Real Analysis I

Sample Solutions

Exercise Suppose that $f : A \to B$ and $g : B \to C$ are bijections. Prove that $g \circ f$ is a bijection.

Solution: To prove that $g \circ f$ is a bijection we need to prove that it is both injective (aka "one-to-one") and surjective (aka "onto").

To prove injective, suppose that $x \neq y$ are arbitrary elements of A. Then, since f is injective, it follows that $f(x) \neq f(y)$. And since g is injective it follows that $g(f(x)) \neq g(f(y))$. Therefore, since $g \circ f(x) = g(f(x))$ and $g \circ f(y) = g(f(y))$, it follows that $g(f(x)) \neq g(f(y))$ as required. And since x and y were arbitrary, it follows that $g \circ f$ is injective.

To prove that $g \circ f$ is surjective, suppose that $c \in C$ is arbitrary. Since g is surjective, we may fix $b \in B$ such that $g(b) = c$. And since f is surjective, we may fix $a \in A$ such that $f(a) = b$. Thus, $g \circ f(a) = g(f(a)) = g(b) = c$. And since c was arbitrary, it follows that $g \circ f$ is surjective. $\square$

Exercise Prove by induction that every natural numbers are either even or odd.

Solution: To prove that every natural number is either even or odd we first prove the base case:
BASE CASE. 1 is odd since $1 = 2 \cdot 0 + 1$.

Next we prove the induction step:
INDUCTIVE STEP. Suppose that n is a natural number and n is either even or odd. To prove the same for $n + 1$ we consider cases
CASE 1. n is even. Therefore there is a k such that $n = 2 \cdot k$. But then $n+1 = 2 \cdot k+1$ and so $n + 1$ is odd.
CASE 2. n is odd. Therefore there is a k such that $n = 2 \cdot k + 1$. But then $n + 1 = (2 \cdot k + 1) + 1 = 2 \cdot (k + 1)$ and so $n + 1$ is even.

Therefore, by the Principle of Induction, every natural number is either even or odd. $\square$

Exercise The number $\sqrt{2}$ is not rational.

Solution. The main fact we will use is that if the square of an integer is even then the number must be even. I.e., If a is an integer and if a^2 is even, then a is even. I leave this for you to prove.

Now, suppose by way of contradiction that $\sqrt{2}$ is a rational number and write $\sqrt{2} = \frac{n}{m}$ so that n and m have no common factors. Therefore, either m or n must be odd.

So then, $2 = \frac{m^2}{n^2}$ and hence $m^2 = 2n^2$.

Therefore m^2 is even and so, as mentioned above, m is even. So we can fix k such that $m = 2k$. Thus $m^2 = 4k^2$ and so we have that $4k^2 = 2n^2$. Simplifying this expression leaves $n^2 = 2k^2$. And, as above, we conclude that n is even. But this contradicts our assumption that n and m have no common factors. $\square$.