The new SDPRX

1 Likelihood

$$
\begin{aligned}
\widehat{\beta}_1 \mid \eta, \beta_1 &\sim N\left(R_1 \eta \beta_1, R_1/N_1 + aI\right) \\
\widehat{\beta}_2 \mid \eta, \beta_2 &\sim N\left(R_2 \eta \beta_2, R_2/N_2 + aI\right) \\
\widehat{\beta}_3 \mid \eta, \beta_3 &\sim N\left(R_3 \eta \beta_3, R_3/N_3 + aI\right)
\end{aligned}
$$

2 Priors

$$
\left(\begin{array}{c} \beta_1 \\ \beta_2 \\ \beta_3 \end{array}\right) \sim \theta \left(\begin{array}{c} \delta_0 \\ \delta_0 \\ \delta_0 \end{array}\right) + (1-\theta) \sum_k^K \pi_k N\left(\left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right), \sigma_k^2 \left(\begin{array}{ccc} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{array}\right)\right)
$$

$$
\begin{aligned}
&\theta \sim \operatorname{unif}(0,1)(\text{Also Beta}(1,1)) \\
&V_k \sim \operatorname{Beta}(1, \alpha) \\
&\pi_1 = V_1 \\
&\pi_k = \prod_{m=1}^{k-1}\left(1 - V_m\right) V_k \\
&\sigma_k^2 \sim IG(.5, .5) \\
&\alpha \sim \operatorname{Gamma}(0.1, 0.1) \\
&\eta \sim (0, 10^6)
\end{aligned}
$$

We will have correlation as

1. ρ_1: correlation between population 1 and 2
2. ρ_2: correlation between population 1 and 3
3. ρ_3: correlation between population 2 and 3

let

$$
\begin{gathered}
\beta = (\beta_1, \beta_2, \beta_3)^T \\
\beta_1 = (\beta_{11}, \beta_{12}, \beta_{13}, \ldots, \beta_{1n})^T
\end{gathered}
$$

and

$$
\beta_j = (\beta_{1j}, \beta_{2j}, \beta_{3j})^T
$$

that represents the individual i and note that R_1, R_2 and R_3 are symmetric

3 Posterior

with

$$S_0 = \sigma_k^2 \begin{pmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{pmatrix}$$

and we note

$$|S_0| = \sigma_k^6 s_0 = \sigma_k^6(-\rho_1^2 - \rho_2^2 - \rho_3^2 + 2\rho_1\rho_2\rho_3 + 1)$$

and

$$s_0 = -\rho_1^2 - \rho_2^2 - \rho_3^2 + 2\rho_1\rho_2\rho_3 + 1$$

Also

$$S_0^{-1} = \frac{1}{s_0\sigma_k^2} \begin{pmatrix} 1-\rho_3^2 & \rho_2\rho_3-\rho_1 & \rho_1\rho_3-\rho_2 \\ \rho_2\rho_3-\rho_1 & 1-\rho_2^2 & \rho_1\rho_2-\rho_3 \\ \rho_1\rho_3-\rho_2 & \rho_1\rho_2-\rho_3 & 1-\rho_1^2 \end{pmatrix} = \frac{S}{\sigma_k^2}$$

We will have

$$\beta^T S\beta = ((1-\rho_3^2)\beta_1^2+(1-\rho_2^2)\beta_2^2+(1-\rho_1^2)\beta_3^2+2(\rho_2\rho_3-\rho_1)\beta_1\beta_2+2(\rho_1\rho_3-\rho_2)\beta_1\beta_3+2(\rho_1\rho_2-\rho_3)\beta_2\beta_3)/s_0$$

3.1 Assignment

$$A_1 = (R_1 + N_1 aI)^{-1}R_1, A_2 = (R_2 + N_2 aI)^{-1}R_2, A_3 = (R_3 + N_3 aI)^{-1}R_3$$

$$B_1 = R_1A_1, B_2 = R_2A_2, B_3 = R_3A_3$$

We will still have the assignment posterior z as:

$$P(z = (k, 1) \mid .)$$

Where $(k, 1)$ means the kth cluster in the non-null population. So we have

$$P(z = (k, 1) \mid .) = \int\int\int P(\hat{\beta}_1 \mid \eta\beta_1)P(\hat{\beta}_2 \mid \eta\beta_2)P(\hat{\beta}_3 \mid \eta\beta_3)P(\beta_1, \beta_2, \beta_3 \mid z = (k, 1), \sigma_k^2)d\beta_1 d\beta_2 d\beta_3$$

$$\times P(z = (k, 1))$$

It is

$$\begin{aligned}&\int\int\int \exp\{-\frac{1}{2}(\hat{\beta}_1 - \eta R_1\beta_1)^T(R_1/N_1 + aI)^{-1}(\hat{\beta}_1 - \eta R_1\beta_1)\}\\ &\exp\{-\frac{1}{2}(\hat{\beta}_2 - \eta R_2\beta_2)^T(R_2/N_2 + aI)^{-1}(\hat{\beta}_2 - \eta R_2\beta_2)\}\\ &\exp\{-\frac{1}{2}(\hat{\beta}_3 - \eta R_3\beta_3)^T(R_3/N_3 + aI)^{-1}(\hat{\beta}_3 - \eta R_3\beta_3)\}\\ &\exp\{-\frac{1}{2}\beta_j^T\frac{S}{\sigma_k^2}\beta_j\}\frac{1}{\sqrt{(2\pi)^3\sigma_k^6 s_0}}d\beta_1 d\beta_2 d\beta_3 \times (1-\theta)\pi_k\end{aligned}$$

For the term

$$-\frac{1}{2}(\hat{\beta}_1 - \eta R_1\beta_1)^T(R_1/N_1 + aI)^{-1}(\hat{\beta}_1 - \eta R_1\beta_1)$$

We can assume that $M = (R_1/N_1 + aI)^{-1}$ and then it is

$$-\frac{1}{2}(\hat{\beta_1}^T M \hat{\beta_1} - 2\hat{\beta_1}^T M \eta R_1 \beta_1 + \eta^2 \beta_1^T R_1 M R_1 \beta_1)$$

where $A_1 = (MR_1)/N_1$ and $B_1 = (R_1 M R_1)/N_1$ Retaining terms only related to variable $\beta_{1j}, \beta_{2j}, \beta_{3j}$ we have

$$\begin{aligned}
&\int\int\int \exp\{-\frac{N_1}{2}\eta^2 B_{1,jj}\beta_{1j}^2 - N_1\eta^2 \sum_{i \neq j} B_{1,ij}\beta_{1i}\beta_{1j} + N_1\eta \sum_i A_{1,ij}\hat{\beta}_{1i}\beta_{1j}\} \\
&\exp\{-\frac{N_2}{2}\eta^2 B_{2,jj}\beta_{2j}^2 - N_2\eta^2 \sum_{i \neq j} B_{2,ij}\beta_{2i}\beta_{2j} + N_2\eta \sum_i A_{2,ij}\hat{\beta}_{2i}\beta_{2j}\} \\
&\exp\{-\frac{N_3}{2}\eta^2 B_{3,jj}\beta_{3j}^2 - N_3\eta^2 \sum_{i \neq j} B_{3,ij}\beta_{3i}\beta_{3j} + N_3\eta \sum_i A_{3,ij}\hat{\beta}_{3i}\beta_{3j}\} \\
&\exp\{-\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2}\beta_j\} \frac{1}{\sqrt{(2\pi)^3\sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1-\theta)\pi_k
\end{aligned}$$

Following the steps given in SDPRX, we then have

$$\begin{aligned}
b_{1j} &= \eta \sum_i A_{1,ij}\hat{\beta}_{1i} - \eta^2 \sum_{i \neq j} B_{1,ij}\beta_{1i} \\
b_{2j} &= \eta \sum_i A_{2,ij}\hat{\beta}_{2i} - \eta^2 \sum_{i \neq j} B_{2,ij}\beta_{2i} \\
b_{3j} &= \eta \sum_i A_{3,ij}\hat{\beta}_{3i} - \eta^2 \sum_{i \neq j} B_{3,ij}\beta_{3i}
\end{aligned}$$

which makes the formula

$$\begin{aligned}
&\int\int\int \exp\{-\frac{N_1}{2}\eta^2 B_{1,jj}\beta_{1j}^2 + N_1 b_{1j}\beta_{1j}\} \\
&\exp\{-\frac{N_2}{2}\eta^2 B_{2,jj}\beta_{2j}^2 + N_2 b_{2j}\beta_{2j}\} \\
&\exp\{-\frac{N_3}{2}\eta^2 B_{3,jj}\beta_{3j}^2 + N_3 b_{3j}\beta_{3j}\} \\
&\exp\{\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2}\beta_j\} \frac{1}{\sqrt{(2\pi)^3\sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1-\theta)\pi_k \\
&= \int\int\int \exp\{-\frac{N_1}{2}\eta^2 B_{1,jj}\beta_{1j}^2 + N_1 b_{1j}\beta_{1j}\} \\
&\exp\{-\frac{N_2}{2}\eta^2 B_{2,jj}\beta_{2j}^2 + N_2 b_{2j}\beta_{2j}\} \\
&\exp\{-\frac{N_3}{2}\eta^2 B_{3,jj}\beta_{3j}^2 + N_3 b_{3j}\beta_{3j}\} \\
&\exp\{-\frac{(1-\rho_3^2)\beta_{1j}^2 + (1-\rho_2^2)\beta_{2j}^2 + (1-\rho_1^2)\beta_{3j}^2 + 2(\rho_2\rho_3-\rho_1)\beta_{1j}\beta_{2j} + 2(\rho_1\rho_3-\rho_2)\beta_{1j}\beta_{3j} + 2(\rho_1\rho_2-\rho_3)\beta_{2j}\beta_{3j}}{2s_0\sigma_k^2}\} \\
&\frac{1}{\sqrt{(2\pi)^3\sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1-\theta)\pi_k
\end{aligned}$$

We then denote

$$
\begin{aligned}
a_{jk1} &= N_1\eta^2 B_{1,jj} + \frac{1-\rho_3^2}{s_0\sigma_k^2} \\
a_{jk2} &= N_2\eta^2 B_{2,jj} + \frac{1-\rho_2^2}{s_0\sigma_k^2} \\
a_{jk3} &= N_3\eta^2 B_{3,jj} + \frac{1-\rho_1^2}{s_0\sigma_k^2} \\
c_{k1} &= \frac{\rho_2\rho_3-\rho_1}{s_0\sigma_k^2} \\
c_{k2} &= \frac{\rho_1\rho_3-\rho_2}{s_0\sigma_k^2} \\
c_{k3} &= \frac{\rho_1\rho_2-\rho_3}{s_0\sigma_k^2}
\end{aligned}
$$

this makes the formula

$$
\begin{aligned}
\int\int\int & \exp\{-\frac{1}{2}a_{jk1}\beta_{1j}^2 + N_1 b_{1j}\beta_{1j} - \frac{1}{2}a_{jk2}\beta_{2j}^2 + N_2 b_{2j}\beta_{2j} - \frac{1}{2}a_{jk3}\beta_{3j}^2 + N_3 b_{3j}\beta_{3j}\} \\
& \exp\{-c_{k1}\beta_{1j}\beta_{2j} - c_{k2}\beta_{1j}\beta_{3j} - c_{k3}\beta_{2j}\beta_{3j}\}\frac{1}{\sqrt{(2\pi)^3\sigma_k^6 s_0}} d\beta_1 d\beta_2 d\beta_3 \times (1-\theta)\pi_k
\end{aligned}
$$

To solve the integration we will use the formula for Gaussian Integration

$$
\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}x^\top Ax + b^\top x + c\right) d^n x = \sqrt{\det\left(2\pi A^{-1}\right)} e^{\frac{1}{2}b^\top A^{-1} b + c}
$$

where A is supposed to be a symmetric matrix.

3.1.1 SDPRX case

In this case we have

$$
\iint \exp\left\{-a_{jk1}\beta_{1j}^2 - a_{jk2}\beta_{2j}^2 + N_1 b_{1j}\beta_{1j} + N_2 b_{2j}\beta_{2j} + c_k\beta_{1j}\beta_{2j}\right\} d\beta_{1j} d\beta_{2j}
$$

So here to use the formula we have

$$
x = (\beta_{1j}, \beta_{2j})^T, b = (N_1 b_{1j}, N_2 b_{2j})^T
$$

and

$$
A = \begin{pmatrix} 2a_{jk1} & -c_k \\ -c_k & 2a_{jk2} \end{pmatrix}
$$

We will have

$$
A^{-1} = \frac{1}{4a_{jk1}a_{jk2} - c_k^2}\begin{pmatrix} 2a_{jk2} & c_k \\ c_k & 2a_{jk1} \end{pmatrix}
$$

and also

$$
b^T A^{-1} b = \frac{2a_{jk2}N_1^2 b_1^2 + 2a_{jk1}N_2^2 b_2^2 + 2c_k N_1 b_1 N_2 b_2}{4a_{jk1}a_{jk2} - c_k^2}
$$

and

$$\frac{1}{2}b^T A^{-1} b = \frac{a_{jk2}N_1^2 b_1^2 + a_{jk1}N_2^2 b_2^2 + c_k N_1 b_1 N_2 b_2}{4a_{jk1}a_{jk2} - c_k^2}$$

In this case, we will have

$$x = (\beta_{1j}, \beta_{2j}, \beta_{3j})^T, b = (N_1 b_{1j}, N_2 b_{2j}, N_3 b_{3j})^T$$

and also

$$A = \begin{pmatrix} a_{jk1} & c_{k1} & c_{k2} \\ c_{k1} & a_{jk2} & c_{k3} \\ c_{k2} & c_{k3} & a_{jk3} \end{pmatrix}$$

where

$$|A| = a_{jk1}a_{jk2}a_{jk3} + 2c_{k1}c_{k2}c_{k3} - a_{jk1}c_{k3}^2 - a_{jk3}c_{k1}^2 - a_{jk2}c_{k2}^2 = s_1$$

and

$$A^{-1} = \frac{1}{s_1}\begin{pmatrix} a_{jk2}a_{jk3} - c_{k3}^2 & c_{k2}c_{k3} - c_{k1}a_{jk3} & c_{k1}c_{k3} - c_{k2}a_{jk2} \\ c_{k2}c_{k3} - c_{k1}a_{jk3} & a_{jk1}a_{jk3} - c_{k2}^2 & c_{k1}c_{k2} - c_{k3}a_{jk1} \\ c_{k1}c_{k3} - c_{k2}a_{jk2} & c_{k1}c_{k2} - c_{k3}a_{jk1} & a_{jk1}a_{jk2} - c_{k1}^2 \end{pmatrix} = L$$

Here we have

$$\sqrt{\det(2\pi A^{-1})} = \sqrt{\frac{(2\pi)^3}{s_1}}$$

and

$$b^T A^{-1} b = L_{11}N_1^2 b_{1j}^2 + L_{22}N_2^2 b_{2j}^2 + L_{33}N_3^2 b_{3j}^2 + 2L_{12}N_1 b_{1j}N_2 b_{2j} + 2L_{13}N_1 b_{1j}N_3 b_{3j} + 2L_{23}N_2 b_{2j}N_3 b_{3j}$$

where

$$L_{11} = \frac{a_{jk2}a_{jk3} - c_{k3}^2}{s_1}$$

$$L_{22} = \frac{a_{jk1}a_{jk3} - c_{k2}^2}{s_1}$$

$$L_{33} = \frac{a_{jk1}a_{jk2} - c_{k1}^2}{s_1}$$

$$L_{12} = \frac{c_{k2}c_{k3} - c_{k1}a_{jk3}}{s_1}$$

$$L_{13} = \frac{c_{k1}c_{k3} - c_{k2}a_{jk2}}{s_1}$$

$$L_{23} = \frac{c_{k1}c_{k2} - c_{k3}a_{jk1}}{s_1}$$

We finally have the results that

$$P(z_j = (1,k)|.) \propto$$

$$\frac{\pi_k(1-\theta)}{\sqrt{s_1 s_0 \sigma_k^6}} \exp\{\frac{1}{2}(L_{11}N_1^2 b_{1j}^2 + L_{22}N_2^2 b_{2j}^2 + L_{33}N_3^2 b_{3j}^2 + 2L_{12}N_1 b_{1j}N_2 b_{2j} + 2L_{13}N_1 b_{1j}N_3 b_{3j} + 2L_{23}N_2 b_{2j}N_3 b_{3j})\}$$

3.1.2 Results validation

For the case where $\rho_1 = \rho_2 = \rho_3 = 0$ we will have

$$c_{k0} = c_{k1} = c_{k2} = 0$$

and

$$s_0 = 1, s_1 = a_{jk1}a_{jk2}a_{jk3}$$

also

$$L = \frac{1}{s_1}\begin{pmatrix} a_{jk2}a_{jk3} & 0 & 0 \\ 0 & a_{jk1}a_{jk3} & 0 \\ 0 & 0 & a_{jk1}a_{jk2} \end{pmatrix} = \begin{pmatrix} \frac{1}{a_{jk1}} & 0 & 0 \\ 0 & \frac{1}{a_{jk2}} & 0 \\ 0 & 0 & \frac{1}{a_{jk3}} \end{pmatrix}$$

where $L_{12} = L_{23} = L_{13} = 0$ and

$$\begin{aligned} &P(z_j = (1,k)|.) \propto \\ &\frac{\pi_k(1-\theta)}{\sqrt{a_{jk1}a_{jk2}a_{jk3}\sigma_k^6}} \exp\{\frac{1}{2}(L_{11}N_1^2b_{1j}^2 + L_{22}N_2^2b_{2j}^2 + L_{33}N_3^2b_{3j}^2)\} \end{aligned}$$

We then have

$$\begin{aligned} a_{jk1} &= N_1\eta^2B_{1,jj} + \frac{1}{\sigma_k^2} \\ a_{jk2} &= N_2\eta^2B_{2,jj} + \frac{1}{\sigma_k^2} \\ a_{jk3} &= N_3\eta^2B_{3,jj} + \frac{1}{\sigma_k^2} \end{aligned}$$

so we have it as

$$\begin{aligned} &P(z_j = (1,k)|.) \propto \\ &\frac{\pi_k(1-\theta)}{\sqrt{(N_1\eta^2B_{1,jj} + \frac{1}{\sigma_k^2})\sigma_k^2(N_2\eta^2B_{2,jj} + \frac{1}{\sigma_k^2})\sigma_k^2(N_3\eta^2B_{3,jj} + \frac{1}{\sigma_k^2})\sigma_k^2}} \\ &\exp\{\frac{1}{2}(\frac{N_1^2b_{1j}^2}{N_1\eta^2B_{1,jj} + (\sigma_k^2)^{-1}} + \frac{N_2^2b_{2j}^2}{N_2\eta^2B_{2,jj} + (\sigma_k^2)^{-1}} + \frac{N_3^2b_{3j}^2}{N_3\eta^2B_{3,jj} + (\sigma_k^2)^{-1}}\} \\ &= \frac{\pi_k(1-\theta)}{\sqrt{(N_1\eta^2B_{1,jj}\sigma_k^2 + 1)(N_2\eta^2B_{2,jj}\sigma_k^2 + 1)(N_3\eta^2B_{3,jj}\sigma_k^2 + 1)}} \\ &\exp\{\frac{1}{2}(\frac{N_1^2b_{1j}^2}{N_1\eta^2B_{1,jj} + (\sigma_k^2)^{-1}} + \frac{N_2^2b_{2j}^2}{N_2\eta^2B_{2,jj} + (\sigma_k^2)^{-1}} + \frac{N_3^2b_{3j}^2}{N_3\eta^2B_{3,jj} + (\sigma_k^2)^{-1}})\} \end{aligned}$$

This corresponds well to the results given by SDPRX for single cell population. Then we assume

$$\rho_1 = \rho, \rho_2 = \rho_3 = 0$$

and we will have

$$s_0 = 1 - \rho^2$$

and

$$
\begin{aligned}
a_{jk1} &= N_1\eta^2 B_{1,jj} + \frac{1}{(1-\rho^2)\sigma_k^2} \\
a_{jk2} &= N_2\eta^2 B_{2,jj} + \frac{1}{(1-\rho^2)\sigma_k^2} \\
a_{jk3} &= N_3\eta^2 B_{3,jj} + \frac{1}{\sigma_k^2} \\
c_{k1} &= -\frac{\rho}{(1-\rho^2)\sigma_k^2} \\
c_{k2} &= 0 \\
c_{k3} &= 0 \\
s_1 &= a_{jk1}a_{jk2}a_{jk3} - a_{jk3}c_{k1}^2
\end{aligned}
$$

and and

$$
\begin{aligned}
L_{11} &= \frac{a_{jk2}a_{jk3}}{a_{jk1}a_{jk2}a_{jk3} - a_{jk3}c_{k1}^2} = \frac{a_{jk2}}{a_{jk1}a_{jk2} - c_{k1}^2} \\
L_{22} &= \frac{a_{jk1}a_{jk3}}{a_{jk1}a_{jk2}a_{jk3} - a_{jk3}c_{k1}^2} = \frac{a_{jk1}}{a_{jk1}a_{jk2} - c_{k1}^2} \\
L_{33} &= \frac{a_{jk2}a_{jk1} - c_{k1}^2}{a_{jk1}a_{jk2}a_{jk3} - a_{jk3}c_{k1}^2} = \frac{1}{a_{jk3}} \\
L_{12} &= \frac{c_{k1}^2 a_{jk3}}{a_{jk1}a_{jk2}a_{jk3} - a_{jk3}c_{k1}^2} = \frac{c_{k1}^2}{a_{jk1}a_{jk2} - c_{k1}^2} \\
L_{13} &= 0 \\
L_{23} &= 0
\end{aligned}
$$

and the final results will be

$$
\begin{aligned}
&P(z_j = (1,k)|.) \propto \\
&\frac{\pi_k(1-\theta)}{\sqrt{s_1 s_0 \sigma_k^6}} \exp\{\frac{1}{2}(L_{11}N_1^2 b_{1j}^2 + L_{22}N_2^2 b_{2j}^2 + L_{33}N_3^2 b_{3j}^2 + 2L_{12}N_1 b_{1j} N_2 b_{2j})\} \\
&\frac{\pi_k(1-\theta)}{\sqrt{a_{jk3}(a_{jk1}a_{jk2} - c_{k1}^2)(1-\rho^2)\sigma_k^6}} \exp\{\frac{1}{2}(L_{11}N_1^2 b_{1j}^2 + L_{22}N_2^2 b_{2j}^2 + L_{33}N_3^2 b_{3j}^2 + 2L_{12}N_1 b_{1j} N_2 b_{2j})\} \\
&= \pi_k(1-\theta)\frac{1}{\sqrt{N_3\eta^2 B_{3,jj}\sigma_k^2 + 1}} \exp\{\frac{1}{2}\frac{N_3^2 b_{3j}^2}{N_3\eta^2 B_{3,jj} + (\sigma_k^2)^{-1}}\} \\
&\frac{1}{\sqrt{a_{jk1}a_{jk2} - c_{k1}^2}\sigma_k^2(1-\rho^2)} \exp\{\frac{1}{2}(\frac{N_1^2 b_{1j}^2 a_{jk2} + N_2^2 b_{2j}^2 a_{jk1} + 2N_1 b_{1j} N_2 b_{2j} c_{k1}^2}{a_{jk1}a_{jk2} - c_{k1}^2})\}
\end{aligned}
$$

We have $c_{k1} = c_k$ for SDPRX cases and $a_{jk1} = 2a_{jk1}$, $a_{jk2} = 2a_{jk2}$, $a_{jk3} = 2a_{jk3}$ for SDPRX cases. We can see that both results corresponds well.

3.2 beta

The posterior of β should be similar to SDPRX, we select those with non-zero value as $\gamma_1, \gamma_2, \gamma_3$, we will have

$$
\begin{aligned}
&P(\beta_{1,\gamma_1}, \beta_{2,\gamma_2}, \beta_{3,\gamma_3}|.)\\
&\propto \exp\{-\frac{N_1}{2}\eta^2\beta_1^T B_1\beta_1 + \eta\hat{\beta_1}^T A_1\beta_1\}\\
&\exp\{-\frac{N_2}{2}\eta^2\beta_2^T B_2\beta_2 + \eta\hat{\beta_2}^T A_2\beta_2\}\\
&\exp\{-\frac{N_3}{2}\eta^2\beta_3^T B_3\beta_3 + \eta\hat{\beta_3}^T A_3\beta_3\}\\
&\exp\{-\frac{1}{2}\beta^T\Sigma_0^{-1}\beta\}\\
&\propto \exp\{\frac{1}{2}\eta^2\beta_\gamma^T B_\gamma\beta_\gamma + \eta\hat{\beta}^T A_\gamma\beta_\gamma\}\exp\{-\frac{1}{2}\beta_\gamma^T\Sigma_{0\gamma}^{-1}\beta_\gamma\}
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_\gamma &= (\beta_{1,\gamma_1}, \beta_{2,\gamma_2}, \beta_{3,\gamma_3})^T\\
A_\gamma &= (N_1\hat{\beta}_1^T A_{1,\gamma_1}, N_2\hat{\beta}_2^T A_{2,\gamma_2}, N_3\hat{\beta}_3^T A_{3,\gamma_3})\\
B_\gamma &= \begin{pmatrix} N_1 B_{1,\gamma_1} & 0 & 0\\ 0 & N_2 B_{2,\gamma_2} & 0\\ 0 & 0 & N_3 B_{3,\gamma_3}\end{pmatrix}
\end{aligned}
$$

It is

$$MVN(\eta\Sigma A_\gamma^T\hat{\beta}_\gamma, \Sigma)$$

where

$$\Sigma = (\eta^2 B_\gamma + \Sigma_0^{-1})^{-1}$$

We will try to derive the inverse of Σ_0 where

$$\Sigma_{0\gamma} = \begin{pmatrix} \Sigma_{00} & \rho_1\Sigma_{00} & \rho_2\Sigma_{00}\\ \rho_1\Sigma_{00} & \Sigma_{00} & \rho_3\Sigma_{00}\\ \rho_2\Sigma_{00} & \rho_3\Sigma_{00} & \Sigma_{00}\end{pmatrix}$$

as a $3n\times 3n$ matrix where n is the number of individuals in the γ population. that Σ_{00} is a diagonal matrix with $\sigma_{z_j}^2$ on the diagonal. Therefore, for its inverse Σ_0^{-1}, we will be having the jth column of the matrix ($j\le n$), having all elements to be 0 except for the position of j, $j+n$ and $j+2n$, if we name them as (a,b,c). To know what they are, we will be having the following equations

$$
\begin{aligned}
a\sigma + b\rho_1\sigma + c\rho_2\sigma = 1\\
a\rho_1\sigma + b\sigma + c\rho_3\sigma = 0\\
a\rho_2\sigma + b\rho_3\sigma + c\sigma = 0
\end{aligned}
$$

assuming σ to be the σ_{z_j} at the jth diagonal. We will have

$$\sigma\begin{pmatrix} 1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_3\\ \rho_2 & \rho_3 & 1\end{pmatrix}\begin{pmatrix} a\\ b\\ c\end{pmatrix} = \begin{pmatrix} 1\\ 0\\ 0\end{pmatrix}$$

So by solving this we will have

$$a = (1-\rho_3^2)/(s_0\sigma)$$
$$b = (\rho_2\rho_3-\rho_1)/(s_0\sigma)$$
$$c = (\rho_1\rho_3-\rho_2)/(s_0\sigma)$$

that will be positioned at the $j, j+n, j+2n$ position of the column j, likewise

$$a = (\rho_2\rho_3-\rho_1))/(s_0\sigma)$$
$$b = (1-\rho_2^2)/(s_0\sigma)$$
$$c = (\rho_1\rho_2-\rho_3)/(s_0\sigma)$$

for $n < j \leq 2n$

$$a = (\rho_1\rho_3-\rho_2))/(s_0\sigma)$$
$$b = (\rho_1\rho_2-\rho_3)/(s_0\sigma)$$
$$c = (1-\rho_1^2)/(s_0\sigma)$$

for $j \geq 2n$ Finally we will have

$$\Sigma_{0\gamma}^{-1} = \begin{pmatrix} (1-\rho_3^2)/s_0\Sigma_{00}^{-1} & (\rho_2\rho_3-\rho_1)/s_0\Sigma_{00}^{-1} & (\rho_1\rho_3-\rho_2)/s_0\Sigma_{00}^{-1} \\ (\rho_2\rho_3-\rho_1)/s_0\Sigma_{00}^{-1} & (1-\rho_2^2)/s_0\Sigma_{00}^{-1} & (\rho_1\rho_2-\rho_3)/s_0\Sigma_{00}^{-1} \\ (\rho_1\rho_3-\rho_2)/s_0\Sigma_{00}^{-1} & (\rho_1\rho_2-\rho_3)/s_0\Sigma_{00}^{-1} & (1-\rho_1^2)/s_0\Sigma_{00}^{-1} \end{pmatrix} = \begin{pmatrix} S_{11}\Sigma_{00}^{-1} & S_{12}\Sigma_{00}^{-1} & S_{13}\Sigma_{00}^{-1} \\ S_{21}\Sigma_{00}^{-1} & S_{22}\Sigma_{00}^{-1} & S_{23}\Sigma_{00}^{-1} \\ S_{31}\Sigma_{00}^{-1} & S_{32}\Sigma_{00}^{-1} & S_{33}\Sigma_{00}^{-1} \end{pmatrix}$$

that Σ_{00}^{-1} is a diagonal matrix having the $1/\sigma_{z_j}^2$ over its diagonal

3.3 Sigma

We have

$$P(\sigma_k^2|.) \propto \prod_{j:z_j=(1,k)} \frac{1}{\sigma_k} \times \exp\{-\frac{1}{2}\beta_j^T \frac{S}{\sigma_k^2}\beta_j\} \times {\sigma_k}^{-2(.5-1)} \times \exp\left\{-\frac{.5}{\sigma_k^2}\right\}$$

denote

$$T_j = S_{11}\beta_{1j}^2 + S_{22}\beta_{2j}^2 + S_{33}\beta_{3j}^2 + 2S_{12}\beta_{1j}\beta_{2j} + 2S_{13}\beta_{1j}\beta_{3j} + 2S_{23}\beta_{2j}\beta_{3j}$$

and the posterior is

$$IG(\frac{M_{1k}}{2} + 0.5, \frac{\sum_{j:z_j=(1,k)} T_j}{2} + 0.5)$$

3.4 theta

The population distribution θ is

$$P(\theta|.) = \text{Beta}(M_0+1, M_1+1)$$

where M_0 is the value of null population and M_1 is the value of non-null population

3.5 stick-breaking parameters

$$P(V_k|.) = \text{Beta}(1 + M_k, \alpha + \sum_{l=k+1}^{K} M_l)$$

$$\pi_1 = V_1, \pi_k = \prod_{l=1}^{k-1} (1 - V_l)\, V_k (k \geq 2)$$

$$P(\alpha|.) = \text{Gamma}(0.1 + K - 1, 0.1 - \sum_{k=1}^{K-1} \log(1 - V_k))$$

where K is the total number of clusters

3.6 eta

$$\begin{aligned}
&P(\eta|.) \\
&\propto \exp\{-\frac{N_1}{2}\eta^2 \sum_{LD} \beta_1^T B_1 \beta_1 + \eta \sum_{LD} \hat{\beta_1}^T A_1 \beta_1\} \\
&\exp\{-\frac{N_2}{2}\eta^2 \sum_{LD} \beta_2^T B_2 \beta_2 + \eta \sum_{LD} \hat{\beta_2}^T A_2 \beta_2\} \\
&\exp\{-\frac{N_3}{2}\eta^2 \sum_{LD} \beta_3^T B_3 \beta_3 + \eta \sum_{LD} \hat{\beta_3}^T A_3 \beta_3\} \\
&\exp\{-\frac{\eta^2}{2 \times 10^{-6}}\} \\
&= N(\mu_1, \sigma_1)
\end{aligned}$$

where LD is the sum across all LD matrices and

$$\begin{aligned}
\mu_1 &= \frac{N_1 \sum_{LD} \hat{\beta_1}^T A_1 \beta_1 + N_2 \sum_{LD} \hat{\beta_2}^T A_2 \beta_2 + N_3 \sum_{LD} \hat{\beta_3}^T A_3 \beta_3}{N_1 \sum_{LD} \beta_1^T B_1 \beta_1 + N_2 \sum_{LD} \beta_2^T B_2 \beta_2 + N_3 \sum_{LD} \beta_3^T B_3 \beta_3 + 10^{-6}} \\
\sigma_1^2 &= \frac{1}{N_1 \sum_{LD} \beta_1^T B_1 \beta_1 + N_2 \sum_{LD} \beta_2^T B_2 \beta_2 + N_3 \sum_{LD} \beta_3^T B_3 \beta_3 + 10^{-6}}
\end{aligned}$$