№11 М.А. Некрасов Артём 216

Вычислить интегралы:

1. $\iint_{\substack{1-x\le y\le 3-x\\ \frac{x}{2}\le y\le 2x}} \frac{(x+y)^2}{xy}\,dxdy$

<u>Решение.</u>

$$\begin{cases} 1-x \le y \le 3-x \\ \dfrac{x}{2} \le y \le 2x \end{cases} \Rightarrow \begin{cases} 1 \le x+y \le 3 \\ \dfrac{1}{2} \le \dfrac{y}{x} \le 2 \end{cases} \Rightarrow \begin{bmatrix} x+y=u \\ \dfrac{y}{x}=v \end{bmatrix} \Rightarrow \begin{cases} u \in [1,3] \\ v \in \left[\dfrac{1}{2},2\right] \end{cases}$$

$$J^{-1} = \begin{vmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{u}{x^2} \Rightarrow \quad J = \frac{x^2}{u}$$

$$\iint_{\substack{1-x\le y\le 3-x\\ \frac{x}{2}\le y\le 2x}} \frac{(x+y)^2}{xy}dxdy = \int_1^3 du \int_{\frac{1}{2}}^2 \frac{u^2}{xy}\cdot\frac{x^2}{u}\,dv = \int_1^3 du \int_{\frac{1}{2}}^2 \frac{ux}{y}dv$$

$$\int_{\frac{1}{2}}^2 \frac{u}{v}dv = u\cdot ln|v|\Big|_{\frac{1}{2}}^2 = u\ln 2 - u\ln\frac{1}{2} = 2u\ln 2$$

$$\int_1^3 2u\ln 2\,du = 2\ln 2\int_1^3 u\,du = ln\,2\,u^2\Big|_1^3 = \ln 2\cdot 3^2 - \ln 2\cdot 1^2 = \boxed{8\ln 2}$$

2. $\iint_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 4\\ x>0,y>0}} (x^2+y^2)dxdy \ (a,b>0)$

<u>Решение.</u>

$$\begin{cases} x = ar\,cos\,\varphi > 0 \\ y = br\,sin\,\varphi > 0 \end{cases} \Rightarrow\Rightarrow \begin{cases} \dfrac{a^2r^2\,cos^2\,\varphi}{a^2} + \dfrac{b^2r^2\,sin^2\,\varphi}{b^2} \le 4 \\ r > 0 \\ \varphi \in [0,2\pi) \end{cases} \Rightarrow \begin{cases} r^2 \le 4 \\ \cos\varphi > 0 \\ \sin\varphi > 0 \\ \varphi \in [0,2\pi) \\ r > 0 \end{cases} \Rightarrow \begin{cases} r \in [0,2] \\ \varphi \in [0,\dfrac{\pi}{2}] \end{cases}$$

$$J = \begin{vmatrix} a\cos\varphi & -ar\sin\varphi \\ b\sin\varphi & br\cos\varphi \end{vmatrix} = abr$$

$$\iint_{\substack{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 4\\ x>0,y>0}} (x^2+y^2)dxdy\ (a,b>0) = \int_0^{\frac{\pi}{2}} d\varphi \int_0^2 (a^2r^2\cos^2\varphi + b^2r^2\sin^2\varphi)abr\,dr$$

$$\int_0^2 (a^2r^2\cos^2\varphi + b^2r^2\sin^2\varphi)abr\,dr = ab\int_0^2 (a^2r^3\cos^2\varphi + b^2r^3\sin^2\varphi)dr =$$

$$= ab\left(\underbrace{\int_0^2 a^2r^3\,cos^2\,\varphi\,dr}_{\left.\frac{a^2r^4\,cos^2\,\varphi}{4}\right|_0^2} + \underbrace{\int_0^2 b^2r^3\,sin^2\,\varphi\,dr}_{\left.\frac{b^2r^4\,sin^2\,\varphi}{4}\right|_0^2}\right) = \left.\frac{a^3br^4\,cos^2\,\varphi + ab^3r^4\,sin^2\,\varphi}{4}\right|_0^2 = \frac{a^3b2^4\,cos^2\,\varphi + ab^3 2^4\,sin^2\,\varphi}{4} - 0 =$$

$$= 4a^3b\cos^2\varphi + 4ab^3\sin^2\varphi$$

$$\int_0^{\frac{\pi}{2}}(4a^3b\cos^2\varphi + 4ab^3\sin^2\varphi)d\varphi = \underbrace{\int_0^{\frac{\pi}{2}} 4a^3b\,cos^2\,\varphi\,d\varphi}_{2a^3b\varphi + a^3b\cdot\sin 2\varphi|_0^{\frac{\pi}{2}}} + \underbrace{\int_0^{\frac{\pi}{2}} 4ab^3\,sin^2\,\varphi\,d\varphi}_{2ab^3\varphi - ab^3\cdot\sin 2\varphi|_0^{\frac{\pi}{2}}} =$$

$$= 2a^3b\varphi + a^3b\cdot\sin 2\varphi + 2ab^3\varphi - ab^3\cdot\sin 2\varphi\Big|_0^{\frac{\pi}{2}} = a^3b\pi + ab^3\pi - 0 = \boxed{a^3b\pi + ab^3\pi}$$

3. $\iint_{[0,a]^n} \dots \int e^{c_1x_1+\cdots c_nx_n}dx_1 \dots dx_n \qquad (c_i \neq 0)$

<u>Решение.</u>

$$\int_0^a dx_1 \cdot \int_0^a dx_2 \cdot \dots \cdot \int_0^a e^{c_1x_1+c_2x_2+\cdots+c_nx_n}dx_n = \int_0^a e^{c_1x_1}dx_1 \cdot \int_0^a e^{c_2x_2}dx_2 \cdot \dots \cdot \int_0^a e^{c_{n-1}x_{n-1}}dx_{n-1} \cdot \underbrace{\left.\frac{e^{c_nx_n}}{c_n}\right|_0^a}_{\frac{e^{c_na}-1}{c_n}} \Rightarrow \cdots \Rightarrow$$

$$\Rightarrow \frac{e^{c_1a}-1}{c_1}\cdot\frac{e^{c_2a}-1}{c_2}\cdot\dots\cdot\frac{e^{c_{n-1}a}-1}{c_{n-1}}\cdot\frac{e^{c_na}-1}{c_n} = \boxed{\frac{(e^{c_1a}-1)\cdot\dots\cdot(e^{c_na}-1)}{c_1\cdot\dots\cdot c_n}}$$

4. $\iint_{\substack{x_1\geq 0,\dots x_n\geq 0,\\ x_1+\cdots+x_n\leq 1}} \dots \int (x_1+\cdots+x_n)dx_1 \dots dx_n$

<u>Решение.</u>