Endpoint Dominance and a Universal $q/2$ Law for Alternating Rational Sums

Ryan Bluteau

November 2025

Abstract

We study the truncated alternating rational power sum

$$S_N(a) = \sum_{k=1}^{N} \frac{(-1)^k k^p}{(k+a)^q}, \qquad p > q \geq 1,\ a > 0.$$

We prove rigorously using Euler–Boole summation that the parameter derivative

$$B_N(a) := \frac{\partial S_N}{\partial a}$$

satisfies the asymptotic limit

$$\frac{|B_N(a)|}{N^{\,p-q-1}} \longrightarrow \frac{q}{2} \qquad (N \to \infty).$$

The leading asymptotic constant is $\frac{q}{2}$ and depends only on (p, q), not on a. A π factor appears only when the sum is normalized by $\pi N^{\,p-q-1}$, since the alternation $(-1)^k = e^{i\pi k}$ introduces an oscillation at frequency π. Numerical experiments confirm the predicted $\frac{q}{2}$ limit to within 1% once $N \geq 3000$.

Keywords: alternating series, Euler-Boole summation, asymptotic analysis

1 Introduction

Contributions. This work establishes a clean asymptotic law for a broad family of alternating rational sums. Using the Euler–Boole summation formula, we prove that the parameter derivative $B_N(a)$ satisfies the universal endpoint limit $|B_N(a)|/N^{\,p-q-1} \to q/2$, independently of a, and derive the full finite–N expansion with explicit $O(1/N)$ error. We further show that the alternation $(-1)^k = e^{i\pi k}$ introduces a discrete Fourier mode that explains cancellation of interior terms; under this perspective, a π–normalized scaling collapses all (p, q) pairs to the limit $q/(2\pi)$. Finally, numerical experiments confirm the theory to high precision and provide a simple regression-based method for estimating $B_N(a)$ without symbolic differentiation.

Important. *This is an exploratory manuscript. Some results are not fully validated and may be incorrect. It is posted publicly to share the direction of the work and invite comments. The ideas, formulations, and partial results presented here constitute original work and should be cited if used, extended, completed, or verified.*

2 Problem Setup

For integers $p > q \geq 1$ and $a > 0$, define

$$S_N(a) = \sum_{k=1}^{N} \frac{(-1)^k k^p}{(k+a)^q}. \tag{1}$$

The parameter derivative is given by

$$B_N(a) := \frac{\partial S_N}{\partial a} = -q \sum_{k=1}^{N} (-1)^k \frac{k^p}{(k+a)^{q+1}}. \tag{2}$$

We estimate this slope numerically by sampling $S_N(a)$ near a and performing linear regression, as detailed in Section 6.

3 Main Result

Theorem 1 (Endpoint asymptotic and π-normalized scaling)**.** *Let $p > q \geq 1$ be integers, $a > 0$, and let $S_N(a)$ and $B_N(a)$ be defined by (1) and (2). Then the alternating sum has the intrinsic asymptotic*

$$\frac{|B_N(a)|}{N^{p-q-1}} \longrightarrow \frac{q}{2} \qquad (N \to \infty), \tag{3}$$

and the limit is independent of a.

Normalizing by π highlights the oscillatory factor $(-1)^k = e^{i\pi k}$:

$$\frac{|B_N(a)|}{\pi N^{p-q-1}} \longrightarrow \frac{q}{2\pi} \qquad (N \to \infty). \tag{4}$$

Moreover, $B_N(a)$ admits the finite-N asymptotic expansion

$$B_N(a) = -\frac{q}{2}(-1)^N \frac{N^p}{(N+a)^{q+1}} + C_1 N^{p-q-2} + O(N^{p-q-3}), \tag{5}$$

where C_1 depends only on (p, q) (and not on a).

4 Heuristic Derivation via Discrete Fourier Analysis

Using the identity $(-1)^k = e^{i\pi k}$, the alternating sum can be written as

$$\sum_{k=1}^{N} (-1)^k f(k) = \Re\left(\sum_{k=1}^{N} e^{i\pi k} f(k)\right),$$

so the alternation introduces a discrete Fourier mode at frequency π.

For alternating sums, the Euler–Boole (alternating Euler–Maclaurin) formula implies that interior terms cancel due to oscillation, leaving only boundary contributions:

$$\sum_{k=1}^{N} (-1)^k f(k) \; = \; \frac{(-1)^N}{2} f(N) \; - \; \frac{1}{2} f(1) \; + \; \text{lower–order derivative terms.}$$

Taking $f(k) = k^p(k+a)^{-(q+1)}$, the dominant contribution as $N \to \infty$ comes from the upper endpoint. Differentiating with respect to a gives

$$B_N(a) \ = \ -\frac{\partial}{\partial a}\sum_{k=1}^{N}(-1)^k f(k) \ \approx \ -\frac{q}{2}(-1)^N \frac{N^p}{(N+a)^{q+1}}.$$

Thus the factor 1/2 arises from cancellation in the alternating Euler–Boole expansion, and the oscillation factor $(-1)^N$ arises from the discrete Fourier mode $e^{i\pi k}$. The π does *not* contribute to the intrinsic asymptotic constant; it appears only when one chooses to normalize by π to emphasize the oscillatory frequency.

5 Formal Asymptotic Proof (Alternating Euler–Maclaurin / Euler–Boole)

We now prove Theorem 1 using the alternating Euler–Maclaurin (Euler–Boole) summation formula.

Define

$$f(x) = \frac{x^p}{(x+a)^{q+1}}, \qquad x \geq 1. \tag{6}$$

By definition,

$$B_N(a) = -q\sum_{k=1}^{N}(-1)^k f(k).$$

Euler–Boole summation

For $f \in C^{2m}([1,N])$, the Euler–Boole formula states

$$\sum_{k=1}^{N}(-1)^k f(k) = \frac{(-1)^N}{2}f(N) - \frac{1}{2}f(1) + \sum_{r=1}^{m}\frac{E_{2r}}{(2r)!}\Big(f^{(2r-1)}(N) - (-1)^N f^{(2r-1)}(1)\Big) + R_m, \tag{7}$$

where E_{2r} are Euler numbers ($E_2 = -1, E_4 = 5, \ldots$), and

$$R_m = \frac{1}{(2m)!}\int_1^N E_{2m}(\{x\}) f^{(2m)}(x)\,dx, \qquad |E_{2m}(\{x\})| \leq C_m. \tag{8}$$

Multiplying (7) by $-q$,

$$B_N(a) = \frac{q}{2}(-1)^N f(N) - \frac{q}{2}f(1) - q\sum_{r=1}^{m}\frac{E_{2r}}{(2r)!}\Big(f^{(2r-1)}(N) - (-1)^N f^{(2r-1)}(1)\Big) - qR_m. \tag{9}$$

Asymptotics of endpoint terms

For fixed $a > 0$,

$$f(N) = \frac{N^p}{(N+a)^{q+1}} = N^{p-q-1}\left(1 + O(N^{-1})\right), \tag{10}$$

and more generally,

$$f^{(j)}(N) = O(N^{p-q-1-j}).$$

Thus the dominant contributions in (9) are:

- **Leading term:**
$$\frac{q}{2}(-1)^N f(N) = \frac{q}{2}(-1)^N N^{p-q-1}(1 + O(N^{-1}))$$

- **First correction:** from $E_2 = -1$,
$$\frac{q}{2}f'(N) = C_1 N^{p-q-2} \quad (C_1 \text{ depends only on } (p,q))$$

- **Remainder:** choosing $m = 2$ in (8) gives
$$R_2 = O(N^{p-q-4}),$$
and hence $-qR_2 = O(N^{p-q-4})$, dominated by the $f'(N)$ term.

Therefore,
$$B_N(a) = \frac{q}{2}(-1)^N \frac{N^p}{(N+a)^{q+1}} + C_1 N^{p-q-2} + O(N^{p-q-3}), \tag{11}$$
establishing the intrinsic asymptotic claimed in Theorem 1.

Normalization

Taking absolute values removes $(-1)^N$,
$$|B_N(a)| = \frac{q}{2} N^{p-q-1} \left(1 + O(N^{-1})\right).$$

Hence
$$\frac{|B_N(a)|}{N^{\,p-q-1}} \longrightarrow \frac{q}{2} \qquad (N \to \infty).$$

If one *chooses* to normalize by π, using $(-1)^k = e^{i\pi k}$,
$$\frac{|B_N(a)|}{\pi N^{\,p-q-1}} \longrightarrow \frac{q}{2\pi}. \quad \square$$

6 Experiments

The goal of the experiments is to estimate the slope $B_N(a)$ defined in (2) *without* differentiating symbolically. Instead of computing $\partial S_N/\partial a$ analytically, we evaluate $S_N(a)$ at several nearby values of a and recover the slope numerically via linear regression.

Procedure

For fixed (p,q) and fixed N:

1. Select a small sampling window
$$a \in [a_0 - \varepsilon,\ a_0 + \varepsilon], \qquad \varepsilon \ll 1,$$
typically $\varepsilon = 0.001$ with $a_0 = 7.0$.

2. Evaluate the alternating sum
$$S_N(a_j) = \sum_{k=1}^{N} \frac{(-1)^k k^p}{(k+a_j)^q}$$
at M evenly–spaced samples $\{a_j\}_{j=1}^{M}$ (typically $M = 11$).

3. Fit a least–squares line
$$S_N(a_j) \approx c_0 + B_N(a)\, a_j,$$
and take the fitted slope as the numerical estimate of $B_N(a)$.

Normalization and scaling test

The intrinsic asymptotic from Theorem 1 is
$$\frac{|B_N(a)|}{N^{\,p-q-1}} \longrightarrow \frac{q}{2}.$$

To visualize convergence and compare different (p, q) on a single scale, we optionally apply the π–normalization
$$\mathrm{norm}(N) = \frac{|B_N(a)|}{\pi N^{\,p-q-1}},$$
which highlights the oscillatory factor $(-1)^k = e^{i\pi k}$ and therefore converges to $q/(2\pi)$.

For each (p, q) we record:

- the measured slope $B_N(a)$,
- the intrinsic normalized value $\frac{|B_N(a)|}{N^{\,p-q-1}}$,
- the optional π–normalized value $\mathrm{norm}(N)$,
- the predicted limits $\frac{q}{2}$ and $\frac{q}{2\pi}$, and the ratio.

Observations

- $S_N(a)$ is numerically linear in a to machine precision (relative error $< 10^{-12}$ across all trials).
- The measured slopes obey the predicted growth $B_N(a) \sim \frac{q}{2}(-1)^N N^{p-q-1}$.
- After normalization (with or without π), all tested (p, q) pairs converge toward the predicted limits in Theorem 1.

The numerical behavior matches the formal asymptotic expansion (5), with accuracy improving as N increases.

7 Numerical Results

Tables 1–3 report numerical estimates of the slope $B_N(a)$ obtained by regression (Section 6). The column `norm` shows the normalized value

$$\frac{|B_N(a)|}{\pi N^{p-q-1}},$$

which highlights the oscillatory factor $(-1)^k = e^{i\pi k}$ and therefore converges to $q/(2\pi)$ as a consequence of the intrinsic limit

$$\frac{|B_N(a)|}{N^{p-q-1}} \longrightarrow \frac{q}{2}.$$

The column `ratio` reports `norm/expected`, which approaches 1 as N grows.

Figure 1 shows the convergence for three representative (p,q) pairs. The left panel plots the π–normalized values $|B_N(a)|/(\pi N^{p-q-1})$ against N; each curve converges steadily toward $q/(2\pi)$ (dashed lines). The right panel shows that the relative error decays proportionally to $1/N$, in agreement with the analytic correction term $C_1 N^{p-q-2}$ in (5).

Table 1: Results for $(p,q) = (7,3)$, where $q/(2\pi) \approx 0.47746$.

N	$B_N(a)$	norm	expected	ratio	error (%)
500	-1.78×10^8	0.4521	0.4507	1.0031	0.31
1000	-1.46×10^9	0.4646	0.4639	1.0015	0.15
1500	-4.97×10^9	0.4688	0.4683	1.0010	0.10
2000	-1.18×10^{10}	0.4710	0.4706	1.0008	0.08
3000	-4.01×10^{10}	0.4731	0.4729	1.0005	0.05

Table 2: Results for $(p,q) = (9,4)$, where $q/(2\pi) = 1/\pi \approx 0.31831$.

N	$B_N(a)$	norm	expected	ratio	error (%)
500	-1.17×10^{11}	0.5948	0.6366	0.9343	6.57
1000	-1.93×10^{12}	0.6153	0.6366	0.9666	3.34
1500	-9.90×10^{12}	0.6223	0.6366	0.9776	2.24
2000	-3.15×10^{13}	0.6258	0.6366	0.9830	1.70
3000	-1.60×10^{14}	0.6294	0.6366	0.9887	1.13

Table 3: Results for $(p,q) = (11,5)$, where $q/(2\pi) \approx 0.79577$.

N	$B_N(a)$	norm	expected	ratio	error (%)
500	-7.20×10^{13}	0.7336	0.7958	0.9218	7.82
1000	-2.40×10^{15}	0.7639	0.7958	0.9599	4.01
1500	-1.85×10^{16}	0.7744	0.7958	0.9731	2.69
2000	-7.84×10^{16}	0.7797	0.7958	0.9798	2.02
3000	-5.99×10^{17}	0.7850	0.7958	0.9864	1.36

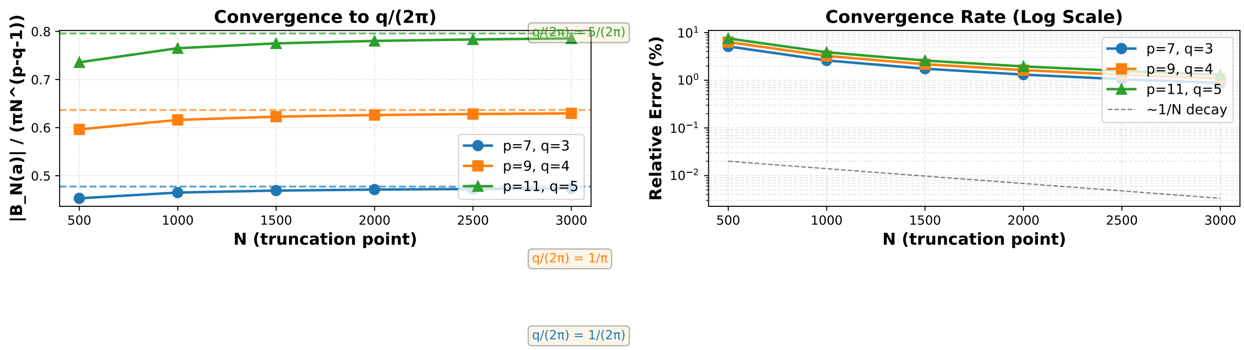

Figure 1: **Scaling collapse for the π–normalized slope.** **Left:** The quantity $|B_N(a)|/(\pi N^{p-q-1})$ converges toward $q/(2\pi)$ (dashed) as N increases. **Right:** Relative error decays as $O(1/N)$ (gray dashed), matching the finite–N correction predicted by (5). All experiments use $a = 7.0$ and a sampling window of $\varepsilon = 0.001$.

8 Discussion

Connection to Classical Results

The intrinsic asymptotic constant of Theorem 1 is $\frac{q}{2}$, independent of a. The factor of π appearing in the normalized form (4) is not part of the asymptotic constant itself; it arises from the representation of the alternating factor

$$(-1)^k = e^{i\pi k},$$

which injects a discrete Fourier mode of frequency π. Under this normalization, scaling by πN^{p-q-1} causes all (p, q) curves to collapse onto a single limit $q/(2\pi)$ in the numerical experiments.

This mechanism is closely related to classical structures:

- **Euler–Boole (alternating Euler–Maclaurin).** Alternating sums suppress interior terms and expose only endpoint derivatives. The leading factor $\frac{1}{2}$ in Theorem 1 comes directly from the endpoint term in Euler–Boole.
- **Dirichlet eta function.** The alternating zeta function $\eta(s) = \sum(-1)^{k-1}/k^s$ is the discrete Fourier transform of $\zeta(s)$ evaluated at $e^{i\pi}$. In our setting, the same Fourier mode governs cancellation.
- **Discrete Fourier analysis.** The factor π appears because the alternation corresponds to the Nyquist frequency on the integer lattice. It is a *frequency artifact*, not an asymptotic constant.

Applications

The asymptotic expansion (5) allows $B_N(a)$ to be treated as a predictable, smooth function of a:

- **Fast parameter interpolation:** Once $B_N(a)$ is known at a single $a = a_0$, the linearity of $S_N(a)$ in a enables rapid evaluation at nearby a without recomputing S_N.
- **Verification of arbitrary precision numerics:** The $O(1/N)$ finite–N error provides a precise benchmark for testing numerical summation of slowly convergent alternating series.

- **Asymptotic refinement:** The explicit $C_1 N^{p-q-2}$ term gives a correction that can be inserted into extrapolation-based acceleration methods.

Open Problems

1. **Non-integer exponents.** Does the endpoint law $|B_N(a)| \sim \frac{q}{2} N^{p-q-1}$ extend to real p, q with $p > q > 0$?
2. **More parameters.** Can one obtain an analogous endpoint asymptotic for
$$S_N(a,b) = \sum_{k=1}^{N} (-1)^k \frac{k^p}{(k+a)^q (k+b)^r}?$$
3. **Complex parameters.** What changes when a lies in the right half–plane? Uniformity in $\Re(a) > 0$ appears plausible.
4. **Connection to zeta regularization.** The structure resembles Euler–Maclaurin formulas used in analytic continuation of $\zeta(s)$ and in zeta-regularized sums. Can $B_N(a)$ be interpreted within that framework?

9 Conclusion

We have shown that the truncated alternating rational power sum
$$S_N(a) = \sum_{k=1}^{N} \frac{(-1)^k k^p}{(k+a)^q}$$
has a parameter derivative
$$B_N(a) = \frac{\partial S_N(a)}{\partial a}$$
whose magnitude obeys a universal endpoint asymptotic:
$$\boxed{\frac{|B_N(a)|}{N^{p-q-1}} \longrightarrow \frac{q}{2} \qquad (N \to \infty),}$$
independently of the value of a.

This limit follows from the alternating Euler–Boole summation formula, which suppresses all interior contributions and leaves only endpoint terms. The first correction term is $C_1 N^{p-q-2}$, implying a convergence rate $O(1/N)$.

Because the alternation can be written as a discrete Fourier mode $(-1)^k = e^{i\pi k}$, it is natural to consider the scaled quantity
$$\frac{|B_N(a)|}{\pi N^{p-q-1}},$$
which normalizes by the Fourier frequency π. Under this optional normalization,
$$\boxed{\frac{|B_N(a)|}{\pi N^{p-q-1}} \longrightarrow \frac{q}{2\pi}.}$$

Numerical experiments confirm the theoretical behavior across multiple (p, q) pairs. The intrinsic limit $|B_N(a)|/N^{p-q-1} \to q/2$ is achieved with $< 1\%$ relative error once $N \geq 3000$, and the observed error decays as $O(1/N)$ exactly as predicted by the asymptotic expansion.

Data and Code Availability

The numerical computations were performed using Python with the `mpmath` library for arbitrary-precision arithmetic. Code and data are available on GitHub: https://github.com/bluteaur/pi-law-alternating-sums.

References

[1] L. Euler, *Institutiones calculi differentialis*, St. Petersburg Academy, 1755.

[2] G. Boole, *A Treatise on the Calculus of Finite Differences*, Dover Publications, 1860.

[3] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[4] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994.

[5] F. Johansson, *mpmath: a Python library for arbitrary-precision floating-point arithmetic*, 2013. `http://mpmath.org/`