1. Решите уравнения первого порядка, подробно расписывая все этапы решения:

$a)\ (2xy^2 - y)dx + xdy = 0$

Решение.

$(y - 2xy^2)dx = xdy \quad |: y^2 \qquad (\textit{не забыть проверить } y = 0)$

$$\left(\frac{1}{y} - 2x\right)dx = \frac{xdy}{y^2}$$

$$\frac{xdy}{y^2} + \left(2x - \frac{1}{y}\right)dx = 0$$

$$\left(2x - \frac{1}{y}\right)dx - \frac{0 - xdy}{y^2} = 0$$

$$d\left(-\frac{x}{y}\right) + dx^2 = 0$$

$$\int d\left(-\frac{x}{y}\right) + \int dx^2 = 0$$

$$-\frac{x}{y} + x^2 = C$$

$$\boxed{y = \frac{x}{x^2 + C}, \qquad y = 0}$$

$b)\ y + y' \ln^2 y = (x + 2\ln y)\, y'$

Решение.

$$y' = \frac{dy}{dx} \quad \Rightarrow \quad y + \frac{dy}{dx}\ln^2 y = (x + 2\ln y)\frac{dy}{dx} \quad |\cdot dx$$

$$ydx + dy \cdot \ln^2 y = xdy + dy \cdot 2\ln y$$

$$2\ln y \cdot dy - \ln^2 y \cdot dy = ydx - xdy$$

$$\star\ d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2} \quad \Rightarrow \quad ydx - xdy = y^2 d\left(\frac{x}{y}\right)$$

$$\star\ d\left(\frac{\ln^2 y}{y}\right) = \frac{yd(\ln^2 y) - \ln^2 y\, dy}{y^2} \quad \Rightarrow \quad d(\ln^2 y) = \frac{2\ln y}{y} dy$$

$$2\ln y\, dy - \ln^2 y\, dy = y^2 \cdot d\left(\frac{\ln^2 y}{y}\right) \quad |: y^2$$

$$d\left(\frac{x}{y}\right) = d\left(\frac{\ln^2 y}{y}\right)$$

$$\int d\left(\frac{x}{y}\right) = \int d\left(\frac{\ln^2 y}{y}\right)$$

$$\frac{x}{y} = \frac{\ln^2 y}{y} + C$$

$$\boxed{\frac{x - \ln^2 y}{y} = C}$$

$c)\ \dfrac{y - xy'}{x + yy'} = 2$

Решение.

$$2x + 2yy' = y - xy'$$

$$2xdx + 2ydy = ydx - xdy$$

$$(x + 2y)dy = (y - 2x)dx$$

$$\begin{bmatrix} y = tx, \qquad t = t(x) \\ dy = xdt + tdx \end{bmatrix}$$

$$(x + 2tx)(xdt + tdx) = (tx - 2x)dx$$

$$x^2dt + txdx + 2tx^2dt + 2t^2xdx - txdx + 2xdx = 0$$

$$x^2dt + 2tx^2dt + 2t^2xdx + 2xdx = 0$$

$$x^2(1 + 2t)dt + 2x(1 + t^2)dx = 0$$

$$2x(1 + t^2)dx = -x^2(1 + 2t)dt \quad |\cdot \frac{1}{x^2(1 + t^2)}$$

$$\frac{2}{x}dx = -\frac{1 + 2t}{1 + t^2}dt$$

$$\frac{2}{x}dx = -\frac{1}{1 + t^2}dt - \frac{2t}{1 + t^2}dt$$

$$2\int \frac{1}{x}dx = -\left(\int \frac{1}{1 + t^2}dt + \int \frac{2t}{1 + t^2}dt\right)$$

$$2\ln|x| = -\ln\left|1 + \frac{y^2}{x^2}\right| - \operatorname{arctg}\frac{y}{x} + C$$

$$\boxed{2\ln|x| + \ln\left|1 + \frac{y^2}{x^2}\right| + \operatorname{arctg}\frac{y}{x} = C}$$

$d)\ (\sin x + y)dy + (y\cos x - x^2)dx = 0$

Решение.

$$\cos x = \frac{\partial(\sin x + y)}{\partial x} = \frac{\partial(y\cos x - x^2)}{\partial y} - \textit{уравнение в полных дифференциалах}$$

Пусть верно $\begin{cases} F'_x = y\cos x - x^2 \\ F'_y = \sin x + y \end{cases}$, тогда

$$F = \int (y\cos x - x^2)dx = y\sin x - \frac{x^3}{3} + C_1(y)$$

$$\left(y\sin x - \frac{x^3}{3} + C_1(y)\right)'_y = \sin x + C_1'(y) = \sin x + y, \qquad C_1'(y) = y, \qquad C_1(y) = \frac{y^2}{2}$$

$$\boxed{y\sin x + \frac{y^2}{2} - \frac{x^3}{3} + C = 0}$$

$e)\ y' = \dfrac{x}{y}e^{2x} + y$

Решение.

$$y' - y = \frac{x}{y}e^{2x}$$

$$y' - y = xy^{-1}e^{2x} \quad |: y^{-1}$$

$$\begin{bmatrix} t = \dfrac{1}{y^{-1-1}} = y^2 \\ dy = \dfrac{dt}{2\sqrt{t}} \end{bmatrix}$$

$$y \cdot \frac{dy}{dx} - y^2 = x \cdot e^{2x}$$

$$\frac{dt}{dx} - 2t = 2x \cdot e^{2x} \ \text{ - } \textit{линейное уравнение}$$

$$\frac{dt}{dx} = 2t$$

$$\frac{dt}{t} = 2dx$$

$$\int \frac{dt}{t} = 2\int dx$$

$$\ln|t| = 2x + C$$

$$t = e^{2x} \cdot C(x), \qquad t' = 2e^{2x} \cdot C(x) + e^{2x} \cdot C'(x)$$

$$2e^{2x} \cdot C(x) + e^{2x} \cdot C'(x) - 2e^{x} \cdot C(x) = 2x \cdot e^{2x}$$

$$C'(x) = 2x, \qquad C(x) = x^2 + D, \qquad t = e^{2x} \cdot (x^2 + D)$$

$$y^2 = e^{2x} \cdot (x^2 + D)$$

$$\boxed{y = \pm e^x \cdot \sqrt{x^2 + D}}$$

$f)\ xy' = x\sqrt{y - x^2} + 2y$

Решение.

$$\begin{bmatrix} z = y - x^2 \\ dz + 2xdx = dy \end{bmatrix}$$

$$x \cdot \frac{dy}{dx} = x\sqrt{z} + 2y$$

$$x \cdot \frac{dz + 2xdx}{dx} = x\sqrt{z} + 2z + 2x^2$$

$$x \cdot z' + 2x^2 = x\sqrt{z} + 2z + 2x^2$$

$$x \cdot z' - 2z = x\sqrt{z}$$

$$z' - \frac{2z}{x} = \sqrt{z} \quad |: z^{\frac{1}{2}}$$

$$\begin{bmatrix} t = \dfrac{1}{z^{\frac{1}{2}-1}} = \sqrt{z} \\ dz = 2tdt \end{bmatrix}$$

$$\frac{1}{t} \cdot \frac{dz}{dx} - \frac{2t}{x} = 1$$

$$2 \cdot \frac{dt}{dx} - \frac{2t}{x} = 1 \ \text{ - } \textit{линейное уравнение}$$

$$\frac{dt}{dx} = \frac{t}{x}$$

$$\frac{dt}{t} = \frac{dx}{x}$$

$$\int \frac{dt}{t} = \int \frac{dx}{x}$$

$$\ln|t| = \ln|x| + C$$

$$t = x \cdot C(x), \qquad t' = C(x) + x \cdot C'(x)$$

$$2 \cdot C(x) + 2x \cdot C'(x) - 2x \cdot C(x) = 1$$

$$C'(x) = \frac{1}{2x}, \qquad C(x) = \frac{1}{2}\ln x + D$$

$$t = x \cdot \left(\frac{1}{2}\ln x + D\right), \qquad z = x^2\left(\frac{1}{2}\ln x + D\right)^2, \qquad y = x^2\left(\left(\frac{1}{2}\ln x + D\right)^2 + 1\right)$$

Проверим ноль:

$$z^{\frac{1}{2}} = 0 \ \Rightarrow \ \sqrt{y - x^2} = 0 \ \Rightarrow \ y = x^2$$

$$\boxed{y = x^2\left(\left(\frac{1}{2}\ln x + D\right)^2 + 1\right), \qquad y = x^2}$$

$g)\ xy'(\ln y - \ln x) = y$

Решение.

$$y' \ln\frac{y}{x} = \frac{y}{x}$$

$$\begin{bmatrix} y = tx, \qquad t = t(x) \\ dy = xdt + tdx \end{bmatrix}$$

$$\left(t + x \cdot \frac{dt}{dx}\right)\ln t = t$$

$$x \cdot \frac{dt}{dx} = \frac{t - t \cdot \ln t}{\ln t}$$

$$\int \frac{dx}{x} = \int \frac{\ln t\, dt}{t \cdot (1 - \ln t)}$$

$$\int \frac{\ln t\, dt}{t \cdot (1 - \ln t)} = -\int \frac{\ln t - 1 + 1}{t \cdot (\ln t - 1)}dt \Rightarrow \begin{bmatrix} u = \ln t - 1 \\ dt = tdu \end{bmatrix} \Rightarrow -\int \frac{u + 1}{u}du = -u - \ln|u| + C = -\ln t - \ln|1 - \ln t| + C$$

$$\int \frac{dx}{x} = \ln|x| + C$$

$$\ln|x| + C = -\ln t - \ln|\ln t - 1| + C$$

$$\ln|x| = \ln\frac{C_1}{t \cdot |\ln t - 1|} \ \Rightarrow \ x = \frac{C_1}{t(\ln t - 1)} \ \Rightarrow \ x = \frac{C_1}{\frac{y}{x} \cdot (\ln\frac{y}{x} - 1)}$$

$$\boxed{y \cdot (\ln y - \ln x - 1) = C_1}$$

$h)\ xdy - 2ydx + xy^2(2xdy + ydx) = 0$

Решение.

$$2xdy + ydx = \frac{d(xy^2)}{y}, \qquad xdy - 2ydx = x^3 d\left(\frac{y}{x^2}\right)$$

$$x^3 d\left(\frac{y}{x^2}\right) + \frac{d(xy^2)}{y} = 0$$

$$\begin{bmatrix} u = \dfrac{y}{x^2} \\ v = xy^2 \end{bmatrix}$$

$$\left(\frac{v}{u^2}\right)^{\frac{3}{5}} du + (u^2 v)^{-\frac{1}{5}} dv = 0$$

$$v^{\frac{3}{5}} u^{-\frac{6}{5}} du + v^{-\frac{1}{5}} u^{-\frac{2}{5}} dv = 0$$

$$v^{\frac{4}{5}} u^{-\frac{6}{5}} du + u^{-\frac{2}{5}} dv = 0$$

$$v^{\frac{4}{5}} du + u^{\frac{4}{5}} dv = 0$$

$$\int u^{-\frac{4}{5}} du = \int v^{-\frac{4}{5}} dv$$

$$u^{\frac{1}{5}} = v^{\frac{1}{5}} + C$$

$$\left(\frac{y}{x^2}\right)^{\frac{1}{5}} - (xy^2)^{\frac{1}{5}} + C = 0$$

Не забываем про нулевые решения.

$$\boxed{\left(\frac{y}{x^2}\right)^{\frac{1}{5}} - (xy^2)^{\frac{1}{5}} + C = 0, \qquad y = 0, \qquad x = 0}$$

$i)\ y' - 8x\sqrt{y} = \dfrac{4xy}{x^2 - 1}$

Решение.

$y' - \dfrac{4xy}{x^2 - 1} = 8x\sqrt{y} \quad |: y^{\frac{1}{2}} \qquad (\textit{не забыть проверить } y = 0)$

$$\begin{bmatrix} z = \dfrac{1}{\square} \\ dy = 2zdz \end{bmatrix}$$

$$\frac{dy}{\sqrt{y}dx} - \frac{4x}{x^2 - 1}\sqrt{y} = 8x$$

$$\frac{2dz}{dx} - \frac{4x}{x^2 - 1} \cdot z = 8x \ \text{ - } \textit{линейное уравнение}$$

$$\frac{dz}{dx} = \frac{2x}{x^2 - 1} \cdot z$$

$$\int \frac{dz}{z} = \int \frac{2x}{x^2 - 1}dx$$

$$\ln|z| = \ln|x^2 - 1| + C \ \Rightarrow \ z = C(x) \cdot (x - 1) \ \Rightarrow \ z' = C'(x) \cdot (x^2 - 1) + C(x) \cdot 2x$$

$$C'(x)(x^2 - 1) + C(x) \cdot 2x - \frac{2x}{x^2 - 1} \cdot C(x)(x^2 - 1) = 4x$$

$$C'(x)(x^2 - 1) = 4x, \qquad C'(x) = \frac{4x}{x^2 - 1}, \qquad C(x) = 2\ln|x^2 - 1| + D$$

$$z = (x^2 - 1) \cdot (2\ln|x^2 - 1| + D), \qquad y = z^2$$

$$\boxed{y = (x^2 - 1)^2 \cdot (2\ln|x^2 - 1| + D)^2, \qquad y = 0}$$

2. Решите уравнение $Riccati \quad y' = y^2 - 2e^x y + e^{2x} + e^x$, если частное решение e^x известно.
Затем введите новую неизвестную функцию z, так чтобы $y = z + e^x$.

Решение.

Пусть $z = y - e^x$, тогда $dz + e^x dx = dy$

$$\frac{dy}{dx} = y^2 - 2e^x y + e^{2x} + e^x$$

$$\frac{dz + e^x dx}{dx} = (z + e^x)^2 - 2e^x(z + e^x) + e^{2x} + e^x$$

$$z' + e^x = z^2 + e^{2x} + 2ze^x - 2ze^x - 2e^{2x} + e^{2x} + e^x \quad |: z^2$$

$$\begin{bmatrix} t = \dfrac{1}{z} \\ -\dfrac{1}{z^2}dz = dt \end{bmatrix}$$

$$\frac{dz}{z^2 dx} = 1 \ \Rightarrow \ -\frac{dt}{dx} = 1 \ \Rightarrow \ t' = 1 \ \Rightarrow \ t = -x + C \ \Rightarrow \ z = \frac{1}{-x + c}$$

$$\boxed{y = \frac{1}{-x + C} + e^x}$$