# CEE262c HW5

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March 6, 2017

In this assignment we work with the 2D-V linearized shallow water equations:

$$\frac{\partial u}{\partial t} = -g\frac{\partial h}{\partial x} - \frac{\partial q}{\partial x} \tag{1}$$

$$\frac{\partial w}{\partial t} = -\frac{\partial q}{\partial z} \tag{2}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x}\int_{-D}^{0} u \, dz \tag{3}$$

## 1

Each term in (1) is discretized as follows:

$$\frac{\partial u}{\partial t} = \frac{u_{i,k}^{n+1} - u_{i,k}^{*}}{\Delta t} + \frac{u_{i,k}^{*} - u_{i,k}^{n}}{\Delta t}$$

$$-g\frac{\partial h}{\partial x} = -\frac{g\theta}{\Delta x}(h_i^{n+1} - h_{i-1}^{n+1}) - \frac{g(1-\theta)}{\Delta x}(h_i^n - h_{i-1}^n)$$

$$-\frac{\partial q}{\partial x} = -\frac{q_{i,k}^c - q_{i-1,k}^c}{\Delta x} - \frac{q_{i,k}^{n-1/2} - q_{i-1,k}^{n-1/2}}{\Delta x}$$

Where the pressure field is to be updated as:

$$q_{i,k}^{n+1/2} = q_{i,k}^c + q_{i,k}^{n-1/2}$$

And define $u^*$ such that:

$$\frac{u_{i,k}^{n+1} - u_{i,k}^{*}}{\Delta t} = -\frac{q_{i,k}^c - q_{i-1,k}^c}{\Delta x} \tag{4}$$

$$u_{i,k}^{n+1} = u_{i,k}^{*} - \frac{\Delta t}{\Delta x}(q_{i,k}^c - q_{i-1,k}^c) \tag{5}$$

Using the above discretizations in (1) and subtracting (4):

$$\frac{u_{i,k}^{*} - u_{i,k}^{n}}{\Delta t} = -\frac{g\theta}{\Delta x}(h_i^{n+1} - h_{i-1}^{n+1}) - \frac{g(1-\theta)}{\Delta x}(h_i^n - h_{i-1}^n) - \frac{1}{\Delta x}(q_{i,k}^{n-1/2} - q_{i-1,k}^{n-1/2})$$

$$u^*_{i,k} = S_{i,k} - \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_i - h^{n+1}_{i-1}) \tag{6}$$

In (6), $S_{i,k}$ is given by:

$$S_{i,k} = u^n_{i,k} - \frac{g\Delta t(1-\theta)}{\Delta x}(h^n_i - h^n_{i-1}) - \frac{\Delta t}{\Delta x}(q^{n-1/2}_{i,k} - q^{n-1/2}_{i-1,k})$$

To discretize (2) we use the same pressure update given by (1) and discretize each term as follows:

$$\frac{\partial w}{\partial t} = \frac{w^{n+1}_{i,k} - w^*_{i,k}}{\Delta t} + \frac{w^*_{i,k} - w^n_{i,k}}{\Delta t}$$

$$-\frac{\partial q}{\partial z} = -\frac{q^c_{i,k} - q^c_{i,k-1}}{\Delta z} - \frac{q^{n-1/2}_{i,k} - q^{n-1/2}_{i,k-1}}{\Delta z}$$

Define $w^*$ such that:

$$\frac{w^{n+1}_{i,k} - w^*_{i,k}}{\Delta t} = -\frac{q^c_{i,k} - q^c_{i,k-1}}{\Delta z} \tag{7}$$

Plugging the above discretizations into (2) and subtracting (7):

$$\frac{w^*_{i,k} - w^n_{i,k}}{\Delta t} = -\frac{q^{n-1/2}_{i,k} - q^{n-1/2}_{i,k-1}}{\Delta z}$$

$$w^*_{i,k} = w^n_{i,k} - \frac{\Delta t}{\Delta z}(q^{n-1/2}_{i,k} - q^{n-1/2}_{i,k-1}) \tag{8}$$

The u-velocity field is bounded by walls on both sides such that at all timesteps $u_{0,k} = u_{N_i,k} = 0$. The w-velocity field is bounded by a wall on the bottom such that for all timesteps $w_{i,0} = 0$. At the free surface we must assume $q^c = 0$ such that $q_{i,N_k} = -q_{i,N_k-1}$. This allows (8) to be solved at the free surface:

$$w^*_{i,N_k} = w^n_{i,N_k} - \frac{\Delta t}{\Delta z}(q^{n-1/2}_{i,N_k} - q^{n-1/2}_{i,N_k-1})$$

$$w^*_{i,N_k} = w^n_{i,N_k} + \frac{2\Delta t}{\Delta z}q^{n-1/2}_{i,N_k-1} \tag{9}$$

In equation (3) we discretize each term as follows:

$$\frac{\partial h}{\partial t} = \frac{h^{n+1}_i - h^n_i}{\Delta t}$$

$$\begin{aligned}
-\frac{\partial}{\partial x}\int_{-D}^{0} u\ dz &= -\theta\frac{\partial}{\partial x}\int_{-D}^{0} u^*\ dz - (1-\theta)\frac{\partial}{\partial x}\int_{-D}^{0} u^n\ dz \\
&= -\theta\frac{\partial}{\partial x}\sum_{k=1}^{N_k} u^*_{i,k}\Delta z - (1-\theta)\frac{\partial}{\partial x}\sum_{k=1}^{N_k} u^n_{i,k}\Delta z \\
&= -\frac{\theta\Delta z}{\Delta x}\sum_{k=1}^{N_k}(u^*_{i+1,k} - u^*_{i,k}) - \frac{(1-\theta)\Delta z}{\Delta x}\sum_{k=1}^{N_k}(u^n_{i+1,k} - u^n_{i,k})
\end{aligned}$$

Note in the above expression we have assumed that $u^{n+1} - u^*$ is small and replaced $u^{n+1}$ by $u^*$ in the theta method. Equation (3) becomes:

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} = -\frac{\theta \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^*_{i+1,k} - u^*_{i,k}) - \frac{(1-\theta)\Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{i+1,k} - u^n_{i,k}) \tag{10}$$

$$h_i^{n+1} = h_i^n - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^*_{i+1,k} - u^*_{i,k}) - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{i+1,k} - u^n_{i,k}) \tag{11}$$

Substituting in (6) for $u^*$:

$$\begin{aligned} h_i^{n+1} = h_i^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{i+1,k} - u^n_{i,k}) \ldots \\ - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} \left( S_{i+1,k} - \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_{i+1} - h^{n+1}_i) - S_{i,k} + \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_i - h^{n+1}_{i-1}) \right) \end{aligned}$$

$$\begin{aligned} h_i^{n+1} - \frac{\theta \Delta t D}{\Delta x} \left( \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_{i+1} - h^{n+1}_i) - \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_i - h^{n+1}_{i-1}) \right) \ldots \\ = h_i^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{i+1,k} - u^n_{i,k}) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{i+1,k} - S_{i,k}) \end{aligned}$$

$$-a h^{n+1}_{i-1} + (1+2a) h^{n+1}_i - a h^{n+1}_{i+1} = R_i$$

Where:

$$a = \frac{\theta^2 g D \Delta t^2}{\Delta x^2}$$

$$R_i = h_i^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{i+1,k} - u^n_{i,k}) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{i+1,k} - S_{i,k})$$

At the boundary cells we use the no-flux boundary condition to complete the system of equations. Starting from (11) we derive the equations for $h_0$ and $h_{N_i - 1}$:

$$h_0^{n+1} = h_0^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{1,k} - u^n_{0,k}) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^*_{1,k} - u^*_{0,k})$$

$$h_0^{n+1} = h_0^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{1,k}) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{1,k} - \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_1 - h^{n+1}_0))$$

$$(1+a) h_0^{n+1} - a h_1^{n+1} = h_0^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u^n_{1,k}) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{1,k}) = R_0$$

$$h^{n+1}_{N_i-1} = h^n_{N_i-1} - \frac{(1-\theta)\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}(u^n_{N_i,k} - u^n_{N_i-1,k}) - \frac{\theta\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}(u^*_{N_i,k} - u^*_{N_i-1,k})$$

$$h^{n+1}_{N_i-1} = h^n_{N_i-1} + \frac{(1-\theta)\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}(u^n_{N_i-1,k}) + \frac{\theta\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}(S_{N_i-1,k} - \frac{g\theta\Delta t}{\Delta x}(h^{n+1}_{N_i-1} - h^{n+1}_{i-2}))$$

$$(1+a)h^{n+1}_{N_i-1} - ah^{n+1}_{N_i-2} = h^n_{N_i-1} + \frac{(1-\theta)\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}(u^n_{N_i-1,k}) + \frac{\theta\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}(S_{N_i-1,k}) = R_{N_i-1}$$

To solve for $h^{n+1}$ solve the following tridiagonal system:

$$\begin{pmatrix} (1+a) & -a & & & \\ & \ddots & \ddots & & \\ & & -a & (1+2a) & -a & \\ & & & \ddots & \ddots & \\ & & & & -a & (1+a) \end{pmatrix} \begin{pmatrix} \\ \\ h \\ \\ \\ \end{pmatrix}^{n+1} = \begin{pmatrix} R_0 \\ \vdots \\ R_j \\ \vdots \\ R_{N_i-1} \end{pmatrix}$$

# 2

In (5) and (7) we dicretely defined $\vec{u}^{n+1} = \vec{u}^* - \Delta t\vec{\nabla}q^c$. Taking the divergence and using $\vec{\nabla}\cdot\vec{u}^{n+1} = 0$:

$$\vec{u}^{n+1} = \vec{u}^* - \Delta t\vec{\nabla}q^c$$
$$\nabla^2 q^c = \frac{1}{\Delta t}\vec{\nabla}\cdot\vec{u}^*$$

This can be written discretely as:

$$\frac{1}{\Delta x}\left(\left.\frac{\partial q^c}{\partial x}\right|_{i+1,k} - \left.\frac{\partial q^c}{\partial x}\right|_{i,k}\right) + \frac{1}{\Delta z}\left(\left.\frac{\partial q^c}{\partial z}\right|_{i,k+1} - \left.\frac{\partial q^c}{\partial z}\right|_{i,k}\right) = b_{i,k} \tag{12}$$

$$\frac{1}{\Delta x^2}(q^c_{i+1,k} - 2q^c_{i,k} + q^c_{i-1,k}) + \frac{1}{\Delta z^2}(q^c_{i,k+1} - 2q^c_{i,k} + q^c_{i,k-1}) = b_{i,k} \tag{13}$$

$$b_{i,k} = \frac{1}{\Delta t}\left(\frac{u^*_{i+1,k} - u^*_{i,k}}{\Delta x} + \frac{w^*_{i,k+1} - w^*_{i,k}}{\Delta z}\right)$$

Using the no-flux condition at the walls ($u_{0,k} = u_{N_i,k} = 0$ and $w_{i,0} = 0$) in (4):

$$\frac{u^{n+1}_{i=\text{wall},k} - u^*_{i=\text{wall},k}}{\Delta t} = -\frac{q^c_{i,k} - q^c_{i-1,k}}{\Delta x}$$
$$0 = \left.\frac{\partial q^c}{\partial x}\right|_{\text{wall},k}$$

The result is analagous in (7). With this information and with the free surface condition $q_{i,N_k} = -q_{i,N_k-1}$ as in (9) we can rewrite (13) to work at the 8 different types of boundaries:

At $(0,k)$:

$$\frac{1}{\Delta x^2}(q^c_{1,k} - q^c_{0,k}) + \frac{1}{\Delta z^2}(q^c_{0,k+1} - 2q^c_{0,k} + q^c_{0,k-1}) = \frac{1}{\Delta t}\left(\frac{u^*_{1,k}}{\Delta x} + \frac{w^*_{0,k+1} - w^*_{0,k}}{\Delta z}\right)$$

At $(0,0)$:

$$\frac{1}{\Delta x^2}(q^c_{1,0} - q^c_{0,0}) + \frac{1}{\Delta z^2}(q^c_{0,1} - q^c_{0,0}) = \frac{1}{\Delta t}\left(\frac{u^*_{1,0}}{\Delta x} + \frac{w^*_{0,1}}{\Delta z}\right)$$

At $(i,0)$:

$$\frac{1}{\Delta x^2}(q^c_{i+1,0} - 2q^c_{i,0} + q^c_{i-1,0}) + \frac{1}{\Delta z^2}(q^c_{i,1} - q^c_{i,0}) = \frac{1}{\Delta t}\left(\frac{u^*_{i+1,0} - u^*_{i,0}}{\Delta x} + \frac{w^*_{i,1}}{\Delta z}\right)$$

At $(N_i - 1, 0)$:

$$\frac{1}{\Delta x^2}(-q^c_{N_i-1,0} + q^c_{N_i-2,0}) + \frac{1}{\Delta z^2}(q^c_{N_i-1,1} - q^c_{N_i-1,0}) = \frac{1}{\Delta t}\left(\frac{-u^*_{N_i-1,0}}{\Delta x} + \frac{w^*_{N_i-1,1}}{\Delta z}\right)$$

At $(N_i - 1, k)$:

$$\frac{1}{\Delta x^2}(-2q^c_{N_i-1,k} + q^c_{N_i-2,k}) + \frac{1}{\Delta z^2}(q^c_{N_i-1,k+1} - 2q^c_{N_i-1,k} + q^c_{N_i-1,k-1}) = \frac{1}{\Delta t}\left(\frac{-u^*_{N_i-1,k}}{\Delta x} + \frac{w^*_{N_i-1,k+1} - w^*_{N_i-1,k}}{\Delta z}\right)$$

At $(N_i - 1, N_k - 1)$:

$$\frac{1}{\Delta x^2}(-q^c_{N_i-1,N_k-1} + q^c_{N_i-2,N_k-1}) + \frac{1}{\Delta z^2}(-3q^c_{N_i-1,N_k-1} + q^c_{N_i-1,N_k-2}) = \frac{1}{\Delta t}\left(\frac{-u^*_{N_i-1,N_k-1}}{\Delta x} + \frac{w^*_{i,k+1} - w^*_{i,k}}{\Delta z}\right)$$

At $(i, N_k - 1)$:

$$\frac{1}{\Delta x^2}(q^c_{i+1,N_k-1} - 2q^c_{i,N_k-1} + q^c_{i-1,N_k-1}) + \frac{1}{\Delta z^2}(-3q^c_{i,N_k-1} + q^c_{i,N_k-2}) = \frac{1}{\Delta t}\left(\frac{u^*_{i+1,N_k-1} - u^*_{i,N_k-1}}{\Delta x} + \frac{w^*_{i,N_k} - w^*_{i,N_k-1}}{\Delta z}\right)$$

At $(0, N_k - 1)$:

$$\frac{1}{\Delta x^2}(q^c_{1,N_k-1} - q^c_{0,N_k-1}) + \frac{1}{\Delta z^2}(-3q^c_{0,N_k-1} + q^c_{0,N_k-2}) = \frac{1}{\Delta t}\left(\frac{u^*_{1,N_k-1}}{\Delta x} + \frac{w^*_{i,k+1} - w^*_{i,k}}{\Delta z}\right)$$

Using these boundary conditions and (13) a pentadiagonal matrix can be constructed and $q^c$ can be computed using a conjugate gradient solver.

After solving the system of equations for $q^c$ use (1) to calculate $q^{n+1/2}$. Then use (5) to update the u-velocity field $u^{n+1}$. To update the w-velocity field use the fact that the flow is divergence free:

$$\frac{w^{n+1}_{i,k+1} - w^{n+1}_{i,k}}{\Delta z} = -\frac{u^{n+1}_{i+1,k} - u^{n+1}_{i,k}}{\Delta x}$$

$$w^{n+1}_{i,k+1} = w^{n+1}_{i,k} - \frac{\Delta z}{\Delta x}(u^{n+1}_{i+1,k} - u^{n+1}_{i,k})$$

Starting with the bottom boundary condition $w_{i,0} = 0$ this system of equations can be solved by back substitution.

## 3

Figure 1 shows the time evolution of the free surface comparing the analytical result to the non-hydrostatic and hydrostatic numerical results. The comparison is considered for three domains with varying depth to length ratios: $D/L \in \{1, 1/4, 1/8\}$. For $D/L = 1$ the non-hydrostatic model result is out of phase with the analytical. The wavelength from the hydrostatic model is much shorter than the analytical. As $D/L$ decreases, the wavelength from the hydrostatic model increases and the non-hydrostatic phase shift decreases. For $D/L = 8$ the hydrostatic and non-hydrostatic model results both nearly match the analytical. The results of the hydrostatic model in particular are significantly improved because neglecting the non-hydrostatic pressure is a more appropriate assumption for the lower $D/L$ ratio.

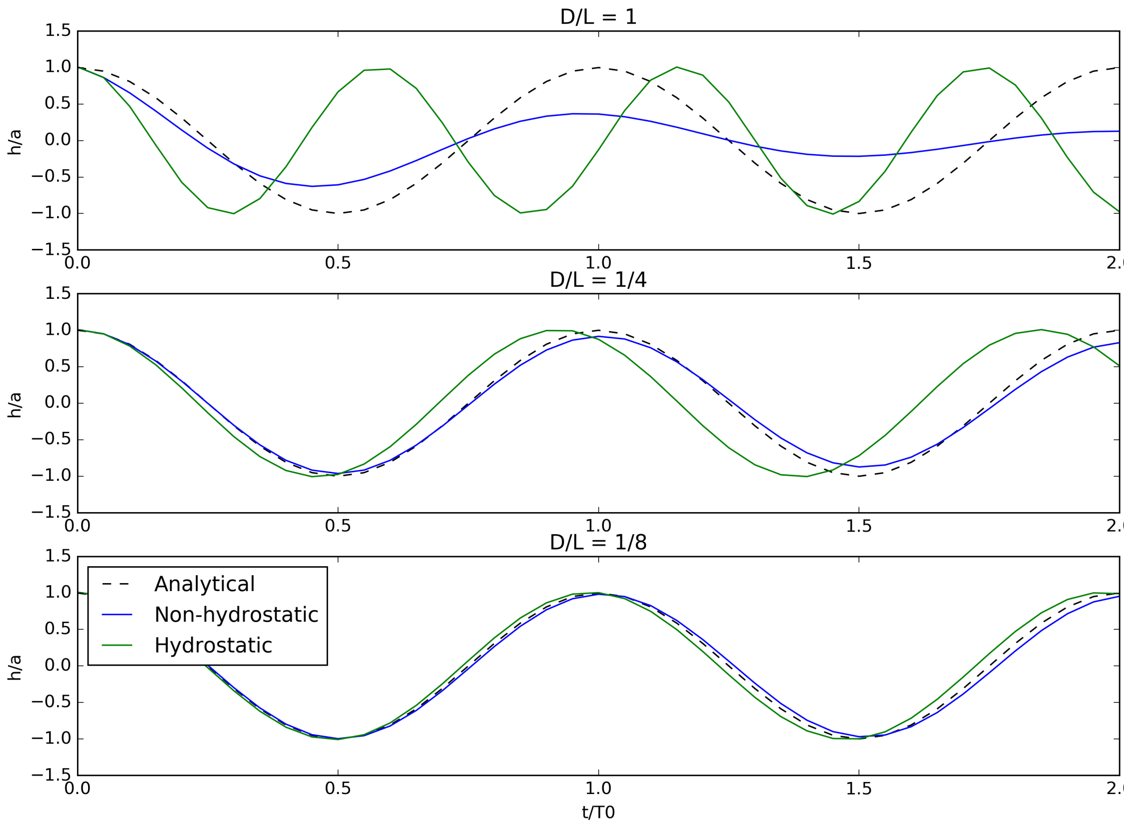


Figure 1: Timeseries of free surface height normalized by initial condition amplitude at $x/L = 0$. Comparison of theoretical, non-hydrostatic and hydrostatic results for $D/L \in \{1, 1/4, 1/8\}$.

Figure 2 compares the vertical velocity profiles for the hydrostatic and non-hydrostatic models for the same three $D/L$ ratios. The hydrostatic profiles remain linear for all $D/L$ ratios. The u velocity is roughly constant with depth and the w velocity decreases from 1 at the surface to zero at the bottom. The non-hydrostatic profiles change significantly as $D/L$ varies. For $D/L = 1$ the velocity profiles for both u and w decrease non-linearly, with approximately 50% of the velocity decrease occuring in the top 20% of the water column. As $D/L$ is decreased the velocity profiles converge to roughly match the linear hydrostatic profiles (the w velocity converges first). This occurs because for lower $D/L$ the hydrostatic assumption is a more appropriate model for the physical characteristics of the problem.

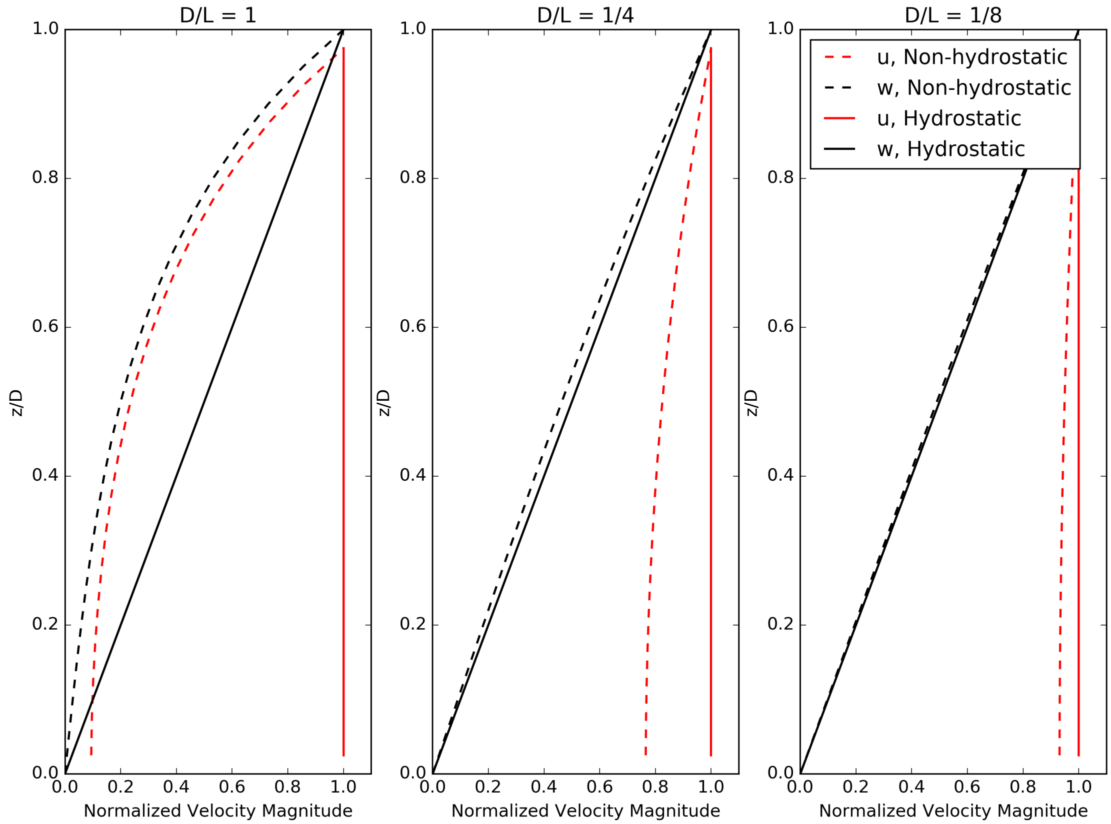


Figure 2: Vertical profiles of velocity for $u$ and $w$ normalized by their maximum value along the profile. Comparison of non-hydrostatic and hydrostatic velocity profiles for $D/L \in \{1, 1/4, 1/8\}$.

# 4

With $k = 2\pi/\lambda$ and $\lambda = 2L$, the theoretical wave phase speed is:

$$c = \sqrt{\frac{g}{k}\tanh(kD)} \tag{14}$$

The deep-water limit of the dispersion relation is:

$$c_0 = \sqrt{\frac{g}{k}} \tag{15}$$

The shallow-water limit is:

$$c_s = \sqrt{gD} \tag{16}$$

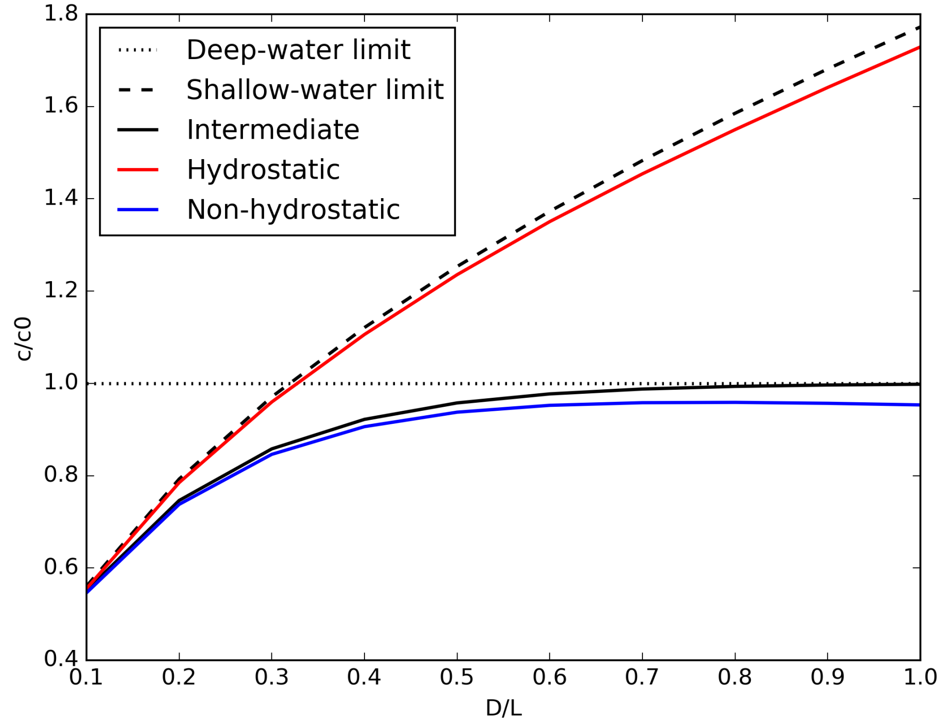


Figure 3: Theoretical and numerical dispersion relations for wave phase speed normalized by the deep-water limit. The theoretical intermediate value is given by (14). The theoretical deep-water limit is given by (15) and the theoretical shallow-water limit is given by (16).

# 5

Figure 4 ($\Delta t = T/20$) and Figure 5 ($\Delta t = T/80$) compare the freesurface evolution of the projection and correction methods for the non-hydrostatic pressure field. The projection method has a lot of diffusion, for $\Delta t = T/20$ the diffusion takes the free surface fluctuations to essentially zero by about 5 wave periods. For $\Delta t = T/20$ the correction method also shows significant diffusion, this comes from the assumption in (10) that $u^{n+1}$ can be replaced by $u^*$. The correction method also suffers from dispersion, which is expected. For $\Delta t = T/20$ the numerical dispersion causes the phase of the free surface cosine wave shifts by over a half period over 10 wave periods. For $\Delta t = T/80$ the phase of the free surface cosine wave shifts by only a small fraction of a period.

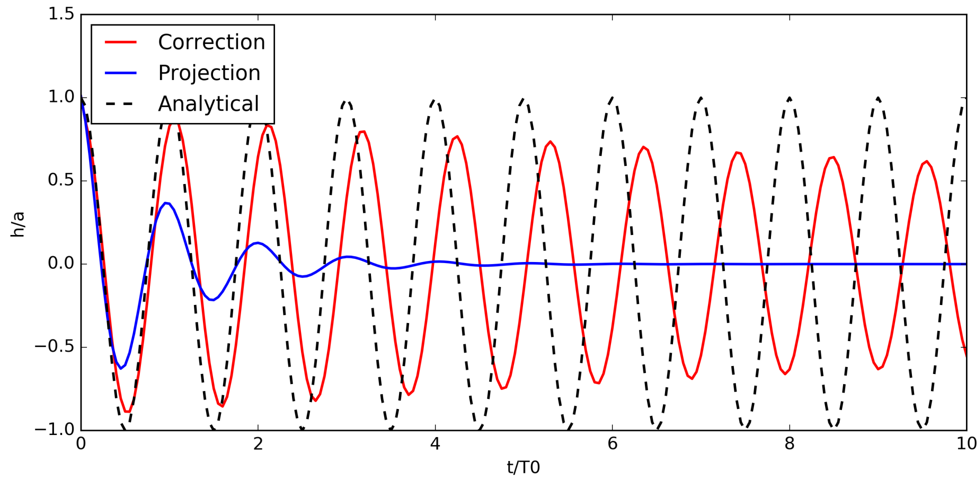


Figure 4: $\Delta t = T/20$. Timeseries of free surface height normalized by initial condition amplitude at $x/L = 0$. Comparison of analytical solution to projection and correction methods for non-hydrostatic pressure calculation.

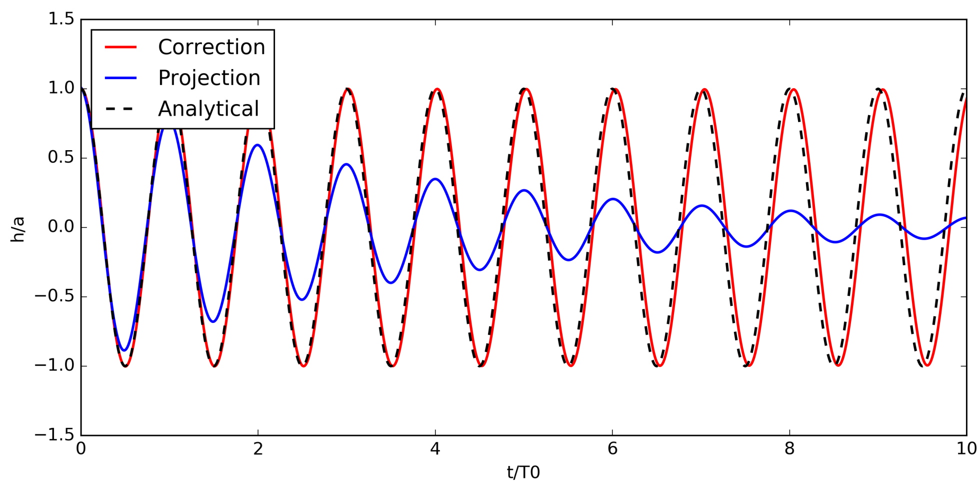


Figure 5: $\Delta t = T/80$. Timeseries of free surface height normalized by initial condition amplitude at $x/L = 0$. Comparison of analytical solution to projection and correction methods for non-hydrostatic pressure calculation.

Figures (6) and (7) show the time accuracy of the correction and projection methods respectively. Error is computed as $(||f(\Delta t)-f(\Delta t/2)||_2)_{t=t_{\max}}$, the relative error. The correction method is second order accurate in time. The projection method is first order accurate in time.

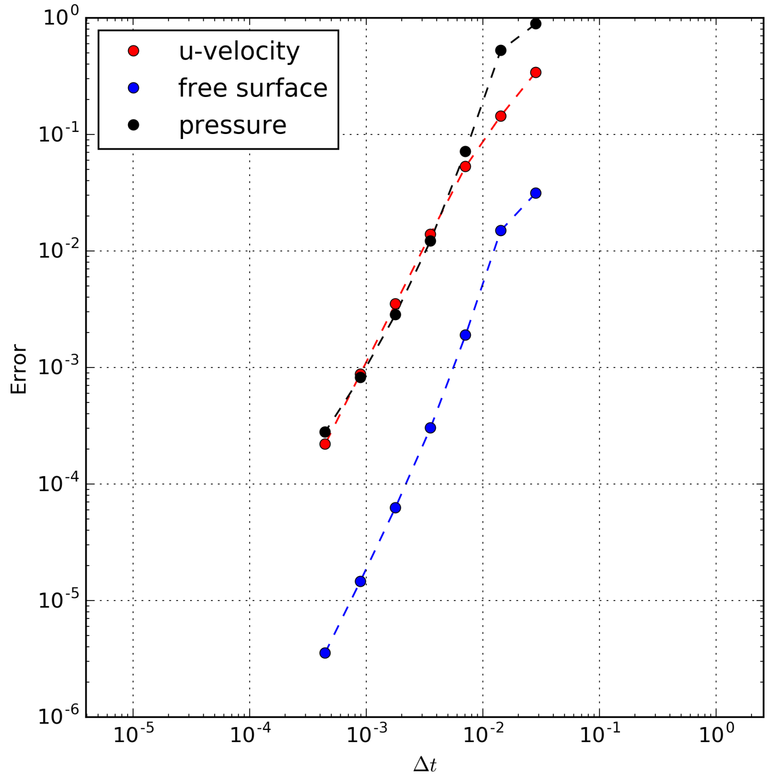


Figure 6: Time accuracy of the correction method for non-hydrostatic pressure calculation

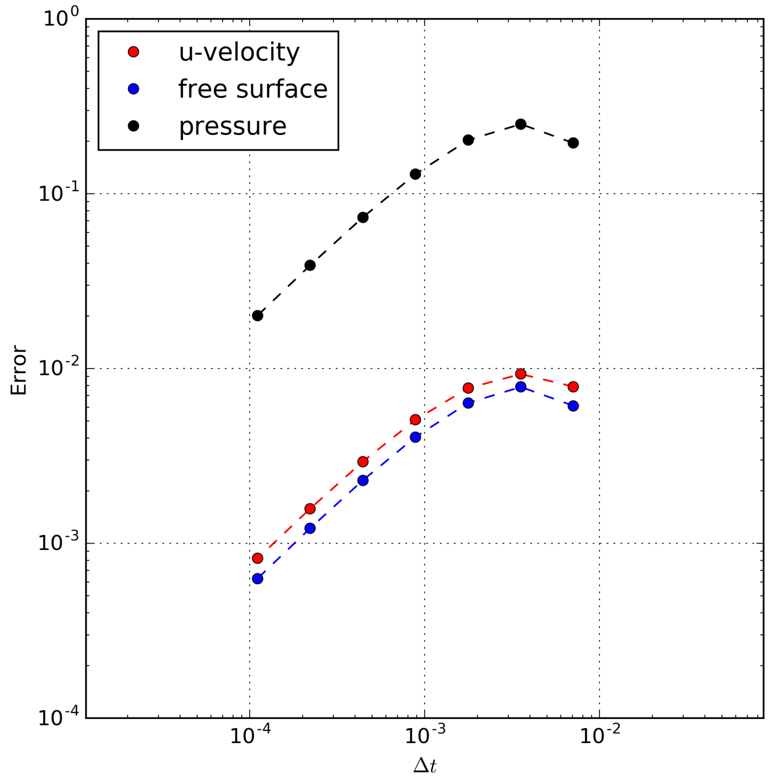


Figure 7: Time accuracy of the projection method for non-hydrostatic pressure calculation