Randomness in Biology [2014 Aug Term]

Solutions to Homework 6

1. The most general one-step master equation in two variables is:

$$\frac{dp_{i,j}}{dt} = -\left(a_{i,j} + b_{i,j} + c_{i,j} + d_{i,j}\right)p_{i,j} + a_{i-1,j}p_{i-1,j} + b_{i+1,j}p_{i+1,j} + c_{i,j-1}p_{i,j-1} + d_{i,j+1}p_{i,j+1}$$

a. Write down the deterministic ODE this would correspond to. I.e. an equation of the form

$$\frac{d}{dt}i = f(i,j), \quad \frac{d}{dt}j = g(i,j)$$

where *f* and *g* are written in terms of the *a,b,c,d* and i, j are considered continuous.

Ans:

$$\frac{d}{dt}i = a(i,j) - b(i,j) \quad \frac{d}{dt}j = c(i,j) - d(i,j)$$

b. Suppose now that the coefficients all have the following linear (or affine) form:

$$x_{i,j} = x_0 + x_1 i + x_2 j$$

where this applies separately for $x = a, b, c, d$.

Derive an ODE for the means $< i >, < j >$. You should find it is identical to the deterministic case. This argument is easily extended to multiple variables.

Ans:

E.g. let us check

$$\frac{d}{dt} < i > = \frac{d}{dt}\sum i p_{i,j} = -\sum i a_{i,j}p_{i,j} + \sum i a_{i-1,j}p_{i-1,j} - \sum i b_{i,j}p_{i,j} + \sum i b_{i+1,j}p_{i+1,j}$$

(as discussed in class, the other terms cancel out).

$$\begin{aligned} &= \sum(-i + i + 1)a_{i,j}p_{i,j} + \sum(-i + i - 1)b_{i,j}p_{i,j} \\ &= \sum(a_0 + a_1 i + a_2 j)p_{i,j} - \sum(b_0 + b_1 i + b_2 j)p_{i,j} \\ &= a_0 + a_1 < i > + a_2 < j > - b_0 - b_1 < i > - b_2 < j > \end{aligned}$$

So $\quad \frac{d}{dt} < i > = a(< i >, < j >) - b(< i >, < j >)$

and similarly for the equation for $< j >$. We recover the deterministic ODE.