EE 261 The Fourier Transform and its Applications
Fall 2006
Final Exam, December 13, 2006

Notes:
There are 7 questions for a total of 120 points

Write all your answers in your exam booklets

When there are several parts to a problem, in many cases the parts can be done independently, or the result of one part can be used in another part.

Please be neat and indicate clearly the main parts of your solutions

1. (15 points) Let $f(t)$ be a periodic signal of period 1. One says that $f(t)$ has *half-wave symmetry* if

$$f(t - \frac{1}{2}) = -f(t)\,.$$

   (a) Sketch an example of a signal that has half-wave symmetry.

   (b) If $f(t)$ has half-wave symmetry and its Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t}$$

   show that $c_n = 0$ if n is even.
   Hint: $-c_n = -\int_0^1 e^{-2\pi i n t} f(t)\, dt = \int_0^1 e^{-2\pi i n t} f(t - \frac{1}{2})\, dt$.

2. (20 points) *Sampling using the derivative* Suppose that $f(t)$ is a bandlimited signal with $\mathcal{F}f(s) = 0$ for $|s| \geq 1$ (bandwidth 2). According to the sampling theorem, knowing the values $f(n)$ for all integers n (sampling rate of 1) is not sufficient to interpolate the values $f(t)$ for all t. However, if *in addition* one knows the values of the derivative $f'(n)$ at the integers then there is an interpolation formula with a sampling rate of 1. In this problem you will derive that result.

   Let $F(s) = \mathcal{F}f(s)$ and let $G(s) = \frac{1}{2\pi i}(\mathcal{F}f')(s) = sF(s)$.

   (a) For $0 \leq s \leq 1$ show that

   $$\begin{aligned}(Ш * F)(s) &= F(s) + F(s-1)\\ (Ш * G)(s) &= sF(s) + (s-1)F(s-1)\end{aligned}$$

   and then show that

   $$F(s) = (1-s)(Ш * F)(s) + (Ш * G)(s)\,.$$

   (b) For $-1 \leq s \leq 0$ show that

   $$\begin{aligned}(Ш * F)(s) &= F(s) + F(s+1)\\ (Ш * G)(s) &= sF(s) + (s+1)F(s+1)\end{aligned}$$

   and then show that

   $$F(s) = (1+s)(Ш * F)(s) - (Ш * G)(s)\,.$$

   (c) Using parts (a) and (b) show that for all s, $-\infty < s < \infty$,

   $$F(s) = \Lambda(s)(Ш * F)(s) - \Lambda'(s)(Ш * G)(s)\,,$$

   where $\Lambda(s)$ is the triangle function

   $$\Lambda(s) = \begin{cases} 1-|s|, & |s| \leq 1\,,\\ 0, & |s| \geq 1\,.\end{cases}$$

   (d) From part (c) derive the interpolation formula

   $$f(t) = \sum_{n=-\infty}^{\infty} f(n)\mathrm{sinc}^2(t-n) + \sum_{n=-\infty}^{\infty} f'(n)(t-n)\mathrm{sinc}^2(t-n)\,.$$

3. (20 points) *The DFT and linear interpolation*

(a) Let $\underline{\mathrm{y}}$ be the discrete signal, periodic of order M,

$$\underline{\mathrm{y}} = (1\,,\, \frac{1}{2}\,,\, 0\,,\, \ldots\,,\, 0\,,\, \frac{1}{2})$$

Show that its DFT is

$$\underline{Y}[m] = 1 + \cos(2\pi m/M)\,.$$

(b) Let $\underline{\mathrm{f}} = (\underline{\mathrm{f}}[0]\,,\, \underline{\mathrm{f}}[1]\,,\, \ldots\,,\, \underline{\mathrm{f}}[N-1])$ be a discrete signal and let $\underline{\mathrm{F}} = (\underline{\mathrm{F}}[0]\,,\, \underline{\mathrm{F}}[1]\,,\, \ldots\,,\, \underline{\mathrm{F}}[N-1])$ be its DFT. Recall that the upsampled version of $\underline{\mathrm{f}}$ is the signal $\underline{\mathrm{h}}$ of order $2N$ obtained by inserting zeros between the values of $\underline{\mathrm{f}}$, i.e.,

$$\underline{\mathrm{h}} = (\underline{\mathrm{f}}[0]\,,\, 0\,,\, \underline{\mathrm{f}}[1]\,,\, 0\,,\, \underline{\mathrm{f}}[2]\, \ldots\,,\, 0\,,\, \underline{\mathrm{f}}[N-1]\,,\, 0)\,.$$

Show that $\underline{\widetilde{\mathrm{f}}} = \underline{\mathrm{h}} * \underline{\mathrm{y}}$ is the 'linearly interpolated' version of $\underline{\mathrm{f}}$:

$$(\underline{\mathrm{f}}[0]\,,\, \frac{\underline{\mathrm{f}}[0] + \underline{\mathrm{f}}[1]}{2}\,,\, \underline{\mathrm{f}}[1]\,,\, \frac{\underline{\mathrm{f}}[1] + \underline{\mathrm{f}}[2]}{2}\,,\, \underline{\mathrm{f}}[2]\,, \ldots,\, \underline{\mathrm{f}}[N-1]\,,\, \frac{\underline{\mathrm{f}}[N-1] + \underline{\mathrm{f}}[0]}{2})\,.$$

Hint: Here we take $M = 2N$ for the period of $\underline{\mathrm{y}}$. Note that

$$\underline{\mathrm{y}} = \underline{\delta}_0 + \frac{1}{2}\underline{\delta}_1 + \frac{1}{2}\underline{\delta}_{2N-1}$$

and remember the effect of convolving with a shifted discrete δ. Line up the $2N$-tuples.

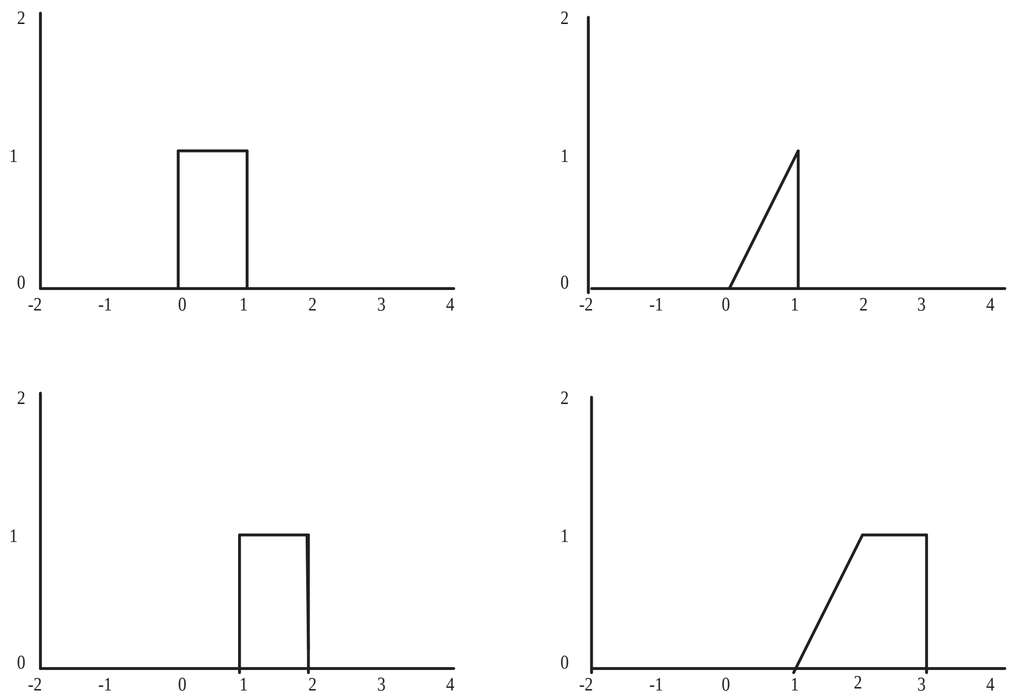
(c) In a problem set you showed that the DFT of $\underline{\mathrm{h}}$ is a replicated form of $\underline{\mathrm{F}}$,

$$\underline{H}[m] = \underline{\mathrm{F}}[m] \qquad m = 0, 1, \ldots, 2N-1$$

$$\underline{H} = (\overbrace{\underbrace{\underline{\mathrm{F}}[0]\,,\, \underline{\mathrm{F}}[1]\,,\, \ldots\,,\, \underline{\mathrm{F}}[N-1]}_{\underline{\mathrm{F}}}\,,\, \underbrace{\underline{\mathrm{F}}[0]\,,\, \underline{\mathrm{F}}[1]\,,\, \ldots\,,\, \underline{\mathrm{F}}[N-1]}_{\underline{\mathrm{F}}}}^{\underline{H}})$$

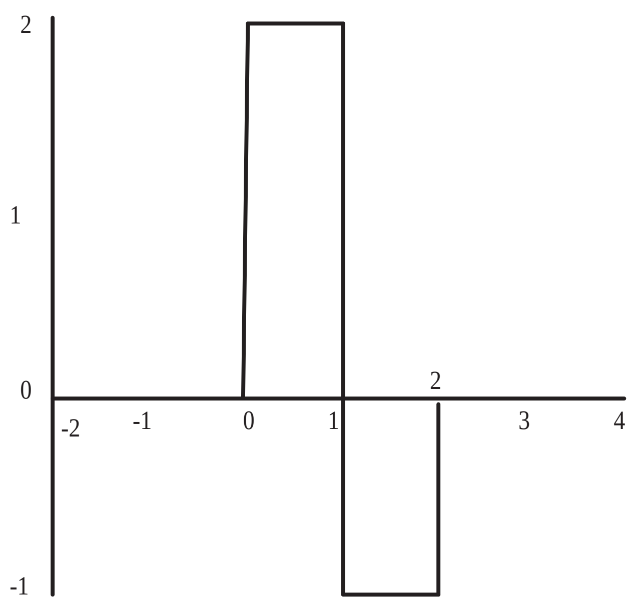
Assuming this, find the DFT of $\underline{\widetilde{\mathrm{f}}}$.

4. (10 points) A linear system L has the inputs (on the left) and outputs (on the right) shown below.



(a) Is L time-invariant? Justify your answer.

(b) Sketch the output of L given the input below.



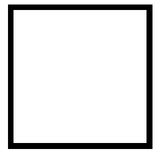
5. (15 points) Suppose we model the Stanford Clock Tower bells as a system, where the hammer (to hit the bell) is the input, the bell is the system, and the ringing sound is the output.

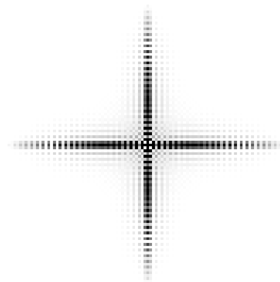
   (a) Is the system linear (approximately)? Is it time invariant?

   (b) Is this system stable? (Recall that 'stable' means bounded inputs result in bounded outputs.)

   (c) Give an analytic expression that might represent the impulse response, $h(t)$, of the system. Justify your answer.

6. (30 points) Consider the square shown below to be represented by a functions $f(x_1, x_2)$ of x_1 and x_2. The gray level shows the value at a given point, with black being 1 and white being 0. Next to $f(x_1, x_2)$ is a plot of the magnitude of its Fourier transform $|\mathcal{F}f(\xi_1, \xi_2)|$.





Let $a > 1$ be a fixed constant, and let A be the matrix

$$A = \begin{pmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{pmatrix}$$

The problem involves a number of figures and is stated on the next page.

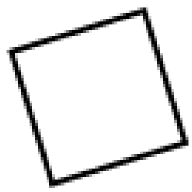
Consider the following modifications of $f(x_1, x_2)$:

1. $f(ax_1, x_2)$ 2. $f(x_1, ax_2)$ 3. $f(x_1 + a, x_2)$

4. $f\left(A\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)$ 5. $f\left(A^{-1}\begin{pmatrix}x_1\\x_2\end{pmatrix}\right)$ 6. $f(x_1, x_2) * \operatorname{sinc}(ax_1)\operatorname{sinc}(ax_2)$

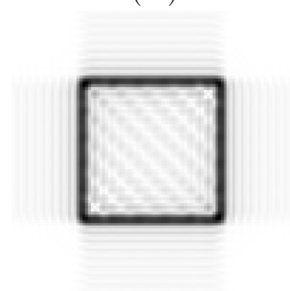
Match the modification 1 – 6 of $f(x_1, x_2)$ with the corresponding plot (i) – (vi) *and* with the plot of the corresponding Fourier transform (A) – (F). Give brief explanations.

(i)

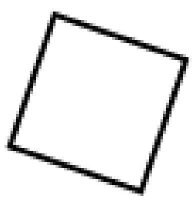


(ii)

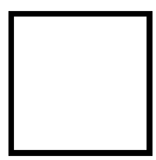
(iii)



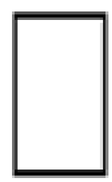
(iv)



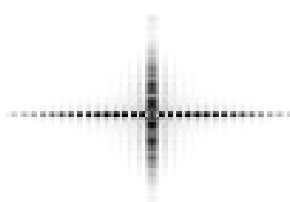
(v)



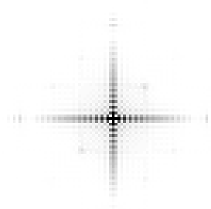
(vi)



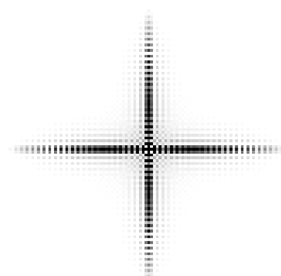
(A)



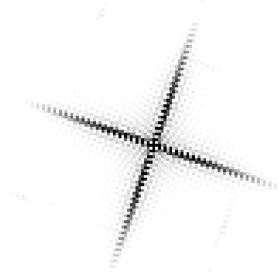
(B)



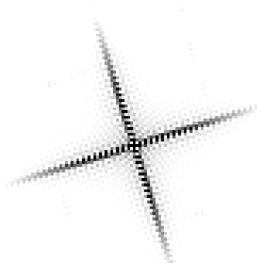
(C)



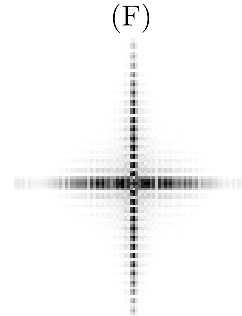
(D)



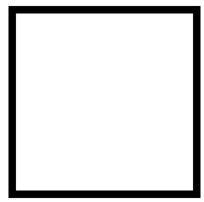
(E)



(F)



7. (10 points) Again consider the square shown below to be represented by a functions $f(x_1, x_2)$ of x_1 and x_2. The gray level shows the value at a given point, with black being 1 and white being 0. Thus the drawing shows precisely where $f(x_1, x_2) = 1$.



Assume the outside dimension of the square is 1 and the inside dimension of the square is .9. Find the Fourier transform of $f(x_1, x_2)$.