Sampling Based Stability Analysis of Switched Linear Systems

1 Introduction

As our computational resources have increased, so is the complexity of the models we use for the analysis of dynamical systems. Today, the industrial models do not only consist of simple differential or difference equations; these models are multimodal, hybrid, and contain a variety of subcomponents such as lookup tables, delay differential equations, actuation delays, and thermodynamic models. The current modeling paradigm is also highly distributed, in the sense that, the model subcomponents are developed by different parties. Therefore, their internal structure is partially, if not completely unknown to the end user. Hence, it is often virtually impossible to obtain simple analytical formulas for today's industrial scale models. However, once again thanks to the advances in our computational power, in spite of this increased complexity, it might still be possible to efficiently obtain simulation data from these models. Therefore, it is a natural question to ask whether we can provide formal analyses about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, given a dynamical system as in:

$$x_{k+1} = f(k, x_k), \tag{1}$$

where, $x_k \in \mathbb{R}^n$, k is index of time. We ask the following question: *given N input-output pairs, (x_1, y_1), (x_2, y_2), ..., (x_N, y_N) such that $y_k = f(k, x_k)$, what can we say about the stability of the system* (1)*?* Note that, one approach to this problem is firstly identifying the dynamics, i.e., f and then applying the existing techniques in the model- based stability analysis literature. However, there are two main reasons behind our quest to directly work on input-output pairs and bypassing the identification phase: (1) Even when the function f is known, the stability analysis is still hard. (2) Except a few simple subclasses

such as linear systems, the existing identification techniques can only identify f up to an approximation error. How to relate this identification error to an error in the stability of the system (1) is still a nontrivial problem.

The initial ideas behind this paper was born with the recent efforts on using simulation traces to find Lyapunov functions in [**?**], [**?**] and [**?**]. In all of these work, the main idea is that if one can construct a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(y_i) \leq V(x_i)$, for all $i \in \{1, 2, \ldots, N\}$.

. Topcu et.al. [**?**], Kapinski et.al. [**?**]construct Lyapunov function candidates using the simulation traces, however to be able to formally verify the constructed Lyapunov function, they require the knowledge of the full dynamics. In [**?**] and [**?**] Bobiti and Lazar address this and provide sampling based probabilistic and deterministic guarantees of a given Lyapunov function candidate.

2   Preliminaries

We consider the usual Hilbert finite normed vector space $(\mathbb{R}^n, \ell_2)$, $n \in \mathbb{N}_{>0}$, ℓ_2 the classical euclidean norm. We denote a unit ball in $\mathbb{R}^n$ with B and unit sphere in $\mathbb{R}^n$ of radius r as S. We only denote the radius r explicitly as in B_r and S_r, when r is different than 1. We denote the set of real symmetric matrices of size n by $\mathbb{S}^n$, and the set of linear functions in $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$. We denote the ellipsoid described by the matrix $P \in \mathbb{S}^n$ as E_P. We denote the homothety of ratio λ by $\mathcal{H}_\lambda$.

We denote by M the set $\{1, 2, \ldots, m\}$, $(m \in \mathbb{N}_{>0})$ and consider a switched linear system of modes $\mathcal{M} = \{A_1, A_2, \ldots, A_m\} \subset \mathbb{R}^{n \times n}$ indexed by M. We observe the system:

$$x_{k+1} = A_{\sigma(k)}x_k, \tag{2}$$

where, $x_k \in \mathbb{R}^n$, k is index of time and $\sigma : \mathbb{N} \to M$ is the switching sequence. Let $y_k := x_{k+1}$. We assume that we only know the number of modes m, the input x_k and the output y_k at each time k. We ignore what are the matrices and the index of the matrix which is applied at every time. We consider the following problem.

3   Optimization Problem

We are interested in solving the following optimization problem for a given $\gamma \in (0,1)$:

$$
\begin{aligned}
&\text{find} &&P\\
&\text{subject to} &&(A_i s)^T P(A_i s) \leq \gamma^2 s^T P s, \quad \forall i = \{1,2,\ldots,m\}, \forall\, s \in \mathrm{S}, \quad (3)\\
& &&P \succ 0.
\end{aligned}
$$

Note that if P is a solution to (3), then so is αP for any $\alpha \in \mathbb{R}_{>0}$. Therefore, we can rewrite (3) as the following optimization problem:

$$
\begin{aligned}
&\text{find} &&P\\
&\text{subject to} &&(A_k s)^T P(A_k s) \leq \gamma^2 s^T P s, \quad \forall i = \{1,2,\ldots,m\}, \forall\, s \in \mathrm{S}, \quad (4)\\
& &&P \succeq I.
\end{aligned}
$$

We define the linear isomorphism Φ as the natural mapping $\Phi : \mathbb{R}^{\frac{n(n+1)}{2}} \to \mathbb{S}^n$. Using this mapping, for a fixed $\gamma \in (0,1]$ we can rewrite (4) as:

$$
\begin{aligned}
&\text{find} &&p\\
&\text{subject to} &&f(p,x) \leq 0, \forall x \in X.
\end{aligned}
\tag{5}
$$

where $f(p,x) = \max(f_1(p,x), f_2(p))$, and

$$
\begin{aligned}
f_1(p,x) &:= (A_k s)^T \Phi(p)(A_k s) - \gamma^2 s^T \Phi(p)s\\
f_2(p) &:= \lambda_{\max}(\Phi(-p)) + 1.
\end{aligned}
$$

Proposition 3.1. *The optimization problem* (5) *is convex.*

Proof. The function $f_1(p,x)$ is clearly convex in p for a fixed $x \in X$. The function $\lambda_{max} : \mathbb{S}^n \to \mathbb{R}$ maps a symmetric positive matrix to its maximum eigenvalue. It is well-known that the function $\lambda_{\max}$ is a convex function of P. [?]. This means that, $p \mapsto \Phi(\lambda_{\max}(p))$ is convex in p. Moreover, maximum of convex functions is also convex, which shows that $f(p,x)$ is convex in p. $\square$

Note that the optimization problem (5) has infinitely many constraints. We next consider the following optimization problem where we sample N constraints of (5) independently and identically with the probability measure $\mathbb{P}(A) = \frac{\mu(A)}{\mu(X)}, \forall\, A \in \Sigma$, where $N \geq d+1$, and $d := \frac{n(n+1)}{2}$. We denote this sampling by $\omega := \{x_1, x_2, \ldots, x_N\} \subset X$, and obtain the following convex optimization problem $\mathrm{Opt}(\omega)$:

$$\begin{aligned} \text{find} \quad & p \\ \text{subject to} \quad & f(p, x) \leq 0, \forall x \in \omega. \end{aligned} \tag{6}$$

Let $p^*(\omega)$ be the solution of $\text{Opt}(\omega)$. We are interested in the probability of $p^*(\omega)$ violating at least one constraint in the original problem (5). Therefore, we define constraint violation property next.

Constraint violation probability [?] The constraint violation probability is defined as:

$$\mathcal{V}^*(\omega) = \begin{cases} \mathbb{P}\{x \in X : f(p^*(\omega), x) > 0\} & \text{if } \omega \in X^{N*}, \\ 1, & \text{otherwise} \end{cases}$$

where $X^{N*} := \{\omega \in X^N : \text{the solution of } \text{Opt}(\omega) \text{ exists}\}$. Note that, since we have $\mathbb{P}(A) = \frac{\mu(A)}{\mu(X)}$, we can rewrite this as:

$$\mathcal{V}^*(\omega) = \begin{cases} \frac{\mu\{x \in X : f(p^*(\omega), x) > 0\}}{\mu(X)} & \text{if } \omega \in X^{N*}, \\ 1, & \text{otherwise} \end{cases}$$

We make the following assumptions on the problem $\text{Opt}(\omega)$:

1. Uniqueness of solution: Note that this can be enforced by adding a tie-break rule of at most $\frac{n(n-1)}{2}$ convex conditions discriminating our solutions.

2. Nondegenaracy: with probability 1, there is no redundancy in the constraint obtained from the sampling.

The following theorem from [?] explicitly gives a relationship between $V^*(\omega)$ and N, n.

Theorem 3.2 (from [?]). *Consider the optimization problem $\text{Opt}(\omega)$ given in (6). Let Assumption 1 and Assumption 2 hold. Then, for all $\epsilon \in (0, 1)$ the following holds:*

$$\mathbb{P}^N\{\{\mathcal{V}^*(\omega) \leq \epsilon\} \cap X^{N*}\} \geq 1 - \sum_{j=0}^{d} \binom{N}{j} \epsilon^j (1 - \epsilon)^{N-j}.$$

Note that $\epsilon = 1 - I^{-1}(\beta, N - d, d + 1)$ and can be interpreted as the ratio of the measure of points in X that might violate at least one of the constraints in (4) to the measure of all points in X.

We now state our main theorem, which is based on Theorem 3.2 and devote the next section to proving it step by step. We denote by γ^*, the optimum value of the following optimization problem:

$$\min_{P,\gamma} \quad \gamma$$
$$\text{subject to} \quad (A_i s)^T P(A_i s) \leq \gamma^2 s^T P s, \quad \forall i = \{1, 2, \ldots, m\}, \forall s \in \mathrm{S}, \quad (7)$$
$$P \succ 0.$$

Theorem 3.3 (Main Theorem). *For any $\eta > 0$, given $N \geq n + 1$ and $\beta \in [0, 1)$, we can compute $\delta < \infty$ such that with probability at least β, $\rho \leq \delta(1 + \eta)\gamma^*$. Moreover, as $N \to \infty$, $\delta \to 1$.*

4 Relating the measure of bad sets

For a given sampling $\omega \in X^{N*}$, let $V := \{x \in X : f(p^*(\omega), x) > 0\}$, i.e., the set of points for which at least one constraint is violated, and V_S, V_M be its projections on S and M, respectively.

Lemma 4.1. $\mu_S(V_S) \leq \frac{\mu(V)}{m_1}$, *where* $m_1 = \min\{\mu_M(\{k\}), k \in M\}$.

Proof. Let $A \subset X$, $A_\mathrm{S} = \pi_\mathrm{S}(A)$ and $A_M = \pi_M(A)$. We notice that Σ_M is the disjoint union of its 2^m elements $\{B_i, i \in \{1, 2, \ldots 2^m\}\}$. Then A is the disjoint union $A = \sqcup_{1 \leq i \leq 2^m}(A_i, B_i)$ where $A_i = \pi_M^{-1}(B_i) \in \mathrm{S}$. We notice that $A_\mathrm{S} = \sqcup_{1 \leq i \leq 2^m} A_i$, and

$$\mu_\mathrm{S}(A_\mathrm{S}) = \sum_{1 \leq i \leq 2^m} \mu_\mathrm{S}(A_i).$$

We have

$$\mu(A) = \mu(\sqcup_{1 \leq i \leq 2^m}(A_i, B_i)) \;\; = \;\; \sum_{1 \leq i \leq 2^m} \mu((A_i, B_i))$$
$$= \;\; \sum_{1 \leq i \leq 2^m} \mu_\mathrm{S} \otimes \mu_M((A_i, B_i))$$
$$= \;\; \sum_{1 \leq i \leq 2^m} \mu_\mathrm{S}(A_i)\mu_M(B_i).$$

Let m_1 be the minimum value of μ_M on its atoms: $m_1 = \min\{\mu_M(\{k\}), k \in M\}$ (recall that $m_1 > 0$). Then since $\forall i$, $\mu_M(B_i) \geq m_1$, we have

$$\mu_\mathrm{S}(A_\mathrm{S}) \leq \frac{\mu(A)}{m_1}. \tag{8}$$

This proves our statement by taking $A = V_\mathrm{S}$. $\qquad\square$

Corollary 4.2. *When the modes are sampled from the set M uniformly random,*

$$\mu_S(V_S) \leq m\mu(V).$$

We consider the linear transformation mapping S to E_P that denoted by $L \in \mathcal{L}(\mathbb{R}^n)$. Note that since $P \in \mathbb{S}^n$, it can be written in its Choleski form $P = UDU^{-1}$, where D diagonal matrix of its eigenvalues, and $U \in O_n(\mathbb{R})$. We define $D^{1/2}$ the positive square root of D as the matrix $\mathrm{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$. Then, the positive square root of P is $VD^{1/2}V$. This means that, $L = P^{1/2}$. For the rest of the write-up, we denote

$$V' := \Pi_S(L^{-1}(V_S)),$$

and show how to upper bound $\sigma^{n-1}(V')$ in terms of $\mu(V)$.

Lemma 4.3. *Let ψ a smooth change of coordinates in $\mathbb{R}^n$ and $\mathcal{D} \subset S$, whose image under ψ is $\mathcal{D}' \subset \psi(S)$. Let μ_S be a positive spherical measure induced by a measure μ on $\mathbb{R}^n$. Let Σ_E and μ_E be the σ-algebra and the measure induced from Σ_S and μ_S on the ellipsoid $E = \psi(S)$. Then*

$$\mu_E(\psi(V_S)) = |\det(\psi)|\mu_S(V_S), \tag{9}$$

where $\psi \in \mathcal{L}(\mathbb{R}^n)$.

Proof. We have $\mu_S(\mathcal{D}) = \int_{x \in \mathcal{D}} \mathbb{1}_D(x) \, d\mu_S(x)$, $\mu_S = \{\phi_{i*}(\mu)\}_i$ and

$$\mu(\mathcal{D}') = \int_{y \in \mathcal{D}'} \mathbb{1}_{D'}(y) \, d\mu(y) = \int_{x \in \mathcal{D}} \mathbb{1}_{x \in \mathcal{D}} |\det J(\phi(x))| \, d\mu(x).$$

This gives

$$\mu_E(\mathcal{D}') = \int_{y \in \mathcal{D}'} \mathbb{1}_{D'}(y) \, d\mu_E(y) = \int_{x \in \mathcal{D}} \mathbb{1}_{x \in \mathcal{D}} |\det J(\psi(x))| \, d\mu_S(x).$$

In particular, if $\psi \in \mathcal{L}(\mathbb{R}^n)$, then $\forall x \in \mathbb{R}^n$, $\det(J(\psi(x)) = \det(\psi)$ and

$$\mu_E(\mathcal{D}') = \int_{y \in \mathcal{D}'} \mathbb{1}_{D'}(y) \, d\mu_E(y) = |\det(\psi)| \int_{x \in \mathcal{D}} \mathbb{1}_{x \in \mathcal{D}} \, d\mu_S(x).$$

This proves the statement of the lemma when $\mathcal{D} = V_S$. $\square$

Definition Let X be a Hilbert space, A a nonempty subset of X and $\psi : A \to X$. Then ψ is called firmly nonexpansive if

$$\forall x, y \in A, \|\psi(x) - \psi(y)\|^2 + \|(\mathrm{Id} - \psi)(x) - (\mathrm{Id} - \psi)(y)\|^2 \leq \|x - y\|^2,$$

where Id denotes the identity function from X to X.

Theorem 4.4 (from [?]). *Let C be a nonempty closed convex subset of X, then the convex projector on C, Π_C, is firmly nonexpansive.*

Corollary 4.5.

$$\|\Pi_C(x) - \Pi_C(y)\| \leq \|x - y\| \qquad \forall\, x, y \in C. \tag{10}$$

Lemma 4.6.

$$\mu_S(\Pi_S(L^{-1}(V_S))) \leq \det(L^{-1}) \left(\frac{1}{\lambda_{\min}(L^{-1})}\right)^n \mu_S(V_S). \tag{11}$$

Proof. Note that the mapping Π_S can be seen as the composition of the Π_{S_r} for some $r > 0$, and $\mathcal{H}_{\frac{1}{r}}$. Let $E' := L^{-1}(S)$, then when $r < \min_{x \in E'} \|x\| = \lambda_{\min}(L^{-1})$ we have

$$\Pi_{S_{\lambda_{\min}}}(x) = \Pi_{B_{\lambda_{\min}}}(x) \qquad \forall\, x \in E'.$$

This shows that the restriction of $\Pi_{S_{\lambda_{\min}}}$ to E' is a convex projector.

Then by Corollary 4.5

$$\|\Pi_{S_{\lambda_{\min}}}(x) - \Pi_{S_{\lambda_{\min}}}(y)\| \leq \|x - y\|, \qquad \forall\, x, y \in E'. \tag{12}$$

This shows that 1 is a Lipschitz constant of the function $\Pi_{S_{\lambda_{\min}}}$ on E'.

By composing $\Pi_{S_{\lambda_{\min}}}$ with $\mathcal{H}_{\frac{1}{\lambda_{\min}}}$, we obtain Π_S. Since the Lipschitz constant of composition of two functions can be bounded by the multiplication of Lipschitz constants of each function, the Lipschitz constant of Π_S on E' is $\frac{1}{\lambda_{\min}}$, which means that:

$$\|\Pi_S(x) - \Pi_S(y)\| \leq \frac{1}{\lambda_{\min}} \|x - y\|, \qquad \forall\, x, y \in E'. \tag{13}$$

Note that, the inequality in (13) is an equality when x is in the eigenspace of $\lambda_{\min}$ and $y = -x$.

Recall that for any smooth Lipschitz function ϕ with Lipschitz constant, $\text{Lip}(\phi)$, we have for all x, $|\det(J(\phi(x))| \leq \text{Lip}(\phi)^n$. Combining this with (13) and Lemma 4.6, we get the statement of the lemma. □

Theorem 4.7. $\sigma^{n-1}(V') \leq m\epsilon \sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, *where* $\mu(V) = \epsilon$.

Proof. By taking μ_S as the uniform spherical measure σ^{n-1}, and combining Corollary 4.2 with Lemma 4.6 we get the statement of the theorem. □

5   Relating ϵ to δ

We denote $\epsilon' := \frac{\epsilon}{2}\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, where the additional factor $\frac{1}{2}$ follows from the homogeneity of the dynamics. In this section, we show how to relate ϵ' to δ in the statement of the Theorem 3.3. We start by a few definitions that will help us along the way. Let d be a distance on $\mathbb{R}^n$. We define the distance between a set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ is $d(X, p) := \inf_{x \in X} d(x, p)$.

Spherical Cap We define the *spherical cap* on S for a given hyperplane $c^T x = k$ as:
$$\mathcal{C}_{c,k} := \{x \in \mathrm{S} : c^T x > k\}.$$

Proposition 5.1 (see e.g. [?]). *The distance between the point $x = 0$ and the hyperplane $c^T x = k$ is $\frac{|k|}{\|c\|}$.*

We define the function $\Delta : 2^{\mathrm{S}} \to [0, 1]$ as:
$$\Delta(X) := \sup\{r : \mathrm{B}_r \subseteq \mathrm{convhull}\ (\mathrm{S} \setminus X)\}. \tag{14}$$

Note that, $\Delta(X)$ can be rewritten as in:
$$\Delta(X) = d(\partial\mathrm{convhull}\ (\mathrm{S} \setminus X), 0). \tag{15}$$

Lemma 5.2. $\Delta(\mathcal{C}_{c,k}) = \min\left(1, \frac{|k|}{\|c\|}\right)$.

Proof. Note that $\mathrm{convhull}\ (\mathrm{S} \setminus X) = \{x \in \mathrm{B} : c^T x \leq k\}$.
$$
\begin{aligned}
\Delta(X) &= d(\partial\mathrm{convhull}\ (\mathrm{S} \setminus X), 0) \\
&= \min(d(\partial\mathrm{B}, 0), d(\partial\{x : c^T x \leq k\}, 0)) \\
&= \min(d(\mathrm{S}, 0), d(\{x : c^T x = k\}, 0)) \\
&= \min\left(1, \frac{|k|}{\|c\|}\right).
\end{aligned}
$$
$\square$

Corollary 5.3. $\Delta(\mathcal{C}_{c,k_1}) < \Delta(\mathcal{C}_{c,k_2})$ *when* $k_1 < k_2$.

Lemma 5.4. $\sigma^{n-1}(\mathcal{C}_{c,k_1}) < \sigma^{n-1}(\mathcal{C}_{c,k_2})$, *for* $k_1 > k_2$.

Proof. $\mathrm{convhull}\ (\mathrm{S} \setminus \{x \in \mathrm{S} : c^T x > k_1\}) \subseteq \mathrm{convhull}\ (\mathrm{S} \setminus \{x \in \mathrm{S} : c^T x > k_2\})$, for $k_1 > k_2$. $\square$

Now we are ready to present the following lemma which is the key to proving our main result.

Lemma 5.5. *For any set $X \subseteq S$, there exists c and k such that $\mathcal{C}_{c,k}$ satisfies:*

$$\mathcal{C}_{c,k} \subseteq X,$$

and

$$\Delta(\mathcal{C}_{c,k}) = \Delta(X). \tag{16}$$

Proof. Let $X_S := \text{convhull}\,(S \backslash X)$. Since the distance function d is continuous and the set ∂X_S is compact there exists a point $x^* \in \partial X_S$, such that:

$$\Delta(X) = d(\partial X_S, 0) = \inf_{x \in \partial X_S} d(x, 0) = \min_{x \in \partial X_S} d(x, 0) = d(x^*, 0). \tag{17}$$

Next, consider the supporting hyperplane of X_S at x^*, which we denote by $\left\{x : c^T x = k\right\}$. Note that, this supporting hyperplane is unique because it is also a supporting hyperplane of the ball $\mathrm{B}_{\Delta(X)}$ at x^* as well, which is unique. This can be seen from the fact that:

$$\partial \mathrm{B}_{\Delta(X)} \subseteq \partial X_S \subseteq \left\{x : c^T x = k\right\}.$$

Then we have:

$$\Delta(X) = d(x^*, 0) = d(\{x : c^T x = k\}, 0) = \frac{|k|}{\|c\|}.$$

Now, consider the spherical cap $\mathcal{C}_{c,k}$. Then, by Lemma we have $\Delta(\mathcal{C}_{c,k}) = \frac{|k|}{\|c\|}$. Therefore, $\Delta(X) = \Delta(\mathcal{C}_{c,k})$.

We next show $\mathcal{C}_{c,k} \subseteq X$. We prove this by contradiction. Assume $x \in \mathcal{C}_{c,k}$ and $x \notin X$. Note that, if $x \notin X$, then $x \in \mathrm{S} \setminus X \subseteq \text{convhull}\,(\mathrm{S} \setminus X)$. Since $x \in \mathcal{C}_{c,k}$ we have $c^T x > k$. But due to the fact that $x \in \text{convhull}\,(\mathrm{S} \setminus X)$, we also have $c^T x \leq k$, which leads to a contradiction. Therefore, $\mathcal{C}_{c,k} \subseteq X$. $\quad\square$

We now prove our main result.

Theorem 5.6. *Let $X_{\epsilon'} = \{X \subset S : \sigma^{n-1}(X) = \epsilon'\}$. Then, for any $\epsilon' \in (0,1)$, the function $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ for some X which is a spherical cap.*

Proof. We prove this via contradiction. Assume that there exists no spherical cap in $X_{\epsilon'}$ such that $\Delta(X)$ attains its minimum. This means there exists an $X^* \in X_{\epsilon'}$, where X^* is not a spherical cap and $\arg\min_{X \in X_{\epsilon'}}(\Delta(X)) = X^*$.

By Lemma 5.5 we can construct a spherical cap $\mathcal{C}_{c,k}$ such that $\mathcal{C}_{c,k} \subseteq X^*$ and $\mathcal{C}_{c,k} = \Delta(X^*)$. Note that, we further have $\mathcal{C}_{c,k} \subset X^*$, since X^* is assumed not to be a spherical cap. This means that, there exists a spherical cap $\sigma^{n-1}(\mathcal{C}_{c,k})$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) < \epsilon'$.

Then, the spherical cap $\mathcal{C}_{c,\tilde{k}}$ with $\sigma^{n-1}(\mathcal{C}_{c,\tilde{k}}) = \epsilon'$, satisfies $\tilde{k} < k$, due to Lemma 5.4. This implies $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$ due to Lemma 5.3. Therefore, $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(X^*)$. This is a contradiction since we initially assumed that $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ at X^*. $\square$

Theorem 5.7. *Given a spherical cap $\mathcal{C}_{c,k} \subseteq S$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon'$,*

$$\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)},$$

where $\alpha := I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Here I^{-1} is the inverse incomplete beta function, i.e., $I^{-1}(y, a, b) = x$ where $I_x(a, b) = y$.

Proof. Let $h := 1 - \Delta(\mathcal{C}_{c,k})$. It is well known [?] that the area of the spherical cap $\mathcal{C}_{c,k} \subseteq S$ is given by the equation:

$$\epsilon' = \sigma^{n-1}(\mathcal{C}_{c,k}) = \frac{\pi^{d/2}}{\Gamma[\frac{d}{2}]}I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right), \tag{18}$$

where I is the incomplete beta function. From this, we get the following set of equations:

$$\frac{\epsilon'\Gamma[\frac{d}{2}]}{\pi^{d/2}} = I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right)$$

$$2h - h^2 = I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right)$$

$$2h - h^2 = \alpha$$

$$h^2 - 2h + \alpha = 0. \tag{19}$$

From (19), we get $h = 1 \pm \sqrt{(1-\alpha)}$. Since $h \leq 1$, we conclude that $\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)}$. Note that, $\Delta(\mathcal{C}_{c,k})$ only depends on ϵ for fixed n. $\square$

Corollary 5.8. *For a fixed $\beta \in (0, 1)$, $\lim_{N\to\infty} \delta_\beta(N) = 1$.*

Proof. We first prove that $\lim_{N\to\infty} \epsilon_\beta(N) = 0$. Note that, we can upper bound $1 - \beta$ as follows:

$$\binom{1-\beta = \sum_{j=0}^d N}{j\epsilon^j(1-\epsilon)^{N-j} \leq (d+1)N^d(1-\epsilon)^{N-d}.} \tag{20}$$

We prove $\lim_{N\to\infty} \epsilon_\beta(N) = 0$ by contradiction and assume that $\lim_{N\to\infty} \epsilon_\beta(N) \neq 0$. This means that, there exists some $\delta > 0$ such that $\epsilon_\beta(N) > \delta$ infinitely often. Then, consider the subsequence N_k such that $\epsilon_\beta(N_k) > \delta$, $\forall\, k$. By (20) we have:

$$1 - \beta \leq (d+1)N_k^d(1-\epsilon)^{N_k-d} \leq (d+1)N_k^d(1-\delta)^{N_k-d} \,\forall\, k \in \mathbb{N}.$$

Note that $\lim_{k\to\infty}(d+1)N_k^d(1-\delta)^{N_k-d} = 0$. Therefore, there exists a k' such that, we have $(d+1)N_{k'}^d(1-\delta)^{N_k'-d} < 1 - \beta$, which is a contradiction. Therefore, we must have $\lim_{N\to\infty} \epsilon_\beta(N) = 0$.

Showing I^{-1} in its first parameter, $\delta = \sqrt{1 - \alpha}$ tends to 1 as $\epsilon \to 0$.

$\square$