Price and greeks of European binary option

Consider an asset with spot price $S = \{S_t; 0 \le t \le T\}$ following geometric Brownian motion of volatility σ.

A European binary call option with strike K and maturity T pays off

$$\text{Payoff}_{\text{Call}} = 1_{S_T \ge K}, \tag{1}$$

where $1_{S_T \ge K}$ is an indicator function. A European binary put option with the same strike and maturity pays off

$$\text{Payoff}_{\text{Put}} = 1_{S_T \le K}. \tag{2}$$

Price

The price of the European binary call option is given by

$$\text{Price}_{\text{Call}} = \mathbb{E}[1_{S_T \ge K}] = N(d_2), \tag{3}$$

where N is the cumulative distribution function of the normal distribution and

$$d_2 = \frac{\log(S_0/K)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}. \tag{4}$$

The price of a European binary put option is given by $1 - \text{Price}_{\text{Call}}$ because a relation $\text{Payoff}_{\text{Call}} + \text{Payoff}_{\text{Put}} = 1$ holds almost surely.

Delta

Delta is given by

$$\text{Delta}_{\text{Call}} = \frac{N'(d_2)}{S_0\sigma\sqrt{T}}, \tag{5}$$

where we used a derivative $\partial d_2/\partial S_0 = 1/(S_0\sigma\sqrt{T})$.

Delta of a European binary put option is $\text{Delta}_{\text{Put}} = -\text{Delta}_{\text{Call}}$.

Gamma

Gamma of the European binary option is given by

$$\text{Gamma} = \frac{N''(d_2)}{S_0^2\sigma^2 T} - \frac{N'(d_2)}{S_0^2\sigma\sqrt{T}} = -\frac{N'(d_2)}{S_0^2\sigma\sqrt{T}}\left(\frac{d_2}{\sigma\sqrt{T}} + 1\right), \tag{6}$$

where we used a relation $N''(x) = -xN'(x)$ to show the second equality.

Gamma of a European binary put option is $\text{Gamma}_{\text{Put}} = -\text{Gamma}_{\text{Call}}$.