Project 5: A Pascal-Like Triangle of Eulerian Numbers

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1. Show algebraically and geometrically that $\binom{n}{2} + \binom{n+1}{2} = n^2$.

$$\binom{n}{2} = \frac{(n)(n-1)}{2}$$

and

$$\binom{n+1}{2} = \frac{(n+1)(n)}{2}.$$

Thus,

$$\begin{aligned}&\binom{n}{2} + \binom{n+1}{2} \\ &= \frac{(n)(n-1)}{2} + \frac{(n+1)(n)}{2} \\ &= \frac{n^2-n}{2} + \frac{n^2+n}{2} \\ &= \frac{2n^2}{2} = n^2. \quad \square\end{aligned}$$

Geometrically:

Imagine an $n * n$ square made out of n^2 dots. Now, draw a "set of stairs" partitioning it into two right isosceles triangles with side lengths of n and $n-1$. Those two triangles correspond to triangular numbers, also known as $\binom{i+1}{2}$. Thus, the two triangles correspond to $\binom{n}{2}$ and $\binom{n+1}{2}$ and add up to n^2. $\square$

2. Show algebraically and geometrically that $\binom{n}{3} + 4\binom{n+1}{3} + \binom{n+2}{3} = n^3$.

$$\binom{n}{3} = \frac{(n)(n-1)(n-2)}{6}$$
$$\binom{n+1}{3} = \frac{(n+1)(n)(n-1)}{6}$$

and

$$\binom{n+2}{3} = \frac{(n+2)(n+1)(n)}{6}.$$

Thus,

$$\binom{n}{3} + 4\binom{n+1}{3} + \binom{n+2}{3}$$
$$= \frac{(n)(n-1)(n-2)}{6} + \frac{4(n+1)(n)(n-1)}{6} + \frac{(n+2)(n+1)(n)}{6}$$
$$= \frac{n^3 - 3n^2 + 2n}{6} + \frac{4(n^3 - n)}{6} + \frac{n^3 + 3n^2 + 2n}{6}$$
$$= \frac{6n^3}{6} = n^3. \quad \square$$

Geometrically:

Imagine an $n * n * n$ cube made out of n^3 dots. Now, create one triangular pyramid in the bottom left corner with side length n. Those two triangles correspond to triangular numbers, also known as $\binom{i+1}{2}$. Thus, the two triangles correspond to $\binom{n}{2}$ and $\binom{n+1}{2}$ and add up to n^2. $\square$

3. Use Question 1 and the hockey stick theorem to find a nice formula for $1^2+2^2+3^2+...+n^2$. Contrast with the version usually seen in calculus.

The hockey stick identity: $\sum_{i=r}^{n}\binom{i}{r} = \binom{n+1}{r+1}$.

$$1^2 + 2^2 + 3^2 + ... + n^2$$

can be written as

$$\sum_{x=1}^{n} x^2.$$

This equals

$$\sum_{x=1}^{n} [\binom{x}{2} + \binom{x+1}{2}]$$

by (Q1). That is equivalent to

$$\sum_{x=1}^{n} \binom{x}{2} + \sum_{x=2}^{n+1} \binom{x}{2}.$$

Note that $\binom{1}{2} = 0$. This means that the left summation is equivalent to $\sum_{x=2}^{n}\binom{x}{2}$. Using the hockey stick identity, we get

$$\binom{n+1}{3} + \binom{n+2}{3}.$$

This form is actually equivalent to the calculus form of $\frac{(n)(n+1)(2n+1)}{6}$ after expanding the binomial coefficients. $\square$

4. Use Question 2 to find a nice formula for $1^3 + 2^3 + 3^3 + ... + n^3$.

$$1^3 + 2^3 + 3^3 + ... + n^3$$

can be written as

$$\sum_{x=1}^{n} x^3$$

$$= \sum_{x=1}^{n} [\binom{x}{3} + 4\binom{x+1}{3} + \binom{x+2}{3}]$$

by (Q2). That is equivalent to

$$\sum_{x=1}^{n} \binom{x}{3} + 4\sum_{x=2}^{n+1} \binom{x}{3} + \sum_{x=3}^{n+2} \binom{x}{3}.$$

By the hockey stick identity:

$$\binom{n+1}{4} + 4\binom{n+2}{4} + \binom{n+3}{4}.$$

□

5. Express n^4 as in Questions 1 and 2.

$$n^4 = \binom{n}{4} + 11\binom{n+1}{4} + 11\binom{n+2}{4} + \binom{n+3}{4}$$

(Method: Polynomial Fitting. See Question 7.) $\square$

6. Use Question 5 to find a nice formula for $1^4 + 2^4 + 3^4 + ... + n^4$.

$$1^4 + 2^4 + 3^4 + ... + n^4$$

can be written as

$$\sum_{x=1}^{n} x^4$$

$$= \sum_{x=1}^{n} [\binom{x}{4} + 11\binom{x+1}{4} + 11\binom{x+2}{4} + \binom{x+3}{4}]$$

by Q5. That is equivalent to

$$\sum_{x=1}^{n} \binom{x}{4} + 11\sum_{x=2}^{n+1} \binom{x}{4} + 11\sum_{x=3}^{n+2} \binom{x}{4} + \sum_{x=4}^{n+3} \binom{x}{4}.$$

By the hockey stick identity:

$$\binom{n+1}{5} + 11\binom{n+2}{5} + 11\binom{n+3}{5} + \binom{n+4}{5}.$$

□

It is easy to see that the coefficients of the binomial coefficients are identical in Questions 5 and 6. The pattern $\{1, 11, 11, 1\}$ appears in the formulae of both n^4 and of $\sum n^4$. In fact, these coefficients will always be the same for both of them, even for different powers. The process of splitting the sum into multiple sums and then using the hockey stick identity on all of them will work on every set of coefficients for an n^m, and we will use this fact in Question 8 in reverse to find the coefficients of n^5 from $\sum n^5$.

7. Here is an alternate way of accomplishing Question 6:

Find integers a, b, c and d so that
$1^4+2^4+3^4+...+n^4 = a\binom{n+1}{5}+b\binom{n+2}{5}+c\binom{n+3}{5}+d\binom{n+4}{5}$

using $n = 1, 2, 3, 4$. This sometimes is referred to as "Polynomial Fitting."

For $n = 1$, the equation in the question becomes

$$1 = a * 0 + b * 0 + c * 0 + d * 1.$$

d must equal 1 for this equation to be correct.

For $n = 2$, the equation becomes

$$1 + 16 = a * 0 + b * 0 + c * 1 + 1 * 6.$$

c must equal 11 for this equation to be correct.

For $n = 3$, the equation becomes

$$1 + 16 + 81 = a * 0 + b * 1 + 11 * 6 + 1 * 21.$$

b must equal 11 for this equation to be correct.

For $n = 4$, the equation becomes

$$1 + 16 + 81 + 256 = a * 1 + 11 * 6 + 11 * 21 + 1 * 56.$$

a must equal 1 for this equation to be correct.

Thus the set $\{a, b, c, d\}$ is equal to $\{1, 11, 11, 1\}$. $\square$

8. Express n^5 as in Questions 1 and 2.

Use Polynomial Fitting.

$1^5+2^5+3^5+...+n^5 = a\binom{n+1}{6}+b\binom{n+2}{6}+c\binom{n+3}{6}+d\binom{n+4}{6}+e\binom{n+5}{6}$

For $n=1$, the equation becomes

$$1 = a*0+b*0+c*0+d*0+e*1.$$

e must equal 1 for this equation to be correct.

For $n=2$, the equation becomes

$$1+32 = a*0+b*0+c*0+d*1+1*7.$$

d must equal 26 for this equation to be correct.

For $n=3$, the equation becomes

$$1+32+243 = a*0+b*0+c*1+26*7+1*28.$$

c must equal 66 for this equation to be correct.

For $n=4$, the equation becomes

$$1+32+243+1024 = a*0+b*1+66*7+26*28+1*84.$$

b must equal 26 for this equation to be correct.

For $n=5$, the equation becomes

$$1+32+243+1024+3125 = a*1+26*7+66*28+26*84+1*210.$$

a must equal 1 for this equation to be correct.

Thus the set $\{a, b, c, d, e\}$ is equal to $\{1, 26, 66, 26, 1\}$ and n^5 is equal to

$$\binom{n}{5} + 26\binom{n+1}{5} + 66\binom{n+2}{5} + 26\binom{n+3}{5} + \binom{n+4}{5}.$$

$\square$

9. Express $1^5 + 2^5 + 3^5 + ... + n^5$ as a sum of multiples of binomial coefficients.

From Q8 we know that the coefficients are

$$\{1, 26, 66, 26, 1\},$$

so the $\sum n^5$ is equal to

$$\binom{n+1}{6} + 26\binom{n+2}{6} + 66\binom{n+3}{6} + 26\binom{n+4}{6} + \binom{n+5}{6}.$$

□

10. In Questions 1, 2, 5, and 7, n^2, n^3, n^4 and n^5 were expressed as sums of multiples of binomial coefficients with integer coefficients. Arrange these expressions in a Pascal-like triangle, concentrating on these integer coefficients. The coefficients are called Eulerian numbers.

Triangle:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 4 \quad 1 \\
1 \quad 11 \quad 11 \quad 1 \\
1 \quad 26 \quad 66 \quad 26 \quad 1 \\
1 \quad 57 \quad 302 \quad 302 \quad 57 \quad 1 \\
1 \quad 120 \quad 1191 \quad 2416 \quad 1191 \quad 120 \quad 1 \\
1 \quad 247 \quad 4293 \quad 15619 \quad 15619 \quad 4293 \quad 247 \quad 1
\end{array}
$$

11. Investigate the properties of this triangle. Include a discussion of row sums, how entries are found, and a possible recursion. Use $\begin{bmatrix} n \\ k \end{bmatrix}$ for notation.

Row sums in the Eulerian triangle are the factorial numbers. I will return to a discussion of why this is so after I propose a recursion.

Recursion:
I propose that

$$\begin{bmatrix} n \\ k \end{bmatrix} = (n-k+1)\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + k\begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

Similarly to Pascal's triangle, each element is calculated from the two elements directly above it. In this recursion, the two elements are both multiplied by particular multipliers depending on where in the triangle the element is.

With this recursion, when creating row $n+1$ from row n, it is clear that each entry in row n is multiplied by $(n-k+1)+(k)=n+1$. (Each entry is both the "left side" and the "right side" of the recursion, as it can be either above and to the left of an entry in the next row or it can be above and to the right.) Thus, each row sum is $n+1$ times the previous, leading to the row sums being factorials. Induction can be used to prove this – the base case being row 0 summing to 1!.

I also explored the Eulerian triangle to see if any other interesting patterns were present. However, many patterns which existed in Pascal's did not exist in the Eulerian triangle. For example, in Pascal's triangle multiples of numbers form upside-down, smaller triangles, sums of semi-diagonal slices add to the Fibonacci numbers, and sums of squares of entries in a row create entries in the middle column further down in the triangle. None of these patterns were present in the Eulerian triangle. $\square$