

# Data Driven Stability Analysis of Black-box Switched Linear Systems with Probabilistic Guarantees

## I. INTRODUCTION

Today's complex engineered systems (you can replace complex engineered systems with CPS) are characterized by the interaction of a large number of heterogeneous components. Consequently, the models used to analyze these systems are equally complex and consist of heterogeneous sub-models relying on different modeling assumptions and based on principles from different scientific disciplines. It is not uncommon to encounter a patchwork of differential equations, difference equations, hybrid automata, lookup tables, custom switching logic, low-level legacy code, etc. To further compound the difficulty in analyzing these systems, different components of a complex engineered system are typically designed by different suppliers. Although a high-level specification for these components may be known, detailed models are not available for intellectual property reasons. We are thus faced with a tremendous gap between the existing analysis techniques that rely on closed-form models and the models available in industry. It is, therefore, not surprising the emphasis that industry places on simulation since despite the complexity of models, it is always possible to simulate them. Therefore, it is a natural question to ask whether we can provide formal analyses about certain properties of these complex systems based solely on the information obtained via their simulations. In this paper, we focus on one of the most important of such properties in the context of control theory: stability.

More formally, we consider a dynamical system as in:

$$x_{k+1} = f(k, x_k), \quad (1)$$

where,  $x_k \in X$ ,  $k \in \mathbb{N}$  is used to index time. We start with the following question to serve as a stepping stone: Given  $N$  pairs,  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $\dots$ ,  $(x_N, y_N)$  such that  $y_k = f(k, x_k)$ , what can we say about the stability of the system (1)? For the rest of the paper, we use the term *black-box* to refer to models where we do not have access to its dynamics, yet we can observe  $f(x)$  by exciting it with different initial conditions  $x$ . **We should say that this is non-ideal when we don't know what the state is, but it is a start, and it makes sense in certain situations.** One approach to this problem is firstly identifying the dynamics, i.e.,  $f$  and then applying the existing techniques in the model-based stability analysis literature. However, unless  $f$  is a linear function, there are two main reasons behind our quest to directly work on input-output pairs and bypassing the identification phase: (1) Even when the function  $f$  is known, in general, the stability analysis is a very difficult problem [?], [?]. (2) **Paulo**

**wants to change this: The existing identification techniques can only identify  $f$  up to an approximation error. How to relate this identification error to an error in the stability of the system (1) is still a nontrivial problem.**

The initial idea behind this paper was influenced by the recent efforts in [7], [5] and [1] in using simulation traces to find Lyapunov functions for systems with known dynamics. **Will put Liberzon here.** In these works, the main idea is that if one can construct a Lyapunov function candidate decreasing along many finite trajectories starting from different initial conditions, it should also decrease along every other trajectory. Then, once a Lyapunov function candidate is constructed, this intuition is put to test by verifying the candidate function either via off-the-shelf tools as in [7] and [5], or via sampling based techniques as in [1]. Note that, since we do not have access to the dynamics, the second step cannot be directly applied to black-box systems. However, these sampling based ideas raise the following question that we address in this paper: By observing that a candidate Lyapunov function decreases on a large number of simulations we empirically build a certain confidence that such candidate Lyapunov function is a bona-fide Lyapunov function. *Can we translate this confidence into a confidence in the stability of the underlying system?*

Note that, even in the case of a 2D linear system the connection between these two beliefs is nontrivial. In fact, one can easily construct an example with one stable and one unstable eigenvalue for which even though almost all trajectories diverge to the infinity, it is possible to construct a Lyapunov function candidate whose level sets are contracting everywhere except a small set. **Should we give a specific example here, and put a figure?** Moreover, the size of this "violating set" can be arbitrarily small based on the magnitude of the unstable eigenvalue.

In this paper, we take the first step to close this gap. Since the identification and stability analysis of linear systems are well understood, we do so by focusing on switched linear systems. Note that identification and deciding the stability of arbitrary switched linear systems is NP-hard [4]. Aside from their theoretical value, switched systems model the behavior of dynamical systems in the presence of known or unknown varying parameters. These parameters can model internal properties of the dynamical system such as uncertainties, look-up tables, values in a discrete register as well as exogenous inputs provided by a controller in a closed-loop control system. **Need to make these examples more specific.**

The stability of switched systems closely is closely related to the *joint spectral radius* (JSR) of the matrices appearing in (2). Under certain conditions deciding stability amounts

to deciding whether JSR is less than one [4]. In this paper, we present an algorithm to bound the JSR of a switched linear system from  $N$  observations. This algorithm is based on tools from the random convex optimization literature [2], and provides an upper bound on the JSR with a user-defined confidence level. As  $N$  increases, this bound gets tighter. Moreover, with a closed form expression, we characterize what the exact trade-off between the tightness of this bound and the number of samples is. In order to understand the quality of our upper bound, the algorithm also provides a deterministic lower bound.

The organization of the paper is as follows: **TO BE FILLED**.

## II. PRELIMINARIES

### A. Notation

We consider the usual Hilbert finite normed vector space  $(\mathbb{R}^n, \ell_2)$ ,  $n \in \mathbb{N}_{>0}$ , with  $\ell_2$  the classical Euclidean norm. We denote the set of linear functions in  $\mathbb{R}^n$  by  $\mathcal{L}(\mathbb{R}^n)$ , and the set of real symmetric matrices of size  $n$  by  $\mathcal{S}^n$ . We denote a unit ball in  $\mathbb{R}^n$  with  $B$  and unit sphere in  $\mathbb{R}^n$  as  $S$ . We only explicitly state the radius  $r$  as in  $B_r$  and  $S_r$ , when  $r$  is different than 1. We denote the ellipsoid described by the matrix  $P \in \mathcal{S}^n$  as  $E_P$ , i.e.,  $E_P := \{x \in \mathbb{R}^n : x^T P x = 1\}$ , and we denote by  $\tilde{E}_P$  the volume in  $\mathbb{R}^n$  defined by  $E_P$ :  $\tilde{E}_P = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$ . We denote the (convex) projector on  $S$  by  $\Pi_S$ . We denote the homothety of ratio  $r$  by  $\mathcal{H}_r$ .

For an ellipsoid centered at the origin, and for any of its subsets  $\mathcal{A}$ , the sector defined by  $\mathcal{A}$  is the subset

$$\{t\mathcal{A}, t \in [0, 1]\} \subset \mathbb{R}^n.$$

A sector induced by  $\mathcal{A} \subset E_P$  will be denoted by  $E_P^{\mathcal{A}}$ . In the particular case of the unit sphere, we instead write  $S^{\mathcal{A}}$ .

We consider in this work the classical unsigned and finite uniform spherical measure on  $S$ , denoted by  $\sigma^{n-1}$ . It is associated to  $\mathcal{B}_S$ , the spherical Borelian  $\sigma$ -algebra, and is derived from the Lebesgue measure  $\lambda$ . We have  $\mathcal{B}_S$  defined by  $\mathcal{A} \in \mathcal{B}_S$  if and only if  $S^{\mathcal{A}} \in \mathcal{B}_{\mathbb{R}^n}$ . The spherical measure  $\sigma^{n-1}$  is defined by

$$\forall \mathcal{A} \in \mathcal{B}_S, \sigma(\mathcal{A}) = \frac{\lambda(S^{\mathcal{A}})}{\lambda(B)}.$$

In other words, the spherical measure of a subset of the sphere is related to the Lebesgue measure of the sector of the unit ball it induces. Notice that  $\sigma^{n-1}(S) = 1$ . Similarly, we have a measure on the ellipsoid  $\sigma_P$  defined on the  $\sigma$ -algebra  $\mathcal{B}_{E_P}$  by:  $\forall \mathcal{A} \in \mathcal{B}_{E_P}, \sigma_P(\mathcal{A}) = \frac{\lambda(E_P^{\mathcal{A}})}{\lambda(E_P)}$ .

### B. Stability of Linear Switched Systems

A switched linear system is of the form:

$$x_{k+1} = A_{\sigma(k)} x_k, \quad (2)$$

with switching sequence  $\sigma : \mathbb{N} \rightarrow M$ , where  $M := \{1, 2, \dots, m\}$  and  $A_{\sigma(k)} \in \mathcal{M}$ , for all  $\sigma$  and  $k \in M$ . There are two important properties of linear switched systems that we exploit in this paper.

*Property 2.1:* Let  $\xi(x, k)$  denote the state of the system (2) at time  $k$  starting from the initial condition  $x$ . The dynamical system (2) is homogeneous:

$$\xi(\gamma x, k) = \gamma \xi(x, k).$$

*Property 2.2:* The dynamics given in (2) is convexity-preserving, meaning that for any set of points  $X \subset \mathbb{R}^n$  we have:

$$f(\text{convhull } X) \subset \text{convhull } \{f(X)\}.$$

We now introduce the *Lyapunov exponent* of the system, which is a numerical quantity describing its stability.

**Definition** Given a dynamical system as in (1), its *Lyapunov exponent* is given by

$$\rho = \inf \{r : \forall x_0, \exists C \in \mathbb{R}^+ : \xi(x_0, k) \leq C r^k\}.$$

**@Raphael:** We need a theorem about the stability and  $\rho < 1$  here.

In the case of switched linear systems, the Lyapunov exponent is known as the joint spectral radius of the set of matrices, which can be alternatively defined as follows:

**Definition** [3] Given a set of matrices  $\mathcal{M} \subset \mathbb{R}^{n \times n}$ , its *joint spectral radius* (JSR) is given by

$$\rho(\mathcal{M}) = \lim_{k \rightarrow \infty} \max_{i_1, \dots, i_k} \{\|A_{i_1} \dots A_{i_k}\|^{1/k} : A_{i_j} \in \mathcal{M}\}.$$

*Remark 2.1:* Note that, JSR is a homogeneous function:

$$\rho\left(\frac{\mathcal{M}}{\gamma}\right) = \frac{\rho(\mathcal{M})}{\gamma}, \forall \gamma > 0.$$

## III. A DETERMINISTIC LOWER BOUND FOR JSR

We start by computing a lower bound for  $\rho$  which is based on the following theorem from the switched linear systems literature.

*Theorem 3.1:* [3, Theorem 2.11] For any bounded set of matrices such that  $\rho(\mathcal{M}) < \frac{1}{\sqrt{n}}$ , there exists a Common Quadratic Lyapunov Function (CQLF) for  $\mathcal{M}$ , that is, a  $P \succ 0$  such that:

$$\forall A \in \mathcal{M}, A^T P A \preceq P.$$

The following theorem shows that the existence of a CQLF for (2) can be checked by considering  $N$  pairs  $(x_i, j_i) \in \mathbb{R}^n \times M$ , where  $i \in \{1, \dots, N\}$ .

*Theorem 3.2:* For a given sampling:

$$\omega_N := \{(x_1, j_1), (x_2, j_2), \dots, (x_N, j_N)\},$$

let  $\gamma^*(\omega_N)$  be the optimal solution of the following optimization problem:

$$\begin{aligned} \min \quad & \gamma \\ \text{s.t.} \quad & (A_{j_i} x_i)^T P (A_{j_i} x_i) \leq \gamma^2 x_i^T P x_i, \forall i : 1 \leq i \leq N \\ & P \succ 0. \end{aligned} \quad (3)$$

If  $\gamma^*(\omega_N) < \infty$ , we have:

$$\rho(\mathcal{M}) \geq \frac{\gamma^*(\omega_N)}{\sqrt{n}}.$$

Note that, (3) can be solved by bisection on  $\gamma$ .

*Proof:* Using Remark 2.1, for any  $\epsilon > 0$ ,  $\frac{\mathcal{M}}{(\gamma^*(\omega_N) - \epsilon)}$  has no CQLF. Then, applying Theorem 3.1 we get

$$\frac{\rho(\mathcal{M})}{\gamma^*(\omega_N)} \geq \frac{1}{\sqrt{n}}.$$

■

#### IV. A PROBABILISTIC STABILITY GUARANTEE

In this section, we show how to compute an upper bound on  $\rho$ , with a user-defined confidence  $\beta \in (0, 1)$ . We do this by constructing a CQLF which is valid with probability at least  $\beta$ . Note that, the existence of a CQLF on  $\mathcal{A}$  implies  $\rho < 1$  due to Theorem ?? and due to Property 2.1, it is enough to show that the CQLF is decreasing on a set enclosing the origin, e.g.  $\mathcal{S}$ . Therefore, to obtain an upper bound on  $\rho$ , we consider the following optimization problem:

$$\begin{aligned} \min_{\gamma, P} \quad & \gamma \\ \text{s.t.} \quad & (Ax)^T P(Ax) \leq \gamma^2 x^T P x, \forall A \in \mathcal{M}, \forall x \in \mathcal{S}, \\ & P \succ 0. \end{aligned} \quad (4)$$

Note that, for all  $A \in \mathcal{M}$  the optimal  $P$  and  $\gamma^*$  satisfies  $\frac{A^T}{\gamma^*} P \frac{A}{\gamma^*} \preceq P$ . Therefore,  $\rho\left(\frac{\mathcal{M}}{\gamma^*}\right) \leq 1$ , which leads to the upper bound  $\rho(\mathcal{M}) \leq \gamma^*$ . However, solving the optimization problem (4) is hard since it involves infinitely many constraints. Therefore, we instead sample  $N$  initial states and modes uniformly random from the set  $\mathcal{S} \times \mathcal{M}$ , and solve the following optimization problem with finitely many constraints instead:

$$\begin{aligned} \min_{\gamma, P} \quad & \gamma \\ \text{s.t.} \quad & (Ax)^T P(Ax) \leq \gamma^2 x^T P x, \forall (x, j) \in \omega_N, \\ & P \succ 0. \end{aligned} \quad (5)$$

where  $\omega_N$  is a  $N$ -uniform random sampling of the set  $Z := \mathcal{S} \times \mathcal{M}$ . Note that, since (5) is convex for a fixed  $\gamma$ , we can perform bisection on  $\gamma$  and solve a series of feasibility problems in  $P$  instead. We now analyze the relationship between the solutions of the optimization problem (4) and the following optimization problem:

$$\begin{aligned} \min_{P \in \mathcal{P}} \quad & \lambda_{\max}(P) \\ \text{s.t.} \quad & (A_j x)^T P(A_j x) \leq \eta(N) \gamma^{*2} x^T P x, \forall (x, j) \in \omega_N, \\ & P \succeq I. \end{aligned} \quad (6)$$

where  $\omega_N$  is a  $N$ -uniform random sampling of the set  $Z := \mathcal{S} \times \mathcal{M}$ ,  $\mathcal{P}$  is a closed convex set,  $\eta(N) := (1 + \frac{\eta}{N})$  for some  $\eta > 0$ , and  $\gamma^*(\omega_N)$  is the optimal solution to the optimization problem (5). For the rest of the discussion, we refer to the optimization problem (6) by  $\text{Opt}(\omega_N)$ . We denote its optimal solution by  $P(\omega_N)$ , where we drop the explicit dependence on  $\omega_N$ , when it is clear from the context. There are a few points that are worth noting about (6). Firstly, due to Property 2.1, we are able to replace the constraint  $P \succ 0$  with the constraint  $P \succeq I$ . Moreover, for reasons that will become clear later in the discussion, we chose the objective function as  $\lambda_{\max}(P)$ , instead of solving a feasibility problem in  $P$ . Lastly, the additional factor is needed to ensure strict

feasibility of (6), which will be helpful in the preceding discussion.

The curious question whether the optimal solution of the sampled problem  $\text{Opt}(\omega_N)$  is a feasible solution to (4) has been widely studied in the random convex optimization literature [2]. It turns out that under certain technical assumptions, the optimal solution of (6) is feasible for the original problem (4), with some probability which is a function of the sample size  $N$ . To formalize this discussion, we define the constraint violation probability next.

**Definition** (from [2]) For all  $\omega_N$  for which a solution to  $\text{Opt}(\omega_N)$  exists, the *constraint violation probability* is defined as:

$$\mathcal{V}^*(\omega_N) = \mathbb{P}\{z \in Z : f(P(\omega_N), z) > 0\}. \quad (7)$$

Note that, since we have  $\mathbb{P}(\mathcal{A}) = \frac{\mu(\mathcal{A})}{\mu(Z)}$ , we can rewrite (7) as:

$$\mathcal{V}^*(\omega_N) = \frac{\mu(V)}{\mu(Z)},$$

where  $\mu$  is a measure on  $Z$  and

$$V(\omega_N) := \{z \in Z : f(P(\omega_N), z) > 0\},$$

i.e., the set of points for which at least one constraint is violated for the sampling  $\omega_N$ .

The following theorem from [2] gives an explicit relationship between  $\mathcal{V}^*(\omega_N)$ ,  $N$ , and  $n$ .

**Theorem 4.1:** Let  $d := \frac{n(n+1)}{2} + 1$  and  $N \geq d + 1$ . Consider the optimization problem  $\text{Opt}(\omega_N)$  given in (6). If  $\text{Opt}(\omega_N)$  satisfies the following technical assumptions:

- 1) When the problem  $\text{Opt}(\omega_N)$  admits an optimal solution, this solution is unique.
- 2) Problem  $\text{Opt}(\omega_N)$  is nondegenerate<sup>1</sup> with probability one.

Then, for all  $\epsilon \in (0, 1)$  the following holds:

$$\mathbb{P}^N\{\mathcal{V}^*(\omega_N) \leq \epsilon\} \geq 1 - \sum_{j=0}^d \binom{N}{j} \epsilon^j (1 - \epsilon)^{N-j}.$$

Note that,  $\epsilon$  is a constant can be interpreted as the ratio of the measure of points in  $Z$  that might violate at least one of the constraints in (6) to the measure of all points in  $Z$ .

*Proof:* The proof is an immediate application of Theorem ?? in [2], since the  $\text{Opt}(\omega_N)$  can be written as:

$$\begin{aligned} \min_{P \in \mathcal{P}, t} \quad & t \\ \text{s.t.} \quad & f_{\gamma^*}(P, z) \leq 0, \forall z \in Z \end{aligned} \quad (8)$$

where  $g_{\gamma^*}(P, z) = \max(g_1(P, z), g_2(P), g_3(P))$ , and

$$\begin{aligned} g_1(P, z) &:= (A_j z)^T P(A_j z) - \gamma^{*2} z^T P z \\ g_2(P) &:= \lambda_{\max}(-P) + 1. \\ g_3(P) &:= \lambda_{\max}(P) - t. \end{aligned}$$

■

We first note that, both of the assumptions in the statement of the theorem are technical and even if they do not hold can

<sup>1</sup>Explain this in a footnote maybe?

circumvented by slightly modifying the optimization problem 8. We refer the interested reader to [2] for a more detailed discussion of such techniques. The objective function of (8) is linear while each constraint is convex in  $P$  for all  $z \in Z$ . Moreover, the set  $\mathcal{P}$  being closed and convex, without limiting the feasible region further we can assume  $t$  lies in a closed interval as well. Also note that, the set of decision variables are in  $\mathbb{R}^{\frac{n(n+1)}{2}+1}$ . Then, we can invoke Theorem ?? in [2] with the optimization problem (8) to conclude the statement of the theorem.

Theorem 4.1 states that the optimal solution of the sampled problem  $\text{Opt}(\omega_N)$  violates an  $\epsilon$  fraction of the constraints in the original optimization problem (4) with probability  $\beta$ , where  $\beta$  goes to 1 as  $N$  goes to infinity.

We next show that the measure of constraints that are violated in (4)

*Theorem 4.2:* Consider a set of matrices  $A \in \mathcal{M}$ , and an ellipsoid  $E_P$ , where with probability at least  $\beta \in (0, 1]$ ,  $P$  satisfies:

$$(A_j x)^T P (A_j x) \leq \gamma^* x^T P x, \forall x \in E_P \setminus Z_P, \forall j \in M \quad (9)$$

where  $Z_P \subset E_P$  and  $\sigma^{n-1}(Z_P) = \epsilon$ , we can compute  $\delta$  such that:

$$A_j E_P \subset \delta E_P. \quad (10)$$

*Theorem 4.3:* Consider an  $n$ -dimensional switching system as in (2). For any given  $\beta \in (0, 1)$  and a uniform random sampling  $\omega_N \subset Z$ , with  $N \geq \frac{n(n+1)}{2} + 1$ , and let  $\gamma^*(\omega_N)$  be the optimal solution to (6). Then, we can compute  $\delta$  such that with probability at least  $\beta$ , we have:

$$\rho \leq \frac{\gamma^*(\omega_N)}{\delta}.$$

## V. EXPERIMENTAL RESULTS

### VI. FUTURE WORK

### VII. CONCLUSIONS

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