

CEE262c HW5

Brent Lughino
lughino@stanford.edu

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In this assignment we work with the 2D-V linearized shallow water equations:

$$\frac{\partial u}{\partial t} = -g \frac{\partial h}{\partial x} - \frac{\partial q}{\partial x} \quad (1)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial q}{\partial z} \quad (2)$$

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial x} \int_{-D}^0 u \, dz \quad (3)$$

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Each term in (1) is discretized as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{u_{i,k}^{n+1} - u_{i,k}^*}{\Delta t} + \frac{u_{i,k}^* - u_{i,k}^n}{\Delta t} \\ -g \frac{\partial h}{\partial x} &= -\frac{g\theta}{\Delta x} (h_i^{n+1} - h_{i-1}^{n+1}) - \frac{g(1-\theta)}{\Delta x} (h_i^n - h_{i-1}^n) \\ -\frac{\partial q}{\partial x} &= -\frac{q_{i,k}^c - q_{i-1,k}^c}{\Delta x} - \frac{q_{i,k}^{n-1/2} - q_{i-1,k}^{n-1/2}}{\Delta x} \end{aligned}$$

Where the pressure field is to be updated as:

$$q_{i,k}^{n+1/2} = q_{i,k}^c + q_{i,k}^{n-1/2}$$

And define u^* such that:

$$\frac{u_{i,k}^{n+1} - u_{i,k}^*}{\Delta t} = -\frac{q_{i,k}^c - q_{i-1,k}^c}{\Delta x} \quad (4)$$

$$u_{i,k}^{n+1} = u_{i,k}^* - \frac{\Delta t}{\Delta x} (q_{i,k}^c - q_{i-1,k}^c) \quad (5)$$

Using the above discretizations in (1) and subtracting (4):

$$\frac{u_{i,k}^* - u_{i,k}^n}{\Delta t} = -\frac{g\theta}{\Delta x} (h_i^{n+1} - h_{i-1}^{n+1}) - \frac{g(1-\theta)}{\Delta x} (h_i^n - h_{i-1}^n) - \frac{1}{\Delta x} (q_{i,k}^{n-1/2} - q_{i-1,k}^{n-1/2})$$

$$u_{i,k}^* = S_{i,k} - \frac{g\theta\Delta t}{\Delta x}(h_i^{n+1} - h_{i-1}^{n+1}) \quad (6)$$

In (6), $S_{i,k}$ is given by:

$$S_{i,k} = u_{i,k}^n - \frac{g\Delta t(1-\theta)}{\Delta x}(h_i^n - h_{i-1}^n) - \frac{\Delta t}{\Delta x}(q_{i,k}^{n-1/2} - q_{i-1,k}^{n-1/2})$$

To discretize (2) we use the same pressure update given by (1) and discretize each term as follows:

$$\begin{aligned} \frac{\partial w}{\partial t} &= \frac{w_{i,k}^{n+1} - w_{i,k}^*}{\Delta t} + \frac{w_{i,k}^* - w_{i,k}^n}{\Delta t} \\ -\frac{\partial q}{\partial z} &= -\frac{q_{i,k}^c - q_{i,k-1}^c}{\Delta z} - \frac{q_{i,k}^{n-1/2} - q_{i,k-1}^{n-1/2}}{\Delta z} \end{aligned}$$

Define w^* such that:

$$\frac{w_{i,k}^{n+1} - w_{i,k}^*}{\Delta t} = -\frac{q_{i,k}^c - q_{i,k-1}^c}{\Delta z} \quad (7)$$

Plugging the above discretizations into (2) and subtracting (7):

$$\begin{aligned} \frac{w_{i,k}^* - w_{i,k}^n}{\Delta t} &= -\frac{q_{i,k}^{n-1/2} - q_{i,k-1}^{n-1/2}}{\Delta z} \\ w_{i,k}^* &= w_{i,k}^n - \frac{\Delta t}{\Delta z}(q_{i,k}^{n-1/2} - q_{i,k-1}^{n-1/2}) \end{aligned} \quad (8)$$

The u-velocity field is bounded by walls on both sides such that at all timesteps $u_{0,k} = u_{N_i,k} = 0$. The w-velocity field is bounded by a wall on the bottom such that for all timesteps $w_{i,0} = 0$. At the free surface we must assume $q^c = 0$ such that $q_{i,N_k} = -q_{i,N_k-1}$. This allows (8) to be solved at the free surface:

$$\begin{aligned} w_{i,N_k}^* &= w_{i,N_k}^n - \frac{\Delta t}{\Delta z}(q_{i,N_k}^{n-1/2} - q_{i,N_k-1}^{n-1/2}) \\ w_{i,N_k}^* &= w_{i,N_k}^n + \frac{2\Delta t}{\Delta z}q_{i,N_k-1}^{n-1/2} \end{aligned} \quad (9)$$

In equation (3) we discretize each term as follows:

$$\begin{aligned} \frac{\partial h}{\partial t} &= \frac{h_i^{n+1} - h_i^n}{\Delta t} \\ -\frac{\partial}{\partial x} \int_{-D}^0 u \, dz &= -\theta \frac{\partial}{\partial x} \int_{-D}^0 u^* \, dz - (1-\theta) \frac{\partial}{\partial x} \int_{-D}^0 u^n \, dz \\ &= -\theta \frac{\partial}{\partial x} \sum_{k=1}^{N_k} u_{i,k}^* \Delta z - (1-\theta) \frac{\partial}{\partial x} \sum_{k=1}^{N_k} u_{i,k}^n \Delta z \\ &= -\frac{\theta \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^* - u_{i,k}^*) - \frac{(1-\theta) \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^n - u_{i,k}^n) \end{aligned}$$

Note in the above expression we have assumed that $u^{n+1} - u^*$ is small and replaced u^{n+1} by u^* in the theta method. Equation (3) becomes:

$$\frac{h_i^{n+1} - h_i^n}{\Delta t} = -\frac{\theta \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^* - u_{i,k}^*) - \frac{(1-\theta) \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^n - u_{i,k}^n) \quad (10)$$

$$h_i^{n+1} = h_i^n - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^* - u_{i,k}^*) - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^n - u_{i,k}^n) \quad (11)$$

Substituting in (6) for u^* :

$$\begin{aligned} h_i^{n+1} &= h_i^n - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^n - u_{i,k}^n) \dots \\ &\quad - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} \left(S_{i+1,k} - \frac{g \theta \Delta t}{\Delta x} (h_{i+1}^{n+1} - h_i^{n+1}) - S_{i,k} + \frac{g \theta \Delta t}{\Delta x} (h_i^{n+1} - h_{i-1}^{n+1}) \right) \\ h_i^{n+1} - \frac{\theta \Delta t D}{\Delta x} \left(\frac{g \theta \Delta t}{\Delta x} (h_{i+1}^{n+1} - h_i^{n+1}) - \frac{g \theta \Delta t}{\Delta x} (h_i^{n+1} - h_{i-1}^{n+1}) \right) \dots \\ &= h_i^n - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^n - u_{i,k}^n) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{i+1,k} - S_{i,k}) \\ &\quad - a h_{i-1}^{n+1} + (1+2a) h_i^{n+1} - a h_{i+1}^{n+1} = R_i \end{aligned}$$

Where:

$$a = \frac{\theta^2 g D \Delta t^2}{\Delta x^2}$$

$$R_i = h_i^n - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{i+1,k}^n - u_{i,k}^n) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{i+1,k} - S_{i,k})$$

At the boundary cells we use the no-flux boundary condition to complete the system of equations. Starting from (11) we derive the equations for h_0 and h_{N_i-1} :

$$\begin{aligned} h_0^{n+1} &= h_0^n - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{1,k}^n - u_{0,k}^n) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{1,k}^* - u_{0,k}^*) \\ h_0^{n+1} &= h_0^n - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{1,k}^n) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{1,k} - \frac{g \theta \Delta t}{\Delta x} (h_1^{n+1} - h_0^{n+1})) \\ (1+a) h_0^{n+1} - a h_1^{n+1} &= h_0^n - \frac{(1-\theta) \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{1,k}^n) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{1,k}) = R_0 \end{aligned}$$

$$\begin{aligned}
h_{N_i-1}^{n+1} &= h_{N_i-1}^n - \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{N_i,k}^n - u_{N_i-1,k}^n) - \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{N_i,k}^* - u_{N_i-1,k}^*) \\
h_{N_i-1}^{n+1} &= h_{N_i-1}^n + \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{N_i-1,k}^n) + \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{N_i-1,k} - \frac{g\theta \Delta t}{\Delta x} (h_{N_i-1}^{n+1} - h_{i-2}^{n+1})) \\
(1+a)h_{N_i-1}^{n+1} - ah_{N_i-2}^{n+1} &= h_{N_i-1}^n + \frac{(1-\theta)\Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (u_{N_i-1,k}^n) + \frac{\theta \Delta t \Delta z}{\Delta x} \sum_{k=1}^{N_k} (S_{N_i-1,k}) = R_{N_i-1}
\end{aligned}$$

To solve for h^{n+1} solve the following tridiagonal system:

$$\begin{pmatrix}
(1+a) & -a & & & \\
& \ddots & \ddots & & \\
& & -a & (1+2a) & -a \\
& & & \ddots & \ddots \\
& & & & -a & (1+a)
\end{pmatrix} \begin{pmatrix} h \\ \vdots \\ h \end{pmatrix}^{n+1} = \begin{pmatrix} R_0 \\ \vdots \\ R_j \\ \vdots \\ R_{N_i-1} \end{pmatrix}$$

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In (5) and (7) we discretely defined $\vec{u}^{n+1} = \vec{u}^* - \Delta t \vec{\nabla} q^c$. Taking the divergence and using $\vec{\nabla} \cdot \vec{u}^{n+1} = 0$:

$$\begin{aligned}
\vec{u}^{n+1} &= \vec{u}^* - \Delta t \vec{\nabla} q^c \\
\nabla^2 q^c &= \frac{1}{\Delta t} \vec{\nabla} \cdot \vec{u}^*
\end{aligned}$$

This can be written discretely as:

$$\frac{1}{\Delta x} \left(\frac{\partial q^c}{\partial x} \Big|_{i+1,k} - \frac{\partial q^c}{\partial x} \Big|_{i,k} \right) + \frac{1}{\Delta z} \left(\frac{\partial q^c}{\partial z} \Big|_{i,k+1} - \frac{\partial q^c}{\partial z} \Big|_{i,k} \right) = b_{i,k} \quad (12)$$

$$\frac{1}{\Delta x^2} (q_{i+1,k}^c - 2q_{i,k}^c + q_{i-1,k}^c) + \frac{1}{\Delta z^2} (q_{i,k+1}^c - 2q_{i,k}^c + q_{i,k-1}^c) = b_{i,k} \quad (13)$$

$$b_{i,k} = \frac{1}{\Delta t} \left(\frac{u_{i+1,k}^* - u_{i,k}^*}{\Delta x} + \frac{w_{i,k+1}^* - w_{i,k}^*}{\Delta z} \right)$$

Using the no-flux condition at the walls ($u_{0,k} = u_{N_i,k} = 0$ and $w_{i,0} = 0$) in (4):

$$\begin{aligned}
\frac{u_{i=\text{wall},k}^{n+1} - u_{i=\text{wall},k}^*}{\Delta t} &= -\frac{q_{i,k}^c - q_{i-1,k}^c}{\Delta x} \\
0 &= \frac{\partial q^c}{\partial x} \Big|_{\text{wall},k}
\end{aligned}$$

The result is analagous in (7). With this information and with the free surface condition $q_{i,N_k} = -q_{i,N_k-1}$ as in (9) we can rewrite (13) to work at the 8 different types of boundaries:

At $(0, k)$:

$$\frac{1}{\Delta x^2}(q_{1,k}^c - q_{0,k}^c) + \frac{1}{\Delta z^2}(q_{0,k+1}^c - 2q_{0,k}^c + q_{0,k-1}^c) = \frac{1}{\Delta t} \left(\frac{u_{1,k}^*}{\Delta x} + \frac{w_{0,k+1}^* - w_{0,k}^*}{\Delta z} \right)$$

At $(0, 0)$:

$$\frac{1}{\Delta x^2}(q_{1,0}^c - q_{0,0}^c) + \frac{1}{\Delta z^2}(q_{0,1}^c - q_{0,0}^c) = \frac{1}{\Delta t} \left(\frac{u_{1,0}^*}{\Delta x} + \frac{w_{0,1}^*}{\Delta z} \right)$$

At $(i, 0)$:

$$\frac{1}{\Delta x^2}(q_{i+1,0}^c - 2q_{i,0}^c + q_{i-1,0}^c) + \frac{1}{\Delta z^2}(q_{i,1}^c - q_{i,0}^c) = \frac{1}{\Delta t} \left(\frac{u_{i+1,0}^* - u_{i,0}^*}{\Delta x} + \frac{w_{i,1}^*}{\Delta z} \right)$$

At $(N_i - 1, 0)$:

$$\frac{1}{\Delta x^2}(-q_{N_i-1,0}^c + q_{N_i-2,0}^c) + \frac{1}{\Delta z^2}(q_{N_i-1,1}^c - q_{N_i-1,0}^c) = \frac{1}{\Delta t} \left(\frac{-u_{N_i-1,0}^*}{\Delta x} + \frac{w_{N_i-1,1}^*}{\Delta z} \right)$$

At $(N_i - 1, k)$:

$$\frac{1}{\Delta x^2}(-2q_{N_i-1,k}^c + q_{N_i-2,k}^c) + \frac{1}{\Delta z^2}(q_{N_i-1,k+1}^c - 2q_{N_i-1,k}^c + q_{N_i-1,k-1}^c) = \frac{1}{\Delta t} \left(\frac{-u_{N_i-1,k}^*}{\Delta x} + \frac{w_{N_i-1,k+1}^* - w_{N_i-1,k}^*}{\Delta z} \right)$$

At $(N_i - 1, N_k - 1)$:

$$\frac{1}{\Delta x^2}(-q_{N_i-1,N_k-1}^c + q_{N_i-2,N_k-1}^c) + \frac{1}{\Delta z^2}(-3q_{N_i-1,N_k-1}^c + q_{N_i-1,N_k-2}^c) = \frac{1}{\Delta t} \left(\frac{-u_{N_i-1,N_k-1}^*}{\Delta x} + \frac{w_{i,k+1}^* - w_{i,k}^*}{\Delta z} \right)$$

At $(i, N_k - 1)$:

$$\frac{1}{\Delta x^2}(q_{i+1,N_k-1}^c - 2q_{i,N_k-1}^c + q_{i-1,N_k-1}^c) + \frac{1}{\Delta z^2}(-3q_{i,N_k-1}^c + q_{i,N_k-2}^c) = \frac{1}{\Delta t} \left(\frac{u_{i+1,N_k-1}^* - u_{i,N_k-1}^*}{\Delta x} + \frac{w_{i,N_k}^* - w_{i,N_k-1}^*}{\Delta z} \right)$$

At $(0, N_k - 1)$:

$$\frac{1}{\Delta x^2}(q_{1,N_k-1}^c - q_{0,N_k-1}^c) + \frac{1}{\Delta z^2}(-3q_{0,N_k-1}^c + q_{0,N_k-2}^c) = \frac{1}{\Delta t} \left(\frac{u_{1,N_k-1}^*}{\Delta x} + \frac{w_{i,k+1}^* - w_{i,k}^*}{\Delta z} \right)$$

Using these boundary conditions and (13) a pentadiagonal matrix can be constructed and q^c can be computed using a conjugate gradient solver.

After solving the system of equations for q^c use (1) to calculate $q^{n+1/2}$. Then use (5) to update the u-velocity field u^{n+1} . To update the w-velocity field use the fact that the flow is divergence free:

$$\frac{w_{i,k+1}^{n+1} - w_{i,k}^{n+1}}{\Delta z} = -\frac{u_{i+1,k}^{n+1} - u_{i,k}^{n+1}}{\Delta x}$$

$$w_{i,k+1}^{n+1} = w_{i,k}^{n+1} - \frac{\Delta z}{\Delta x}(u_{i+1,k}^{n+1} - u_{i,k}^{n+1})$$

Starting with the bottom boundary condition $w_{i,0} = 0$ this system of equations can be solved by back substitution.

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Figure 1 shows the time evolution of the free surface comparing the theoretical

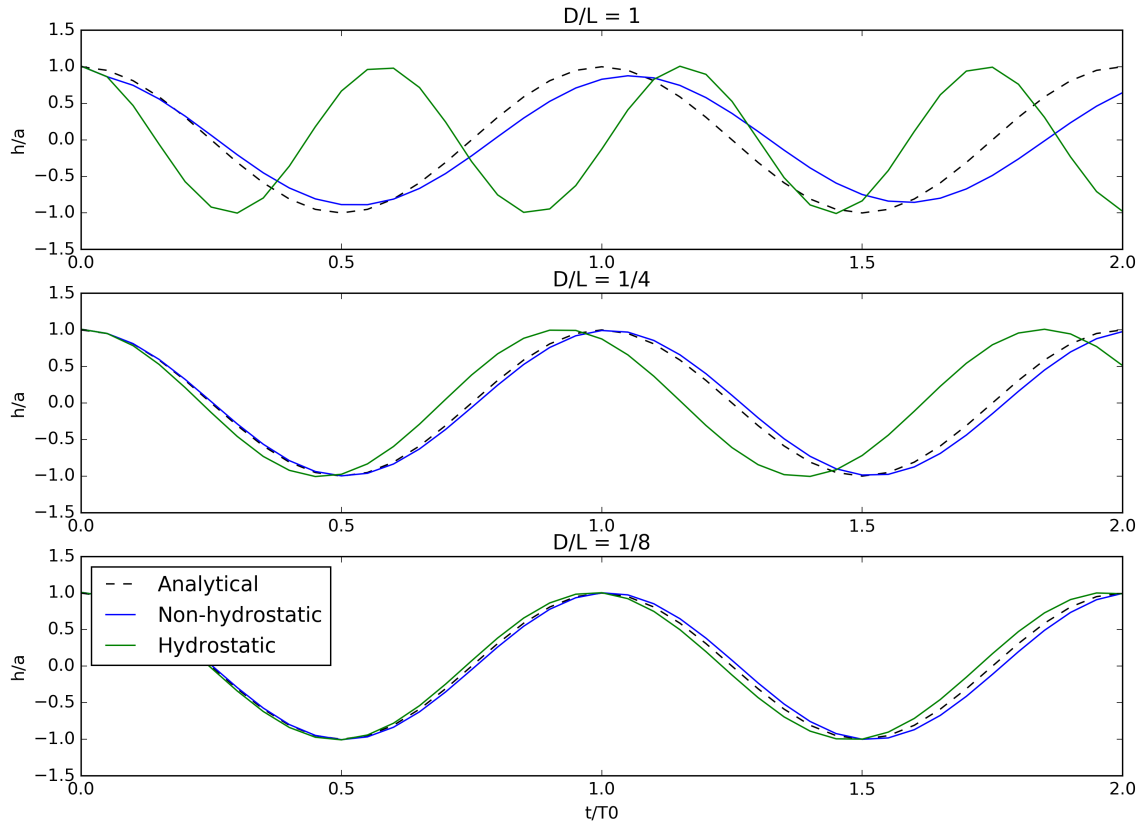


Figure 1: Timeseries of free surface height normalized by initial condition amplitude at $x/L = 0$. Comparison of theoretical, non-hydrostatic and hydrostatic results for $D/L \in \{1, 1/4, 1/8\}$.

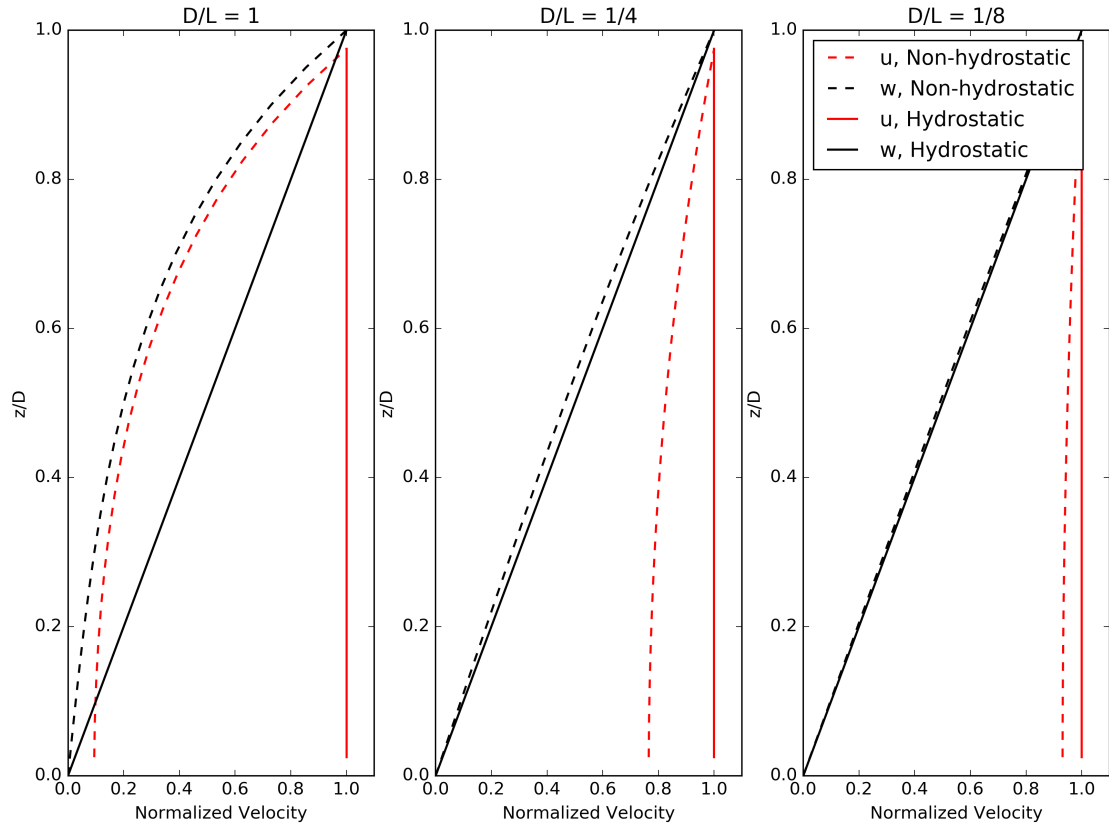


Figure 2: Vertical profiles of velocity for u and w normalized by their maximum value along the profile. Comparison of non-hydrostatic and hydrostatic velocity profiles for $D/L \in \{1, 1/4, 1/8\}$.

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With $k = 2\pi/\lambda$ and $\lambda = 2L$, the theoretical wave phase speed is:

$$c = \sqrt{\frac{g}{k} \tanh(kD)} \quad (14)$$

The deep-water limit of the dispersion relation is:

$$c_0 = \sqrt{\frac{g}{k}} \quad (15)$$

The shallow-water limit is:

$$c_s = \sqrt{gD} \quad (16)$$

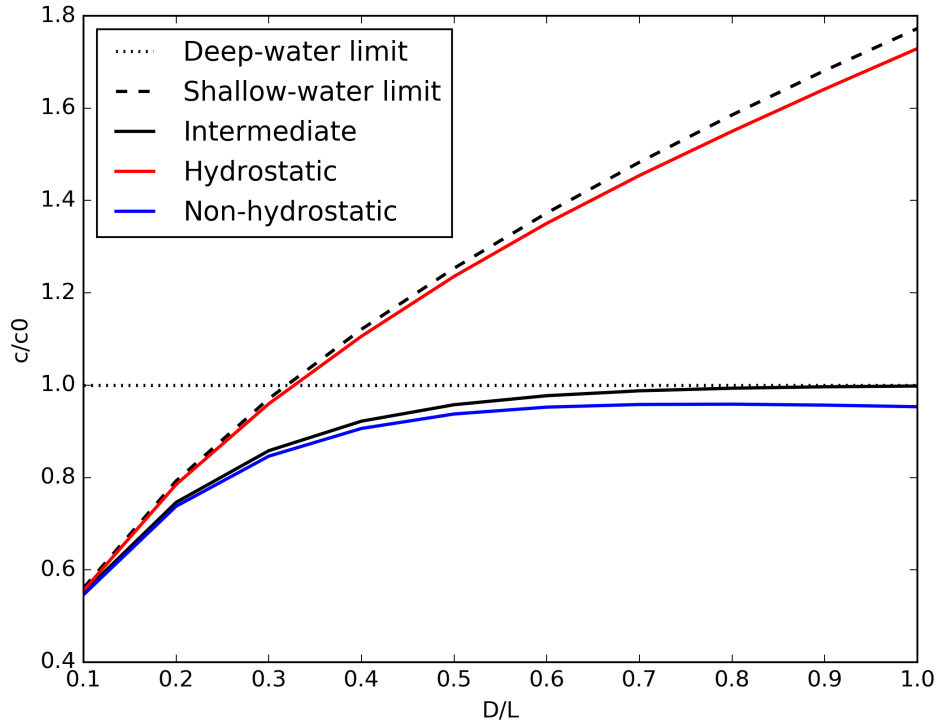


Figure 3: Theoretical and numerical dispersion relations for wave phase speed normalized by the deep-water limit. The theoretical intermediate value is given by (14). The theoretical deep-water limit is given by (15) and the theoretical shallow-water limit is given by (16).

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Figure 4 ($\Delta t = T/20$) and Figure 5 ($\Delta t = T/80$) compare the freesurface evolution of the projection and correction methods for the non-hydrostatic pressure field. The projection method has a lot of diffusion, for $\Delta t = T/20$ the diffusion takes the free surface fluctuations to essentially zero by about

5 wave periods. For $\Delta t = T/20$ the correction method also shows significant diffusion, this comes from the assumption in (10) that $u^{n+1} \simeq u^*$. The correction method also suffers from dispersion, which is expected.

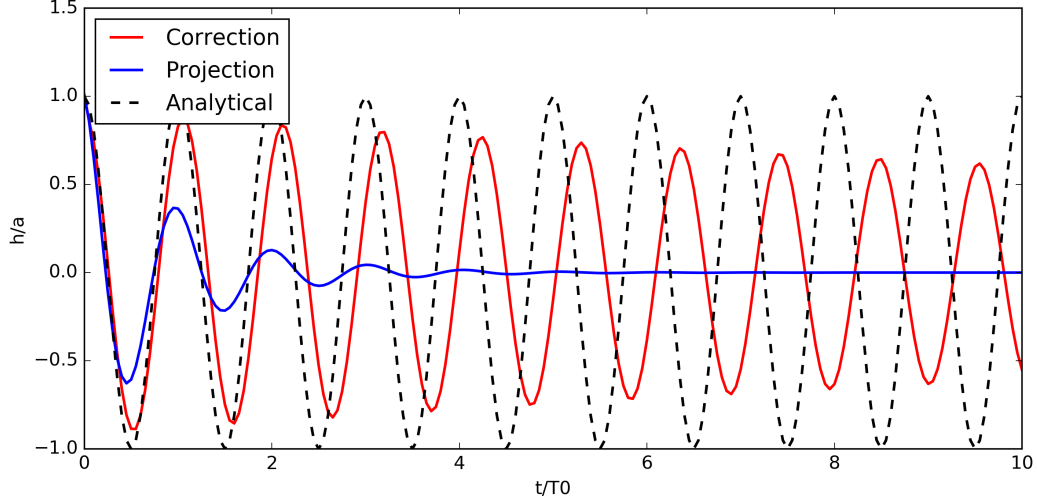


Figure 4: $\Delta t = T/20$. Timeseries of free surface height normalized by initial condition amplitude at $x/L = 0$. Comparison of analytical solution to projection and correction methods for non-hydrostatic pressure calculation.

Figures (6) and (7) show the accuracy of the correction and projection methods respectively. Error is computed as $(\|f(\Delta t) - f(\Delta t/2)\|_2)_{t=t_{\max}}$, the relative error. The

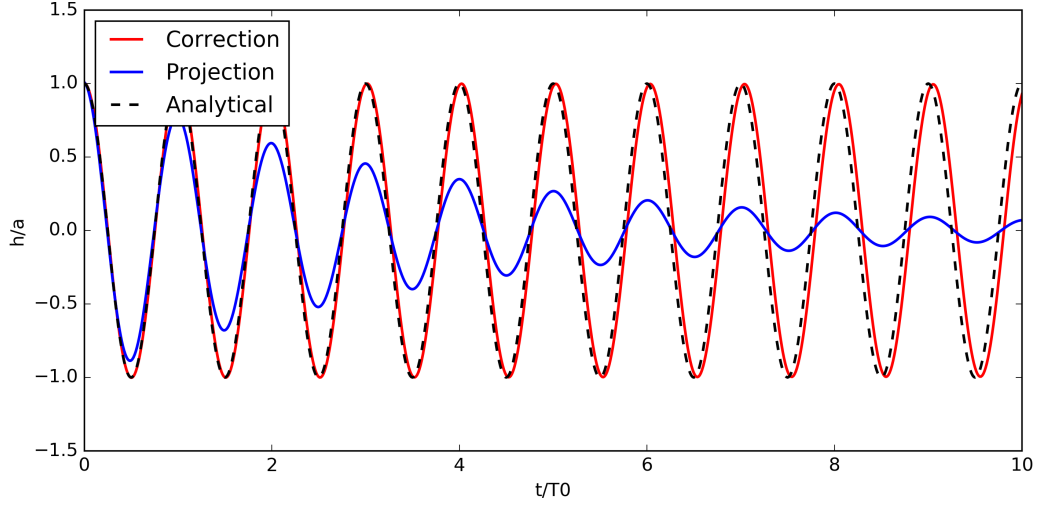


Figure 5: $\Delta t = T/80$. Timeseries of free surface height normalized by initial condition amplitude at $x/L = 0$. Comparison of analytical solution to projection and correction methods for non-hydrostatic pressure calculation.

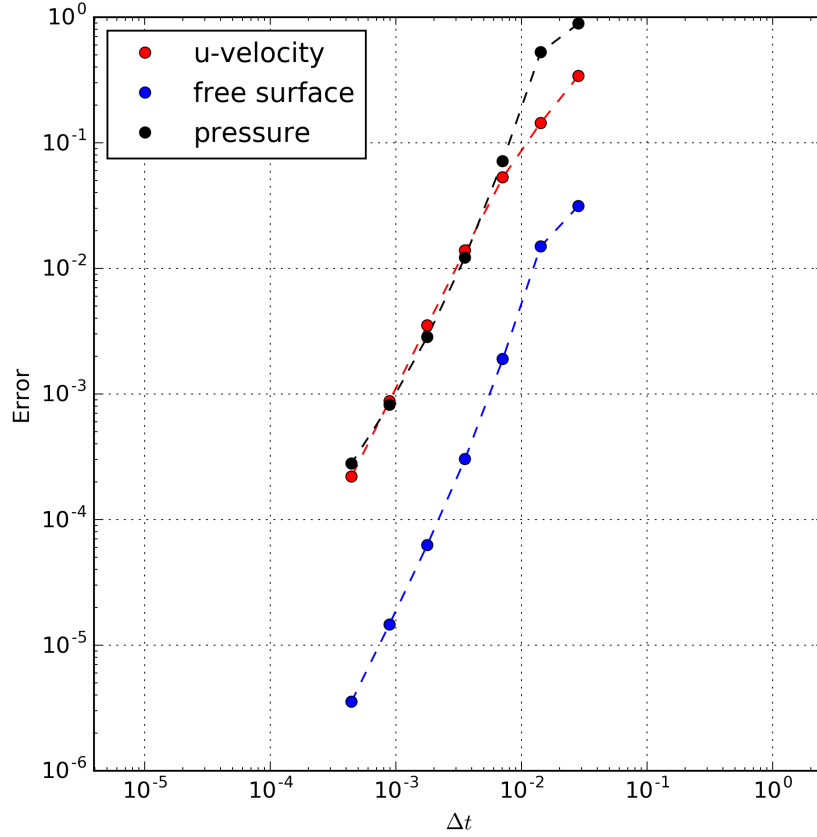


Figure 6: Accuracy of correction method for non-hydrostatic pressure calculation

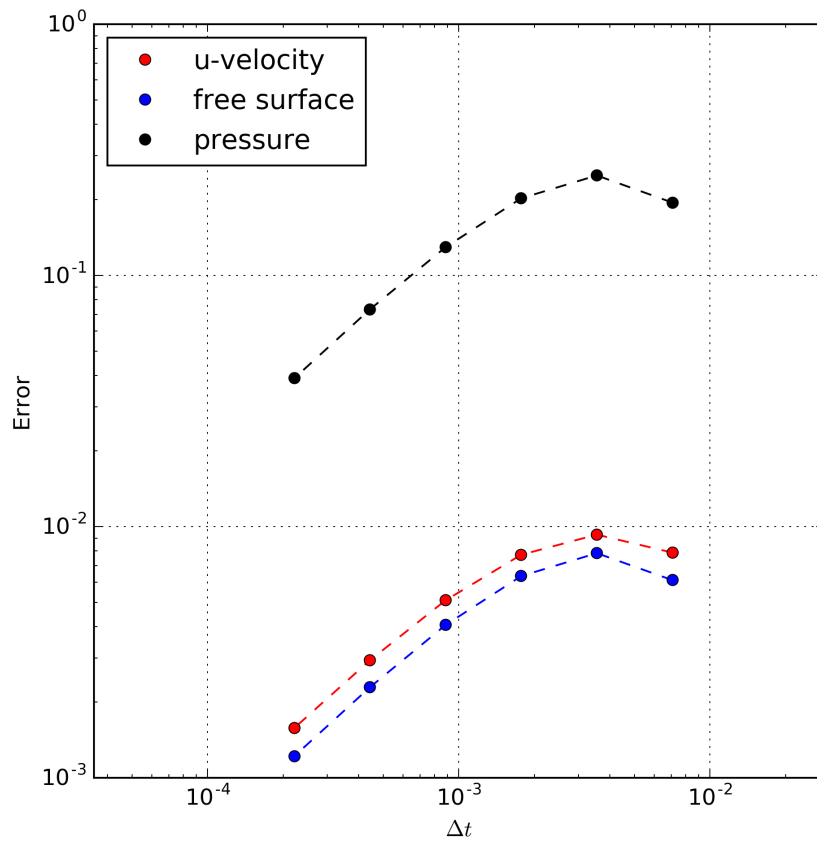


Figure 7: Accuracy of projection method for non-hydrostatic pressure calculation