The Safe λ-Calculus

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Overview

- Safety is a restriction for higher-order grammars.
- It can be transposed to the λ-calculus, giving rise to the Safe λ-calculus.
- Safety has nice algorithmic properties, automata-theoretic and game-semantic characterizations.

What is the Safety Restriction?

- First appeared under the name "restriction of derived types" in "IO and OI Hierarchies" by W. Damm, TCS 1982
- It is a syntactic restriction for higher-order grammars that constrains the occurrences of the variables in the grammar equations according to their orders.

Theorem (Knapik, Niwiński and Urzyczyn (2001,2002))

1. *The Monadic Second Order (MSO) model checking problem for trees generated by safe higher-order grammars of any order is decidable.*
2. *Automata-theoretic characterization: Safe grammars of order n are as expressive as pushdown automata of order n.*

- Aehlig, de Miranda, Ong (2004) introduced the Safe λ-calculus.

Simply Typed λ-Calculus

- Simple types $A := o \mid A \to A$.
- Type order given by $\mathrm{order}(o) = 0$, $\mathrm{order}(A \to B) = \max(\mathrm{order}(A) + 1, \mathrm{order}(B))$.
- Judgements of the form $\Gamma \vdash M : T$

$$(var)\ \frac{}{x : A \vdash x : A} \qquad (wk)\ \frac{\Gamma \vdash M : A}{\Delta \vdash M : A}\ \Gamma \subset \Delta$$

$$(app)\ \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \quad (abs)\ \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \to B}$$

- Example: $f : o \to o \to o, x : o \vdash (\lambda \varphi^{o \to o} x^o.\varphi\ x)(f\ x)$
- A single rule: β-reduction. e.g. $(\lambda x.M)N \to_\beta M[N/x]$

Variable Capture

The usual "problem" in λ-calculus: avoid variable capture when performing substitution: $(\lambda x.(\lambda y.x))y \rightarrow_\beta (\lambda \underline{y}.x)[\underline{y}/x] \neq \lambda y.y$

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1. Standard solution: Barendregt's convention. Variables are renamed so that free variables and bound variables have different names. Eg. $(\lambda x.(\lambda y.x))y$ becomes $(\lambda x.(\lambda z.x))y$ which reduces to $(\lambda z.x)[y/x] = \lambda z.y$
   Drawback: requires to have access to an unbounded supply of names to perform a given sequence of β-reductions.

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Safety avoids the need for variable renaming!

The Safe λ-Calculus

The formation rules

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$$(app)\ \frac{\Gamma \vdash M : (A, \ldots, A_l, B) \quad \Gamma \vdash_s N_1 : A_1 \quad \ldots \quad \Gamma \vdash_s N_l : A_l}{\Gamma \vdash_s M N_1 \ldots N_l : B}$$

with the side-condition $\forall y \in \Gamma : \mathrm{ord}(y) \geq \mathrm{ord}(B)$

$$(abs)\ \frac{\Gamma, x_1 : A_1 \ldots x_n : A_n \vdash_s M : B}{\Gamma \vdash_s \lambda x_1 : A_1 \ldots x_n : A_n . M : A_1 \to \ldots \to A_n \to B}$$

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Property

In the Safe λ-calculus there is no need to rename variables when performing β-reduction.

Example

- Contracting the β-redex in the following term

$$f : o \to o \to o, x : o \vdash (\lambda \varphi^{o \to o} x^{o}.\varphi\ x)(f\ x)$$

leads to variable capture:

$$(\lambda \varphi x.\varphi\ x)(f\ x) \not\to_{\beta} (\lambda x.(f\ x)x).$$

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  Hence the term is unsafe.
  Indeed, $\mathrm{ord}(x) = 0 \leq 1 = \mathrm{ord}(f\ x)$.
- The term $(\lambda \varphi^{o \to o} x^{o}.\varphi\ x)(\lambda y^{o}.y)$ is safe.

The Correspondence Theorem

Let $\Gamma \vdash M : T$ be a pure simply typed term.

- Let $[\![M]\!]$ denote the game-semantic denotation of M.
- $[\![M]\!]$ is a strategy on the game induced by T. It is represented by a set of sequences of moves together with links.

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Theorem

The game semantics of a term can be represented on the computation tree:

$$\mathcal{T}rav(M) \cong \langle\!\langle M \rangle\!\rangle$$

$$Reduction(\mathcal{T}rav(M)) \cong [\![M]\!]$$

where $\langle\!\langle M \rangle\!\rangle$ is the revealed game-semantic denotion (i.e. internal moves are uncovered).

Game-semantic Characterisation of Safety

- Computation tree of safe terms are incrementally-bound : each variable x is bound by the first λ node occurring in *the path to the root* with order $> \mathrm{ord}(x)$.

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- Hence, by the Correspondence theorem, for a play $s \cdot m$ in the game denotation of a safe term, m is justified by the first move in $\ulcorner s \urcorner$ with order $> \mathrm{ord}(m)$. We say that safe term are denoted by incrementally justified strategies.

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Corollary

Justification pointers are redundant in the game-semantics of safe terms.

Recent result

Theorem

Functions definable in the safe λ-calculus are exactly the polynomials.

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Corollary

The conditional operator $C : I \to I \to I \to I$ verifying:

$$\begin{aligned} Ctyz &\to_\beta y, \textit{ if } t \to_\beta \ulcorner 0 \urcorner \\ Ctyz &\to_\beta z, \textit{ if } t \to_\beta \ulcorner n+1 \urcorner \end{aligned}$$

is not definable in the safe simply typed λ-calculus.

E.g.

$$C = \lambda FGH\alpha x.H(\underline{\lambda y.G\alpha x})(F\alpha x) \ .$$

Conclusion and Further Work

Conclusion:

Safety is a syntactic constraint with nice algorithmic and game-semantic properties.

Related works:

- Forthcoming thesis of Jolie G. de Miranda about unsafety.
- Ong introduced computation trees in LICS2006 to prove decidability of MSO theory on infinite trees generated by higher-order grammars (whether safe or not).
- Stirling recently proved decidability of higher-order pattern matching with a game-semantic approach relying on equivalent notions of computation tree and traversal.

Open questions:

- Complexity classes characterised with the Safe λ-calculus?
- Does the pointer economy extend to Safe Idealized Algol? Decidability of contextual equivalence?

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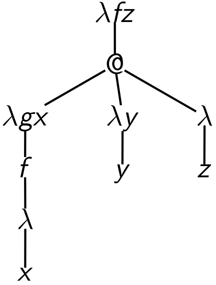
Correspondence (1) : computation trees and traversals

Computation tree = tree representation of the η-long normal form of a term.

Example $M \equiv \lambda fz.(\lambda gx.fx)(\lambda y.y)z$ of type $(o \to o) \to o \to o$.



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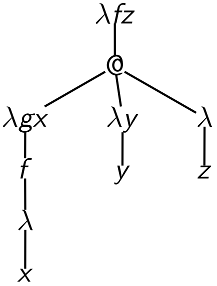

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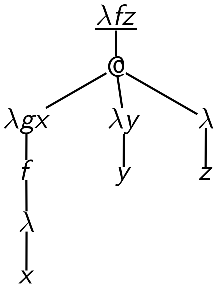

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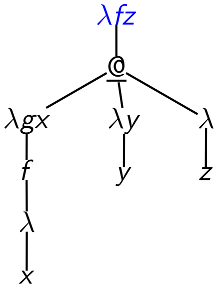

$t = \lambda fz \cdot @$

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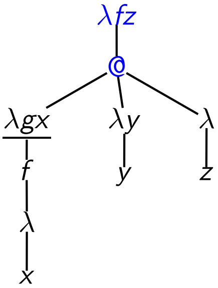
Example $M \equiv \lambda fz.(\lambda gx.fx)(\lambda y.y)z$ of type $(o \rightarrow o) \rightarrow o \rightarrow o$.



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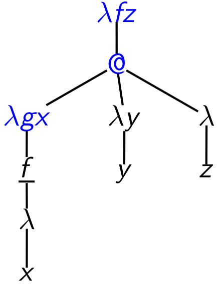

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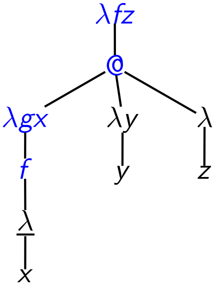

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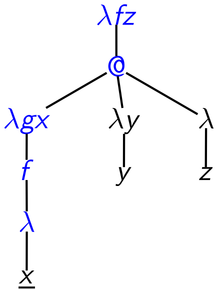

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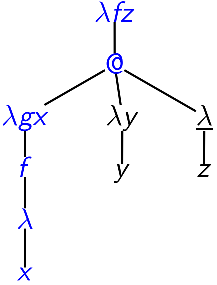

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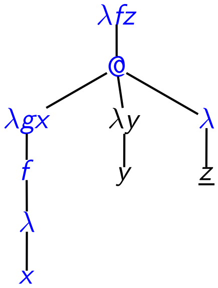

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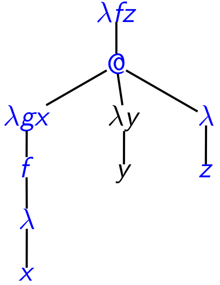

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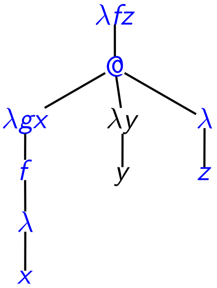

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$$t - @ = \lambda fz \cdot \lambda gx \cdot f \cdot \lambda \cdot x \cdot \lambda \cdot z$$

Correspondence (2) : the theorem

Let M be a pure simply typed term of type T.

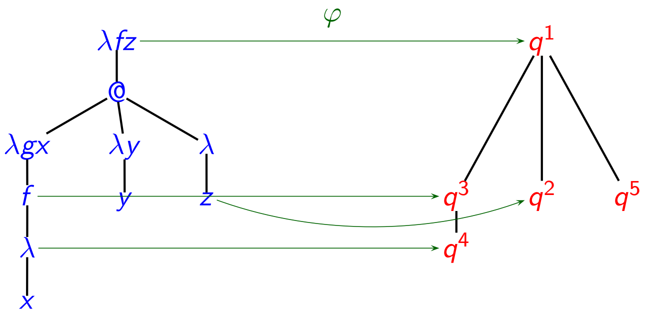
- $\mathcal{T}rav(M)$ = set of traversals of the computation tree of M
- $\mathcal{T}rav(M)^{\upharpoonright r} = \{t \upharpoonright r \mid t \in \mathcal{T}rav(M)\}$
- $\mathcal{T}rav(M)^{-@} = \{t - @ \mid t \in \mathcal{T}rav(M)\}$
- $[\![M]\!]$ = game-semantic denotation of M
- $\langle\!\langle M \rangle\!\rangle$ = revealed denotion (i.e. internal moves are not hidden)

There exists a partial function φ from the nodes of the computation tree to the moves of the arena for T such that

$$\varphi : \mathcal{T}rav(M)^{-@} \xrightarrow{\cong} \langle\!\langle M \rangle\!\rangle$$

$$\varphi : \mathcal{T}rav(M)^{\upharpoonright r} \xrightarrow{\cong} [\![M]\!].$$

Correspondence (3) : example



Take the traversal $t = \lambda fz \cdot @ \cdot \lambda gx \cdot f \cdot \lambda \cdot x \cdot \lambda \cdot z$.
The image by φ of the reduction of t is the play:

$$\varphi(t \upharpoonright r) = \varphi(\lambda fz \cdot f \cdot \lambda \cdot z) = q^1\ q^3\ q^4\ q^2 \in [\![M]\!].$$

Correspondence (4) : Summary

computation tree	arena
traversals	uncovered plays
reduced traversal	plays
paths in the computation tree	P-views of uncovered plays