Price and greeks of American binary option

Consider an asset with spot price $S = \{S_t; 0 \leq t \leq T\}$ following geometric Brownian motion of volatility σ.

An American binary call option with strike K and maturity T pays off

$$\text{Payoff} = 1_{\max\{S_t; 0 \leq t \leq T\} \geq K}, \tag{1}$$

where $1_{\max\{S_t; 0 \leq t \leq T\} \geq K}$ is an indicator function.

Price

Let $W = \{W_t; 0 \leq t \leq T\}$ be Brownian motion driving S. We define the cumulative maximum of W by $M_t = \max\{W_u; 0 \leq u \leq t\}$ and consider $\hat{M}_t = M_t - \frac{1}{2}\sigma t$. According to Corollary 7.2.2 of Ref. [1], if $S_0 < K$,

$$\mathbb{P}[\hat{M}_t \geq m] = 1 - N\left(\frac{m}{\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right) + e^{-\sigma m} N\left(-\frac{m}{\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}\right), \tag{2}$$

where N is the cumulative distribution function of the normal distribution. A condition to get the payoff of unity, $\max\{S_t; 0 \leq t \leq T\} \geq K$, is equivalent to $\hat{M}_T \geq -\sigma^{-1}\log(S_0/K)$. Therefore, the price of the American binary call option is given by 1 if $S_0 \geq K$ and otherwise

$$\begin{aligned}
\text{Price} &= \mathbb{E}[1_{\max\{S_t; 0 \leq t \leq T\} \geq K}] \\
&= \mathbb{P}\left[\hat{M}_T \geq -\frac{1}{\sigma}\log\left(\frac{S_0}{K}\right)\right] \\
&= 1 - N\left(-\frac{\log(S_0/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) + N\left(\frac{\log(S_0/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) \\
&= N(d_2) + \frac{S_0}{K}N(d_1),
\end{aligned} \tag{3}$$

where

$$d_1 = \frac{\log(S_0/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, \quad d_2 = \frac{\log(S_0/K)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}. \tag{4}$$

Delta

Delta is given by 0 if $S_0 \geq K$ and otherwise

$$\text{Delta} = \frac{N'(d_2)}{S_0\sigma\sqrt{T}} + \frac{N(d_1)}{K} + \frac{N'(d_1)}{K\sigma\sqrt{T}}, \tag{5}$$

where we used a derivative $\partial d_1/\partial S_0 = \partial d_2/\partial S_0 = 1/(S_0\sigma\sqrt{T})$.

Gamma

Gamma is given by 0 if $S_0 \geq K$ and otherwise

$$\begin{aligned} \text{Gamma} &= -\frac{N'(d_2)}{S_0^2\sigma\sqrt{T}} + \frac{N''(d_2)}{S_0^2\sigma^2 T} + \frac{N'(d_1)}{S_0K\sigma\sqrt{T}} + \frac{N''(d_1)}{S_0K\sigma^2 T} \\ &= -\frac{N'(d_2)}{S_0^2\sigma\sqrt{T}} - \frac{d_2N'(d_2)}{S_0^2\sigma^2 T} + \frac{N'(d_1)}{S_0K\sigma\sqrt{T}} - \frac{N'(d_1)}{S_0K\sigma^2 T}, \end{aligned} \tag{6}$$

where we used a relation $N''(x) = -xN'(x)$ to show the second equality.

References

[1] Shreve, S.E., 2004. Stochastic calculus for finance II: Continuous-time models (Vol. 11). New York: springer.