2\beta_\gamma^T B_\gamma\beta_\gamma + \eta\hat{\beta}_\gamma^T A_\gamma\beta_\gamma\right\} \exp\left\{-\frac{1}{2}\beta_\gamma^T \Sigma_{0\gamma}^{-1}\beta_\gamma\right\}
\end{aligned}$$

where

$$\begin{aligned}
\beta_\gamma &= (\beta_{1,\gamma_1}, \beta_{2,\gamma_2}, \beta_{3,\gamma_3})^T \\
A_\gamma &= (N_1\hat{\beta}_1^T A_{1,\gamma_1}, N_2\hat{\beta}_2^T A_{2,\gamma_2}, N_3\hat{\beta}_3^T A_{3,\gamma_3}) \\
B_\gamma &= \begin{pmatrix} N_1 B_{1,\gamma_1} & 0 & 0 \\ 0 & N_2 B_{2,\gamma_2} & 0 \\ 0 & 0 & N_3 B_{3,\gamma_3} \end{pmatrix}
\end{aligned}$$

It is

$$MVN(\eta \Sigma A_\gamma^T \hat{\beta}_\gamma, \Sigma)$$

where

$$\Sigma = (\eta^2 B_\gamma + \Sigma_0^{-1})^{-1}$$

We will try to derive the inverse of Σ_0 where

$$\Sigma_{0\gamma} = \begin{pmatrix} \Sigma_{00} & \rho_1 \Sigma_{00} & \rho_2 \Sigma_{00} \\ \rho_1 \Sigma_{00} & \Sigma_{00} & \rho_3 \Sigma_{00} \\ \rho_2 \Sigma_{00} & \rho_3 \Sigma_{00} & \Sigma_{00} \end{pmatrix}$$

we will have

$$\Sigma_{0\gamma}^{-1} = \begin{pmatrix} (1 - \rho_3^2)/s_0 \Sigma_{00}^{-1} & (\rho_2 \rho_3 - \rho_1)/s_0 \Sigma_{00}^{-1} & (\rho_1 \rho_3 - \rho_2)/s_0 \Sigma_{00}^{-1} \\ (\rho_2 \rho_3 - \rho_1)/s_0 \Sigma_{00}^{-1} & (1 - \rho_2^2)/s_0 \Sigma_{00}^{-1} & (\rho_1 \rho_2 - \rho_3)/s_0 \Sigma_{00}^{-1} \\ (\rho_1 \rho_3 - \rho_2)/s_0 \Sigma_{00}^{-1} & (\rho_1 \rho_2 - \rho_3)/s_0 \Sigma_{00}^{-1} & (1 - \rho_1^2)/s_0 \Sigma_{00}^{-1} \end{pmatrix} = \begin{pmatrix} S_{11} \Sigma_{00}^{-1} & S_{12} \Sigma_{00}^{-1} & S_{13} \Sigma_{00}^{-1} \\ S_{21} \Sigma_{00}^{-1} & S_{22} \Sigma_{00}^{-1} & S_{23} \Sigma_{00}^{-1} \\ S_{31} \Sigma_{00}^{-1} & S_{32} \Sigma_{00}^{-1} & S_{33} \Sigma_{00}^{-1} \end{pmatrix}$$

that Σ_{00}^{-1} is a diagonal matrix having the $1/\sigma_{z_j}^2$ over its diagonal

3.3 Sigma

We have

$$P(\sigma_k^2 | \cdot) \propto \prod_{j: z_j=(1,k)} \frac{1}{\sigma_k} \times \exp\left\{-\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2} \beta_j\right\} \times \sigma_k^{-2(.5-1)} \times \exp\left\{-\frac{.5}{\sigma_k^2}\right\}$$

denote

$$T_j = S_{11}\beta_{1j}^2 + S_{22}\beta_{2j}^2 + S_{33}\beta_{3j}^2 + 2S_{12}\beta_{1j}\beta_{2j} + 2S_{13}\beta_{1j}\beta_{3j} + 2S_{23}\beta_{2j}\beta_{3j}$$

and the posterior is

$$IG(\frac{M_{1k}}{2} + 0.5, \frac{\sum_{j:z_j=(1,k)} T_j}{2} + 0.5)$$

3.4 theta

The population distribution θ is

$$P(\theta|\cdot) = \text{Beta}(M_0 + 1, M_1 + 1)$$

where M_0 is the value of null population and M_1 is the value of non-null population

3.5 stick-breaking parameters

$$P(V_k|\cdot) = \text{Beta}(1 + M_k, \alpha + \sum_{l=k+1}^K M_l)$$

$$\pi_1 = V_1, \pi_k = \prod_{l=1}^{k-1} (1 - V_l) V_k (k \geq 2)$$

$$P(\alpha|\cdot) = \text{Gamma}(0.1 + K - 1, 0.1 - \sum_{k=1}^{K-1} \log(1 - V_k))$$

where K is the total number of clusters

3.6 eta

$$\begin{aligned} & P(\eta|\cdot) \\ & \propto \exp\{-\frac{N_1}{2}\eta^2 \sum_{LD} \beta_1^T B_1 \beta_1 + \eta \sum_{LD} \hat{\beta}_1^T A_1 \beta_1\} \\ & \exp\{-\frac{N_2}{2}\eta^2 \sum_{LD} \beta_2^T B_2 \beta_2 + \eta \sum_{LD} \hat{\beta}_2^T A_2 \beta_2\} \\ & \exp\{-\frac{N_3}{2}\eta^2 \sum_{LD} \beta_3^T B_3 \beta_3 + \eta \sum_{LD} \hat{\beta}_3^T A_3 \beta_3\} \\ & \exp\{-\frac{\eta^2}{2 \times 10^{-6}}\} \\ & = N(\mu_1, \sigma_1) \end{aligned}$$

where LD is the sum across all LD matrices and

$$\begin{aligned} \mu_1 &= \frac{N_1 \sum_{LD} \hat{\beta}_1^T A_1 \beta_1 + N_2 \sum_{LD} \hat{\beta}_2^T A_2 \beta_2 + N_3 \sum_{LD} \hat{\beta}_3^T A_3 \beta_3}{N_1 \sum_{LD} \beta_1^T B_1 \beta_1 + N_2 \sum_{LD} \beta_2^T B_2 \beta_2 + N_3 \sum_{LD} \beta_3^T B_3 \beta_3 + 10^{-6}} \\ \sigma_1^2 &= \frac{1}{N_1 \sum_{LD} \beta_1^T B_1 \beta_1 + N_2 \sum_{LD} \beta_2^T B_2 \beta_2 + N_3 \sum_{LD} \beta_3^T B_3 \beta_3 + 10^{-6}} \end{aligned}$$