Sampling Based Stability Analysis of Black-box Switched Linear Systems with Probabilistic Guarantees

I. Introduction

As our computational resources have increased, so is the complexity of the models we use for the analysis of dynamical systems. Today, the industrial models do not only consist of simple differential or difference equations; these models are multimodal, hybrid, and contain a variety of subcomponents such as lookup tables, delay differential equations, and thermodynamic models. The current modeling paradigm is also highly distributed, in the sense that, the model subcomponents are developed by different parties. Therefore, their internal structure is partially or completely unknown to the end user. Hence, it is often hard, if not impossible to obtain analytical formulas for today's industrial scale models. On the other hand, performing simulations is a common way of validating these models via readily available tools. Therefore, it is a natural question to ask whether we can provide formal analyses about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, we consider a dynamical system as in:

$$x_{k+1} = f(k, x_k), \tag{1}$$

where, $x_k \in \mathbb{R}^n$, k is index of time. We start with the following question to serve as a stepping stone: Given N input-output pairs, (x_1, y_1), (x_2, y_2), ..., (x_N, y_N) such that $y_k = f(k, x_k)$, what can we say about the stability of the system (1)? For the rest of the paper, we use the term *black-box* to refer to models where we do not have access to its dynamics (f), yet we can observe its outputs (y) by exciting it with inputs (x). Note that, one approach to this problem is firstly identifying the dynamics, i.e., f and then applying the existing techniques in the model-based stability analysis literature. However, unless f is a linear function, there are two main reasons behind our quest to directly work on input-output pairs and bypassing the identification phase: (1) Even when the function f is known, in general, the stability analysis is still hard [**?**], [**?**]. (2) The existing identification techniques can only identify f up to an approximation error. How to relate this identification error to an error in the stability of the system (1) is still a nontrivial problem.

The initial idea behind this paper was born based on the recent efforts in [7], [5] and [1] in using simulation traces to find Lyapunov functions for systems with known dynamics. In these work, the main idea is that if one can construct a Lyapunov function candidate decreasing along many finite trajectories starting from different initial conditions, then it should decrease along the remaining trajectories as well. Then, once a Lyapunov function candidate is constructed, the presented algorithms are based on verifying it either via off-the-shelf tools as in [7] and [5], or via sampling based techniques as in [1]. Note that, since we do not have access to the dynamics, the second step cannot be directly applied to black-box systems. However, these sampling based ideas trigger the following question that we address in this paper: *Can we translate the confidence we gain in the decrement of a candidate Lyapunov function, into a confidence in the stability of the underlying system?*

Note that, even in the case of a 2D linear system the connection between these two confidence levels is nontrivial. In fact, one can easily construct an example with one stable and one unstable eigenvalue for which even though almost all trajectories diverge to the infinity, it is possible to construct a Lyapunov function candidate whose level sets are contracting everywhere except a small set. Should we give a specific example here, and put a figure? Moreover, the size of this "violating set" can be arbitrarily small based on the magnitude of the unstable eigenvalue. In this paper, we take the first step to close this gap. Since the identification and stability analysis of linear systems are well understood, we do so by focusing on switched linear systems.

Note that identification and deciding the stability of arbitrary switched linear systems is NP-hard [4]. Aside from their theoretical value, switched systems model the behavior of dynamical systems in the presence of known or unknown varying parameters. These parameters can model internal properties of the dynamical system such as uncertainties, look-up tables, values in a discrete register as well as exogenous inputs provided by a controller in a closed-loop control system. Need to make these examples more specific.

The stability of switched systems closely relates to the *joint spectral radius* (JSR) of the matrices appearing in (2). Under certain conditions deciding stability amounts to deciding whether JSR is less than one or not [4]. In this paper, we present an algorithm to approximate the JSR of a switched linear system from N input-output pairs. This algorithm is based on tools from the random convex optimization literature [2], and provides an upper bound on the JSR with a user-defined confidence level. As N increases, this bound gets tighter. Moreover, with a closed form expression, we characterize what the exact trade-off between the tightness of this bound and the number of samples is. In order to understand the quality of our technique, the algorithm also provides a deterministic lower-bound.

The organization of the paper is as follows: TO BE FILLED.

II. Preliminaries

A. Notation

We consider the usual Hilbert finite normed vector space $(\mathbb{R}^n, \ell_2)$, $n \in \mathbb{N}_{>0}$, ℓ_2 the classical euclidean norm. We denote a unit ball in $\mathbb{R}^n$ with B and unit sphere in $\mathbb{R}^n$ of radius r as S. We only denote the radius r explicitly as in B_r and S_r, when r is different than 1. We denote by Π_{S} the (convex) projector on S. We denote the set of real symmetric matrices of size n by $\mathcal{S}^n(\mathbb{R})$, and the set of linear functions in $\mathbb{R}^n$ by $\mathcal{L}(\mathbb{R}^n)$. We denote the ellipsoid described by the matrix $P \in \mathbb{S}^n$ as E_P, i.e., $E_P := \{x \in \mathbb{R}^n : x^T P x = 1\}$. We denote the homothety of ratio r by $\mathcal{H}_r$.

We consider the classical unsigned and finite uniform spherical measure on S, commonly denoted by σ^{n-1}. It is associated to the spherical Borelian σ-algebra and is derived from the Lebesgue measure λ. We have $\mathcal{B}_{\mathrm{S}}$ with $\mathcal{A} \in \mathcal{B}_{\mathrm{S}}$ if and only if the sector $t\mathcal{A}$, $t \in [0,1]$ is in $\mathcal{B}_{\mathbb{R}^n}$. The spherical measure σ^{n-1} is defined by

$$\forall\ \mathcal{A} \in \mathcal{B}_{\mathrm{S}}, \sigma(A) = \frac{\lambda(tA)}{\lambda(B)}.$$

In other words, the spherical measure of a subset of the sphere is related to the Lebesgue measure of the sector of the unit ball it induces. Notice that $\sigma^{n-1}(\mathrm{S}) = 1$.

B. Stability of Linear Switched Systems

A switched linear system is in the form:

$$x_{k+1} = A_{\sigma(k)} x_k, \tag{2}$$

where, $\sigma : \mathbb{N} \to \{1, 2, \ldots, m\}$ is the switching sequence and $A_{\sigma(k)} \in \mathcal{M}$, for all σ and k. There are two important properties of switched systems that we exploit in this paper.

Property 2.1: Let $\xi(x, k)$ denote the state of the system (2) at time k, starting from the initial condition x. The dynamical system (2) is a homogeneous:

$$\xi(\gamma x, k) = \gamma \xi(x, k).$$

Property 2.2: The dynamics given in (2) is convexity-preserving, meaning that for any set of points $X \subseteq \mathbb{R}^n$ we have:

$$f(\text{convhull } X) \subset \text{convhull } \{f(X)\}.$$

To make our reasoning clearer, we introduce the *Lyapunov exponent* of the system, which is a numerical quantity describing its stability.

Definition Given a dynamical system as in (**??**) its *Lyapunov exponent* is given by

$$\rho = \inf\{r : \forall x_0, \exists C \in \mathbb{R}^+ : \quad x(0) = x_0 \Rightarrow x(t) \leq C r^t\}.$$

Under certain conditions, deciding stability amounts to decide whether $\rho < 1$. In order to understand the quality of our techniques, we will actually try to prove lower and upper bounds on ρ.

In the case of switched linear systems, the Lyapunov exponent is known as the Joint Spectral Radius of the set of matrices, which can be alternatively defined as follows:

Definition [3] Given a set of matrices $\mathcal{M} \subset \mathbb{R}^{n \times n}$, its *joint spectral radius* (JSR) is given by

$$\rho(\mathcal{M}) = \lim_{t \to \infty} \max_{i_1, \ldots, i_t} \{||A_{i_1} \ldots A_{i_t}||^{1/t} : A_i \in \mathcal{M}\}.$$

Remark 2.1: Note that one can scale the problem because the JSR is homogeneous:

$$\rho(\mathcal{M}/\gamma) = \rho(\mathcal{M})/\gamma, \forall \lambda > 0,$$

and $\mathcal{M}/\lambda$ can be studied by studying the scaled inputs

$$(x_t, y_t/\gamma, \sigma(t)).$$

Under certain conditions Where is this? deciding stability amounts to decide whether $\rho < 1$. In order to understand the quality of our techniques, we will actually try to prove lower and upper bounds on ρ.

III. A Deterministic Lower Bound for JSR

We start by proving the lower bound, which is based on the following theorem from the switched linear system literature.

Theorem 3.1: [3, Theorem 2.11] For any bounded set of matrices such that $\rho(\mathcal{M}) < \frac{1}{\sqrt{n}}$, there exists a Common Quadratic Lyapunov Function (CQLF) for $\mathcal{M}$, that is, a $P \succ 0$ such that:

$$\forall A \in \mathcal{M},\ A^T P A \preceq P.$$

The following theorem shows that the existence of a CQLF for (2) can be checked by collecting N observation pairs.

Theorem 3.2: For a given sampling:

$$\omega_N = \{x_1, x_2, \ldots, x_N\},$$

let $\gamma^*(\omega_N)$ be the optimal solution of the following optimization problem:

$$\begin{array}{ll} \min & \gamma \\ s.t. & \dfrac{(y_i^T P y_i)}{\gamma^2} \leq x_i^T P x_i, \forall\, i : 1 \leq i \leq N \\ & P \succ 0. \end{array} \tag{3}$$

If $\gamma^*(\omega_N) < \infty$, we have:

$$\rho(\mathcal{M}) \geq \frac{\gamma^*}{\sqrt{n}}.$$

Proof: Using Remark 2.1, for any $\epsilon > 0$, $\frac{\mathcal{M}}{(\gamma^* - \epsilon),}$ has no CQLF. Then, applying Theorem 3.1 we get

$$\frac{\rho(\mathcal{M})}{\gamma^*} \geq \frac{1}{\sqrt{n}}.$$

∎

IV. A Probabilistic Upper Bound for JSR

In this section, we show that using the Property 2.2 and Property 2.1, by sampling finitely many points on a level set of a candidate CQLF, we can compute an upper bound on ρ. The main theorem in this section which formalizes this is based on the following theorem, for whose proof we devote most of this section:

Theorem 4.1: Let $\epsilon \in (0,1)$. For a given a sampling of the switched linear system (2), ω_N and $\beta \in [0,1)$, we can compute $\delta(\beta, N) < \infty$ such that:

$$E_{\delta^2 P} \subset \text{convhull } (E_P \setminus S_\epsilon), \tag{4}$$

for some $P \succ 0$.

Before proving Theorem 4.1,

Theorem 4.2 (Main Theorem): For a given a sampling of the switched linear system (2), ω_N, where $N \geq d+1$ and $\beta \in [0,1)$, we can compute $\delta(\beta, N) < \infty$ such that with probability at least β, $\rho \leq \frac{\gamma^*(\omega_N)}{\delta}$. Moreover, $\lim_{N\to\infty} \delta(N) = 1$.

Proof: Fix $\delta < 1$ and denote E_P the ellipsoid described by P (i.e., $\{x : x^T P x = 1\}$), and denote ϵ such that for any subset S_ϵ of measure ϵ,

$$E_{\delta^2 P} \subset \text{convhull } (E_P \setminus S_\epsilon).$$

Now, denoting N the number of observations available, compute $0 < \beta < 1$ such that

$$N = N(\epsilon, \beta)$$

in Theorem 4.3 above.

Summarizing, the equation above means that with high probability, one has that (3) is satisfied for all $x \in \mathbb{R}^n$, except for a set of measure ϵ. Let us denote S_ϵ this set of violated constraints. Thus,

$$(\mathcal{M}/\lambda^*)\text{convhull } (E_P \setminus S_\epsilon) \subset \text{convhull } (E_P \setminus S_\epsilon).$$

Now, by definition of ϵ, one has

$$E_{\delta^2 P} \subset \text{convhull } (E_P \setminus S_\epsilon),$$

and so

$$(\mathcal{M}/\delta\lambda^*)\text{convhull } (E_P \setminus S_\epsilon) \subset \text{convhull } (E_P \setminus S_\epsilon).$$

Then, $\delta\lambda$ is un upper bound on ρ, with a confidence β. ■

Here, we need to talk about proving the Lyapunov decrement in finitely, many points, talk about campi and then make a transition In this section we suppose that we do not observe the modes, but only the pairs (x_k, y_k). The idea is that if we have an ellipsoid that is contracted for all our observed pairs (x, y), it is not enough to imply stability, because we observed only some possible behaviours of the dynamics. However, suppose that we observed that the $\lambda-$contraction of the polytope $\mathcal{P}$ is satisfied for many values of x, then, it is tempting to extrapolate our observation, and claim that (noting $y = f(x)$ as always),

Let us consider $X = \mathrm{S} \times M$ the Cartesian product of the unit sphere S with M. Every element of X can be written as $x = (s_x, k_x)$ with $s_x \in \mathrm{S}$ and $k_x \in M$. For notational simplicity, we drop the subscript x whenever it is clear from the context. We define the classical projections of X on the sphere and M by $\pi_{\mathrm{S}} : X \to \mathrm{S}$ and $\pi_M : X \to M$.

We are interested in solving the following optimization problem for a given $\gamma \in (0,1)$:

$$\begin{array}{ll} \text{find} & P \\ \text{subject to} & (A_i s)^T P (A_i s) \leq \gamma^2 s^T P s, \, \forall A_i \in \mathcal{M}, \, \forall s \in \mathrm{S}, \\ & P \succ 0. \end{array} \tag{5}$$

Note that, if such a P exists, $\rho \leq \lambda$. Note that if P is a solution to (5), then so is αP for any $\alpha \in \mathbb{R}_{>0}$. Therefore, we can rewrite (5) as the following optimization problem:

$$\begin{array}{ll} \text{find} & P \\ \text{subject to} & (A_k s)^T P (A_k s) \leq \gamma^2 s^T P s, \, \forall A_i \in \mathcal{M}, \, \forall s \in \mathrm{S}, \\ & P \succeq I. \end{array} \tag{6}$$

We have to change the objective function!

Note that the optimization problem (6) has infinitely many constraints. We next consider the following optimization problem where we sample N constraints of (6) independently and identically with the probability measure $\mathbb{P}(A) = \frac{\mu(A)}{\mu(X)}, \forall A \in \Sigma$, where $N \geq d+1$, and $d := \frac{n(n+1)}{2}$. We denote this sampling by $\omega := \{x_1, x_2, \ldots, x_N\} \subset X$, and obtain the following convex optimization problem Opt(ω):

$$\begin{array}{ll} \text{find} & p \\ \text{subject to} & f(p, x) \leq 0, \forall x \in \omega. \end{array} \tag{7}$$

Let $p^*(\omega)$ be the solution of Opt(ω). We are interested in the probability of $p^*(\omega)$ violating at least one constraint in the original problem (6). Therefore, we define constraint violation property next.

Definition The constraint violation probability is defined as:

$$\mathcal{V}^*(\omega) = \mathbb{P}\{x \in X : f(p^*(\omega), x) > 0\}, \, \forall \omega \in X^N.$$

where $X^{N*} := \{\omega \in X^N :$ the solution of Opt(ω) exists$\}$ ([2]). Note that, since we have $\mathbb{P}(A) = \frac{\mu(A)}{\mu(X)}$, we can rewrite this as:

$$\mathcal{V}^*(\omega) = \frac{\mu\{V\}}{\mu(X)}, \, \forall \omega \in X^N,$$

where $V := \{x \in X : f(p^*(\omega), x) > 0\}$, i.e., the set of points for which at least one constraint is violated.

The following theorem from [2] explicitly gives a relationship between $V^*(\omega)$ and N, n.

Theorem 4.3 (from [2]): Consider the optimization problem Opt(ω) given in (7). How to talk about assumptions here? Then, for all $\epsilon \in (0,1)$ the following holds:

$$\mathbb{P}^N\{\{\mathcal{V}^*(\omega) \leq \epsilon\}\} \geq 1 - \sum_{j=0}^{d} \binom{N}{j} \epsilon^j (1-\epsilon)^{N-j}.$$

Note that $\epsilon = 1 - I^{-1}(\beta, N-d, d+1)$ and can be interpreted as the ratio of the measure of points in X that might violate at least one of the constraints in (**??**) to the measure of all points in X.

We now state our main theorem which gives a probabilistic upper bound on JSR, and devote the next section to proving it step by step.

Theorem 4.4 (Main Theorem): For any $\eta > 0$, given $N \geq n+1$ and $\beta \in [0,1)$, we can compute $\delta(N) < \infty$ such that with probability at least β, $\rho \leq \frac{\gamma^*}{\delta}$. Moreover, $\lim_{N\to\infty} \delta(N) = 1$.

A. Proving Theorem 4.4

$V_{\mathrm{S}} := \Pi_S(V), V_M := \Pi_M(V)$. Some transition here!

We assume from now that S is provided with $(\mathcal{B}_{\mathrm{S}}, \sigma^{n-1})$ and that M is provided with the classical σ-algebra associated to finite sets: $\Sigma_M = \wp(M)$, where $\wp(M)$ is the power set of M. We consider an unsigned finite measure[1] μ_M on (M, Σ_M) with $\mathrm{supp}(\mu_M) = M$. In other words, $\forall\, k \in M$, $\mu_M(\{k\}) > 0$. We denote the product σ-algebra $\mathcal{B}_{\mathrm{S}} \bigotimes \Sigma_M$ engendered by $\mathcal{B}_{\mathrm{S}}$ and Σ_M: $\Sigma = \sigma(\pi_{\mathrm{S}}^{-1}(\mathcal{B}_{\mathrm{S}}), \pi_M^{-1}(\Sigma_M))$. On this set, we define the product measure $\mu = \sigma^{n-1} \otimes \mu_M$ which is an unsigned finite measure on X.

Lemma 4.5: $\sigma^{n-1}(V_{\mathrm{S}}) \leq \frac{\mu(V)}{m_1}$, where $m_1 = \min\{\mu_M(\{k\}), k \in M\}$.

Proof:

Let $\mathcal{A} \subset \Sigma$, $\mathcal{A}_{\mathrm{S}} = \pi_{\mathrm{S}}(\mathcal{A})$ and $\mathcal{A}_M = \pi_M(\mathcal{A})$. We notice that Σ_M is the disjoint union of its 2^m elements $\{B_i, i \in \{1, 2, \ldots 2^m\}\}$. Then $\mathcal{A}$ is the disjoint union $\mathcal{A} = \sqcup_{1\leq i\leq 2^m}(\mathcal{A}_i, B_i)$ where $\mathcal{A}_i = \pi_M^{-1}(B_i) \in \mathrm{S}$. We notice that $\mathcal{A}_{\mathrm{S}} = \sqcup_{1\leq i\leq 2^m}\mathcal{A}_i$, and

$$\sigma^{n-1}(\mathcal{A}_{\mathrm{S}}) = \sum_{1\leq i\leq 2^m} \sigma^{n-1}(\mathcal{A}_i).$$

We have

$$\begin{aligned}
\mu(\mathcal{A}) &= \mu(\sqcup_{1\leq i\leq 2^m}(\mathcal{A}_i, B_i)) \\
&= \sum_{1\leq i\leq 2^m} \mu((\mathcal{A}_i, B_i)) \\
&= \sum_{1\leq i\leq 2^m} \sigma^{n-1} \otimes \mu_M((\mathcal{A}_i, B_i)) \\
&= \sum_{1\leq i\leq 2^m} \sigma^{n-1}(\mathcal{A}_i)\mu_M(B_i).
\end{aligned}$$

Let m_1 be the minimum value of μ_M on its atoms: $m_1 = \min\{\mu_M(\{k\}), k \in M\}$ (recall that $m_1 > 0$). Then since $\forall\, i$, $\mu_M(B_i) \geq m_1$, we have

$$\sigma^{n-1}(\mathcal{A}_{\mathrm{S}}) \leq \frac{\mu(\mathcal{A})}{m_1}. \tag{8}$$

This proves our statement by taking $\mathcal{A} = V$. ∎

Corollary 4.6: When the modes are sampled from the set M uniformly random,

$$\sigma^{n-1}(V_{\mathrm{S}}) \leq m\mu(V).$$

We consider the linear transformation mapping S to E_P, denoted by $L \in \mathcal{L}(\mathbb{R}^n)$. Note that since $P \in \mathcal{S}^n(\mathbb{R})$, it can be written in its Choleski form $P = UDU^{-1}$, with D diagonal matrix of its eigenvalues, and $U \in O_n(\mathbb{R})$. We define $D^{1/2}$ the positive square root of D as the matrix $\mathrm{diag}(\sqrt{d_1}, \ldots, \sqrt{d_n})$. Then, the positive square root of P is $UD^{1/2}U^{-1}$. This means that, $L = P^{1/2}$. For the rest of the write-up, we denote

$$V' := \Pi_{\mathrm{S}}(L^{-1}(V_{\mathrm{S}})),$$

and show how to upper bound $\sigma^{n-1}(V')$ in terms of $\mu(V)$.

Remark 4.1: If ψ is a smooth change of coordinates in $\mathbb{R}^n$ and $\mathcal{D} \subset \mathbb{R}^n$, whose image under ψ is $\mathcal{D}' \subset \mathbb{R}^n$, then

$$\lambda(\mathcal{D}') = \int_{x\in\mathcal{D}} 1_{x\in\mathcal{D}} |\det J(\psi(x))| d\lambda(x), \tag{9}$$

which becomes when $\psi \in \mathcal{L}(\mathbb{R}^n)$ (and thus $\forall\ x \in \mathbb{R}^n,\ \det J(\psi(x)) = \det(\psi)$)

$$\lambda(\mathcal{D}') = |\det(\psi)|\lambda(\mathcal{D}). \tag{10}$$

Theorem 4.7:

$$\sigma^{n-1}(\Pi_{\mathrm{S}}(L^{-1}(V_{\mathrm{S}}))) \leq \frac{\det(L^{-1})}{(\lambda_{\min}(L^{-1}))^n}\sigma^{n-1}(V_{\mathrm{S}}). \tag{11}$$

Proof: Let us denote by $B_{V_{\mathrm{S}}}$ the sector of B defined by V_{S}. We denote $C := L^{-1}(B_{V_{\mathrm{S}}})$. We have $\Pi_{\mathrm{S}}(C) = V'$ and $B_{V'} \subset \mathcal{H}_{\lambda_{\min}(L^{-1})}(C)$. We have then

$$\sigma^{n-1}(V') = \lambda(B_{V'}) \leq \lambda(\mathcal{H}_{\lambda_{\min}(L^{-1})}(C)),$$

which means: $\sigma^{n-1}(V') \leq \frac{1}{\lambda_{\min}(L^{-1})^n}\lambda(C)$. Using Remark 4.1, we have the result of the theorem. ∎

Corollary 4.8: $\sigma^{n-1}(V') \leq m\epsilon\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, where $\mu(V) = \epsilon$.

We denote $\epsilon' := \frac{\epsilon}{2}\sqrt{\frac{\lambda_{\max}(P)^n}{\det(P)}}$, where the additional factor $\frac{1}{2}$ follows from the homogeneity of the dynamics. In this section, we show how to relate ϵ' to δ in the statement of the Theorem 4.4. We start by a few definitions that will help us along the way. Let d be a distance on $\mathbb{R}^n$. We define the distance between a set $X \subset \mathbb{R}^n$ and a point $p \in \mathbb{R}^n$ is $d(X,p) := \inf_{x\in X} d(x,p)$.

Definition We define the *spherical cap* on S for a given hyperplane $c^T x = k$ as:

$$\mathcal{C}_{c,k} := \{x \in \mathrm{S} : c^T x > k\}.$$

*Proposition 4.9 (see e.g. [**?**]):* The distance between the point $x = 0$ and the hyperplane $c^T x = k$ is $\frac{|k|}{\|c\|}$.

We define the function $\Delta : 2^{\mathrm{S}} \to [0,1]$ as:

$$\Delta(X) := \sup\{r : \mathrm{B}_r \subseteq \mathrm{convhull}\ (\mathrm{S} \setminus X)\}. \tag{12}$$

Note that, $\Delta(X)$ can be rewritten as in:

$$\Delta(X) = d(\partial \mathrm{convhull}\ (\mathrm{S} \setminus X), 0). \tag{13}$$

Lemma 4.10: $\Delta(\mathcal{C}_{c,k}) = \min\left(1, \frac{|k|}{\|c\|}\right)$.

[1] Recall that the support of a measure μ defined on a measurable space (X, Σ) is $\mathrm{supp}(\mu) = \{A \in \Sigma | \mu(A) > 0\}$

Proof: Note that convhull $(\mathrm{S} \setminus X) = \{x \in \mathrm{B} : c^T x \le k\}$.

$$\begin{aligned}\Delta(X) &= d(\partial\text{convhull } (\mathrm{S}\setminus X), 0)\\ &= \min(d(\partial \mathrm{B}, 0), d(\partial\{x : c^T x \le k\}, 0))\\ &= \min(d(\mathrm{S}, 0), d(\{x : c^T x = k\}, 0))\\ &= \min\left(1, \frac{|k|}{\|c\|}\right).\end{aligned}$$

■

Corollary 4.11: $\Delta(\mathcal{C}_{c,k_1}) < \Delta(\mathcal{C}_{c,k_2})$ when $k_1 < k_2$.

Lemma 4.12: $\sigma^{n-1}(\mathcal{C}_{c,k_1}) < \sigma^{n-1}(\mathcal{C}_{c,k_2})$, for $k_1 > k_2$.

Proof: convhull $(\mathrm{S} \setminus \{x \in \mathrm{S} : c^T x > k_1\}) \subseteq$ convhull $(\mathrm{S} \setminus \{x \in \mathrm{S} : c^T x > k_2\})$, for $k_1 > k_2$. ■

Now we are ready to present the following lemma which is the key to proving our main result.

Lemma 4.13: For any set $X \subseteq \mathrm{S}$, there exists c and k such that $\mathcal{C}_{c,k}$ satisfies:

$$\mathcal{C}_{c,k} \subseteq X,$$

and

$$\Delta(\mathcal{C}_{c,k}) = \Delta(X). \tag{14}$$

Proof: Let $X_S :=$ convhull $(S \setminus X)$. Since the distance function d is continuous and the set ∂X_S is compact there exists a point $x^* \in \partial X_S$, such that:

$$\begin{aligned}\Delta(X) &= d(\partial X_S, 0) = \inf_{x\in\partial X_S} d(x, 0)\\ &= \min_{x\in\partial X_S} d(x, 0) = d(x^*, 0).\end{aligned} \tag{15}$$

Next, consider the supporting hyperplane of X_S at x^*, which we denote by $\{x : c^T x = k\}$. Note that, this supporting hyperplane is unique because it is also a supporting hyperplane of the ball $\mathrm{B}_{\Delta(X)}$ at x^* as well, which is unique. This can be seen from the fact that:

$$\partial \mathrm{B}_{\Delta(X)} \subseteq \partial X_S \subseteq \{x : c^T x = k\}.$$

Then we have:

$$\Delta(X) = d(x^*, 0) = d(\{x : c^T x = k\}, 0) = \frac{|k|}{\|c\|}.$$

Now, consider the spherical cap $\mathcal{C}_{c,k}$. Then, by Lemma we have $\Delta(\mathcal{C}_{c,k}) = \frac{|k|}{\|c\|}$. Therefore, $\Delta(X) = \Delta(\mathcal{C}_{c,k})$.

We next show $\mathcal{C}_{c,k} \subseteq X$. We prove this by contradiction. Assume $x \in \mathcal{C}_{c,k}$ and $x \notin X$. Note that, if $x \notin X$, then $x \in \mathrm{S} \setminus X \subseteq$ convhull $(\mathrm{S} \setminus X)$. Since $x \in \mathcal{C}_{c,k}$ we have $c^T x > k$. But due to the fact that $x \in$ convhull $(\mathrm{S} \setminus X)$, we also have $c^T x \le k$, which leads to a contradiction. Therefore, $\mathcal{C}_{c,k} \subseteq X$. ■

We now prove our main result.

Theorem 4.14: Let $X_{\epsilon'} = \{X \subset \mathrm{S} : \sigma^{n-1}(X) = \epsilon'\}$. Then, for any $\epsilon' \in (0, 1)$, the function $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ for some X which is a spherical cap.

Proof: We prove this via contradiction. Assume that there exists no spherical cap in $X_{\epsilon'}$ such that $\Delta(X)$ attains its minimum. This means there exists an $X^* \in X_{\epsilon'}$, where X^* is not a spherical cap and $\arg\min_{X\in X_{\epsilon'}}(\Delta(X)) = X^*$. By Lemma 4.13 we can construct a spherical cap $\mathcal{C}_{c,k}$ such that $\mathcal{C}_{c,k} \subseteq X^*$ and $\mathcal{C}_{c,k} = \Delta(X^*)$. Note that, we further have $\mathcal{C}_{c,k} \subset X^*$, since X^* is assumed not to be a spherical cap. This means that, there exists a spherical cap $\sigma^{n-1}(\mathcal{C}_{c,k})$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) < \epsilon'$.

Then, the spherical cap $\mathcal{C}_{c,\tilde{k}}$ with $\sigma^{n-1}(\mathcal{C}_{c,\tilde{k}}) = \epsilon'$, satisfies $\tilde{k} < k$, due to Lemma 4.12. This implies $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(\mathcal{C}_{c,k}) = \Delta(X^*)$ due to Lemma 4.11. Therefore, $\Delta(\mathcal{C}_{c,\tilde{k}}) < \Delta(X^*)$. This is a contradiction since we initially assumed that $\Delta(X)$ attains its minimum over $X_{\epsilon'}$ at X^*. ■

Theorem 4.15: Given a spherical cap $\mathcal{C}_{c,k} \subseteq \mathrm{S}$ such that $\sigma^{n-1}(\mathcal{C}_{c,k}) = \epsilon'$,

$$\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)},$$

where $\alpha := I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$. Here I^{-1} is the inverse incomplete beta function, i.e., $I^{-1}(y, a, b) = x$ where $I_x(a, b) = y$.

Proof: Let $h := 1-\Delta(\mathcal{C}_{c,k})$. It is well known [6] that the area of the spherical cap $\mathcal{C}_{c,k} \subseteq \mathrm{S}$ is given by the equation:

$$\epsilon' = \sigma^{n-1}(\mathcal{C}_{c,k}) = \frac{\pi^{d/2}}{\Gamma[\frac{d}{2}]} I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right), \tag{16}$$

where I is the incomplete beta function. From this, we get the following set of equations:

$$\begin{aligned}\frac{\epsilon'\Gamma[\frac{d}{2}]}{\pi^{d/2}} &= I_{2h-h^2}\left(\frac{d-1}{2}, \frac{1}{2}\right)\\ 2h - h^2 &= I^{-1}\left(\frac{\epsilon'\Gamma(\frac{d}{2})}{\pi^{d/2}}, \frac{d-1}{2}, \frac{1}{2}\right)\\ 2h - h^2 &= \alpha\\ h^2 - 2h + \alpha &= 0.\end{aligned} \tag{17}$$

From (17), we get $h = 1 \pm \sqrt{(1-\alpha)}$. Since $h \le 1$, we conclude that $\Delta(\mathcal{C}_{c,k}) = \sqrt{(1-\alpha)}$. Note that, $\Delta(\mathcal{C}_{c,k})$ only depends on ϵ for fixed n. ■

Corollary 4.16: For a fixed $\beta \in (0, 1)$, $\lim_{N\to\infty} \delta_\beta(N) = 1$.

Proof: We first prove that $\lim_{N\to\infty} \epsilon_\beta(N) = 0$. Note that, we can upper bound $1 - \beta$ as follows:

$$\begin{pmatrix} 1-\beta = \sum_{j=0}^{d} N \\ j\epsilon^j(1-\epsilon)^{N-j} \le (d+1)N^d(1-\epsilon)^{N-d}. \end{pmatrix} \tag{18}$$

We prove $\lim_{N\to\infty} \epsilon_\beta(N) = 0$ by contradiction and assume that $\lim_{N\to\infty} \epsilon_\beta(N) \neq 0$. This means that, there exists some $\delta > 0$ such that $\epsilon_\beta(N) > \delta$ infinitely often. Then, consider the subsequence N_k such that $\epsilon_\beta(N_k) > \delta$, $\forall\, k$. By (18) we have:

$$1-\beta \le (d+1)N_k^d(1-\epsilon)^{N_k-d} \le (d+1)N_k^d(1-\delta)^{N_k-d} \forall\, k \in \mathbb{N}.$$

Note that $\lim_{k\to\infty}(d+1)N_k^d(1-\delta)^{N_k-d} = 0$. Therefore, there exists a k' such that, we have $(d+1)N_{k'}^d(1-\delta)^{N'_k-d} < 1-\beta$, which is a contradiction. Therefore, we must have $\lim_{N\to\infty} \epsilon_\beta(N) = 0$.

Showing I^{-1} in its first parameter, $\delta = \sqrt{1-\alpha}$ tends to 1 as $\epsilon \to 0$.

■

V. Experimental Results

VI. Future Work

VII. Conclusions

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