Filling in the Gaps in Raphael's Idea

1 Preliminaries

We consider the usual Hilbert finite normed vector space $(\mathbb{R}^n, \ell_2)$, $n \in \mathbb{N}_{>0}$, ℓ_2 the classical euclidean norm. We denote a unit ball in $\mathbb{R}^n$ with B and unit sphere in $\mathbb{R}^n$ of radius r as S. We only denote the radius r explicitly as in B_r and S_r, when r is different than 1. We denote the set of real symmetric matrices of size n by $\mathbb{S}^n$, and the set of linear functions in $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$. We denote the ellipsoid described by the matrix $P \in \mathbb{S}^n$ as E_P. We denote the homothety of ratio λ by $\mathcal{H}_\lambda$.

For the rest of the write-up, we denote the set of indices of the modes as $M = \{1, 2, \ldots, m\}$, where $m \in \mathbb{N}_{>0}$ is the number of the modes. We denote the joint spectral radius of the set of matrices $\{A_1, A_2, \ldots A_m\}$ by ρ. Let us consider $X = \mathrm{S} \times M$ the Cartesian product of the unit sphere S with M. Every element of X can be written as $x = (s_x, k_x)$ with $s_x \in \mathrm{S}$ and $k_x \in M$. For notational simplicity, we drop the subscript x whenever it is clear from the context.

We define the projections:

$$\pi_{\mathrm{S}} : \mathrm{S} \times M \to \mathrm{S}, (s, k) \mapsto s$$

$$\pi_M : \mathrm{S} \times M \to M, (s, k) \mapsto k.$$

It is well-known that S is a $n-1$ embedded submanifold of $\mathbb{R}^n$, and can thus be seen as an image of an atlas (collection) of smooth maps $\phi_i : U \to \mathrm{S}$, $U \in \mathbb{R}^n$ called charts. It has the topology inherited from its ambient space $\mathbb{R}^n$. If $\mathbb{R}^n$ is provided with a σ-algebra Σ, this parametrization also induces a σ-algebra on S, Σ_{S}. Hence, a measure μ on the measurable space $(\mathbb{R}^n, \Sigma)$ defines a measure μ_{S} on the measurable space $(\mathrm{S}, \Sigma_{\mathrm{S}})$. This measure can be seen as push-forward $\phi_{i*}(\mu)$ of μ by the charts, i.e., $\phi_{i*}(\mu)(A) = \mu(\phi_i^{-1}(A))$ for any $A \in \Sigma_{\mathrm{S}}$. In particular, with the classical Borel σ-algebra and Lebesgue measure in $\mathbb{R}^n$, we obtain a σ-algebra $\mathcal{B}_{\mathrm{S}}$ with $A \in \mathcal{B}_{\mathrm{S}}$ if and only if the sector tA, $t \in [0, 1]$ is in $\mathcal{B}_{\mathbb{R}^n}$; and the classical spherical measure commonly denoted by σ^{n-1} and defined by

$$\forall A \in \mathcal{B}_{\mathrm{S}}, \sigma(A) = \frac{\lambda(tA)}{\lambda(B)}.$$

We can notice that $\sigma^{n-1}(\mathrm{S}) = 1$.

We assume now that S is provided with a σ-algebra Σ_{S} and M with the classical σ-algebra associated to finite sets: $\Sigma_M = \wp(M)$, where $\wp(M)$ is the power set of M.

We consider an unsigned finite spherical measure μ_{S} on $(\mathrm{S}, \Sigma_{\mathrm{S}})$ and an unsigned finite measure[1] μ_M on (M, Σ_M) with $\mathrm{supp}(\mu_M) = M$. In other words, $\forall\, k \in M$, $\mu_M(\{k\}) > 0$.

We denote the product σ-algebra $\Sigma_{\mathrm{S}} \bigotimes \Sigma_M$ engendered by Σ_{S} and Σ_M: $\Sigma = \sigma(\pi_{\mathrm{S}}^{-1}(\Sigma_{\mathrm{S}}), \pi_M^{-1}(\Sigma_M))$. On this set, we define the product measure $\mu = \mu_{\mathrm{S}} \otimes \mu_M$ which is an unsigned finite measure on X.

2 Optimization Problem

We are interested in solving the following optimization problem for a given $\gamma \in (0, 1)$:

$$\begin{array}{ll} \text{find} & P \\ \text{subject to} & (A_i s)^T P (A_i s) \leq \gamma^2 s^T P s, \quad \forall i = \{1, 2, \ldots, m\}, \forall\, s \in \mathrm{S}, \\ & P \succ 0. \end{array} \tag{1}$$

Note that if P is a solution to (1), then so is αP for any $\alpha \in \mathbb{R}_{>0}$.Therefore, we can rewrite (1) as the following optimization problem:

$$\begin{array}{ll} \text{find} & P \\ \text{subject to} & (A_k s)^T P (A_k s) \leq \gamma^2 s^T P s, \quad \forall i = \{1, 2, \ldots, m\}, \forall\, s \in \mathrm{S}, \\ & P \succeq I. \end{array} \tag{2}$$

We define the linear isomorphism Φ as the natural mapping $\Phi : \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{S}^n$. Using this mapping, for a fixed $\gamma \in (0, 1]$ we can rewrite (2) as:

$$\begin{array}{ll} \text{find} & p \\ \text{subject to} & f(p, x) \leq 0, \forall x \in X. \end{array} \tag{3}$$

[1] Recall that the support of a measure μ defined on a measurable space (X, Σ) is $\mathrm{supp}(\mu) = \overline{\{A \in \Sigma | \mu(A) > 0\}}$

where $f(p,x) = \max(f_1(p,x), f_2(p))$, and

$$\begin{aligned} f_1(p,x) &:= (A_k s)^T \Phi(p)(A_k s) - \gamma^2 s^T \Phi(p) s \\ f_2(p) &:= \lambda_{\max}(\Phi(-p)) + 1. \end{aligned}$$

Proposition 2.1. *The optimization problem* (3) *is convex.*

Proof. The function $f_1(p,x)$ is clearly convex in p for a fixed $x \in X$. The function $\lambda_{max} : \mathbb{S}^n \to \mathbb{R}$ maps a symmetric positive matrix to its maximum eigenvalue. It is well-known that the function $\lambda_{\max}$ is a convex function of P. [**?**]. This means that, $p \mapsto \Phi(\lambda_{\max}(p))$ is convex in p. Moreover, maximum of convex functions is also convex, which shows that $f(p,x)$ is convex in p. □

Note that the optimization problem (3) has infinitely many constraints. We next consider the following optimization problem where we sample N constraints of (3) independently and identically with the probability measure $\mathbb{P}(A) = \frac{\mu(A)}{\mu(X)}, \forall\, A \in \Sigma$, where $N \geq d+1$, and $d := \frac{n(n+1)}{2}$. We denote this sampling by $\omega := \{x_1, x_2, \ldots, x_N\} \subset X$, and obtain the following convex optimization problem $\mathrm{Opt}(\omega)$:

$$\begin{array}{ll} \text{find} & p \\ \text{subject to} & f(p,x) \leq 0, \forall x \in \omega. \end{array} \tag{4}$$

Let $p^*(\omega)$ be the solution of $\mathrm{Opt}(\omega)$. We are interested in the probability of $p^*(\omega)$ violating at least one constraint in the original problem (3). Therefore, we define constraint violation property next.

Constraint violation probability [2] The constraint violation probability is defined as:

$$\mathcal{V}^*(\omega) = \begin{cases} \mathbb{P}\{x \in X : f(p^*(\omega), x) > 0\} & \text{if } \omega \in X^{N*}, \\ 1, & \text{otherwise} \end{cases}$$

where $X^{N*} := \{\omega \in X^N : \text{the solution of } \mathrm{Opt}(\omega) \text{ exists}\}$. Note that, since we have $\mathbb{P}(A) = \frac{\mu(A)}{\mu(X)}$, we can rewrite this as:

$$\mathcal{V}^*(\omega) = \begin{cases} \frac{\mu\{x \in X : f(p^*(\omega), x) > 0\}}{\mu(X)} & \text{if } \omega \in X^{N*}, \\ 1, & \text{otherwise} \end{cases}$$

We make the following assumptions on the problem $\mathrm{Opt}(\omega)$:

1. Uniqueness of solution: Note that this can be enforced by adding a tie-break rule of at most $\frac{n(n-1)}{2}$ convex conditions discriminating our solutions.

2. Nondegenaracy: with probability 1, there is no redundancy in the constraint obtained from the sampling.

The following theorem from [2] explicitly gives a relationship between $V^*(\omega)$ and N, n.

Theorem 2.2 (from [2]). *Consider the optimization problem Opt(ω) given in (4). Let Assumption 1 and Assumption 2 hold. Then, for all $\epsilon \in (0,1)$ the following holds:*

$$\mathbb{P}^N\{\{\mathcal{V}^*(\omega) \le \epsilon\} \cap X^{N*}\} \ge 1 - \sum_{j=0}^{d} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j}.$$

Note that $\epsilon = 1 - I_\beta^{-1}(N-d, d+1)$ and can be interpreted as the ratio of the measure of points in X that might violate at least one of the constraints in (2) to the measure of all points in X.

We now state our main theorem, which is based on Theorem 2.2 and devote the next section to proving it step by step. We denote by γ^*, the optimum value of the following optimization problem:

$$\begin{aligned} \min_{P,\gamma} \quad & \gamma \\ \text{subject to} \quad & (A_i s)^T P (A_i s) \le \gamma^2 s^T P s, \quad \forall i = \{1,2,\dots,m\}, \forall\, s \in \mathrm{S}, \\ & P \succ 0. \end{aligned} \tag{5}$$

Theorem 2.3 (Main Theorem). *For any $\eta > 0$, given $N \ge n+1$ and $\beta \in [0,1)$, we can compute $\delta < \infty$ such that with probability at least β, $\rho \le \delta(1+\eta)\gamma^*$. Moreover, as $N \to \infty$, $\delta \to 1$.*

3 Relating the measure of bad sets

For a given sampling $\omega \in X^{N*}$, let $V := \{x \in X : f(p^*(\omega), x) > 0\}$, i.e., the set of points for which at least one constraint is violated, and V_{S}, V_M be its projections on S and M, respectively.

Lemma 3.1. $\mu_S(V_S) \le \frac{\mu(V)}{m_1}$*, where* $m_1 = \min\{\mu_M(\{k\}), k \in M\}$.

Proof. Let $A \subset X$, $A_{\mathrm{S}} = \pi_{\mathrm{S}}(A)$ and $A_M = \pi_M(A)$. We notice that Σ_M is the disjoint union of its 2^m elements $\{B_i, i \in \{1, 2, \dots 2^m\}\}$. Then A is the disjoint union $A = \sqcup_{1 \le i \le 2^m}(A_i, B_i)$ where $A_i = \pi_M^{-1}(B_i) \in \mathrm{S}$. We notice that $A_{\mathrm{S}} = \sqcup_{1 \le i \le 2^m} A_i$, and

$$\mu_{\mathrm{S}}(A_{\mathrm{S}}) = \sum_{1 \le i \le 2^m} \mu_{\mathrm{S}}(A_i).$$

We have

$$\begin{aligned} \mu(A) = \mu(\sqcup_{1 \le i \le 2^m}(A_i, B_i)) &= \sum_{1 \le i \le 2^m} \mu((A_i, B_i)) \\ &= \sum_{1 \le i \le 2^m} \mu_{\mathrm{S}} \otimes \mu_M((A_i, B_i)) \\ &= \sum_{1 \le i \le 2^m} \mu_{\mathrm{S}}(A_i)\mu_M(B_i). \end{aligned}$$

Let m_1 be the minimum value of μ_M on its atoms: $m_1 = \min\{\mu_M(\{k\}), k \in M\}$ (recall that $m_1 > 0$). Then since $\forall\ i$, $\mu_M(B_i) \ge m_1$, we have

$$\mu_{\mathrm{S}}(A_{\mathrm{S}}) \le \frac{\mu(A)}{m_1}. \tag{6}$$

This proves our statement by taking $A = V_{\mathrm{S}}$. ☐

Corollary 3.2. *When the modes are sampled from the set M uniformly random,*

$$\mu_S(V_S) \le m\mu(V).$$

We consider the linear transformation mapping S to E_P that denoted by $L \in \mathcal{L}(\mathbb{R}^n)$. Note that since $P \in \mathbb{S}^n$, it can be written in its Choleski form $P = UDU^{-1}$, where D diagonal matrix of its eigenvalues, and $U \in O_n(\mathbb{R})$. We define $D^{1/2}$ the positive square root of D as the matrix $\mathrm{diag}(\sqrt{d_1}, \dots, \sqrt{d_n})$. Then, the positive square root of P is $VD^{1/2}V$. This means that, $L = P^{1/2}$. For the rest of the write-up, we denote

$$V' := \Pi_{\mathrm{S}}(L^{-1}(V_{\mathrm{S}})),$$

and show how to upper bound $\sigma^{n-1}(V')$ in terms of $\mu(V)$.

Lemma 3.3. *Let ψ a smooth change of coordinates in $\mathbb{R}^n$ and $\mathcal{D} \subset S$, whose image under ψ is $\mathcal{D}' \subset \psi(S)$. Let μ_S be a positive spherical measure induced* *by a measure μ on $\mathbb{R}^n$. Let Σ_E and μ_E be the σ-algebra and the measure induced from Σ_S and μ_S on the ellipsoid $E = \psi(S)$. Then*

$$\mu_E(\psi(V_S)) = |\det(\psi)|\mu_S(V_S), \tag{7}$$

where $\psi \in \mathcal{L}(\mathbb{R}^n)$.

Proof. We have $\mu_\mathrm{S}(\mathcal{D}) = \int_{x\in\mathcal{D}} \mathbb{1}_D(x)\ d\mu_\mathrm{S}(x)$, $\mu_\mathrm{S} = \{\phi_{i*}(\mu)\}_i$ and

$$\mu(\mathcal{D}') = \int_{y\in\mathcal{D}'} \mathbb{1}_{D'}(y)\ d\mu(y) = \int_{x\in\mathcal{D}} \mathbb{1}_{x\in\mathcal{D}} |\det J(\phi(x))|\ d\mu(x).$$

This gives

$$\mu_E(\mathcal{D}') = \int_{y\in\mathcal{D}'} \mathbb{1}_{D'}(y)\ d\mu_E(y) = \int_{x\in\mathcal{D}} \mathbb{1}_{x\in\mathcal{D}} |\det J(\psi(x))|\ d\mu_\mathrm{S}(x).$$

In particular, if $\psi \in \mathcal{L}(\mathbb{R}^n)$, then $\forall x \in \mathbb{R}^n$, $\det(J(\psi(x)) = \det(\psi)$ and

$$\mu_E(\mathcal{D}') = \int_{y\in\mathcal{D}'} \mathbb{1}_{D'}(y)\ d\mu_E(y) = |\det(\psi)| \int_{x\in\mathcal{D}} \mathbb{1}_{x\in\mathcal{D}}\ d\mu_\mathrm{S}(x).$$

This proves the statement of the lemma when $\mathcal{D} = V_\mathrm{S}$. □

Definition Let X be a Hilbert space, A a nonempty subset of X and $\psi : A \to X$. Then ψ is called firmly nonexpansive if

$$\forall x, y \in A, \|\psi(x) - \psi(y)\|^2 + \|(\mathrm{Id} - \psi)(x) - (\mathrm{Id} - \psi)(y)\|^2 \leq \|x - y\|^2,$$

where Id denotes the identity function from X to X.

Theorem 3.4 (from [1])**.** *Let C be a nonempty closed convex subset of X, then the convex projector on C, Π_C, is firmly nonexpansive.*

Corollary 3.5.

$$\|\Pi_C(x) - \Pi_C(y)\| \leq \|x - y\| \qquad \forall\, x, y \in C. \tag{8}$$

Lemma 3.6.

$$\mu_S(\Pi_S(L^{-1}(V_S))) \leq \det(L^{-1}) \left(\frac{1}{\lambda_{\min}(L^{-1})}\right)^n \mu_S(V_S). \tag{9}$$

Proof. Note that the mapping Π_{S} can be seen as the composition of the Π_{S_r} for some $r > 0$, and $\mathcal{H}_{\frac{1}{r}}$. Let $E' := L^{-1}(\mathrm{S})$, then when $r < \min_{x \in E'} \|x\| = \lambda_{\min}(L^{-1})$ we have

$$\Pi_{\mathrm{S}_{\lambda_{\min}}}(x) = \Pi_{\mathrm{B}_{\lambda_{\min}}}(x) \qquad \forall\, x \in E'.$$

This shows that the restriction of $\Pi_{\mathrm{S}_{\lambda_{\min}}}$ to E' is a convex projector.

Then by Corollary 3.5

$$\|\Pi_{\mathrm{S}_{\lambda_{\min}}}(x) - \Pi_{\mathrm{S}_{\lambda_{\min}}}(y)\| \leq \|x - y\|, \qquad \forall\; x, y \in E'. \tag{10}$$

This shows that 1 is a Lipschitz constant of the function $\Pi_{\mathrm{S}_{\lambda_{\min}}}$ on E'.

By composing $\Pi_{\mathrm{S}_{\lambda_{\min}}}$ with $\mathcal{H}_{\frac{1}{\lambda_{\min}}}$, we obtain Π_{S}. Since the Lipschitz constant of composition of two functions can be bounded by the multiplication of Lipschitz constants of each function, the Lipschitz constant of Π_S on E' is $\frac{1}{\lambda_{\min}}$, which means that:

$$\|\Pi_{\mathrm{S}}(x) - \Pi_{\mathrm{S}}(y)\| \leq \frac{1}{\lambda_{\min}} \|x - y\|, \qquad \forall\, x, y \in E'. \tag{11}$$

Note that, the inequality in (11) is an equality when x is in the eigenspace of $\lambda_{\min}$ and $y = -x$.

Recall that for any smooth Lipschitz function ϕ with Lipschitz constant, $\mathrm{Lip}(\phi)$, we have for all x, $|\det(J(\phi(x))| \leq \mathrm{Lip}(\phi)^n$. Combining this with (11) and Lemma 3.6, we get the statement of the lemma. □

Theorem 3.7. $\sigma^{n-1}(V') \leq m\epsilon\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$*, where* $\mu(V) = \epsilon$*.*

Proof. By taking μ_{S} as the uniform spherical measure σ^{n-1}, and combining Corollary 3.2 with Lemma 3.6 we get the statement of the theorem. □

4 Relating ϵ to δ

We denote $\epsilon' := \frac{\epsilon}{2}\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, where the additional factor $\frac{1}{2}$ follows from the homogeneity of the dynamics. In this section, we show how to relate ϵ' to δ in the statement of the Theorem 2.3. We start by a few definitions that will help us along the way.

Spherical Cap We define the *spherical cap* on S for a given hyperplane $c^T x = k$ as:

$$\mathcal{C}_{c,k} := \{x \in \mathrm{S} : c^T x > k\}.$$

Proposition 4.1. *Let d be a distance on $\mathbb{R}^n$. The distance between a set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ is $d(X,p) := \inf_{x\in X} d(x,p)$. Then if X is compact, we have:*

$$d(X,p) = \min_{x\in X} d(x,p).$$

Proof. This is due to the fact that the function d is continuous and therefore attains its minimum on the compact set X. □

Proposition 4.2 (see e.g. [?]). *The distance between the point $x = 0$ and the hyperplane $c^T x = k$ is $\frac{|k|}{\|c\|}$.*

Lemma 4.3. *Consider the boundary of the compact convex set*

$$X := \bigcap_{i\in I} \{x : c_i^T x \leq k_i\},$$

denoted by ∂X, then:

$$d(\partial X, 0) = \min_{i\in I} \frac{|k_i|}{\|c_i\|}.$$

Proof. Due to Proposition 4.2 we have:

$$d(\partial X, 0) = \inf_{i\in I} d\left(\{x : c_i^T x = k_i\}, 0\right) = \inf_{i\in I} \frac{|k_i|}{\|c_i\|}. \tag{12}$$

Note that due to Proposition 4.1, the $\inf_{x\in\partial X} d(x,0)$ is attained by a point $x^* \in \partial X$ due to compactness of ∂X. Hence, there exists an $i \in I$ such that $c_i^T x^* = k_i$ and this leads to: $\inf_{i\in I} \frac{|k_i|}{\|c_i\|} = \min_{i\in I} \frac{|k_i|}{\|c_i\|}$. □

We define the function $\Delta : 2^{\mathrm{S}} \to [0,1]$ as:

$$\Delta(X) = \sup\{r : \mathrm{B}_r \subseteq \text{convhull } (\mathrm{S} \setminus X)\}. \tag{13}$$

Lemma 4.4. *Given a set $X \subseteq S$, let convhull $(S \setminus X) := B \cap \mathcal{P}$, where $\mathcal{P} := \bigcap_{i\in I} \{x : c_i^T x \leq k_i\}$. Then we have:*

$$\Delta(X) = \min_{i\in I} \frac{|k_i|}{\|c_i\|}.$$

Proof.

$$\begin{aligned}
\Delta(X) &= \sup\{r : \mathrm{B}_r \subseteq \text{convhull } (\mathrm{S} \setminus X)\} \\
&= \min\left(d(\partial\mathcal{P}, 0), d(\mathrm{S}, 0)\right) \\
&= \min\left(\min_{i\in I} \frac{|k_i|}{\|c_i\|}, 1\right) \qquad \text{due to Lemma 4.3.}
\end{aligned} \tag{14}$$

□

Lemma 4.5. *$\Delta(\mathcal{C}_{c,k_1}) < \Delta(\mathcal{C}_{c,k_2})$ when $k_1 < k_2$.*

Proof.

$$\Delta(\mathcal{C}_{c,k}) = \text{convhull } (\mathrm{S}\backslash\mathcal{C}_{c,k}) = \text{convhull } (\mathrm{S}\backslash\{x \in \mathrm{S} : c^T x > k\}) = \{x \in \mathrm{B} : c^T x \leq k\}.$$

Due to Lemma 4.4, we have $\Delta(X) = \frac{|k|}{\|c\|}$. Then since $\frac{|k_1|}{\|c\|} < \frac{|k_2|}{\|c\|}$ when $k_1 < k_2$, the result of the lemma follows. □

Lemma 4.6. *$\sigma^{n-1}(\mathcal{C}_{c,k_1}) < \sigma^{n-1}(\mathcal{C}_{c,k_2})$, for $k_1 > k_2$.*

Proof. convhull $(\mathrm{S} \setminus \{x \in \mathrm{S} : c^T x > k_1\}) \subseteq$ convhull $(\mathrm{S} \setminus \{x \in \mathrm{S} : c^T x > k_2\})$, for $k_1 > k_2$. □

Now we are ready to present the following lemma which is the key to proving our main result.

Lemma 4.7. *For any set $X \subseteq S$, there exists c and k such that $\mathcal{C}_{c,k}$ satisfies:*

$$\mathcal{C}_{c,k} \subseteq X,$$

and

$$\Delta(\mathcal{C}_{c,k}) = \Delta(X). \tag{15}$$

Proof. As discussed previously, we can represent convhull $(\mathrm{S} \setminus X)$ as

$$\text{convhull } (\mathrm{S} \setminus X) = \bigcap_{i \in I} \{x \in \mathrm{B} : c_i^T x \leq k_i\}.$$

Note that when $X = \emptyset$, the statement of the lemma trivially holds since we can always find a c and k such that $\mathcal{C}_{c,k} = \emptyset$, hence we assume $X \neq \emptyset$ for the rest of the proof. This implies $I \neq \emptyset$. Then due to Lemma 4.4, there exists $\ell \in I$ such that $\Delta(X) = \frac{|k_\ell|}{\|c_\ell\|}$. Now, consider the spherical cap $\mathcal{C}_{c_\ell,k_\ell}$. Note that again due to Lemma 4.4, we have $\Delta(\mathcal{C}_{c_\ell,k_\ell}) = \frac{|k_\ell|}{\|c_\ell\|} = \Delta(X)$.

We next show $\mathcal{C}_{c_\ell,k_\ell} \subseteq X$. We prove this by contradiction. Assume $x \in \mathcal{C}_{c_\ell,k_\ell}$ and $x \notin X$. Note that, if $x \notin X$, then $x \in \mathrm{S}\backslash X \subseteq$ convhull $(\mathrm{S}\backslash X)$. Since $x \in \mathcal{C}_{c_\ell,k_\ell}$ we have $c_\ell^T x > k_\ell$, but due to the fact that $x \in$ convhull $(\mathrm{S} \setminus X)$, we also have $c_\ell^T x \leq k_\ell$, which leads to a contradiction. Therefore, $\mathcal{C}_{c_\ell,k_\ell} \subseteq X$. □

We now prove our main result.

Theorem 4.8. *Let $X_{\epsilon'} = \{X \subset S : \sigma^{n-1}(X) = \epsilon'\}$. Then, for any $\epsilon' \in (0,1)$, the function $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ when X is a spherical cap.*

Proof. We prove this via contradiction. Assume that $X^* \in X_{\epsilon'}$, X^* is not a spherical cap and $\arg\min_{X \in X_{\epsilon'}}(\Delta(X)) = X^*$. Due to Lemma 4.7 we can construct a spherical cap $\mathcal{C}_{c,k}$ such that $\mathcal{C}_{c,k} \subseteq X^*$ and $\mathcal{C}_{c,k} = \Delta(X^*)$. Note that, we further have $\mathcal{C}_{c,k} \subset X^*$, since X^* is assumed not to be a spherical cap. This means that $\sigma^{n-1}(\mathcal{C}_{c,k}) < \epsilon'$.

Then, the spherical cap $\mathcal{C}_{c,\tilde{k}}$ with $\sigma^{n-1}(\mathcal{C}_{c,\tilde{k}}) = \epsilon'$, satisfies $\tilde{k} < k$, due to Lemma 4.6. This implies $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$ due to Lemma 4.5. Therefore, $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(X^*)$. This is a contradiction since we initially assumed that $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ at X^*. □

Theorem 4.9. *Given a spherical cap $\mathcal{C}_{c,k} \subseteq S$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon'$,*

$$\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)},$$

where $\alpha := I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Here I^{-1} is the inverse incomplete beta function, i.e., $I^{-1}(y,a,b) = x$ where $I_x(a,b) = y$.

Proof. Let $h := 1 - \Delta(\mathcal{C}_{c,k})$. It is well known [3] that the area of the spherical cap $\mathcal{C}_{c,k} \subseteq \mathrm{S}$ is given by the equation:

$$\epsilon' = \sigma^{n-1}(\mathcal{C}_{c,k}) = \frac{\pi^{d/2}}{\Gamma[\frac{d}{2}]} I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right), \tag{16}$$

where I is the incomplete beta function. From this, we get the following set of equations:

$$\begin{aligned} \frac{\epsilon'\Gamma[\frac{d}{2}]}{\pi^{d/2}} &= I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right) \\ 2h - h^2 &= I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right) \\ 2h - h^2 &= \alpha \\ h^2 - 2h + \alpha &= 0. \end{aligned} \tag{17}$$

From (17), we get $h = 1 \pm \sqrt{(1-\alpha)}$. Since $h \leq 1$, we conclude that $\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)}$. Note that, $\Delta(\mathcal{C}_{c,k})$ only depends on ϵ for fixed n. □

Corollary 4.10. *When $N \to \infty$, $\Delta(X) \to 1$.*

Proof. To be proved: $\epsilon \to 0$.

Then by our assumption $\frac{\lambda_{\max}}{\lambda_{\min}}$ is bounded so ϵ' tends to 0. By continuity and monotonicity of I^{-1} in its first parameter, Δ tends to 1.

□

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