CEE262c Assignment 5: 2-D (x-z) Linearized nonhydrostatic solver

Due: In class Wednesday, Mar 2, 2016

In this assignment you will implement a discretization of the linearized nonhydrostatic equations of motion and show that the discretization correctly captures the dispersive behavior of irrotational water waves, for which the wave phase speed is given by

$$c^2 = \frac{g}{k}\tanh(kD)\,, \tag{1}$$

where $k = 2\pi/\lambda$ is the wavenumber and D is the depth. In a closed domain of length L, a first-mode standing wave with a wavelength of $\lambda = 2L$ oscillates with a free-surface profile given by

$$h(x,t) = a\cos(kx)\cos(\omega t)\,,$$

where a is the wave amplitude and $\omega = ck$ is the wave frequency associated with the fundamental wavenumber $k = 2\pi/\lambda = \pi/L$. In the limit as $D \to \infty$, we recover the deep-water phase speed

$$c_0^2 = \frac{g}{k}\,, \tag{2}$$

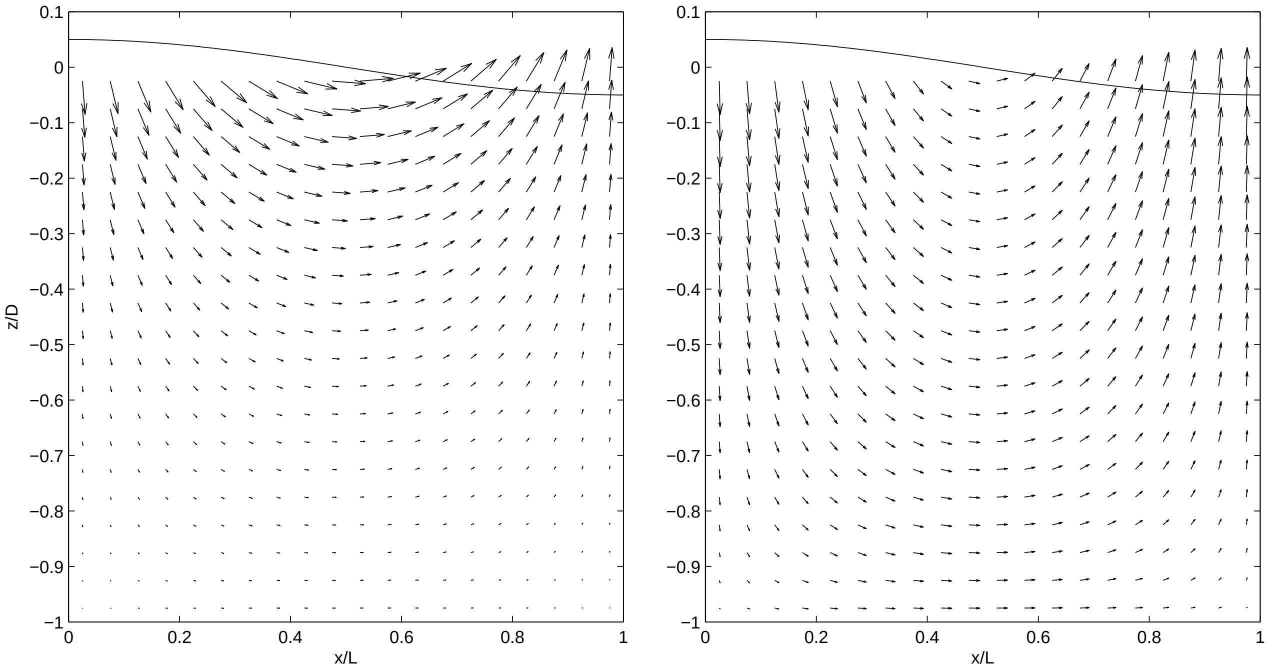
while in the limit of shallow water, we recover the shallow water phase speed

$$c_s^2 = gD\,. \tag{3}$$

Because the hydrostatic approximation assumes shallow-water behavior, the wave phase speed will always be given by equation (3), while inclusion of the nonhydrostatic pressure will correctly capture the dispersion relation (1). Hydrostatic models generally overpredict phase speeds and vertical velocities because of the lack of vertical inertia. The velocity fields in a free-surface seiche for a nonhydrostatic and hydrostatic calculation are depicted in Figure 1.

1. Beginning with the linearized nonhydrostatic equations of motion

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -g\frac{\partial h}{\partial x} - \frac{\partial q}{\partial x}\,,\\
\frac{\partial w}{\partial t} &= -\frac{\partial q}{\partial z}\,,\\
\frac{\partial h}{\partial t} &= -\frac{\partial}{\partial x}\int_{-D}^{0} u\,dz\,,
\end{aligned}$$



(a) (b)

Figure 1: Nonhydrostatic (a) and hydrostatic (b) velocity vectors associated with a free-surface seiche. The free-surface amplitude is exaggerated by a factor of 5.

discretize these equations on a staggered grid with $N_i \times N_k$ cells with $\Delta x = L/N_i$ and $\Delta z = D/N_k$. Use the theta method for the free-surface and the pressure correction method for the nonhydrostatic pressure and approximate the integral as

$$\int_{-D}^{0} u\,dz = \sum_{k=1}^{N_k} u_{i,k}\Delta z = \Delta z \sum_{k=1}^{N_k} u_{i,k}\,,$$

and, using $D = \sum_{k=1}^{N_k} \Delta z$, show that, if the predictor velocity field is obtained with

$$u^*_{i,k} = S_{i,k} - \frac{g\theta\Delta t}{\Delta x}\left(h_i^{n+1} - h_{i-1}^{n+1}\right)\,, \tag{4}$$

$$w^*_{i,k} = w^n_{i,k} - \frac{\Delta t}{\Delta z}\left(q_{i,k}^{n-1/2} - q_{i,k-1}^{n-1/2}\right)\,, \tag{5}$$

where the explicit term for momentum is given by

$$S_{i,k} = u^n_{i,k} - \frac{g(1-\theta)\Delta t}{\Delta x}\left(h_i^n - h_{i-1}^n\right) - \frac{\Delta t}{\Delta x}\left(q_{i,k}^{n-1/2} - q_{i-1,k}^{n-1/2}\right)\,, \tag{6}$$

then the tridiagonal system for the free surface is given by

$$-ah_{i-1}^{n+1} + (1+2a)h_i^{n+1} - ah_{i+1}^{n+1} = R_i\,, \tag{7}$$

where $a = \theta^2 C^2$, $C^2 = gD\Delta t^2/\Delta x^2$, and

$$R_i = h_i^n - \frac{(1-\theta)\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}\left(u^n_{i+1,k} - u^n_{i,k}\right) - \frac{\theta\Delta t\Delta z}{\Delta x}\sum_{k=1}^{N_k}\left(S_{i+1,k} - S_{i,k}\right)\,. \tag{8}$$

Use no-flux boundary conditions given by $u_{1,j} = u_{N_i+1,j} = 0$ to derive the free-surface equation for h_1^{n+1} and $h_{N_i}^{n+1}$.

2. Next, use the discrete continuity equation to derive the Poisson equation for the non-hydrostatic pressure-correction field that enforces continuity with

$$u_{i,k}^{n+1} = u_{i,k}^* - \frac{\Delta t}{\Delta x}\left(qc_{i,k} - qc_{i-1,k}\right) , \tag{9}$$

$$w_{i,k}^{n+1} = w_{i,k}^* - \frac{\Delta t}{\Delta z}\left(qc_{i,k} - qc_{i,k-1}\right) . \tag{10}$$

Derive the Poisson equation in the form

$$L(qc_{i,k}) = b_{i,k} , \tag{11}$$

where

$$b_{i,k} = \frac{1}{\Delta t}\left(\frac{u_{i+1,k}^* - u_{i,k}^*}{\Delta x} + \frac{w_{i,k+1}^* - w_{i,k}^*}{\Delta z}\right) , \tag{12}$$

and employ no-flux boundary conditions at the sidewalls and corners to derive the operator at the boundaries. At the free-surface, assume $qc = 0$ such that the ghost value above the free-surface is obtained with $qc_{i,N_k+1} = -qc_{i,N_k}$. Using this same condition on the old pressure, the predictor velocity at the free surface should be obtained with

$$w_{i,N_k+1}^* = w_{i,N_k+1}^n - \frac{\Delta t}{\Delta z}\left(q_{i,N_k+1}^{n-1/2} - q_{i,N_k}^{n-1/2}\right) , \tag{13}$$

$$= w_{i,N_k+1}^n + \frac{2\Delta t}{\Delta z} q_{i,N_k}^{n-1/2} . \tag{14}$$

3. With an initial free-surface profile given by

$$h(x, t = 0) = a\cos(\pi x/L) ,$$

where $a = 0.01$ m, compute the position of the free surface at $x = 0$ as a function of time in a square domain with 20×20 grid points and $L = D = 1$ m over a time period of $t_{max} = 2T$, where $T = 2\pi/\omega$ and $\omega = ck$, with c defined by the dispersion relation (1). Plot $h(x = 0)/a$ as a function of t/T using the nonhydrostatic and the hydrostatic solvers, and remember that you need to perform the Adams-Bashforth extrapolation

$$h_{x=0} = \frac{1}{2}\left(3h_1 - h_2\right) .$$

Employ a time step size of $\Delta t = T/20$ and let $\theta = 0.5$ and compare results for $D/L = 1$, $1/4$, and $1/8$, and compare to the analytical solution $h(x = 0, t) = a\cos(\omega t)$. Also compare the horizontal and vertical velocity profiles as a function of z/D and show that the nonhydrostatic profiles converge to the hydrostatic profiles as $D/L \to 0$. For these plots compare the horizontal velocity at the center and the vertical velocity at the right wall with

$$
\begin{aligned}
\widehat{u}_k &= \frac{|u_{c,k}|}{\max|u_{c,k}|}\,,\\
\widehat{w}_k &= \frac{|w_{r,k}|}{\max|w_{r,k}|}\,,
\end{aligned}
$$

where

$$
\begin{aligned}
u_{c,k} &= \frac{1}{2}\left(u_{N_i/2,k} + u_{N_i/2+1,k}\right)\,,\\
w_{r,k} &= \frac{1}{2}\left(3w_{N_i,k} - w_{N_i-1,k}\right)\,.
\end{aligned}
$$

The steps required to solve for the free-surface and velocity field at each time step are as follows

(a) Compute the explicit term $S_{i,k}$ from equation (6).

(b) Compute R_i, the right-hand side for the free surface, in equation (8).

(c) Invert the tridiagonal system given by equation (7) to obtain the new free surface, h_i^{n+1}. Note that since the diagonals do not change in time you can compute them before you start the time loop.

(d) Update the predictor velocity field with equations (4) and (5) using the old pressure. Only use these to update the interior cells, since the velocity on all three walls is zero and the free-surface velocity is updated in step (e) below.

(e) Update the predictor velocity at the free surface using equation (14).

(f) Compute the right-hand side of the pressure-Poisson equation using equation (12). Note that this will be modified for all cells near the walls but will still use w^*_{i,N_k+1} computed at the free-surface.

(g) Solve the pressure-Poisson equation (11). You can use the 2d conjugate gradient solver on coursework called `cg2.m`. Use a tolerance for the normalized residual of 10^{-10}.

(h) Correct the **horizontal** velocity with equation (9).

(i) Update the vertical velocity with the continuity equation. Starting at $k = 1$, $w_{i,1}^{n+1} = 0$, and we can integrate upwards in the water column to obtain $w_{i,k+1}^{n+1}$, $k = 1, \dots, N_k$ with

$$
w_{i,k+1}^{n+1} = w_{i,k}^{n+1} - \frac{\Delta z}{\Delta x}\left(u_{i+1,k}^{n+1} - u_{i,k}^{n+1}\right)\,.
$$

This will yield the vertical velocity at the free-surface, at the new time step, i.e. w_{i,N_k+1}^{n+1}.

(j) Update the new pressure with

$$q_{i,k}^{n+1/2} = q_{i,k}^{n-1/2} + qc_{i,k}\,.$$

Note that the hydrostatic calculation is identical except in place of step (g) we set $qc = 0$ and proceed with the rest of the calculation with $q = 0$.

4. Using 20×20 grid cells repeat part 3 but this time show that the phase speed computed by the hydrostatic code follows the shallow water phase speed (3) while that computed by the nonhydrostatic code follows relation (1). Run your simulation in hydrostatic and nonhydrostatic mode with $L = 1$ and `D=[0.1:0.1:1]`, and plot the resulting wave speeds for each case normalized by $c_0 = \sqrt{g/k}$ as a function of D/L, and overlay plots of the theoretical results. Figure 4 depicts the analytical dispersion relations, and you should overlay your results on this plot. For each case, run for a total of 40 time steps over the estimated analytical period and determine the time step at which the free-surface at $x = 0$ first changes sign, then use interpolation to find the time at which it changed sign with

$$t_c = t^n - \Delta t \frac{h^n}{h^{n+1} - h^n}\,,$$

where in this case n is the time step for which $h^n > 0$ and $h^{n+1} < 0$. This is one-quarter the wave period, $T = 4t_c$, from which the phase speed can be computed with $c = 2L/T$.

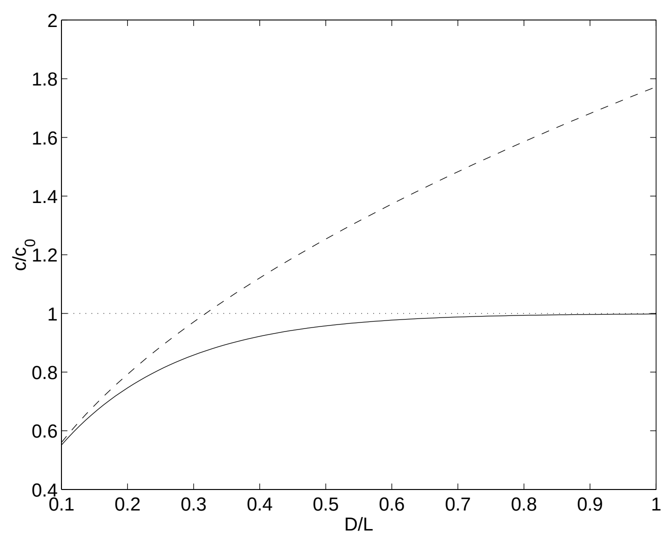


Figure 2: Dispersion relations for intermediate ($-$), shallow ($--$), and deep-water ($\cdots$) waves, normalized by $c_0 = \sqrt{g/k}$.

5. Employ the pressure projection method and compare the time-history of $h_{x=0}$ with the projection and correction methods over 10 wave periods. Use the same parameters as in part 3 and set $D = L = 1$ m. Next demonstrate and discuss the time-accuracy of the methods (for u,h, and q).