Data Driven Stability Analysis of Black-box Switched Linear Systems with Probabilistic Guarantees

***Abstract*— We explore the general problem of deciding stability of a 'black-box system', that is, a system whose model equations are not known. The only information at our possession is a set of observations, that is, couples of vectors of the type $(x(t), x(t+1))$. We adopt a probabilistic approach, and focus on switching systems, which are a widely used model for many complex systems, and are well known to be hard to analyze, even in a non-'black-box setting'.**

We show that, for a given (randomly generated) set of observations, one can give a stability guarantee on the system, for some level of confidence, with a trade-off between the quality of the guarantee and the level of confidence. We provide an explicit way of computing the best stability guarantee, as a function of the number of observations, and the required level of confidence. Our results rely on a geometrical analysis, combining chance-constrained optimization theory with stability analysis tools for switching systems.

I. Introduction

Today's complex cyber-physical systems are characterized by the interaction of a large number of heterogeneous components. Consequently, the models used to analyze these systems are equally complex and consist of heterogeneous sub-models relying on different assumptions and based on principles from different scientific disciplines. It is not uncommon to encounter a patchwork of differential equations, difference equations, hybrid automata, lookup tables, custom switching logic, low-level legacy code, etc. To further compound the difficulty in analyzing these systems, different components of a complex engineered system are typically designed by different suppliers. Although a high-level specification for these components may be known, detailed models are not available for intellectual property reasons. We are thus faced with a tremendous gap between the existing analysis techniques that rely on closed-form models and the models available in industry. It is, therefore, not surprising the emphasis that industry places on simulation since despite the complexity of models, it is always possible to simulate them. This raises the question of whether we can provide formal guarantees about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, we consider a dynamical system as in:

$$x_{k+1} = f(k, x_k), \tag{1}$$

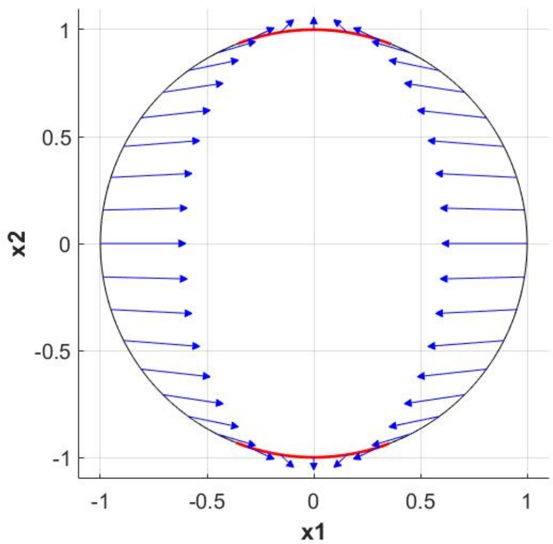
where, $x_k \in X$ is the state and $k \in \mathbb{N}$ is the time index. We start with the following question to serve as a stepping stone: Given N pairs, (x_1, y_1), (x_2, y_2), $\ldots$, (x_N, y_N) such that $y_k = f(k, x_k)$, what can we say about the stability of the system (1)? For the rest of the paper, we use the term *black-box* to refer to models where we do not have access to its dynamics, yet we can observe $f(x)$ by exciting it with different initial conditions x. We should say that this is non-ideal when we don't know what the state is, but it is a start, and it makes sense in certain situations. Paulo's comment: If we don't have a model, then we don't know what the state is. But the issue I raised is even deeper. Why can we assume that we can initialize the state randomly? Joris' comment: we do not necessarily assume that WE initialize the set randomly. We have random observations of the state, we do not know the process that picks the state, and we model this process with a random distribution. By default we take it uniform since we cannot say some states are privileged a priori. But a future step would be to consider different distributions and extend our guarantees to them. One approach to this problem is firstly identifying the dynamics, i.e., f and then applying the existing techniques in the model-based stability analysis literature. However, unless f is a linear function, there are two main reasons behind our quest to directly work on input-output pairs and bypassing the identification phase: (1) Even when the function f is known, in general, the stability analysis is a very difficult problem [**?**], [**?**]. (2) Paulo wants to change this: The existing identification techniques can only identify f up to an approximation error. How to relate this identification error to an error in the stability of the system (1) is still a nontrivial problem.

The initial idea behind this paper was influenced by the recent efforts in [8], [6] and [1] in using simulation traces to find Lyapunov functions for systems with known dynamics. Will put Liberzon and Sayan Mitra here. In these works, the main idea is that if one can construct a Lyapunov function candidate decreasing along several finite trajectories starting from different initial conditions, it should also decrease along every other trajectory. Then, once a Lyapunov function candidate is constructed, this intuition is put to test by verifying the candidate function either via off-the-shelf tools as in [8] and [6], or via sampling based techniques as in [1]. Note that, since we do not have access to the dynamics, the second step cannot be directly applied to black-box systems. However, these sampling based ideas raise the following question that we address in this paper: By observing that a candidate Lyapunov function decreases on a large number of simulations we empirically build a certain confidence that such candidate Lyapunov function is a bona-fide Lyapunov function. *Can we translate this confidence into a confidence in the stability of the underlying system?*

Note that, even in the case of a 2D linear system the connection between these two beliefs is nontrivial. In fact, one can easily construct an example with one stable and one unstable eigenvalue for which even though almost all trajectories diverge to the infinity, it is possible to construct a Lyapunov function candidate whose level sets are contracting everywhere except a small set. The system

$$\begin{bmatrix} -2 & 0 \\ 0 & 0.3 \end{bmatrix}$$

admits a Lyapunov function candidate on the unit circle except the two red areas.



Moreover, the size of this "violating set" can be arbitrarily small based on the magnitude of the unstable eigenvalue.

In this paper, we take the first step to close this gap. Since the identification and stability analysis of linear systems are well understood, we do so by focusing on switched linear systems. Note that identification and deciding the stability of arbitrary switched linear systems is NP-hard [5]. Aside from their theoretical value, switched systems model the behavior of dynamical systems in the presence of known or unknown varying parameters. These parameters can model internal properties of the dynamical system such as uncertainties, look-up tables, values in a discrete register as well as exogenous inputs provided by a controller in a closed-loop control system. Need to make these examples more specific.

The stability of switched systems is closely related to the *joint spectral radius* (JSR) of the matrices appearing in (3). Under certain conditions deciding stability amounts to deciding whether the JSR is less than one [5]. In this paper, we present an algorithm to bound the JSR of a switched linear system from N observations. This algorithm is based on tools from the random convex optimization literature [3], and provides an upper bound on the JSR with a user-defined confidence level. As N increases, this bound gets tighter. Moreover, with a closed form expression, we characterize what is the exact trade-off between the tightness of this bound and the number of samples. In order to understand the quality of our upper bound, the algorithm also provides a deterministic lower bound.

The organization of the paper is as follows: TO BE FILLED.

II. Preliminaries

A. Notation

We consider the usual finite normed vector space $(\mathbb{R}^n, \ell_2)$, $n \in \mathbb{N}_{>0}$, with ℓ_2 the classical Euclidean norm. We denote the set of linear functions in $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$, the set of orthogonal matrices of size n by $\mathcal{O}_n$, and the set of real symmetric matrices of size n by $\mathcal{S}^n$. In particular, the set of positive definite matrices, which are matrices $P \in \mathcal{S}^n$ such that $\forall\, x \in \mathbb{R}^n - \{0\}$, $x^T P x > 0$, is denoted by $\mathcal{S}^n_{++}$. We also use the usual notation $P \succ 0$ to say that P is positive definite. Given a set $X \subset \mathbb{R}^n$, and $r \in \mathbb{R}_{>0}$ we write $rX := \{x \in X : rx\}$ to denote the scaling of this set. We denote by B_r (respectively S_r) the ball (respectively sphere)of radius r centered at the origin. When $r = 1$, we omit the index for the sake of simplicity. We denote the ellipsoid described by the matrix $P \in \mathcal{S}^n_{++}$ as E_P, i.e., $E_P := \{x \in \mathbb{R}^n : x^T P x = 1\}$, and we denote by $\tilde{E}_P$ the volume in $\mathbb{R}^n$ defined by E_P: $\tilde{E}_P = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$. We denote the spherical projector on S by Π_{S}. We denote the homothety of ratio r by $\mathcal{H}_r$.

For an ellipsoid centered at the origin, and for any of its subsets $\mathcal{A}$, the sector defined by $\mathcal{A}$ is the subset

$$\{t\mathcal{A},\ t \in [0,1]\} \subset\ \mathbb{R}^n.$$

A sector induced by $\mathcal{A} \subset E_P$ will be denoted by $E_P^{\mathcal{A}}$. In the particular case of the unit sphere, we instead write $S^{\mathcal{A}}$.

We consider in this work the classical unsigned and finite uniform spherical measure on S, denoted by σ^{n-1}. It is associated to $\mathcal{B}_{\mathrm{S}}$, the spherical Borelian σ-algebra, and is derived from the Lebesgue measure λ. We have $\mathcal{B}_{\mathrm{S}}$ defined by $\mathcal{A} \in \mathcal{B}_{\mathrm{S}}$ if and only if $S^{\mathcal{A}} \in \mathcal{B}_{\mathbb{R}^n}$. The spherical measure σ^{n-1} is defined by

$$\forall\, \mathcal{A} \in \mathcal{B}_{\mathrm{S}},\, \sigma(\mathcal{A}) = \frac{\lambda(S^{\mathcal{A}})}{\lambda(B)}.$$

In other words, the spherical measure of a subset of the sphere is related to the Lebesgue measure of the sector of the unit ball it induces. Notice that $\sigma^{n-1}(\mathrm{S}) = 1$. Since $P \in \mathcal{S}^n_{++}$, it can be written in its Choleski form $P = UDU^{-1}$, where D is the diagonal matrix of its eigenvalues and $U \in O_n$. Let

$$L := UD^{1/2}U^{-1}. \tag{2}$$

Note that, L^{-1} maps the elements of S to E_P. Then, we define the measure on the ellipsoid σ_P on the σ-algebra $\mathcal{B}_{E_P} := L^{-1}\mathcal{B}_{\mathrm{S}}$, where $\forall\, \mathcal{A} \in \mathcal{B}_{E_P}$, $\sigma_P(\mathcal{A}) = \sigma^{n-1}(L\mathcal{A})$.

For $m \in \mathbb{N}_{>0}$, we denote by M the set $M = \{1, 2, \ldots, m\}$. Set M is provided with the classical σ-algebra associated to finite sets: $\Sigma_M = \wp(M)$, where $\wp(M)$ is the power set of M. We consider the uniform measure μ_M on (M, Σ_M).

We define $Z = \mathrm{S} \times M$ as the Cartesian product of the unit sphere and M. We denote the product σ-algebra $\mathcal{B}_{\mathrm{S}} \bigotimes \Sigma_M$

generated by $\mathcal{B}_{\mathrm{S}}$ and Σ_M: $\Sigma = \sigma(\pi_{\mathrm{S}}^{-1}(\mathcal{B}_{\mathrm{S}}), \pi_M^{-1}(\Sigma_M))$. On this set, we define the product measure $\mu = \sigma^{n-1} \otimes \mu_M$. We have μ uniform and $\mu(Z) = 1$.

B. Stability of Linear Switched Systems

A switched linear system related to a set of modes $\mathcal{M} = \{A_i, i \in M\}$ is of the form:

$$x_{k+1} = A_{\tau(k)} x_k, \tag{3}$$

with switching sequence $\tau : \mathbb{N} \to M$. There are two important properties of linear switched systems that we exploit in this paper.

Property 2.1: Let $\xi(x, k, \tau)$ denote the state of the system (3) at time k starting from the initial condition x and with switching sequence τ. The dynamical system (3) is homogeneous:

$$\xi(\gamma x, k, \tau) = \gamma \xi(x, k, \tau).$$

Property 2.2: The dynamics given in (3) is convexity-preserving, meaning that for any set of points $X \subset \mathbb{R}^n$ we have:

$$f(\text{convhull } X) \subset \text{convhull } \{f(X)\}.$$

We now introduce the *Lyapunov exponent* of the system, which is a numerical quantity describing its stability.

Definition Given a dynamical system as in (1), its *Lyapunov exponent* is given by

$$\rho = \inf \{r : \forall x_0, \exists C \in \mathbb{R}^+ : \xi(x_0, k) \leq C r^k\}.$$

In the case of switched linear systems, the Lyapunov exponent is known as the joint spectral radius of the set of matrices, which can be alternatively defined as follows:

Definition [4] Given a set of matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}$, its *joint spectral radius* (JSR) is given by

$$\rho(\mathcal{M}) = \lim_{k \to \infty} \max_{i_1, \ldots, i_k} \{||A_{i_1} \ldots A_{i_k}||^{1/k} : A_{i_j} \in \mathcal{M}\}.$$

Property 2.3 (Corollary 1.1, [5]): For any bounded set of matrices $\mathcal{M}$, the corresponding switched dynamical system is stable if and only if $\rho(\mathcal{M}) < 1$.

Property 2.4 (Proposition 1.3, [5]): For any bounded set of matrices $\mathcal{M}$, and any invertible matrix T,

$$\rho(\mathcal{M}) = \rho(T \mathcal{M} T^{-1}).$$

Note that this last result implies that the JSR is invariant under similarity transformations (and is a fortiori a homogeneous function: $\rho\left(\frac{\mathcal{M}}{\gamma}\right) = \frac{\rho(\mathcal{M})}{\gamma}, \forall \gamma > 0$).

III. A Deterministic Lower Bound for JSR

We start by computing a lower bound for ρ which is based on the following theorem from the switched linear systems literature.

Theorem 3.1: [4, Theorem 2.11] For any bounded set of matrices such that $\rho(\mathcal{M}) < \frac{1}{\sqrt{n}}$, there exists a Common Quadratic Lyapunov Function (CQLF) for $\mathcal{M}$, that is, a $P \succ 0$ such that:

$$\forall A \in \mathcal{M},\, A^T P A \preceq P.$$

The following theorem shows that the existence of a CQLF for (3) can be checked by considering N pairs $(x_i, j_i) \in \mathbb{R}^n \times M$, where $i \in \{1, \ldots N\}$.

Theorem 3.2: For a given sampling:

$$\omega_N := \{(x_1, j_1), (x_2, j_2), \ldots, (x_N, j_N)\},$$

let $\gamma^*(\omega_N)$ be the optimal solution of the following optimization problem:

$$\begin{array}{ll} \min & \gamma \\ \text{s.t.} & (y_i)^T P (y_i) \leq \gamma^2 x_i^T P x_i,\, \forall\, i : 1 \leq i \leq N \\ & P \succ 0 \end{array} \tag{4}$$

where y_i is the value obtained by applying the unkown mode of index j_i on the point x_i. If $\gamma^*(\omega_N) < \infty$, we have:

$$\rho(\mathcal{M}) \geq \frac{\gamma^*(\omega_N)}{\sqrt{n}}.$$

Note that, (4) can be solved by bisection on γ.

Proof: Using Remark [property ???], for any $\epsilon > 0$, $\frac{\mathcal{M}}{(\gamma^*(\omega_N) - \epsilon)}$ has no CQLF. Then, applying Theorem 3.1 we get

$$\frac{\rho(\mathcal{M})}{\gamma^*(\omega_N)} \geq \frac{1}{\sqrt{n}}.$$

■

IV. A Probabilistic Stability Guarantee

In this section, we show how to compute an upper bound on ρ, with a user-defined confidence $\beta \in [0, 1)$. We do this by constructing a CQLF which is valid with probability at least β. Note that, the existence of a CQLF on a implies $\rho < 1$ due to Theorem **??** and due to Property 2.1, it is enough to show that the CQLF is decreasing on a set enclosing the origin, e.g. S. Therefore, to obtain an upper bound on ρ, we consider the following optimization problem:

$$\begin{array}{ll} \min_{\gamma, P} & \gamma \\ \text{s.t.} & (Ax)^T P (Ax) \leq \gamma^2 x^T P x,\, \forall A \in \mathcal{M},\, \forall\, x \in \mathrm{S}, \\ & P \succ 0. \end{array} \tag{5}$$

Note that, for all $A \in \mathcal{M}$ the optimal P and the optimal γ for the problem (5) satisfies: $\frac{A}{\gamma}^T P \frac{A}{\gamma} \preceq P$. Therefore, $\rho\left(\frac{\mathcal{M}}{\gamma}\right) \leq 1$, which leads to the upper bound $\rho\left(\mathcal{M}\right) \leq \gamma$. However, solving the optimization problem (5) is hard since it involves infinitely many constraints. Therefore, we instead sample N initial states and modes uniformly random from the set $Z := \mathrm{S} \times M$, and solve the following optimization problem with finitely many constraints instead:

$$\begin{array}{ll} \min_{\gamma, P} & \gamma \\ \text{s.t.} & (Ax)^T P (Ax) \leq \gamma^2 x^T P x,\, \forall (x, j) \in \omega_N, \\ & P \succ 0. \end{array} \tag{6}$$

where ω_N is a N-uniform random sampling of the set $Z := \mathrm{S} \times M$. Note that, since (6) is convex for a fixed γ, we can perform bisection on γ and solve a series of feasibility problems in P instead. Therefore, we now analyze the relationship between the solutions of the optimization problem (5) and the following optimization problem:

$$\begin{array}{ll} \min_P & \lambda_{\max}(P) \\ \text{s.t.} & (A_j x)^T P (A_j x) \leq ((1+\eta)\gamma^*)^2 x^T P x, \, \forall (x,j) \in \omega_N, \\ & P \succeq I. \end{array} \tag{7}$$

where ω_N is a N-uniform random sampling of the set Z, $\eta > 0$, and γ^* is the optimal solution to the optimization problem (6). For the rest of the discussion, we refer to the optimization problem (7) by $\text{Opt}(\omega_N)$. We denote its optimal solution by $P(\omega_N)$ and $\gamma^*(\omega_N)$. We drop the explicit dependence of P on ω_N when it is clear from the context. There are a few points that are worth noting about (7). Firstly, due to Property 2.1, we are able to replace the constraint $P \succ 0$ with the constraint $P \succeq I$. Moreover, for reasons that will become clear later in the discussion, we chose the objective function as $\lambda_{\max}(P)$, instead of solving a feasibility problem in P. Lastly, the additional η factor is introduced to ensure strict feasibility of (7), which will be helpful in the preceding discussion.

The curious question whether the optimal solution of the sampled problem $\text{Opt}(\omega_N)$ is a feasible solution to (5) has been widely studied in the random convex optimization literature [3]. It turns out that under certain technical assumptions, the optimal solution of (7) is feasible for the original problem (5), with some probability which is a function of the sample size N. To formalize this discussion, we define the constraint violation probability next.

Definition (from [3]) For all ω_N for which a solution to $\text{Opt}(\omega_N)$ exists, the *constraint violation probability* is defined as:

$$\mathcal{V}^*(\omega_N) = \mathbb{P}\{z \in Z : f(P(\omega_N), z) > 0\}. \tag{8}$$

Note that, since we have $\mathbb{P}(\mathcal{A}) = \frac{\mu(\mathcal{A})}{\mu(Z)}$, we can rewrite (8) as:

$$\mathcal{V}^*(\omega_N) = \frac{\mu(V(\omega_N))}{\mu(Z)},$$

where $V(\omega_N) := \{z \in Z : f(P(\omega_N), z) > 0\}$, i.e., the set of points for which at least one constraint is violated for the sampling ω_N.

Theorem 4.1: Let $d := \frac{n(n+1)}{2} + 1$ and $N \geq d+1$. Consider the optimization problem $\text{Opt}(\omega_N)$ given in (7), where ω_N is a uniform random sampling of the set Z. If $\text{Opt}(\omega_N)$ satisfies the following technical assumptions:

1) When the problem $\text{Opt}(\omega_N)$ admits an optimal solution, this solution is unique.
2) Problem $\text{Opt}(\omega_N)$ is nondegenerate[1] with probability one.

Then, for all $\epsilon \in (0,1)$ the following holds:

$$\mathbb{P}^N \left\{ \mu(V(\omega_N)) \leq \epsilon \right\} \geq 1 - \sum_{j=0}^{d} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j}. \tag{9}$$

[1]Explain this in a footnote maybe?

Proof: The proof is an immediate application of Theorem ?? in [3], since the $\text{Opt}(\omega_N)$ can be written as:

$$\begin{array}{ll} \min_{P,t} & t \\ \text{s.t.} & f_{\gamma^*}(P,z) \leq 0, \, \forall \, z \in Z \end{array} \tag{10}$$

where $g_{\gamma^*}(P,z) = \max(g_1(P,z), g_2(P), g_3(P))$, and

$$\begin{aligned} g_1(P,z) &:= (A_j z)^T P (A_j z) - {\gamma^*}^2 z^T P z \\ g_2(P) &:= \lambda_{\max}(-P) + 1. \\ g_3(P) &:= \lambda_{\max}(P) - t. \end{aligned}$$

We first note that, both of the assumptions in the statement of the theorem are technical and even if they do not hold for the optimization problem 10, it is possible to obtain . We refer the interested reader to [3] for a more detailed discussion of such techniques. The objective function of (10) is linear while each constraint is convex in P for all $z \in Z$. Also note that, the set of decision variables are in $\mathbb{R}^{\frac{n(n+1)}{2}+1}$. Then, we can invoke Theorem ?? in [3] with the optimization problem (10) to conclude the statement of the theorem. ■

Theorem 4.1 states that the optimal solution of the sampled problem $\text{Opt}(\omega_N)$ violates an ϵ fraction of the constraints in the original optimization problem (5) with probability β, where β goes to 1 as N goes to infinity.

Before proceeding to the next proof, we present the following lemma which will be helpful.

Remark 4.1: (see e.g. [**?**]) Let $X \subset \mathbb{R}^n$ and $L \in \mathcal{L}(\mathbb{R}^n)$. Then we have:

$$\lambda(\mathcal{L}(X)) = |\det(L)| \lambda(X).$$

Theorem 4.2: Let $\gamma \in \mathbb{R}_{>0}$. Consider a set of matrices $A \in \mathcal{M}$, and the matrix P, satisfying:

$$(A_j x)^T P (A_j x) \leq \gamma^2 x^T P x, \; \forall (x,j) \in Z \setminus V, \tag{11}$$

for some $V \subset Z$ where $\mu(V) \leq \epsilon$. Then, the following also holds:

$$(\bar{A}_j x)^T (\bar{A}_j x) \leq \gamma^2 x^T x, \; \forall x \in \mathrm{S} \setminus \mathrm{S}', \forall j \in M,$$

for some $\mathrm{S}' \subset \mathrm{S}$ where $\bar{A}_j = L^{-1} A_j L$, where L is defined as in (2) and

$$\sigma(\mathrm{S}') \leq m\epsilon \sqrt{\frac{\lambda_{\max}(P)}{\det(P)^n}}.$$

Proof: Note that $V \subset \Sigma$. Let $V_{\mathrm{S}} = \pi_{\mathrm{S}}(V)$ and $V_M = \pi_M(V)$. We notice that Σ_M is the disjoint union of its 2^m elements $\{\mathcal{M}_i, i \in \{1, 2, \ldots 2^m\}\}$. Then V can be written as the disjoint union $V = \sqcup_{1 \leq i \leq 2^m} (\mathcal{S}_i, \mathcal{M}_i)$ where $\mathcal{S}_i \in \Sigma(\mathrm{S})$. We notice that $V_{\mathrm{S}} = \sqcup_{1 \leq i \leq 2^m} \mathcal{S}_i$, and

$$\sigma^{n-1}(V_{\mathrm{S}}) = \sum_{1 \leq i \leq 2^m} \sigma^{n-1}(\mathcal{S}_i).$$

We have

$$\begin{aligned} \mu(V) &= \mu(\sqcup_{1\le i\le 2^m}(\mathcal{S}_i,\mathcal{M}_i)) \\ &= \sum_{1\le i\le 2^m} \mu(\mathcal{S}_i,\mathcal{M}_i) \\ &= \sum_{1\le i\le 2^m} \sigma^{n-1}\otimes\mu_M(\mathcal{S}_i,\mathcal{M}_i) \\ &= \sum_{1\le i\le 2^m} \sigma^{n-1}(\mathcal{S}_i)\mu_M(\mathcal{M}_i). \end{aligned}$$

Note that we have

$$\min_{j\in M}\mu_M(\{j\}) = \frac{1}{m}.$$

Then since $\forall\ i,\ \mu_M(\mathcal{M}_i)\ge \frac{1}{m}$, we get:

$$\sigma^{n-1}(V_{\mathrm{S}}) \le \frac{\mu(V)}{\frac{1}{m}} \le m\epsilon. \tag{12}$$

This means that

$$(A_jx)^TP(A_jx)\le\gamma^2x^TPx,\ \forall\, x\in \mathrm{S}\setminus V_S,\ \forall\, m\in M, \tag{13}$$

where $\sigma^{n-1}(V_S)\le m\epsilon$.

We then perform the change of coordinates defined by $L^{-1}\in\mathcal{L}(\mathbb{R}^n)$ which maps S to P defined as in (2). We can then rewrite (13) in this new coordinates system as in:

$$(\bar{A}_jx)^T(\bar{A}_jx)\le\gamma^2x^Tx,\ \forall\, x\in E_P\setminus L^{-1}(V_S),\ \forall\, m\in M. \tag{14}$$

Due to the the homogeneity Property 2.1, this implies

$$(\bar{A}_jx)^T(\bar{A}_jx)\le\gamma^2x^Tx,\ \forall\, x\in \mathrm{S}\backslash\Pi_S(L^{-1}(V_S)),\ \forall\, m\in M. \tag{15}$$

We now show how to relate $\sigma^{n-1}(V_S)$ to $\sigma^{n-1}(\Pi_S(L^{-1}(V_S)))$. Consider $S^{V_{\mathrm{S}}}$ the sector of B defined by V_{S}. We denote $C:=L^{-1}(S^{V_{\mathrm{S}}})$ and $V':=\Pi_S(L^{-1}(V_S))$. We have $\Pi_{\mathrm{S}}(C)=V'$ and $S^{V'}\subset\mathcal{H}_{1/\lambda_{\min}(L^{-1})}(C)$. This leads to:

$$\sigma^{n-1}(V')=\lambda(S^{V'})\le\lambda(\mathcal{H}_{1/\lambda_{\min}(L^{-1})}(C)),$$

which means the following holds:

$$\begin{aligned} \sigma^{n-1}(V') &\le \frac{1}{\lambda_{\min}(L^{-1})^n}\lambda(L^{-1}(S^{V_{\mathrm{S}}})) \\ &= \frac{|\det(L^{-1})|}{\lambda_{\min}(L^{-1})^n}\sigma(S^{V_S}), \\ &= \sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}\sigma^{n-1}(V_S) \end{aligned} \tag{16}$$

where the first equality follows from Remark 4.1. Putting together (16), (13), , we get the statement of the theorem where $S'=\Pi_S(L^{-1}(V_S))$. ∎

Lemma 4.3: Let $\epsilon\in(0,1)$. Then, we can compute $\alpha(\epsilon)$ satisfying:

$$\alpha(\epsilon)=\sup\{r: r\mathrm{B}\subset \text{convhull }(\mathrm{S}\setminus X_\epsilon), \tag{17}$$

for all $X_\epsilon\in\{X\subset \mathrm{S}:\sigma^{n-1}(X)\le\epsilon\}$.

Proof: See Appendix. ∎

Lemma 4.4: Let $\epsilon\in(0,1)$ and $\gamma\in\mathbb{R}_{>0}$. Consider the set of matrices and $A\in\mathcal{M}$ satisfying:

$$(A_jx)^T(A_jx)\le\gamma x^Tx,\quad \forall\, x\in \mathrm{S}\setminus\mathrm{S}',\forall\, j\in,M, \tag{18}$$

where $\mathrm{S}'\subset\mathrm{S}$ and $\sigma^{n-1}(\mathrm{S}')\le\epsilon$, then we have:

$$\rho(\mathcal{M})\le\frac{\gamma}{\alpha(\epsilon)}$$

where $\alpha(\epsilon)$ is defined as in (17).

Proof: Note that, (18) implies that:

$$A_j(\mathrm{S}\setminus\mathrm{S}')\subset\gamma\mathrm{B}.$$

Using Property 2.2 this also implies:

$$A_j\text{convhull }(\mathrm{S}\setminus\mathrm{S}')\subset\text{convhull }(A_j(\mathrm{S}\setminus\mathrm{S}'))\subset\gamma^*\mathrm{B}.$$

By Lemma 4.3 we have:

$$A_j(\alpha(\epsilon)\mathrm{B})\subset A_j(\text{convhull }(\mathrm{S}\setminus\mathrm{S}'))\subset\mathrm{B},\quad\forall\, j\in M,$$

where $\alpha(\epsilon)$ is defined as in (27). Therefore, we get:

$$\alpha(\epsilon)A_j(\mathrm{B})\subset\gamma\mathrm{B}.$$

which implies that $\rho(\mathcal{M})\le\frac{\gamma}{\alpha(\epsilon)}$. ∎

Theorem 4.5: Consider an n-dimensional switching system as in (3). For any given $\beta\in(0,1]$, $\eta>0$ and a uniform random sampling $\omega_N\subset Z$, with $N\ge\frac{n(n+1)}{2}+1$, and let $\gamma^*(\omega_N)$ be the optimal solution to (7). Then, we can compute $\delta(\beta,\omega_N)$, such that with probability at least β we have:

$$\rho\le\frac{\gamma^*(\omega_N)(1+\eta)}{\delta(\beta,\omega_N)},$$

where $\lim_{N\to\infty}\delta(\beta,\omega_N)=1$.

Proof: Note that, by definition of $\gamma^*(\omega_N)$ we have:

$$(A_jx)^TP(A_jx)\le(\gamma^*(1+\eta))^2x^TPx,\quad\forall\,(x,j)\in\omega_N$$

for some $P\succ 0$. Note that the inequality (9) in Theorem 4.1 can be also written as:

$$\mathbb{P}^N\{\mu(V(\omega_N))\le\epsilon\}\ge 1-I(1-\epsilon;N-d,d+1), \tag{19}$$

where $I(\ell;a,b)$ is the regularized in complete beta function. Then, for all $\epsilon\in(0,1)$ satisfying:

$$\epsilon\le 1-I^{-1}(1-\beta;N-d,d+1), \tag{20}$$

we have $\mathbb{P}^N\{\mu(V(\omega_N))\le\epsilon\}\ge\beta$. Then, by Theorem 4.1 for all ϵ satisfying (20), with probability at least β the following holds:

$$(A_jx)^TP(A_jx)\le(\gamma^*(1+\eta))^2x^TPx,\quad\forall(x,j)\in Z\setminus V.$$

By Theorem 4.2, this implies that with probability at least β the following also holds:

$$(\bar{A}_jx)^T(\bar{A}_jx)\le\gamma^2x^Tx,\ \forall\, x\in\mathrm{S}\setminus\mathrm{S}',\forall\, j\in M,$$

for some S' where $\sigma^{n-1}(S')\le m\epsilon\sqrt{\frac{\lambda_{\max}(P)}{\det(P)^n}}$. Then, applying Lemma 4.4, we can compute

$$\delta(\beta,\omega_N)=\bar{\alpha}(m(1-I^{-1}(1-\beta;N-d,d+1)))$$

such that with probability at least β we have:

$$\bar{A}_j \mathrm{B} \subset \frac{\gamma^*(\omega_N)(1+\eta)}{\delta(\beta,\omega_N)}\mathrm{B},\ \forall j \in M,$$

By Property 2.4, this means that with probability at least β:

$$\rho \leq \frac{\gamma^*(\omega_N)(1+\eta)}{\delta(\beta,\omega_N)},$$

which completes the proof of the first part of the theorem. We show that $\lim_{N\to\infty}\delta(\beta,\omega_N) = 1$ in the Appendix D. ■

V. Experimental Results

VI. Conclusions

Appendix

A. Notation and Background

Before proceeding to the main lemmas we use to prove Lemma 4.3, we first introduce the necessary preliminary definitions and related background.

Let d be a distance on $\mathbb{R}^n$. The distance between a set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ is $d(X,p) := \inf_{x\in X} d(x,p)$. Note that the map $p \mapsto d(X,p)$ is continuous on $\mathbb{R}^n$.

Definition We define the *spherical cap* on S for a given hyperplane $c^T x = k$ as:

$$\mathcal{C}_{c,k} := \{x \in \mathrm{S} : c^T x > k\}.$$

Remark 1.1: Consider the spherical caps $\mathcal{C}_{c,k_1}$ and $\mathcal{C}_{c,k_2}$ such that $k_1 > k_2$, then we have:

$$\sigma^{n-1}(\mathcal{C}_{c,k_1}) < \sigma^{n-1}(\mathcal{C}_{c,k_2}).$$

Definition A *supporting hyperplane* of a set $X \subset \mathbb{R}^n$ is a hyperplane $\{x : c^T x = k\}$ that has the following two properties:

- $X \subset \{x : c^T x \leq k\}$ or $X \subset \{x : c^T x \geq k\}$.
- $X \cap \{x : c^T x = k\} \neq \emptyset$.

Remark 1.2: [2] Consider a convex set $X \subset \mathbb{R}^n$. For every $x \in \partial X$, there exists a supporting hyperplane containing x. Moreover, if X is smooth, then this supporting hyperplane is unique.

Remark 1.3: The distance between the point $x = 0$ and the hyperplane $c^T x = k$ is $\frac{|k|}{\|c\|}$.

We now define the function $\Delta : \wp(\mathrm{S}) \to [0,1]$ as:

$$\Delta(X) := \sup\{r : r\mathrm{B} \subset \text{convhull } (\mathrm{S} \setminus X)\}. \tag{21}$$

Note that, $\Delta(X)$ can be rewritten as:

$$\Delta(X) = d(\partial \text{convhull } (\mathrm{S}\setminus X), 0). \tag{22}$$

Lemma 1.1: Consider the spherical cap $\mathcal{C}_{c,k}$. We have:

$$\Delta(\mathcal{C}_{c,k}) = \min\left(1, \frac{|k|}{\|c\|}\right).$$

Proof: Note that:

$$\text{convhull } (\mathrm{S}\setminus X) = \left\{x \in \mathrm{B} : c^T x \leq k\right\}.$$

Then the following equalities hold:

$$\begin{aligned}\Delta(X) &= d(\partial\text{convhull } (\mathrm{S}\setminus X), 0)\\ &= \min(d(\partial \mathrm{B}, 0), d(\partial\{x : c^T x \leq k\}, 0))\\ &= \min(d(\mathrm{S}, 0), d(\{x : c^T x = k\}, 0))\\ &= \min\left(1, \frac{|k|}{\|c\|}\right).\end{aligned}$$

■

Corollary 1.2: Consider the spherical caps $\mathcal{C}_{c,k_1}$ and $\mathcal{C}_{c,k_2}$ such that $k_1 \leq k_2$. Then we have:

$$\Delta(\mathcal{C}_{c,k_1}) \leq \Delta(\mathcal{C}_{c,k_2}).$$

B. Preliminary Results

Lemma 1.3: For any set $X \subset \mathrm{S}$, there exist c and k such that $\mathcal{C}_{c,k}$ satisfies:

$$\mathcal{C}_{c,k} \subset X,$$

and

$$\Delta(\mathcal{C}_{c,k}) = \Delta(X). \tag{23}$$

Proof: Let $\tilde{X} :=$ convhull $(S\backslash X)$. Since d is continuous and the set $\partial\tilde{X}$ is compact, there exists a point $x^* \in \partial\tilde{X}$, such that:

$$\Delta(X) = d(\partial X_S, 0) = \min_{x\in\partial\tilde{X}} d(x,0) = d(x^*, 0).$$

Next, consider the supporting hyperplane of $\tilde{X}$ at x^*, which we denote by $\left\{x : c^T x = k\right\}$. Note that this supporting hyperplane is a supporting hyperplane of the ball $(\Delta(X)\mathrm{B})$ at x^* since we have:

$$\partial(\Delta(X)\mathrm{B}) \subset \partial\tilde{X} \subset \left\{x : c^T x = k\right\}.$$

By Remark 1.2, this implies that $\left\{x : c^T x = k\right\}$ is in fact the unique supporting hyperplane at x^*. Then we have:

$$\Delta(X) = d(x^*, 0) = d(\{x : c^T x = k\}, 0) = \min\left(1, \frac{|k|}{\|c\|}\right).$$

Now, consider the spherical cap $\mathcal{C}_{c,k}$. Then, by Lemma 1.1 we have $\Delta(\mathcal{C}_{c,k}) = \min\left(1, \frac{|k|}{\|c\|}\right)$. Therefore, $\Delta(X) = \Delta(\mathcal{C}_{c,k})$.

We next show $\mathcal{C}_{c,k} \subset X$. We prove this by contradiction. Assume $x \in \mathcal{C}_{c,k}$ and $x \notin X$. Note that, if $x \notin X$, then $x \in \mathrm{S}\setminus X \subset$ convhull $(\mathrm{S}\setminus X)$. Since $x \in \mathcal{C}_{c,k}$, we have $c^T x > k$. But due to the fact that $x \in$ convhull $(\mathrm{S}\setminus X)$, we also have $c^T x \leq k$, which leads to a contradiction. Therefore, $\mathcal{C}_{c,k} \subset X$. ■

Lemma 1.4: Let $\mathcal{X}_\epsilon = \{X \subset \mathrm{S} : \sigma^{n-1}(X) = \epsilon\}$. Then, for any $\epsilon \in (0,1)$, the function $\Delta(X)$ attains its minimum over X_ϵ for some X which is a spherical cap.

Proof: We prove this via contradiction. Assume that there exists no spherical cap in $\mathcal{X}_\epsilon$ such that $\Delta(X)$ attains its minimum. This means there exists an $X^* \in \mathcal{X}_\epsilon$, where X^* is not a spherical cap and $\arg\min_{X\in\mathcal{X}_\epsilon}(\Delta(X)) = X^*$. By Lemma 1.3, we can construct a spherical cap $\mathcal{C}_{c,k}$ such that $\mathcal{C}_{c,k} \subset X^*$ and $\mathcal{C}_{c,k} = \Delta(X^*)$. Note that, we further have $\mathcal{C}_{c,k} \subsetneq X^*$, since X^* is assumed not to be a spherical cap. This means that, there exists a spherical cap $\sigma^{n-1}(\mathcal{C}_{c,k})$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) < \epsilon$.

Then, the spherical cap $\mathcal{C}_{c,\tilde{k}}$ with $\sigma^{n-1}(\mathcal{C}_{c,\tilde{k}}) = \epsilon$, satisfies $\tilde{k} < k$ by Remark 1.1. This implies

$$\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$$

by Corollary 1.2. Therefore, $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(X^*)$. This is a contradiction since we initially assumed that $\Delta(X)$ attains its minimum over $\mathcal{X}_\epsilon$ at X^*. ∎

C. Proof of Lemma 4.3

Proof: Let the function $\Delta(X)$ be defined as in (21). Then by Lemma 1.4 we know that:

$$\Delta(X_\epsilon) \geq \Delta(\mathcal{C}_{c,k}), \tag{24}$$

for some spherical cap $\mathcal{C}_{c,k} \subset \mathrm{S}$, where $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon$. It is known (see e.g. [7]) that the area of such $\mathcal{C}_{c,k}$, is given by the equation:

$$\sigma^{n-1}(\mathcal{C}_{c,k}) = \frac{\pi^{d/2}}{\Gamma[\frac{d}{2}]} I\left(1 - \Delta(X_\epsilon)^2; \frac{d-1}{2}, \frac{1}{2}\right), \tag{25}$$

where I is the regularized incomplete beta function. Since, $\sigma^{n-1}(X_\epsilon) \leq \epsilon$, we get the following set of equations:

$$\begin{aligned} \frac{\epsilon\Gamma[\frac{d}{2}]}{\pi^{d/2}} &\leq I\left(1 - \Delta(X_\epsilon)^2; \frac{d-1}{2}, \frac{1}{2}\right) \\ 1 - \Delta(\mathcal{C}_{c,k})^2 &\leq I^{-1}\left(\frac{\epsilon\Gamma(\frac{d}{2})}{\pi^{d/2}}; \frac{d-1}{2}, \frac{1}{2}\right) \\ \Delta(\mathcal{C}_{c,k})^2 &\geq 1 - I^{-1}\left(\frac{\epsilon\Gamma(\frac{d}{2})}{\pi^{d/2}}; \frac{d-1}{2}, \frac{1}{2}\right) \end{aligned} \tag{26}$$

The inequalities (26) and (24) imply the inclusion given in (17), where

$$\alpha(\epsilon) = \sqrt{1 - I^{-1}\left(\frac{\epsilon\Gamma(\frac{d}{2})}{\pi^{d/2}}; \frac{d-1}{2}, \frac{1}{2}\right)}. \tag{27}$$

∎

D. Proof of $\lim_{N\to\infty} \delta(\beta, \omega_N) = 1$ in Theorem 4.5

We start with the following lemma.

Lemma 1.5: Consider the sequential sampling ω_N. Let $d = \frac{n(n+1)}{2}$ and $P(\omega_N)$ be the optimal solution to the optimization problem $\mathrm{Opt}(\omega_N)$ and let $\lambda_{\max}(P(\omega_N))$ be the optimal objective function value for this $P(\omega_N)$. Then, $\lambda_{\max}(P(\omega_N))$ is uniformly bounded in N.

Proof: We first define the following optimization problem:

$$\begin{aligned} \min_P \quad & \lambda_{\max}(P) \\ \text{s.t.} \quad & (A_j x)^T P (A_j x) \leq (1+\eta)\gamma^2 x^T P x, \ \forall (x,j) \in \omega_N, \\ & P \succeq I, \end{aligned} \tag{28}$$

where we denote its optimal solution by $\lambda_{\max}(\gamma, \omega_N)$.

Note that, for all $d \in \mathbb{Z}$ such that $0 < d \leq N$ we have $\gamma^*(\omega_d) \leq \gamma^*(\omega_N)$. Also note that,

$$\lambda_{\max}(\gamma^*(\omega_N), \omega_N) \leq \lambda_{\max}(\gamma^*(\omega_d), \omega_N).$$

But note that, there exists a $c > 0$ such that $\lambda_{\max}(\gamma^*(\omega_d), \omega_N) < c$ since the problem (28) is strictly feasible for any γ such that $\gamma \leq \gamma^*$. This implies: $\lambda_{\max}(\gamma^*(\omega_d), \omega_N) \leq c$, which completes the proof of this lemma. ∎

We now prove that $\lim_{N\to\infty} \epsilon(\beta, N) = 0$. Note that, we can upper bound $1-\beta$ as follows:

$$1 - \beta = \sum_{j=0}^{d} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j} \leq (d+1) N^d (1-\epsilon)^{N-d}. \tag{29}$$

We prove $\lim_{N\to\infty} \epsilon(\beta, N) = 0$ by contradiction. Assume that $\lim_{N\to\infty} \epsilon(\beta, N) \neq 0$. This means that, there exists some $\delta > 0$ such that $\epsilon(\beta, N) > \delta$ infinitely often. Then, consider the subsequence N_k such that $\epsilon(\beta, N_k) > \delta$, $\forall\, k$. Then, by (29) we have:

$$1-\beta \leq (d+1) N_k^d (1-\epsilon)^{N_k - d} \leq (d+1) N_k^d (1-\delta)^{N_k - d} \, \forall\, k \in \mathbb{N}.$$

Note that $\lim_{k\to\infty} (d+1) N_k^d (1-\delta)^{N_k - d} = 0$. Therefore, there exists a k' such that, we have

$$(d+1) N_{k'}^d (1-\delta)^{N_k' - d} < 1 - \beta,$$

which is a contradiction. Therefore, we must have $\lim_{N\to\infty} \epsilon(\beta, N) = 0$.

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