Solution to HW2

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1. (20 points) Refer to homework statement.

Solution:

Except for the last subplot, these plots shows how each function will transform given functions in homework. These two functions are in first subplot. They are called `fun1` and `fun2`.

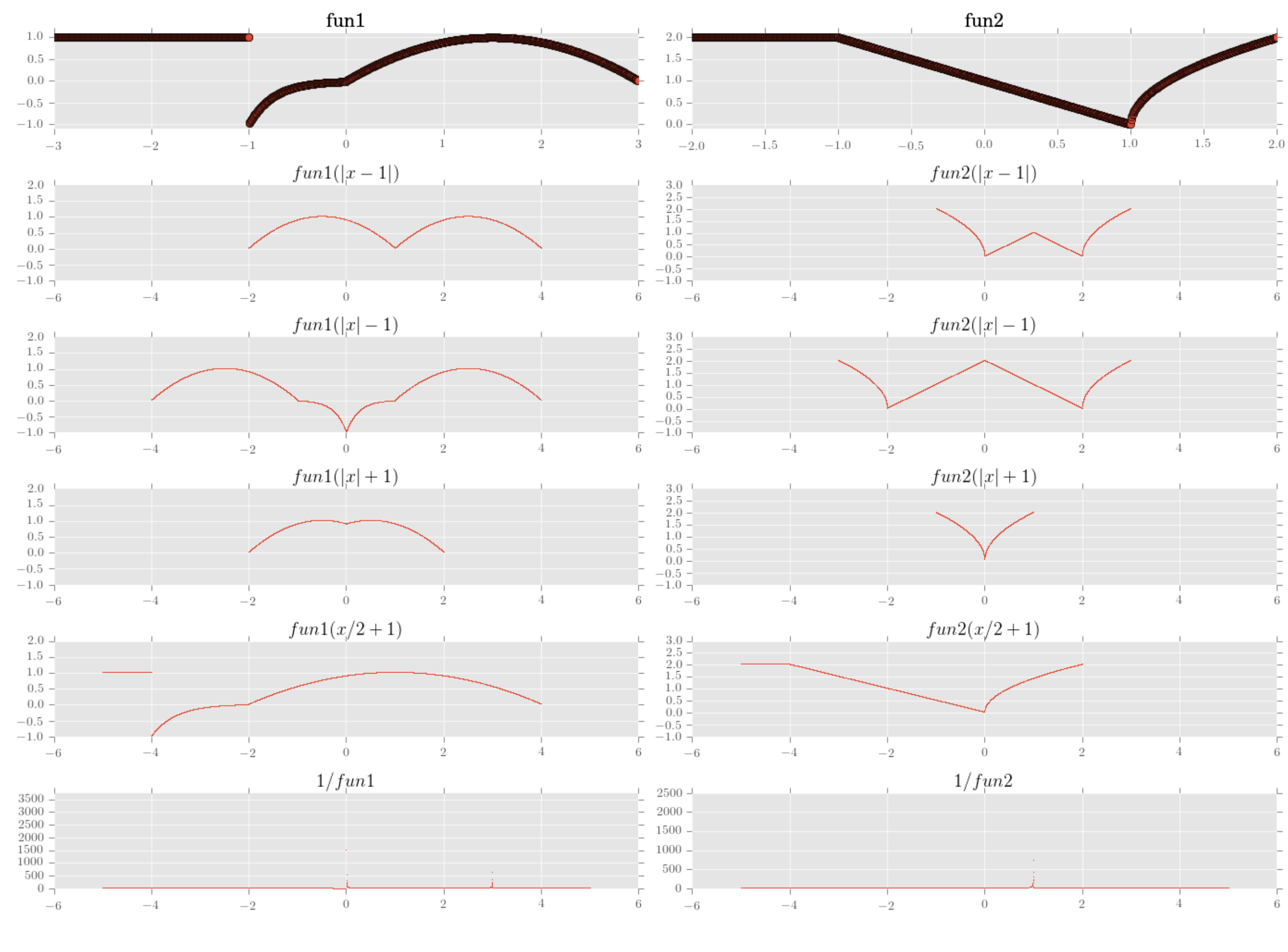

Figure 1: Solution to problem 1

2. (12 points) Refer to homework statement.

Solution:

See figure 2.

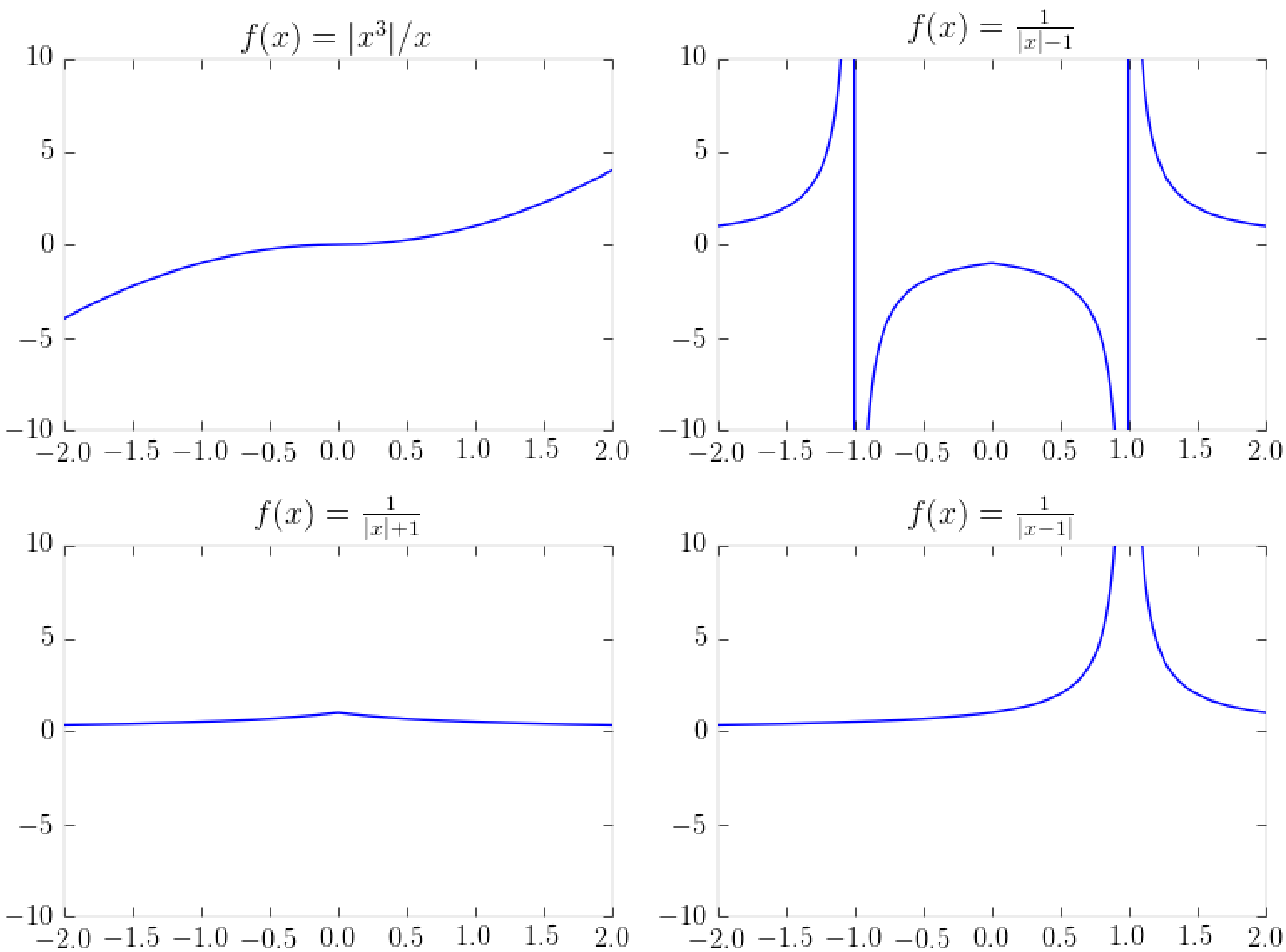

Figure 2: Solution 2

3. (5 points) Show that if $f(x) = \sqrt[3]{x}$, then for any $a \in \mathbb{R} \setminus \{0\}$ $f'(a) = 1/3a^{-2/3}$.

Solution:

We start with $f(a+h) = f(a) + hf'(a) + R(h)$ and $R(h)/h \to 0$ when $h \to 0$. We can write the following:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

This leads to,

$$
\begin{aligned}
f(\sqrt[3]{x}) &= \lim_{h\to 0} \frac{\sqrt[3]{x+h}-\sqrt[3]{x}}{h} && (2)\\
&= \lim_{h\to 0} \frac{(x+h)^{1/3}-x^{1/3}}{h} && (3)\\
&= \lim_{h\to 0} \frac{x^{1/3}\left(\left(1+\frac{h}{x}\right)^{1/3}-1\right)}{h} \qquad = \lim_{h\to 0} \frac{x^{1/3}\left(1+\frac{h}{3x}+\ldots-1\right)}{h} && (4)\\
&= x^{1/3}\left(\frac{h}{3xh}\right) && (5)\\
&= x^{-2/3}\left(\frac{1}{3}\right) && (6)
\end{aligned}
$$

4. (5 points) Show that if $g(x) = \frac{1}{f(x)}$, then for any $a \in \mathbb{R}$ such that $f(a) \neq 0, g'(a) = -\frac{f'(a)}{f(a)^2}$.

Solution:

$$
\begin{aligned}
g'(a) &= \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} && (7)\\
&= \lim_{h\to 0} \frac{\frac{1}{f(a+h)}-\frac{1}{f(a)}}{h} && (8)\\
&= \lim_{h\to 0} \frac{f(a)-f(a+h)}{hf(a)f(a+h)} && (9)\\
&= \lim_{\mathbf{h}\to \mathbf{0}} -\frac{\mathbf{f(a+h)-f(h)}}{\mathbf{h}}\frac{1}{f(a)f(a+h)} && (10)\\
&= -f'(a)\frac{1}{f(a)f(a)} && (11)
\end{aligned}
$$

5. (5 points) Write the approximation for $\sqrt{99.8}$ and $\sqrt[3]{5.01^2+2}$.

Solution:

6. (8 points) Differentiate each of the following ....

Solution:

7. (8 points) Inverse function theorem

(a) (4 points) **Solution:**

$$f(x) = \sqrt{1+x^3} \text{ and } f(f^{-1}(x)) = x \tag{12}$$

$$\implies \sqrt{1+(f^{-1}(x))^3} = x \tag{13}$$

$$\implies f^{-1}(x) = (x^2-1)^{1/3} \tag{14}$$

$$\implies (f^{-1}(x))' = \frac{1}{3}(x^2-1)^{-2/3}2x \tag{15}$$

(b) (4 points) Using inverse function theorem $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$.

Solution:

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)} \tag{16}$$

$$= \frac{2f(x)}{3x^2} = \frac{2f(x)}{3(f^2(x)-1)^{2/3}} \tag{17}$$

$$\implies (f^{-1})'(a) = \frac{2a}{3(a^2-1)^{2/3}} \tag{18}$$

Which is same as equation (15).