

Data Driven Stability Analysis of Black-box Switched Linear Systems

Abstract

We address the problem of deciding stability of a “black-box” dynamical system (i.e., a system whose model is not known) from a set of observations. The only assumption we make on the black-box system is that it can be described by a switched linear system.

We show that, for a given (randomly generated) set of observations, one can give a stability guarantee, for some level of confidence, with a trade-off between the quality of the guarantee and the level of confidence. We provide an explicit way of computing the best stability guarantee, as a function of both the number of observations and the required level of confidence. Our results rely on geometrical analysis and combining chance-constrained optimization theory with stability analysis techniques for switched systems.

1 Introduction

Today’s complex cyber-physical systems are characterized by the interaction of a large number of heterogeneous components. Consequently, the models used to analyze these systems are equally complex and consist of heterogeneous sub-models relying on different assumptions and based on principles from different scientific disciplines. It is not uncommon to encounter a patchwork of differential equations, difference equations, hybrid automata, lookup tables, custom switching logic, low-level legacy code, etc. To further compound the difficulty in analyzing these systems, different components of a complex engineered system are typically designed by different suppliers. Although a high-level specification for these components may be known, detailed models are not available for intellectual property reasons. We are thus faced with a tremendous gap between the existing analysis techniques that rely on closed-form models and the models available in industry. It is, therefore, not surprising the emphasis that industry places on simulation since despite the complexity of models, it is always possible to simulate them. This raises

the question of whether we can provide formal guarantees about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, we consider a dynamical system as in:

$$x_{k+1} = f(k, x_k), \quad (1)$$

where, $x_k \in X$ is the state and $k \in \mathbb{N}$ is the time index. In this paper, we focus on switched systems, but we believe that the presented results can be extended to more general classes of dynamical systems. We start with the following question to serve as a stepping stone: For some $l \in \mathbb{N}_{>0}$, given N traces of length l , $(x_0^1, x_1^1, \dots, x_l^1)$, $(x_0^2, x_1^2, \dots, x_l^2)$, \dots , $(x_0^N, x_1^N, \dots, x_l^N)$ belonging to the behavior of the system (1), (i.e., $x_{k+1}^i = f(k, x_k^i)$ for any $k \in \{0, \dots, l-1\}$ and any $i \in \{1, \dots, N\}$), what can we say about the stability of the system (1)? For the rest of the paper, we use the term *black-box* to refer to models where we do not have access to the model, i.e., to f , yet we can indirectly learn information about f by observing traces of length l (becoming, in the particular case where $l = 1$, couples of points (x_k, y_k) as defined in (1)).

A potential approach to this problem is to first identify the dynamics, i.e., the function f , and then apply existing techniques from the model-based stability analysis literature. However, unless f is a linear function, there are two main reasons behind our quest to directly work on system behaviors and bypass the identification phase: 1) Even when the function f is known, in general, stability analysis is a very difficult problem [?]; 2) Identification can potentially introduce approximation errors, and can be algorithmically hard as well. Again, this is the case for switched systems [?]. A fortiori, the combination of these two steps in an efficient and robust way seems far from obvious.

In recent years, increasing number of researchers started addressing various verification and design problems in control of black-box systems [?, ?, ?, ?]. In particular, the initial idea behind this paper was influenced by the recent efforts in [?, ?], and [?] on using simulation traces to find Lyapunov functions for systems with known dynamics. In these works, the main idea is that if one can construct a Lyapunov function candidate decreasing along several finite trajectories starting from different initial conditions, it should also decrease along every other trajectory. Then, once a Lyapunov function candidate is constructed, this intuition is put to test by verifying the candidate function either via off-the-shelf tools as in [?] and [?], or via sampling-based techniques as in [?]. This also relates to almost-Lyapunov functions introduced in [?],

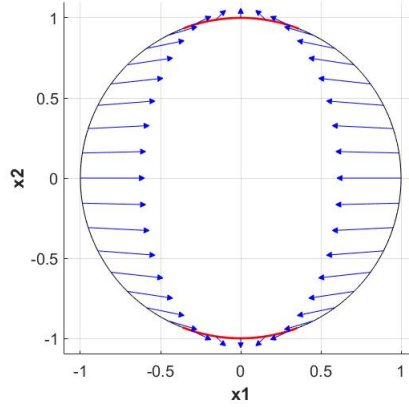


Fig. 1: A simple dynamics and the level set of an “almost Lyapunov function”. Even though this function decreases at almost all points in its level set, almost all trajectories diverge to infinity.

which presents a relaxed notion of stability proved via Lyapunov functions decreasing everywhere except on a small set. Note that, since we do not have access to the dynamics, these approaches cannot be directly applied to black-box systems. However, these ideas raise the following problem that we address in this paper: By observing that a candidate Lyapunov function decreases on a large number of observations, we empirically build a certain confidence that such candidate Lyapunov function is a bona-fide Lyapunov function. *Can we translate this confidence into a confidence that this Lyapunov function decreases at most of the points in the state space?*

Note that, even in the case of a 2D linear system, the connection between these two beliefs is nontrivial. In fact, one can easily construct an example where a candidate Lyapunov function decreases everywhere on its levels sets, except for an arbitrarily small subset, yet, almost all trajectories diverge to infinity. For example, the system

$$x^+ = \begin{bmatrix} 0.14 & 0 \\ 0 & 1.35 \end{bmatrix} x,$$

admits a Lyapunov function candidate on the unit circle except on the two red areas shown in Fig. 1. Moreover, the size of this “violating set” can be made arbitrarily small by changing the magnitude of the unstable eigenvalue. Nevertheless, the only trajectories that do not diverge to infinity are those starting on the stable eigenspace that has zero measure.

In this paper, we take the first steps to infer stability from observations

of switched linear systems. In addition to the preceding example, there are other reasons to temper our expectations for proving stability from data: identifying and deciding stability from arbitrary switched linear systems is NP-hard [?].

The stability of switched systems is closely related to the *joint spectral radius* (JSR) of the matrices modeling the dynamics in each mode. Deciding stability amounts to deciding whether the JSR is less than one [?]. In this paper, we present an algorithm to bound the JSR of a switched linear system from a finite number N of observations. This algorithm partly relies on tools from the random convex optimization literature (also known as chance-constrained optimization, see [?, ?, ?]), and provides an upper bound on the JSR with a user-defined confidence level. As N increases, this bound gets tighter. Moreover, with a closed form expression, we characterize what is the exact trade-off between the tightness of this bound and the number of samples. In order to understand the quality of our upper bound, the algorithm also provides a deterministic lower bound. Finally, we provide an asymptotic guarantee on the gap between the upper and the lower bound, for large N .

The organization of the paper is as follows: In Section 2, we introduce the notations and provide the necessary background in stability of switched systems. In Section 4, we present a deterministic lower bound for the JSR. Section 3 presents the main contribution of this paper where we provide a probabilistic stability guarantee for a given switched system, based on finite observations. We experimentally demonstrate the performance of the presented techniques in Section 5 and conclude in Section 6, while hinting at our related future work.

2 Preliminaries

2.1 Notation

We consider the usual finite normed vector space (\mathbb{R}^n, ℓ_2) , $n \in \mathbb{N}_{>0}$, with ℓ_2 the classical Euclidean norm. We denote the set of linear functions in \mathbb{R}^n by $\mathcal{L}(\mathbb{R}^n)$, and the set of real symmetric matrices of size n by \mathcal{S}^n . In particular, the set of positive definite matrices is denoted by \mathcal{S}_{++}^n . We write $P \succ 0$ to state that P is positive definite, and $P \succeq 0$ to state that P is positive semi-definite. Given a set $X \subset \mathbb{R}^n$, and $r \in \mathbb{R}_{>0}$ we write $rX := \{x \in X : rx\}$ to denote the scaling of ratio r of this set. We denote by \mathbb{B} (respectively \mathbb{S}) the ball (respectively sphere) of unit radius centered at the origin. We denote the ellipsoid described by the matrix $P \in \mathcal{S}_{++}^n$ as E_P , i.e., $E_P := \{x \in \mathbb{R}^n : x^T P x = 1\}$. Finally, we denote the spherical projector

on \mathbb{S} by $\Pi_{\mathbb{S}} := x/\|x\|$.

We consider in this work the classical uniform spherical measure on \mathbb{S} , denoted by σ^{n-1} , and derived from the Lebesgue measure λ (see the appendix for precise definitions). For $m \in \mathbb{N}_{>0}$, we denote by M the set $M = \{1, 2, \dots, m\}$ and we provide it with the uniform measure μ_M . For any $l \in \mathbb{N}_{>0}$, we denote by M^l the l -Cartesian product of M : $M^l = M \times M \times \dots \times M$, and similarly we provide it with the uniform measure μ_{M^l} . Finally, we define $Z_l = \mathbb{S} \times M^l$ as the Cartesian product of the unit sphere \mathbb{S} and M^l . On the set Z_l , we define the product measure $\mu_l = \sigma^{n-1} \otimes \mu_{M^l}$. Note that, μ_l is a uniform measure on Z_l and $\mu_l(Z_l) = 1$.

2.2 Stability of Switched Linear Systems

A *switched linear system* with a set of modes $\mathcal{M} = \{A_i, i \in M\}$ is of the form:

$$x_{k+1} = f(k, x_k), \quad (2)$$

with $f(k, x_k) = A_{\tau(k)}x_k$ and switching sequence $\tau : \mathbb{N} \rightarrow M$. There are two important properties of switched linear systems that we exploit in this paper.

Property 2.1. *Let $\xi(x, k, \tau)$ denote the state of the system (2) at time k starting from the initial condition x and with switching sequence τ . The dynamical system (2) is homogeneous: $\xi(\gamma x, k, \tau) = \gamma \xi(x, k, \tau)$.*

The joint spectral radius of the set of matrices \mathcal{M} closely relates to the stability of the system (2) and is defined as follows:

Definition 2.1 (from [?]). *Given a finite set of matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}$, its joint spectral radius (JSR) is given by*

$$\rho(\mathcal{M}) = \lim_{k \rightarrow \infty} \max_{i_1, \dots, i_k} \left\{ \|A_{i_1} \dots A_{i_k}\|^{1/k} : A_{i_j} \in \mathcal{M} \right\}.$$

Property 2.2 (Corollary 1.1, [?]). *Given a finite set of matrices \mathcal{M} , the corresponding switched dynamical system is stable if and only if $\rho(\mathcal{M}) < 1$.*

Property 2.3 (Proposition 1.3, [?]). *Given a finite set of matrices \mathcal{M} , and any invertible matrix T ,*

$$\rho(\mathcal{M}) = \rho(T\mathcal{M}T^{-1}),$$

i.e., the JSR is invariant under similarity transformations (and is a fortiori a homogeneous function: $\forall \gamma > 0$, $\rho(\mathcal{M}/\gamma) = \mathcal{M}/\gamma$).

Theorem 2.1. [?, Theorem 2.11] *For any finite set of matrices such that $\rho(\mathcal{M}) < \frac{1}{\sqrt{n}}$, there exists a Common Quadratic Lyapunov Function (CQLF) for \mathcal{M} , that is, a $P \succ 0$ such that:*

$$\forall A \in \mathcal{M}, A^T P A \preceq P.$$

CQLFs are useful because they can be computed (if they exist) with semidefinite programming (see [?]), and they constitute a stability guarantee for switched systems as we formalize next.

Theorem 2.2. [?, Prop. 2.8] *Consider a finite set of matrices \mathcal{M} . If there exist some $\gamma \geq 0$ and $P \succ 0$ such that*

$$\forall A \in \mathcal{M}, A^T P A \preceq \gamma^2 P,$$

then $\rho(\mathcal{M}) \leq \gamma$.

Note that the smaller α is, the tighter is the upper bound we get on $\rho(\mathcal{M})$. Therefore, we could consider, in particular, the optimal solution γ^* of the following optimization problem:

$$\begin{aligned} \min_{\gamma, P} \quad & \gamma \\ \text{s.t.} \quad & (Ax)^T P (Ax) \leq \gamma^2 x^T P x, \forall A \in \mathcal{M}, \forall x \in \mathbb{R}^n, \\ & P \succ 0. \end{aligned} \tag{3}$$

Note that, instead of considering the set of constraints x as being all \mathbb{R}^n , we could restrict it to the unit sphere \mathbb{S} . This is due to Property 2.1, since it implies that it is sufficient to show the decrease of a CQLF on an arbitrary set enclosing the origin, e.g., \mathbb{S} . Hence, we consider from now the following optimization problem, with its optimal solution that will also be γ^* :

$$\begin{aligned} \min_{\gamma, P} \quad & \gamma \\ \text{s.t.} \quad & (Ax)^T P (Ax) \leq \gamma^2 x^T P x, \forall A \in \mathcal{M}, \forall x \in \mathbb{S}, \\ & P \succ 0. \end{aligned} \tag{4}$$

The two theorems above provide us with a *converse Lyapunov result*: if there exists a CQLF, then our system is stable. If, on the contrary, there is no such stability guarantee, one may conclude a lower bound on the JSR. By combining these two results, one obtains an approximation algorithm for the JSR: the upper bound γ^* obtained above is within an error factor of $\frac{1}{\sqrt{n}}$ of the true value. It turns out that one can still refine this technique, in order to improve the error factor, and asymptotically get rid of it. This is a well-known technique for the “white-box” computation of the JSR, which we summarize in the following corollary:

Corollary 2.3. *For any finite set of matrices such that $\rho(\mathcal{M}) < \frac{1}{\sqrt[2l]{n}}$, there exists a CQLF for*

$$\mathcal{M}^l := \{A_{i_1}, \dots, A_{i_l} : A_i \in \mathcal{M}\},$$

that is, a $P \succ 0$ such that:

$$\forall A \in \mathcal{M}^l, A^T P A \preceq P.$$

Proof. It is easy to see from the definition of the JSR that

$$\rho(\mathcal{M}^l) = \rho(\mathcal{M})^l.$$

Thus, applying Theorem 2.1 to \mathcal{M}^l , one directly obtains the corollary. \square

We can then consider from now, for any $l \in \mathbb{N}_{>0}$, the following optimization problem, whose solution γ^* will be an upper bound on $\rho(\mathcal{M})$:

$$\begin{aligned} \min_{\alpha, P} \quad & \gamma \\ \text{s.t.} \quad & (Ax)^T P (Ax) \leq \gamma^{2l} x^T P x, \forall A \in \mathcal{M}^l, \forall x \in \mathbb{S}, \\ & P \succ 0. \end{aligned} \tag{5}$$

2.3 Problem Formulation

We now formally present the problem we will be considering from now. We recall that our observations are traces of the form $(x_k, x_{k+1}, \dots, x_{k+l})$ for some arbitrary $l \in \mathbb{N}_{>0}$, and that we do not have access to the mode applied to the system at each time step.

To generate these traces, we assume that we can randomly pick a finite number of initial conditions $x_0^i \in \mathbb{R}^n$, and that we can randomly generate a sequence of l modes applied to each of these points. Hence, the probability event corresponding to a given observed trace $(x_k, x_{k+1}, \dots, x_{k+l})$ is another $(l+1)$ -tuple (x_k, j_1, \dots, j_l) . More precisely, we assume that we can uniformly sample such $(l+1)$ -tuples in $Z_l = \mathbb{S} \times M^l$, giving us a sample denoted by

$$\omega_N := \{(x_0^1, j_{1,1}, \dots, j_{1,l}), (x_0^2, j_{2,1}, \dots, j_{2,l}), \dots, (x_0^N, j_{N,1}, \dots, j_{N,l})\} \subset Z_l.$$

By uniformly sampling, we mean that the points in ω_N are drawn according to the measure μ_l , i.e., the points x_0^i are drawn from \mathbb{S} according to the classical spherical measure σ^{n-1} , and the modes are drawn from M according to the classical uniform measure μ_M at each time step. The space

of sampling for the initial conditions is restricted to \mathbb{S} since, as we recall, by Property 2.1, the system is homogeneous. **Motivate why uniform sampling for modes.**

From a sample ω_N , we obtain the set of corresponding available observations $W_{\omega_N} := \{(x_0^1, x_1^1, \dots, x_l^1), (x_0^2, x_1^2, \dots, x_l^2), \dots, (x_0^N, x_1^N, \dots, x_l^N)\}$, which satisfy

$$x_l^i = A_{j_{i,1}} \dots A_{j_{i,l}} x_0^i, \quad \forall (x_0^i, x_1^i, \dots, x_l^i) \in W_{\omega_N}.$$

In this work, we aim at understanding what type of guarantees one can obtain on the stability of System (2) (that is, on the JSR of \mathcal{M}) from a finite, uniformly sampled, set of data. More precisely, we answer the following problem:

Problem 2.1. *Consider a finite set of matrices \mathcal{M} , describing a switched system (2), and suppose that one has a set of N observations*

$$\omega_N = \{(x_0^1, j_{1,1}, \dots, j_{1,l}), (x_0^2, j_{2,1}, \dots, j_{2,l}), \dots, (x_0^N, j_{N,1}, \dots, j_{N,l})\},$$

sampled according to the uniform measure μ_l on Z_l .

- *For a given number $\beta \in (0, 1)$, provide an upper bound $\overline{\rho(\omega_N)}$ on $\rho(\mathcal{M})$, together with a level of confidence β , that is, a number $\rho(\omega_N)$ such that*

$$\mu_l \left(\{\omega_N : \rho(\mathcal{M}) \leq \overline{\rho(\omega_N)}\} \right) \geq \beta.$$

- *For the same given level of confidence β , provide a lower bound $\underline{\rho(\mathcal{M})}$ on $\rho(\mathcal{M})$.*

In fact, we will see in Section 4 that there is no need for a such level of confidence β in the case of the lower bound; indeed, in Theorem 4.1, we derive a deterministic lower bound for a given (sufficiently high) number of observations.

The idea from now will be to leverage the fact that conditions for the existence of a CQLF for (2) can be obtained by considering a finite number of traces of the form $(x_k, x_{k+1}, \dots, x_l) \in \mathbb{R}^n$.

3 A Probabilistic Stability Guarantee

In this section, we show how to compute an upper bound on ρ , with a user-defined confidence $\beta \in (0, 1)$. We do this by constructing a l -step CQLF

which is valid with probability at least β . Note that, the existence of a l -step CQLF implies $\rho \leq 1$ due to Theorem 2.2.

In Section 2.2, we have given an optimization problem, (5), that provides a stability guarantee. Nevertheless, solving this problem as stated is not possible since it involves infinitely many constraints. We consider then the following optimization problem:

$$\begin{aligned} \min_P \quad & \gamma \\ \text{s.t.} \quad & (x_l^i)^T P(x_l^i) \leq \gamma^{2l} (x_0^i)^T P x_0^i, \quad \forall (x_0^i, \dots, x_l^i) \in W_{\omega_N} \\ & P \succ 0, \quad \gamma \geq 0. \end{aligned} \quad (6)$$

Note that, (6) can be efficiently solved by semidefinite programming and bisection on the variable γ (see [?]). Let us denote from now by $\gamma^*(\omega_N)$ the optimal solution of this problem, which we will use to compute a (probabilistic) upper bound on the JSR.

Let us analyze the relationship between the solutions of the theoretical optimization problem (5) and the practical version (6), with finitely many constraints. Even though in practice, one would solve the optimization problem (6) as suggested above, for the sake of rigor and clarity of our proofs, we introduce a slightly different optimization problem. In this new optimization problem, an objective function is considered, and a “regularization parameter”, $\eta > 0$, is added. As the reader will see, we will derive results valid for arbitrarily small values of η , and so this will not hamper the practical accuracy of our technique, while allowing us to derive a theoretical asymptotic guarantee (i.e. for large number of observations).

$$\begin{aligned} \min_P \quad & \lambda_{\max}(P) \\ \text{s.t.} \quad & (A_{j_1} \dots A_{j_l} x)^T P (A_{j_1} \dots A_{j_l} x) \leq ((1 + \eta) \gamma^*(\omega_N))^{2l} x^T P x, \\ & \quad \quad \quad \forall (x, j_1 \dots j_l) \in \omega_N \subset Z_l, \\ & P \succeq I, \end{aligned} \quad (7)$$

where $Z_l := \mathbb{S} \times M^l$, $\eta > 0$, and $\gamma^*(\omega_N)$ is the optimal solution to the optimization problem (6). Recall that ω_N is a given uniform random sample on the set Z_l and of size N .

For the rest of the discussion, we refer to the optimization problem (7) by $\text{Opt}(\omega_N)$. We denote its optimal solution by $P(\omega_N)$. We drop the explicit dependence of P on ω_N when it is clear from the context. There are a few points that are worth noting about (7). Firstly, due to Property 2.1, we can replace the constraint $P \succ 0$ with the constraint $P \succeq I$. Moreover, for

reasons that will become clear later in the discussion, we chose the objective function as $\lambda_{\max}(P)$, instead of solving a feasibility problem in P . Lastly, the additional η factor is introduced to ensure strict feasibility of (7), which will be helpful in the following discussion.

The curious question whether the optimal solution of the sampled problem $\text{Opt}(\omega_N)$ is a feasible solution to (4) has been widely studied in the literature [?]. It turns out that under certain technical assumptions, one can bound the proportion of the constraints of the original problem (4) that are violated by the optimal solution of (??), with some probability which is a function of the sample size N .

In the following theorem, we adapt a classical result from random convex optimization literature to our problem.

Shoud I write there the 2 assumptions uniqueness+nondegeneracy

Theorem 3.1 (adapted from Theorem 3.3, [?]). *Let d be the dimension of $\text{Opt}(\omega_N)$ and $N \geq d + 1$. Consider the optimization problem $\text{Opt}(\omega_N)$ given in (??), where ω_N is a uniform random sample drawn from the set Z_l . Then, for all $\epsilon \in (0, 1)$ the following holds:*

$$\mu_l^N \{ \omega_N \in Z_l^N : \mu_l(V(\omega_N)) \leq \epsilon \} \geq 1 - \sum_{j=0}^d \binom{N}{j} \epsilon^j (1 - \epsilon)^{N-j}, \quad (8)$$

where μ_l^N denotes the product probability measure on Z_l^N , and $V(\omega_N)$ is defined by

$$V(\omega_N) := \{ z = (x, j_1, \dots, j_l) \in Z_l : (A_{j_1} \dots A_{j_l} x)^T P(\omega_N) (A_{j_1} \dots A_{j_l} x) > \gamma^* x^T P(\omega_N) x \},$$

i.e., it is the set of constraints that are violated by the optimal solution of $\text{Opt}(\omega_N)$.

Theorem 3.1 states that the optimal solution of the sampled problem $\text{Opt}(\omega_N)$ violates no more than an ϵ fraction of the constraints in the original optimization problem (4) with probability β , where β goes to 1 as N goes to infinity. This means that, the ellipsoid computed by $\text{Opt}(\omega_N)$ is “almost invariant” except on a set of measure ϵ . The rest of this section builds on this, and it has two important intermediate results leading us to our main theorem. By exploiting Property 3.1 below, we first show in Lemma 3.2 how one can compute an upper bound on the JSR when this “almost invariant” set is the unit sphere \mathbb{S} .

Property 3.1. *The dynamics given in (2) is convexity-preserving, meaning that for any set of points $X \subset \mathbb{R}^n$ we have:*

$$f(\text{convhull}(X)) \subset \text{convhull}(f(X)).$$

Lemma 3.2. *Let $\epsilon \in (0, 1)$ and $\gamma \in \mathbb{R}_{>0}$. Consider the set of matrices \mathcal{M} and $A \in \mathcal{M}$ satisfying:*

$$(A_{j_1} \dots A_{j_l} x)^T (A_{j_1} \dots A_{j_l} x) \leq \gamma^{2l} x^T x, \quad \forall x \in \mathbb{S} \setminus \mathbb{S}', \forall j \in M, \quad (9)$$

where $\mathbb{S}' \subset \mathbb{S}$ and $\sigma^{n-1}(\mathbb{S}') \leq \epsilon$, then we have:

$$\rho(\mathcal{M}) \leq \frac{\gamma}{\sqrt[l]{\alpha(\epsilon)}}$$

where $\alpha(\epsilon)$ is given in (24).

Proof. Note that, (9) implies that: $A_j(\mathbb{S} \setminus \mathbb{S}') \subset \gamma^l \mathbb{B}$. Using Property 3.1 this also implies:

$$A_j \text{convhull}(\mathbb{S} \setminus \mathbb{S}') \subset \text{convhull}(A_j(\mathbb{S} \setminus \mathbb{S}')) \subset \gamma^l \mathbb{B}.$$

Then, by Lemma C.1 in Appendix C, we have:

$$A_j(\alpha(\epsilon)\mathbb{B}) \subset A_j(\text{convhull}(\mathbb{S} \setminus \mathbb{S}')) \subset \gamma^l \mathbb{B}, \quad \forall j \in M,$$

by definition of $\alpha(\epsilon)$ given in (23). Therefore, we get:

$$\alpha(\epsilon)A_j(\mathbb{B}) \subset \gamma^l \mathbb{B},$$

which implies that $\rho(\mathcal{M}^l) \leq \frac{\gamma^l}{\alpha(\epsilon)}$ and hence $\rho(\mathcal{M}) \leq \frac{\gamma}{\sqrt[l]{\alpha(\epsilon)}}$. \square

We now know how to compute an upper bound on the JSR when the "almost invariant" ellipsoid is \mathbb{S} . Thanks to Property 2.3, if this is not the case, we can simply perform a change of coordinates mapping this ellipsoid to \mathbb{S} and compute the JSR in the new coordinates system instead. To do this, in the next theorem, we bound the measure of violating constraints on \mathbb{S} after the change of coordinates, in terms of the measure of the violated constraints on $\mathbb{S} \times M$ in the original coordinates.

Proposition 3.3. *Let $\gamma \in \mathbb{R}_{>0}$. Consider a set of matrices $A \in \mathcal{M}$, and a matrix $P \succ 0$ satisfying:*

$$(A_j x)^T P (A_j x) \leq \gamma^2 x^T P x, \quad \forall (x, j) \in Z \setminus V, \quad (10)$$

for some $V \subset Z$ where $\mu(V) \leq \epsilon$. Then, by defining L as in (30) and $\bar{A}_j = L^{-1} A_j L$, one also has:

$$(\bar{A}_j x)^T (\bar{A}_j x) \leq \gamma^2 x^T x, \quad \forall x \in \mathbb{S} \setminus \mathbb{S}', \forall j \in M,$$

for some $\mathbb{S}' \subset \mathbb{S}$ such that: $\sigma(\mathbb{S}') \leq m\epsilon\kappa(P)$, where

$$\kappa(P) = \sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}.$$

Proof. Let $V_{\mathbb{S}} = \pi_{\mathbb{S}}(V)$. We know that Σ_M is the disjoint union of its 2^m elements $\{\mathcal{M}_i, i \in \{1, 2, \dots, 2^m\}\}$. Then V can be written as the disjoint union $V = \sqcup_{1 \leq i \leq 2^m} (\mathcal{S}_i, \mathcal{M}_i)$ where $\mathcal{S}_i \in \Sigma_{\mathbb{S}}$. We notice that $V_{\mathbb{S}} = \sqcup_{1 \leq i \leq 2^m} \mathcal{S}_i$, and

$$\sigma^{n-1}(V_{\mathbb{S}}) = \sum_{1 \leq i \leq 2^m} \sigma^{n-1}(\mathcal{S}_i).$$

We have

$$\begin{aligned} \mu(V) &= \mu(\sqcup_{1 \leq i \leq 2^m} (\mathcal{S}_i, \mathcal{M}_i)) = \sum_{1 \leq i \leq 2^m} \mu(\mathcal{S}_i, \mathcal{M}_i) \\ &= \sum_{1 \leq i \leq 2^m} \sigma^{n-1} \otimes \mu_M(\mathcal{S}_i, \mathcal{M}_i) \\ &= \sum_{1 \leq i \leq 2^m} \sigma^{n-1}(\mathcal{S}_i) \mu_M(\mathcal{M}_i). \end{aligned}$$

Note that we have $\min_{j \in M} \mu_M(\{j\}) = \frac{1}{m}$. Then since $\forall i, \mu_M(\mathcal{M}_i) \geq \min_{j \in M} \mu_M(\{j\}) = \frac{1}{m}$, we get:

$$\sigma^{n-1}(V_{\mathbb{S}}) \leq \frac{\mu(V)}{\frac{1}{m}} \leq m\epsilon. \quad (11)$$

This means that

$$(A_j x)^T P(A_j x) \leq \gamma^2 x^T P x, \forall x \in \mathbb{S} \setminus V_{\mathbb{S}}, \forall m \in M, \quad (12)$$

where $\sigma^{n-1}(V_{\mathbb{S}}) \leq m\epsilon$.

We then perform the change of coordinates defined by $L^{-1} \in \mathcal{L}(\mathbb{R}^n)$ which maps \mathbb{S} to E_P , defined as in (30). We can then rewrite (12) in this new coordinates system as in:

$$(\bar{A}_j x)^T (\bar{A}_j x) \leq \gamma^2 x^T x, \forall x \in E_P \setminus L^{-1}(V_{\mathbb{S}}), \forall m \in M. \quad (13)$$

Due to the the homogeneity of the dynamics described in Property 2.1, this implies:

$$(\bar{A}_j x)^T (\bar{A}_j x) \leq \gamma^2 x^T x, \forall x \in \mathbb{S} \setminus \Pi_{\mathbb{S}}(L^{-1}(V_{\mathbb{S}})), \forall m \in M. \quad (14)$$

We now show how to relate $\sigma^{n-1}(V_{\mathbb{S}})$ to $\sigma^{n-1}(\Pi_{\mathbb{S}}(L^{-1}(V_{\mathbb{S}})))$. Consider $\mathbb{S}^{V_{\mathbb{S}}}$, the sector of \mathbb{B} defined by $V_{\mathbb{S}}$. We denote $C := L^{-1}(\mathbb{S}^{V_{\mathbb{S}}})$ and $V' := \Pi_{\mathbb{S}}(L^{-1}(V_{\mathbb{S}}))$. We have $\Pi_{\mathbb{S}}(C) = V'$ and $\mathbb{S}^{V'} \subset \mathcal{H}_{1/\lambda_{\min}(L^{-1})}(C)$, where \mathcal{H} is the homothety of ratio $1/\lambda_{\min}(L^{-1})$. This leads to:

$$\sigma^{n-1}(V') = \lambda(\mathbb{S}^{V'}) \leq \lambda(\mathcal{H}_{1/\lambda_{\min}(L^{-1})}(C)).$$

Then, the following holds:

$$\begin{aligned}
\sigma^{n-1}(V') &\leq \frac{1}{\lambda_{\min}(L^{-1})^n} \lambda(C) \\
&\leq \frac{1}{\lambda_{\min}(L^{-1})^n} \lambda(L^{-1}(\mathbb{S}^{V_{\mathbb{S}}})) \\
&= \frac{|\det(L^{-1})|}{\lambda_{\min}(L^{-1})^n} \lambda(\mathbb{S}^{V_{\mathbb{S}}}), \tag{15}
\end{aligned}$$

$$= \sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}} \sigma^{n-1}(V_{\mathbb{S}}) \tag{16}$$

where (15) follows from the fact that

$$\lambda(Q(X)) = |\det(Q)|\lambda(X),$$

for any set $X \subset \mathbb{R}^n$ and $Q \in \mathcal{L}(\mathbb{R}^n)$ (see e.g. [?]). Putting together (12), (14), and (16) we get the statement of the theorem where $\mathbb{S}' = \Pi_{\mathbb{S}}(L^{-1}(V_{\mathbb{S}}))$. \square

Remark 3.1.

We are now ready to prove our main theorem by putting together all the above pieces. For a given level of confidence β , we prove that the upper bound $\gamma^*(\omega_N)$, which is valid solely on finitely many observations, is in fact a true upper bound, at the price of increasing it by the factor $\frac{1}{\delta(\beta, \omega_N)}$. Moreover, as expected, this factor gets smaller as we increase N and decrease β .

Theorem 3.4. *Consider an n -dimensional switched linear system as in (2) and a uniform random sampling $\omega_N \subset Z$, where $N \geq \frac{n(n+1)}{2} + 1$. Let $\gamma^*(\omega_N)$ be the optimal solution to (??). Then, for any given $\beta \in (0, 1)$ and $\eta > 0$, we can compute $\delta(\beta, \omega_N)$, such that with probability at least β we have:*

$$\rho \leq \frac{\gamma^*(\omega_N)(1 + \eta)}{\delta(\beta, \omega_N)},$$

where $\lim_{N \rightarrow \infty} \delta(\beta, \omega_N) = 1$ with probability 1.

Proof. By definition of $\gamma^*(\omega_N)$ we have:

$$(A_j x)^T P (A_j x) \leq (\gamma^*(1 + \eta))^2 x^T P x, \quad \forall (x, j) \in \omega_N$$

for some $P \succ 0$. Then, by rewriting Theorem 3.1 we also have:

$$\mu^N \{ \omega_N \in Z^N : \mu(V(\omega_N)) \leq \epsilon \} \geq 1 - I(1 - \epsilon; N - d, d + 1), \tag{17}$$

where $I(\ell; a, b)$ is the regularized incomplete beta function. Let $\epsilon(\beta, N) = 1 - I^{-1}(1 - \beta; N - d, d + 1)$. Then, by Theorem 3.1, with probability at least β the following holds:

$$(A_j x)^T P(A_j x) \leq (\gamma^*(1 + \eta))^2 x^T P x, \quad \forall (x, j) \in Z \setminus V.$$

By Theorem 3.3, this implies that with probability at least β the following also holds:

$$(\bar{A}_j x)^T (\bar{A}_j x) \leq \gamma^2 x^T x, \quad \forall x \in \mathbb{S} \setminus \mathbb{S}', \forall j \in M,$$

for some \mathbb{S}' where $\sigma^{n-1}(\mathbb{S}') \leq m\epsilon\kappa(P)$. Then, applying Lemma 3.2, we can compute

$$\delta(\beta, \omega_N) = \alpha(\epsilon'(\beta, N)),$$

where

$$\epsilon'(\beta, N) = \frac{1}{2} m\kappa(P) \epsilon(\beta, N) \tag{18}$$

such that with probability at least β we have:

$$\bar{A}_j \mathbb{B} \subset \frac{\gamma^*(\omega_N)(1 + \eta)}{\delta(\beta, \omega_N)} \mathbb{B}, \quad \forall j \in M,$$

By Property 2.3, this means that with probability at least β :

$$\rho \leq \frac{\gamma^*(\omega_N)(1 + \eta)}{\delta(\beta, \omega_N)},$$

which completes the proof of the first part of the theorem. Note that, the ratio $\frac{1}{2}$ introduced in the expression of ϵ' is due to the homogeneity of the system described in Property 2.1, which implies that $x \in V_{\mathbb{S}} \iff -x \in V_{\mathbb{S}}$. We refer the interested reader to Appendix C.1 for the second part of this proof, namely showing that $\lim_{N \rightarrow \infty} \delta(\beta, \omega_N) = 1$ with probability 1. \square

4 A Deterministic Lower Bound for the JSR

In this section, we provide a theorem for a deterministic lower bound based on the observations W_{ω_N} , whose accuracy depends on the “horizon” l .

Theorem 4.1. *For an arbitrary $l \in \mathbb{N}$, and a given uniform sample*

$$\omega_N := \{(x_1, j_{1,1}, \dots, j_{1,l}), (x_2, j_{2,1}, \dots, j_{2,l}), \dots, (x_N, j_{N,1}, \dots, j_{N,l})\} \subset Z_l,$$

by considering $\gamma^*(\omega_N)$ the optimal solution of the optimization problem (??):

$$\begin{aligned} \min_P \quad & \gamma \\ \text{s.t.} \quad & (y_i)^T P(y_i) \leq \gamma^{2l} x_i^T P x_i, \forall (x_i, y_i) \in W_{\omega_N} \\ & P \succ 0, \gamma \geq 0, \end{aligned} \quad (19)$$

we have:

$$\rho(\mathcal{M}) \geq \frac{\gamma^*(\omega_N)}{\sqrt[2l]{n}}.$$

Proof. Let $\epsilon > 0$. By definition of γ^* , there exists no matrix $P \in \mathcal{S}_{++}^n$ such that:

$$(Ax)^T P(Ax) \leq (\gamma^*(\omega_N) - \epsilon)^{2l} x^T P x, \quad \forall x \in \mathbb{R}^n, \forall A \in \mathcal{M}^l.$$

By Property 2.3, this means that there exists no CQLF for the scaled set of matrices $\frac{\mathcal{M}^l}{(\gamma^*(\omega_N) - \epsilon)^l}$. Since the above inequality is true for every $\epsilon \geq 0$, using Theorem 2.1, and the fact that $\rho(\mathcal{M}^l) = \rho(\mathcal{M})^l$, we conclude:

$$\frac{\rho(\mathcal{M})}{\gamma^*(\omega_N)} \geq \frac{1}{\sqrt[2l]{n}}.$$

□

Remark 4.1. In the above discussion, we introduce the concept of ‘ l -step CQLF’, and showed that it allows to refine the initial $1/\sqrt{(n)}$ approximation provided by the CQLF method. In the switching systems literature, there are other techniques for refining this approximation, as for instance replacing the LMIs in Theorem 2.1 by Sum-Of-Squares (SOS) constraints; see [?] or [?, Theorem 2.16]. It seems that the concept of l -step CQLF is better suited for our purpose, as we briefly discuss below. We leave for further work a more systematic analysis of the behaviours of the different refining techniques.

5 Experimental Results

We illustrate our technique on a two-dimensional switched system with 4 modes. We fix the confidence level, $\beta = 0.92$, and compute the lower and upper bounds on the JSR for $N := 15 + 15k$, $k \in \{0, \dots, 23\}$, according to Theorem 4.1 and Theorem 3.4, respectively. We illustrate the average performance of our algorithm over 10 different runs in Fig. 2 and Fig. 3.

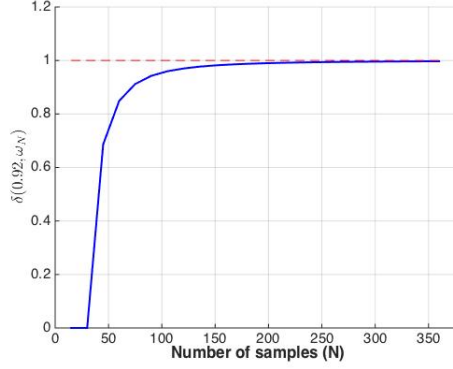


Fig. 2: Evolution of δ with increasing N .

Fig. 2 shows the evolution of $\delta(\beta, N)$ as N increases. We illustrate that δ converges to 1 as expected. In Fig. 3, we plot the upper bound and lower bound for the JSR of the system computed by Theorem 3.4 and Theorem 4.1, respectively. To demonstrate the performance of our technique, we also provide the JSR approximated by the JSR toolbox [?], which turns out to be 0.7727. Note that, the plot for the upper bound starts from $N = 45$. This is due to the fact for $N = 15$, and $N = 30$, $\delta(\beta, \omega_N) = 0$, hence it is not possible to compute a nontrivial upper bound for these small values of N . As can be seen, the upper bound approaches to a close vicinity of the real JSR with approximately 200 samples. In addition, the gap between the upper and lower bound converges to a multiplicative factor of $\frac{\rho}{\sqrt{n}}$ as expected.

Note that, if we increase the dimension of the switched system, the convergence of δ to 1 will become much slower. We confirmed this via experiments up to dimension $n = 6$. For example, for dimension $n = 4$, it took $N = 5,000$ to $N = 10,000$ points to reach $\delta = 0.9$. We nevertheless observe convergence of the upper bound to $\rho(\mathcal{M})$, and convergence of the lower bound to $\frac{\rho(\mathcal{M})}{\sqrt{n}}$. The gap between these two limits is $\frac{\rho}{\sqrt{n}}$ and could be improved by considering a more general class of common Lyapunov functions, such as those that can be described by sum-of-squares polynomials [?]. We leave this for future work.

Finally, we randomly generate 10,000 test cases with systems of dimension between 2 and 7, number of modes between 2 and 5, and size of samples N between 30 and 800. We take $\beta = 0.92$ and we check if the upper bound computed by our technique is greater than the actual JSR of the system. We get 9873 positive tests, out of 10,000, which gives us a probability of 0.9873

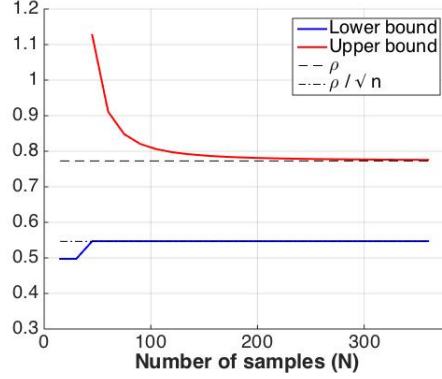


Fig. 3: Evolution of the upper and lower bounds on the JSR with increasing N , for $\beta = 0.92$.

of the correctness of the upper bound computed. Note that, this probability is significantly above the provided β . This is expected, since our techniques are based on worst-case analysis and thus fairly conservative.

5.1 Networked Control System

We now consider a linear time-invariant control system given as $x_{k+1} = Ax_k + Bu_k$, where we do not have access to its dynamics given by the matrices A and B . The control law is of the form $u_k = Kx_k$, where K is also unknown. The open-loop system is unstable with eigenvalues at $\{0.45, 1.1\}$. The controller stabilizes the system by bringing its eigenvalues to $\{0.8, -0.7\}$. The control input is transmitted over a wireless communication channel that is utilized by ℓ users, including the controller. The communication between the users and the recipients is performed based on the IEEE 802.15.4 MAC layer protocol [?], which is used in some of the proposed standards for control over wireless, e.g., WirelessHART [?]. This MAC layer integrates both guaranteed slots and contention based slots. In this example, we consider a beacon-enabled mode of the MAC protocol. In this setup, a centralized control user periodically synchronizes and configures all the users. This period is named Beacon Interval. This interval is divided into two intervals: active and inactive period. The active period is divided into 6 slots. The first 2 slots correspond to the contention access period (CAP), and the next 4 slots correspond to the collision free period (CFP). In the CAP, the users can only send their message if the channel is “idle” with carrier-sense multiple access with collision avoidance (CSMA/CA). In the CFP however, each user

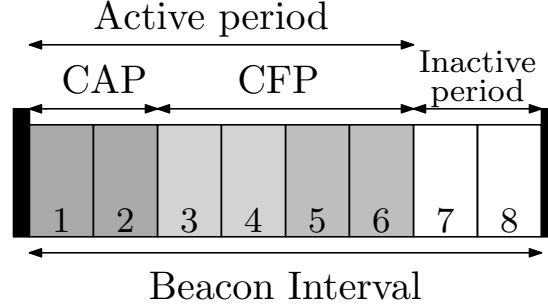


Fig. 4: The time allocation structure of the modified IEEE 802.15.4 MAC layer.

has guaranteed time slots, during which there are no packet losses. In our example, the third and fourth slots are designated for the controller, while the fifth and sixth slots are allocated to the other users. Finally, during the inactive period, all users enter a low-power mode to save energy. We illustrate the overall structure of this communication protocol in Fig. 4. We now want to decide whether the resulting closed-loop network control system is stable by simulating it starting from different initial conditions.

Note that, the closed-loop dynamics of the underlying system when the controller is active is $A_c = A + BK$. Then, we can model the overall networked control system by the linear switched system $x_{k+1} = \bar{A}x_k$, where $\bar{A} \in \mathcal{M}$ and

$$\mathcal{M} = \{A^2 A_c^2 A^4, A_c A A_c^2 A^4, A A_c^3 A^4, A_c^4 A^4\}. \quad (20)$$

Note that, in (20), each mode corresponds to a different utilization of the CFP by the users. For example, the mode defined by $A_c A A_c^2 A^4$ is active when the first slot in the CFP is assigned to the controller and the second slot is assigned to the other users. We assume that all of the users using the channel have an equal probability of being assigned to a time slot during the CFP. Therefore, the probability of each mode in \mathcal{M} being active is $\left\{ \frac{1}{(\ell-1)^2}, \frac{1}{\ell(\ell-1)}, \frac{1}{(\ell-1)\ell}, \frac{1}{\ell^2} \right\}$, i.e., the modes given in (20) will not be active with the same probability. Hence, we make use of Remark ?? and update our bounds accordingly. Fig. 5 shows the computed bounds. As can be seen, approximately after 500 samples, the upper bound on the JSR drops below 1, which lets us decide that the given closed-loop networked control system is stable, with probability 0.95.

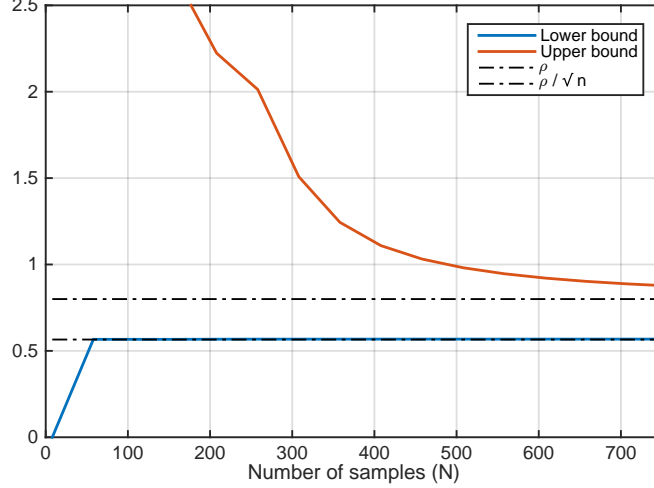


Fig. 5: The evolution of the computed upper and lower bounds on the JSR with respect to the number of simulations collected from the networked control system.

6 Conclusions

In this paper, we investigated the question of how one can conclude stability of a dynamical system when a model is not available and, instead, we have randomly generated state measurements. Our goal is to understand how the observation of well-behaved trajectories *intrinsically* implies stability of a system. It is not surprising that we need some standing assumptions on the system, in order to allow for any sort of nontrivial stability certificate solely from a finite number of observations.

The novelty of our contribution is twofold: First, we use as standing assumption that the unknown system can be described by a switching linear system. This assumption covers a wide range of systems of interest, and to our knowledge no such “black-box” result has been available so far on switched systems. Second, we apply powerful techniques from chance constrained optimization. The application is not obvious, and relies on geometric properties of linear switched systems.

We believe that this guarantee is quite powerful, in view of the hardness of the general problem. In the future, we plan to investigate how to generalize our results to more complex or realistic systems. We are also improving the numerical properties of our technique by incorporating sum-of-squares

optimization, and relaxing the sampling assumptions on the observations.

A Notation and Background

Before proceeding to the main lemmas we use to prove Lemma C.1, we first introduce some necessary definitions and related background.

Let d be a distance on \mathbb{R}^n . The distance between a set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ is $d(X, p) := \inf_{x \in X} d(x, p)$. Note that the map $p \mapsto d(X, p)$ is continuous on \mathbb{R}^n . Given a set $X \subset \mathbb{R}^n$, ∂X denotes the boundary of set X .

Definition A.1. We define the spherical cap on \mathbb{S} for a given hyperplane $c^T x = k$ as:

$$\mathcal{C}_{c,k} := \{x \in \mathbb{S} : c^T x > k\}.$$

Remark A.1. Consider the spherical caps \mathcal{C}_{c,k_1} and \mathcal{C}_{c,k_2} such that $k_1 > k_2$, then we have:

$$\sigma^{n-1}(\mathcal{C}_{c,k_1}) < \sigma^{n-1}(\mathcal{C}_{c,k_2}).$$

Remark A.2. The distance between the point $x = 0$ and the hyperplane $c^T x = k$ is $\frac{|k|}{\|c\|}$.

We now define the function $\Delta : \wp(\mathbb{S}) \rightarrow [0, 1]$ as:

$$\Delta(X) := \sup\{r : r\mathbb{B} \subset \text{convhull}(\mathbb{S} \setminus X)\}. \quad (21)$$

Note that, $\Delta(X)$ can be rewritten as:

$$\Delta(X) = d(\partial \text{convhull}(\mathbb{S} \setminus X), 0). \quad (22)$$

Lemma A.1. Consider the spherical cap $\mathcal{C}_{c,k}$. We have:

$$\Delta(\mathcal{C}_{c,k}) = \min\left(1, \frac{|k|}{\|c\|}\right).$$

Proof. Note that:

$$\text{convhull}(\mathbb{S} \setminus X) = \{x \in \mathbb{B} : c^T x \leq k\}.$$

Then the following equalities hold:

$$\begin{aligned} \Delta(X) &= d(\partial \text{convhull}(\mathbb{S} \setminus X), 0) \\ &= \min(d(\partial \mathbb{B}, 0), d(\partial\{x : c^T x \leq k\}, 0)) \\ &= \min(d(\mathbb{S}, 0), d(\{x : c^T x = k\}, 0)) \\ &= \min\left(1, \frac{|k|}{\|c\|}\right). \end{aligned}$$

□

Corollary A.2. *Consider the spherical caps \mathcal{C}_{c,k_1} and \mathcal{C}_{c,k_2} such that $k_1 \leq k_2$. Then we have:*

$$\Delta(\mathcal{C}_{c,k_1}) \leq \Delta(\mathcal{C}_{c,k_2}).$$

B Preliminary Results

Lemma B.1. *For any set $X \subset \mathbb{S}$, there exist c and k such that $\mathcal{C}_{c,k}$ satisfies: $\mathcal{C}_{c,k} \subset X$, and $\Delta(\mathcal{C}_{c,k}) = \Delta(X)$.*

Proof. Let $\tilde{X} := \text{convhull}(\mathbb{S} \setminus X)$. Since d is continuous and the set $\partial\tilde{X}$ is compact, there exists a point $x^* \in \partial\tilde{X}$, such that:

$$\Delta(X) = d(\partial\tilde{X}, 0) = \min_{x \in \partial\tilde{X}} d(x, 0) = d(x^*, 0).$$

Next, consider the hyperplane which is tangent to the ball $\Delta(X)\mathbb{B}$ at x^* , which we denote by $\{x : c^T x = k\}$. Then we have:

$$\Delta(X) = d(x^*, 0) = d(\{x : c^T x = k\}, 0) = \min \left(1, \frac{|k|}{\|c\|} \right).$$

Now, consider the spherical cap defined by this tangent plane i.e., $\mathcal{C}_{c,k}$. Then, by Lemma A.1 we have $\Delta(\mathcal{C}_{c,k}) = \min \left(1, \frac{|k|}{\|c\|} \right)$. Therefore, $\Delta(X) = \Delta(\mathcal{C}_{c,k})$.

We next show $\mathcal{C}_{c,k} \subset X$. We prove this by contradiction. Assume $x \in \mathcal{C}_{c,k}$ and $x \notin X$. Note that, if $x \notin X$, then $x \in \mathbb{S} \setminus X \subset \text{convhull}(\mathbb{S} \setminus X)$. Since $x \in \mathcal{C}_{c,k}$, we have $c^T x > k$. But due to the fact that $x \in \text{convhull}(\mathbb{S} \setminus X)$, we also have $c^T x \leq k$, which leads to a contradiction. Therefore, $\mathcal{C}_{c,k} \subset X$. \square

Proposition B.2. *Let $\mathcal{X}_\epsilon = \{X \subset \mathbb{S} : \sigma^{n-1}(X) \leq \epsilon\}$. Then, for any $\epsilon \in (0, 1)$, the function $\Delta(X)$ attains its minimum over \mathcal{X}_ϵ for some X which is a spherical cap.*

Proof. We prove this via contradiction. Assume that there exists no spherical cap in \mathcal{X}_ϵ such that $\Delta(X)$ attains its minimum. This means there exists an $X^* \in \mathcal{X}_\epsilon$, where X^* is not a spherical cap and $\arg \min_{X \in \mathcal{X}_\epsilon} (\Delta(X)) = X^*$. By Lemma B.1, we can construct a spherical cap $\mathcal{C}_{c,k}$ such that $\mathcal{C}_{c,k} \subset X^*$ and $\Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$. Note that, we further have $\mathcal{C}_{c,k} \subsetneq X^*$, since X^* is assumed not to be a spherical cap. This means that, there exists a spherical cap $\sigma^{n-1}(\mathcal{C}_{c,k})$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) < \epsilon$.

Then, the spherical cap $\mathcal{C}_{c,\tilde{k}}$ with $\sigma^{n-1}(\mathcal{C}_{c,\tilde{k}}) = \epsilon$, satisfies $\tilde{k} < k$ by Remark A.1. This implies

$$\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$$

by Corollary A.2. Therefore, $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(X^*)$. This is a contradiction since we initially assumed that $\Delta(X)$ attains its minimum over \mathcal{X}_ϵ at X^* . \square

C Main Lemma

Lemma C.1. *Let $\epsilon \in (0, \frac{1}{2})$ and $\alpha : (0, 1) \rightarrow \mathbb{R}_{\geq 0}$ be defined by:*

$$\alpha(\epsilon) := \inf_{X \in \mathcal{X}_\epsilon} \sup\{r : r\mathbb{B} \subset \text{convhull}(\mathbb{S} \setminus X)\}, \quad (23)$$

where $\mathcal{X}_\epsilon = \{X \subset \mathbb{S} : \sigma^{n-1}(X) \leq \epsilon\}$. Then, $\alpha(\epsilon)$ is given by the formula:

$$\alpha(\epsilon) = \sqrt{1 - I^{-1}\left(2\epsilon; \frac{n-1}{2}, \frac{1}{2}\right)}, \quad (24)$$

where I is the regularized incomplete beta function.

Proof. By Proposition B.2 we know that:

$$\alpha(\epsilon) = \Delta(\mathcal{C}_{c,k}), \quad (25)$$

for some spherical cap $\mathcal{C}_{c,k} \subset \mathbb{S}$, where $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon$. It is known (see e.g. [?]) that the area of such $\mathcal{C}_{c,k}$, is given by the equation:

$$\sigma^{n-1}(\mathcal{C}_{c,k}) = \frac{I\left(1 - \Delta(\mathcal{C}_{c,k})^2; \frac{n-1}{2}, \frac{1}{2}\right)}{2}. \quad (26)$$

Since, $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon$, we get the following set of equations:

$$\begin{aligned} \epsilon &= \frac{I\left(1 - \Delta(\mathcal{C}_{c,k})^2; \frac{n-1}{2}, \frac{1}{2}\right)}{2} \\ 1 - \Delta(\mathcal{C}_{c,k})^2 &= I^{-1}\left(2\epsilon; \frac{n-1}{2}, \frac{1}{2}\right) \\ \Delta(\mathcal{C}_{c,k})^2 &= 1 - I^{-1}\left(2\epsilon; \frac{n-1}{2}, \frac{1}{2}\right). \end{aligned} \quad (27)$$

Then, the inequalities (25) and (27) imply the inclusion given in (23). \square

C.1 Proof of $\lim_{N \rightarrow \infty} \delta(\beta, \omega_N) = 1$

Recall that, $\delta(\beta, \omega_N) = \alpha(m\kappa(P(\omega_N))\epsilon(\beta, \omega_N))$. We first show that $\kappa(P(\omega_N))$ is uniformly bounded in N . The optimization problem $\text{Opt}(\omega_N)$ given in (??), with $\gamma^*(\omega_N)$ replaced by $\gamma^*(Z)(1 + \frac{\eta}{2})$ is strictly feasible, and

thus admits a finite optimal value K for some solution $P_{\eta/2}$. Note that, $\lim_{N \rightarrow \infty} \gamma^*(\omega_N) = \gamma^*(Z)$ with probability 1. Thus, for large enough N , $\gamma^*(\omega_N)(1 + \eta) > \gamma^*(Z)(1 + \frac{\eta}{2})$. This also means that, for large enough N , $\text{Opt}(\omega_N)$ admits $P_{\eta/2}$ as a feasible solution and thus the optimal value of $\text{Opt}(\omega_N)$ is bounded by K . In other words, $\lambda_{\max}(P(\omega_N)) \leq K$. Moreover, since $\lambda_{\max}(P(\omega_N)) \geq 1$, we also have $\det(P(\omega_N)) \geq 1$, which means that

$$\kappa(P(\omega_N)) = \sqrt{\frac{\lambda_{\max}(P(\omega_N))^n}{\det(P(\omega_N))}} \leq \sqrt{K^n}. \quad (28)$$

We next show that for a fixed $\beta \in (0, 1)$ $\lim_{N \rightarrow \infty} \epsilon(\beta, N) = 0$. Note that, $\epsilon(\beta, N)$ is intrinsically defined by the following equation:

$$1 - \beta = \sum_{j=0}^d \binom{N}{j} \epsilon^j (1 - \epsilon)^{N-j}.$$

We can then upper bound the term $1 - \beta$ as in:

$$1 - \beta \leq (d + 1)N^d(1 - \epsilon)^{N-d}. \quad (29)$$

We prove $\lim_{N \rightarrow \infty} \epsilon(\beta, N) = 0$ by contradiction. Assume that $\lim_{N \rightarrow \infty} \epsilon(\beta, N) \neq 0$. This means that, there exists some $c > 0$ such that $\epsilon(\beta, N) > c$ infinitely often. Then, consider the subsequence N_k such that $\epsilon(\beta, N_k) > c, \forall k$. Then, by (29) we have:

$$1 - \beta \leq (d + 1)N_k^d(1 - \epsilon)^{N_k-d} \leq (d + 1)N_k^d(1 - c)^{N_k-d} \forall k \in \mathbb{N}.$$

Note that $\lim_{k \rightarrow \infty} (d + 1)N_k^d(1 - c)^{N_k-d} = 0$. Therefore, there exists a k' such that:

$$(d + 1)N_{k'}^d(1 - c)^{N_{k'}-d} < 1 - \beta,$$

which is a contradiction. Therefore, we must have $\lim_{N \rightarrow \infty} \epsilon(\beta, N) = 0$.

Putting this together with (28), we get:

$$\lim_{N \rightarrow \infty} m\kappa(P(\omega_N))\epsilon(\beta, \omega_N) = 0.$$

By the continuity of the function I^{-1} this also implies: $\lim_{N \rightarrow \infty} \alpha(m\kappa(P(\omega_N))\epsilon(\beta, \omega_N)) = 1$.

D Precisions on measures definitions

For an ellipsoid centered at the origin, and for any of its subsets \mathcal{A} , the *sector* defined by \mathcal{A} is the subset

$$\{t\mathcal{A}, t \in [0, 1]\} \subset \mathbb{R}^n.$$

We denote by $E_P^{\mathcal{A}}$ the sector induced by $\mathcal{A} \subset E_P$. In the particular case of the unit sphere, we instead write $\mathbb{S}^{\mathcal{A}}$. We can notice that $E_P^{E_P}$ is the volume in \mathbb{R}^n defined by E_P : $E_P^{E_P} = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$.

The spherical Borelian σ -algebra, denoted by $\mathcal{B}_{\mathbb{S}}$, is defined by

$$\mathcal{A} \in \mathcal{B}_{\mathbb{S}} \iff \mathbb{S}^{\mathcal{A}} \in \mathcal{B}_{\mathbb{R}^n}.$$

We provide $(\mathbb{S}, \mathcal{B}_{\mathbb{S}})$ with the classical, unsigned and finite uniform spherical measure σ^{n-1} defined by

$$\forall \mathcal{A} \in \mathcal{B}_{\mathbb{S}}, \sigma(\mathcal{A}) = \frac{\lambda(\mathbb{S}^{\mathcal{A}})}{\lambda(\mathbb{B})}.$$

In other words, the spherical measure of a subset of the sphere is related to the Lebesgue measure of the sector of the unit ball it induces. Notice that $\sigma^{n-1}(\mathbb{S}) = 1$.

Moreover, since $P \in \mathcal{S}_{++}^n$, it can be written in its Choleski form

$$P = L^T L, \tag{30}$$

where L is an upper triangular matrix. Note that, L^{-1} maps the elements of \mathbb{S} to E_P . Then, we define the measure on the ellipsoid σ_P on the σ -algebra $\mathcal{B}_{E_P} := L^{-1}\mathcal{B}_{\mathbb{S}}$, where $\forall \mathcal{A} \in \mathcal{B}_{E_P}$, $\sigma_P(\mathcal{A}) = \sigma^{n-1}(L\mathcal{A})$.

Set M is provided with the classical σ -algebra associated to the finite sets: $\Sigma_M = \wp(M)$, where $\wp(M)$ is the set of subsets of M . We provide (M, Σ_M) with the uniform measure μ_M . Similarly, we define Σ_{M^l} as the product $\bigotimes^l \Sigma_M$ (which is here equal to $\wp(M)^l$), and we provide (M^l, Σ_{M^l}) with the uniform product measure $\mu_{M^l} = \bigotimes^l \mu_M$.

We can now denote the product σ -algebra $\mathcal{B}_{\mathbb{S}} \bigotimes (\Sigma_{M^l})$ generated by $\mathcal{B}_{\mathbb{S}}$ and Σ_{M^l} : $\Sigma = \sigma(\pi_{\mathbb{S}}^{-1}(\mathcal{B}_{\mathbb{S}}), \pi_{M^l}^{-1}(\Sigma_{M^l}))$, where $\pi_{\mathbb{S}} : Z_l \rightarrow \mathbb{S}$ and $\pi_{M^l} : Z_l \rightarrow M^l$ are the standard projections. On $(Z_l, \mathcal{B}_{\mathbb{S}} \bigotimes (\Sigma_{M^l}))$, we define the product measure $\mu_l = \sigma^{n-1} \otimes \mu_{M^l}$. Note that, μ_l is a uniform measure on Z_l and $\mu_l(Z_l) = 1$.