Data Driven Stability Analysis of Black-box Switched Linear Systems with Probabilistic Guarantees

I. Introduction

Today's complex engineered systems (you can replace complex engineered systems with CPS) are characterized by the interaction of a large number of heterogeneous components. Consequently, the models used to analyze these systems are equally complex and consist of heterogeneous sub-models replying on different modeling assumptions and based on principles from different scientific disciplines. It is not uncommon to encounter a patchwork of differential equations, difference equations, hybrid automata, lookup tables, custom switching logic, low-level legacy code, etc. To further compound the difficulty in analyzing these systems, different components of a complex engineered system are typically designed by different suppliers. Although a high-level specification for these components may be known, detailed models are not available for intellectual property reasons. We are thus faced with a tremendous gap between the existing analysis techniques that rely on closed-form models and the models available in industry. It is, therefore, not surprising the emphasis that industry places on simulation since despite the complexity of models, it is always possible to simulate them.Therefore, it is a natural question to ask whether we can provide formal analyses about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, we consider a dynamical system as in:

$$x_{k+1} = f(k, x_k), \tag{1}$$

where, $x_k \in \mathbb{R}^n$, k is index of time. We start with the following question to serve as a stepping stone: Given N input-output pairs, (x_1, y_1), (x_2, y_2), $\ldots$, (x_N, y_N) such that $y_k = f(k, x_k)$, what can we say about the stability of the system (1)? For the rest of the paper, we use the term *black-box* to refer to models where we do not have access to its dynamics (f), yet we can observe its outputs (y) by exciting it with inputs (x). Note that, one approach to this problem is firstly identifying the dynamics, i.e., f and then applying the existing techniques in the model-based stability analysis literature. However, unless f is a linear function, there are two main reasons behind our quest to directly work on input-output pairs and bypassing the identification phase: (1) Even when the function f is known, in general, the stability analysis is still hard [**?**], [**?**]. (2) The existing identification techniques can only identify f up to an approximation error. How to relate this identification error to an error in the stability of the system (1) is still a nontrivial problem.

The initial idea behind this paper was born based on the recent efforts in [7], [5] and [1] in using simulation traces to find Lyapunov functions for systems with known dynamics. In these work, the main idea is that if one can construct a Lyapunov function candidate decreasing along many finite trajectories starting from different initial conditions, then it should decrease along the remaining trajectories as well. Then, once a Lyapunov function candidate is constructed, the presented algorithms are based on verifying it either via off-the-shelf tools as in [7] and [5], or via sampling based techniques as in [1]. Note that, since we do not have access to the dynamics, the second step cannot be directly applied to black-box systems. However, these sampling based ideas trigger the following question that we address in this paper: *Can we translate the confidence we gain in the decrement of a candidate Lyapunov function, into a confidence in the stability of the underlying system?*

Note that, even in the case of a 2D linear system the connection between these two confidence levels is nontrivial. In fact, one can easily construct an example with one stable and one unstable eigenvalue for which even though almost all trajectories diverge to the infinity, it is possible to construct a Lyapunov function candidate whose level sets are contracting everywhere except a small set. Should we give a specific example here, and put a figure? Moreover, the size of this "violating set" can be arbitrarily small based on the magnitude of the unstable eigenvalue. In this paper, we take the first step to close this gap. Since the identification and stability analysis of linear systems are well understood, we do so by focusing on switched linear systems.

Note that identification and deciding the stability of arbitrary switched linear systems is NP-hard [4]. Aside from their theoretical value, switched systems model the behavior of dynamical systems in the presence of known or unknown varying parameters. These parameters can model internal properties of the dynamical system such as uncertainties, look-up tables, values in a discrete register as well as exogenous inputs provided by a controller in a closed-loop control system. Need to make these examples more specific.

The stability of switched systems closely relates to the *joint spectral radius* (JSR) of the matrices appearing in (2). Under certain conditions deciding stability amounts to deciding whether JSR is less than one or not [4]. In this paper, we present an algorithm to approximate the JSR of a switched linear system from N input-output pairs. This algorithm is based on tools from the random convex optimization literature [2], and provides an upper bound on the JSR with a user-defined confidence level. As N increases, this bound gets tighter. Moreover, with a closed form expression, we characterize what the exact trade-off between the tightness of this bound and the number of samples is. In order to understand the quality of our technique, the algorithm also provides a deterministic lower-bound.

The organization of the paper is as follows: TO BE FILLED.

II. Preliminaries

A. Notation

We consider the usual Hilbert finite normed vector space $(\mathbb{R}^n, \ell_2)$, $n \in \mathbb{N}_{>0}$, with ℓ_2 the classical Euclidean norm. We denote the set of linear functions in $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$, and the set of real symmetric matrices of size n by $\mathcal{S}^n$. We denote a unit ball in $\mathbb{R}^n$ with B and unit sphere in $\mathbb{R}^n$ as S. We only explicitly state the radius r as in B_r and S_r, when r is different than 1. We denote the ellipsoid described by the matrix $P \in \mathcal{S}^n$ as E_P, i.e., $E_P := \{x \in \mathbb{R}^n : x^T P x = 1\}$, and we denote by $\tilde{E}_P$ the volume in $\mathbb{R}^n$ defined by E_P: $\tilde{E}_P = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$. We denote the (convex) projector on S by Π_{S}. We denote the homothety of ratio r by $\mathcal{H}_r$.

For an ellipsoid with center at the origin, and for any of its subsets $\mathcal{A}$, the sector defined by $\mathcal{A}$ is the subset $\{t\mathcal{A},\ t \in [0,1]\} \subset \mathbb{R}^n$. A sector induced by $\mathcal{A} \subset E_P$ will be denoted by $E_P^{\mathcal{A}}$. In the particular case of the unit sphere, we have then $S^{\mathcal{A}}$.

We consider in this work the classical unsigned and finite uniform spherical measure on S, denoted by σ^{n-1}. It is associated to $\mathcal{B}_{\mathrm{S}}$, the spherical Borelian σ-algebra, and is derived from the Lebesgue measure λ. We have $\mathcal{B}_{\mathrm{S}}$ defined by $\mathcal{A} \in \mathcal{B}_{\mathrm{S}}$ if and only if $S^{\mathcal{A}} \in \mathcal{B}_{\mathbb{R}^n}$. The spherical measure σ^{n-1} is defined by

$$\forall\ \mathcal{A} \in \mathcal{B}_{\mathrm{S}},\ \sigma(\mathcal{A}) = \frac{\lambda(S^{\mathcal{A}})}{\lambda(B)}.$$

In other words, the spherical measure of a subset of the sphere is related to the Lebesgue measure of the sector of the unit ball it induces. Notice that $\sigma^{n-1}(\mathrm{S}) = 1$. Similarly, we have a measure on the ellipsoid σ_P defined on the σ-algebra $\mathcal{B}_{E_P}$ by: $\forall\ \mathcal{A} \in \mathcal{B}_{E_P}$, $\sigma_P(\mathcal{A}) = \frac{\lambda(E_P^{\mathcal{A}})}{\lambda(\tilde{E}_P)}$.

B. Stability of Linear Switched Systems

A switched linear system is in the form:

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{2}$$

where, $\sigma : \mathbb{N} \to M$ is the switching sequence where $M := \{1, 2, \ldots, m\}$ and $A_{\sigma(k)} \in \mathcal{M}$, for all σ and $k \in M$. There are two important properties of switched systems that we exploit in this paper.

Property 2.1: Let $\xi(x, k)$ denote the state of the system (2) at time k starting from the initial condition x. The dynamical system (2) is homogeneous:

$$\xi(\gamma x, k) = \gamma \xi(x, k).$$

Property 2.2: The dynamics given in (2) is convexity-preserving, meaning that for any set of points $X \subset \mathbb{R}^n$ we have:

$$f(\text{convhull } X) \subset \text{convhull } \{f(X)\}.$$

We now introduce the *Lyapunov exponent* of the system, which is a numerical quantity describing its stability.

Definition Given a dynamical system as in (1), its *Lyapunov exponent* is given by

$$\rho = \inf\{r : \forall x_0, \exists C \in \mathbb{R}^+ : \xi(x_0, k) \leq C r^k\}.$$

In the case of switched linear systems, the Lyapunov exponent is known as the Joint Spectral Radius of the set of matrices, which can be alternatively defined as follows:

Definition [3] Given a set of matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}$, its *joint spectral radius* (JSR) is given by

$$\rho(\mathcal{M}) = \lim_{k \to \infty} \max_{i_1, \ldots, i_k} \{||A_{i_1} \ldots A_{i_k}||^{1/k} : A_i \in \mathcal{M}\}.$$

Remark 2.1: Note that, JSR is a homogeneous function:

$$\rho\left(\frac{\mathcal{M}}{\gamma}\right) = \frac{\rho(\mathcal{M})}{\gamma}, \forall \lambda > 0,$$

and $\mathcal{M}/\lambda$ can be studied by studying the triple:

$$\left(x_k, \frac{y_k}{\gamma}, \sigma(k)\right).$$

We need a theorem about the stability and $\rho < 1$ here.

III. A Deterministic Lower Bound for JSR

We start by computing a lower bound for ρ which is based on the following theorem from the switched linear systems literature.

Theorem 3.1: [3, Theorem 2.11] For any bounded set of matrices such that $\rho(\mathcal{M}) < \frac{1}{\sqrt{n}}$, there exists a Common Quadratic Lyapunov Function (CQLF) for $\mathcal{M}$, that is, a $P \succ 0$ such that:

$$\forall A \in \mathcal{M},\ A^T P A \preceq P.$$

The following theorem shows that the existence of a CQLF for (2) can be checked by considering N pairs $(x_i, j_i) \in \mathbb{R}^n \times M$, and $i \in \{1, \ldots N\}$.

Theorem 3.2: For a given sampling:

$$\omega_N := \{(x_1, j_1), (x_2, j_2), \ldots, (x_N, j_N)\},$$

let $\gamma^*(\omega_N)$ be the optimal solution of the following optimization problem:

$$\begin{aligned} \min \quad & \gamma \\ s.t. \quad & (A_{j_i} x)^T P (A_{j_i} x) \leq \gamma^2 x_i^T P x_i, \forall i : 1 \leq i \leq N \\ & P \succ 0. \end{aligned} \tag{3}$$

If $\gamma^*(\omega_N) < \infty$, we have:

$$\rho(\mathcal{M}) \geq \frac{\gamma^*}{\sqrt{n}}.$$

Note that, (3) can be solved by bisection on γ.

Proof: Using Remark 2.1, for any $\epsilon > 0$, $\frac{\mathcal{M}}{(\gamma^* - \epsilon),}$ has no CQLF. Then, applying Theorem 3.1 we get

$$\frac{\rho(\mathcal{M})}{\gamma^*} \geq \frac{1}{\sqrt{n}}.$$

■

IV. A Probabilistic Upper Bound for JSR

In this section, we show that using the Property 2.1 and Property 2.2, by sampling finitely many points on a level set of a candidate CQLF, we can compute an upper bound on ρ. This section formalizes this discussion. Before proceeding to the main theorem, we motivate the upcoming technical discussion by stating the following theorem to which most of this section is devoted.

Theorem 4.1: Let $\epsilon \in (0,1]$, $\beta \in [0,1)$. Consider a uniform random sampling of $\mathrm{S} \times M$, denoted by ω_N. Let γ^* and P be the optimal solution to:

$$\begin{array}{ll} \min & \gamma \\ \text{s.t.} & (A_j x)^T P (A_j x) \leq \gamma x^T P x, \, \forall (x,j) \in \omega_N \\ & P \succ 0. \end{array} \quad (4)$$

Then for all Z_P with $\sigma_P(Z_P) \leq \epsilon$, we have with probability at least β:

$$(A_j x)^T P (A_j x) \leq \gamma^* x^T P x, \, \forall \, x \in E_P \setminus Z_P, \forall j \in M. \quad (5)$$

Moreover, we can compute $\delta(\beta, \omega_N) < \infty$ such that:

$$E_{\delta^2 P} \subset \text{convhull } (E_P \setminus Z_P), \quad (6)$$

and $\lim_{N \to \infty} \delta(\beta, \omega_N) = 1$.

Once Theorem 4.1 is established, the main result of this section follows.

Theorem 4.2 (Main Theorem): Let ω_N be a uniform sampling of $\mathrm{S} \times M$, where $N \geq \frac{n(n+1)}{2} + 1$ and $\beta \in [0,1)$. We can compute $\delta(\beta, \omega_N) < \infty$ such that with probability at least β we have

$$\rho \leq \frac{\gamma^*(\omega_N)}{\delta}.$$

Moreover, $\lim_{N \to \infty} \delta(\beta, \omega_N) = 1$.

Proof: Note that, by definition of γ^* we have:

$$(A_j x)^T P (A_j x) \leq \gamma^* x^T P x, \quad \forall \, (x,j) \in \omega_N$$

for some $P \succ 0$. Then, by Theorem 4.1 we have:

$$(A_j x)^T P (A_j x) \leq \gamma^* x^T P x, \quad \forall \, x \in E_P \setminus Z_P, \forall j \in M,$$

which can be rewritten as:

$$\frac{\mathcal{M}}{\gamma^*}(E_P \setminus Z_P) \subseteq E_P \setminus Z_P.$$

By Property 2.2, this implies:

$$\frac{\mathcal{M}}{\gamma^*} \text{convhull } (E_P \setminus Z_P) \subset \text{convhull } (E_P \setminus Z_P).$$

Then, due to Theorem 4.1 we also have (6), meaning:

$$\left(\frac{\mathcal{M}}{\delta \gamma^*}\right) \text{convhull } (E_P \setminus Z_P) \subset \text{convhull } (E_P \setminus Z_P).$$

Then, $\delta\gamma^*$ is an upper bound on ρ, with probability at least β. ■

A. Proving Theorem 4.1

Before proceeding to the proof of Theorem 4.1, we introduce some further notation that will help us in the rest of this section. Let us consider $Z = \mathrm{S} \times M$, the Cartesian product of the unit sphere S with M. Every element of Z can be written as $z = (x_z, j_z)$ with $x_z \in \mathrm{S}$ and $j_z \in M$. For notational simplicity, we drop the subscript z whenever it is clear from the context. We define the classical projections of Z on the sphere and M by $\pi_{\mathrm{S}} : Z \to \mathrm{S}$ and $\pi_M : Z \to M$.

To obtain an upper bound on ρ, the optimization problem we consider the following optimization problem:

$$\begin{array}{ll} \min & \gamma \\ \text{s.t.} & (Ax)^T P (Ax) \leq \gamma x^T P x, \, \forall A \in \mathcal{M}, \, \forall \, x \in \mathrm{S}, \\ & P \succ 0. \end{array} \quad (7)$$

Note that if P is a solution of (7), due to Property 2.1, then so is αP for any $\alpha \in \mathbb{R}_{>0}$. Therefore, the constraint $P \succ 0$ can be replaced with the constraint $P \succeq I$. Also note that, the optimal solution γ^* satisfies $\rho \leq \gamma^*$. However, solving (7) is hard since it involves infinitely many constraints. Therefore, for a given sampling ω_N, we instead consider the optimization problem (4) with finitely many constraints sampled from Z.

Let $\gamma^*(\omega_N)$ be the optimal solution of (4). We now analyze the relationship between the solutions of the optimization problem (7) and:

$$\begin{array}{ll} \text{find} & P \\ \text{s.t.} & (A_j x)^T P (A_j x) \leq \gamma^* x^T P x, \, \forall (x,j) \in \omega_N \\ & P \succ 0. \end{array} \quad (8)$$

We can rewrite (8) in the following form:

$$\begin{array}{ll} \text{find} & P \\ \text{s.t.} & f_{\gamma^*}(P, z) \leq 0, \, \forall \, z \in Z \end{array} \quad (9)$$

where $f_{\gamma^*}(P, z) = \max(f_1(P, z), f_2(P))$, and

$$\begin{array}{rcl} f_1(P, z) & := & (A_j z)^T P (A_j z) - {\gamma^*}^2 x^T P x \\ f_2(P) & := & \lambda_{\max}(-P) + 1. \end{array}$$

We denote by $\mathrm{Opt}(\omega_N)$ the optimization problem (8) for the rest of the paper. Let $P(\omega_N)$ be the solution of $\mathrm{Opt}(\omega_N)$. We are interested in the probability of $P(\omega_N)$ violating at least one constraint in the original problem (7). Therefore, we define the constraint violation property next.

Definition (from [2]) The *constraint violation probability* is defined as:

$$\mathcal{V}^*(\omega_N) = \mathbb{P}\{z \in Z : f(P(\omega_N), z) > 0\}, \, \forall \, \omega_N \in Z^{N*}\}. \quad (10)$$

where

$$Z^{N*} := \{\omega_N \in Z^N : \text{the solution of } \mathrm{Opt}(\omega_N) \text{ exists}\}.$$

Due to the definition of γ^*, we know that a solution to $\mathrm{Opt}(\omega_N)$ exists for any ω_N and therefore $Z^{N*} = Z^N$. Also note that, since we have $\mathbb{P}(\mathcal{A}) = \frac{\mu(\mathcal{A})}{\mu(Z)}$, we can rewrite (10) as:

$$\mathcal{V}^*(\omega_N) = \frac{\mu(V)}{\mu(Z)}, \, \forall \, \omega_N \in Z^N,$$

where μ is a measure on Z and $V := \{z \in Z : f(P(\omega_N), z) > 0\}$, i.e., the set of points for which at least one constraint is violated.

The following theorem from [2] gives an explicit relationship between $\mathcal{V}^*(\omega_N)$, N, and n.

Theorem 4.3 (from [2]): Let $d := \frac{n(n+1)}{2}$. Consider the optimization problem $\mathrm{Opt}(\omega_N)$ given in (8). How to talk about assumptions of this theorem here? Then, for all $\epsilon \in (0,1)$ the following holds:

$$\mathbb{P}^N\{\{\mathcal{V}^*(\omega) \leq \epsilon\}\} \geq 1 - \sum_{j=0}^{d} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j}.$$

Note that $\epsilon = 1 - I^{-1}(\beta, N-d, d+1)$, where I is the regularized incomplete beta function. It can be interpreted as the ratio of the measure of points in Z that might violate at least one of the constraints in (8) to the measure of all points in Z.

The proof consists of finding an upper-bound on the measure of points of S violating the constraints of $\mathrm{Opt}(\omega_N)$, and then of computing an upper-bound for δ.

We assume from now that S is provided with $(\mathcal{B}_{\mathrm{S}}, \sigma^{n-1})$ and that M is provided with the classical σ-algebra associated to finite sets: $\Sigma_M = \wp(M)$, where $\wp(M)$ is the power set of M. We consider an unsigned finite measure[1] μ_M on (M, Σ_M) with $\mathrm{supp}(\mu_M) = M$. In other words, $\forall\, j \in M$, $\mu_M(\{j\}) > 0$. We denote the product σ-algebra $\mathcal{B}_{\mathrm{S}} \bigotimes \Sigma_M$ engendered by $\mathcal{B}_{\mathrm{S}}$ and Σ_M: $\Sigma = \sigma(\pi_{\mathrm{S}}^{-1}(\mathcal{B}_{\mathrm{S}}), \pi_M^{-1}(\Sigma_M))$. On this set, we define the product measure $\mu = \sigma^{n-1} \otimes \mu_M$ which is an unsigned finite measure on Z. We also define $V_{\mathrm{S}} := \pi_S(V), V_M := \pi_M(V)$.

Given a set V of points violating the constraints of the optimization problem (8), we look first for an upper bound on the measure of points on S violating at least one constraint.

Lemma 4.4:

$$\sigma^{n-1}(V_{\mathrm{S}}) \leq \frac{\mu(V)}{m_1},$$

where $m_1 = \min\{\mu_M(\{j\}), j \in M\}$.

Proof:

Let $\mathcal{Z} \subset \Sigma$, $\mathcal{Z}_{\mathrm{S}} = \pi_{\mathrm{S}}(\mathcal{Z})$ and $\mathcal{Z}_M = \pi_M(\mathcal{Z})$. We notice that Σ_M is the disjoint union of its 2^m elements $\{\mathcal{M}_i, i \in \{1, 2, \dots 2^m\}\}$. Then $\mathcal{Z}$ can be written as the disjoint union $\mathcal{Z} = \sqcup_{1 \leq i \leq 2^m}(\mathcal{S}_i, \mathcal{M}_i)$ where $\mathcal{S}_i \in \Sigma(\mathrm{S})$. We notice that $\mathcal{Z}_{\mathrm{S}} = \sqcup_{1 \leq i \leq 2^m} \mathcal{S}_i$, and

$$\sigma^{n-1}(\mathcal{Z}_{\mathrm{S}}) = \sum_{1 \leq i \leq 2^m} \sigma^{n-1}(\mathcal{S}_i).$$

[1]Recall that the support of a measure μ defined on a measurable space (X, Σ) is $\mathrm{supp}(\mu) = \overline{\{A \in \Sigma | \mu(A) > 0\}}$

We have

$$\begin{aligned}
\mu(\mathcal{Z}) &= \mu(\sqcup_{1 \leq i \leq 2^m}(\mathcal{S}_i, \mathcal{M}_i)) \\
&= \sum_{1 \leq i \leq 2^m} \mu(\mathcal{S}_i, \mathcal{M}_i) \\
&= \sum_{1 \leq i \leq 2^m} \sigma^{n-1} \otimes \mu_M(\mathcal{S}_i, \mathcal{M}_i) \\
&= \sum_{1 \leq i \leq 2^m} \sigma^{n-1}(\mathcal{S}_i) \mu_M(\mathcal{M}_i).
\end{aligned}$$

Let m_1 be the minimum value of μ_M on its atoms (recall that $m_1 > 0$):

$$m_1 = \min\{\mu_M(\{j\}), j \in M\}.$$

Then since $\forall\, i$, $\mu_M(\mathcal{M}_i) \geq m_1$, we have

$$\sigma^{n-1}(\mathcal{Z}_{\mathrm{S}}) \leq \frac{\mu(\mathcal{Z})}{m_1}. \tag{11}$$

This proves our statement by taking $\mathcal{Z} = V$. ∎

We recall that we consider the specific case of a uniform sampling both on S and on M.

Corollary 4.5: When the modes are sampled from the set M uniformly random,

$$\sigma^{n-1}(V_{\mathrm{S}}) \leq m\mu(V).$$

We now have the measure of points on S that may violate at least one constraint of $\mathrm{Opt}(\omega_N)$. These points correspond to points on E_P where the system may not be γ-contracting. However, the computation of $\delta(\beta, \omega_N)$ on E_P is hard. To circumvent this, we perform a change of coordinates so that the ellipsoid E_P becomes a sphere in this new coordinate system. We now formalize how the measure of violating points in E_P can be mapped to this new coordinate system.

We consider the linear transformation mapping S to E_P, denoted by $L \in \mathcal{L}(\mathbb{R}^n)$. Note that since $P \in \mathcal{S}^n$, it can be written in its Choleski form $P = UDU^{-1}$, with D diagonal matrix of its eigenvalues, and $U \in O_n$. We define $D^{1/2}$ the positive square root of D as the matrix $\mathrm{diag}(\sqrt{d_1}, \dots, \sqrt{d_n})$. Then, the positive square root of P is $UD^{1/2}U^{-1}$. This means that $L = P^{1/2}$. We now examine L^{-1}, which maps points from S to the ellipsoid that is the image of S after the linear transformation. We denote

$$V' := \Pi_{\mathrm{S}}(L^{-1}(V_{\mathrm{S}})).$$

Note that, V' is the projection of the violating points from the ellipsoid corresponding to the image of the unit sphere in the original coordinates to the unit sphere in the new coordinate system. We now show how to over approximate $\sigma^{n-1}(V')$ in terms of $\mu(V)$.

Remark 4.1: If ψ is a smooth change of coordinates in $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$, whose image under ψ is $X \subset \mathbb{R}^n$, then

$$\lambda(X') = \int_{x \in X} 1_{x \in X} |\det J(\psi(x))| d\lambda(x), \tag{12}$$

which becomes when $\psi \in \mathcal{L}(\mathbb{R}^n)$: (and thus $\forall\, x \in \mathbb{R}^n$, $\det J(\psi(x)) = \det(\psi)$)

$$\lambda(X') = |\det(\psi)| \lambda(X). \tag{13}$$

Theorem 4.6:

$$\sigma^{n-1}(\Pi_{\mathrm{S}}(L^{-1}(V_{\mathrm{S}}))) \leq \frac{\det(L^{-1})}{(\lambda_{\min}(L^{-1}))^n}\sigma^{n-1}(V_{\mathrm{S}}). \quad (14)$$

Proof: Let us consider $S^{V_{\mathrm{S}}}$ the sector of B defined by V_{S}. We denote $C := L^{-1}(S^{V_{\mathrm{S}}})$. We have $\Pi_{\mathrm{S}}(C) = V'$ and $S^{V'} \subset \mathcal{H}_{1/\lambda_{\min}(L^{-1})}(C)$. We have then

$$\sigma^{n-1}(V') = \lambda(S^{V'}) \leq \lambda(\mathcal{H}_{1/\lambda_{\min}(L^{-1})}(C)),$$

which means:

$$\sigma^{n-1}(V') \leq \frac{1}{\lambda_{\min}(L^{-1})^n}\lambda(C).$$

Using Remark 4.1, we have the result of the theorem. ∎

Corollary 4.7: $\sigma^{n-1}(V') \leq m\epsilon\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, where $\mu(V) = \epsilon$.

We now show how to relate the upper bound on the measure of violating points with δ. We denote $\epsilon' := \frac{\epsilon}{2}\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, where the additional factor $\frac{1}{2}$ follows from the homogeneity of the dynamics which implies a symmetry of V, i.e.,

$$x \in V_{\mathrm{S}} \implies -x \in V_{\mathrm{S}}.$$

We start by a few definitions that will help us along the way. Let d be a distance on $\mathbb{R}^n$. The distance between a set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ is $d(X,p) := \inf_{x\in X} d(x,p)$. Note that the map $p \mapsto d(X,p)$ is continuous on $\mathbb{R}^n$.

Definition We define the *spherical cap* on S for a given hyperplane $c^Tx = k$ as:

$$\mathcal{C}_{c,k} := \{x \in \mathrm{S} : c^Tx > k\}.$$

Definition A *supporting hyperplane* of a set $X \subset \mathbb{R}^n$ is a hyperplane that has the following two properties:

- X is entirely contained in one of the two closed half-spaces bounded by the hyperplane.
- X has at least one boundary-point on the hyperplane.

Remark 4.2: [**?**] Consider a convex set $X \subset \mathbb{R}^n$. For every $x \in \partial X$, there exists a supporting hyperplane containing x. Moreover, if X is smooth, then this supporting hyperplane is unique.

*Proposition 4.8 (see e.g. [**?**]):* The distance between the point $x = 0$ and the hyperplane $c^Tx = k$ is $\frac{|k|}{\|c\|}$.

We define the function $\Delta : \wp(\mathrm{S}) \to [0,1]$ as:

$$\Delta(X) := \sup\{r : \mathrm{B}_r \subseteq \text{convhull } (\mathrm{S} \setminus X)\}. \quad (15)$$

Note that, $\Delta(X)$ can be rewritten as in:

$$\Delta(X) = d(\partial\text{convhull } (\mathrm{S} \setminus X), 0). \quad (16)$$

Lemma 4.9: $\Delta(\mathcal{C}_{c,k}) = \min\left(1, \frac{|k|}{\|c\|}\right)$.

Proof: Note that convhull $(\mathrm{S} \setminus X) = \left\{x \in \mathrm{B} : c^Tx \leq k\right\}$.

$$\begin{aligned}\Delta(X) &= d(\partial\text{convhull } (\mathrm{S} \setminus X), 0)\\ &= \min(d(\partial\mathrm{B}, 0), d(\partial\{x : c^Tx \leq k\}, 0))\\ &= \min(d(\mathrm{S}, 0), d(\{x : c^Tx = k\}, 0))\\ &= \min\left(1, \frac{|k|}{\|c\|}\right).\end{aligned}$$

∎

Corollary 4.10: $\Delta(\mathcal{C}_{c,k_1}) < \Delta(\mathcal{C}_{c,k_2})$ when $k_1 < k_2$.

Lemma 4.11: $\sigma^{n-1}(\mathcal{C}_{c,k_1}) < \sigma^{n-1}(\mathcal{C}_{c,k_2})$, for $k_1 > k_2$.

Proof: convhull $(\mathrm{S} \setminus \{x \in \mathrm{S} : c^Tx > k_1\}) \subseteq$ convhull $(\mathrm{S} \setminus \{x \in \mathrm{S} : c^Tx > k_2\})$, for $k_1 > k_2$. ∎

Now we are ready to present the following lemma which is the key to proving our main result.

Lemma 4.12: For any set $X \subseteq \mathrm{S}$, there exist c and k such that $\mathcal{C}_{c,k}$ satisfies:

$$\mathcal{C}_{c,k} \subseteq X,$$

and

$$\Delta(\mathcal{C}_{c,k}) = \Delta(X). \quad (17)$$

Proof: Let $X_S :=$ convhull $(S \setminus X)$. Since the distance function d is continuous and the set ∂X_S is compact there exists a point $x^* \in \partial X_S$, such that:

$$\begin{aligned}\Delta(X) &= d(\partial X_S, 0) = \inf_{x\in\partial X_S} d(x,0)\\ &= \min_{x\in\partial X_S} d(x,0) = d(x^*,0). \quad (18)\end{aligned}$$

Next, consider the supporting hyperplane of X_S at x^*, which we denote by $\left\{x : c^Tx = k\right\}$. Note that this supporting hyperplane is a supporting hyperplane of the ball $\mathrm{B}_{\Delta(X)}$ at x^* since we have:

$$\partial\mathrm{B}_{\Delta(X)} \subseteq \partial X_S \subseteq \left\{x : c^Tx = k\right\}.$$

By Remark 4.2, this implies that in fact $\left\{x : c^Tx = k\right\}$ is the unique supporting hyperplane at x^*. Then we have:

$$\Delta(X) = d(x^*, 0) = d(\{x : c^Tx = k\}, 0) = \frac{|k|}{\|c\|}.$$

Now, consider the spherical cap $\mathcal{C}_{c,k}$. Then, by Lemma we have $\Delta(\mathcal{C}_{c,k}) = \frac{|k|}{\|c\|}$. Therefore, $\Delta(X) = \Delta(\mathcal{C}_{c,k})$.

We next show $\mathcal{C}_{c,k} \subseteq X$. We prove this by contradiction. Assume $x \in \mathcal{C}_{c,k}$ and $x \notin X$. Note that, if $x \notin X$, then $x \in \mathrm{S} \setminus X \subseteq$ convhull $(\mathrm{S} \setminus X)$. Since $x \in \mathcal{C}_{c,k}$ we have $c^Tx > k$. But due to the fact that $x \in$ convhull $(\mathrm{S} \setminus X)$, we also have $c^Tx \leq k$, which leads to a contradiction. Therefore, $\mathcal{C}_{c,k} \subseteq X$. ∎

We now prove our main result.

Theorem 4.13: Let $X_{\epsilon'} = \{X \subset \mathrm{S} : \sigma^{n-1}(X) = \epsilon'\}$. Then, for any $\epsilon' \in (0,1)$, the function $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ for some X which is a spherical cap.

Proof: We prove this via contradiction. Assume that there exists no spherical cap in $X_{\epsilon'}$ such that $\Delta(X)$ attains its minimum. This means there exists an $X^* \in X_{\epsilon'}$, where X^* is not a spherical cap and $\arg\min_{X\in X_{\epsilon'}}(\Delta(X)) = X^*$. By Lemma 4.12 we can construct a spherical cap $\mathcal{C}_{c,k}$ such that $\mathcal{C}_{c,k} \subseteq X^*$ and $\mathcal{C}_{c,k} = \Delta(X^*)$. Note that, we further have $\mathcal{C}_{c,k} \subset X^*$, since X^* is assumed not to be a spherical cap. This means that, there exists a spherical cap $\sigma^{n-1}(\mathcal{C}_{c,k})$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) < \epsilon'$.

Then, the spherical cap $\mathcal{C}_{c,\tilde{k}}$ with $\sigma^{n-1}(\mathcal{C}_{c,\tilde{k}}) = \epsilon'$, satisfies $\tilde{k} < k$, due to Lemma 4.11. This implies $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$ due to Lemma 4.10. Therefore, $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(X^*)$. This is a contradiction since we initially assumed that $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ at X^*. ∎

Theorem 4.14: Given a spherical cap $\mathcal{C}_{c,k} \subseteq \mathrm{S}$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon'$,

$$\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)},$$

where $\alpha := I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$. Here I^{-1} is the inverse incomplete beta function, i.e., $I^{-1}(y,a,b) = x$ where $I_x(a,b) = y$.

Proof: Let $h := 1-\Delta(\mathcal{C}_{c,k})$. It is well known [6] that the area of the spherical cap $\mathcal{C}_{c,k} \subseteq \mathrm{S}$ is given by the equation:

$$\epsilon' = \sigma^{n-1}(\mathcal{C}_{c,k}) = \frac{\pi^{d/2}}{\Gamma[\frac{d}{2}]} I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right), \qquad (19)$$

where I is the incomplete beta function. From this, we get the following set of equations:

$$\begin{aligned}
\frac{\epsilon'\Gamma[\frac{d}{2}]}{\pi^{d/2}} &= I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right) \\
2h-h^2 &= I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right) \\
2h-h^2 &= \alpha \\
h^2-2h+\alpha &= 0. \qquad (20)
\end{aligned}$$

From (20), we get $h = 1 \pm \sqrt{(1-\alpha)}$. Since $h \leq 1$, we conclude that $\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)}$. Note that, $\Delta(\mathcal{C}_{c,k})$ only depends on ϵ for fixed n. ∎

Corollary 4.15: For a fixed $\beta \in (0,1)$, $\lim_{N\to\infty} \delta_\beta(N) = 1$.

Proof: We first prove that $\lim_{N\to\infty} \epsilon_\beta(N) = 0$. Note that, we can upper bound $1-\beta$ as follows:

$$\begin{pmatrix} 1-\beta = \sum_{j=0}^{d} N \\ j\epsilon^j(1-\epsilon)^{N-j} \leq (d+1)N^d(1-\epsilon)^{N-d}. \end{pmatrix} \qquad (21)$$

We prove $\lim_{N\to\infty} \epsilon_\beta(N) = 0$ by contradiction and assume that $\lim_{N\to\infty} \epsilon_\beta(N) \neq 0$. This means that, there exists some $\delta > 0$ such that $\epsilon_\beta(N) > \delta$ infinitely often. Then, consider the subsequence N_k such that $\epsilon_\beta(N_k) > \delta$, $\forall\, k$. By (21) we have:

$$1-\beta \leq (d+1)N_k^d(1-\epsilon)^{N_k-d} \leq (d+1)N_k^d(1-\delta)^{N_k-d}\, \forall\, k \in \mathbb{N}.$$

Note that $\lim_{k\to\infty}(d+1)N_k^d(1-\delta)^{N_k-d} = 0$. Therefore, there exists a k' such that, we have $(d+1)N_{k'}^d(1-\delta)^{N_k'-d} < 1-\beta$, which is a contradiction. Therefore, we must have $\lim_{N\to\infty} \epsilon_\beta(N) = 0$.

Showing I^{-1} is monotonic its first parameter, $\delta = \sqrt{1-\alpha}$ tends to 1 as $\epsilon \to 0$.

∎

V. Experimental Results

VI. Future Work

VII. Conclusions

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