0.1 Safe lambda calculus without homogeneous types

We use a set of sequents of the form $\Gamma \vdash^{-k} M : A$ where the meaning is "variables in Γ have orders at least $\mathsf{ord}(A) - k$". The sequents $\vdash^0$ and $\vdash^{-1}$ are also noted $\vdash^+$ and $\vdash^-$ respectively.

$$(\mathbf{seq_{i,k}}) \quad \frac{\Gamma \vdash^{-i} M : A}{\Gamma \vdash^{-(i+k)} M : A} \quad k > 0$$

$$(\mathbf{var}) \quad \frac{}{x : A \vdash^0 x : A}$$

$$(\mathbf{wk^{-i}}) \quad \frac{\Gamma \vdash^{-i} M : A}{\Gamma, x : B \vdash^{-i} M : A} \quad \mathsf{ord}(B) \geq \mathsf{ord}(A) - i$$

$$(\mathbf{app^{-i}}) \quad \frac{\Gamma \vdash^{-i} M : A \to B \qquad \Gamma \vdash^0 N : A,}{\Gamma \vdash^{-i+\delta} MN : B} \qquad \delta = \max\left(0, 1 + \mathsf{ord}(A) - \mathsf{ord}(B)\right)$$

$$(\mathbf{abs^{-i}}) \quad \frac{\Gamma, \overline{x} : \overline{A} \vdash^{i} M : B}{\Gamma \vdash^0 \lambda \overline{x} : \overline{A}.M : \overline{A}, B} \qquad \forall y \in \Gamma : \mathsf{ord}(y) \geq \mathsf{ord}(\overline{A}, B)$$

Note that:

- $\overline{A}, B$ denotes the type $(A_1, A_2, \ldots, A_n, B)$;
- all the types appearing in the rule are not required to be homogeneous. For instance in the rule $(\mathbf{app^{-i}})$, the type $A \to B$ is not necessarily homogeneous;
- the environment $\Gamma, \overline{x}$ is not stratified. In particular, variables in $\overline{x}$ do not necessarily have the same order. Also there may be variable in Γ of order smaller than $\mathsf{ord}(x_i)$ for some i.

Claim: Provided that substitution is done simultaneously (even for variable of different order), there is not variable capture when performing substitution on a safe (non homogeneous) term.

Proof. TO DO! □