Sampling Based Stability Analysis of Black-box Switched Linear Systems with Probabilistic Guarantees

I. Introduction

As our computational resources have increased, so is the complexity of the models we use for the analysis of dynamical systems. Today, the industrial models do not only consist of simple differential or difference equations; these models are multimodal, hybrid, and contain a variety of subcomponents such as lookup tables, delay differential equations, and thermodynamic models. The current modeling paradigm is also highly distributed, in the sense that, the model subcomponents are developed by different parties. Therefore, their internal structure is partially or completely unknown to the end user. Hence, it is often hard, if not impossible to obtain analytical formulas for today's industrial scale models. On the other hand, performing simulations is a common way of validating these models via readily available tools. Therefore, it is a natural question to ask whether we can provide formal analyses about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, we consider a dynamical system as in:

$$x_{k+1} = f(k, x_k), \tag{1}$$

where, $x_k \in \mathbb{R}^n$, k is index of time. We start with the following question to serve as a stepping stone: Given N input-output pairs, (x_1, y_1), (x_2, y_2), ..., (x_N, y_N) such that $y_k = f(k, x_k)$, what can we say about the stability of the system (1)? For the rest of the paper, we use the term *black-box* to refer to models where we do not have access to its dynamics (f), yet we can observe its outputs (y) by exciting it with inputs (x). Note that, one approach to this problem is firstly identifying the dynamics, i.e., f and then applying the existing techniques in the model-based stability analysis literature. However, unless f is a linear function, there are two main reasons behind our quest to directly work on input-output pairs and bypassing the identification phase: (1) Even when the function f is known, in general, the stability analysis is still hard [**?**], [**?**]. (2) The existing identification techniques can only identify f up to an approximation error. How to relate this identification error to an error in the stability of the system (1) is still a nontrivial problem.

The initial idea behind this paper was born based on the recent efforts in [6], [4] and [1] in using simulation traces to find Lyapunov functions for systems with known dynamics. In these work, the main idea is that if one can construct a Lyapunov function candidate decreasing along many finite trajectories starting from different initial conditions, then the Lyapunov function should decrease along the remaining trajectories as well. Then, once a Lyapunov function candidate is constructed, the presented algorithms are based on verifying it either via off-the-shelf tools as in [6] and [4], or via sampling based techniques as in [1]. Note that, since we do not have access to the dynamics, the second step cannot be directly applied to black-box systems. However, these sampling based ideas trigger the following question that we address in this paper: *Can we translate the confidence we gain in the decrement of a candidate Lyapunov function, into a confidence in the stability of the underlying system?*

Note that, even in the case of a 2D linear system the connection between these two confidence levels is nontrivial. In fact, one can easily construct an example with one stable and one unstable eigenvalue for which even though almost all trajectories diverge to the infinity, it is possible to construct a Lyapunov function candidate whose level sets are contracting everywhere except a small set. Should we give a specific example here, and put a figure? Moreover, the size of this "violating set" can be arbitrarily small based on the magnitude of the unstable eigenvalue. In this paper, we take the first step to close this gap. Since the identification and stability analysis of linear systems are well understood, we do so by focusing on switched linear systems.

A switched linear system is in the form:

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{2}$$

where, $\sigma : \mathbb{N} \to \{1, 2, \ldots, m\}$ is the switching sequence and $A_{\sigma(k)} \in \mathcal{M}$, for all σ and k. Note that identification and deciding the stability of arbitrary switched linear systems is NP-hard [3]. Aside from their theoretical value, switched systems model the behavior of dynamical systems in the presence of known or unknown varying parameters. These parameters can model internal properties of the dynamical system such as uncertainties, look-up tables, values in a discrete register as well as exogenous inputs provided by a controller in a closed-loop control system. Need to make these examples more specific.

The stability of switched systems closely relates to the *joint spectral radius* (JSR) of the matrices appearing in (2). Under certain conditions deciding stability amounts to deciding whether JSR is less than one or not [3]. In this paper, we present an algorithm to approximate the JSR of a switched linear system from N input-output pairs. This algorithm is based on tools from the random convex optimization literature [2], and provides an upper bound on the JSR with a user-defined confidence level. As N increases, this bound gets tighter. Moreover, with a closed form expression, we characterize what the exact trade-off between the tightness of this bound and the number of samples is. In order to understand the quality of our technique, the algorithm also provides a deterministic lower-bound.

The organization of the paper is as follows: TO BE FILLED.

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